Université Denis Diderot **2012-2013**
Mathématiques M1 **Spectral Theory**

Examen du 16 janvier 2013
Durée 3 heures. Documents interdits.

---

1. Let $\mathcal{H}$ be complex Hilbert space, $A \in \mathcal{L}(\mathcal{H})$ and $P \in \mathbb{C}[X]$.
   (a) Recall the relation which exists between $\mathrm{Sp}(P(A))$ and $\mathrm{Sp}A$ for a self-adjoint operator A. Show that the same result holds for an arbitrary bounded operator.
   (b) Prove that if $P(A) = 0$, then $\mathrm{Sp}A$ is contained in the set of roots of P.
2. Let $H = L^2(\mathbb{R}, \lambda)$ be the Hilbert space of complex-valued square integrable functions with respect to the Lebesgue measure. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable and λ-locally square integrable function on $\mathbb{R}$ (i.e. $\int_K |f|^2 d\lambda < \infty$ for all $K \subset \mathbb{R}$ compact). Define the unbounded operator M on H :

$$\mathcal{D}(M) = \{\xi \in H : \int |f\xi|^2 d\lambda < \infty\} \quad (M\xi)(x) = f(x)\xi(x) \quad \xi \in \mathcal{D}(M),\ x \in \mathbb{R}.$$

   (a) Show that M is densely defined.
   (b) Let T be the unbounded operator on H defined by

$$\mathcal{D}(T) = \mathcal{D}(M) \quad (T\xi)(x) = \overline{f(x)}\xi(x) \quad \xi \in \mathcal{D}(T),\ x \in \mathbb{R}.$$

   Show that T is densely defined and $M = T^*$. Deduce that M is closed.
   (c) Compute M^*.
   (d) Show that $\mathrm{sp}(M) = \mathrm{EssIm}(f)$ where

$$\mathrm{EssIm}(f) = \{\lambda \in \mathbb{C} : \forall \epsilon > 0\ \lambda(f^{-1}(B(\lambda, \epsilon))) > 0\}.$$

   (e) Show that if f is continuous then $\mathrm{sp}(M) = \overline{f(\mathbb{R})}$.
   (f) Let $A \subset \mathbb{R}$ be a Borel subset such that $\lambda(A) < \infty$, ν be the finite measure on $\mathbb{R}$ defined by $\nu(B) = \lambda(A \cap B)$ for all B Borel subset of $\mathbb{R}$. Observe that the integral with respect to ν is

$$\int g d\nu = \int_A g d\lambda.$$

   Let $f_*(\nu)$ the measure image of ν by f i.e., $f_*(\nu)(B) = \nu(f^{-1}(B))$ for all Borel subset $B \subset \mathbb{C}$. Let $\xi = 1_A \in H$. Suppose that f is bounded and continuous. Show that M is bounded and the spectral measure μ_ξ of M associated to ξ is $\mu_\xi = f_*(\nu)$.
3. Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be *self-adjoint*.
   (a) Show that : $\forall n \in \mathbb{N}$, $||T^{2^n}|| = ||T||^{2^n}$.
   (b) Show by recursion that the following property $\mathcal{P}(n)$ is true for any $n \in \mathbb{N}$:

$$\mathcal{P}(n) : \quad \forall k \in \mathbb{N} \text{ such that } 0 \le k \le 2^n, \quad ||P^k|| = ||P||^k.$$

   [Hint : for $2^n < k \le 2^{n+1}$, consider $\ell := 2^{n+1} - k$ and $P^\ell P^k$.]

   Conclusion : we have shown that, for any **self-adjoint** operator $P \in \mathcal{L}(\mathcal{H})$, $||P^n|| = ||P||^n$, $\forall n \in \mathbb{N}$.

4. Let $\mathcal{H}$ be a complex Hilbert space. The goal of this exercise is to show that there are no *bounded* operators $P, Q \in \mathcal{L}(\mathcal{H})$ such that $[P, Q] = 1_{\mathcal{H}}$ (where $1_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$ is the identity operator) and $[P, Q] := PQ - QP$.

   (a) Assume that $\mathcal{H}$ has a finite dimension n. Show, by using a simple argument, that there are no operators $P, Q \in \mathcal{L}(\mathcal{H})$ such that $[P, Q] = 1_{\mathcal{H}}$.

   (b) In the following we assume that the dimension of $\mathcal{H}$ is infinite. Show that, for any pair of operators $P, Q \in \mathcal{L}(\mathcal{H})$, we have :

   $$\forall n \in \mathbb{N}^*, \quad [P^n, Q] = \sum_{j=1}^{n} P^{n-j}[P, Q]P^{j-1}.$$

   (c) We argue by contradiction and we assume that there exists operators $P, Q \in \mathcal{L}(\mathcal{H})$ such that $[P, Q] = 1_{\mathcal{H}}$. By using the result of the previous question and of the previous exercise, show that, for any $n \in \mathbb{N}^*$,

   $$2||P||\ ||Q|| \geq n$$

   and conclude to a contradiction.

   (d) Give an example of a complex Hilbert space $\mathcal{H}$ and two *non bounded* operators P and Q such that there exists a dense vector subspace $V \subset \mathcal{H}$ such that $\forall \varphi \in V$, $[P, Q]\varphi = \varphi$

5. Let $\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C}) \simeq \ell^2(\mathbb{Z})$. We denote by $(\epsilon_n)_{n \in \mathbb{Z}}$ the canonical Hilbertian Hermitian orthogonal basis of $\mathcal{H}$ (i.e. ϵ_n is the sequence which vanishes for all relative integer, excepted for n, for which it takes the value 1). We note $L, R \in \mathcal{L}(\mathcal{H})$ the operators defined by :

   $$L\epsilon_n = \epsilon_{n-1} \quad \text{et} \quad R\epsilon_n = \epsilon_{n+1}, \quad \forall n \in \mathbb{Z}$$

   and $A := L + R \in \mathcal{L}(\mathcal{H})$.

   (a) Compute L^* and R^*. Deduce that A is self-adjoint.

   (b) We note $U : \ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{R}/\mathbb{Z})$ the unitary operator defined by : $(U\epsilon_n)(\theta) = e^{i2\pi n\theta}$, $\forall n \in \mathbb{Z}$ (Fourier series isomorphism). For all function $m \in L^\infty(\mathbb{R}/\mathbb{Z})$ we note $\widehat{m} \in \mathcal{L}\left(L^2(\mathbb{R}/\mathbb{Z})\right)$ the multiplication operator defined by :

   $$\forall f \in L^2(\mathbb{R}/\mathbb{Z}), \quad (\widehat{m}f)(\theta) = m(\theta)f(\theta), \quad \text{p.p.}$$

   Find ULU^{-1} and URU^{-1} and show that they coincide with multiplication operators by functions to be precised. Deduce UAU^{-1}.

   (c) Let $\psi \in \mathcal{H}$ be different of 0 and $g := U\psi$. We note $F_\psi := \{P(A)\psi |\ P \in \mathbb{C}[X]\} = \mathrm{Vect}_{\mathbb{C}}\{A^n\psi |\ n \in \mathbb{N}\}$.

   Show that the UF_ψ, the image of F_ψ by U, is equal to : $\mathcal{P}_+ g := \{fg |\ f \in \mathcal{P}_+\}$, where $\mathcal{P}_+$ is the subspace of $L^2(\mathbb{R}/\mathbb{Z})$ of polynomials in $e^{i2\pi\theta}$ and $e^{-i2\pi\theta}$ which are *even* function of θ.

   (d) Let ψ and g be as in the preceding question. We define $h \in L^2(\mathbb{R}/\mathbb{Z})$ by :

   $$h(\theta) = 2i \sin(2\pi\theta)\overline{g(-\theta)}.$$

   Show that, $\forall f_1, f_2 \in \mathcal{P}_+$, $\langle f_1 g, f_2 h\rangle_{L^2} = 0$.

   (e) Deduce from the preceding questions that A does not admit a cyclic vector.