Université Denis Diderot — **2012-2013**
Mathématiques M1 — **Spectral Theory**

Partiel

Durée 2 heures. Documents interdits.

Notations. Let H be a Hilbert space. We denote by $\mathrm{sp}(T)$ the spectrum of T and by $\mathrm{sp}_p(T)$ the point spectrum of T.

1. Let $a = (a_n)_{n\in\mathbb{N}}$ be a bounded sequence of complex numbers. Define $T_a e_n = a_n e_n$, where $(e_n)_{n\in\mathbb{N}}$ is the canonical orthonormal basis of $l^2(\mathbb{N})$.
   (a) Show that T_a extends uniquely to a linear bounded operator on $l^2(\mathbb{N})$.
   (b) Compute $\mathrm{Sp}_p(T_a)$.
   (c) Give a necessary and sufficient condition on a for T_a to be invertible.
   (d) Compute $\mathrm{Sp}(T_a)$.
   (e) Give a necessary and sufficient condition on a for T_a to be compact.
2. Let q be a continuous function on $[0,1]^2$ and μ a probability measure on $[0,1]$. We define a bounded linear operator T_q from $C([0,1])$ to $C([0,1])$ by

$$(T_q f)(x) = \int_0^1 q(x,y)f(y)d\mu(y) \quad \text{for all } f \in C([0,1]).$$

   (a) Show that if q is a polynomial function, then T_q is a finite rank operator on $C([0,1])$.
   (b) Deduce that T_q is compact.
   (c) Show that

$$\|T_q\| = \max_{x\in[0,1]} \int_0^1 |q(x,y)|d\mu(y)$$

3. Let $H = \mathrm{L}^2([0,1],\lambda)$ where λ is the Lebesgue measure. Define, for $\xi \in H$,

$$(T\xi)(x) = \int_0^x y(1-x)\xi(y)dy + \int_x^1 x(1-y)\xi(y)dy.$$

   (a) Show that T is a compact self-adjoint operator on H.
   (b) Show that if ξ is continuous, then $G = T\xi \in C^2$ and G satisfies the differential equation $G'' = -\xi$ with the boundary conditions $G(0) = G(1) = 0$.
   (c) Compute $\mathrm{sp}_p(T)$ and $\mathrm{sp}(T)$.