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HARMONIC MAPS

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Introduction

The subject of harmonic maps is vast and has found many applications, and it would require a very long book to cover all aspects, even superficially. Hence, we have made a choice; in particular, highlighting the key questions of *existence*, *uniqueness* and *regularity* of harmonic maps between given manifolds. Thus we shall survey some of the main methods of global analysis for answering these questions.

We first consider relevant aspects of harmonic functions on Euclidean space; then we give a general introduction to harmonic maps. The core of our work is in Chapters 3–6 where we present the analytical methods. We round of the article by describing how twistor theory and integrable systems can be used to construct many more harmonic maps. On the way, we mention harmonic morphisms: maps between Riemannian manifolds which preserve Laplace's equation; these turn out to be a particular class of harmonic maps and exhibit some properties dual to those of harmonic maps.

More information on harmonic maps can be found in the following articles and books; for generalities: [61, 62, 63, 219], analytical aspects: [21, 88, 103, 118, 131, 133, 135, 189, 204, 194], integrable systems methods: [73, 94, 117], applications to complex and Kähler geometry: [63, 135], harmonic morphisms: [7], and other topics: [64, 231].

Contents

1 Harmonic functions on Euclidean spaces

Harmonic functions on an open domain Ω of $\mathbb{R}^m$ are solutions of the *Laplace equation*

$$\Delta f = 0, \quad \text{where } \Delta := \frac{\partial^2}{(\partial x^1)^2} + \cdots + \frac{\partial^2}{(\partial x^m)^2} \qquad \big((x^1, \cdots, x^m) \in \Omega\big). \tag{1}$$

The operator Δ is called the *Laplace operator* or *Laplacian* after P.-S. Laplace. Equation (1) and the *Poisson equation*[1] $-\Delta f = g$ play a fundamental role in mathematical physics: the Laplacian occurs in *Newton's law of*

[1] We prefer to put a minus sign in front of Δ, since the operator $-\Delta$ has many positivity properties.

gravitation (the gravitational potential U obeys the law $-\Delta U = -4\pi G\rho$, where ρ is the mass density), *electromagnetism* (the electric potential V is a solution of $-\varepsilon_0 \Delta V = \rho$, where ρ is the electric charge distribution), *fluid mechanics* (the right hand side term in the Navier–Stokes system $\frac{\partial u^i}{\partial t} + u^j \frac{\partial u^i}{\partial x^j} + \frac{\partial p}{\partial x^i} = \nu \Delta u^i$ models the effect of the viscosity), and the *heat equation* $\frac{\partial f}{\partial t} = \Delta f$.

The *fundamental solution* $G = G_m$ of the Laplacian is the solution of the Poisson equation $-\Delta G = \delta$ on $\mathbb{R}^m$, where δ is the Dirac mass at the origin, that has the mildest growth at infinity, i.e. $G_2(x) = (2\pi)^{-1}\log(1/r)$ if $m = 2$ and $G_m(x) = 1/\{(m-2)\,|S^{m-1}|\, r^{m-2}\}$ if[2] $m \geq 1$ and $m \neq 2$.

1.1 The Dirichlet principle

The harmonic functions are critical points (also called *extremals*) of the Dirichlet functional

$$E_\Omega(f) := \frac{1}{2}\int_\Omega \sum_{\alpha=1}^{m} \left(\frac{\partial f}{\partial x^\alpha}(x)\right)^2 d^m x = \frac{1}{2}\int_\Omega |df_x|^2 d^m x,$$

where $d^m x := dx^1 \cdots dx^m$. This comes from the fact that, for any smooth function g with compact support in Ω, the *first variation* $(\delta E_\Omega)_f(g) := \lim_{\varepsilon\to 0}\{E_\Omega(f+\varepsilon g) - E_\Omega(f)\}/\varepsilon$ reads

$$(\delta E_\Omega)_f(g) = \int_\Omega \sum_{\alpha=1}^{m} \frac{\partial f}{\partial x^\alpha}\frac{\partial g}{\partial x^\alpha} d^m x = \int_\Omega (-\Delta f) g \, d^m x. \tag{2}$$

This variational formulation (G. Green, 1833; K.F. Gauss, 1837; W. Thomson, 1847; B. Riemann, 1853) reveals that the Laplace operator depends on the (canonical) metric on $\mathbb{R}^m$, since $|df_x|$ is nothing but the Euclidean norm of $df_x \in (\mathbb{R}^m)^*$.

This leads to a strategy to solve the **Dirichlet problem**: given an open bounded subset Ω of $\mathbb{R}^m$ with smooth boundary $\partial\Omega$ and a continuous function $\gamma : \partial\Omega \longrightarrow \mathbb{R}$, find a continuous function $f : \overline{\Omega} \longrightarrow \mathbb{R}$, smooth in Ω, such that

$$\Delta f = 0 \quad \text{in}\, \Omega\,, \quad \text{and} \quad f = \gamma \quad \text{on}\, \partial\Omega\,. \tag{3}$$

The idea to solve (3), named the **Dirichlet principle** by Riemann or the **direct method of the calculus of variations**, is the following: we consider the class of functions $\mathcal{D}_\gamma(\Omega) := \{f \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^0(\overline{\Omega}) |\, f = \gamma \text{ on } \partial\Omega\}$ and we look for a map $\underline{f} \in \mathcal{D}_\gamma(\Omega)$ which minimizes E_Ω among all maps in $\mathcal{D}_\gamma(\Omega)$. *If we can prove the existence of a such a minimizer* $\underline{f}$ in $\mathcal{D}_\gamma(\Omega)$, then by (2), $\underline{f}$ is a critical point of E_Ω and is a solution of the Dirichlet problem (3). The difficulty was to prove *the existence* of a minimizer. Riemann was confident that there was such a minimizer, although K. Weierstrass proved that the method proposed at that time had a gap and many people had given up with this formal idea. Then D. Hilbert proposed in 1900 to replace $\mathcal{D}_\gamma(\Omega)$ by a larger class and this led to a definitive solution formulated by H. Weyl in 1940 [223].

1.2 Existence of solutions to the Dirichlet problem

Several methods may be used to solve the Dirichlet problem including the 'balayage' method by H. Poincaré [173], and the use of sub- and super-solutions by O. Perron [166], see [90]. But the variational approach seems to be the most robust one to generalize to finding harmonic maps between manifolds.

The modern variational proof for the existence of solutions to (3) uses the *Sobolev space* $W^{1,2}(\Omega)$: the set of (classes of) functions f in $L^2(\Omega)$ whose derivatives $\partial f/\partial x^j$ in the distribution sense are in $L^2(\Omega)$. When endowed with the inner product $\langle f,g\rangle_{W^{1,2}} := \int_\Omega \big(fg + \langle df, dg\rangle\big) d^m x$ and norm $||f||^2_{W^{1,2}} := \langle f,f\rangle_{W^{1,2}}$, the space $W^{1,2}(\Omega)$ is a Hilbert space. An important technical point is that $\mathcal{C}^\infty(\overline{\Omega})$ is dense in $W^{1,2}(\Omega)$. Assuming that the

[2]Here $|S^{m-1}| = 2\pi^{m/2}/\,\Gamma(m/2)$ is the $(m-1)$-dimensional Hausdorff measure of the unit sphere S^{m-1}.

boundary $\partial\Omega$ is smooth, there is a unique linear continuous operator defined on $W^{1,2}(\Omega)$ which extends the trace operator $f \longmapsto f|_{\partial\Omega}$ from $\mathcal{C}^\infty(\overline{\Omega})$ to $\mathcal{C}^\infty(\partial\Omega)$. Its image is the Hilbert space $W^{\frac{1}{2},2}(\partial\Omega)$ of (classes of) functions γ in $L^2(\partial\Omega)$ such that $\int_{\partial\Omega}\int_{\partial\Omega}\left(\gamma(x)-\gamma(y)\right)^2/|x-y|^m d\mu(x)d\mu(y) < +\infty$, where $d\mu$ denotes the measure on $\partial\Omega$. So the Dirichlet problem makes sense if the boundary data γ belongs to $W^{\frac{1}{2},2}(\partial\Omega)$, and if we look for f in $W^{1,2}(\Omega)$. Inspired by the Dirichlet principle we define the class $W^{1,2}_\gamma(\Omega) := \{f \in W^{1,2}(\Omega)|\, u|_{\partial\Omega} = \gamma\}$ and we look for a map $f \in W^{1,2}_\gamma(\Omega)$ which minimizes E_Ω: it will be a *weak solution* of the Dirichlet problem.

The solution of this problem when Ω is *bounded* comes from the following. First one chooses a map $f_\gamma \in W^{1,2}_\gamma(\Omega)$, so that $\forall f \in W^{1,2}_\gamma(\Omega)$, $f - f_\gamma \in W^{1,2}_0(\Omega)$. But since Ω is bounded, functions g in $W^{1,2}_0(\Omega)$ obey the *Poincaré inequality* $||g||_{W^{1,2}} \leq C_P||dg||_{L^2}$. This implies the bound $||f||_{W^{1,2}} \leq ||f_\gamma||_{W^{1,2}} + C_P\sqrt{2E_\Omega(f_\gamma)}$ for any $f \in W^{1,2}_\gamma(\Omega)$. A consequence is that $||f||_{W^{1,2}}$ *is bounded as soon as* $E_\Omega(f)$ *is bounded*. Now we are ready to study a minimizing sequence $(f_k)_{k\in\mathbb{N}}$, i.e. a sequence in $W^{1,2}_\gamma(\Omega)$ such that

$$\lim_{k\to\infty} E_\Omega(f_k) = \inf_{W^{1,2}_\gamma(\Omega)} E_\Omega. \tag{4}$$

Because $E_\Omega(f_k)$ is obviously bounded, $||f_k||_{W^{1,2}}$ is also bounded, so that f_k takes values in a compact subset of $W^{1,2}_\gamma(\Omega)$ for the weak $W^{1,2}$-topology. Hence, because of the compactness[3] of the embedding $W^{1,2}(\Omega) \subset L^2(\Omega)$, we can assume that, after extracting a subsequence if necessary, there exists $\underline{f} \in W^{1,2}_\gamma(\Omega)$ such that $f_k \to \underline{f}$ *weakly* in $W^{1,2}$, strongly in L^2 and a.e. on Ω. We write $f_k = \underline{f} + g_k$, so that $g_k \to 0$ *weakly* in $W^{1,2}$, and from the identity $E_\Omega(f_k) = E_\Omega(\underline{f}) + E_\Omega(g_k) + \int_\Omega \langle d\underline{f}, dg_k\rangle$ we obtain

$$\limsup_{k\to\infty} E_\Omega(f_k) = E_\Omega(\underline{f}) + \limsup_{k\to\infty} E_\Omega(g_k). \tag{5}$$

Hence $\limsup_{k\to\infty} E_\Omega(f_k) \geq E_\Omega(\underline{f})$, i.e. E_Ω is *lower semi-continuous*. Comparing (4) and (5) we obtain

$$\left(E_\Omega(\underline{f}) - \inf_{W^{1,2}_\gamma(\Omega)} E_\Omega\right) + \limsup_{k\to\infty} E_\Omega(g_k) = 0.$$

Both terms in this equation are non-negative, hence must vanish: this tells us that $\underline{f}$ is a minimizer of E_Ω in $W^{1,2}_\gamma(\Omega)$ and *a posteriori* that $g_k \to 0$ *strongly* in $W^{1,2}$, i.e. $f_k \to \underline{f}$ *strongly* in $W^{1,2}$.

Hence we have obtained a *weak solution* to the Dirichlet problem. It remains to show that this solution is *classical*, i.e. that $\underline{f}$ is smooth in Ω and that, if γ is continuous, then $\underline{f}$ is continuous on $\overline{\Omega}$ and agrees with γ on $\partial\Omega$. This is the *regularity problem*. Several methods are possible: one may for instance deduce the interior regularity from the identity $\underline{f} = \underline{f} * \chi_\rho$ which holds on $\{x \in \Omega|\, B(x,\rho) \subset \Omega\}$, where $\chi_\rho \in \mathcal{C}^\infty_c(\mathbb{R}^m)$ is rotationally symmetric, has support in $B(0,\rho)$ and satisfies $\int_{\mathbb{R}^m}\chi_\rho = 1$ and $*$ denotes the *convolution* operator given by $f * g(x) = \int_{\mathbb{R}^m} f(x-y)g(y)d^my$. This identity is actually a version of the mean value property (see the next paragraph) valid for weak solutions.

1.3 The mean value property and the maximum principle

Let f be a harmonic function on an open subset Ω, $x_0 \in \Omega$ and $\rho > 0$ such that $B(x_0,\rho) \subset \Omega$. Stokes' theorem gives: $\forall r \in (0,\rho]$, $\int_{\partial B(x_0,r)}(\partial f/\partial r)\, d\mu(x) = \int_{B(x_0,r)} \Delta f d^mx = 0$, where $r = |x - x_0|$. It implies that $⨍_{\partial B(x_0,r)} f := 1/(|S^{m-1}|\, r^{m-1}) \int_{\partial B(x_0,r)} f\, d\mu(x)$ is independent of r. Hence, since f is continuous at x_0, we have $f(x_0) = ⨍_{\partial B(x_0,r)} f$. By averaging further over all spheres $\partial B(x_0,r)$ with $0 < r < \rho$, one deduces that $f(x_0) = ⨍_{B(x_0,r)} f$.

A similar argument works for superharmonic or subharmonic functions: a smooth function $f : \Omega \longrightarrow \mathbb{R}$ is *superharmonic* (resp. *subharmonic*) if and only if $-\Delta f \geq 0$ (resp. $-\Delta f \leq 0$). Then, if f superharmonic (resp. subharmonic) and $B(x_0,\rho) \subset \Omega$, we have $f(x_0) \geq ⨍_{B(x_0,r)} f$ (resp. $f(x_0) \leq ⨍_{B(x_0,\rho)} f$).

[3]By the Rellich–Kondrakov theorem, valid here because Ω is bounded.

The mean value property implies the *maximum* and *minimum principles*: assume that Ω is open, bounded and connected and that f is harmonic on Ω, continuous on $\overline{\Omega}$ and that $x_0 \in \Omega$ is an *interior*(!) point where f is maximal, i.e., $\forall x \in \Omega$, $f(x) \leq f(x_0)$. Then we choose $B(x_0, \rho) \subset \Omega$ and, by the mean value property, $f(x_0) = \fint_{B(x_0,\rho)} f$ or, equivalently, $\int_{B(x_0,\rho)} (f(x_0) - f(x))\, d^m x = 0$. But since f is maximal at x_0 the integrand in this last integral is non-negative and hence must vanish. Thus $f(x) = f(x_0)$ on $B(x_0, \rho)$. So we have shown that $(f|_\Omega)^{-1}(\sup_\Omega f) := \{x \in \Omega |\ f(x) = \sup_\Omega f\}$ is open. It is also closed because f is continuous. Hence since Ω is connected, **either** $(f|_\Omega)^{-1}(\sup_\Omega f) = \Omega$ **and** f **is constant, or** $(f|_\Omega)^{-1}(\sup_\Omega f) = \emptyset$**, which means that** $\sup_\Omega f$ **is achieved on the boundary** $\partial\Omega$ (since $\overline{\Omega}$ is compact). This is the *(strong) maximum principle*.

One sees that the preceding argument still works if we replace the property $f(x_0) = \fint_{B(x_0,\rho)} f$ by $f(x_0) \leq \fint_{B(x_0,\rho)} f$, i.e. if we only assume that f is *subharmonic*. Similarly the *minimum principle* works for *superharmonic* functions.

1.4 Uniqueness and minimality

The uniqueness of solutions to the Dirichlet problem can be obtained as a consequence of the maximum principle: let f_1 and f_2 be two solutions of the Dirichlet problem and let $f := f_2 - f_1$. Since f_1 agrees with f_2 on $\partial\Omega$, the trace of f on $\partial\Omega$ vanishes. But f is also harmonic, and hence satisfies the maximum principle: this implies that $\sup_\Omega f = \sup_{\partial\Omega} f = 0$, so $f \leq 0$ on Ω. Similarly, the minimum principle implies $f \geq 0$ on Ω. Hence $f = 0$, which means that f_1 coincides with f_2.

A straightforward consequence of this uniqueness result is that any solution f of (3) actually coincides with **the** minimizer of E_Ω in $W^{1,2}_\gamma(\Omega)$. One can recover this minimality property directly from the identity

$$\forall g \in W^{1,2}_\gamma(\Omega), \quad E_\Omega(g) = E_\Omega(g - f) + \int_\Omega \langle df, dg\rangle - E_\Omega(f).$$

On using Stokes' theorem twice, $\Delta f = 0$ and the fact that $f|_{\partial\Omega} = g|_{\partial\Omega}$, we obtain

$$\int_\Omega \langle df, dg\rangle = \int_\Omega \text{div}(g\nabla f) = \int_{\partial\Omega} g\frac{\partial f}{\partial n} = \int_{\partial\Omega} f\frac{\partial f}{\partial n} = \int_\Omega \text{div}(f\nabla f) = 2E_\Omega(f). \tag{6}$$

Hence $E_\Omega(g) = E_\Omega(g - f) + E_\Omega(f)$, which implies that f minimizes E_Ω in $W^{1,2}_\gamma(\Omega)$.

1.5 Relation with holomorphic functions

In dimension 2, harmonic functions are closely linked with holomorphic functions. Throughout this article, we shall use the identification $\mathbb{R}^2 \simeq \mathbb{C}$, $(x, y) \longmapsto x + iy$ and the operators

$$\frac{\partial}{\partial z} := \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \overline{z}} := \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

If Ω is an open subset of $\mathbb{C}$, recall that a smooth function $\varphi : \Omega \longrightarrow \mathbb{C}$ is *holomorphic* (rep. *antiholomorphic*) if and only if $\partial\varphi/\partial\overline{z} = 0$ (resp. $\partial\varphi/\partial z = 0$). Then because of the identity $\partial^2/\partial z\partial\overline{z} = \partial^2/\partial\overline{z}\partial z = (1/4)\Delta$ it is clear that, if $\varphi : \Omega \longrightarrow \mathbb{C}$ is holomorphic or antiholomorphic, then $\text{Re}\,\varphi$ and $\text{Im}\,\varphi$ are harmonic functions. Conversely, if we are given a harmonic function $f : \Omega \longrightarrow \mathbb{R}$, then $\partial f/\partial z$ is holomorphic. Moreover if Ω is simply connected the holomorphic function φ defined by $\varphi(z) = 2\displaystyle\int_{z_0}^z \frac{\partial f}{\partial z}(\zeta)\, d\zeta$ satisfies $\dfrac{\partial\varphi}{\partial z} = 2\dfrac{\partial f}{\partial z}$ and $f = \text{Re}\,\varphi + C$, where $C \in \mathbb{R}$ is a constant. The imaginary part of φ provides us with another harmonic function $g := \text{Im}\,\varphi$, the *harmonic conjugate function* of f. Note that some representation formulas for harmonic functions in terms of holomorphic data have been found in dimension three (E.T. Whitakker [224]) and in dimension four (H. Bateman and R. Penrose [8, 165]).

2 Harmonic maps between Riemannian manifolds

2.1 Definition

Throughout the rest of this article, $\mathcal{M} = (\mathcal{M}, g)$ and $\mathcal{N} = (\mathcal{N}, h)$ will denote smooth Riemannian manifolds, without boundary unless otherwise indicated, of arbitrary (finite) dimensions m and n respectively. We denote their Levi-Civita connnections by ${}^g\nabla$ and ${}^h\nabla$ respectively. By an *(open) domain* of $\mathcal{M}$ we mean a non-empty connected open subset of $\mathcal{M}$; if a domain has compact closure, we shall call that closure a *compact domain*. We use the *Einstein summation convention* where summation over repeated subscript-superscript pairs is understood.

We define harmonic maps as the solution to a variational problem which generalizes that in Chapter 1 as follows. Let $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ be a smooth map. Let Ω be a domain of $\mathcal{M}$ with a piecewise $\mathcal{C}^1$ boundary $\partial\Omega$. The *energy* or *Dirichlet integral* of ϕ over Ω is defined by

$$E_\Omega(\phi) = \frac{1}{2}\int_\Omega |d\phi|^2\,\omega_g\,. \tag{7}$$

Here ω_g is the volume measure on $\mathcal{M}$ defined by the metric g, and $|d\phi|$ is the Hilbert–Schmidt norm of $d\phi$ given at each point $x \in \mathcal{M}$ by

$$|d\phi_x|^2 = \sum_{i=1}^m h_{\phi(x)}\big(d\phi_x(e_i), d\phi_x(e_i)\big) \tag{8}$$

where $\{e_i\}$ is an orthonormal basis for $T_x\mathcal{M}$. In local coordinates $(x^1,\dots,x^m)$ on $\mathcal{M}$, $(y^1,\dots,y^n)$ on $\mathcal{N}$,

$$|d\phi_x|^2 = g^{ij}(x)h_{\alpha\beta}(\phi(x))\,\phi_i^\alpha\phi_j^\beta \quad \text{and} \quad \omega_g = \sqrt{|g|}\,dx^1\cdots dx^m\,; \tag{9}$$

here ϕ_i^α denotes the partial derivative $\partial\phi^\alpha/\partial x^i$ where $\phi^\alpha := y^\alpha \circ \phi$, (g_{ij}) denotes the metric tensor on $\mathcal{M}$ with determinant $|g|$ and inverse (g^{ij}), and $(h_{\alpha\beta})$ denotes the metric tensor on $\mathcal{N}$.

By a *smooth (one-parameter) variation* $\Phi = \{\phi_t\}$ *of* ϕ we mean a smooth map $\Phi : \mathcal{M} \times (-\varepsilon, \varepsilon) \to \mathcal{N}$, $\Phi(x,t) = \phi_t(x)$, where $\varepsilon > 0$ and $\phi_0 = \phi$. We say that it is *supported in* Ω if $\phi_t = \phi\ \forall\, t$ on the complement of the interior of Ω. A smooth map $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is called *harmonic* if it is a *critical point* (or *extremal*) of the energy integral, i.e., for all compact domains Ω and all smooth one-parameter variations $\{\phi_t\}$ of ϕ supported in Ω, the *first variation* $\frac{d}{dt}E_\Omega(\phi_t)\big|_{t=0}$ is zero. The first variation is given by

$$(\delta E_\Omega)_\phi(v) := \frac{d}{dt}E_\Omega(\phi_t)\Big|_{t=0} = -\int_{\mathcal{M}} \langle \tau(\phi), v\rangle\,\omega_g\,. \tag{10}$$

Here v denotes the *variation vector field* of $\{\phi_t\}$ defined by $v = \partial\phi_t/\partial t|_{t=0}$, $\langle\cdot,\cdot\rangle$ denotes the inner product on $\phi^{-1}T\mathcal{N}$ induced from the metric on $\mathcal{N}$, and $\tau(\phi)$ denotes the *tension field of* ϕ defined by

$$\tau(\phi) \quad = \quad \text{Trace}\,{}^W\nabla d\phi = \sum_{i=1}^m {}^W\nabla d\phi(e_i,e_i) = \sum_{i=1}^m \big\{{}^\phi\nabla_{e_i}\big(d\phi(e_i)\big) - d\phi({}^g\nabla_{e_i}e_i)\big\}\,. \tag{11}$$

Here ${}^\phi\nabla$ the pull-back of the Levi-Civita connection on $\mathcal{N}$ to the bundle $\phi^{-1}T\mathcal{N}$, and ${}^W\nabla$ the tensor product connection on the bundle $W = T^*\mathcal{M} \otimes \phi^{-1}T\mathcal{N}$ induced from these connections. We see that the tension field is the trace of the *second fundamental form* of ϕ defined by $\beta(\phi) = {}^W\nabla d\phi$, more explicitly, $\beta(\phi)(X,Y) = {}^\phi\nabla_X\big(d\phi(Y)\big) - d\phi({}^g\nabla_X Y)$ for any vector fields X, Y on $\mathcal{M}$. In local coordinates,

$$\tau(\phi)^\gamma \quad = \quad g^{ij}\left(\frac{\partial^2\phi^\gamma}{\partial x^i\partial x^j} - {}^g\Gamma_{ij}^k\frac{\partial\phi^\gamma}{\partial x^k} + {}^h\Gamma_{\alpha\beta}^\gamma(\phi)\frac{\partial\phi^\alpha}{\partial x^i}\frac{\partial\phi^\beta}{\partial x^j}\right) \tag{12}$$

$$= \quad \Delta_g\phi^\gamma + g(\text{grad}\,\phi^\alpha,\,\text{grad}\,\phi^\beta)\,{}^h\Gamma_{\alpha\beta}^\gamma\,. \tag{13}$$

Here ${}^g\Gamma^k_{ij}$ and ${}^h\Gamma^\gamma_{\alpha\beta}$ denote the Christoffel symbols on $(\mathcal{M}, g)$ and $(\mathcal{N}, h)$, respectively, and Δ_g denotes the Laplace–Beltrami operator on functions $f : \mathcal{M} \to \mathbb{R}$ given by

$$\Delta_g f = \text{Trace}\ {}^W\nabla df = \sum_{i=1}^{m} \{e_i(e_i(f)) - ({}^g\nabla_{e_i} e_i) f\}, \tag{14}$$

or, in local coordinates,

$$\Delta_g f = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|}\, g^{ij} \frac{\partial f}{\partial x^j} \right) = g^{ij} \left(\frac{\partial^2 f}{\partial x^i \partial x^j} - {}^g\Gamma^k_{ij} \frac{\partial f}{\partial x^k} \right). \tag{15}$$

Note that $\tau(\phi)$ can be interpreted as the negative of the gradient at ϕ of the energy functional E on a suitable space of mappings, i.e., it points in the direction in which E decreases most rapidly [61, (3.5)]. In local coordinates, the *harmonic equation*

$$\tau(\phi) = 0 \tag{16}$$

is a *semilinear* second-order elliptic system of partial differential equations.

2.2 Examples

We list some important examples of harmonic maps. See, for example, [66, 61, 63, 7] for many more.

1. **Constant maps** $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ and **identity maps** $\text{Id} : (\mathcal{M}, g) \to (\mathcal{M}, g)$ are clearly always harmonic maps

2. **Isometries** are harmonic maps. Further, composing a harmonic map with an isometry on its domain or codomain preserves harmonicity.

3. **Harmonic maps between Euclidean spaces**. A smooth map $\phi : A \to \mathbb{R}^n$ from an open subset A of $\mathbb{R}^m$ is harmonic if and only if each component is a harmonic function, as discussed in the first Chapter.

4. **Harmonic maps to a Euclidean space**. A smooth map $\phi : (\mathcal{M}, g) \to \mathbb{R}^n$ is harmonic if and only if each of its components is a *harmonic function* on $(\mathcal{M}, g)$, as in first chapter. See [46] for recent references.

5. **Harmonic maps to the circle** S^1 are given by integrating harmonic 1-forms with integral periods. Hence, when the domain $\mathcal{M}$ is compact, there are non-constant harmonic maps to the circle if and only if the first Betti number of $\mathcal{M}$ is non-zero. In fact, there is a harmonic map in every homotopy class (see, [7, Example 3.3.8]).

6. **Geodesics**. For a smooth curve, i.e. smooth map $\phi : A \to \mathcal{N}$ from an open subset A of $\mathbb{R}$ or from the circle S^1, the tension field is just the acceleration vector of the curve; hence ϕ *is harmonic if and only if it defines a geodesic* parametrized linearly (i.e., parametrized by a constant multiple of arc length). More generally, a map $\phi : \mathcal{M} \to \mathcal{N}$ is called *totally geodesic* if it maps linearly parametrized geodesics of $\mathcal{M}$ to linearly parametrized geodesics of $\mathcal{N}$, such maps are characterized by the vanishing of their second fundamental form. Since (11) exhibits the tension field as the trace of the second fundamental form, *totally geodesic maps are harmonic*.

7. **Isometric immersions** Let $\phi : (\mathcal{N}, h) \to (\mathcal{P}, k)$ be an isometric immersion. Then its second fundamental form $\beta(\phi)$ of ϕ has values in the normal space and coincides with the usual second fundamental form $A \in \Gamma(S^2T^*\mathcal{N} \otimes N\mathcal{N})$ of $\mathcal{N}$ as an (immersed) submanifold of $\mathcal{P}$ defined on vector fields X, Y on $\mathcal{M}$ by $A(X, Y) = -$ normal component of ${}^h\nabla_X Y$.[4] (Here, by $S^2T^*\mathcal{N}$ we denote the symmetrized tensor product of $T^*\mathcal{N}$ with itself and $N\mathcal{N}$ is the normal bundle of N in P.) In particular, the tension field $\tau(\phi)$ is m times the mean curvature of $\mathcal{M}$ in $\mathcal{N}$ so that ϕ *is harmonic if and only if* $\mathcal{M}$ *is a minimal submanifold of* $\mathcal{N}$.

8. **Compositions** The composition of two harmonic maps is not, in general, harmonic. In fact, the tension field of the composition of two smooth maps $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ and $f : (\mathcal{N}, h) \to (\mathcal{P}, k)$ is given by

$$\tau(f \circ \phi) = df(\tau(\phi)) + \beta(f)(d\phi, d\phi) = df(\tau(\phi)) + \sum_{i=1}^{m} \beta(f)(d\phi(e_i), d\phi(e_i)) \tag{17}$$

[4] The minus sign is often omitted

where $\{e_i\}$ is an orthonormal frame on $\mathcal{N}$. From this we see that *if ϕ is harmonic and f totally geodesic, then $f \circ \phi$ is harmonic*.

9. **Maps into submanifolds**. Suppose that $j : (\mathcal{N}, h) \to (\mathcal{P}, k)$ is an isometric immersion. Then, as above, its second fundamental form A has values in the normal space of $\mathcal{N}$ in P and so from the composition law just discussed, $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is harmonic if and only if $\tau(j \circ \phi)$ is normal to M, and this holds if and only if

$$\tau(j \circ \phi) + \text{Trace}\, A(d\phi, d\phi) = 0 \tag{18}$$

10. **Holomorphic maps**. By writing the tension field in complex coordinates, it is easy to see that *holomorphic (or antiholomorphic) maps $\phi : (\mathcal{M}, g, J^{\mathcal{M}}) \to (\mathcal{N}, h, J^{\mathcal{N}})$ between Kähler manifolds are harmonic* [66].[5]

11. **Maps between surfaces**. Let $\mathcal{M} = (\mathcal{M}^2, g)$ be a *surface*, i.e., two-dimensional Riemannian manifold. Assume it is oriented and let $J^{\mathcal{M}}$ be rotation by $+\pi/2$ on each tangent space. Then $(\mathcal{M}^2, g, J^{\mathcal{M}})$ defines a complex structure on $\mathcal{M}$ so that it becomes a *Riemann surface*; this structure is automatically Kähler. Let $\mathcal{N}$ be another oriented surface. Then from the last paragraph, we see that *any holomorphic or antiholomorphic map from $\mathcal{M}$ to $\mathcal{N}$ is harmonic*.

A smooth map $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ between Riemannian manifolds is called *weakly conformal* if its differential preserves angles at *regular points*—points where the differential is non-zero. Points where the differential is zero are called *branch points*. In local coordinates, a smooth map ϕ is weakly conformal if and only if there exists a function $\lambda : \mathcal{M} \to [0, \infty)$ such that

$$h_{\alpha\beta}\phi_i^\alpha \phi_j^\beta = \lambda^2 g_{ij} \tag{19}$$

Weakly conformal maps between surfaces are locally the same as holomorphic maps and so *weakly conformal maps of surfaces are harmonic*.

12. **Maps from surfaces**. (i) Let $\mathcal{M} = (\mathcal{M}^2, g)$ be a surface and let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map to an arbitrary Riemannian manifold. Then the energy integral (7) is clearly invariant under conformal changes of the metric, and thus so is harmonicity of ϕ. To see this last invariance another way, let (x, y) be *conformal local coordinates*, i.e., coordinates on an open set of $\mathcal{M}$ in which $g = \mu^2(dx^2 + dy^2)$ for some real-valued function μ. Write $z = x + iy$. Then the harmonic equation reads

$$^{\phi}\nabla_{\partial/\partial\bar{z}} \frac{\partial\phi}{\partial z} \equiv {}^{\phi}\nabla_{\partial/\partial z} \frac{\partial\phi}{\partial\bar{z}} = 0 \tag{20}$$

If $\mathcal{M}$ is oriented, then we may take (x, y) to be oriented; the the coordinates $z = x + iy$ give $\mathcal{M}$ the complex structure of the last paragraph. Hence, harmonicity of a map from a Riemann surface is well defined.

Alternatively, from (17) we obtain the slightly more general statement that *the composition of a weakly conformal map $\phi : \mathcal{M} \to \mathcal{N}$ of surfaces with a harmonic map $f : \mathcal{N} \to \mathcal{P}$ from a surface to an arbitrary Riemannian manifold is harmonic*.

For any smooth map $\phi : \mathcal{M}^2 \to (\mathcal{N}, h)$ from an oriented surface, define the Hopf differential by

$$\mathcal{H} = (\phi^* h)^{(2,0)} = h\Big(\frac{\partial\phi}{\partial z}, \frac{\partial\phi}{\partial z}\Big) dz^2 = \frac{1}{4}\left\{ h\Big(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial x}\Big) - h\Big(\frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial y}\Big) + 2i\, h\Big(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\Big)\right\} dz^2 \tag{21}$$

Here we use the complex eigenspace decomposition $\phi^* h = (\phi^* h)^{(2,0)} + (\phi^* h)^{(1,1)} + (\phi^* h)^{(0,2)}$ under the action of $J^{\mathcal{M}}$ on quadratic forms on $T\mathcal{M}$. Note that (i) if *ϕ is harmonic, then $\mathcal{H}$ is a holomorphic quadratic differential*, i.e., a holomorphic section of $\otimes^2 T^*_{1,0}M$;[6]

(ii) *ϕ is conformal if and only if $\mathcal{H}$ vanishes*. It follows that *any harmonic map from the 2-sphere is weakly conformal* [144, 88, 117]. Indeed, when $\mathcal{M}$ is the 2-sphere, $\otimes^2 T^*_{1,0}\mathcal{M}$ has negative degree so that any holomorphic section of it is zero.

[5] A. Lichnerowicz relaxes the conditions on $\mathcal{M}$ and $\mathcal{N}$ for which this is true; see, for example, [61] or [7, Chapter 8].

[6] This is an example of a *conservation law*, see §3.1 for more details and the generalization to higher dimensions.

13. **Minimal branched immersions**. For a weakly conformal map from a surface $(\mathcal{M}^2, g)$, comparing definitions shows that the tension field is a multiple of its mean curvature vector, so that *a weakly conformal map $\phi : (\mathcal{M}^2, g) \to (\mathcal{N}, h)$ is harmonic if and only if its image is minimal at regular points*; such maps are called *minimal branched immersions.* In suitable coordinates, the branch points have the form $z \mapsto \big(z^k + O(z^{k+1}), O(z^{k+1})\big)$ for some $k \in \{2, 3, \dots\}$ [95].

Note also that (ii) the energy of a weakly conformal map $\phi : (\mathcal{M}^2, g) \to (\mathcal{N}, h)$ from a compact surface is equal to its *area*:

$$\mathcal{A}(\phi) = \int_{\mathcal{M}} |d\phi(e_1) \wedge d\phi(e_2)|\, \omega_g \qquad (\{e_1, e_2\} \text{ orthonormal frame}). \tag{22}$$

14. **Harmonic morphisms** are a special sort of harmonic map; we turn to those now.

2.3 Harmonic morphisms

A continuous map $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is called a *harmonic morphism* if, for every harmonic function $f : V \to \mathbb{R}$ defined on an open subset V of $\mathcal{N}$ with $\phi^{-1}(V)$ non-empty, the composition $f \circ \phi$ is harmonic on $\phi^{-1}(V)$. It follows that ϕ is smooth, since harmonic functions have that property, by a classical result of Schwartz [195, Chapter VI, Théorème XXIX]. Further, since any harmonic function on a real-analytic manifold is real analytic [168], harmonic morphisms between real-analytic Riemannian manifolds are, in fact, real analytic.

The subject of harmonic morphisms began with a paper of C. G. J. Jacobi [125], published in 1848. Jacobi investigated when complex-valued solutions to Laplace's equation on domains of Euclidean 3-space remain solutions under post-composition with holomorphic functions in the plane. It follows quickly that such solutions pull back locally defined harmonic functions to harmonic functions, i.e., are harmonic morphisms. A hundred years later came the axiomatic formulation of *Brelot harmonic space*. This is a topological space endowed with a sheaf of 'harmonic' functions characterized by a number of axioms. The morphisms of such spaces, i.e. mappings which pull back germs of harmonic functions to germs of harmonic functions, were confusingly called *harmonic maps* [48]; the term *harmonic morphisms* was coined by B. Fuglede [77].

To keep the number of references manageable, in the sequel we shall often refer to the book [7] which gives a systematic account of the subject, and which may be consulted for a list of original references.

A smooth map $\phi : (\mathcal{M}, g) \to (\mathcal{N}, h)$ is called *horizontally (weakly) conformal* (or *semiconformal*) if, for each $p \in \mathcal{M}$, *either*, (i) $d\phi_p = 0$, in which case we call p a *critical point*, or, (ii) $d\phi_p$ maps the *horizontal space* $\mathcal{H}_p = \{\ker(d\phi_p)\}^\perp$ conformally onto $T_{\phi(p)}\mathcal{N}$, i.e., $d\phi_p$ is surjective and there exists a number $\lambda(p) \neq 0$ such that

$$h\big(d\phi_p(X), d\phi_p(Y)\big) = \lambda(p)^2\, g(X, Y) \qquad (X, Y \in \mathcal{H}_p)\,,$$

in which case we call p a *regular point*. On setting $\lambda = 0$ at critical points, we obtain a continuous function $\lambda : \mathcal{M} \to [0, \infty)$ called the *dilation* of ϕ; note that λ^2 is smooth since it equals $|d\phi|^2/n$. In local coordinates, the condition for horizontal weak conformality is

$$g^{ij} \phi_i^\alpha \phi_j^\beta = \lambda^2 h^{\alpha\beta}\,. \tag{23}$$

Note that this condition is dual to condition (19) weak conformality, see also [7]. We have the following characterization [77, 124]: *a smooth map $\phi : \mathcal{M} \to \mathcal{N}$ between Riemannian manifolds is a harmonic morphism if and only if it is both harmonic and horizontally weakly conformal.* This is proved by (i) showing that there is a harmonic function $f : \mathcal{N} \supset V \to \mathbb{R}$ with any prescribed (traceless) 2-jet; see [7, §4.2]; (ii) applying the formula (17) for the tension field of the composition of ϕ with such harmonic functions f. It follows that *a non-constant harmonic morphism is **(i)** an open mapping, **(ii)** a submersion on a dense open set* — in fact the complement of this, the set of critical points, is a *polar set*.

Regarding the behaviour of a harmonic morphism at a critical point, the *symbol*, i.e. the first non-zero term of the Taylor expansion is a harmonic morphism between Euclidean spaces given by homogeneous polynomials; by studying these it follows that (i) *if $\dim \mathcal{M} < 2 \dim \mathcal{N} - 2$, the harmonic morphism has no critical points, i.e.,* *is submersive*; (ii) *if* $\dim \mathcal{M} = 2 \dim \mathcal{N} - 2$, *the symbol is the cone on a Hopf map* [7, Theorem 5.7.3]. When $\dim \mathcal{M} = 3$ and $\dim \mathcal{N} = 2$, locally [7, Proposition 6.1.5], and often globally [7, Lemma 6.6.3], a harmonic morphism looks like a submersion followed by a holomorphic map of surfaces; the critical set is the union of geodesics. When $\dim \mathcal{M} = 4$ and $\dim \mathcal{N} = 3$, critical points are isolated and the harmonic morphism looks like the cone on the Hopf map $S^3 \to S^2$ [7, §12.1]. In both these cases, there are *global factorization theorems*. In other cases, little is known about the critical points.

The system (16, 23) for a harmonic morphism is, in general, *overdetermined*, so there are no general existence results. However, in many cases, we can establish existence or non-existence as we now detail.

1. **When** $\dim \mathcal{N} = 1$, the equation (23) is automatic, so that *a harmonic morphism is exactly a harmonic map*. If $\mathcal{N} = \mathbb{R}$, it is thus a harmonic function; for $\mathcal{N} = S^1$, see Example 5 of §2.2.

2. **When** $\dim \mathcal{M} = \dim \mathcal{N} = 2$, the equation (16) is implied by the equation (23), so that *the harmonic morphisms are precisely the weakly conformal maps*; see Example 9 of §2.2 for a discussion of such maps.

3. **When** $\dim \mathcal{N} = 2$ **and** $\dim \mathcal{M}$ **is arbitrary**, we have a number of special properties which are dual to those for (weakly conformal) harmonic maps *from* surfaces: (i) *conformal invariance in the codomain*: if we replace the metric on the codomain by a conformally equivalent metric, or post-compose the map with a (weakly) conformal map of surfaces, then it remains a harmonic morphism; (ii) a *variational characterization*: harmonic morphisms are the critical points of the energy when both the map and the metric on the horizontal space are varied, see [7, Corollary 4.3.14]; (iii) a non-constant map is a harmonic morphism if and only if it is horizontally weakly conformal and, at regular points, its fibres are minimal [6], i.e., at regular points, the fibres form a *conformal foliation by minimal submanifolds*.

4. **When** $\dim \mathcal{N} = 2$ **and** $\dim \mathcal{M} = 3$, if $\mathcal{M}$ has constant curvature, there are many harmonic morphisms locally given by a sort of *Weierstrass formula* [7, Chapter 6]. Globally, there are few, for example, when $\mathcal{M} = \mathbb{R}^3$, only orthogonal projection from $\mathbb{R}^3$ to $\mathbb{R}^2$ followed by a weakly conformal map. If $\mathcal{M}$ does not have constant curvature, the presence of a harmonic morphism implies some symmetry of the Ricci tensor and, locally, there can be at most two non-constant harmonic morphisms (up to post-composition with weakly conformal maps), and, none for most metrics including that of the Lie group Sol. As for *global* topological obstructions, a harmonic morphism from a compact 3-manifold gives it the structure of a *Seifert fibre space* [7, §10.3].

5. **When** $\dim \mathcal{N} = 2$ **and** $\dim \mathcal{M} = 4$, if $\mathcal{M}$ is Einstein, there is a *twistor correspondence* between harmonic morphisms to surfaces and *Hermitian structures on* $\mathcal{M}$. There are curvature obstructions for the local existence of such Hermitian structures. See [7, Chapter 7].

6. **When** $\dim \mathcal{N} = 2$ **and** $\mathcal{M}$ **is a symmetric space**, by finding suitably orthogonal families of complex-valued harmonic functions and composing these with holomorphic maps, Gudmundsson and collaborators construct harmonic morphisms from many compact and non-compact classical symmetric spaces [93], see also [7, §8.2].

7. **Riemannian submersions** *are harmonic, and so are harmonic morphisms, if and only if their fibres are minimal*. The *Hopf maps* from $S^3 \to S^2$, $S^7 \to S^4$, $S^{15} \to S^8$, $S^{2n+1} \to \mathbb{C}P^n$, $S^{4n+3} \to \mathbb{H}P^n$ are examples of such harmonic morphisms. See also [7, §4.5].

8. The natural projection of a **warped product** $\mathcal{M} = F \times_{f^2} \mathcal{N} \to \mathcal{N}$ onto its second factor is a horizontally conformal map with grad λ vertical, totally geodesic fibres and integrable horizontal distribution; in particular is a harmonic morphism. The *radial projections* $\mathbb{R}^m \setminus \{\mathbf{0}\} \to S^{m-1}$ $(m = 2, 3, \ldots)$, given by $\mathbf{x} \mapsto \mathbf{x}/|\mathbf{x}|$, are such maps. See also [7, §12.4]

9. When $\dim \mathcal{M} - \dim \mathcal{N} = 1$, i.e., the map $\phi : \mathcal{M} \to \mathcal{N}$ has **one-dimensional fibres**, R. Bryant [29] gives the following *normal form* for the metric g on the domain of a submersive harmonic morphism ϕ in terms of the pull-back $\phi^* h$ of the metric on the codomain and the dilation λ of the map, namely,

$$g = \lambda^{-2} \phi^* h + \lambda^{2n-4} \theta^2$$

where θ is a connection 1-form; thus *locally* such a harmonic morphism is a *principal* S^1*-bundle with* S^1*-connection*; this holds *globally* if the fibres are all compact, see [7, §10.5].

10. It follows that given a **Killing field** V (or isometric action) on $(\mathcal{M}, g)$, there are locally harmonic morphisms with fibres tangent to V.

By analysing the overdetermined system (16, 23) using *exterior differential systems*, Bryant [29] shows that there any harmonic morphism with one-dimensional fibres from a space form is of warped product type or comes from a Killing field (this has been generalized to Einstein manifolds by R. Pantilie and Pantilie & Wood, see [7, Chapter 12]). It follows that *the only harmonic morphisms from Euclidean spheres with one-dimensional fibres are the Hopf maps* $S^{2n+1} \to \mathbb{C}P^n$.

11. There are **topological restrictions** on the existence of harmonic morphisms, for example, since harmonic morphisms preserve the harmonicity of 1-forms, Eells and Lemare showed that *the Betti number of the domain cannot be less than that of the codomain*, see [7, Proposition 4.3.11]. Pantilie and Wood show that the *Euler characteristic* must vanish for a harmonic morphism with fibres of dimension one from a compact domain of dimension not equal to 4. In particular *there is no non-constant harmonic morphism from a sphere* S^{2n} $(n \neq 2)$ *to a Riemannian manifold of dimension* $2n-1$, *whatever the metrics*. Further the Pontryagin numbers and the signature vanish, see [7, §12.1].

12. **When** $\dim \mathcal{M} = 4$, *the Euler characteristic is even and equals the the critical points of the harmonic morphism*, so that we cannot rule out the existence of a harmonic morphism from S^4. By Bryant's result in item 8 above, there is no harmonic morphism from the *Euclidean* 4-sphere with one-dimensional fibres; however, there is one if the metric on S^4 is changed by a suitable conformal factor. This map is given by suspending the Hopf map, first finding a suspension which is horizontally conformal, then changing the metric conformally on the domain to 'render' it harmonic. At both stages, the problem is reduced to solving an ordinary differential equation for the suspension function with suitable boundary values, and the method applies to find many more harmonic morphisms, see [7, Chapter 13].

13. Finally note that J.-Y. Chen shows that **stable harmonic maps** *from compact Riemannian manifolds to* S^2 *are all harmonic morphisms*. This is shown by calculating the second variation and showing that its non-negativity forces the map to be horizontally weakly conformal, see [7, §8.7].

3 Weakly harmonic maps and Sobolev spaces between manifolds

3.1 Weakly harmonic maps

An extension of the Dirichlet principle or, more generally, the use of variational methods requires the introduction of a class of distributional maps endowed with a topology which is sufficiently coarse to ensure the *compactness* of sequences of maps which we hope will converge to a solution. On the other hand, the energy functional should be defined on this class and we should be able to make sense of its Euler–Lagrange equation (16). These two requirements are somewhat in conflict, and will lead us to model the class of maps on the Sobolev space $W^{1,2}(\mathcal{M})$. But that will force us to work with weak solutions of (16), i.e., *weakly harmonic maps*. However, as soon as $m := \dim \mathcal{M} \geq 2$, a map $f \in W^{1,2}(\mathcal{M})$ is not continuous in general. Hence, even if $W^{1,2}(\mathcal{M}, \mathcal{N})$ makes sense, there is no reason for a map $\phi \in W^{1,2}(\mathcal{M}, \mathcal{N})$ to take values in any open subset, in general. This makes it difficult to study ϕ by using local charts on the *target manifold* $\mathcal{N}$. Today[7] most authors avoid these difficulties by using the Nash–Moser embedding theorem (see, for example, [91]) as follows. In the following **we shall assume that** $\mathcal{N}$ **is compact**. Then there exist an *isometric embbeding* $j : (\mathcal{N}, g) \longrightarrow (\mathbb{R}^N, \langle \cdot, \cdot \rangle)$. And we define (temporarily), for any open subset $\Omega \subset \mathcal{M}$,

$$W^{1,2}_j(\Omega, \mathcal{N}) := \{u \in W^{1,2}(\Omega, \mathbb{R}^N) |\, u(x) \in j(\mathcal{N}) \text{ a.e.}\}. \tag{24}$$

On this set the *energy* or *Dirichlet functional* defined by (7) now reads

$$E_\Omega(u) := \frac{1}{2} \int_\Omega g^{ij}(x) \left\langle \frac{\partial u}{\partial x^j}, \frac{\partial u}{\partial x^i} \right\rangle \omega_g.$$

[7] In his 1948 paper [156], C. B. Morrey had to work *without* the Nash–Moser theorem which was not yet proved.

But, if we assume that $\mathcal{M}$ is also compact, then for any two isometric embeddings j_1, j_2, the spaces $W^{1,2}_{j_1}(\mathcal{M},\mathcal{N})$ and $W^{1,2}_{j_2}(\mathcal{M},\mathcal{N})$ are homeomorphic and $E_\Omega(j_2 \circ j_1^{-1} \circ u) = E_\Omega(u)$. Hence we simply[8] write $W^{1,2}(\mathcal{M},\mathcal{N}) := W^{1,2}_{j}(\mathcal{M},\mathcal{N})$.

Weakly harmonic maps

In order to define weakly harmonic maps as extremals of $E_\mathcal{M}$ we have to specify which infinitesimal deformations of a map $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ we will consider. Consider a neighbourhood $\mathcal{V}$ of $\mathcal{N}$ in $\mathbb{R}^N$ such that the projection map $P : \mathcal{V} \longrightarrow \mathcal{N}$ which sends each $y \in \mathcal{V}$ to the nearest point in $\mathcal{N}$ is well defined and smooth[9]. Now let $u \in W^{1,2}(\mathcal{M},\mathcal{N})$. For any map $v \in W^{1,2}(\mathcal{M},\mathbb{R}^N) \cap L^\infty(\mathcal{M},\mathbb{R}^N)$ we observe that, for ε sufficiently small, $u + \varepsilon v \in \mathcal{V}$, so that $u^v_\varepsilon := P(u+\varepsilon v) \in W^{1,2}(\mathcal{M},\mathcal{N})$. We set $\dot{u}^v_0 := \lim_{\varepsilon\to 0}(u^v_\varepsilon - u)/\varepsilon = dP_u(v)$ a.e. and

$$(\delta E_\mathcal{M})_u(\dot{u}^v_0) := \lim_{\varepsilon\to 0} \frac{E_\mathcal{M}(u_\varepsilon) - E_\mathcal{M}(u)}{\varepsilon}.$$

(What is important in this definition is that $\varepsilon \longmapsto u^v_\varepsilon$ is a differentiable curve into $W^{1,2}(\mathcal{M},\mathbb{R}^N)$ such that $\forall\varepsilon$, $u^v_\varepsilon \in W^{1,2}(\mathcal{M},\mathcal{N})$, $du^v_\varepsilon/d\varepsilon \in W^{1,2}(\mathcal{M},\mathbb{R}^N)$ and $u^v_0 = u$.) And u is **weakly harmonic** if and only if $(\delta E_\mathcal{M})_u(\dot{u}^v_0) = 0$ for all $v \in W^{1,2} \cap L^\infty(\mathcal{M},\mathbb{R}^N)$. Equivalently u is a *solution in the distribution sense* of a *system of N coupled scalar elliptic PDEs*, i.e. an *$\mathbb{R}^N$-valued elliptic PDE*

$$\Delta_g u + g^{ij}(x)\, A_{u(x)}\left(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\right) = 0 \tag{25}$$

where $A \in \Gamma(S^2T^*\mathcal{N} \otimes N\mathcal{N})$ is the second fundamental form of the embedding j as in §2.2 ($N\mathcal{N}$ now denotes the normal bundle of $\mathcal{N}$ in $\mathbb{R}^N$);[10] one can check that this condition is independent of the embedding j [118]. Indeed, it is just the equation (18).

Example. ($\mathcal{N} = S^n$, *the unit sphere*) The n-dimensional sphere S^n is the submanifold $\{y \in \mathbb{R}^{n+1} |\, |y| = 1\}$, its metric is the pull-back of the standard Euclidean metric by the embedding $j : S^n \longrightarrow \mathbb{R}^{n+1}$. The second fundamental form of j is given by $A_y(X,Y) = \langle X,Y\rangle y$, so that the weakly harmonic maps are the maps in $W^{1,2}(\mathcal{M},S^n)$ such that

$$\Delta_g u + |du|^2 u = 0 \quad \text{in a distribution sense.} \tag{26}$$

Remarks (i) In (25), $\Delta_g u \in W^{-1,2}(\mathcal{M},\mathbb{R}^N)$ is defined in the distribution sense, the coefficients of $A_{u(x)}$ are in L^∞ because $\mathcal{N}$ is compact and so $g^{ij}(x)A_{u(x)}(\partial u/\partial x^i, \partial u/\partial x^j) \in L^1(\mathcal{M},\mathbb{R}^N)$.

(ii) The system (25) is an example of a *semilinear elliptic system with a nonlinearity which is quadratic in the first derivatives*, for which a general regularity theory has been developed (see [143, 229, 121, 83]). This nonlinearity is the reason why most of analytical properties valid for harmonic functions are lost: *existence*, *regularity*, *uniqueness* and *minimality* may fail in general, unless some extra hypotheses are added.

(iii) A difficulty particular to this theory is that $W^{1,2}(\mathcal{M},\mathcal{N})$ is not a $\mathcal{C}^1$-manifold. One can only say that $W^{1,2}(\mathcal{M},\mathcal{N})$ is a Banach manifold, which is not separable if $m \geq 2$, and that $\mathcal{C}^0 \cap W^{1,2}(\mathcal{M},\mathcal{N})$ is a closed separable submanifold of $W^{1,2}(\mathcal{M},\mathcal{N})$ (see [31]). Moreover, $W^{1,2}(\mathcal{M},\mathcal{N})$ does not have the same topology as $\mathcal{C}^0(\mathcal{M},\mathcal{N})$ in general (see §3.2 and 3.3).

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[8] In the case where $\mathcal{M}$ is not compact, we may not have $W^{1,2}_{j_1}(\mathcal{M},\mathcal{N}) \simeq W^{1,2}_{j_2}(\mathcal{M},\mathcal{N})$ because the L^2 norms of $j_1\circ\phi$ and $j_2\circ\phi$ may be different (indeed, one of the two norms may be bounded whereas the other one may be infinite). This suggests that perhaps a more satisfactory (but less used) definition of $W^{1,2}_{j}(\mathcal{M},\mathcal{N})$ would be: *the set of measurable distributions on $\mathcal{M}$ with values in $\mathbb{R}^N$ such that $du \in L^2(\mathcal{M})$ and $u(x) \in j(\mathcal{N})$ a.e.*

[9] We may use other projection maps, not necessarily Euclidean projections, see [118].

[10] An equivalent formula for A is, as follows: if for any $y \in \mathcal{N}$, we denote by $P^\perp_y : \mathbb{R}^N \longrightarrow N_y\mathcal{N}$ the orthonormal projection, then A can be defined by $A_y(X,Y) := (D_X P^\perp_y)(Y)$, $\forall X,Y \in \Gamma(T\mathcal{N})$, where D is the (flat) Levi-Civita connection on $\mathbb{R}^N$.

Minimizing maps

A map $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ is called an **energy minimizing map** if any map $v \in W^{1,2}(\mathcal{M},\mathcal{N})$ which coincides with u outside a compact subset $K \subset \mathcal{M}$ has an energy greater than or equal to that of u, i.e. $E_{\mathcal{M}}(v) \geq E_{\mathcal{M}}(u)$. A weaker notion is that $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ is called **locally energy minimizing** if, for any point $x \in \mathcal{M}$, there exists a neighbourhood $U \subset \mathcal{M}$ of x such that any map $v \in W^{1,2}(\mathcal{M},\mathcal{N})$ which coincides with u outside a compact subset $K \subset U$ has an energy greater than or equal to that of u.

Stationary maps

The family $\{u^v_\varepsilon |\ v \in W^{1,2}(\mathcal{M},\mathbb{R}^N) \cap L^\infty(\mathcal{M},\mathbb{R}^N)\}$ of infinitesimal deformations used for the definition of a weakly harmonic map u does not contain some significant deformations. For example, consider *radial projection* $u_\odot \in W^{1,2}(B^3,S^2)$ defined by[11]

$$u_\odot(x) = x/|x|\,; \tag{27}$$

it seems natural to *move* the singularity of this map along some smooth path. For example we let $a \in \mathcal{C}^1((-1,1),B^3)$ parametrize such a path in B^3 such that $a(0) = 0$ and we consider the family of maps $u_\varepsilon \in W^{1,2}(B^3,S^2)$ defined by $u_\varepsilon(x) = (x - a(\varepsilon))/|x - a(\varepsilon)|$. Then $du_\varepsilon/d\varepsilon$ is **not** in $W^{1,2}(B^3,\mathbb{R}^3)$, and hence we cannot take this infinitesimal variation of $u_\odot$ into account for weakly harmonic maps. This is the reason for considering a second type of variation: we let $(\varphi_t)_{t\in I}$ (where $I \subset \mathbb{R}$ is some open interval which contains 0) be a $\mathcal{C}^1$ family of smooth diffeomorphisms $\varphi_t : \mathcal{M} \longrightarrow \mathcal{M}$ such that φ_0 is the identity. Then for any $u \in W^{1,2}(\mathcal{M},\mathcal{N})$, $(u\circ\varphi_t)_{t\in I}$ is a $\mathcal{C}^1$ family of maps in $W^{1,2}(\mathcal{M},\mathcal{N})$ such that $u\circ\varphi_0 = u$. Following [189] we say that u **is stationary if** (i) u **is weakly harmonic, and** (ii) **for any family of diffeomorphisms** $(\varphi_t)_{t\in I}$ **with** $\varphi_0 = \mathbf{Id}_{\mathcal{M}}$,

$$\lim_{t\to 0}\big(E_{\mathcal{M}}(u\circ\varphi_t) - E_{\mathcal{M}}(u)\big)/t = 0. \tag{28}$$

Note that, without loss of generality, we can assume that the diffeomorphisms φ_t have of the form $\varphi_t = e^{tX}$, where X is a smooth tangent vector field with compact support on $\mathcal{M}$. Maps u which satisfies (28) can be characterized by the following local condition derived by P. Baird and J. Eells, and by A. I. Pluzhnikov independently [6, 170, 118]. Let us stress temporarily the dependence of the Dirichlet energy on the metric g on $\mathcal{M}$ by writing $E_{\mathcal{M}} = E_{(\mathcal{M},g)}$. Then we remark that, by the change of variable $\widetilde{x} = e^{tX}(x)$ in the Dirichlet integral, we have:

$$E_{(\mathcal{M},g)}(u\circ e^{tX}) = E_{(\mathcal{M},(e^{-tX})^*g)}(u\circ e^{tX}\circ e^{-tX}) = E_{(\mathcal{M},(e^{-tX})^*g)}(u),$$

where $(e^{-tX})^*g$ is the pull-back of the metric g by e^{-tX}. But we compute:

$$E_{(\mathcal{M},(e^{-tX})^*g)}(u) = E_{(\mathcal{M},g)}(u) + t\int_{\mathcal{M}} \left(L_X g^{ij}\right) S_{ij}(u)\,\omega_g + o(t),$$

where

$$S_{ij}(u) := \frac{1}{2}|du|^2_g g_{ij} - (u^*h)_{ij} = \frac{1}{2}g^{kl}(x)\left\langle \frac{\partial u}{\partial x^k},\frac{\partial u}{\partial x^l}\right\rangle g_{ij} - \left\langle \frac{\partial u}{\partial x^i},\frac{\partial u}{\partial x^j}\right\rangle,$$

where $|du|^2_g = g^{ij}(x)\,\langle \partial u/\partial x^i\,,\,\partial u/\partial x^j\rangle$, is called the **stress-energy tensor**. If $u \in W^{1,2}(\mathcal{M},\mathcal{N})$, then its components are in $L^1(\mathcal{M})$. Hence condition (28) is equivalent to the fact that $\int_{\mathcal{M}} \left(L_X g^{ij}\right) S_{ij}(u)\,\omega_g = 0$, for all smooth tangent vector fields X with compact support on $\mathcal{M}$. Moreover, since the stress-energy tensor is symmetric we have the identity $\left(2g^{ik}\nabla_k X^j + L_X g^{ij}\right) S_{ij}(u) = 0$, from which we can deduce by an integration by parts that u **satisfies** (28) **if and only if** $S_{ij}(u)$ **is covariantly divergence-free**, i.e.

$$\forall j,\ \nabla_i S^i_j(u) = 0 \ \text{ in the distribution sense } \big(\text{where } S^i_j(u) := g^{ik}S_{kj}(u) \text{ and } \nabla_i := \nabla_{\partial/\partial x^i}\big). \tag{29}$$

[11] For $a \in \mathbb{R}^m$ and $r > 0$ we write $B^m(a,r) := \{x \in \mathbb{R}^m |\ |x - a| < r\}$; also, for brevity write $B^m := B^m(0,1)$.

Remarks (i) If the metric g on $\mathcal{M}$ is Euclidean, i.e. if we can write $g_{ij} = \delta_{ij}$ in some coordinate system, then the covariant conservation law (29) becomes a system of m conservations laws.

(ii) if $m = 2$ then S_{ij} is trace free. Furthermore we can use conformal local coordinates $z = x^1 + ix^2$ on $\mathcal{M}$. Then if we identify S_{ij} with the quadratic form $S := S_{ij}dx^i dx^j$, we easily compute:

$$-2S = \mathrm{Re}\left\{\left(\Big|\frac{\partial u}{\partial x^1}\Big|^2 - \Big|\frac{\partial u}{\partial x^2}\Big|^2 - 2i\Big\langle\frac{\partial u}{\partial x^1},\frac{\partial u}{\partial x^2}\Big\rangle\right)(dz)^2\right\} = 4\mathcal{H},$$

where $\mathcal{H}$ is the Hopf differential of u as defined in (21). We note that: (i) u **is conformal if and only if** $\mathcal{H}$ **or equivalently** S **vanishes** and (ii) **the stress-energy tensor is divergence free, i.e.** (29) **holds, if and only if** $\mathcal{H}$ **is holomorphic**.

Relationship between the different notions of critical points

It is easy to prove the inclusions:

$$\{\text{minimizing maps}\} \subset \{\text{locally minimizing maps}\} \subset \{\text{stationary maps}\} \subset \{\text{weakly harmonic maps}\}\,;$$

these inclusions are strict in general. For example, the identity map $\mathrm{Id} : S^3 \longrightarrow S^3$ is *locally minimizing* (see §6.2) but *not globally minimizing* (see §3.3). The map $u^{(2)} \in W^{1,2}(B^3, S^2)$, defined by $u^{(2)}(x) = P^{-1}\circ Z^2 \circ P(x/|x|)$, where $P : S^2 \longrightarrow \mathbb{C}P = \mathbb{C}\cup\{\infty\} = \mathbb{R}^2 \cup \{\infty\}$ defined by

$$P(y^1, y^2, y^3) = (y^1 + iy^2)\big/(1 + y^3) \tag{30}$$

is the stereographic projection and $Z^2(z) = z^2$ is *stationary* but *not locally minimizing* (see §4.3 and [24]). The map $v_\lambda :\in W^{1,2}(B^3, S^2)$, defined by $v_\lambda(x) = P^{-1}\circ\lambda\circ P(x/|x|)$, where λ is the multiplication by some $\lambda \in \mathbb{C}^*$ is *weakly harmonic* but *not stationary if* $|\lambda| \neq 1$ (see [118], §1.4). However, *smooth harmonic maps* are stationary: one can check by a direct computation that, if u is a map of class $\mathcal{C}^2$, (25) implies (29).

3.2 The density of smooth maps in $W^{1,p}(\mathcal{M},\mathcal{N})$

In this section and the following, it may clarify the discussion to consider the more general family of spaces

$$W^{1,p}(\mathcal{M},\mathcal{N}) := \{u \in W^{1,p}(\mathcal{M},\mathbb{R}^N)|\ u(x)\in\mathcal{N}\text{ a.e.}\},$$

$W^{1,p}(\mathcal{M}) := \{u \in L^p(\mathcal{M})|\ du \in L^p(\mathcal{M})\}$ and $W^{1,p}(\mathcal{M},\mathbb{R}^N) := W^{1,p}(\mathcal{M})\otimes\mathbb{R}^N$, where $1 \leq p < \infty$. An interesting functional on $W^{1,p}(\mathcal{M},\mathcal{N})$ is the *p-energy*

$$E^{(p)}_{\mathcal{M}}(u) := \frac{1}{p}\int_{\mathcal{M}}\left(g^{ij}(x)\left\langle\frac{\partial u}{\partial x^i},\frac{\partial u}{\partial x^j}\right\rangle\right)^{p/2}\omega_g. \tag{31}$$

For any Riemannian manifold $\mathcal{M}$ of dimension m and for any compact manifold $\mathcal{N}$ of dimension n, let us define

- $H^{1,p}_s(\mathcal{M},\mathcal{N})$:= the closure of $\mathcal{C}^1(\mathcal{M},\mathcal{N})\cap W^{1,p}(\mathcal{M},\mathcal{N})$ in the *strong* $W^{1,p}$-topology;
- $H^{1,p}_w(\mathcal{M},\mathcal{N})$:= the closure of $\mathcal{C}^1(\mathcal{M},\mathcal{N})\cap W^{1,p}(\mathcal{M},\mathcal{N})$ in the *sequential weak*[12] $W^{1,p}$-topology: a map $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ belongs to $H^{1,p}_w(\mathcal{M},\mathcal{N})$ if and only if there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of maps in $\mathcal{C}^1(\mathcal{M},\mathcal{N})\cap W^{1,p}(\mathcal{M},\mathcal{N})$ such that v_k converges weakly to u as $k\to\infty$.

[12] The space $H^{1,p}_w(\mathcal{M},\mathcal{N})$ plays an important role when using variational methods. For example, if we minimize the p-energy *among smooth maps*, the minimizing sequence converges weakly. Hence the weak solution that we obtain is naturally in $H^{1,p}_w(\mathcal{M},\mathcal{N})$.

Note that we have always the inclusions

$$H^{1,p}_s(\mathcal{M},\mathcal{N}) \subset H^{1,p}_w(\mathcal{M},\mathcal{N}) \subset W^{1,p}(\mathcal{M},\mathcal{N}). \tag{32}$$

The easy case $m \leq p$. We first observe that, *if* $p > m$*, the Sobolev embedding theorem implies that* $W^{1,p}(\mathcal{M},\mathcal{N}) \subset \mathcal{C}^0(\mathcal{M},\mathcal{N})$, so that by a standard regularization in $W^{1,p}(\mathcal{M},\mathbb{R}^N)$ followed by a projection onto $\mathcal{N}$, one can prove easily that $H^{1,p}_s(\mathcal{M},\mathcal{N})$ is dense in $W^{1,p}(\mathcal{M},\mathcal{N})$. This result has been extended in [191] **to the critical exponent** $p = m$, by using the Poincaré inequality:[13] $⨍_{B^m(x,r)} |\varphi - ⨍_{B^m(x,r)} \varphi|^m \leq C \int_{B^m(x,r)} |d\varphi|^m$. In conclusion, **if** $p \in [m,\infty)$**, all inclusions in** (32) are equalities.

The hard case $1 \leq p < m$. One of the more instructive example is radial projection $u_\odot : B^m \to S^{m-1}$ given by (27). This map has a point singularity at 0, but is in $W^{1,p}(B^m, S^{m-1})$ if $p < m$. We shall see later that $u_\odot$ cannot be approximated by smooth maps with values in S^{m-1} for $m-1 < p < m$. Variants of $u_\odot$ are the maps $u^s_\odot : B^m \longrightarrow S^{m-s-1}$, for $s \in \mathbb{N}$ such that $0 \leq s \leq m-1$, defined by $u^s_\odot(x,y) = x/|x|$, for $(x,y) \in \mathbb{R}^{m-s} \times \mathbb{R}^s$: this map is singular along the s-dimensional subspace $x = 0$.

Approximation by smooth maps with singularities. The following result by F. Bethuel [12, 102] shows that the structure of the singularities of the maps $u^s_\odot$ is somehow generic. Let

$\mathcal{R}^{p,k}(\mathcal{M},\mathcal{N})$: the set of maps $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ such that $\exists\, \Sigma_u \subset \mathcal{M}$ with $u \in \mathcal{C}^k(\mathcal{M} \setminus \Sigma_u, \mathcal{N})$, $\Sigma_u = \bigcup_{i=1}^r \Sigma_i,\ \Sigma_i$ is a subset of a manifold of dimension $m - [p] - 1,\ \partial\Sigma_i$ is $\mathcal{C}^k$

(note that, if $m-1 \leq p < m$, each Σ_i is a point). Then

$$\textit{if } 1 < p < m, \quad \textit{then } \mathcal{R}^{p,k}(\mathcal{M},\mathcal{N}) \textit{ is dense in } W^{1,p}(\mathcal{M},\mathcal{N}). \tag{33}$$

Moreover, F. B. Hang and F. H. Lin [102] proved that the singular set Σ_u can be chosen as the $(m-[p]-1)$-skeleton of a smooth rectilinear cell decomposition.

The case of maps into the sphere. The idea of the proof of (33) in the case where $\mathcal{N} = S^n$ and $n \leq p < n+1$ is the following (see also [17, 88]). Let $u \in W^{1,p}(\mathcal{M}, S^n)$. Then by convolution with mollifiers we first produce a sequence of smooth maps $(u_\rho)_\rho$ which converges strongly to u as $\rho \to 0$, but *has values in* $B^{n+1}(0,1)$. However, for any $\varepsilon > 0$ the measure of $V^\varepsilon_\rho := u_\rho^{-1}(B^{n+1}(0,1-\varepsilon))$ tends to 0 as $\rho \to 0$. The main task is to compose the restriction $(u_\rho)|_{V^\varepsilon_\rho}$ of u_ρ to V^ε_ρ with a projection map from $B^{n+1}(0,1-\varepsilon)$ to its boundary in order to obtain a map into $B^{n+1} \setminus B^{n+1}(0,1-\varepsilon)$. The naive projection $x \longmapsto (1-\varepsilon)x/|x|$ fails because $u_\rho/|u_\rho|$ has infinite $W^{1,p}$-norm in general. The trick, inspired by [107], consists of using a different projection map $\Pi_a : x \longmapsto (1-\varepsilon)(x-a)/|x-a|$, where $a \in B^{n+1}(0,\frac{1}{2})$: by averaging over $a \in B^{n+1}(0,\frac{1}{2})$ and using Fubini's theorem one finds that there exists some a such that the $W^{1,p}$-norm of $(\Pi_a \circ u_\rho)|_{V^\varepsilon_\rho}$ is bounded in terms of the $W^{1,p}$-norm of $(u_\rho)|_{V^\varepsilon_\rho}$. Moreover, Sard's theorem ensures us that for a generic a, $u_\rho^{-1}(a)$, i.e. the singular set of $\Pi_a \circ u_\rho$ is a smooth submanifold of codimension $n+1 = [p]+1$.

The property (33) shows that questions of density rely on approximating maps in $\mathcal{R}^{p,k}(\mathcal{M},\mathcal{N})$ by smooth maps. Again it is instructive to look at the example of the map $u_\odot \in W^{1,p}(B^m, S^{m-1})$: a way to approximate $u_\odot$ is *to move the topological singularity through a path joining the origin 0 to the boundary* ∂B^m. Consider such a path (for example, $[-1,0] \times \{0\}^{m-1} \subset \mathbb{R}^m$), then by modifying $u_\odot$ inside a small tube around this path in such a way that the topological degree on each sphere $S^{m-1}_r := \partial B^m(0,r)$ cancels, we obtain a continuous map into the sphere. For instance, for $\varepsilon > 0$ sufficiently small, we construct a map u_ε by replacing, for any $r \in [0,1]$, the

[13] Note that similar arguments show that *maps such that* $\lim_{r\to 0} E_{x,r}(u) = 0$ *for all* $x \in \mathcal{M}$ *can be approximated by smooth maps* (see §4.3 for a definition of $E_{x,r}$). This result is a key ingredient in the regularity theory for harmonic maps by R. Schoen and K. Uhlenbeck [190], see again §4.3. All these results fit in the framework of a theory of maps into manifolds with *vanishing mean oscillation*, developed by H. Brezis and L. Nirenberg [27]: for any locally integrable function f on $\mathbb{R}^m$, for any $x \in \mathbb{R}^m$ and any $r > 0$, set $f_{x,r} := ⨍_{B^m(x,r)} f$, then let $||f||_{BMO} := \sup_{x\in\mathbb{R}^m} \sup_{r>0} (⨍_{B^m(x,r)} |f - f_{x,r}|^p)^{1/p}$, for some $p \in [1,\infty)$. Then the space of functions of *bounded mean oscillation* (BMO) on $\mathbb{R}^m$ is the set of locally integrable functions f on $\mathbb{R}^m$ such that $||f||_{BMO}$ is bounded and this definition does *not* depend on p [128]. The subspace of functions of *vanishing mean oscillation* (VMO) on $\mathbb{R}^m$ is composed of maps such that $\lim_{r\to 0}(⨍_{B^m(x,r)} |f - f_{x,r}|^p)^{1/p} = 0$ for any $x \in \mathbb{R}^m$ (see [128, 118])

restriction $u_\odot|_{S_r^{m-1}}$ of $u_\odot$ to S_r^{m-1} by its left composition with the map $T^{-1}_{\lambda(r,\varepsilon)} \circ U \circ T_{\lambda(r,\varepsilon)} : S^{m-1} \longrightarrow S^{m-1}$, where $U(y) = (|y^1|, y^2, \cdots, y^m)$ and $T_\lambda(y) = (\cosh\lambda + y^1 \sinh\lambda)^{-1}(\sinh\lambda + y^1\cosh\lambda, y^2, \cdots, y^m)$ $(\lambda \in \mathbb{R})$ and we choose $\lambda(r,\varepsilon)$ in such a way that u_ε coincides with $u_\odot$ outside the tubular neighbourhood of the path of radius ε. Then, inside the small tube, $|du_\varepsilon| \leq (C/\varepsilon)|du_\odot|$ so that the extra cost in $W^{1,p}$-norm of this modification is of order $\varepsilon^{m-1}/\varepsilon^p$ (note that ε^{m-1} controls the volume of the tube). We see that

(i) if $1 \leq p < m-1$, this p-energy cost can be as small as we want;
(ii) if $p = m-1$, the p-energy costs does not tend to zero as $\varepsilon \to 0$ but is bounded;
(iii) if $m-1 < p < m$, the cost tends to ∞ as $\varepsilon \to 0$.

These heuristic considerations are behind a series of results proved by F. Bethuel [12] and summarized in the following table:

Inclusions	$H_s^{1,p}(B^m,S^{m-1})$ $\subset$ $H_w^{1,p}(B^m,S^{m-1})$	$\subset$ $W^{1,p}(B^m,S^{m-1})$
$1 \leq p < m-1$	$=$	$=$
$p = m-1$	$\subsetneq$	$=$
$m-1 < p < m$	$=$	$\subsetneq$

In particular, we see that $p = m-1$ is another critical exponent: $u_\odot$ can be approximated by smooth maps weakly in $W^{1,m-1}$ but not strongly.

The fact that $H_s^{1,p}(B^m,S^{m-1}) \neq W^{1,p}(B^m,S^{m-1})$ for $m-1 \leq p < m$ can be checked by using a degree argument. Here is a proof for $m=3$ and $2 \leq p < 3$. Let $\omega_{S^2} := j^*(y^1dy^2\wedge dy^3 + y^2dy^3\wedge dy^1 + y^3dy^1\wedge dy^2)$ be the volume form on S^2 (j is the embedding $S^2 \subset \mathbb{R}^3$). Let $\chi \in \mathcal{C}^\infty(B^3,\mathbb{R})$ be a function which depends only on $r = |x|$, such that $\chi(1) = 0$ (i.e. $\chi = 0$ on ∂B^3) and $\chi(0) = -1$. Assume that there exists a sequence $(u_k)_{k\in\mathbb{N}}$ of functions in $\mathcal{C}^2(B^3,S^2)$ such that $u_k \to u_\odot$ strongly in $W^{1,2}(B^3,S^2)$. Then since $u_k^*\omega_{S^2}$ is quadratic in the first derivatives of u, $\int_{B^3} d\chi\wedge u_k^*\omega_{S^2}$ converges to $\int_{B^3} d\chi\wedge u_\odot^*\omega_{S^2} = \int_0^1 4\pi(d\chi/dr)dr = 4\pi$. On the other hand, since u_k is smooth, $d\left(u_k^*\omega_{S^2}\right) = u_k^*(d\omega_{S^2}) = 0$ and so

$$0 = \int_{\partial B^3}\chi\, u_k^*\omega_{S^2} = \int_{B^3} d\left(\chi\, u_k^*\omega_{S^2}\right) = \int_{B^3} d\chi\wedge u_k^*\omega_{S^2}.$$

Hence we also deduce that $\int_{B^3} d\chi\wedge u_k^*\omega_{S^2} \to 0$, a contradiction. In §5.4 another proof is given for $p = 2$.

In fact, a nice characterization of $H_s^{1,2}(B^3,S^2)$ was given by Bethuel [10] in terms of the pull-back of the volume form ω_{S^2} on S^2: **a map $u \in W^{1,2}(B^3,S^2)$ can be approximated by smooth maps in the strong $W^{1,2}$-topology if and only if** $d(u^*\omega_{S^2}) = 0$ (see also §5.4 for more results about $u^*\omega_{S^2}$.) This may be generalized to some situations (see [11]) but not all: indeed it is not clear whether such a cohomological criterion can be found to recognize maps in $H_s^{1,3}(B^4,S^2)$ — for example, the singular map defined by $h_\odot^{\mathbb{C}}(x) = H^{\mathbb{C}}(x/|x|)$, where $H^{\mathbb{C}} : S^3 \longrightarrow S^2$ is the Hopf fibration, is in $W^{1,3}(B^4,S^2)$ but not in $H_s^{1,3}(B^4,S^2)$ — see [110] for more details on this delicate situation.

The role of the topology of $\mathcal{M}$ and $\mathcal{N}$. We have seen that, when $\mathcal{N} = S^n$, the topology of $\mathcal{N}$ may cause obstructions to the density of smooth maps in $W^{1,p}(\mathcal{M},\mathcal{N})$. The first general statement in this direction is due to F. Bethuel and X. Zheng [17] and Bethuel [12] in terms of the $[p]$-th homotopy group of $\mathcal{N}$; namely, for $\mathcal{M} = B^m$ we have

$$\textit{if } 1 < p < m, \textit{ then } \quad H_s^{1,p}(B^m,\mathcal{N}) = W^{1,p}(B^m,\mathcal{N}) \quad \Longleftrightarrow \quad \pi_{[p]}(\mathcal{N}) = 0.$$

However, for an *arbitrary* manifold $\mathcal{M}$, the condition that $\pi_{[p]}(\mathcal{N}) = 0$ is *not sufficient* to ensure that $H_s^{1,p}(\mathcal{M},\mathcal{N}) = W^{1,p}(\mathcal{M},\mathcal{N})$, in general. This was pointed out in [102]. An example is the map $v_\odot \in W^{1,2}(\mathbb{R}P^4,\mathbb{R}P^3)$ defined by $v_\odot[x^0 : x^1 : x^2 : x^3 : x^4] = [x^1 : x^2 : x^3 : x^4]$, with a singularity at $[1:0:0:0:0]$; there is no way to remove this singularity[14], so, *there is no sequence of smooth maps converging weakly to $v_\odot$*. Hence

[14] In contrast with the map $u_\odot \in W^{1,2}(B^4,S^3)$ where the topological singularity can be moved to the boundary with an arbitrary low energy cost.

$H^{1,2}_w(\mathbb{R}P^4,\mathbb{R}P^3) \neq W^{1,2}(\mathbb{R}P^4,\mathbb{R}P^3)$, although $\pi_2(\mathbb{R}P^3) = 0$. A result due to P. Hajłasz [96] is valid for an arbitrary manifold $\mathcal{M}$:

if $1 \leq p < m$, *then* $\quad \pi_1(\mathcal{N}) = \cdots = \pi_{[p]}(\mathcal{N}) = 0 \quad \Longrightarrow \quad H^{1,p}_s(\mathcal{M},\mathcal{N}) = W^{1,p}(\mathcal{M},\mathcal{N}).$

The general result is due to F. B. Hang and F. H. Lin [102] and, in the case where $\mathcal{M}$ has no boundary, is the following. First we say that $\mathcal{M}$ *satisfies the k-extension property with respect to* $\mathcal{N}$ if, for any CW complex structure $(X^j)_{j\in\mathbb{N}}$ on $\mathcal{M}$ and for any $f \in \mathcal{C}^0(X^{k+1},\mathcal{N})$, the restriction $f|_{X^k}$ of f on X^k has a continuous extension on $\mathcal{M}$. Then, if $1 < p < m$, we have [102]:

$$H^{1,p}_s(\mathcal{M},\mathcal{N}) = W^{1,p}(\mathcal{M},\mathcal{N}) \quad \Longleftrightarrow \quad \begin{cases} \pi_{[p]}(\mathcal{N}) = 0 \textit{ and } \mathcal{M} \textit{ satisfies} \\ \textit{the } [p-1]\textit{-extension property with respect to } \mathcal{N}. \end{cases}$$

The case when p is not an integer. The identity between $H^{1,p}_s(B^m,S^{m-1})$ and $H^{1,p}_w(B^m,S^{m-1})$ for $p \neq m-1$ is actually a particular case of a general phenomenon, as shown by Bethuel [12]: *for any domain $\mathcal{M} \subset \mathcal{M}$ and for any compact manifold $\mathcal{N}$,*

if $p > 1$ *is* **not an integer**, *then* $\quad H^{1,p}_s(\mathcal{M},\mathcal{N}) = H^{1,p}_w(\mathcal{M},\mathcal{N}).$

The case when p is an integer. The question left open is, in cases where $H^{1,p}_s(\mathcal{M},\mathcal{N}) \subsetneq W^{1,p}(\mathcal{M},\mathcal{N})$, to characterize the intermediate space $H^{1,p}_w(\mathcal{M},\mathcal{N})$. A first answer was given in [12] for maps into the sphere:

if $p \in \mathbb{N}$ *satisfies* $p < m$, *then* $\quad H^{1,p}_s(B^m,S^p) \subsetneq H^{1,p}_w(B^m,S^p) = W^{1,p}(B^m,S^p).$

A generalization was proved by P. Hajłasz in [96]:

if $p \in \mathbb{N}$ *satisfies* $p < m$, *then* $\quad \pi_1(\mathcal{N}) = \cdots = \pi_{p-1}(\mathcal{N}) = 0 \quad \Longrightarrow \quad H^{1,p}_w(\mathcal{M},\mathcal{N}) = W^{1,p}(\mathcal{M},\mathcal{N}).$

And the following further result was obtained by M. R. Pakzad and T. Rivière [162]:

for $p = 2, \quad \pi_1(\mathcal{N}) = 0 \quad \Longrightarrow \quad H^{1,2}_w(\mathcal{M},\mathcal{N}) = W^{1,2}(\mathcal{M},\mathcal{N}).$

For more general situations, assuming that $\mathcal{M}$ has no boundary, a *necessary* condition for a map to be in $H^{1,p}_w(\mathcal{M},\mathcal{N})$ was found by F. B. Hang and F. H. Lin in [102]: they proved that **if** $u \in H^{1,p}_w(\mathcal{M},\mathcal{N})$ **then** $u_{\sharp,[p]-1}(h)$ **is extendible to $\mathcal{M}$ with respect to $\mathcal{N}$**. The precise definition of $u_{\sharp,[p]-1}(h)$ is delicate: roughly speaking, by using ideas of B. White (see [226, 227] and §3.3), it is possible to define the homotopy class $u_{\sharp,[p]-1}(h)$ of the restriction of a map $u \in H^{1,p}_w(\mathcal{M},\mathcal{N})$ to a *generic* $([p]-1)$-skeleton of a rectilinear cell decomposition h of $\mathcal{M}$. Furthermore Hang and Lin in [102] *conjectured*[15] *that this condition is also a sufficient one*, i.e., that *if $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ and $u_{\sharp,[p]-1}(h)$ is extendible to $\mathcal{M}$ with respect to $\mathcal{N}$, then $u \in H^{1,p}_w(\mathcal{M},\mathcal{N})$*. In [101] Hang proved that **this conjecture is true for $p = 2$**.

Note that in the special case $p = 1$, Hang proved that $H^{1,1}_s(\mathcal{M},\mathcal{N}) = H^{1,1}_w(\mathcal{M},\mathcal{N})$ [100].

3.3 The topology of $W^{1,p}(\mathcal{M},\mathcal{N})$

The motivation for understanding the topology of $W^{1,p}(\mathcal{M},\mathcal{N})$ is to adapt the *direct method* of the calculus of variations to find a harmonic map in a homotopy class of maps between $\mathcal{M}$ and $\mathcal{N}$, i.e., by minimizing the energy in this homotopy class.

Some difficulties are illustrated by the following question[16] [66]: *What is the infimum of the energy in the homotopy class of the identity map* Id : $S^m \longrightarrow S^m$ *?*

- if $m = 1$, Id is minimizing and all minimizers in its homotopy class are rotations.

[15]They proved this conjecture for maps in $\mathcal{R}^{k,p}_w(\mathcal{M},\mathcal{N})$.

[16]The displayed facts were noticed by C. B. Morrey.

- if $m \geq 3$, **the infimum of the energy is** 0. Indeed, consider, for example, the family of conformal Möbius maps $T_\lambda : S^m \longrightarrow S^m$ for $\lambda \in \mathbb{R}$ defined by $T_\lambda(y) = (\cosh\lambda + y^1 \sinh\lambda)^{-1}(\sinh\lambda + y^1\cosh\lambda, y^2, \cdots, y^m)$; for all $\lambda \in \mathbb{R}$, T_λ is homotopic to the identity (actually T_0 equals the identity map) but as λ goes to $+\infty$, $E_{S^m}(T_\lambda)$ tends to zero and T_λ converges *strongly* to a constant map.
- the intermediate case $m = 2$ corresponds to the *critical dimension*; then *all the maps T_λ have the same energy, are conformal harmonic, and minimize the energy in their homotopy class, but T_λ converges* weakly *to a constant map*[17] *as* $\lambda \to +\infty$. One then speaks of a *bubbling* phenomenon, see §5.3.

Prescribing the action on the first homotopy group. The first positive result in these directions was in the case $m = \dim \mathcal{M} = 2$ and $\partial\mathcal{M} = \emptyset$ studied by R. Schoen and S.T. Yau [193]. Let γ be a smooth immersed path in $\mathcal{M}$ and $u \in W^{1,2}(\mathcal{M},\mathcal{N})$; in general the 'restriction' $u \circ \gamma$ of u to γ is not continuous (just in $W^{\frac{1}{2},2}$) but one can prove that, if we change γ to a *generic* path $\widetilde{\gamma}$ which is homotopic to γ, then $u \circ \widetilde{\gamma}$ *is* continuous.

(i) First, we use the following observation[18]: for any map $f \in W^{1,2}(S^1 \times (0,1), \mathbb{R}^N)$, the map $(\theta, s) \longmapsto |df(\theta,s)|^2$ is in $L^1(S^1 \times (0,1))$; hence by using the Fubini–Study theorem on $S^1 \times (0,1)$ one deduces that, for a.e. $s \in (0,1)$, the map $\theta \longmapsto |df(\theta,s)|^2$ belongs to $L^1(S^1)$, so that the restriction of f to $S^1 \times \{s\}$ is in $W^{1,2}(S^1) \subset \mathcal{C}^0(S^1)$. We apply this result to $f = u \circ \Gamma$, where $\Gamma \in \mathcal{C}^1(S^1 \times (0,1), \mathcal{M})$ parametrizes a strip composed of parallel paths $\gamma_s := \Gamma(\cdot, s)$ homotopic to the same path γ.

(ii) Second, if $s_1 < s_2$ are two values in $(0,1)$ such that $u \circ \gamma_{s_1}$ and $u \circ \gamma_{s_2}$ are continuous, then we can use the existence theorem of Morrey [156] to prove that there exists a smooth minimizing harmonic map $U : S^1 \times (s_1, s_2) \longrightarrow \mathcal{N}$ which agree with $u \circ \Gamma$ on $\partial S^1 \times (s_1,s_2) = (S^1 \times \{s_2\}) \cup (S^1 \times \{s_1\})$. We deduce that $u \circ \gamma_{s_1}$ *and* $u \circ \gamma_{s_2}$ *are homotopic*.

This leads to the definition of the image by u of the homotopy class of γ: it is the homotopy class of $u \circ \gamma_s$, where γ_s is a path in the same homotopy class as γ, which is generic in the above sense. We can thus define the induced conjugacy class of homorphisms

$$u_{\sharp 1} : \pi_1(\mathcal{M}) \longrightarrow \pi_1(\mathcal{N}).$$

One can check, moreover, that *this homomorphism is preserved by weak convergence in* $W^{1,2}(\mathcal{M},\mathcal{N})$, i.e., if v_k converges weakly to u in $W^{1,2}$ when $k \to +\infty$, and $\forall k \in \mathbb{N}$, $(v_k)_{\sharp 1} = v_{\sharp 1}$ for some $v \in \mathcal{C}^0(\mathcal{M},\mathcal{N})$, then $u_{\sharp 1} = v_{\sharp 1}$. Eventually this leads to the following **existence result** of Schoen and Yau [193]: *assume that $\mathcal{M}$ is surface without boundary. Then, for any family $\gamma_1, \cdots, \gamma_k$ of loops in $\mathcal{M}$ and for any continuous map $v : \mathcal{M} \longrightarrow \mathcal{M}$, there exists a locally energy-minimizing harmonic map in the class of maps $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ such that* $u_{\sharp 1}([\gamma_i]) = v_{\sharp 1}([\gamma_i])$, $\forall i = 1, \cdots, k$. This result has been generalized to the case where the dimension of $\mathcal{M}$ is arbitrary by F. Burstall [30] and B. White [225].

Remarks (i) Note that, if $\pi_j(\mathcal{N}) = 0$ for $j \geq 2$, then the homotopy class of a continuous map u from $\mathcal{M}$ to $\mathcal{N}$ is completely characterized by the induced conjugacy class of homomorphisms $u_{\sharp 1} : \pi_1(\mathcal{M}) \longrightarrow \pi_1(\mathcal{N})$; thus, when $m = 2$, the existence result of Schoen and Yau amounts to minimizing the energy in a given homotopy class of continuous maps between $\mathcal{M}$ and $\mathcal{N}$ (recall that continuous maps are then dense in $W^{1,2}(\mathcal{M},\mathcal{N})$).

(ii) The definition of $u_{\sharp 1} : \pi_1(\mathcal{M}) \longrightarrow \pi_1(\mathcal{N})$ **does not make sense** if $u \in W^{1\cdot p}(\mathcal{M},\mathcal{N})$ for $1 \leq p < 2$. Indeed, as in step (i), we still have that $u \circ \gamma_s$ is continuous for a generic s, but step (ii) does not work: the homotopy class of $u \circ \gamma_s$ can vary as s changes (see B. White [227] or J. Rubinstein and P. Sternberg [186]).

Defining the d-homotopy class. For any $d \in \mathbb{N}$, we say that *two maps $u, v \in \mathcal{C}^0(\mathcal{M},\mathcal{N})$ are d-homotopic* and we write $u \sim_d v$ *if their restrictions to the d-skeleton of a triangulation of $\mathcal{M}$ are homotopic*. For any map $u \in \mathcal{C}^0(\mathcal{M},\mathcal{N})$ we thus can define the *d-homotopy class* $[u]_d := \{v \in \mathcal{C}^0(\mathcal{M},\mathcal{N}) |\ u \sim_d v\}$. Observe that **if**

[17] *This is a consequence of the following observations: on the one hand by using the standard compactness arguments we can extract a subsequence of $(v_k)_{k\in\mathbb{N}}$ which converges weakly in $W^{1,2}$ and a.e. to some limit v, but on the other hand it is clear that v_k converges a.e. (and more precisely pointwise on $S^m \setminus \{(-1,0,\cdots,0)\}$) to $(1,0,\cdots,0)$, so that $v = (1,0,\cdots,0)$. Since this argument works for* any subsequence *the full sequence $(v_\lambda)_{\lambda>0}$ converges weakly to this constant.*

[18] Which itself is the key ingredient of the classical Courant–Lebesgue lemma, see, for example, [88, 3.3.1]

$u \sim_d v$ **then the induced homomorphisms** $u_{\sharp j}, v_{\sharp j} : \pi_j(\mathcal{M}) \longrightarrow \pi_j(\mathcal{N})$ **coincide for each** $1 \leq j \leq d$, so that this notion extends the previous one. Actually A.I. Pluzhnikov [171] and B. White [227] showed that *it is possible to define the d-homotopy class of a map u in $H^{1,p}_s$, $H^{1,p}_w$ or $W^{1,p}(\mathcal{M},\mathcal{N})$ for certain ranges of values of d and p.* The following table summarizes the results proved in [227]. It gives, for each space $H^{1,p}_s$, $H^{1,p}_w$ or $W^{1,p}$, the values of d for which one can define the d-homotopy class of a map u in this space, and it specifies natural topologies which preserve this d-homotopy class:

Spaces	$H^{1,p}_s(\mathcal{M},\mathcal{N})$	$H^{1,p}_w(\mathcal{M},\mathcal{N})$	$W^{1,p}(\mathcal{M},\mathcal{N})$
Values of d for which $[u]_d$ makes sense:	$\mathbb{N} \cap [1,p]$	$\mathbb{N} \cap [1,p)$	$\mathbb{N} \cap [1,p-1]$
Topology which preserves $[u]_d$:	strong $W^{1,p}$	weak $W^{1,p}$	weak $W^{1,p}$

The definition of $[u]_d$ for $u \in H^{1,p}_w(\mathcal{M},\mathcal{N})$ when $d < p$ follows from the following result [171, 226]: *if $d \in \mathbb{N}$ and $d < p$, then $\forall K > 0$, $\exists \varepsilon > 0$, such that if u_1 and u_2 are two Lipschitz continuous maps such that $||u_1||_{W^{1,p}}, ||u_2||_{W^{1,p}} < K$ and $||u_1 - u_2||_{L^p} < \varepsilon$, then $u_1 \sim_d u_2$.* Hence one can define the d-homotopy class of a given $u \in H^{1,p}_w(\mathcal{M},\mathcal{N})$ by using any sequence of Lipschitz continuous maps $(v_k)_{k\in\mathbb{N}}$ which converges weakly to u in $W^{1,p}$ and setting $[u]_d := [v_k]_d$ for k large enough. For $u \in H^{1,p}_s(\mathcal{M},\mathcal{N})$, the previous argument applies also when defining $[u]_d$ if $d < p$; if $d = p$ we must use a further approximation argument.

In constrast, the definition of $[u]_d$ for $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ and $d \leq p-1$ cannot be obtained by using approximations by smooth maps, but must be done directly. Here the idea consists of proving that the restriction of u on a *generic* d-skeleton is continuous and that the homotopy class of this restriction is independent of the d-skeleton, following a strategy similar to the result of Schoen and Yau. The details of the proof are, however, more involved.

The k-homotopy type helps to characterize the topology of the spaces $H^{1,p}_s$ and $W^{1,p}(\mathcal{M},\mathcal{N})$, as follows.

Connected components of $H^{1,p}_s(\mathcal{M},\mathcal{N})$. For any $u \in H^{1,p}_s(\mathcal{M},\mathcal{N})$ denote by $[u]_{H^{1,p}_s}$ its connected component in $H^{1,p}_s(\mathcal{M},\mathcal{N})$ for the strong $W^{1,p}$-topology. The classes $[u]_{H^{1,p}_s}$ have been characterized by A.I. Pluzhnikov [171] and B. White [226] as follows: *the connected components of $H^{1,p}_s(\mathcal{M},\mathcal{N})$ are exactly the $[p]$-homotopy classes inside $H^{1,p}_s(\mathcal{M},\mathcal{N})$.* In other words, for any $u \in H^{1,p}_s(\mathcal{M},\mathcal{N})$, $[u]_{H^{1,p}_s} = [u]_{[p]}$.

This has the following important consequence: *for any smooth map $v \in \mathcal{C}^1(\mathcal{M},\mathcal{N})$, the infimum of the p-energy among smooth maps in the homotopy class of v depends uniquely on the $[p]$-homotopy type of v.* A further result is: *for a smooth map v, $v \sim_{[p]} C$ (where C is a constant map) if and only if the infimum of the p-energy in $[v]_{[p]}$ is* 0 [171, 226]. Note that the limit of a minimizing sequence of the p-energy in a $[p]$-homotopy class $[v]_{[p]}$ may not be in $[v]_{[p]}$, but only in its closure for the sequential weak topology of $W^{1,p}$ in general. See the example with $\mathcal{M} = \mathcal{N} = S^m$, $v = \mathrm{Id}$ discussed at the beginning of this section.

Connected components of $W^{1,p}(\mathcal{M},\mathcal{N})$. For $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ denote by $[u]_{W^{1,p}}$ its connected component. The study of the connected components of $W^{1,p}(\mathcal{M},\mathcal{N})$ was initiated by H. Brezis and Y. Li [25]. Complete answers were obtained by F. B. Hang and F. H. Lin [102] as follows:

(i) *The connected components of $W^{1,p}(\mathcal{M},\mathcal{N})$ are path-connected.* This is a consequence of the following: $\forall u \in W^{1,p}(\mathcal{M},\mathcal{N})$, $\exists \varepsilon > 0$ such that $\forall v \in W^{1,p}(\mathcal{M},\mathcal{N})$, if $||u - v||_{W^{1,p}} < \varepsilon$, then there exists $U \in \mathcal{C}^0([0,1], W^{1,p}(\mathcal{M},\mathcal{N}))$ such that $U(0,\cdot) = u$ and $U(1,\cdot) = v$. We write $u \sim_{W^{1,p}} v$ for this property.

(ii) **the connected components of** $W^{1,p}(\mathcal{M},\mathcal{N})$ **are exactly the** $([p]-1)$**-homotopy classes inside** $W^{1,p}(\mathcal{M},\mathcal{N})$, i.e. $\forall u, v, \in W^{1,p}(\mathcal{M},\mathcal{N})$, $u \sim_{W^{1,p}} v$ if and only if $u \sim_{[p]-1} v$.

(iii) *as p varies, the quotient space $W^{1,p}(\mathcal{M},\mathcal{N})/\sim_{W^{1,p}}$ changes only for integer values of p*, i.e. if $[p_1] = [p_2] < p_1 < p_2 < [p_1]+1$, the map $\iota_{p_2,p_1} : W^{1,p_2}(\mathcal{M},\mathcal{N})/\sim_{W^{1,p_2}} \longrightarrow W^{1,p_1}(\mathcal{M},\mathcal{N})/\sim_{W^{1,p_1}}$ induced by the inclusion $W^{1,p_2}(\mathcal{M},\mathcal{N}) \subset W^{1,p_1}(\mathcal{M},\mathcal{N})$ is a bijection (this was conjectured in [25]).

Result (ii) has the following corollary: *a map $u \in W^{1,p}(\mathcal{M},\mathcal{N})$ is connected to a smooth map by a path if and only if $u_{\sharp,[p]-1}$ is extendible to $\mathcal{M}$ with respect to $\mathcal{N}$.* This implies, in particular, the results (also proved in [25]):

- if $\forall j \in \mathbb{N}$ such that $1 \leq j \leq [p]-1$ we have $\pi_j(\mathcal{N}) = 0$, then $W^{1,p}(\mathcal{M},\mathcal{N})$ is path-connected;
- if $p < m$, then $W^{1,p}(S^m,\mathcal{N})$ is path-connected.

Concerning (iii), the change in the number of connected components of $W^{1,p}(\mathcal{M},\mathcal{N})$ when p varies can occur in two ways. Indeed, *as p decreases, either* connected components coalesce together — this is, for example, the case for $W^{1,p}(S^m,S^m)$: this space has different connected components classified by the topological degree if $p \geq m$ and is connected if $p < m$; *or*, contradicting a conjecture in [25], new connected components can appear — this is the case for $W^{1,p}(\mathbb{R}P^3,\mathbb{R}P^2)$: for $p \in (2,3)$ connected components appear, forming a subset of maps which cannot be connected by a path to a smooth map (and which hence cannot be approximated by smooth maps), see [21, 102].

The degree. If dim $\mathcal{M} =$ dim $\mathcal{N}$, the homotopy classes of maps $\mathcal{M} \longrightarrow \mathcal{N}$ can sometimes be classified by the topological degree. This is the case if, for instance, $\mathcal{M}$ is *connected, oriented*[19] *and without boundary and if* $\mathcal{N} = S^m$ (by a theorem of H. Hopf)[20]. The degree for a map $u \in \mathcal{C}^1(\mathcal{M},S^m)$ is then given by the formula deg $u = \left(1/|S^m|\right)\int_{\mathcal{M}} \det(du)\omega_{\mathcal{M}} = \left(1/|S^m|\right)\int_{S^m} u^*\omega_{S^m}$. We give this formula explicitly for the case $p = 2$:

$$\text{deg}\, u = \frac{1}{4\pi}\int_{\mathcal{M}} u^*\omega_{S^2} = \frac{1}{4\pi}\int_{\mathcal{M}} \left\langle u, \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y}\right\rangle dx\, dy,$$

where (x,y) are local conformal coordinates on $\mathcal{M}$. This functional, being quadratic in the first derivatives of u, has the following continuity properties:

(i) it is continuous on $\mathcal{C}^1(\mathcal{M},S^2)$ for the *strong* and the *weak* $W^{1,p}$ topology for all $p > 2$, hence for $p > 2$ we can extend deg on $H^{1,p}_s(\mathcal{M},S^2) = W^{1,p}(\mathcal{M},S^2)$;

(ii) it is continuous on $\mathcal{C}^1(\mathcal{M},S^2)$ for the *strong* **but not** for the *weak* $W^{1,p}$ topology for $p = 2$, hence since $H^{1,2}_s(\mathcal{M},S^2) = W^{1,2}(\mathcal{M},S^2)$ we can extend deg on $W^{1,2}(\mathcal{M},S^2)$, but this functional is not continuous with respect to the weak topology;

(iii) it is **not** continuous on $\mathcal{C}^1(\mathcal{M},S^2)$ for the *strong* or the *weak* $W^{1,p}$ topology for all $p < 2$.

In cases (i) and (ii) ($p \geq 2$), the degree functional takes integer values and, $\forall k \in \mathbb{N}$, $\text{deg}^{-1}(k)$ is a connected component of $W^{1,p}(\mathcal{M},S^2)$ for its strong topology. In case (i), the continuity for the weak topology follows from the fact that, on the one hand, for a sequence $(u_k)_{k\in N}$ which converges weakly to some u in $W^{1,p}(\mathcal{M},S^2)$, $f_k := (\partial_x u_k)\times(\partial_y u_k)$ converges weakly in $L^{p/2}$ to $f := (\partial_x u)\times(\partial_y u)$, because of a phenomenon of *compensated compactness*, based on writing $f_k = \partial_x\left(u_k(\partial_y u_k)\right) - \partial_y\left(u_k(\partial_x u_k)\right)$ (see [159, 211]). On the other hand, by the Rellich–Kondrakov theorem, we can assume that $u_k \to u$ strongly in $L^{2p/p-2}$ and hence in $L^{p/p-2} = \left(L^{p/2}\right)^*$. It follows that the integral $\int_{\mathcal{M}}\langle u_k, f_k\rangle\omega_{\mathcal{M}}$ converges to $\int_{\mathcal{M}}\langle u, f\rangle\omega_{\mathcal{M}}$. This delicate argument breaks down[21] for $p = 2$: we still have that f_k converges in the weak-$\star$ topology of L^1, but we *cannot find*, in general, a subsequence of u_k which converges strongly in L^∞ (otherwise we would have an embedding of $W^{1,2}(\mathcal{M})$ in $\mathcal{C}^0(\mathcal{M}) \subset L^\infty(\mathcal{M})$!). Indeed, in the case where $\mathcal{M} = S^2$, the family of (degree 1) Möbius maps $(T_\lambda)_{\lambda\in\mathbb{R}}$ converges weakly to a constant map in $W^{1,2}(S^2,S^2)$ as $\lambda \to +\infty$ (a *bubbling* phenomenon, see §5.3). Lastly (iii) can seen by considering the family of maps $(u_t)_{t\in[0,1]}$ from S^2 to S^2 defined by $u_t(x) = (x-ta)/|x-ta|$, where $a \in \mathbb{R}^3$ has $|a| = 2$; for $1 \leq p < 2$, this defines a continuous path in $W^{1,p}(S^2,S^2)$, which connects the smooth map $u_\odot = u_0$ of degree 1 to the smooth map u_1 of degree 0 (see [27, 21]).

Lastly, in [26] H. Brezis, Y. Li, P. Mironescu and L. Nirenberg defined a notion of *degree for maps* $u \in W^{1,p}(S^n\times\Lambda^{m-n},S^n)$, *where* $m \geq n$ *and* Λ^{m-n} *is an open connected subset of* $\mathbb{R}^{m-n}$, *assuming that* $p \geq n+1$ (note that, in the special case $m = n$, the condition $p \geq n$ is enough). In the case $n = 1$, we recover from this result the conclusions of [30, 227, 186]. Furthermore, *two maps u and v in $W^{1,p}(S^n\times\Lambda^{m-n},S^n)$ are in the same connected component if and only if deg f = deg g* (see [26, 21]). See [27] for further results concerning the degree.

[19] If $\mathcal{M}$ is connected, without boundary but *not oriented*, the homotopy classes are classified by the degree mod 2.

[20] But if $\mathcal{M}$ and $\mathcal{N}$ are spheres with different dimensions, this is not so, for example, maps from S^3 to S^2 are classified according to their Hopf degree, see [110].

[21] A rich interplay between cohomology and compensated compactness theory occurs here: for any smooth function $\psi \in \mathcal{C}^1(\mathcal{M})$ and any 2-form β on S^2 which is *exact*, i.e., $\beta = d\alpha$ for some 1-form α, the functional $u \longmapsto \int_{\mathcal{M}}\psi\, u^*\beta$ is continuous for the weak $W^{1,2}$ topology because of the relation $u^*\beta = d(u^*\alpha)$, so that a compensated compactness argument is possible; however, if β is *closed but not exact*, this argument does not work. See [98] for a detailed study of these phenomena.

3.4 The trace of Sobolev maps

For any domain $\Omega \subset \mathbb{R}^m$ with smooth boundary and for any $p \in [1, +\infty)$, the trace operator $\mathrm{tr} : \mathcal{C}^1(\Omega, \mathbb{R}^N) \longrightarrow \mathcal{C}^1(\partial\Omega, \mathbb{R}^N)$ can be extended to a continuous and surjective operator $\mathrm{tr} : W^{1,p}(\Omega, \mathbb{R}^N) \longrightarrow W^{1-\frac{1}{p},p}(\partial\Omega, \mathbb{R}^N) := \left\{ g \in L^p(\partial\Omega, \mathbb{R}^N) |\ ||g||_{W^{1-\frac{1}{p},p}(\partial\Omega)} < +\infty \right\}$ if $p > 1$, where:

$$||g||_{W^{1-\frac{1}{p},p}(\partial\Omega)} := ||g||_{L^p(\partial\Omega)} + \left(\int_{\partial\Omega} \int_{\partial\Omega} \frac{|g(x) - g(y)|^p}{|x - y|^{p+m-2}} \, dx \, dy \right)^{1/p}.$$

(If $p = 1$, the image of the trace operator is $L^1(\partial\Omega, \mathbb{R}^N)$.) This definition can be extended to the case of a manifold $\mathcal{M}$ with a smooth boundary, by using local charts to define $W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathbb{R}^N)$ and the trace operator $\mathrm{tr} : W^{1,p}(\mathcal{M}, \mathbb{R}^N) \longrightarrow W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathbb{R}^N)$. Similarly the trace $\mathrm{tr}\, u$ of a map $u \in W^{1,p}(\mathcal{M}, \mathcal{N})$ is always contained in:

$$W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N}) := \left\{ g \in W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathbb{R}^N) |\ g(x) \in \mathcal{N}, \text{ for a.e. } x \in \partial\mathcal{M} \right\}.$$

However, the map $\mathrm{tr} : W^{1,p}(\mathcal{M}, \mathcal{N}) \longrightarrow W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$ is not onto in general, i.e., it is not true in general that any map $g \in W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$ is the trace of a map in $W^{1,p}(\mathcal{M}, \mathcal{N})$. Obstructions occur even for continuous maps: for instance, the trace operator $\mathrm{tr} : \mathcal{C}^1(B^m, \mathcal{N}) \longrightarrow \mathcal{C}^1(\partial B^m, \mathcal{N})$ is onto if and only if $\pi_{m-1}(\mathcal{N}) = 0$. In the following we define $T^p(\partial\mathcal{M}, \mathcal{N}) := \left\{ g \in W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N}) |\ \exists u \in W^{1,p}(\mathcal{M}, \mathcal{N}) \text{ such that } u|_{\partial\mathcal{M}} = g \right\}$. The question whether $T^p(\partial\mathcal{M}, \mathcal{N}) = W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$ for given $\mathcal{M}, \mathcal{N}$ and p is largely open. Here are some results:

- If $p \geq m$, F. Bethuel and F. Demengel [16] proved that $T^p(\partial\mathcal{M}, \mathcal{N}) = W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$ **if and only if any continuous map** $g \in \mathcal{C}^0(\partial\mathcal{M}, \mathcal{N})$ **can be extended to a map** $u \in \mathcal{C}^0(\mathcal{M}, \mathcal{N})$.
- For $1 \leq p < m$, R. Hardt and F. H. Lin [107] proved that

$$\text{if} \quad \pi_1(\mathcal{N}) = \cdots = \pi_{[p]-1}(\mathcal{N}) = 0, \quad \text{then} \quad T^p(\partial\mathcal{M}, \mathcal{N}) = W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N}).$$

- Conversely Bethuel and Demengel [16] proved that, *if* $1 \leq p < m$, *then* $\pi_{[p]-1}(\mathcal{N}) = 0$ *is a necessary condition for having* $T^p(\partial\mathcal{M}, \mathcal{N}) = W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$. Moreover, they proved that, *if* $1 < p < m$, *then, for any* $\mathcal{N}$ *such that* $\pi_j(\mathcal{N}) \neq 0$ *for some integer* $j \leq [p] - 1$, *one can construct a manifold with boundary* $\mathcal{M}$ *such that* $T^p(\partial\mathcal{M}, \mathcal{N}) \neq W^{1-\frac{1}{p},p}(\partial\mathcal{M}, \mathcal{N})$.

Furthermore it is proved in [16] that, in the case where $\mathcal{M} = B^m$ and $\mathcal{N} = S^1$, *if* $3 \leq p < m$ *then* $T^p(\partial B^m, S^1) \neq W^{1-\frac{1}{p},p}(\partial B^m, S^1)$. For more results on fractional Sobolev spaces into S^1, see the report of P. Mironescu [155] or the papers [20, 183].

4 Regularity

4.1 Regularity of *continuous* weakly harmonic maps

Note that *as soon as we know that a (weakly) harmonic map* ϕ *is continuous*, then we can *localize* its image, i.e. by restricting ϕ to a sufficiently small ball in $\mathcal{M}$ *we can assume that the image of* ϕ *is contained in an arbitrary small subset of* $\mathcal{N}$ *with good convexity properties or with a convenient coordinate system*. Thus the main results concern the **higher regularity** of *continuous weakly harmonic maps*. The hard step here is to prove that the weak solution ϕ is Lischiptz continuous, i.e. that $d\phi$ is bounded a.e.[22]. This was proved by O. Ladyzhenskaya, N. Ural'tseva in

[22]Once we know that $d\phi \in L^\infty_{loc}$, it then follows from (25) that $\Delta\phi \in L^\infty_{loc}$, which implies by standard estimates on the inverse of the Laplacian (see [157], 6.2.5) that $\phi \in W^{2,p}_{loc}$, for all $p < \infty$. Hence we deduce that $\Delta\phi \in W^{1,p}_{loc}$ and hence that $\phi \in W^{3,p}_{loc}$ for all $p > 0$. We can then repeat this argument to show that $\phi \in W^{r,p}_{loc}, \forall r$ and so the smoothness of the solution follows (it is called a *bootstrap* argument).

[143] in a more general context, by using contributions of C. B. Morrey [157], a proof can be found in [135]. In [189], a proof is given in the case when the weakly harmonic map is *Hölder* continuous. Estimates of the Hölder norms of higher derivatives of ϕ in terms of $|d\phi|$ were obtained by J. Jost and H. Karcher [136] for harmonic maps with values in a geodesically convex ball: on such balls they construct and use *almost linear functions* (which are based on *harmonic coordinates*, in which the Hölder norm of Christoffel symbols are bounded in terms of the curvature).

4.2 Regularity results in dimension two

If dim $\mathcal{M} = 2$ and $\mathcal{N}$ can be embedded isometrically in some Euclidean space, **all weakly harmonic maps in** $W^{1,2}(\mathcal{M}, \mathcal{N})$ **are continuous** and hence, by the results of §4.1, smooth. This was proved first for *minimizing maps* by C. B. Morrey [156] (see also [88, p. 304] for an exposition of the original proof of Morrey).

This was extended to *conformal weakly harmonic maps* by M. Grüter [92] (see also [133]). Grüter's proof works also for conformal weak solutions of the *H-system* $\Delta_g u + A(u)(du, du) = 2H(u)(\partial u/\partial x^1 \times \partial u/\partial x^2)$ in an oriented 3-dimensional manifold $\mathcal{N}$, where $H(u)$ is a L^∞ bounded function on $\mathcal{N}$. Conformal solutions to this problem parametrize surfaces with prescribed mean curvature H. The proof in [92] uses the conformality assumption in an essential way. Then R. Schoen [189] proved that *all stationary maps on a surface are smooth*. The proof is based on the following trick. Let $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$ be a stationary map; since the Hopf differential $\mathcal{H}$ is holomorphic (see §3.1), *either* it vanishes everywhere and then u is conformal and we apply directly the result of Grüter, *or* $\mathcal{H} = h(dz)^2$ vanishes only at isolated points. If so, outside the zeros of h we can locally define the harmonic function $f(z) := \mathrm{Re}\big(2i\int_{z_0}^z \sqrt{h(\zeta)}d\zeta\big)$. Then the map $U := (u, f)$ with values in $\mathcal{N} \times \mathbb{R}$ is weakly harmonic and conformal and hence is smooth. Thus u is smooth outside the zeros of h, and hence is smooth everywhere by the result of J. Sacks and K. Uhlenbeck [188] (see §5.3).

The regularity of weakly harmonic maps on a surface in *the general case* was proved by F. Hélein, first in the special case where $\mathcal{N} = S^n$ [113], and then in the case where $\mathcal{N}$ is an arbitrary compact Riemannian manifold without boundary [116]. The proof for $\mathcal{N} = S^n$ is simpler and relies on a previous work by H. Wente [221] on the solutions $X \in W^{1,2}(B^2, \mathbb{R}^3)$ on the unit ball[23] of $\mathbb{R}^2$ of the H-system

$$\Delta X = 2H\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}\,, \tag{34}$$

for a constant $H \neq 0$. Wente proved that any weak solution of this system is continuous and hence, thanks again to the general theory of quasilinear elliptic systems, smooth. It is based on the special structure of (34) which reads, for example for the first component of X, $\Delta X^1 = 2H\{X^2, X^3\}$, where we introduce the notation

$$\{a, b\} := \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x} \qquad \text{for } a, b \in W^{1,2}(\Omega), \quad \text{where } \Omega \subset \mathbb{R}^2.$$

Since $\{a, b\}$ is quadratic in the first derivatives of a and b, it sits naturally in $L^1(B^2)$. Also, we know from the standard theory of singular integrals that, for any function $f \in L^1(B^2)$, a solution ψ of $-\Delta\psi = f$ is necessarily in all spaces $L^p_{loc}(B^2)$, for $1 \leq p < \infty$, but fails to be in $L^\infty(B^2)$. Here the key result is that a solution φ of the equation $-\Delta\varphi = \{a, b\}$ on B^2 is slightly more regular; in particular, we can locally estimate the L^∞ norm of φ in terms of $||a||_{W^{1,2}}$ and $||b||_{W^{1,2}}$. This is due to the special structure of $\{a, b\}$, which is a *Jacobian determinant*, and is connected to the theory of compensated compactness [159, 211]. These properties were expressed by H. Brezis and J.-M. Coron [23] as a *Wente inequality*,

$$||\varphi||_{L^\infty} + ||d\varphi||_{L^2} \leq C||a||_{W^{1,2}}||a||_{W^{1,2}}\,, \tag{35}$$

valid for any solution φ of $-\Delta\varphi = \{a, b\}$ on B^2 which satisfies $\varphi = 0$ on ∂B^2. This inequality was subsequently extended to arbitrary surfaces and the best constants for estimating $||\varphi||_{L^\infty}$ or $||d\varphi||_{L^2}$ were found, see [118]. The

[23] Since the regularity problem is local, and every ball in a Riemannian surface is conformally equivalent to the Euclidean ball B^2, there is no loss of generality in working on B^2.

point here is that, once we have (35), we can easily deduce, by approximating by smooth maps, that **solutions to** $-\Delta\varphi = \{a, b\}$ **are continuous.** Hence the result of Wente follows.

For harmonic maps the key observation is that a u *is weakly harmonic if and only if the following conservation laws hold*

$$d\left(\star(u^i du^j - u^j du^i)\right) = 0 \qquad \forall\, i, j \ \text{ such that } \quad 1 \leq i, j \leq n+1, \tag{36}$$

where $\star$ is the Hodge operator on B^2. This was remarked and exploited for evolution problems [41, 197]. One can either check (36) directly by using (26) or derive it as a consequence of Noether's theorem, due to the invariance of the Dirichlet functional under the action of $SO(n+1)$ on $W^{1,2}(\Omega, S^n)$ [118]. From (36) we deduce that there exist maps $b^i_j \in W^{1,2}(\Omega)$ such that $db^i_j = -\star(u^i du^j - u^j du^i)$. Then we note that $\Delta b^i_j dx \wedge dy = d(\star(db^i_j)) = d(u^i du^j - u^j du^i) = 2\{u^i, u^j\} dx \wedge dy$ so that, by a Hodge decomposition of db^i_j and by using Wente inequality, we can deduce the continuity of u. This was the approach in [113]. A more direct proof[24] is the following: since $2\langle u, du\rangle = d(|u|^2) = 0$, we can rewrite the harmonic map equation (26) as

$$-\Delta u^i = u^i |du|^2 = \left(u^i \frac{\partial u_j}{\partial x} - u_j \frac{\partial u^i}{\partial x}\right)\frac{\partial u^j}{\partial x} + \left(u^i \frac{\partial u_j}{\partial y} - u_j \frac{\partial u^i}{\partial y}\right)\frac{\partial u^j}{\partial y} = \{b^i_j, u^j\}\,, \tag{37}$$

where, as usual we sum over repeated indices, $u_i := \delta_{ij} u^j$ and we have used the relation $db^i_j = -\star(u^i du^j - u^j du^i)$. Note that an alternative way to write (37) is

$$d(\star du^i) + db^i_j \wedge du^j = 0. \tag{38}$$

We deduce that u is continuous. This method can be extended without difficulty if we replace the target S^n by any homogeneous manifold $\mathcal{N}$, since then Noether's theorem provides us with the conservations laws that we need [114].

In the case where $\mathcal{N}$ has no symmetry we need to refine the results on the quantities $\{a, b\}$. In [45] R. Coifman, P.-L. Lions, Y. Meyer and S. Semmes proved that, **if** $a, b \in W^{1,2}(\mathbb{R}^2)$ **then** $\{a, b\}$ **belongs to the generalized Hardy space** $\mathcal{H}^1(\mathbb{R}^2)$. We do not give here the various and slightly complicated definitions of the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$, which was introduced by E. Stein and G. Weiss [206], but just list useful properties of it:

a) $\mathcal{H}^1(\mathbb{R}^m)$ is a strict subspace of $L^1(\mathbb{R}^m)$;

b) any function φ on $\Omega \subset \mathbb{R}^m$ such that $\Delta\varphi = f$ on Ω, where $f \in \mathcal{H}^1(\mathbb{R}^m)$, belongs to $W^{2,1}(\Omega)$, i.e. its second partial derivatives are integrable [205];

c) **let** $\alpha \in W^{1,2}(\mathbb{R}^m)$ **and** β **be a closed (in the distribution sense)** $(m-1)$**-form on** $\mathbb{R}^m$ **with coefficients in** $L^2(\mathbb{R}^m)$**; then** $d\alpha \wedge \beta = f\, dx^1 \wedge \cdots \wedge dx^m$**, where** f **belongs to** $\mathcal{H}^1(\mathbb{R}^m)$ [45]. In particular, if $a, b \in W^{1,2}(\mathbb{R}^2)$ then $\{a, b\} \in \mathcal{H}^1(\mathbb{R}^2)$;

d) by a theorem of C. Fefferman [72], $\mathcal{H}^1(\mathbb{R}^m)$ is the dual space of VMO$(\mathbb{R}^m)$ and **the dual space of** $\mathcal{H}^1(\mathbb{R}^m)$ **is BMO**$(\mathbb{R}^m)$ (see footnote 13).

Now we come back to the regularity problem. We now assume that there exists a smooth section $\widetilde{e} := (\widetilde{e}_1, \ldots, \widetilde{e}_n)$ of the bundle $\mathcal{F}$ of orthonormal tangent frames on $\mathcal{N}$. Although there are topological obstructions, there are ways to reduce to this situation, see [118]. For any map $u \in W^{1,2}(B^2, \mathcal{N})$, consider the pull-back bundle $u^*\mathcal{F}$. To any $R \in W^{1,2}(B^2, SO(n))$ we associate the section $e := \widetilde{e} \circ u \cdot R$ of $u^*\mathcal{F}$ defined by $e_a := (\widetilde{e}_b \circ u) R^b_a$, and we minimize over all gauge transformations $R \in W^{1,2}(B^2, SO(n))$ the functional $F(e) := \frac{1}{4}\int_{B^2} \sum_{1 \leq a,b \leq n} |\omega^b_a|^2 dx^1 dx^2$, where $\omega^b_a := \langle de_a, e_b\rangle$. It is easy to show that the infimum is achieved for some *harmonic section* $\underline{e}$ of $u^*\mathcal{F}$ [118, Lemma 4.1.3]. The Euler–Lagrange equation satisfied by $\underline{e}$ can be written as a system of conservation laws (again a consequence of Noether's theorem):

$$d\left(\star\underline{\omega}^b_a\right) = 0 \quad \text{on } \Omega \quad \text{and} \quad \underline{\omega}^b_a(\partial_n) = 0 \quad \text{on } \partial\Omega\,, \tag{39}$$

[24] This was pointed out by P.-L Lions.

which is satisfied by its Maurer–Cartan form $\underline{\omega}_a^b := \langle d\underline{e}_a, \underline{e}_b \rangle$. Thanks to (39), we can construct maps $A_a^b \in W^{1,2}(B^2)$ such that $dA_a^b = \star\underline{\omega}_a^b$ on B^2 and $A_a^b = 0$ on ∂B^2. Then the key observation is that

$$\Delta A_a^b = \left\langle \frac{\partial \underline{e}_a}{\partial x}, \frac{\partial \underline{e}_b}{\partial y} \right\rangle - \left\langle \frac{\partial \underline{e}_a}{\partial y}, \frac{\partial \underline{e}_b}{\partial x} \right\rangle = \sum_{i=1}^N \{\underline{e}_a^i, \underline{e}_b^i\}\,, \tag{40}$$

where $(\underline{e}_a^i(x))_{1\leq i\leq N}$ are the coordinates of $\underline{e}_a(x) \in T_{u(x)}\mathcal{N} \subset \mathbb{R}^N$ in a fixed orthonormal basis of $\mathbb{R}^N$. Hence the right hand side of (40) coincides locally with some function in $\mathcal{H}^1(\mathbb{R}^2)$, thanks to property c) of Hardy spaces above. Hence by property b), the second derivatives of A_a^b are locally integrable. This property implies that the components of dA_a^b are in the *Lorentz space* $L^{2,1}$, a slight improvement on L^2 [118]. But since $dA_a^b = \star\underline{\omega}_a^b$, this improvement is valid also for the connection $\underline{\omega}_a^b$.

Lastly, consider a weakly harmonic map $u \in W^{1,2}(B^2, \mathcal{N})$ and write its Euler–Lagrange equation (25) in the moving frame $\underline{e}$: if we set $\alpha^a := \langle \partial u/\partial z, \underline{e}_a \rangle$ and $\theta_b^a := \underline{\omega}_a^b(\partial/\partial\overline{z})$, we obtain $\partial\alpha^a/\partial\overline{z} = \theta_b^a\alpha^b$. In this equation, α^a is in L^2 whereas, thanks to the choice of a *Coulomb moving frame* $\underline{e}$, the function θ_b^a is in $L^{2,1}$. This slight improvement turns out to be enough to conclude that u is Lipschitz continuous.

Recently T. Rivière [184] proved the regularity of all maps $u \in W^{1,2}(B^2, \mathcal{N})$ which are critical points of the functional $F(u) := \frac{1}{2}\int_{B^2} |du|^2 dxdy + \int_{B^2} u^*\omega$, where ω is a $\mathcal{C}^1$ differential 2-form on $\mathcal{N}$ such that the coefficients of $d\omega$ are in $L^\infty(\mathcal{N})$. This answers positively conjectures of E. Heinz and S. Hildebrandt. The method provides, in particular, an alternative proof of the regularity of weakly harmonic maps with values in an arbitrary manifold without Coulomb moving frames. Instead, it relies on constructing *conservation laws*, as for maps into the sphere, but *without* symmetry. First, let us try to imitate equation (38) for a weakly harmonic map into an arbitrary compact manifold $\mathcal{N}$. We let $A \in \Gamma(S^2T^*\mathcal{N} \otimes N\mathcal{N})$ be the second fundamental form of the embedding $\mathcal{M} \subset \mathbb{R}^N$. For $y \in \mathcal{N}$ denote by $A_{jk}^i(y)$ the components of A_y in a fixed orthonormal basis $(\epsilon_1, \cdots, \epsilon_N)$ of $\mathbb{R}^N$, i.e., $A_y(X,Y) = A_{jk}^i(y)X^jY^k\epsilon_i$ $(X, Y \in T_y\mathcal{N})$. Then we can write the Euler–Lagrange equation (25) for u as

$$d(\star du^i) - (\star A_{kj}^i(u)\, du^k) \wedge du^j = 0.$$

But since A takes values in the normal bundle, we have $\sum_{j=1}^N A_{ki}^j(u)du^j = 0$, so that we can transform the previous equation into

$$d(\star du^i) - (\star\Omega_j^i) \wedge du^j = 0 \quad \text{where} \quad \Omega_j^i := A_{kj}^i(u)\, du^k - A_{ki}^j(u)\, du^k. \tag{41}$$

If we compare with (38), which can also be written $d(\star du^i) - (\star(u^i du_j - u_j du^i)) \wedge du^j = 0$, we see that (38) is a particular case of (41), where $\Omega_j^i = u^i du_j - u_j du^i$. The difference is that we do not have $d(\star\Omega_j^i) = 0$ in general. But *both forms are skew-symmetric in* (i,j). And that property is actually enough. The idea is to substitute for $\star du^i$ another quantity, of the form $A_j^i(\star du^i)$, where $A_j^i \in W^{1,2}(B^2)$. A computation using (41) shows that

$$d\left(A_j^i(\star du^i)\right) = -\star\left(dA_j^i - A_k^i\Omega_j^k\right) \wedge du^j.$$

Hence if we assume that we can find maps $A := (A_j^i)_{1\leq i,j\leq N}$, $B := (B_j^i)_{1\leq i,j\leq N} \in W^{1,2}(B^2, M(N,\mathbb{R}))$ such that A is invertible with a bounded inverse and

$$\star\left(dA_j^i - A_k^i\Omega_j^k\right) = dB_j^i\,, \tag{42}$$

then we obtain an equation similar to (38), i.e.,

$$d\left(\star A_j^i(\star du^j)\right) + dB_j^i \wedge du^j = d\left(\star A_j^i(\star du^j) + B_j^i du^j\right) = 0. \tag{43}$$

Then formulation (43) allows us to prove the continuity of u easily: we use the Hodge decomposition: $A_j^i du^j = dD_j^i - \star dE_j^i$ for some functions $D_j^i, E_j^i \in W^{1,2}(B^2)$, then we deduce $d(\star dE_j^i) = -dA_j^i \wedge du^j$, i.e. $-\Delta E_j^i = \{A_j^i, u^j\}$ from the definition of E_j^i and so we obtain $d(\star dD_j^i) = -dB_j^i \wedge du^j$, i.e. $-\Delta D_j^i = \{B_j^i, u^j\}$ from (43). Hence, from properties b) and c) of Hardy spaces, we deduce that the first derivatives of D_j^i and E_j^i are in the

Lorentz space $L^{2,1}$; since A has a bounded inverse, it follows that the first derivatives of u are also in $L^{2,1}$. Thus u is continuous. To complete the proof one needs to prove the existence of A and B solving (42). For that purpose Rivière adapts a result of K. Uhlenbeck [217] to first prove the existence of some gauge transformation map $P \in W^{1,2}(B^2, SO(N))$ such that $\Omega^P := P^{-1}dP + P^{-1}\Omega P$ satisfies the Coulomb gauge condition $d(\star\Omega^P) = 0$. This implies, in particular, that $P^{-1}dP + P^{-1}\Omega P = \star d\xi$, for some map $\xi \in W^{1,2}(B^2, so(N))$. Then by putting $A := \widetilde{A}P^{-1}$, equation (42) reduces to $d\widetilde{A} - \widetilde{A}(\star d\xi) + (\star dB)P = 0$, a linear elliptic system in $\widetilde{A}$ and B, which can be solved by a fixed point argument.

4.3 Regularity results in dimension greater than two

Preliminary facts

If $m := \dim \mathcal{M} \geq 3$, weakly harmonic maps in $W^{1,2}(\mathcal{M}, \mathcal{N})$ will *not* be regular in general and may even be completely discontinuous as shown by the result of T. Rivière (see §5.4), unless $\mathcal{N}$ has some convexity properties (see §6.3). But *partial* regularity results hold for minimizing or stationary maps. Indeed we are able, in general, to prove that such maps are smooth outside a closed subset Σ that we will call the *singular set*. The size of Σ is estimated in terms of some Hausdorff dimension and corresponding Hausdorff measure. Fix some $s \in [0, m]$. For any covering of Σ by a countable union of balls $(B_j^m)_{j\in J}$ of radius r_j, consider the quantity $\mathcal{H}^s((B_j^m)_{j\in J}, \Sigma) := \alpha(s)\sum_{j\in J} r_j^s$, where $\alpha(s) = 2\pi^{\frac{s}{2}}/s\Gamma(\frac{s}{2})$: this measures *approximately* the s-dimensional volume of Σ. The **s-dimensional Hausdorff measure of** Σ is:

$$\mathcal{H}^s(\Sigma) := \sup_{\delta>0} \inf_{r_j<\delta} \mathcal{H}^s\big((B_j^m)_{j\in J}, \Sigma\big) \quad \text{(in the infimum, } (B_j^m)_{j\in J} \text{ is such that } \Sigma \subset \cup_{j\in J} B_j^m).$$

Then there exists some $d \in [0, m]$ such that $\forall s \in [0, d)$, $\mathcal{H}^s(\Sigma) = 0$ and $\forall s \in (d, m]$, $\mathcal{H}^s(\Sigma) = +\infty$. If $\mathcal{H}^d(\Sigma)$ is finite, d is called the **Hausdorff dimension of** Σ. In the special case when Σ is a smooth submanifold of dimension k, then $d = k$ and $\mathcal{H}^d(\Sigma)$ coincides with the d-dimensional volume of Σ.

Furthermore it is useful to analyze the first consequences of the Euler–Lagrange equation (25) and the conservation law for the stress-energy tensor (29) concerning the regularity of a weak solution $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$. Equation (25) implies that the components of $\Delta_g u$ are in $L^1(\mathcal{M})$, from which one can deduce that the first derivatives of u are locally in L^p for $1 \leq p < m/(m-1)$, which has no interest. However, the conservation law (29) immediately provides the following strong improvement to the regularity of u.

The monotonicity formula. Given a map $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$; to each $x \in \mathcal{M}$ and $r > 0$ such that the geodesic ball $B(x, r)$ is contained in $\mathcal{M}$, we associate the quantity

$$E_{x,r}(u) := \frac{1}{r^{m-2}} \int_{B(x,r)} |du|_g^2 \omega_g.$$

Now *let $B(a, r) \subset \mathcal{M}$ be a geodesic ball centred at a and of radius $r > 0$ such that the distance from a to its cut locus and to $\partial\mathcal{M}$ is greated than r. Then there exist constants C (depending on m) and Λ (depending on a bound of the curvature on $B(a, r)$) such that, if $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$* ***satisfies the relation*** *(29), then for all $x \in B(a, r/2)$ the function $(0, r/2] \ni \rho \longmapsto e^{C\Lambda\rho}E_{x,\rho}(u)$ is non-decreasing* [231]. If the metric on $\mathcal{M}$ is *flat* this holds with $\Lambda = 0$ and this can be proved by integrating over $B^m(x_0, r)$ the relation $(\partial/\partial x^\alpha)\big((x^\beta - x_0^\beta)S_\beta^\alpha(u)\big) = S_\alpha^\alpha(u) = \frac{1}{2}(m-2)|du|_g^2$, a consequence of (29). We then get an identity from which we derive:

$$\text{for } 0 < r_1 < r_2, \quad E_{x,r_2}(u) - E_{x,r_1}(u) = \frac{2}{r^{m-2}} \int_{B^m(x,r_2)\setminus B^m(x,r_1)} \left|\frac{\partial u}{\partial n}\right|^2 d^m x \geq 0, \tag{44}$$

where $\partial u/\partial n$ denotes the normal derivative of u. The monotonicity formula has strong consequences; for simplicity, we expound these in the case where $(\mathcal{M}, g)$ is flat[25]. First, elementary geometric reasoning shows that, for

[25]Since in the regularity theory we are interested in the local properties of weak solutions, the effect of the curvature of $\mathcal{M}$ can be neglected.

$\gamma \in (0,1)$, $E_{x_0,r}(u)$ controls $E_{x,\gamma r}(u)$ for $x \in B^m(x_0,(1-\gamma)r)$ and hence, by (44), $E_{x_0,r}(u)$ controls $E_{x,\rho}(u)$ for $x \in B^m(x_0,(1-\gamma)r)$ and $\rho \leq \gamma r$. Then, by a Poincaré–Sobolev inequality:

$$\frac{1}{\rho^m}\int_{B^m(r,\rho)} |u-u_{x,\rho}|^2\, dx \leq C\, E_{x,\rho}(u), \qquad \text{with } u_{x,\rho} := \frac{1}{|B^m(x,\rho)|}\int_{B^m(x,\rho)} u\, dx,$$

we deduce a bound on $\sup\big\{\rho^{-m}\int_{B^m(r,\rho)} |u-u_{x,\rho}|^2\, dx \mid x \in B^m(x_0,(1-\gamma)r), \rho \leq \gamma r\big\}$, i.e., roughly speaking, on the local BMO-norm of u on $B^m(x_0,(1-\gamma)r)$. The BMO space (see footnote 13) contains all the spaces L^p, for $1 \leq p < \infty$, and hence is very close to L^∞. Thus this is an important gain of regularity.

The ε-regularity. Our task is to put together consequences of (25) and (29) in order to improve the preceding observations. The *(continuous) main step* in most regularity results consists *of showing that there exists some* $\varepsilon_0 > 0$ *such that for any weak solution* u *(for a suitable notion of 'weak'), if* $E_{a,r}(u) < \varepsilon_0$*, then, for* $0 < \sigma < \rho$ *such that* ρ/r *is sufficiently small and for* $x \in \mathcal{M}$ *close to* a,

$$E_{x,\sigma}(u) \leq C\left(\frac{\rho}{\sigma}\right)^{\alpha} E_{x,\rho}(u) \tag{45}$$

for some constants $C > 0$ *and* $\alpha > 0$. If this is true, we are in a position to apply the *Dirichlet growth theorem of Morrey* (see [157, 83]), which implies that u is Hölder continuous with exponent $\alpha/2$ in a neighbourhood of a. This method is the reason for the partial regularity: a covering argument shows that, if $\Sigma := \{a \in \mathcal{M} |\ \lim_{r\to 0}\inf E_{a,r}(u) \geq \varepsilon_0\}$ had a non-vanishing $(m-2)$-dimensional measure, u would have infinite energy, hence $\mathcal{H}^{m-2}(\Sigma) = 0$ by contradiction. The continuous main step itself can be achieved by proving a *discrete* version of it: *there exists some* $\varepsilon_0 > 0$ *and some* $\tau \in (0,1)$ *such that, for any weak solution* u *(here again we stay vague), if* $E_{x,r}(u) < \varepsilon_0$*, then*

$$E_{x,\tau r}(u) \leq \frac{1}{2}E_{x,r}(u). \tag{46}$$

Indeed, by using this result at several scales and concatenating them, one easily deduces (45).

A first attempt. We now describe in a naive way an attempt to prove the *discrete main step* (46). First, we observe that, if u is defined on $B^m(a,r)$, then the map $T_{a,r}u$ defined by $T_{a,r}u(x) := u(rx+a)$ is defined on $B^m := B^m(0,1)$ and, furthermore, $E_{0,1}(T_{a,r}u) = E_{a,r}(u)$, which shows that one can work without loss of generality with a map $u \in W^{1,2}(B^m,\mathcal{N})$. So our aim is to prove that $E_{0,\tau}(u) \leq \frac{1}{2}E_{0,1}(u)$ for some $\tau > 0$ under some smallness assumption on $E_{0,1}(u)$. We split $u = v + w$, where v agrees with u on ∂B^m and is harmonic with values in $\mathbb{R}^N \supset \mathcal{N}$, and w vanishes on ∂B^m and has $\Delta w = \Delta u = -A(u)(du,du)$. Then, for $\tau \in (0,1)$,

$$E_{0,\tau}(u) = \frac{1}{\tau^{m-2}}\int_{B^m(0,\tau)} |du|^2 d^m x \leq \frac{2}{\tau^{m-2}}\int_{B^m(0,\tau)} |dv|^2 d^m x + \frac{2}{\tau^{m-2}}\int_{B^m(0,\tau)} |dw|^2 d^m x.$$

We now estimate separately each term on the right hand side. On the one hand, since v is harmonic, $|dv|^2$ is a subharmonic function (see Chapter 1) and hence

$$\frac{2}{\tau^{m-2}}\int_{B^m(0,\tau)} |dv|^2 d^m x \leq \frac{2}{\tau^{m-2}}\tau^m\int_{B^m(0,1)} |dv|^2 d^m x \leq 2\tau^2 E_{0,1}(u). \tag{47}$$

On the other hand, we have

$$\int_{B^m(0,\tau)} |dw|^2 d^m x \leq \int_{B^m(0,1)} |dw|^2 d^m x = \int_{\partial B^m(0,1)} \left\langle w, \frac{\partial w}{\partial n}\right\rangle d^m x - \int_{B^m(0,1)} \langle w, \Delta w\rangle\, d^m x,$$

which implies, since $w = 0$ on ∂B^m,

$$\frac{2}{\tau^{m-2}}\int_{B^m(0,\tau)} |dw|^2 d^m x \leq \frac{2}{\tau^{m-2}}\int_{B^m(0,1)} \langle u-v, A(u)(du,du)\rangle\, d^m x. \tag{48}$$

We see from (47) that, by choosing τ sufficiently small, the contribution of v in $E_{0,\tau}(u)$ can be as small as we want in comparison to $E_{0,1}(u)$. Hence the difficulty in proving (46) lies in estimating the right-hand side of (48). We may write $\int_{B^m(0,1)}\langle u-v, A(u)(du,du)\rangle d^mx \leq C\sup_{B^m(0,1)}|u-v|\int_{B^m(0,1)}|du|^2d^mx = C\sup_{B^m(0,1)}|u-v|E_{0,1}(u)$ and, by using the maximum principle for v we can estimate $\sup_{B^m(0,1)}|u-v|$ in terms of a bound $\text{osc}_{B^m(0,1)}u := \sup_{x,y\in B^m(0,1)}|u(x)-u(y)|$ on the oscillation of u on $B^m(0,1)$. However, *we have no estimate on these oscillations* but only on the *mean oscillation*, hence our attempt *failed*. Anyway, we see that we are in a borderline situation since, again, an estimate in BMO space is close to an L^∞ estimate. The following partial regularity results can be obtained by filling this gap between BMO and L^∞.

Regularity of *minimizing* maps in dimension greater than two

For minimizing maps, partial regularity results were obtained by R. Schoen and K. Uhlenbeck [190] (and also by M. Giaquinta and E. Giusti [84, 85] under the assumption that the image is contained a a single coordinate chart): **let $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ be a minimizing weakly harmonic map, then there exists a closed singular set $\Sigma \subset \mathcal{M}$ such that u is Hölder continuous on $\mathcal{M}\setminus\Sigma$ and $\mathcal{H}^{m-3}(\Sigma) < \infty$.** This is proved in two steps:

(i) first, one shows that a minimizing map u is smooth outside a singular set Σ such that $\mathcal{H}^{m-2}(\Sigma) = 0$;

(ii) then, one shows that, near a point $x_0 \in \Sigma$, the minimizing map u behaves asymptotically like a homogeneous map, so that, in particular, the singular set looks asymptotically like a cone centred at x_0. This forces a reduction of the dimension of Σ.

Step (i) [190, 84] relies on the ideas expounded in the previous paragraph, since a minimizing map is automatically stationary. A key observation is that, if we have a local BMO bound on a stationary map u, then we can *approximate* u locally by a smooth map $u^{(h)}$ (where $h > 0$ is small) with values in $\mathbb{R}^N \supset \mathcal{N}$, and the estimate on the *mean* oscillation of u becomes an estimate on the oscillations of $u^{(h)}$. Thus the previous attempt works if we replace u by $u^{(h)}$ (with suitable adaptations), leading to an estimate of $E_{0,\tau}(u^{(h)})$ in terms of $E_{0,1}(u)$. Since, again, u has small local mean oscillation, we can compose $u^{(h)}$ with a projection onto $\mathcal{N}$ to get a smooth map u_h *with values in* $\mathcal{N}$ which approximates u, and then deduce an estimate for $E_{0,\tau}(u_h)$ in terms of $E_{0,1}(u)$. But we are interested in estimating $E_{0,\tau}(u)$, and here we use the fact that u is a *minimizer*: by a delicate gluing process we construct a test function U_h which agrees with u on $\partial B^m(0,2\tau)$ and coincides with u_h in $B^m(0,\tau)$, and we obtain (46) by comparing the energy of u and the energy of U_h on suitable balls.

Step (ii) [190, 85] is inspired by a similar work by H. Federer [71]. It is based on the analysis of a *blow-up sequence* $(u_k)_{k\in\mathbb{N}}$ of minimizing maps centred at a point a in the singular set Σ. Each $u_k \in W^{1,2}(B^m,\mathcal{N})$ is defined by $u_k(x) := u(a + r_kx)$, for some decreasing sequence $(r_k)_{k\in\mathbb{N}}$ which converges to 0. It is not difficult to prove that, after extraction of a subsequence if necessary, $(u_k)_{k\in\mathbb{N}}$ converges weakly in $W^{1,2}$ to a map $\underline{u}_a \in W^{1,2}(B^3,\mathcal{N})$, called the *tangent map at* a. However, one can prove that, actually, $(u_k)_{k\in\mathbb{N}}$ converges *strongly* in $W^{1,2}$ to $\underline{u}_a$ and that $\underline{u}_a$ is weakly harmonic[26]. Hence we can pass to the limit in (44) and deduce that $\partial\underline{u}_a/\partial n = 0$, i.e., $\underline{u}_a$ is homogeneous.

Remarks (i) A variant of the proof of step (i) has been proposed by S. Luckhaus [151], with applications to a much larger class of functionals on maps with values in manifolds. Also, in the special case $\mathcal{N} = S^2$, simpler proofs are available: by R. Hardt, D. Kinderlehrer and F. H. Lin [105], and by Y. Chen and Lin [42].

(ii) In step (ii) it is not clear a priori whether the tangent map $\underline{u}_a$ at a singularity a depends on the choice of the blow-up sequence $(u_k)_{k\in\mathbb{N}}$. It is actually a deep and difficult question. L. Simon [198] (see also [199] for simplifications) proved that *if $\mathcal{N}$ is real analytic, for any map $u \in W^{1,2}(\mathcal{M},\mathcal{N})$ which is a minimizer and is singular at $a \in \mathcal{M}$, if the tangent map $\underline{u}_a$ is smooth outside 0, then this tangent map is unique*. In constrast, B. White [228] constructed a harmonic map into a smooth *non-analytic* Riemannian manifold with a one-parameter *family of tangent maps* having an isolated singularity at the same point, hence proving that the analyticity assumption in the result of Simon is crucial. See the survey by Hardt [103] for a discussion of these questions.

[26] $\underline{u}_a$ is actually minimizing, as shown by S. Luckhaus [152].

Reduction of the singular set. These results can be improved if we assume some further conditions on $\mathcal{N}$: for instance, if $\mathcal{N}$ is non-negatively curved or if the image of a minimizing map is contained in a geodesically convex ball, then minimizing maps are smooth (see §6.3). Optimal examples of such convex targets are the compact subsets of $S^n_+ := \{y \in \mathbb{R}^{n+1} |\ y^{n+1} > 0\}$. These examples are close to the borderline case where the target is $\overline{S^n_+} := \{y \in \mathbb{R}^{n+1} |\ y^{n+1} \geq 0\}$, since minimizing maps into $\overline{S^n_+}$ may not be smooth (see §6.2). In order to estimate the size of the critical set outside these situations, one possible approach is to try to classify the *minimizing tangent maps* $u \in W^{1,2}(B^m, \mathcal{N})$, i.e. maps of the form $u(x) = \psi(x/|x|)$, where $\psi : S^{m-1} \longrightarrow \mathcal{N}$. This relies on proving kinds of *Bernstein theorems* for *minimizing tangent maps* into $\mathcal{N}$. These questions have been investigated by R. Schoen and K. Uhlenbeck [192] and M. Giaquinta and J. Souček [89] in two cases:

(i) in the limit case, where $\mathcal{N} = \overline{S^n_+}$: **a minimizing map** $u \in W^{1,2}(\mathcal{M}, \overline{S^n_+})$ **is smooth if** $n \leq 6$ **and has a closed singular set of Hausdorff dimension less or equal to** $n - 7$ **for** $n \geq 7$ [192, 89]. This is based on results in [120, 126] (see also §6.3).

(ii) beyond the limit case, if $\mathcal{N} = S^n$: **a minimizing map** $u \in W^{1,2}(\mathcal{M}, S^n)$ **is smooth if** $m :=$ **dim** $\mathcal{M} \leq \overline{m}(n)$**, where** $\overline{m}(n)$ **is given by the following table** [192]:

n	2	3	4	5	6	7	8	9	$[10, \infty)$
$\overline{m}(n)$	2	3	3	3	4	4	5	5	6

See also §6.2. Lastly, *extra results on reduction of the singular set* were proved for *stationary* maps by F. H. Lin and, in particular, are valid for minimizing maps, see below.

The structure of the singular set. The singular set Σ has a simple structure in dimension 3, since then it is composed of isolated point. However, in higher dimensions, Σ has a positive Hausdorff dimension in general and the analysis of its regularity requires the use of techniques from geometric measure theory. For maps u in $W^{1,2}(B^4, S^2)$ R. Hardt and F. H. Lin [108] proved that *the singular set* Σ *of a minimizer in* $W^{1,2}(B^4, S^2)$ *with a smooth trace on* ∂B^4 *is the union of a finite set and of finitely many Hölder continous closed curves with only finitely many crossings.* For more general situations L. Simon [200] proved that *if* $\mathcal{N}$ *is compact and real analytic, for any minimizer* $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$ *with singular set* Σ *and any ball* $\mathcal{B} \subset \mathcal{M}$, $\Sigma \cap \mathcal{B}$ *is the union of a finite pairwise disjoint collection of locally* $(m-3)$*-rectifiable locally compact sets.*[27] See [103] for a survey; see also the book of Simon [201].

Minimizing maps from the unit ball B^3 **to** S^2. H. Brezis, J.-M. Coron et E. H. Lieb [24] found further results in the special case $\mathcal{M} = B^3 \subset \mathbb{R}^3$ and $\mathcal{N} = S^2$. They prove that a *minimizing harmonic map can only have singularities of degree* ± 1; more precisely, **the only homogeneous minimizing maps** $B^3 \ni x \longmapsto \psi(x/|x|) \in S^2$ **are of the form** $\psi(x/|x|) = \pm Rx/|x|$**, where** $R \in SO(3)$ **is a rotation** (similar results holds for $\mathcal{N} = \mathbb{R}P^2$). The minimality of the radial projection $u_\odot(x) = x/|x|$ is obtained by establishing the lower bound $E_{B^3}(u) \geq E_{N^3}(u_\odot) = 4\pi$ for any minimizing map $u \in W^{1,2}_{u_\odot}(B^3, S^2)$, by using the following idea. By the partial regularity result [190] any such map u is smooth outside a finite singular set $\{a_1, \cdots, a_p\}$ with respective degrees $\{d_1, \cdots, d_p\}$. Then, from the local inequality $\frac{1}{2}|du|^2 \geq |u^*\omega_{S^2}|$, which holds a.e., one deduces that

$$E_{B^3}(u) \geq \int_{B^3} |u^*\omega_{S^2}|\, dx^1 dx^2 dx^3 \geq \int_{B^3} d\zeta \wedge (u^*\omega_{S^2}) = \int_{\partial B^3} \zeta\, u^*\omega_{S^2} - \int_{B^3} \zeta\, d(u^*\omega_{S^2})$$

for all $\zeta \in Lip(\Omega)$ such that $|\nabla\zeta|_{L^\infty} \leq 1$. But the condition: $u = u_\odot$ on ∂B^3 implies that $\int_{\partial B^3} \zeta u^*\omega_{S^2} = \int_{\partial B^3} \zeta\omega_{S^2}$. Furthermore, by using $d(u^*\omega_{S^2}) = \sum_{i=1}^p d_i \delta_{a_i}$ (see also §5.4 and (53)), we finally get

$$E_{B^3}(u) \geq \sup_{\zeta \in Lip(\Omega), |\nabla\zeta|_{L^\infty} \leq 1} \Big(\int_{\partial B^3} \zeta\, \omega_{S^2} - \sum_{i=1}^p d_i\, \zeta(a_i) \Big).$$

[27] More can be said when all *Jacobi fields* along (i.e., infinitesimal deformations of) the harmonic maps are *integrable*, i.e., come from genuine deformations through harmonic maps, see [200, 201, 145].

Then the proof can be reduced to an optimization problem on the set of configurations of the type $\{(a_1,d_1),\cdots,(a_p,d_p)\}$, which can be solved by adapting a theorem of Birkhoff.

Still for the case of minimizing harmonic maps u from B^3 to S^2, F. Almgren and E. H. Lieb [4] found a bound on the number $N(u)$ of singularities of u: $N(u)$ is certainly not bounded in terms of its energy $E_{B^3}(u)$, but it is in terms of the energy of its trace on ∂B^3. Indeed, **there exists a universal constant** $C>0$ **such that, for any** $\varphi\in W^{1,2}(\partial B^3,S^2)$,

$$\textbf{for any } u\in W^{1,2}_\varphi(B^3,S^2) \textbf{ which is a minimizer of } E_{B^3}\textbf{, we have } N(u)\leq CE_{\partial B^3}(\varphi)\,.$$

The precise value of C is not known but examples constructed in [4] show that we must have $C\geq 1/(4\pi)$. It is also shown that a similar result where the energy $E_{\partial B^3}(\varphi)$ is replaced by the area covered by φ *cannot hold*.

Minimizers of the relaxed energy. The regularity of the minimizers in $W^{1,2}(B^3,S^2)$ of the functional $E^\lambda_{B^3}=E_{B^3}+4\lambda\pi L$ (see §5.4) has been investigated by H. Brezis and F. Bethuel [14] who proved that, **if** $\lambda\in[0,1)$**, any minimizer of** $E^\lambda_{B^3}$ **is smooth on** $B^3\setminus\Sigma$**, where** $\mathcal{H}^0(\Sigma)<\infty$**, i.e.** Σ **is a finite collection of points**. The case $\lambda=1$ corresponds to the *relaxed energy* $E^{rel}_{B^3}=E_{B^3}+4\pi L$, which is harder to deal with: the only partial regularity result that we know is due to M. Giaquinta, G. Modica and J. Souček [87, 88] who showed that *minimizers of* $E^{rel}_{B^3}$ *are smooth on* $B^3\setminus\Sigma$*, where* $\mathcal{H}^1(\Sigma)<\infty$. It is a paradox that the regularity theory for minimizers of the relaxed energy, which was designed for producing continuous harmonic maps, is less understood than the theory of minimizers of the standard energy functional.

Minimizers of the p**-energy**. The previous results have been extended to minimizers of the p-energy in various cases by S. Luckhaus [151], R. Hardt and F. H. Lin [107], M. Fuchs [75, 76], and by F. H. Lin in the important paper [148].

Regularity of *stationary* maps in dimension greater than two

For stationary maps, we have the following partial regularity result: **let** $u\in W^{1,2}(\mathcal{M},\mathcal{N})$ **be a stationary map; then there exists a closed singular set** $\Sigma\subset\mathcal{M}$ **such that** u **is Hölder continuous on** $\mathcal{M}\setminus\Sigma$ **and** $\mathcal{H}^{m-2}(\Sigma)=0$**.** This was proved by L. C. Evans [70] in the case where $\mathcal{N}=S^n$ and by F. Bethuel [13] in the general case.

The proof of Evans [70, 118, 88] is based on the discovery that the attempt expounded above really works for maps into a sphere S^n. Recall that the difficulty was to estimate a quantity of the type $\int_B\langle u-v,A(u)(du,du)\rangle\, d^mx$ and that only the *mean* oscillation of $u-v$ can be estimated in terms of $E_{0,1}(u)$. However we can use the same observations as in dimension two, i.e. write the harmonic map equation in the form $d(\star du^i)+du^j\wedge\star(u^idu_j-u_jdu^i)=0$, and use the conservation law $d(\star(u^idu_j-u_jdu^i))=0$. This implies, by using the property c) of Hardy spaces, that $A^i(u)(du,du)d^mx=u^i|du|^2d^mx=du^j\wedge\star(u^idu_j-u_jdu^i)$ coincides locally with a function in the Hardy space $\mathcal{H}^1(\mathbb{R}^m)$. Thus, by property d) of Hardy spaces, we can estimate $\int_B\big\langle u-v,A(u)(du,du)\big\rangle d^mx$ as a *duality product between the (local) BMO norm of* $u-v$ *and the (local) Hardy norm of* $A(u)(du,du)$, and hence complete the proof.

The proof of F. Bethuel [13, 118] uses a *Coulomb moving frame* $(\underline{e}_1,\cdots,\underline{e}_n)$ as in [116]. The strategy is somewhat parallel to the proof of Evans, but the realization is much more delicate. The idea for estimating $|du|$ on a small ball consists of using a Hodge decomposition $\langle d\left(\zeta(u-u_{0,1})\right),\underline{e}_a\rangle=dw^a+\star dv^a$, where $u_{0,1}:=|B^m(a,r)|^{-1}\int_{B^m(a,r)}u$ and $\zeta\in\mathcal{C}^\infty_c(B^m(a,r))$ is a cut-off function. Then both terms in the decomposition are estimated separately. However, because the system is not as simple as in the case treated by Evans, we need to replace Morrey's rescaled energy $E_{a,r}(u)$ by $M_{a,r}(u):=\sup\big\{\rho^{1-m}\int_{B^m(x,\rho)}|du|\mid B^m(x,\rho)\subset B^m(a,r)\big\}$ (which also controls the local bounded mean oscillation of u).

Remarks (i) Several variants of the proof by Evans exist: one can avoid the use of the Fefferman–Stein theorem on the duality between $\mathcal{H}^1$ and BMO, as done by S. Chanillo [40], or even avoid the use of the Hardy space, as done by S.-Y. A. Chang, L. Wang and P. C. Yang [39].

(ii) Using the conservation laws discovered by T. Rivière in [184], Rivière and M. Struwe [185] derived a simplified proof of the result of Bethuel, without using Coulomb moving frames.

Reduction of the singular set. The question of whether $\mathcal{H}^{m-3}(\Sigma)$ is finite is still open. The reason is that the blow-up technique used by Schoen and Uhlenbeck does not work here, since we are not able to prove that, after extracting a subsequence if necessary, a blow-up sequence $u_k(x) = u(a + r_k x)$ at a point a converges *strongly* when $r_k \longrightarrow 0$. Indeed we can only prove that it converges *weakly*. This leads to the more general question of understanding *a sequence* $(v_k)_{k\in\mathbb{N}}$ *of stationary maps which converges* weakly *to some limit* v: after extracting a subsequence is necessary, we can assume that the energy density $|dv_k|^2 d^m x$ converges weakly in the sense of Radon measures to a non-negative Radon measure μ which can be decomposed as $\mu = |dv|^2 + \nu$; the measure ν detects the defect of strong convergence, i.e. *the sequence converges strongly if and only if* $\nu = 0$. By a careful analysis of such sequences, F. H. Lin [147] proved that *the singular support*[28] Γ *of* μ *is a rectifiable subset with a finite* $(m-2)$*-dimensional Hausdorff measure*. Moreover ν is supported by Γ and, more precisely, is equal to the $(m-2)$-dimensional measure supported by Γ times an $\mathcal{H}^{m-2}$-measurable density $\Theta(x)$. This result is optimal as shown by the following example: assume that there exists a non-trivial harmonic map $\phi : S^2 \longrightarrow \mathcal{N}$ and, for any $\lambda \in \mathbb{R}$, let $u_\lambda \in \mathcal{C}^\infty(B^2 \times B^{m-2}, \mathcal{N})$ be defined by $u_\lambda(x,y) = \phi \circ P^{-1}(\lambda x)$, where $P : S^2 \longrightarrow \mathbb{R}^2$ is the stereographic projection (30). Then each u_λ is stationary and $|du_\lambda|^2\, d^m x$ converges weakly to a Radon measure ν supported by $\{0\} \times B^{m-2}$ when $\lambda \longrightarrow +\infty$. Moreover Lin and Rivière [149] proved that, *in the case where* $\mathcal{N} = S^n$*, for a.e. point* $x \in \Gamma$ *(in the sense of* $(m-2)$*-dimensional measure) the density* $\Theta(x)$ *is a finite sum of energies of harmonic maps from* S^2 *to* S^m (this result generalizes the identity (49) for maps of surfaces) and, in particular, if $\mathcal{N} = S^2$, $\Theta(x)$ is a integer multiple of 8π. For a general target manifold, a further result by Lin [147] is that, for a given $\mathcal{N}$, *any sequence of weakly converging stationary maps converges strongly (i.e. satisfies* $\nu = 0$*) if and only if there is no smooth non-constant harmonic map from* S^2 *to* $\mathcal{N}$. Applying his results to a blow-up sequence of stationary maps, Lin [147] proved that **if $\mathcal{N}$ does not carry any harmonic S^2, then the singular set Σ of a stationary map with values in $\mathcal{N}$ has Hausdorff dimension $s \leq m-4$. If, furthermore, $\mathcal{N}$ is real analytic, then Σ is s-rectifiable.** On the other hand a consequence of the work by Lin and Rivière [149] is that, *for a stationary map* u *into* S^2*, if* $\liminf_{k\to\infty} E_{x,r}(u) < 8\pi$*, then* u *is continuous at* x.

Stationary critical points of the p-energy. A notion similar to the notion of stationary maps for critical points of the p-energy makes sense, and the previous regularity results has been extended to this case by L. Mou and P. Yang [158].

5 Existence methods

5.1 Existence of harmonic maps by the direct method

The general strategy for proving existence of harmonic maps consists of choosing a non-empty class $\mathcal{E} \subset W^{1,p}(\mathcal{M},\mathcal{N})$ of maps which is defined, for example, by some Dirichlet boundary conditions or some topological constraints, and then to consider a sequence $(u_k)_{k\in\mathbb{N}}$ minimizing the energy $E_\mathcal{M}$ in $\mathcal{E}$. Here we assume for simplicity that $\mathcal{M}$ is compact. One can repeat the arguments given in Chapter 1 for the solution to the classical Dirichlet problem: since $\mathcal{E}$ is non-empty it contains maps of finite energy and so, in particular, the minimizing sequence has bounded energy. Thus, there is a subsequence $(\varphi(k))_{k\in\mathbb{N}} \subset (k)_{k\in\mathbb{N}}$ such that $\left(u_{\varphi(k)}\right)_{k\in\mathbb{N}}$ converges *weakly* in $W^{1,p}(\mathcal{M},\mathbb{R}^N)$ to some map $\underline{u} \in W^{1,p}(\mathcal{M},\mathbb{R}^N)$. An extra task is to check that $\underline{u}(x) \in \mathcal{N}$ a.e. This is a consequence of the fact that, because of the Rellich–Kondrakov theorem, the subsequence $(\varphi(k))_{k\in\mathbb{N}}$ converges *strongly* to $\underline{u}$ in $L^p(\mathcal{M},\mathbb{R}^N)$ for all $p < 2m/(m-2)$. Hence we can extract a further subsequence $(\varphi_1(k))_{k\in\mathbb{N}} \subset (\varphi(k))_{k\in\mathbb{N}}$ such that $\left(u_{\varphi_1(k)}\right)_{k\in\mathbb{N}}$ converges a.e. on $\mathcal{M}$ to $\underline{u}$, by a standard result of Lebesgue theory. This implies $\underline{u}(x) \in \mathcal{N}$ a.e. on $\mathcal{M}$. Hence $\underline{u} \in W^{1,2}(\mathcal{M},\mathcal{N})$. Then two cases can occur:

(i) **$\mathcal{E}$ is closed with respect to the weak topology of** $W^{1,2}(\mathcal{M},\mathcal{N})$. Then we know that $\underline{u} \in \mathcal{E}$ and, using the fact that $E_\mathcal{M}$ is lower semi-continuous for the weak $W^{1,2}$-topology as in the classical case (see §1), we prove that $\underline{u}$ is actually an energy minimizing map in $\mathcal{E}$, and so is weakly harmonic. In the special case when $\mathcal{M}$ is two-dimensional, the classical regularity result of C. B. Morrey [156] ensures that $\underline{u}$ is smooth. In

[28] The singular set Γ also coincides with $\cap_{r>0}\{x \in B^m |\ \liminf_{k\to\infty} E_{x,r}(u_k) \geq \varepsilon_0\}$.

higher dimensions, the minimizers are only partially regular, as shown by the regularity theory of R. Schoen and K. Uhlenbeck [190] (see §4.3).

(ii) $\mathcal{E}$ **is not closed with respect to the weak topology of** $W^{1,2}(\mathcal{M},\mathcal{N})$. Then no general argument guarantees that $\underline{u} \in \mathcal{E}$ or that $\underline{u}$ is an energy minimizer.

5.2 The direct method in a class of maps closed for the weak topology

The class $\mathcal{E}$ is closed with respect to the weak topology of $W^{1,2}(\mathcal{M},\mathcal{N})$ in the following situations:

1. $\mathcal{E}$ **is defined through Dirichlet boundary conditions**, because the trace operator given by tr : $W^{1,2}(\mathcal{M},\mathbb{R}^N) \longrightarrow W^{\frac{1}{2},2}(\partial\mathcal{M},\mathbb{R}^N)$ is continuous for the weak topologies[29]. The first application was the solution of the Plateau problem for a surface in a Riemannian manifold by C. B. Morrey [156].

2. $\mathcal{E}$ **is defined by prescribing the action of maps in** $W^{1,2}(\mathcal{M},\mathcal{N})$ **on** $\pi_1(\mathcal{M})$ (see also §3.3). The first application was the following result by L. Lemaire [144]: **let** $\mathcal{M}$ **and** $\mathcal{N}$ **be two Riemannian manifolds of dimension 2, with** $\partial\mathcal{N} = \emptyset$**, and assume that** genus $\mathcal{M} \geq 1$ **and** genus $\mathcal{N} \geq 1$**. Then any homotopy class of maps between** $\mathcal{M}$ **and** $\mathcal{N}$ **contains a minimizing harmonic representative**. In the proof of this result, the fundamental groups $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$ are seen as the automorphisms groups of the universal covers $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ of $\mathcal{M}$ and $\mathcal{N}$, respectively. Then, to any homotopy class represented by a map $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$, we associate the class of equivariant maps $\widetilde{u} : \widetilde{\mathcal{M}} \longrightarrow \widetilde{\mathcal{N}}$ such that $\forall\gamma \in \pi_1(\mathcal{M})$, $\widetilde{u} \circ \gamma = \varphi_{\sharp 1}(\gamma) \circ \widetilde{u}$, and we minimize the energy integral over a fundamental domain of $\widetilde{\mathcal{M}}$ in this class. This result was subsequentely generalized by R. Schoen and S. T. Yau [193] to the case when the dimension of the target $\mathcal{N}$ is arbitrary, and then to higher dimensions in [30, 225].

3. $\mathcal{E}$ **is a family of maps which are *equivariant* with respect to a symmetry group.** This means that we are given a group G which acts by isometries $x \longmapsto g \cdot x$ and $y \longmapsto g \cdot y$, $(x \in \mathcal{M},\ y \in \mathcal{N},\ g \in G)$ on $\mathcal{M}$ and $\mathcal{N}$, respectively, and then $\mathcal{E} := \{u : \mathcal{M} \longrightarrow \mathcal{N} | \ \forall g \in G, \forall x \in \mathcal{M}, u(g \cdot x) = g \cdot u(x)\}$. That a critical point under such a symmetry constraint (assuming some extra hypotheses) is also a critical point without the symmetry constraint is the content of a general principle by R. Palais [163]. For a discrete group this approach was used, for example, by L. Lemaire [144] to prove the existence of harmonic maps between a surface $\mathcal{M}$ without boundary of genus $g \geq 2$ and the sphere S^2 which are equivariant with respect to a finite group spanned by reflections with respect to planes in $\mathbb{R}^3$. For continuous groups, this principle is expounded in [64] and the regularity of equivariant minimizing maps is studied by A. Gastel [78]. Many applications concern the reduction of the harmonic map problem to an ODE [55, 64] or to a system in two variables [79, 80], see §5.5.

4. $\mathcal{N}$ **is a manifold with *non-positive curvature*.** This improves strongly the behaviour of minimizing sequences (see §6.3). One instance is the following result [189, Theorem 2.12]: *assume that $\mathcal{E}$ is a homotopy class of maps between two compact manifolds $\mathcal{M}$ and $\mathcal{N}$ of arbitrary dimensions and that $\mathcal{N}$ has* non-positive curvature *and let $v \in \mathcal{C}^3(\mathcal{M},\mathcal{N})$. Then there exists a harmonic map $u \in \mathcal{C}^2(\mathcal{M},\mathcal{N})$ such that $u = v$ on $\partial\mathcal{M}$ and u is homotopic to v through maps with fixed values on $\partial\mathcal{M}$.*

5. $\mathcal{E}$ **is a class of diffeomorphisms** between two Riemannian surfaces $\mathcal{M}$ and $\mathcal{N}$: a result by J. Jost and R. Schoen [137, 131] asserts that if $\partial\mathcal{M} = \partial\mathcal{N} = \emptyset$, if $\mathcal{M}$ and $\mathcal{N}$ have the same genus and if $\varphi : \mathcal{M} \longrightarrow \mathcal{N}$ is a diffeomorphism, then *there exists a harmonic diffeomorphism u homotopic to φ which has the least energy among all diffeomorphisms homotopic to φ.* Actually, the difficulty here is not to get the existence of the minimizer u, but rather to prove that u is weakly harmonic, as not all first variations are allowed.

6. **The target has non-empty boundary.** Again this condition does not cause particular problems when finding a minimizer, but does when proving that this minimizer satisfies, at least weakly, the harmonic maps equation, since, as in the previous example, we are not allowed to use all first variations. However, if $\mathcal{B}$ is a HJW-convex ball of $\mathcal{N}$ (see §6.3 for the definition), and if, for example, $\partial\mathcal{M} \neq \emptyset$ and we fix a Dirichlet boundary condition with values in $\mathcal{B}$, then S. Hildebrandt, W. Jäger and K.-O. Widman [120] prove the existence of a minimizing solution

[29] Any *linear* operator between Banach spaces continuous for the strong topologies is continuous for the weak topologies.

ot the Dirichlet problem with values in $\mathcal{B}$ which is weakly harmonic (in particular the image of the minimizing map does not meet $\partial\mathcal{B}$). A variant of this result was proved by J. Jost [132] in the case dim $\mathcal{M} = 2$: if we fix a boundary condition with values in a sufficiently small ball $\mathcal{B} \subset \mathcal{N}$ and we minimize the energy with this Dirichlet boundary condition *among those maps with values in* $\mathcal{N}$, then the minimizer takes values in $\mathcal{B}$.

In the results [156] in 1. and [193] in 2., by further minimizing over all Dirichlet boundary conditions which parametrizes a Jordan curve in $\mathcal{N}$ in the case of [156], or the conformal structures of $\mathcal{M}$ in the case of [193], the minimizing harmonic map becomes a minimal branched immersion in the sense of §2.2.

5.3 The direct method in a class of maps <u>not closed</u> for the weak topology: case dim $\mathcal{M} = 2$

This case holds in situations where the definition of $\mathcal{E}$ relies partially or completely on the action of maps $u : \mathcal{M} \longrightarrow \mathcal{N}$ on $\pi_2(\mathcal{M})$ or on the degree of maps between two surfaces. See also §3.2 and 3.3.

- For example, consider the case when $\mathcal{M}$ is the 2-dimensional ball B^2 and $\mathcal{N}$ any manifold such that $\pi_2(\mathcal{N})$ is non-trivial, and choose a smooth map $\varphi : B^2 \longrightarrow \mathcal{N}$ which is constant on ∂B^2 and covers a (non-zero) generator of $\pi_2(\mathcal{N})$. Then, as observed in [144], there is no minimizer in the class of maps homotopic to φ which shares the same Dirichlet boundary condition. This is a consequence of the more general result that *any harmonic maps on a ball B^m which agrees with a constant on the boundary is a constant map*, proved[30] by L. Lemaire [144] for $m = 2$.
- J. Eells and J. C. Wood [67] proved that any harmonic map of a given degree d between two Riemannian surfaces $\mathcal{M}$ and $\mathcal{N}$ is holomorphic or antiholomorphic if genus $\mathcal{M} + |d$ genus $\mathcal{N}| > 0$. This implies, for example, that *there is no harmonic map of degree ± 1 from a 2-torus to a 2-sphere whatever metrics they are given*, since there is no holomorphic map of degree 1 from a torus to $\mathbb{C}P^1 = S^2$. Hence in particular *the minimum of the energy among degree 1 (or -1) maps between a torus and a sphere is not achieved*. This last conclusion remains true if we replace the torus by a higher genus surface, as shown by Lemaire [144] and K. Uhlenbeck independently: a minimizing sequence necessarily converges weakly to a constant map. Furthermore Y. Ge [82] showed that, after extracting a subsequence if necessary, the energy density of such a sequence concentrates at one point.

Bubbles

The first general analysis of the situation when dim $\mathcal{M} = 2$ was done by J. Sacks and K. Uhlenbeck [188] who addressed the question of finding harmonic maps inside a homotopy class $\mathcal{E}$ of maps between a surface $\mathcal{M}$ without boundary and an arbitrary compact manifold $\mathcal{N}$. One of the reasons why $\mathcal{E}$ is not closed with respect to the weak topology, in general, is the conformal invariance of the Dirichlet energy and of the harmonic maps problem in two dimensions (see §2.2). For example, when $\mathcal{M} = S^2$, the group of conformal transformations of S^2 is the group of homographies $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] : z \longmapsto (az+b)/(cz+d)$ acting on S^2 through the stereographic projection (30). Using the action of this group, it is easy to produce minimizing sequences in a homotopy class $\mathcal{E}$ of maps $S^2 \longrightarrow \mathcal{N}$ whose weak limit escapes from the homotopy class (see §3.3). This instability of minimizing sequences can be cured as in [188] by working with the perturbed functional $E^\alpha_{\mathcal{M}}(u) := \int_{\mathcal{M}} (1 + |du|^2)^\alpha \mu$, for $\alpha > 1$ which is not conformally invariant anymore (here $\mu := \omega_g / \int_{\mathcal{M}} \omega_g$ is an area 2-form of total integral 1), and then letting $\alpha \to 1$. However a more serious difficulty is the following: imagine that $\pi_2(\mathcal{N})$ has at least two generators γ_1, γ_2 and that, for instance, we know that there exist minimizing harmonic maps $u_1, u_2 : S^2 \longrightarrow \mathcal{N}$ where u_1 (resp. u_2) is a representative of γ_1 (resp. γ_2). Then it may happen that there is no minimizer in the class $\gamma_1 + \gamma_2$: indeed maps in a minimizing sequence could look asymptotically like a map covering the image of u_1 in a neighbourhood of some point $p_1 \in S^2$ and the image of u_2 in a neighbourhood of another point $p_2 \in S^2$ (two *bubbles*), all the other points of S^2 (inside a domain conformally equivalent to a long cylinder) being mapped harmonically to a geodesic connecting a point of $u_1(S^2)$ to a point $u_2(S^2)$ (a *neck*). Then the limit may be either u_1 or u_2 (up to the composition with some

[30] For $m \geq 3$ this result was extended by J. C. Wood [230] and by H. Karcher and Wood [141].

conformal map of S^2) or a constant map (mapping S^2 to a point of the geodesic), depending how randomly the instability effects of the conformal group acts. Again by replacing an arbitrary minimizing sequence by a sequence $(u^\alpha)_{\alpha>1}$ of minimizers of $E^\alpha_{\mathcal{M}}$ in $\mathcal{E}$ we can possibly avoid the instability effects of the conformal group, but *we cannot avoid the possible bubblings, i.e. prevent the limit u of $(u^\alpha)_{\alpha>1}$ as $\alpha\to 1$ escaping from $\mathcal{E}$ in general*.

J. Sacks and K. Uhlenbeck prove the following results [188]. They first establish that, if $\alpha>1$, the functional $E^\alpha_{\mathcal{M}}$ achieves its minimum in each connected component of $W^{1,2\alpha}(\mathcal{M},\mathcal{N})$ at a smooth map u_α which satisfies the (elliptic) Euler–Lagrange equation of $E^\alpha_{\mathcal{M}}$. Then they prove three basic results:

(i) **The main estimate**. There exists $\varepsilon>0$ and $\alpha_0>1$ such that, for any geodesic ball $\mathcal{B}\subset\mathcal{M}$, any map $u:\mathcal{B}\longrightarrow\mathcal{N}$ with $E^2_{\mathcal{B}}(u)<\varepsilon$ which is a smooth critical point of $E^\alpha_{\mathcal{M}}$ for some $\alpha\in[1,\alpha_0)$, we have a uniform family of estimates $||du||_{W^{1,p}(\mathcal{B}')}\leq C(p,\mathcal{B}')||du||_{L^2(\mathcal{B})}$ for any $p\in(1,\infty)$ and any smaller disk $\mathcal{B}'\subset\mathcal{B}$.

(ii) **The removability of isolated singularities for weakly harmonic maps**. This says that, for any map $u\in W^{1,2}(\mathcal{M},\mathcal{N})$, and any finite family of points $\{z_1,\cdots,z_k\}\subset\mathcal{M}$ such that u is smooth and harmonic on $\mathcal{M}\setminus\{z_1,\cdots,z_k\}$, there exists a smooth extension of u to $\mathcal{M}$ which is harmonic.

(iii) **An energy gap**. $\exists\varepsilon>0$, $\exists\alpha_0>1$ such that for any map $u\in W^{1,2}(\mathcal{M},\mathcal{N})$ which is a critical point of $E^\alpha_{\mathcal{M}}$ for some $\alpha\in[1,\alpha_0)$, if $E_{\mathcal{M}}(u)<\varepsilon$, then u is constant.

Note that the proofs of (ii) and (iii) use (i). Thanks to the *main estimate* and a covering argument, Sacks and Uhlenbeck prove that a subsequence of the family of $E^\alpha_{\mathcal{M}}$-minimizers $(u_\alpha)_{\alpha>1}$ converges to some map $u\in W^{1,2}(\mathcal{M},\mathcal{N})$ in the weak $W^{1,2}$ topology and in $\mathcal{C}^1(\mathcal{M}\setminus\{z_1,\cdots,z_k\},\mathcal{N})$, where $\{z_1,\cdots,z_k\}$ is a finite collection of points of $\mathcal{M}$ where possible bubblings occur. Then, by the result of *removability of isolated singularities* (ii), we deduce that u extends to a smooth harmonic map. However nothing guarantees that this map is non-constant. On the other hand, an analysis of the behaviour of u_α near the bubbling points z_j reveals that, *if $|du_\alpha|$ is not bounded in a neighbourhood of z_j*, then we can find a subsequence of maps $v_{j,\alpha}:B^2(0,R_\alpha)\longrightarrow\mathcal{N}$ (where $\lim_{\alpha\to1}R_\alpha=+\infty$), defined by $v_{j,\alpha}(x)=u_\alpha(\exp_{x_\alpha}(\lambda_\alpha x))$, where (x_α) is a sequence of points of $\mathcal{M}$ which converges to z_j and $\lim_{\alpha\to1}\lambda_\alpha=0$, such that for any ball $B^2(0,R)\subset\mathbb{R}^2$, the restriction of $v_{j,\alpha}$ to $B^2(0,R)$ converges in $\mathcal{C}^1(B^2(0,R))$ to the restriction to $B^2(0,R)$ of some map v_j as $\alpha\to1$, and $v_j:\mathbb{R}^2\longrightarrow\mathcal{N}$ is a harmonic map of finite energy. Since $\mathbb{R}^2$ is conformally equivalent to S^2 minus a point and thanks again to the *removability of isolated singularities* result, we can extend v_j to a harmonic map $S^2\longrightarrow\mathcal{N}$ (moreover we know that any harmonic map on S^2 is conformal, i.e. holomorphic or antiholomorphic, see §2.2). Hence we can picture the limit of u_α as the collection of harmonic maps $u:\mathcal{M}\longrightarrow\mathcal{N}$ and $v_j:S^2\longrightarrow\mathcal{N}$, for $1\leq j\leq k$, with the extra (lost) information that the image of each map v_j is connected by a geodesic to the point $u(z_j)$ (a so-called *bubble tree*). We have moreover[31]:

$$E_{\mathcal{M}}(u)+\sum_{j=1}^k E_{S^2}(v_j)=\lim_{\alpha\to1}\sup E(u_\alpha). \tag{49}$$

By using this analysis and the *energy gap* property, Sacks and Uhlenbeck deduce the following results:

- if $\pi_2(\mathcal{N})=0$, or if we minimize in a conjugacy class of homomorphisms $\pi_1(\mathcal{M})\longrightarrow\pi_1(\mathcal{N})$, the maps u_α converge strongly to u, hence u is a minimizer of the energy in the same class as u_α. We thus recover the results of L. Lemaire [144] or R. Schoen and S. T. Yau [193]. Here the conclusion is achieved by constructing test maps $\widehat{u}_\alpha$ which coincide with u_α away from the bubbling points and with the weak limit u near the bubbling points: because of the topological hypotheses, $\widehat{u}_\alpha$ is in the same topological class as u_α and hence we can exploit the inequality $E_\alpha(u_\alpha)\leq E_\alpha(\widehat{u}_\alpha)$.
- if $\pi_2(\mathcal{N})\neq0$, choose $\mathcal{M}=S^2$ and a non-trivial *free 2-homotopy class* Γ of $\mathcal{N}$, i.e. a connected component of $\mathcal{C}^1(S^2,\mathcal{N})$ which does not contain the constant maps. Then **either γ contains a minimizing harmonic map or, for all $\delta>0$, there exists non-trivial free 2-homotopy classes Γ_1 and Γ_2 such that $\Gamma\subset\Gamma_1+\Gamma_2$ and $\inf_{v\in\Gamma_1}E_{\mathcal{M}}(v)+\inf_{v\in\Gamma_2}E_{\mathcal{M}}(v)<\inf_{v\in\Gamma}E_{\mathcal{M}}(v)+\delta$.**

[31] Sacks and Uhlenbeck just proved the inequality $\leq$ in (49); the equality in (49) was established by J. Jost [132] and T. H. Parker [164]

- **if** $\pi_2(\mathcal{N}) \neq 0$ **and** $\mathcal{M} = S^2$**, there exist a set of free homotopy classes** $\Lambda = \{\Gamma_i | \, i \in I\} \subset \pi_0 \mathcal{C}^1(S^2, \mathcal{N})$ **which forms a generating set for** $\pi_2(\mathcal{N})$ **under the action**[32] **of** $\pi_1(\mathcal{N})$ **such that each** $\Gamma_i \in \Lambda$ **contains a minimizing harmonic map**.

Note that the last result implies that **there exists a non-trivial harmonic map** $S^2 \longrightarrow \mathcal{N}$ **as soon as** $\pi_2(\mathcal{N}) \neq 0$. The second result can be translated into the following: *if* $\pi_2(\mathcal{N}) \neq 0$, $\mathcal{M} = S^2$ *and* Γ *is a non-trivial* free 2-homotopy class *of* $\mathcal{N}$*, then if there exists* $\delta > 0$ *such that, for any non-trivial free 2-homotopy classes* Γ_1 *and* Γ_2 *with* $\Gamma \subset \Gamma_1 + \Gamma_2$ *we have*

$$\inf_{v \in \Gamma} E_{\mathcal{M}}(v) \leq \inf_{v \in \Gamma_1} E_{\mathcal{M}}(v) + \inf_{v \in \Gamma_2} E_{\mathcal{M}}(v) - \delta \, , \tag{50}$$

then the minimum of $E_{\mathcal{M}}$ *is achieved in* Γ. This important property is connected with a similar observation made previously by T. Aubin for the Yamabe problem [5] and with further subsequent developments like the results by C. Taubes [212] for the Yang–Mills connections on a 4-dimensional manifold or the concentration compactness principle of P.-L. Lions [150].

Remarks (i) An alternative analysis with improvements to the understanding of the bubbling phenomenon have been obtained by J. Jost [130, 131, 133] by using a method reminiscent of the *balayage* technique of H. Poincaré.

(ii) Further refinements to the analysis of bubbling were made by T. H. Parker [164] by using the notion of *bubble tree*, which was introduced previously by Parker and J. Wolfson in the study of *pseudo-holomorphic curves*, and by W. Y. Ding and G. Tian [58]. The heat flow equation also provides another approach, which was used by M. Struwe to recover the theory of Sacks and Uhlenbeck (see below).

(iii) The influence of bubbling phenomena is not confined to harmonic maps of surfaces, but plays a major role in the existence theory of harmonic maps in higher dimensions, as expounded in §5.4, and in regularity theory (see the results on reduction of the singular set of stationary maps in §4.3).

Applications of the theory of bubbling

In some cases a precise analysis to decide whether (50) holds is possible: this was done first by H. Brezis and J.-M. Coron [22] and J. Jost [131] independently. We set $\mathcal{M} = B^2$, the unit ball in $\mathbb{R}^2$, and $\mathcal{N} = S^2$ and we let $\gamma \in T^2(\partial B^2, S^2) :=$ the set of maps $\gamma : \partial B^2 \longrightarrow S^2$ such that there exists $u \in W^{1,2}(B^2, S^2)$ with $u|_{\partial B^2} = \gamma$. Then the class $\mathcal{E} := W^{1,2}_\gamma(B^2, S^2) := \{u \in W^{1,2}(B^2, S^2) | \, u|_{\partial B^2} = \gamma\}$ is non-empty and closed for the weak $W^{1,2}$ topology. Hence application of the direct method provides us with a smooth harmonic map $\underline{u}$ which minimizes E_{B^2} in $\mathcal{E}$. We now consider the functional on $W^{1,2}_\gamma(B^2, S^2)$ defined by

$$Q(u) := \frac{1}{4\pi} \int_{B^2} \Big\langle u \, , \, \frac{\partial u}{\partial x} \times \frac{\partial u}{\partial y} \Big\rangle d^2x = \frac{1}{4\pi} \int_{B^2} u^* \omega_{S^2} .$$

We observe that Q takes discrete values on $W^{1,2}_\gamma(B^2, S^2)$, more precisely: for all $u \in W^{1,2}_\gamma(B^2, S^2)$, $Q(u) - Q(\underline{u}) \in \mathbb{Z}$. The geometric interpretation of this is that, if we consider the map $u \sharp \underline{u} : S^2 \longrightarrow S^2$ defined via the identification $\mathbb{C} \cup \{\infty\} \simeq S^2$ by setting $u \sharp \underline{u} = u$ on B^2 and $(u \sharp \underline{u})(z) = \underline{u}(z/|z|^2)$ on $\mathbb{C} \setminus B^2$, then $Q(u) - Q(\underline{u})$ is the degree of $u \sharp \underline{u}$. Then for any $k \in \mathbb{Z}$, the classes $\mathcal{E}_k := \{u \in \mathcal{E} | \, Q(u) - Q(\underline{u}) = k\}$ are the connected components of $\mathcal{E}$ for the strong $W^{1,2}$ topology. So they are the free 2-homotopy classes of S^2. But they are not closed for the weak topology; hence it is not clear whether $\inf_{\mathcal{E}_k} E_{B^2}$ is achieved. However, one can prove that, *if* γ *is not constant, then there exists some* $v \in \mathcal{E}$ *such that* $|Q(v) - Q(\underline{u})| = 1$ *and* $E_{B^2}(v) < E_{B^2}(\underline{u}) + 4\pi$. But since the minimum of the energy in any *non-trivial homotopy class* of maps $S^2 \longrightarrow S^2$ is greater or equal to 4π, this shows that (50) holds, hence it follows that **there is a harmonic map** $\overline{u}$ **which minimizes** E_{B^2} **in its homotopy class, the latter being either** $\mathcal{E}_1$ **or** $\mathcal{E}_{-1}$. Moreover, as proved in [22], in the case when γ is the restriction of the inverse $P^{-1} : \mathbb{R}^2 \longrightarrow S^2$ of stereographic projection (30) to $\partial B^2 \subset \mathbb{R}^2$, the constructed solutions $\underline{u}$ and $\overline{u}$ (which here are restrictions to B^2 of stereographic projections) are the only miminizers in their respective class and moreover there are no minimizers in the other classes.

[32] The set $\pi_0 \mathcal{C}^1(S^2, \mathcal{N})$ of free homotopy classes can be identified with the set of orbits of the natural action of $\pi_1(\mathcal{N})$ on $\pi_2(\mathcal{N})$.

The following generalization was obtained partially by A. Soyeur [203] and later completed by E. Kuwert [142] and J. Qing [174] independently, see also [88] for an exposition. We first associate two degrees d^- and d^+ to the boundary data $\gamma \in T^2(\partial B^2, S^2)$: if γ has a *holomorphic (resp. antiholomorphic) extension* u^+ (resp. u^-) inside B^2 with values in $S^2 \simeq \mathbb{C}P$ we let $d^+ := Q(u^+) - Q(\underline{u})$ (resp. $d^- := Q(u^-) - Q(\underline{u})$), if γ has no holomorphic (resp. no antiholomorphic) extension inside B^2, set $d^+ := +\infty$ (resp. $d^- := -\infty$). Note that we always have $d^- \leq d^+$, with equality if and only if γ is a constant. Then

(i) **for** $k \in \mathbb{Z}$ **which satisfies** $k \in (-\infty, d^-) \cup (d^+, \infty)$**, the minimum of** E_{B^2} **is never achieved in** $\mathcal{E}_k$. Furthermore if $k \in (-\infty, d^-] \cup [d^+, \infty)$, $\inf_{\mathcal{E}_k} E_{B^2} = \inf_{\mathcal{E}_{d^\pm}} E_{B^2} + 4\pi|k - d^\pm|$, where $d^\pm = d^-$ if $k \leq d^-$ and $d^\pm = d^+$ if $k \geq d^+$ (so that, in particular, (50) does not hold);

(ii) **for all** $k \in \mathbb{Z}$ **such that** $d^- \leq k \leq d^+$**, the minimum of** E_{B^2} **is achieved in** $\mathcal{E}_k$ **.**

Similar results have been obtained by Qing [176] for maps with values in a Kähler manifold.

The heat flow

Observe that, in the method of J. Sacks and K. Uhlenbeck, the family of minimizers $(u_\alpha)_{\alpha>1}$ of $E^\alpha_{\mathcal{M}}$ produces particular minimizing sequences for $E_{\mathcal{M}}$ as $\alpha \to 1$. One of the advantages of this is that, not only does it help to balance the instability due to the action of the conformal group, but it also gives us some control of the tension field (25). Another natural way to control the tension field for a minimizing sequence is to consider the heat flow equation:

$$\frac{\partial u}{\partial t} = \Delta_g u + g^{ij} A_u\Big(\frac{\partial u}{\partial x^i}, \frac{\partial u}{\partial x^j}\Big) \quad \text{on } [0,T) \times \mathcal{M}. \tag{51}$$

The study of this equation was initiated in [66, 1, 99] in the case when the curvature of the target manifold $\mathcal{N}$ is non-positive (see §6.3). If we remove this hypothesis, the first results[33] were obtained by M. Struwe [207], for the case when dim $\mathcal{M} = 2$ and $\partial\mathcal{M} = \emptyset$: *for any* $u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$*, there exists a* **global weak** *solution* $u : [0, \infty) \times \mathcal{M} \longrightarrow \mathcal{N}$ *of the heat equation (51) which satisfies the energy decay estimate:*

$$E_{\mathcal{M}}(u(T,\cdot)) + \int_0^T \int_{\mathcal{M}} \Big|\frac{\partial u}{\partial t}\Big|^2 \omega_g dt \ \leq\ E_{\mathcal{M}}(u_0), \quad \forall T > 0 \tag{52}$$

and which is smooth outside finitely many singular points $(\overline{t}_j, \overline{x}_j)_{1\leq j\leq k}$*. The solution is unique in this class. Moreover, at each singularity* $(\overline{t}_j, \overline{x}_j)$*, a harmonic sphere* v_j *bubbles off, i.e. there exists a sequence* $(t_{\ell,j}, x_{\ell,j})_{\ell\in\mathbb{N}}$ *which converges to* $(\overline{t}_j, \overline{x}_j)$ *(with* $t_{\ell,j} < \overline{t}_j$*) such that* $u_{\ell,j}(x) := u(t_{\ell,j}, \exp_{x_{\ell,j}}(\lambda_{\ell,j} x))$ *converges to* v_j *in* $W^{2,2}_{loc}(\mathbb{R}^2, \mathcal{N})$*, where* $(\lambda_{\ell,j})_{\ell\in\mathbb{N}}$ *is a sequence of positive numbers such that* $\lim_{\ell\to\infty} \lambda_{\ell,j} = 0$*. The map* v_j *can then be extended to a smooth harmonic map* $S^2 \longrightarrow \mathcal{N}$*. Lastly, there exists a sequence* $(T_\ell)_{\ell\in\mathbb{N}}$ *of times such that* $\lim_{\ell\to\infty} T_\ell = \infty$ *and* $u(T_\ell, \cdot)$ *converges weakly in* $W^{1,2}$ *to a smooth harmonic map* $u_\infty : \mathcal{M} \longrightarrow \mathcal{N}$ *as* $\ell \to \infty$. This result was extended to the case when $\partial\mathcal{M} \neq \emptyset$ by K. C. Chang [37]. These results can be used to recover similar results to those of Sacks and Uhlenbeck, see for example the last chapter of the book of Struwe [209].

The question of whether the solutions to the heat flow equation in two dimensions really develop singularities remained open for some time until K. C. Chang, W. Ding, R. Ye [38] constructed an example of an initial condition

[33] Following the result of Struwe [207], further results on the heat flow when dim $\mathcal{M} \geq 3$ and with no assumption on the curvature of $\mathcal{N}$ were obtained: the first existence results were obtained by Y. M. Chen [41] for $\mathcal{M}$ arbitrary and $\mathcal{N} = S^n$, and by Struwe [208] for $\mathcal{M} = \mathbb{R}^m$ and $\mathcal{N}$ an arbitrary compact manifold. By putting together their ideas, Chen and Struwe [43] obtained the following existence result: *for any map* $u_0 \in W^{1,2}(\mathcal{M}, \mathcal{N})$*, there exists a weak solution to the heat flow equation defined for all time and with Cauchy data* u_0*, i.e. coinciding with* u_0 *at* $t = 0$*. This solution is regular outside a singular set which has locally finite* m*-dimensional Hausdorff measure with respect to the parabolic metric.* Then J.-M. Coron and J.-M. Ghidaglia [50] produced the first examples of weak solutions which blow up at finite time, hence proving that there are no classical solutions in general and Coron [49] built an example of Cauchy data for which there are infinitely many weak solutions to the heat flow equation (actually the Cauchy data is a weakly harmonic map). Later on, similar blow-up and non-uniqueness results were proved for the heat flow on surfaces (see the next paragraph).

$u_0 : S^2 \longrightarrow S^2$ for which the heat flow does blow up in finite time. Note that the inequality in the estimate (52) would be straighforward if the solution were smooth (just multiply the heat equation by u and integrate). Actually the left-hand side of (52) is smooth outside the singular points $(\overline{t}_j, \overline{x}_j)_{1\leq j\leq k}$. In [177] J. Qing proved that, at these bubbling points, the discontinuity of this left-hand side is *just* equal to minus the sum of the energies of the harmonic spheres v_j which separate, i.e., there is no energy loss in the necks connecting u_∞ (the 'body' map) to the v_j's (the 'bubble' maps). He further proved that, if at some time $\overline{t}$ there are p harmonic spheres $v_{j_1}, \cdots v_{j_p}$ bubbling off, then $\lambda_{\ell,i}/\lambda_{\ell,j} + \lambda_{\ell,j}/\lambda_{\ell,i} + |x_{\ell,j} - x_{\ell,i}|^2/(\lambda_{\ell,i}\lambda_{\ell,j}) \to \infty$ as $\ell \to \infty$ for $i, j \in \{j_1, \cdots, j_p\}$ such that $i \neq j$; roughly speaking, this means that each bubble decouples from the other ones in distance or in scale. The analysis of what is happening in the necks was further refined in [178, 215]. In [213] P. Topping proved that *if* $\mathcal{M} = \mathcal{N} = S^2$ *and if one assumes the hypothesis* (H): u_∞ and the v_j's are either all holomorphic or all antiholomorphic, *then* $u(t,\cdot)$ *converges uniformly in time as* $t \to \infty$ *strongly in* $L^p(S^2, \mathbb{R}^3)$ *and in* $W^{1,2}(S^2 \setminus \{\overline{x}_1, \cdots, \overline{x}_k\})$. The latter result depends strongly on the fact that the target is S^2 (see [213]).

The uniqueness of weak solutions to (51) was proved by A. Freire [74], *under the further assumption that* $E_\mathcal{M}(u(t,\cdot))$ *is a non-increasing function of* t. But, in [214], P. Topping constructed solutions of the heat flow from a surface to S^2 *which are different* from Struwe's solution, hence proving the **non-uniqueness of weak solutions to equation** (51), **in general**. The point, however, is that Topping's solutions are obtained by *attaching bubbles*, i.e. have the reverse behaviour of Struwe's solutions, so that the energy $E_\mathcal{M}(u(t,\cdot))$ *increases* by a jump of 4π each time a bubble is attached.

Lastly, in [215], P. Topping performed a very fine analysis of *almost-harmonic* maps from S^2 to S^2, i.e. maps $u \in W^{1,2}(S^2, S^2)$ such that the L^2 norm of the tension field $\tau(u) = \Delta_{S^2} u + |du|^2 u$ is small. Recall that, if $\tau(u) = 0$, then u is harmonic and hence either holomorphic or antiholomorphic, so that its energy is 4π times its degree in $\mathbb{Z}$. P. Topping proved that this quantization of the energy remains true for almost-harmonic maps and more precisely establishes the estimate: $|E_{S^2}(u) - 4\pi k| \leq C||\tau(u)||^2_{L^2(S^2)}$ (for some $k \in \mathbb{Z}$), for all u in $W^{1,2}(S^2, S^2)$ except for some exceptional special cases. This allows him to recover the same conclusions as in [213] concerning the convergence in time and the uniqueness of the location of the singularities of the heat flow, but *without* assuming the hypothesis (H) above. These results are strong in the sense that an almost-harmonic map u may have, for example, a holomorphic body with anti-holomorphic bubbles attached, and then u is *not* close to a harmonic map in the $W^{1,2}(S^2)$ topology. To deal with such cases, Topping established an estimate asserting the existence of a repulsive effect between holomorphic and antiholomorphic components of a bubble tree.

5.4 The direct method in a class of maps not closed for the weak topology: case dim $\mathcal{M} \geq 3$

Some cases where the class $\mathcal{E} \subset W^{1,2}(\mathcal{M}, \mathcal{N})$ chosen for the minimization of $E_\mathcal{M}$ is *not* weakly closed have already been described in §3.3. We will here mainly discuss other situations, starting from the work of H. Brezis, J.-M. Coron and E. H. Lieb [24].

Prescribing singularities

We begin with an example. Let $\Omega \subset \mathbb{R}^3$ and $a \in \Omega$; we will choose a subset $\mathcal{E}$ of $\mathcal{C}^1(\Omega \setminus \{a\}, S^2) \cap W^{1,2}(\Omega, S^2)$. Note that $\mathcal{C}^1(\Omega \setminus \{a\}, S^2) \cap W^{1,2}(\Omega, S^2)$ is not empty since it contains the map u_a defined by $u_a(x) = (x - a)/|x - a|$ (which is even weakly harmonic). Moreover, for each sphere $S^2_{a,r} = \partial B^3(a, r)$ centred on a which is contained in $\overline{\Omega}$, the restriction of u_a to $S^2_{a,r}$ has degree 1. Let us fix

$$\mathcal{E} := \{u \in \mathcal{C}^1(\Omega \setminus \{a\}, S^2) \cap W^{1,2}(\Omega, S^2) \mid \text{deg}\, u|_{S^2_{a,r}} = 1, \text{ for } S^2_{a,r} \subset \Omega\}.$$

Then, in some sense, the minimization of E_Ω in $\mathcal{E}$ extends the problem of minimizing the energy among maps between surfaces of a given degree (see §5.3). Indeed, as shown in [24], after the extraction of a subsequence if necessary, a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ of E_Ω in $\mathcal{E}$ converges weakly to a constant map c, *in all cases except if* Ω *is a ball centred at* a. If we assume, for simplicity, that there exists an unique line segment $[a, b]$ which joins a to the nearest point in $\partial\Omega$ (i.e., such that $b \in \partial\Omega$ and $d(a, \partial\Omega) = |b - a|$) then u_k converges strongly to c on

$\Omega \setminus V_\varepsilon[a,b]$, where $V_\varepsilon[a,b]$ is a neighbourhood of $[a,b]$. Furthermore, the restriction of u_k to a sphere $S^2_{a,r}$ will be almost constant outside the intersection of $S^2_{a,r}$ with $V_\varepsilon[a,b]$, whereas it will almost conformally cover the target S^2 on $S^2_{a,r} \cap V_\varepsilon[a,b]$. Hence a *line of bubbles* separates from u_k along $[a,b]$. Lastly, the infimum of the energy, $\inf_{u\in\mathcal{E}} E_\Omega(u)$, is precisely $4\pi|b-a|$, i.e. the area of S^2 times the length of the line segment. A similar situation, arises if we have a **dipole** as introduced in [24]. Here we assume that

$$\mathcal{E} := \big\{u \in \mathcal{C}^1(\Omega \setminus \{p,n\}, S^2) \cap W^{1,2}(\Omega, S^2) \,|\, \text{deg}\, u|_{S^2_{p,r}} = 1, \text{deg}\, u|_{S^2_{n,r}} = -1 \text{ for } S^2_{p,r}, S^2_{n,r} \subset \Omega\big\},$$

where $p, n \in B^3$ are two distinct points. Then, a minimizing sequence in the class $\mathcal{E}$ converges to a constant outside a neighbourhood of the line segment $[p,n]$, and its energy concentrates along $[p,n]$.

Actually, a more general situation was considered in [24]: let $\{a_1, \cdots, a_p\} \subset \Omega \subset \mathbb{R}^3$ be an arbitrary finite collection of points of $\Omega \subset \mathbb{R}^3$ and $d_1, \cdots, d_p \in \mathbb{Z}$. Then set

$$\mathcal{E} := \big\{u \in \mathcal{C}^1(\Omega \setminus \{a_1, \cdots, a_p\}, S^2) \cap W^{1,2}(\Omega, S^2) \,|\, \forall\, i = 1, \cdots, p,\ \text{deg}\, u|_{S^2_{a_i,r}} = d_i, \text{ for } S^2_{a_i,r} \subset \Omega\big\}.$$

In order to describe the behaviour of a minimizing sequence in $\mathcal{E}$ we need to define the notion of a *minimal connection* as introduced in [24]. For simplicity, we will assume that $\Omega = B^3 := B^3(0,1)$ and that the total degree $Q := \sum_{i=1}^p d_i$ is *zero*. First, call the points a_i such that $d_i > 0$ (resp. $d_i < 0$) *positive* (resp. *negative*) (points a_i such that $d_i = 0$ do not play any role in the following, hence we can forget about them without loss of generality). We list the positive points with each a_i repeated d_i times and write this list as $p_1, \cdots, p_\kappa$. Likewise we list the negative points as $n_1, \cdots, n_{\kappa'}$. Note that $\kappa - \kappa' = Q = 0$. A *connection* C is then a pairing of the two lists $(p_1, n_{\sigma(1)}), \cdots, (p_\kappa, n_{\sigma(\kappa)})$, where σ is a permutation of $\{1, \ldots, \kappa\}$. The *length* of the connection C is $L(C) := \sum_{i=1}^\kappa d(p_i, n_{\sigma(i)})$. Lastly, the *length of the minimal connection* is: $L := \min_C L(C)$ and a *minimal connection* is a connection $\underline{C}$ (which may not be unique) such that $L(\underline{C}) = L$. Then **the infimum**[34] **of** E_{B^3} **on** $\mathcal{E}$ **is** $4\pi L$ and, if we exclude the case when $\{a_1, \cdots, a_p\} = \emptyset$ or $\{0\}$, we have:

- this infimum is never achieved and, after extraction of a subsequence if necessary, a minimizing sequence $(u_k)_{k\in\mathbb{N}}$ of E_{B^3} in $\mathcal{E}$ converges weakly to a constant map;
- again, after extraction of a subsequence if necessary, lines of bubbles separate from u_k along a minimal connection $\underline{C}$. More precisely, the energy density $\frac{1}{2}|du_k|^2$ converges weakly in Radon measures to a measure μ supported by a minimal connection: for all measurable $A \subset B^3$, $\mu(A) = 4\pi\mathcal{H}^1(A \cap \underline{C})$, where $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure (see §4.3 for the definition).

Moreover, the locations and degrees of the singularities of a map $u \in W^{1,2}(\Omega, S^2) \cap \mathcal{C}^1(\Omega \setminus \{a_1, \cdots, a_p\}, S^2)$ can be detected by computing the differential of the 2-form $u^*\omega_{S^2}$ (see §3.2), because of the relation:

$$d(u^*\omega_{S^2}) = \Big(\sum_{i=1}^p d_i \delta_{a_i}\Big) dx^1 \wedge dx^2 \wedge dx^3, \quad \text{where } \delta_{a_i} \text{ is the Dirac mass at } a_i\,. \tag{53}$$

Note that the coefficients of $u^*\omega_{S^2}$ are in $L^1(\Omega)$ and equation (53) holds in the distribution sense, i.e., $\int_{\partial B^3} \zeta(u^*\omega_{S^2}) - \int_{B^3} d\zeta \wedge u^*\omega_{S^2} = \sum_{i=1}^p d_i \zeta(a_i)$, $\forall \zeta \in \mathcal{C}^\infty(\Omega)$. In fact, the latter relation makes sense even if ζ belongs to the set $Lip(\Omega)$ of Lipschitz continuous functions on Ω. This leads to an alternative (dual) formula[35] for the length of the minimal connection:

$$L(u) = \max_{\zeta\in Lip(\Omega),\, |\nabla\zeta|_{L^\infty}\le 1} \sum_{i=1}^p d_i \zeta(a_i) = \max_{\zeta\in Lip(\Omega),\, |\nabla\zeta|_{L^\infty}\le 1} \Big\{\int_{\partial B^3} \zeta(u^*\omega_{S^2}) - \int_{B^3} d\zeta \wedge u^*\omega_{S^2}\Big\}. \tag{54}$$

[34] An alternative proof of the inequality $\inf_{u\in\mathcal{E}} E_{B^3}(u) \ge 4\pi L$ was given by F. Almgren, W. Browder and E. H. Lieb [3] by using the *coarea formula* $\int_{B^3} (J_2 u)(x) dx^1 dx^2 dx^3 = \int_{y\in S^2} \mathcal{H}^1(u^{-1}(y)) d\mathcal{H}^2(y)$, valid for a smooth map $u : B^3 \longrightarrow S^2$. Here $\mathcal{H}^2$ is the 2-dimensional Hausdorff measure on S^2, $\mathcal{H}^1$ is the 1-dimensional Hausdorff measure on a generic fibre $u^{-1}(w)$ of u and $(J_2 u)(x)$ denotes the 2-dimensional Jacobian of u at x. Note that the coarea formula has been extended to Sobolev mappings between manifolds by P. Hajłasz [97], leading to another variant of the proof of the Brezis, Coron and Lieb result.

[35] Note that $Q = 0$ implies that $\int_{\partial B^3} u^*\omega_{S^2} = 0$, so that the maximum in (54) is finite.

But **the right-hand side of** (54) **makes sense for an arbitrary map** $u \in W^{1,2}_\varphi(B^3, S^2)$**, and can be used to extend the definition of** $L(u)$ **to the whole of** $W^{1,2}_\varphi(B^3, S^2)$ **if the degree of** φ **is zero**. Moreover, it was proved by Bethuel, Brezis and Coron [15] that *the functional* $L : W^{1,2}_\varphi(B^3, S^2) \longrightarrow \mathbb{R}$ *is continuous for the strong* $W^{1,2}$ *topology.* Lastly, a result of Brezis and P. Mironescu [20, 21] asserts that, for any $u \in W^{1,2}(B^3, S^2)$ such that $u|_{\partial B^3}$ is a smooth map of degree 0, there exist two sequences $(p_1, p_2, \ldots)$ and $(n_1, n_2, \ldots)$ of points of B^3 such that

$$d(u^*\omega_{S^2}) = 4\pi \sum_{i=1}^{\infty} (\delta_{p_i} - \delta_{n_i}) \tag{55}$$

and $\sum_{i=1}^\infty |p_i - n_i| < \infty$. Then $L(u)$ is equal to the infimum of all sums $\sum_{i=1}^\infty |p_i - n_i|$ such that (55) holds.

Generalizations. Similar situations occur, for instance, if we work in $W^{1,n}(\mathcal{M}, S^n)$, where dim $\mathcal{M} \geq n+1$, and we try to minimize the n-energy among maps which are smooth outside a codimension $n+1$ submanifold Σ and which have prescribed degree around each connected componen of Σ. This case was first considered by F. Almgren, W. Browder and E. H. Lieb [3], who pointed out that the minimal connection has to be replaced by an n-area minimizing integral current. We refer to [88, Chapter 5] for subsequent developments.

The gap phenomenon

An important and surprising observation was made by R. Hardt and F. H. Lin [106] at about the same time: we still assume that $\mathcal{M} = B^3$ and $\mathcal{N} = S^2$ and we let $\varphi : \partial B^3 \longrightarrow S^2$ be a smooth map *of degree* 0. Then $\mathcal{C}^1_\varphi(B^3, S^2) := \{u \in \mathcal{C}^1(B^3, S^2) |\ u = \varphi \text{ on } \partial B^3\}$ is not empty and we may consider its closure $H^1_{\varphi,s}(B^3, S^2)$ in the strong $W^{1,2}$ topology. Another natural class is $W^{1,2}_\varphi(B^3, S^2) := \{u \in W^{1,2}(B^3, S^2) |\ u = \varphi \text{ on } \partial B^3\}$. Then it is proved in [106] that we can choose the boundary conditions φ such that:

$$\inf_{u\in\mathcal{C}^1_\varphi(B^3,S^2)} E_{B^3}(u) = \inf_{u\in H^1_{\varphi,s}(B^3,S^2)} E_{B^3}(u) \ > \inf_{u\in W^{1,2}_\varphi(B^3,S^2)} E_{B^3}(u)\,. \tag{56}$$

This implies that the inclusion $H^1_s(B^3, S^2) \subset W^{1,2}(B^3, S^2)$ is strict, as discussed in §3.2. The construction of φ relies on ideas close to the preceding discussion: imagine that we fix two *dipoles* of length $\ell > 0$, i.e. pairs of points (p_1, n_1) and (p_2, n_2) with opposite degrees ± 1, such that $|p_1 - n_1| = |p_2 - n_2| = \ell$. Place the points p_1 and n_1 very close to the *north* pole $(0, 0, 1)$ of ∂B^3, with p_1 *outside* B^3 but n_1 *inside* B^3, specifically, $p_1 = (0, 0, 1+\ell/2)$ and $n_1 = (0, 0, 1-\ell/2)$. Similarly, place p_2 and n_2 very close to the *south* pole: $p_2 = (0, 0, -1+\ell/2)$ and $n_2 = (0, 0, -1-\ell/2)$. This is all embedded in, say, $B^3(0, 2)$. Now consider how a sequence of maps $(v_k)_{k\in\mathbb{N}}$ in $W^{1,2}(B^3(0, 2), S^2)$ which minimizes $E_{B^2(0,2)}$ in the class of maps v such that $d(v^*\omega_{S^2}) = \delta_{p_1} + \delta_{p_2} - \delta_{n_1} - \delta_{n_2}$ would look: v_k is almost constant outside neighbourhoods of the line segments $[p_1, n_1]$ and $[p_2, n_2]$, and the restriction of v_k to a piece of surface cutting one of these segments transversally covers S^2 almost conformally. Then we take $\varphi = (v_k)|_{\partial B^2}$ for k large enough. We observe that

(i) the degree of φ is equal to the sum of the degrees of the singularities n_1 and p_2 enclosed by ∂B^3, i.e., 0 ;

(ii) $\inf_{u\in W^{1,2}_\varphi(B^3,S^2)} E_{B^3}(u)$ is certainly smaller than $E_{B^3}(v_k)$, which is of order $4\pi\ell$;

(iii) $\inf_{u\in\mathcal{C}^1_\varphi(B^3,S^2)} E_{B^3}(u)$ is of order 8π.

Hence, (56) follows by choosing ℓ sufficiently small. To prove (iii), we estimate the energy of any map $\psi \in \mathcal{C}^1_\varphi(B^3, S^2)$ from below as follows. For any $h \in (-1, 1)$, consider the disk D_h which is the intersection of B^3 with the plane $\{x^3 = h\}$ and the domain $H_h := B^3 \cap \{x^3 < h\}$: its boundary ∂H_h is the union of D_h and the spherical cap $S_h := (\partial B^3) \cap \{x^3 \leq h\}$. On the one hand, the restriction of ψ to S_h is almost constant except in a small neighbourhood of the south pole, where $\psi|_{S_h}$ covers almost all of S^2 with degree 1, and on ∂D_h, the map ψ is nearly constant. On the other hand, since ψ is continuous inside H_h, the degree of its restriction to ∂H_h is 0. These two facts imply that the restriction $\psi|_{D_h}$ should almost cover S^2 with degree -1. Hence $\int_{D_h} \frac{1}{2}|d\psi|^2 d^3x \geq |\int_{D_h} \psi^*\omega_{S^2}| \simeq 4\pi$. By integrating this inequality on $h \in (-1, 1)$ we obtain (iii).

The relaxed energy

Exploiting the fact that $H^1_{\varphi,w}(B^3,S^2)=W^{1,2}_\varphi(B^3,S^2)$ (see §3.2), i.e. that $\forall u\in W^{1,2}_\varphi(B^3,S^2)$ there exists a sequence $(v_k)_{k\in\mathbb{N}}$ of maps in $\mathcal{C}^1_\varphi(B^3,S^2)$ which converges *weakly* in $W^{1,2}$ to u, we can define the **relaxed energy** E^{rel}_Ω on $W^{1,2}_\varphi(B^3,S^2)$ by

$$E^{rel}_{B^3}:=\inf\big\{\lim_{k\to\infty}\inf\int_{B^3}|du_k|^2dx^1dx^2dx^3\mid v_k\in\mathcal{C}^1_\varphi(B^3,S^2),v_k\to u\text{ weakly in }W^{1,2}\big\}.$$

The following expression for $E^{rel}_{B^3}$, valid when the degree of φ is zero, was given by F. Bethuel, H. Brezis and J.-M. Coron [15]:

$$E^{rel}_{B^3}(u)=E_{B^3}(u)+4\pi L(u),$$

where $L(u)$ is length of the the minimal connection associated to u defined by (54). A nice theory was built by M. Giaquinta, G. Modica and J. Souček [88] in order to picture geometrically the relaxed energy and, more generally many bubbling phenomena[36]. The relaxed energy satisfies the properties (i) $\forall\,u\in W^{1,2}_\varphi(B^3,S^2)$, $E^{rel}_{B^3}(u)\geq E_{B^3}(u)$, *with equality if* $u\in\mathcal{C}^1_\varphi(B^3,S^2)$; (ii) $\inf_{u\in W^{1,2}_\varphi(B^3,S^2)}E^{rel}_{B^3}(u)=\inf_{u\in\mathcal{C}^1_\varphi(B^3,S^2)}E_{B^3}(u)$.

Other interesting functionals provided by interpolating between the Dirichlet energy E_{B^3} and the relaxed energy $E^{rel}_{B^3}$ were considered in [15]: for $\lambda\in\mathbb{R}$ consider $E^\lambda_{B^3}(u)=E_{B^3}(u)+4\lambda\pi L(u)$. Then first of all, $\forall\lambda\in\mathbb{R}$, *the critical points of* $E^\lambda_{B^3}$ *on* $W^{1,2}(B^3,S^2)$ *are weakly harmonic*. Second, *for* $0\leq\lambda\leq 1$, $E^\lambda_{B^3}$ *is lower semi-continuous*. This implies that, for $0\leq\lambda\leq1$, the direct method can be used successfully in order to minimize $E^\lambda_{B^3}$ in, say, $W^{1,2}_\varphi(B^3,S^2)$ in order to obtain a family of weakly harmonic maps with the same boundary conditions (see the §4.3 for partial regularity results). This shows the strong *non-uniqueness of solutions* for the Dirichlet problem for harmonic maps in dimensions larger than three.

Minimizing the energy among continuous maps

In view of properties (i) and (ii) of the relaxed energy functional $E^{rel}_{B^3}$, it is tempting to use it in order to answer the following question: *given smooth boundary data* $\varphi:\partial B^3\longrightarrow S^2$ *of degree 0, is there a smooth harmonic map* $B^3\longrightarrow S^2$ *extending* φ *?* One strategy might be to minimize $E^{rel}_{B^3}$ over $W^{1,2}_\varphi(B^3,S^2)$: if we think, for example, of boundary data φ leading to a gap phenomenon described before, and we compare the values of the *relaxed energy* for the smooth and for the singular maps that we can construct, we realize that the gain in energy from allowing singularities is exactly cancelled by the cost due to the length of the minimal connection. But these considerations are only heuristic up to now: for the moment the question of whether minimizers of the relaxed energy are smooth is completely open (see §4.3).

On the other hand a direct approach to the problem of *minimizing the energy functional* $E_\mathcal{M}$ *in a class* $\mathcal{E}$ *of smooth maps in a given homotopy class between two arbitrary compact manifolds without boundary* $\mathcal{M}$ *and* $\mathcal{N}$ has been addressed by F. H. Lin [148]. He proved that *if* $(u_k)_{k\in\mathbb{N}}$ *is a minimizing sequence in* $\mathcal{E}$*, then, after extracting a subsequence if necessary,* u_k *converges weakly in* $W^{1,2}(\mathcal{M},\mathcal{N})$ *to a weakly harmonic map* $u\in W^{1,2}(\mathcal{M},\mathcal{N})$ *and* $|du_k|^2d^mx$ *converges weakly to the Radon measure* $\mu=|du|^2+\nu$*. Moreover,* u *is smooth away from a closed, rectifiable set* Σ *and* $\mathcal{H}^{m-2}(\Sigma)$ *is bounded*. The non-negative Radon measure ν measures the defect of strong convergence: it is the product of the $(m-2)$-dimensional Hausdorff measure supported by Σ times a function Θ on Σ which is measurable with respect to the $(m-2)$-dimensional Hausdorff measure. Lastly, **for almost all** $x\in\Sigma$**,** $\Theta(x)$ **is equal to a finite sum of energies of harmonic non-constant maps from** S^2 **to** $\mathcal{N}$, so that he obtain a higher-dimensional analogue of the results of Sacks and Uhlenbeck discussed in §5.3. Compare also with the results on the reduction of the singular set of a stationary map by Lin and Rivière presented in §4.3.

[36]The basic idea is to represent a map u between manifolds by its graph, which, in the case that u is in a Sobolev space but not continuous, is a *Cartesian current*, i.e. a current in the sense of geometric measure theory which satisfies some special conditions. In the enlarged class of Cartesian currents, we can describe precisely what the weak limit of a minimizing sequence is, keeping track of the *necks* connecting the bubbles in two dimensions, or the *minimal connection* in three dimensions. See [88] for a complete exposition.

Towards completely discontinuous weakly harmonic maps

A notion of *relative relaxed energy* was introduced by F. Bethuel, H. Brezis and J.-M. Coron [15] as follows. Again, we fix smooth boundary data $\varphi : \partial B^3 \longrightarrow S^2$ of degree zero and we first define our functional on $\mathcal{R}^{2,1}_\varphi(B^3,S^2)$, the set of maps $u \in W^{1,2}_\varphi(B^3,S^2)$ which are $\mathcal{C}^1$ outside a finite number of points (see §3.2). For a pair (u,v) of maps in $\mathcal{R}^{2,1}_\varphi(B^3,S^2)$ we define the *length* $L(u,v)$ *of the minimal connection of* u *relative to* v to be the length of the minimal connection connecting the singularities of u and the singularities of v, where the singularities of v are *counted with opposite degrees*. By using the definition of the length of a minimal connection given by the right-hand side of (54), $L(u,v)$ can be expressed as

$$L(u,v) = \max_{\zeta\in Lip(\Omega),\, |\nabla\zeta|_{L^\infty}\leq 1} \int_{\partial B^3} d\zeta \wedge (u^*\omega_{S^2} - v^*\omega_{S^2})\,. \tag{57}$$

Thanks to (57), the functional $L : \mathcal{R}^{2,1}_\varphi(B^3,S^2) \times \mathcal{R}^{2,1}_\varphi(B^3,S^2) \longrightarrow \mathbb{R}$ can be extended to a functional $L : W^{1,2}_\varphi(B^3,S^2) \times W^{1,2}_\varphi(B^3,S^2) \longrightarrow \mathbb{R}$. It is shown in [15] that this functional is continuous on $W^{1,2}_\varphi(B^3,S^2) \times W^{1,2}_\varphi(B^3,S^2)$ and that, for any fixed $v \in W^{1,2}_\varphi(B^3,S^2)$, the functional

$$F_{B^3,v}(u) := E_{B^3}(u) + 4\pi L(u,v)$$

is lower semi-continuous on $W^{1,2}_\varphi(B^3,S^2)$. Moreover, the critical points of $F_{B^3,v}$ are weakly harmonic. This has turned out to be a powerful tool for constructing singular weakly harmonic maps.

First, R. Hardt, F. H. Lin and C. Poon [109] constructed weakly harmonic maps with a finite, but arbitrary, number of prescribed singularities located on a line. In their construction, they first fix a map $v \in \mathcal{R}^{2,1}_\varphi(B^3,S^2)$ which is invariant by rotations around some axis and which has dipoles of singularities along the axis of symmetry. Then they minimize the relative relaxed energy $F_{B^3,v}$ among all maps $u \in W^{1,2}_\varphi(B^3,S^2)$ which are also rotationally symmetric, and they show that the singular set of the minimizer is the same as the singular set of v. This result was improved by F. Rivière [181] who considered a sequence $(v_k)_{k\in\mathbb{N}^*}$ of rotationally symmetric maps in $W^{1,2}_\varphi(B^3,S^2)$ having more and more singularities along the axis of symmetry and the corresponding sequence $(u_k)_{k\in\mathbb{N}^*}$ of minimizers for F_{B^3,v_k} among rotationally symmetric maps in $W^{1,2}_\varphi(B^3,S^2)$. He was able to prove that $(u_k)_{k\in\mathbb{N}^*}$ converges to a weakly harmonic map having a line of singularity. Lastly Rivière [182] proved that, **for any non-constant map $\varphi : \partial B^3 \longrightarrow S^2$, there exists a weakly harmonic map in $W^{1,2}_\varphi(B^3,S^2)$ which is discontinuous everywhere in B^3.** This result rests on the *construction of a dipole* lemma: for any smooth map $w : B^3(a,r) \longrightarrow S^2$ such that $dw(a) \neq 0$ and for any $\rho \in (0,r)$ there exists a pair of points (p,n) inside $B^3(a,\rho)$ and a map $\widetilde{w} \in W^{1,2}(B^3(a,r),S^2)$ which is smooth outside $\{p,n\}$, has a degree 1 singularity at p and a degree -1 singularity at n, coincides with w in $B^3(a,r)\setminus B^3(a,\rho)$, and which satisfies

$$E_{B^3(a,r)}(\widetilde{w}) < E_{B^3(a,r)}(w) + 4\pi|p-n|\,. \tag{58}$$

That the inequality in (58) is strict[37] is crucial, as in the 2-dimensional theory (see §5.3). A second main ingredient in the proof of Rivière is the construction of a sequence $(v_k)_{k\in\mathbb{N}^*}$ of maps in $\mathcal{R}^{2,1}_\varphi(B^3,S^2)$ having more and more singularities. Each map v_{k+1} is constructed from v_k by adding a dipole and using the *construction of a dipole* lemma in order to control the extra cost of energy by (58). The sequence $(v_k)_{k\in\mathbb{N}^*}$ also converges strongly to some completely discontinuous map $v \in W^{1,2}_\varphi(B^3,S^2)$. The last task is then to show that any minimizer of $F_{B^3,v}$ is completely discontinuous.

5.5 Other analytical methods for existence

Morse and Lusternik–Schnirelman theories

A general reference for the ideas in this paragraph is the book of M. Struwe [209]. One of the first applications of these variational methods, devoted to existence proofs of *non-minimal* critical points is the work by G. D. Birkhoff

[37] Note that a weaker, non-strict, analogous inequality was already obtained in [10].

[18] which establishes the existence of closed geodesics on a surface of genus 0, i.e. the image of a harmonic map of a circle, see §2.2. Extensions to higher-dimensional harmonic maps is rather difficult and most of the known results concern the case $m = 2$.

In [188] J. Sacks and K. Uhlenbeck addressed the study of both *minimizing* (see §5.3) and *non-minimizing* harmonic maps from a surface without boundary $\mathcal{M}$ to a compact manifold without boundary $\mathcal{N}$. As for minimizing maps, they first establish the existence of non-minimizing critical points of the functional $E^\alpha_{\mathcal{M}}$ (see §5.3) for $\alpha > 1$, and then study the behaviour of these critical points when $\alpha \to 1$. The Morse theory for critical points of $E^\alpha_{\mathcal{M}}$ has better properties when $\alpha > 1$, since this functional then satisfies the *Palais–Smale condition*[38]. Let $\Omega(\mathcal{M},\mathcal{N})$ be the space of base point preserving (continuous) maps from $\mathcal{M}$ to $\mathcal{N}$ (i.e. we fix some points $x_0 \in \mathcal{M}$ and $y_0 \in \mathcal{N}$ and we consider maps which send x_0 to y_0). First, Sacks and Uhlenbeck proved that *if* $\Omega(\mathcal{M},\mathcal{N})$ *is not contractible, then there exist non-trivial critical points of* $E^\alpha_{\mathcal{M}}$ *between* $\mathcal{M}$ *and* $\mathcal{N}$. This critical point is non-minimizing if $\mathcal{C}^0(\mathcal{M},\mathcal{N})$ is connected. They noticed that the hypothesis that $\Omega(\mathcal{M},\mathcal{N})$ is not contractible is satisfied, in particular, *if* $\mathcal{M} = S^2$ *and if the universal cover of* $\mathcal{N}$ *is not contractible*, since $\pi_{k+2}(\mathcal{N}) = \pi_k(\Omega(S^2,\mathcal{N}))$. Second, they considered a sequence of maps from S^2 to $\mathcal{N}$ which are critical points of $E^\alpha_{\mathcal{M}}$ for $\alpha > 1$, and study its convergence as $\alpha \to 1$. The analysis is similar to the case of minimizing maps, see §5.3. They concluded that, **if the universal cover of $\mathcal{N}$ is not contractible, there exists a non-trivial harmonic map from S^2 to $\mathcal{N}$.** These results were extended by J. Jost in [133] using a different approach. Similar results have been obtained by Jost and Struwe [138], with applications to the Plateau problem for surfaces of arbitrary genus. See [134] for a survey and the papers by G. F. Wang [220] and Y. Ge [82] for recent applications to maps on a surface of genus greater than one with values in S^2.

These methods can also be applied on surfaces with boundary to construct non-minimizing harmonic maps with prescribed Dirichlet boundary condition. An example is the construction of saddle-point harmonic maps from the unit disc to the sphere S^n for $n \geq 3$ by V. Benci and J.-M. Coron [9]. This was extended to maps from a planar domain bounded by several disks by W. Y. Ding [54]. Similar results has been obtained by J. Qing [175] for maps from the unit disc to S^2.

Gauss maps of constant mean curvature surfaces

An important motivation for studying harmonic maps into spheres or, more generally, into a Grassmannian, is the result by E. A. Ruh and J. Vilms [187] on a submanifold Σ of dimension m immersed in the Euclidean space $\mathbb{R}^{m+p}$ and its Gauss map $f : \Sigma \longrightarrow G_m(m+p)$ to the Grassmannian of oriented m-dimensional subspaces of $\mathbb{R}^{m+p}$; this asserts that *the covariant derivative of the mean curvature vector field is equal to the tension field of its Gauss map*. In particular, *an immersion in* $\mathbb{R}^{m+p}$ *has parallel mean curvature if and only if its Gauss map is harmonic*. Note that, if $m = 2$ and $m + p = 3$, then $G_2(3) \simeq S^2$. The consequences of this fact are numerous[39]. For example, any construction of a mean curvature surface in $\mathbb{R}^3$ provides us with a harmonic map from that surfaces to S^2: constant mean curvature surfaces of genus 1 (tori) were first constructed by H. Wente [222] by using a delicate analysis of the $\sinh$–Gordon equation[40], later on N. Kapouleas [139] constructed higher-genus surfaces. The method here relies on gluing together pieces of explicitly known constant mean curvature surfaces (actually, segments of Delaunay surfaces) to produce, first, an approximate solution and then, by a careful use of a fixed point theorem, an exact solution near the approximate one. Since the work of Kapouleas, a huge variety of constructions has been done by following this strategy, see for example [140, 153].

A recent related result is the construction by P. Collin and H. Rosenberg [47] of *a harmonic diffeomorphism from the plane* $\mathbb{R}^2$ *onto the hyperbolic disc* H^2. Note that E. Heinz proved in 1952 that *there is no harmonic diffeomorphism from the hyperbolic disc* H^2 *onto the Euclidean plane* $\mathbb{R}^2$, and it was conjectured by R. Schoen

[38] The Palais–Smale condition reads: *for any sequence of maps* $(u_k)_{k\in\mathbb{N}}$ *such that* $E^\alpha_{\mathcal{M}}(u_k)$ *is bounded and* $\left(\delta E^\alpha_{\mathcal{M}}\right)_{u_k}$ *converges to 0, there is a subsequence which converges strongly*, see [209].

[39] In particular, the structure of the completely integrable system for harmonic maps from a surface to S^2 and for constant mean curvature surfaces in $\mathbb{R}^3$ coincide *locally*, see Chapter 7.

[40] Since the work by Wente, a full classification of constant mean curvature tori has been obtained by using methods of completely integrable systems, see Chapter 7.

that symmetrically there is should be no harmonic diffeomorphism from $\mathbb{R}^2$ to H^2 — the result of Collin and Rosenberg contradicts this conjecture. The proof relies on constructing an entire minimal graph in the product $H^2 \times \mathbb{R}$ which has the same conformal structure as $\mathbb{R}^2$. Hence, the harmonic diffeomorphism is the restriction to this graph of the projection mapping $H^2 \times \mathbb{R} \longrightarrow H^2$.

Ordinary differential equations

Many interesting examples of harmonic maps can be constructed by using reduction techniques. One powerful construction is the *join* of two *eigenmaps* of spheres introduced by R. T. Smith [202]: a map $u : S^m \longrightarrow S^n$ is called an *eigenmap* if and only if it is a harmonic map with a constant energy density; given two eigenmaps $u_1 : S^{m_1} \longrightarrow S^{n_1}$ and $u_2 : S^{m_2} \longrightarrow S^{n_2}$, and a function $\alpha : [0, \pi/2] \longrightarrow [0, \pi/2]$ such that $\alpha(0) = 0$ and $\alpha(\pi/2) = \pi/2$, the *α-join of u_1 and u_2* is the map $u_1 *_\alpha u_2 \longrightarrow S^{m_1+m_2+1} \longrightarrow S^{n_1+n_2+1}$ defined by $(u_1 *_\alpha u_2)(x_1 \sin s\, ,\, x_2 \cos s) = (u_1(x_1) \sin \alpha(s)\, ,\, u_2(x_2) \cos \alpha(s))$. The harmonic map equation on $u_1 *_\alpha u_2$ reduces to an ordinary differential equation for α which can be solved in many cases [202, 64, 167]. A similar ansatz is the *α-Hopf construction* [179] $\varphi : S^{m_1+m_2+1} \longrightarrow S^{n+1}$ on a harmonic bi-eigenmap $f : S^{m_1} \times S^{m_2} \to S^n$: φ defined by $\varphi(x_1 \sin s\, ,\, x_2 \cos s) = (f(x_1, x_2) \sin \alpha(s)\, ,\, \cos \alpha(s))$. This construction leads also to a family of new examples [64, 56, 57, 81]. Similar reductions to systems of equations in more variables have been done [79, 80][41].

6 Other analytical properties

6.1 Uniqueness of and restrictions on harmonic maps

Uniqueness of harmonic maps in a given class of maps does not hold in general. The main case where uniqueness holds, with general methods to prove it, is when the target manifold satisfies strong convexity properties (see §6.3). An example of a result outside this situation requires the *smallness of the scaled energy* $E_{x,r}$ (see §4.3) for maps from $B^3 \subset \mathbb{R}^3$ to a compact manifold $\mathcal{N}$: *There exist some* $\varepsilon_0 > 0$ *and a constant* $C = C(\mathcal{N})$ *such that, for any boundary data* $g \in W^{1,2}(\partial B^3, \mathcal{N})$ *such that* $E_{\partial B^3}(g) < \varepsilon_0$*, there is a unique weakly harmonic map* $u \in W^{1,2}_g(B^3, \mathcal{N})$ *such that* $\sup_{x_0 \in B^3, r>0} \{r^{-1} \int_{B^3(x_0,r)\cap B^3} |du|^2 d^3x\} < C\varepsilon_0$. This was proved by M. Struwe [210] by using the regularity techniques for stationary maps in dimension greater than 2 (see §4.3).

Other restrictions on harmonic maps occur in the case where $\mathcal{M}$ is a surface without boundary and rely on methods of complex analysis (as in the result of Eells and Wood [67], see §5.3) or on the use of twistor theory for maps from the 2-sphere and integrable systems theory for maps from tori (see Chapter 7). See also the non-existence results for harmonic maps on a manifold with a non-empty boundary which are constant on the boundary [144, 230, 141] in §5.3.

6.2 Minimality of harmonic maps

A natural question is the following. *Consider a weakly harmonic map* $u \in W^{1,2}(\mathcal{M}, \mathcal{N})$*; then is* u *an energy minimizer?* If the answer is yes, one of the most efficient methods to prove it is to combine results on existence, regularity and uniqueness. Many such results are available if $\mathcal{N}$ has good convexity properties; these are expounded in §6.3. Here is an example by R. Schoen and K. Uhlenbeck [192] of a result which can be proved without these convexity assumptions. *Let* $S^n_+ := \{y \in S^n \subset \mathbb{R}^{n+1} |\ y^{n+1} > 0\}$ *and* $u : \mathcal{M} \longrightarrow S^n_+$ *be a smooth harmonic map, then* u *is an energy minimizer among maps from* $\mathcal{M}$ *to* S^n. The proof proceeds as follows: let $\Omega \subset \mathcal{M}$ be any bounded domain with smooth boundary and apply the existence theorem of S. Hildebrandt, W. Jäger and K.-O. Widman [120] which asserts that there exists a *smooth* least energy map $\widetilde{u}$ from Ω to S^n_+ which agrees with u in $\partial\Omega$. Then, by the uniqueness result of W. Jäger and H. Kaul [126], we actually have $\widetilde{u} = u$ on Ω. Hence, u is energy minimizing among maps with values in S^n_+. Now let $v \in W^{1,2}(\Omega, S^n)$ be a map

[41] Harmonic *morphisms* can also be found by this method, see [7, Chapter 13].

which agrees with u on $\partial\Omega$ and let $v_+ := \big(v^1,\cdots,v^n,|v^{n+1}|\big)$. We observe that $v_+ \in W^{1,2}(\Omega,S^n)$, v_+ agrees with u on $\partial\Omega$ and $E_{\mathcal{M}}(v_+) = E_{\mathcal{M}}(v)$. Actually v_+ takes values in the closure $\overline{S^n_+}$ of S^n_+, but it is easy to produce a continuous family $(R_\varepsilon)_{\varepsilon\leq 0}$ of retraction maps $R_\varepsilon : \overline{S^n_+} \longrightarrow \overline{S^n_+}$ such that $R_0 = \mathrm{Id}$, the image of R_ε is contained in S^n_+ if $\varepsilon > 0$, and $\lim_{\varepsilon\to 0} E_\Omega(R_\varepsilon\circ v_+) = E_\Omega(v_+)$. Moreover since $u(\overline{\Omega})$ is compact in S^n_+, we can construct R_ε in such a way that $R_\varepsilon\circ v_+$ agrees with u on $\partial\Omega$. Hence, $\forall\varepsilon > 0$, $E_\Omega(R_\varepsilon\circ v_+) \geq E_\Omega(u)$ which gives $E_\Omega(v) = E_\Omega(v_+) \geq E_\Omega(u)$ on letting $\varepsilon\to 0$; the result follows. By similar reasoning, Jäger and Kaul [127] proved also that, if $u_\ominus \in W^{1,2}(B^m,S^m)$ is the map defined by $u_\ominus(x) = (x/|x|,0)$, the minimum in $W^{1,2}_{u_\ominus}(B^m,S^m)$ is achieved by (i) a smooth rotationally symmetric diffeomorphism from B^m to $\overline{S^m_+}$ if $1\leq m\leq 6$, (ii) $u_\ominus$ if $7\leq m$.

Another favorable circumstance for proving the minimality of a harmonic map is if the harmonic map is a *diffeomorphism*. In dimension two the following result was proved by J.-M. Coron and F. Hélein [52]. *Let $(\mathcal{M},g)$ and $(\mathcal{N},h)$ be two Riemannian surfaces, then any harmonic diffeomorphism $\underline{u}$ between $(\mathcal{M},g)$ and $(\mathcal{N},h)$ is an energy minimizer among maps in the same homotopy class and (if $\partial\mathcal{M}\neq\emptyset$) with the same boundary conditions.* The idea is that, thanks to the Hopf differential of $\underline{u}$, one can construct an isometric embedding $(\mathcal{N},h)\subset(\mathcal{M},h_1)\times(\mathcal{M},h_2)$ with two natural projections $\pi_a : (\mathcal{N},h)\longrightarrow(\mathcal{M},h_a)$ (for $a=1,2$) such that $\pi_1\circ\underline{u}$ is harmonic conformal and hence a minimizer and $\pi_2\circ\underline{u}$ is harmonic into $(\mathcal{M},h_2)$. However the curvature of $(\mathcal{M},h_2)$ is non-positive[42]. Thus $\pi_2\circ\underline{u}$ is also a minimizer thanks to results in [2, 111] (see §6.3). Moreover $\underline{u}$ is the *unique minimizer* if there exists a metric g_2 on $\mathcal{M}$ of negative curvature which is conformal to g [52]. Coron and Hélein also proved the minimality of some rotationally symmetric harmonic diffeomorphisms in dimension greater than two. These results were extended by Hélein [112, 115], by using *null Lagrangians*[43].

Because of the partial regularity theory of R. Schoen and K. Uhlenbeck [190] (see §4.3), it is important to identify the *homogeneous* maps u in $W^{1,2}(B^m,\mathcal{N})$ which are minimizing (recall that u is homogeneous if it is of the form $u(x)=\psi(x/|x|)$), since the *minimizing tangent maps*, which model the behaviour of a minimizing map near a singularity, are homogenous. Most known results concern the map $u^s_\odot\in W^{1,2}(B^m,S^{m-s-1})$ defined by $u^s_\odot(x,y) = x/|x|$, for $(x,y)\in\mathbb{R}^{m-s}\times\mathbb{R}^s$ (having an s-dimensional singular set) and, in particular, radial projection $u_\odot := u^0_\odot\in W^{1,2}(B^m,S^{m-1})$: **for any $m\geq 3$ and for any $s\geq 0$, $u^s_\odot$ is a minimizer.** Various proofs exist, depending on the values of m and s:

- for $s=0$ and $m\geq 7$ by Jäger and Kaul [127], as a corollary of the previous results on $u_\ominus$;
- for $s=0$ and $m=3$ by H. Brezis, J.-M. Coron and E. H. Lieb [24] (see §4.3) and $u_\odot$ is the *unique* minimizer;
- for $s=0$ and $m\geq 3$ by F. H. Lin [146];
- for $s\geq 0$ and $m\geq 3$ by J.-M. Coron and R. Gulliver [51] (the general case).

The method of Lin is very short and uses a comparison of the energy functional $E_{B^m}(u)$, for $u\in W^{1,2}_{u_\odot}(B^m,S^{m-1})$, with another functional $F(u) := \int_{B^m} u^*(d\beta) = \int_{B^m} d(u^*\beta)$, where β is the $(m-1)$-form on $B^m\times\mathbb{R}^3$ defined by $\beta := \sum_{1\leq i<j\leq m}(-1)^{i+j+1}(y^idy^j - y^jdy^i)\wedge dx^1\wedge\cdots\wedge\widehat{dx^i}\wedge\cdots\wedge\widehat{dx^j}\wedge\cdots dx^m$. Write $u^*(d\beta) = \lambda(du)\,dx^1\wedge\cdots\wedge dx^m$. First, *from the fact that u takes values in S^{m-1} a.e.*, we show that $\lambda(du)\leq(m-2)|du|^2$ a.e., with equality if $u=u_\odot$. Second, we obtain from Stokes' theorem,

$$2(m-2)E_{B^m}(u) \geq \int_{B^m} d(u^*\beta) = \int_{\partial B^m} u^*\beta = \int_{\partial B^m} u^*_\odot\beta = \int_{B^m} d(u^*_\odot\beta) = 2(m-2)E_{B^m}(u_\odot).$$

The functional $\int_{B^m} u^*(d\beta)$ is an example of a *null Lagrangian*. Lin's method is similar to the use of *calibrations* for minimal surfaces and to the argument used in equation (6) for harmonic functions. The proof of Coron and Gulliver uses two ingredients: (i) a representation of the energy of a map u by an integral over the Grassmannian

[42] This argument does not work if $\mathcal{M}\simeq\mathcal{N}\simeq S^2$ but in this case any harmonic map is conformal and hence minimizing.

[43] The results by Coron and Hélein [52] use methods inspired from the work of Coron and R. Gulliver [51], whereas the use of null Lagrangians for harmonic maps was introduced by F. H. Lin [146], see below.

manifold $G_3(\mathbb{R}^{m-s})$ of 3-planes Y in $\mathbb{R}^{m-s}$ of the energies of $\pi_Y \circ u \in W^{1,2}(B^m, S^2_Y)$, where $\pi_Y : S^{m-s-1} \longrightarrow S^{m-s-1} \cap Y := S^2_Y$ is the natural 'radial' projection and (ii) the coarea formula [44]. They also studied the maps $h^{\mathbb{C}}_{\odot} \in W^{1,2}(B^4, S^2)$ and $h^{\mathbb{H}}_{\odot} \in W^{1,2}(B^8, S^4)$ defined by $h^{\mathbb{C}}_{\odot}(x) = H^{\mathbb{C}}(x/|x|)$ and $h^{\mathbb{H}}_{\odot}(x) = H^{\mathbb{H}}(x/|x|)$, where $H^{\mathbb{C}} : S^3 \to S^2$ and $H^{\mathbb{H}} : S^7 \to S^4$ are the *complex* and *quaternionic Hopf fibrations* (see §2.3), respectively, and they proved by similar methods that $h^{\mathbb{C}}_{\odot}$ and $h^{\mathbb{H}}_{\odot}$ are minimizing.

6.3 Analytic properties according to the geometric structure of $\mathcal{N}$

The target manifold $(\mathcal{N}, h)$ has non-positive Riemannian curvature

In this case, the harmonic map problem has many good convexity properties.

Existence. The first existence result was obtained by J. Eells and J. Sampson [66], and S.I. Al'ber [1] independently through the study of the heat equation $\partial\phi/\partial t = \tau(\phi)$ for a map $\phi : [0,\infty) \times \mathcal{M} \longrightarrow \mathcal{N}$, where $\partial\mathcal{M} = \emptyset$, with the Cauchy condition $\phi(0,\cdot) = \phi_0$ where $\phi_0 : \mathcal{M} \longrightarrow \mathcal{N}$ is a smooth map: if $\mathcal{M}$ and $\mathcal{N}$ are compact there always exists a finite time solution (i.e. defined on $[0,T] \times \mathcal{M}$), but *if $(\mathcal{N}, h)$ has non-positive curvature, this solution can be extended for all time*. Moreover the solution $\phi(t,\cdot)$ *converges*[45] *to a smooth harmonic map* $\underline{\phi}$ when $t \to +\infty$, which is *homotopic to* ϕ_0. When the boundary $\partial\mathcal{M}$ is non-empty and a Dirichlet condition $\phi(t,\cdot) = g$ on $\partial\mathcal{M}$ is imposed, these results were extended by R. Hamilton [99]. The existence conclusion can be recovered by using the Leray–Schauder degree theory [119], the maximum principle [130] or the direct method (see [216, 189] and §5.2).

Regularity. *Weakly harmonic maps into a non-positively curved manifold are smooth* and, moreover, the existence of convex functions on $\mathcal{N}$ allows *higher regularity estimates*: these are consequences of more general results, see §4.1 and below.

Minimality. The harmonic map $\underline{\phi}$ constructed in [66, 1] or [99] is actually energy minimizing [2, 111]. This follows by using the first and the second variation formulae for $E_{\mathcal{M}}$ given in [66]; this implies, in particular, the following identity [2]: let $\phi, \phi_0 : \mathcal{M} \longrightarrow \mathcal{N}$ be two smooth maps, and let $\Phi : [0,1] \times \mathcal{M} \longrightarrow \mathcal{N}$ be a *geodesic homotopy between ϕ_0 and ϕ*, i.e. a smooth homotopy such that $\Phi(0,\cdot) = \phi_0$ and $\Phi(1,\cdot) = \phi$ and, for each fixed $x \in \mathcal{M}$, $s \longmapsto \Phi(s,x)$ is a geodesic; then, if ϕ_0 is harmonic we have

$$E_{\mathcal{M}}(\phi) - E_{\mathcal{M}}(\phi_0) = \int_0^1 d\sigma \int_0^\sigma ds \int_{\mathcal{M}} \Big\{ |\nabla_{\frac{\partial}{\partial s}} d\phi|^2 - g^{ij}\, {}^hR_{\alpha\beta\gamma\delta}(\phi) \frac{\partial\phi^\alpha}{\partial s} \phi^\beta_i \frac{\partial\phi^\gamma}{\partial s} \phi^\delta_j \Big\} \omega_g. \tag{59}$$

Hence if ${}^hR$ is non-positive the right hand side is nonnegative and this implies that any harmonic map is the minimizer in its homotopy class.

Uniqueness. Actually, each homotopy class contains, in most cases,[46] **only one** harmonic map: this was shown by P. Hartman [111] and S.I. Al'ber [2] independently and can be deduced from (59), see also [189]. An alternative method is possible if $\partial\mathcal{M} \neq \emptyset$: if $\mathcal{N}$ is *simply connected*[47] we can use the squared distance function $d^2 : \mathcal{N} \times \mathcal{N} \longrightarrow [0,\infty)$, which is a *strictly convex function* [135], see below.

Other properties. The *Bochner identity for harmonic maps* proved in [66],

$$\frac{1}{2}\Delta_g |d\phi|^2 = |\nabla d\phi|^2 - g^{ij} g^{kl}\, {}^hR_{\alpha\beta\gamma\delta}(\phi) \phi^\alpha_i \phi^\beta_k \phi^\gamma_j \phi^\delta_l + g^{ij}\, {}^gRic(\phi_i, \phi_j)\,, \tag{60}$$

is particularly useful if ${}^hR$ is non-positive and $\mathcal{M}$ is compact, since it then implies [66]:

$$-\Delta_g |d\phi|^2 \leq C |d\phi|^2, \quad \text{so, in particular } |d\phi|^2 \text{ is subharmonic.} \tag{61}$$

[44] See footnote 34. A similar method was used by Hélein in his thesis for proving: let $\phi : D \to \mathcal{N}$ be a submersive harmonic morphism with connected fibres from a compact domain of $\mathbb{R}^3$ with smooth boundary to a Riemann surface, then ϕ is the unique energy minimizer amongst maps with the same boundary values. See also [52].

[45] Eells and Sampson [66] established that $\phi(t,\cdot)$ *sub*converges to $\underline{\phi}$, but Hartman [111] proved that it actually converges.

[46] If $\partial\mathcal{M} = \emptyset$ non uniqueness can occur in two cases: we may have two different *constant* harmonic maps or two different harmonic maps which parametrize the same geodesic.

[47] Then any pair of points $p, q \in \mathcal{N}$ can be joined by a unique geodesic [135], and so $(\mathcal{N}, h)$ is convex.

This inequality can be used to prove: (i) Liouville-type theorems [66]; (ii) the compactness in the $\mathcal{C}^k$-topology of the set $\mathcal{H}_\Lambda$ of maps $u \in \mathcal{C}^\infty(\mathcal{M},\mathcal{N})$ such that u is harmonic and $E_\mathcal{M}(u) < \Lambda$ (see [189]).

The target manifold has weaker convexity properties

The case when there exists a convex function on $\mathcal{N}$. Such functions are abundant on simply connected non-positively curved manifolds, but they also exist on any sufficiently small geodesic ball in $\mathcal{N}$. The basic observation is that *the composition of any harmonic map with a convex function is subharmonic and hence obeys the maximum principle* [124]. For instance, if the squared distance function $d^2 : \mathcal{N} \times \mathcal{N} \longrightarrow [0,\infty)$ exists and is *convex*, we can compose it with a pair $(u_0,u_1) : \mathcal{M} \longrightarrow \mathcal{N}\times\mathcal{N}$ of harmonic maps which agree on $\partial\mathcal{M} \neq \emptyset$ to prove the **uniqueness** result that $u_1 = u_2$, see, for example, [135]. Even more [86], if $g : \mathcal{N} \longrightarrow [0,\infty)$ is *bounded* and *strictly convex*, then for any $\mathcal{C}^2$ harmonic map $\phi : \mathcal{M} \longrightarrow \mathcal{N}$, we have

$$c_1|d\phi|^2 \leq \Delta(g\circ\phi), \quad \text{where } c_1 > 0. \tag{62}$$

Use of inequality (62) together with the monotonicity inequality (see §4.3) leads to the **local estimate** $\sup_{B(a,r/2)} |d\phi|^2 \leq Cr^{-n}\int_{B(a,r)} |d\phi|^2$, see [86, 189]. This can be used as the starting point for higher order estimates, see [86, 135].

The case when the image of ϕ is contained in a geodesically convex ball. The *optimal regularity result for weakly harmonic maps* with this kind of hypothesis is due to S. Hildebrandt, W. Jäger and K.-O. Widman [120]. We will say that a domain $\mathcal{B} \subset \mathcal{N}$ is an *HJW-convex ball* (after Hildebrandt, Jäger and Widman) if $\mathcal{B}$ is a geodesic ball $B(p_0,R) \subset \mathcal{N}$ (where $p_0 \in \mathcal{N}$) such that

(i) $\forall p \in B(p_0,R)$, *the cut-locus of* p *does not intersect* $B(p_0,R)$*;*

(ii) $R \leq 2\pi/\sqrt{\kappa}$*, where* $\kappa > 0$ *is an upper bound of the Riemannian curvature on* $B(p_0,R)$*.*

Then Hildebrandt, Jäger and Widman proved *the existence of a solution to the Dirichlet problem with values in a HJW-convex ball* (see §5.2, f) and §6.2) and that **any weakly harmonic map ϕ with values in a HJW-convex ball is Hölder continuous** [120]. This result is optimal because of the following example: consider the map $u_\ominus \in W^{1,2}(B^m,\overline{S^m_+})$, where $\overline{S^m_+} := \{y \in S^m |\ y^{m+1} \geq 0\}$, defined by $u_\ominus(x) = (x/|x|,0)$, then, if $m \geq 3$ this maps has finite energy and is *weakly harmonic*. However $u_\ominus$ is clearly singular, but the hypothesis (i) of the above theorem is not satisfied.

With exactly the same hypothesis on the target, W. Jäger and H. Kaul in [126] found the following **uniqueness result**: assume that $\mathcal{M}$ is connected and $\partial\mathcal{M} \neq \emptyset$ and let $\phi_1,\phi_2 : \mathcal{M} \longrightarrow \mathcal{B}$ be two smooth harmonic maps which agree on $\partial\mathcal{M}$; then, if $\mathcal{B}$ is a HKW-convex ball, $\phi_1 = \phi_2$. Again this result is optimal since, on the one hand, for $3 \leq m$, $W^{1,2}(B^m,\overline{S^m_+})$ contains the weakly harmonic map $u_\ominus$; on the other hand, for $3 \leq m \leq 6$, the minimum in $W^{1,2}_{u_\ominus}(B^m,\overline{S^m_+})$ is achieved by a smooth diffeomorphism onto $\overline{S^m_+}$, hence providing us with another harmonic map [127] (see [129] for improvements).

Influence of the topology of $\mathcal{N}$. Beyond more or less local assumptions on the curvature or the convexity of the target manifolds, many existence and regularity results are improved if one assumes that *there is no non-constant harmonic map from* S^2 *to* $\mathcal{N}$. This is related to the *bubbling phenomenon* which was discussed at length in §5.3 and 5.4.

7 Twistor theory and completely integrable systems

This is a rapid review of the development of the application of twistor theory and integrable systems to the study of harmonic maps. For further details, see, for example, [63, 94, 117, 73].

7.1 Twistor theory for harmonic maps

The genesis of the twistor theory for harmonic maps can be considered to be the following well-known result:[48] *Let $\phi : \mathcal{M}^2 \to \mathbb{R}^3$ be a conformal immersion from a Riemann surface $(\mathcal{M}^2, J^{\mathcal{M}})$. Then its Gauss map $\gamma : \mathcal{M}^2 \to S^2$ is antiholomorphic if and only if ϕ is harmonic (equivalently, minimal).*

The result was generalized to $\mathbb{R}^n$ by S.-S. Chern [44]. Indeed, let $\phi : \mathcal{M}^2 \to \mathbb{R}^n$ be a weakly conformal map. On identifying the Grassmannian $G_2^{\rm or}(\mathbb{R}^n)$ of oriented 2-planes in $\mathbb{R}^n$ with the complex quadric $Q_{n-2} = \{[z_1 : \cdots : z_n] \in \mathbb{C}P^n : z_1^{\,2} + \ldots + z_n^{\,2} = 0\}$, its Gauss map $\gamma : \mathcal{M}^2 \to G_2^{\rm or}(\mathbb{R}^n) = Q_{n-2}$ is given by the projective class of $\partial\phi/\partial\bar{z}$, where z is any local complex coordinate on $\mathcal{M}^2$, If ϕ is harmonic, γ is antiholomorphic by the harmonic equation, see (20). Note further that this antiholomorphicity implies that the Gauss map of a *weakly* conformal map extends smoothly across the set of branch points. Conversely, if γ is antiholomorphic, $\partial^2\phi/\partial z\partial\bar{z}$ is a multiple of the vector $\partial\phi/\partial\bar{z}$, which is tangential; but it is also a multiple of the mean curvature vector which is normal, thus it must vanish, hence ϕ is harmonic.

Now let $\mathcal{N} = \mathcal{N}^n$ be a general Riemannian manifold of dimension $n \geq 2$. Let $\pi : G_2^{\rm or}(\mathcal{N}) \to \mathcal{N}$ be the Grassmann bundle whose fibre at a point q of $\mathcal{N}^n$ is the Grassmannian of all oriented 2-dimensional subspaces of $T_q\mathcal{N}$. This is an associated bundle of the frame bundle $\mathrm{O}(\mathcal{N})$ of $\mathcal{N}$. Using the Levi-Civita connection, we may decompose the tangent bundle of $G_2^{\rm or}(\mathcal{N})$ into vertical and horizontal subbundles: $TG_2^{\rm or}(\mathcal{N}) = \mathcal{H} \oplus \mathcal{V}$; we denote the projections onto those subbundles by the same letters. Given any conformal immersion $\phi : \mathcal{M}^2 \to \mathcal{N}^n$, we define its *Gauss lift* $\gamma : \mathcal{M}^2 \to G_2^{\rm or}(\mathcal{N})$ by $\gamma(p) =$ the image of $d\phi_p$. Let $J^{\mathcal{V}}$ be the complex structure on the Grassmannian fibres of π. Say that γ is *vertically antiholomorphic* if

$$\mathcal{V} \circ d\gamma \circ J^{\mathcal{M}} = -J^{\mathcal{V}} \circ \mathcal{V} \circ d\gamma\,. \tag{63}$$

Then Chern's result extends to: *γ is vertically antiholomorphic if and only if ϕ is harmonic*. Further, the Gauss lift of a weakly conformal harmonic map extends smoothly over the branch points.

Maps into 4-dimensional manifolds. Suppose that $\mathcal{N} = \mathcal{N}^4$ is an oriented 4-dimensional Riemannian manifold. Then each $w \in G_2^{\rm or}(\mathcal{N}^4)$ defines an almost Hermitian structure J_w on $T_{\pi(w)}\mathcal{N}^4$. Further, if $\phi : \mathcal{M}^2 \to \mathcal{N}^4$ is a conformal immersion, then for any $p \in \mathcal{M}^2$, $d\phi_p$ intertwines $J_p^{\mathcal{M}}$ and $J_{\gamma(p)}$. Equivalently, lift J_w to an almost complex structure $J_w^{\mathcal{H}}$ on $\mathcal{H}_w$; then γ is *horizontally holomorphic* in the sense that

$$\mathcal{H} \circ d\gamma \circ J^{\mathcal{M}} = J^{\mathcal{H}} \circ \mathcal{H} \circ d\gamma\,. \tag{64}$$

We now define two almost complex structures J^1 and J^2 on the manifold $G_2^{\rm or}(\mathcal{N}^4)$ by setting J_w^1 (resp. J_w^2) equal to $J_w^{\mathcal{H}}$ on $\mathcal{H}_w$ and $J_w^{\mathcal{V}}$ (resp. $-J_w^{\mathcal{V}}$) on $\mathcal{V}_w$. Then the results above translate into: *the Gauss lift of a smooth immersion is holomorphic with respect to J^2 if and only if the map is conformal and harmonic.*

In fact, the projection of a J^2-holomorphic map into $G_2^{\rm or}(\mathcal{N})$ is always harmonic. More generally, let (Z, J^Z) be an almost complex manifold. A submersion $\pi : Z \to \mathcal{N}^4$ is called a *twistor fibration (for harmonic maps, with twistor space Z)* if the projection $\pi \circ f$ of any holomorphic map f from a Riemann surface to (Z, J^Z) is harmonic. The Grassmann bundle provides such a twistor fibration; we now find other twistor fibrations.

The Grassmann bundle $G_2^{\rm or}(\mathcal{N}^4)$ can be written as the product of two other bundles as follows. For any even-dimensional Riemannian manifold $\mathcal{N}^{2n}$, let $J(\mathcal{N}) \to \mathcal{N}$ be *the bundle of almost Hermitian structures on $\mathcal{N}$*. This is an associated bundle of $\mathrm{O}(\mathcal{N})$; indeed $J(\mathcal{N}) = \mathrm{O}(\mathcal{N}) \times_{\mathrm{O}(2n)} J(\mathbb{R}^{2n})$ where $J(\mathbb{R}^{2n}) = \mathrm{O}(2n)/\mathrm{U}(n)$ is the space of orthogonal complex structures on $\mathbb{R}^{2n}$. When $\mathcal{N}$ is oriented, $J(\mathcal{N})$ is the disjoint union of $J^+(\mathcal{N})$ and $J^-(\mathcal{N})$, *the bundles of positive* and *negative almost Hermitian structures on $\mathcal{N}^4$*, respectively. Give these bundles almost complex structures J^1 and J^2 in the same way as for $G_2^{\rm or}(\mathcal{N}^4)$. Then, when $\mathcal{N}$ is 4-dimensional, we have a bundle isomorphism $G_2^{\rm or}(\mathcal{N}^4) \to J^+(\mathcal{N}^4) \times J^-(\mathcal{N}^4)$ given by $w \mapsto (J_w^+, J_w^-)$ where J_w^+ (resp. J_w^-) is the unique almost Hermitian structure which is rotation by $+\pi/2$ on w. This isomorphism preserves J^1, J^2 and the horizontal

[48] This result is related to the Weierstrass–Enneper representation formula for a conformal parametrization $X : \Omega \subset \mathbb{C} \longrightarrow \mathbb{R}^3$ of a minimal surface in $\mathbb{R}^3$, which reads $X(z) = X(z_0) + \mathrm{Re}(\int_{z_0}^z (i(w^2-1),\, w^2+1,\, 2iw)(h/2)\, d\zeta)$, where w and h are respectively a meromorphic and a holomorphic function. Indeed, here w represents the Gauss map through an orientation reversing stereographic projection.

spaces. The Gauss lift of an immersion $\phi : \mathcal{M}^2 \to \mathcal{N}^4$ thus decomposes into two *twistor lifts* $\psi_\pm : \mathcal{M}^2 \to J^\pm\mathcal{N}^4$. Both natural projections $J^\pm\mathcal{N}^4 \to \mathcal{N}$ are twistor fibrations; in fact we have the following result of J. Eells and S. Salamon [65]: **There is a bijective correspondence between non-constant weakly conformal harmonic maps $\phi : \mathcal{M}^2 \to \mathcal{N}^4$ and non-vertical J^2-holomorphic maps $\psi_\pm : \mathcal{M}^2 \to J^\pm\mathcal{N}^4$ given by setting $\psi_\pm$ equal to the twistor lift of ϕ.** For some related results in higher dimensions, see [180].

The problem with using this to find harmonic maps is that J^2 *is never integrable*. However, J^1 *is integrable if and only if the Riemannian manifold $\mathcal{N}^4$ is anti-selfdual*. Now a J^2-holomorphic map $\mathcal{M}^2 \to (Z, J^2)$ is also J^1-holomorphic if and only if it is *horizontal*, i.e., its differential has image in the horizontal subbundle $\mathcal{H}$, and horizontal holomorphic maps project to harmonic maps which are *real isotropic* in a sense that we now explain.

Real isotropic harmonic maps. A map $\phi : \mathcal{M}^2 \to \mathcal{N}^n$ from a Riemann surface to an arbitrary Riemannian manifold is called *real isotropic* if, for any complex coordinate z, all the derivatives $\nabla_Z^\alpha(\partial\phi/\partial z)$ lie in some isotropic subspace of $T^{\mathbb{C}}_{\phi(z)}\mathcal{N}$, i.e.

$$\eta_{\alpha,\beta} := \left\langle \nabla_Z^\alpha(\partial\phi/\partial z), \nabla_Z^\beta(\partial\phi/\partial z)\right\rangle = 0 \quad \text{for all } \alpha, \beta \in \{0, 1, 2, \ldots\}. \tag{65}$$

Here, $Z = \partial/\partial z$ and $\langle\ ,\ \rangle$ denotes the inner product on $T\mathcal{N}$ extended to $T^{\mathbb{C}}\mathcal{N}$ by complex bilinearity. For example, a holomorphic map to a Kähler manifold is real isotropic with the isotropic subspace being the $(1,0)$-tangent space. Now, in an extension to the argument showing that all harmonic maps from S^2 are weakly conformal (see §2.2), we show inductively on $k = \alpha + \beta$ that the inner products define holomorphic differentials $\eta_{\alpha,\beta}dz^k$ on S^2; since all holomorphic differentials on S^2 vanish for topological reasons, *all harmonic maps from S^2 to S^n are real isotropic*, and hence are obtained as projections of horizontal holomorphic maps into the twistor space. Such maps are easy to construct from 'totally isotropic' holomorphic maps into $\mathbb{C}P^n$ giving E. Calabi's theorem [36], as follows. Say that a map to a sphere or complex projective space is *full* if its image does not lie in a totally geodesic subsphere or projective subspace. Then **there is a $2:1$ correspondence between full harmonic maps $\pm\phi : S^2 \to S^{2n}$ and full totally isotropic holomorphic maps from S^2 to $\mathbb{C}P^n$.**

For an arbitrary oriented Riemannian manifold $\mathcal{N}$ of even dimension $2n$ greater than four, J^1 is integrable on $J^\pm(\mathcal{N})$ if and only if $\mathcal{N}$ is conformally flat. In order to apply twistor theory to more general manifolds, we need to find *reduced twistor spaces* on which J^1 is integrable. To do this, let $K \in \mathrm{O}(2n)$ be the holonomy group of $\mathcal{N}$ and $\mathcal{P} \to \mathcal{N}$ the corresponding *holonomy bundle* given by reducing the structure group of $\mathrm{O}(\mathcal{N})$ to K. Then $J(\mathcal{N}) = \mathcal{P} \times_K J(\mathbb{R}^{2n})$. The holonomy group K acts on $J(\mathbb{R}^{2n})$ by conjugation, decomposing it into orbits O_i ; it thus acts on $J(\mathcal{N})$, decomposing it into the union of subbundles associated to $\mathcal{P}$ and having fibre one of the orbits O_i . These subbundles are the candidates for our reduced twistor spaces.

For example, if $\mathcal{N}$ is a generic Kähler n-manifold, $K = U(n)$ and we find that the complex $U(n)$-orbits of $J(\mathcal{N})$ can be identified with the Grassmann bundles $G_r(T^{1,0}\mathcal{N}) \to \mathcal{N}$ $(r = 0, \ldots, n)$. These are thus twistor fibrations for harmonic maps. Note that, *for $0 < r < n$, J^1 is integrable on $G_r(T^{1,0}\mathcal{N})$ if and only if the Bochner tensor of $\mathcal{N}$ vanishes*.

Complex isotropic harmonic maps. Horizontal holomorphic maps into the Grassmann bundle project to harmonic maps which are *complex isotropic* in the sense that all the covariant derivatives $\nabla_Z^\alpha(\partial^{1,0}\phi/\partial z)$ are orthogonal in $T'_{\phi(z)}\mathcal{N}$ to all the covariant derivatives $\nabla_{\overline{Z}}^\alpha(\partial^{1,0}\phi/\partial\overline{z})$ with respect to the Hermitian inner product on $T'\mathcal{N}$. In particular, when $\mathcal{N} = \mathbb{C}P^n$, an argument again involving the holomorphicity of differentials constructed from the above inner products shows that all harmonic maps from $S^2 \to \mathbb{C}P^n$ are complex isotropic, and so given by such projections. In this case we can explicitly identify the Grassmann bundles and construct all holomorphic horizontal maps into it from holomorphic maps $S^2 \to \mathbb{C}P^n$ by considering their iterated derivatives. This leads to the result [68]: **There is a one-to-one correspondence between pairs (f, r) where f is a full holomorphic map from S^2 to $\mathbb{C}P^n$ and $r \in \{0, 1, \ldots, n\}$ and full harmonic maps from S^2 to $\mathbb{C}P^n$.**

Maps into symmetric spaces. Now let G be a compact Lie group and $\mathcal{N}^{2n} = G/K$ an irreducible Riemannian symmetric space. Then the natural projection $G \to G/K = \mathcal{N}$ is a reduction of the frame bundle with structure group K. As above, K acts on $J(\mathbb{R}^{2n})$ and thence on $J(\mathcal{N}) = G \times_K J(\mathbb{R}^{2n})$. Any orbit in $J(\mathbb{R}^{2n})$ is of the form K/H for some closed subgroup H; the corresponding orbit in $J(\mathcal{N})$ is the subbundle $\pi : G \times_K K/H \cong$

$G/H \to G/K$ where π is the natural projection. This subbundle can alternatively be thought of an orbit of the action of G on $J(\mathcal{N})$. F. E. Burstall and J. H. Rawnsley [35] showed that such an orbit is almost complex manifold on which J^1 is integrable if and only if is contained in the zero set of the Nijenhuis tensor of J^1, They go on to prove that, if $\mathcal{N} = G/K$ is an *inner* symmetric space[49] of compact type, that zero set consists of finitely many orbits of G with each orbit G/H a *flag manifold* of G and that every flag manifold of G occurs for some inner symmetric space G/K. Further, any flag manifold G/H can be written alternatively as $G^{\mathbb{C}}/P$ for some suitable parabolic subgroup of the complexified group $G^{\mathbb{C}}$, and so has a natural complex structure J^1. On replacing J^1 by $-J^1$ on the fibres, we obtain a non-integrable almost complex structure J^2 and then *the natural projection* $(G/H, J^2) \to G/K = \mathcal{N}$ *is a twistor fibration for harmonic maps*. Further every harmonic map from S^2 to $\mathcal{N}$ is the projection of some J^2-holomorphic map into a suitable flag manifold. Moreover Burstall and Rawnsley exhibit holomorphic differentials;[50]if these vanish then the J^2-holomorphic curve is in fact holomorphic for the complex structure J^1. For the special case of *isotropic* harmonic maps, see below.

7.2 Loop group formulations

Again let G be a compact Lie group, and let ω be its (left) Maurer–Cartan form; this is a 1-form with values in the Lie algebra $\mathfrak{g}$ of G which satisfies the Maurer–Cartan equation $d\omega + \frac{1}{2}[\omega \wedge \omega] = 0$ where $[\omega \wedge \omega](X, Y) = 2[\omega(X), \omega(Y)]$ $(X, Y \in T_\gamma G,\ \gamma \in G)$. Note that ω gives an explicit trivialization $TG \cong G \times \mathfrak{g}$ of the tangent bundle; the Maurer–Cartan equation expresses the condition that the connection $d + \omega$ on this bundle is flat.

Maps into Lie groups. Now let $\phi : \mathcal{M}^m \to G$ be a smooth map from a Riemannian manifold to G. Let A be the $\mathfrak{g}$-valued 1-form given by the pull-back $\phi^*\omega$. Then A represents the differential $d\phi$; indeed, if G is a matrix group, $A = \phi^{-1}d\phi$. Pulling back the Maurer–Cartan equation shows that A satisfies

$$dA + \frac{1}{2}[A \wedge A] = 0. \tag{66}$$

This equation is an *integrability condition*: given a $\mathfrak{g}$-valued 1-form, we can find a smooth map $\phi : \mathcal{M} \to G$ with $A = \phi^{-1}d\phi$ if and only if (66) is satisfied. Further, it is easy to see that ϕ is harmonic if and only if

$$d^*A = 0\,. \tag{67}$$

Now let $\mathcal{M}^2$ be a simply connected Riemann surface and let (U, z) be a complex chart. Writing $A = A_z dz + A_{\bar z} d\bar z$ we may add and subtract the equations (66,67) to obtain the equivalent pair of equations:

$$\frac{\partial A_z}{\partial \bar z} + \frac{1}{2}[A_{\bar z}, A_z] = 0\,, \qquad \frac{\partial A_z}{\partial z} + \frac{1}{2}[A_z, A_{\bar z}] = 0\,. \tag{68}$$

We now introduce a parameter $\lambda \in S^1 := \{\lambda \in \mathbb{C}^* |\ |\lambda| = 1\}$ (called the *spectral parameter*), and consider the loop of 1-forms:

$$A_\lambda = \frac{1}{2}(1 - \lambda^{-1})A_z dz + \frac{1}{2}(1 - \lambda)A_{\bar z} d\bar z\,. \tag{69}$$

K. Uhlenbeck noticed[51] [218] that A satisfies the pair (66,67) if and only if

$$dA_\lambda + \frac{1}{2}[A_\lambda \wedge A_\lambda] = 0 \quad \text{for all} \quad \lambda \in S^1; \tag{70}$$

this equation is a *zero curvature equation*: it says that for each λ, $d + A_\lambda$ is a flat connection on $\mathcal{M} \times \mathfrak{g}$. If satisfied, there is a loop of maps E_λ on $\mathcal{M}$ satisfying $E_\lambda^*(\omega) = A_\lambda$, since $\mathcal{M}$ is simply connected; equivalently, a map

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[49] An inner symmetric space is a Riemannian symmetric space whose involution is inner.

[50] These differentials vanish for harmonic maps from S^2 to S^{2n}, $\mathbb{C}P^n$ and $S^4 \simeq \mathbb{H}P^1$, so that one recovers the previous classification results for such maps [36, 28, 68].

[51] Uhlenbeck's discovery was known previously to several physicists, see for example [172].

$\mathcal{E} : \mathcal{M} \to \Omega G$ into the *(based) loop group* of G: $\Omega G = \{\gamma : S^1 \to G \mid \gamma(1) = \text{identity of } G\}$ (where the loops γ satisfy some regularity assumption such as C^∞).

The map $\mathcal{E} : \mathcal{M} \to \Omega G$ is called the *extended solution* corresponding to ϕ. Now suppose that G is a matrix group, i.e., $G \subset GL(\mathbb{R}^N) \subset \mathbb{R}^{N\times N}$. It can be written as Fourier series

$$\mathcal{E}(z) : \lambda \longmapsto E_\lambda(z) = \sum_{i=-\infty}^{\infty} \lambda^i \widehat{E}_i(z) \qquad (z \in \mathcal{M})$$

for some maps $\widehat{E}_i : \mathcal{M} \to G$. If this is a finite series, we say that ϕ has *finite uniton number*. Uhlenbeck showed that *all harmonic maps from S^2 to the unitary group (and so to all compact groups) have finite uniton number*. She also gave a *Bäcklund-type transform* which gives new harmonic maps from old ones by multiplying their extended solution by a suitable linear factor called *uniton*, and showed that the extended solution of a harmonic map $\phi : S^2 \to \mathrm{U}(n)$ can be *factorized* as the product of unitons, so that ϕ can be obtained from a constant map by *adding a uniton* no more than n times. Another proof was given by G. Segal [196] using a Grassmannian model of $\mathrm{U}(n)$. An extension of the factorization theorem to maps into most other compact groups G was proved by Burstall and Rawnsley [35].

We can also consider the 'free' loop group $\Lambda G = \{\gamma : S^1 \to G\}$ and we may define loop groups $\Omega G^{\mathbb{C}}$ and $\Lambda G^{\mathbb{C}}$ for the complexified group $G^{\mathbb{C}}$ in the same way. Let $\Lambda^+ G^{\mathbb{C}}$ be the subgroup of loops which extend holomorphically to the disk $D^2 := \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$, i.e., have Fourier coefficients $\widehat{\gamma}_i$ zero for negative i. Then, we have an *Iwasawa decomposition* $\Lambda G^{\mathbb{C}} = \Omega G \cdot \Lambda^+ G^{\mathbb{C}}$ so that we can write ΩG as $\Lambda G^{\mathbb{C}}/\Lambda^+ G^{\mathbb{C}}$; this gives ΩG a *complex structure*. Now (69) tells us that the partial derivative $\mathcal{E}_{\bar z}$ lies in $\Lambda^+ \mathfrak{g}^{\mathbb{C}}$ which means that $\mathcal{E}$ is holomorphic. Further $\mathcal{E}_z$ lies in the subspace of $\Lambda \mathfrak{g}^{\mathbb{C}}$ where all Fourier coefficients other than A_{-1} and A_0 are zero; we say that $\mathcal{E}$ is *superhorizontal*. Thus we can interpret the fibration $\pi : \Omega G \longrightarrow G$ given by $\mathcal{E} \longmapsto \mathcal{E}|_{\lambda=-1}$ as a twistor fibration, since *any* harmonic map from $\mathcal{M}$ to G is the image by π of a holomorphic horizontal curve in ΩG.

Maps into Riemannian symmetric spaces. We can apply the above to harmonic maps into symmetric spaces G/K by including G/K by the totally geodesic Cartan embedding $\iota : G/K \longrightarrow G$ defined by $\iota(g\cdot K) = \tau(g)g^{-1}$, where $\tau : G \longrightarrow G$ is the Cartan involution[52] such that $(G^\tau)_0 \subset K \subset G^\tau$; here $G^\tau := \{g \in G \mid \tau(g) = g\}$ and $(G^\tau)_0$ is the connected component of G^τ which contains the identity. However, there is an alternative more geometrical method which we now describe. For any map $\phi : \mathcal{M} \longrightarrow G/K$ choose a lift $f : \mathcal{M} \longrightarrow G$ of it and consider its Maurer–Cartan form $\alpha = f^*\omega \simeq f^{-1}df$. The Cartan involution τ induces a linear involution on the Lie algebra $\mathfrak{g} \simeq T_{\mathrm{Id}}G$ that we denote also by τ. The eigenvalues of τ are ± 1 and we have the eigenspace decomposition $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$, where, for $a = 0, 1$, $\mathfrak{g}_a$ is the $(-1)^a$-eigenspace. Note that $\mathfrak{g}_0 = \mathfrak{k}$ is the Lie algebra of K. Now we can split $\alpha = \alpha_0 + \alpha_1$ according to the eigenspace decomposition of $\mathfrak{g}$ and further split $\alpha_1 = \alpha_1' + \alpha_1''$, where $\alpha_1' := \alpha_1(\partial/\partial z)\, dz$ and $\alpha_1'' := \alpha_1(\partial/\partial \bar z)\, d\bar z$. Then $\phi : \mathcal{M} \longrightarrow G/K$ *is harmonic if and only if, for all* $\lambda \in S^1$ *we have* $d\alpha_\lambda + (1/2)[\alpha_\lambda \wedge \alpha_\lambda] = 0$, where

$$\alpha_\lambda := \lambda^{-1}\alpha_1' + \alpha_0 + \lambda\alpha_1'' \quad \text{for all } \lambda \in S^1. \tag{71}$$

This relation allows us to construct a family of maps $f_\lambda : \mathcal{M} \longrightarrow G$ by integrating the relation $\alpha_\lambda = f_\lambda^*\omega \simeq f_\lambda^{-1}df_\lambda$. Each map f_λ lifts a *harmonic* map $\phi_\lambda : \mathcal{M} \longrightarrow G/K$ given by $\phi_\lambda(z) = f_\lambda(z)K$, hence $(\phi_\lambda)_{\lambda\in S^1}$ is an *associated family* of harmonic maps. Alternatively we can view the family $F = (f_\lambda)_{\lambda\in S^1}$ as a single map from $\mathcal{M}$ to the *twisted loop group* $\Lambda G_\tau := \{\gamma : S^1 \longrightarrow G \mid \gamma(-\lambda) = \tau(\gamma(\lambda))\}$ and the family $\Phi = (\phi_\lambda)_{\lambda\in S^1}$ as a map into $(\Lambda G_\tau)/K$. Given a harmonic map ϕ, the map Φ is unique if we assume for instance the extra condition $f_\lambda(z_0) = \mathrm{Id}$, for some $z_0 \in \mathcal{M}$. The representation of a harmonic map into G/K using twisted loop groups is related to the one using based loop groups through the relations $E_\lambda = f_\lambda f^{-1}$ and $\iota(\phi_\lambda) = \tau(f_\lambda)f_\lambda^{-1} = E_{-\lambda}E_\lambda^{-1}$.

A 'Weierstrass' representation. We denote the complexification of ΛG_τ by $\Lambda G_\tau^{\mathbb{C}}$. We also define $\Lambda^+ G_\tau^{\mathbb{C}}$ as the subgroup of loops $\gamma \in \Lambda G_\tau^{\mathbb{C}}$ which have a holomorphic extension (that we still denote by γ) in the disk D^2 and, if $\mathfrak{B} \subset G^{\mathbb{C}}$ is a solvable Borel subgroup such that the Iwasawa decomposition $G^{\mathbb{C}} = G \cdot \mathfrak{B}$ holds, we let

[52] $\tau(g) = s_o g s_o^{-1}$ where s_o is the point reflection in the base point of $\mathcal{N}$.

$\Lambda^+_{\mathfrak{B}}G^{\mathbb{C}}_\tau$ be the subgroup of loops $\gamma \in \Lambda^+G^{\mathbb{C}}_\tau$ such that $\gamma(0) \in \mathfrak{B}$. Now J. Dorfmeister, F. Pedit and H. Y. Wu [60] proved that an Iwasawa decomposition $\Lambda G^{\mathbb{C}}_\tau = \Lambda G_\tau \cdot \Lambda^+_{\mathfrak{B}}G^{\mathbb{C}}_\tau$ holds, so that we can define a natural fibration $\pi_\tau : \Lambda G^{\mathbb{C}}_\tau \longrightarrow \Lambda G^{\mathbb{C}}_\tau / \Lambda^+_{\mathfrak{B}}G^{\mathbb{C}}_\tau = \Lambda G_\tau$. They show also that if $H : \mathcal{M} \longrightarrow \Lambda G^{\mathbb{C}}_\tau$ is a *holomorphic curve* which satisfies the *superhorizontality condition* $\lambda H^*\omega \simeq \lambda H^{-1}dH \in \Lambda^+\mathfrak{g}^{\mathbb{C}}$, then $F = \pi_\tau \circ H$ (i.e., the unique map F into $\Lambda G^{\mathbb{C}}_\tau$ such that $H = FB$, for some map $B : \mathcal{M} \longrightarrow \Lambda^+_{\mathfrak{B}}G^{\mathbb{C}}_\tau$) lifts an *associated family of harmonic maps*. Conversely Dorfmeister, Pedit and Wu proved that any harmonic map from a simply connected surface to $\mathcal{N}$ arises that way. The superhorizontal holomorphic maps H which covers a given F are *not* unique. However we can use another *Birkhoff decomposition* $\Lambda G^{\mathbb{C}}_\tau \supset \mathcal{C} = \Lambda^-_*G^{\mathbb{C}}_\tau \cdot \Lambda^+G^{\mathbb{C}}_\tau$, where $\Lambda^-_*G^{\mathbb{C}}_\tau$ is the subset of loops $\gamma \in \Lambda G^{\mathbb{C}}_\tau$ which have a holomorphic extension to $\mathbb{C}P^1 \setminus D^2 := \{\lambda \in \mathbb{C} \cup \{\infty\} | \, |\lambda| \geq 1\}$ and such that $\gamma(\infty) = \mathrm{Id}$. Here $\mathcal{C}$ is the *big cell*, a dense subset of the connected component of Id in $\Lambda G^{\mathbb{C}}_\tau$. Further Dorfmeister, Pedit and Wu showed that for any lift F of an associated family of harmonic maps into $\mathcal{N}$, there exist finitely many points $\{a_1, \cdots, a_k\}$ such that F takes values in $\mathcal{C}$ outside $\{a_1, \cdots, a_k\}$. We can hence decompose $F = F^-F^+$ on $\mathcal{M} \setminus \{a_1, \cdots, a_k\}$, where F^- (respectively F^+) takes values in $\Lambda^-_*G^{\mathbb{C}}_\tau$ (respectively $\Lambda^+G^{\mathbb{C}}_\tau$), and then F^- extends to a meromorphic superhorizontal curve on $\mathcal{M}$ with poles at $a_1, \cdots, a_k$. Then the Maurer–Cartan form of F^-, $\mu = (F^-)^*\omega$, reads $\mu_\lambda = \lambda^{-1}\xi dz$, where $\xi : \mathcal{M} \longrightarrow \mathfrak{g}^{\mathbb{C}}_1$ is a meromorhic potential called the *meromorphic potential* of F. This provides *Weierstrass data* for the harmonic map and is known as the 'DPW' method [60].

Pluriharmonic maps. This can be extended to the more general case of 'pluriharmonic' maps: a smooth map from a complex manifold is called *pluriharmonic* if its restriction to every complex one-dimensional submanifold is harmonic. Let $\phi : (\mathcal{M}, J^{\mathcal{M}}) \to \mathcal{N}$ be a smooth map from a simply connected complex manifold to a Riemannian symmetric space $\mathcal{N} = G/K$. For $\lambda = e^{-i\theta} \in S^1$, define an endomorphism of $T\mathcal{M}$ by $R_\lambda = (\cos\theta)I + (\sin\theta)J$. Extending this by complex-linearity to the complexified tangent bundle $T^{\mathbb{C}}\mathcal{M}$, we have that $R_\lambda = \lambda^{-1}I$ on the $(1,0)$-tangent bundle $T'\mathcal{M}$ and $R_\lambda = \lambda I$ on the $(0,1)$-tangent bundle $T''\mathcal{M}$. Note that, if $\mathcal{M}$ is a Riemann surface, R_λ is rotation through θ. J. Dorfmeister and J.-H. Eschenburg [59] show that ϕ is pluriharmonic if and only if there is a parallel bundle isometry $\mathfrak{R}_\lambda : \phi^*T\mathcal{N} \to \phi^*_\lambda T\mathcal{N}$ preserving the curvature such that $\mathfrak{R}_\lambda \circ d\phi \circ R_\lambda = d\phi_\lambda$ for some smooth family of maps $\phi_\lambda \;\; (\lambda \in S^1)$, and that the maps ϕ_λ are all pluriharmonic; thus *pluriharmonic maps again come in associated S^1-families*. Then with similar definitions of superhorizontal and holomorphic to those above, we obtain the result: *there is a one-to-one correspondence between pluriharmonic maps $\phi : \mathcal{M} \to G/K$ and superhorizontal holomorphic maps $\Phi : \mathcal{M} \to \Lambda_\sigma G/K$ with $\phi = \pi \circ \Phi$.*

The twistor theory revisited. Twistor theory appears as a special case: a map is called *isotropic* if the associated family ϕ_λ is trivial, i.e. $\phi_\lambda = \phi$ up to congruence for all $\lambda \in S^1$. Then, for each $z \in \mathcal{M}$, the $\mathfrak{R}_\lambda(z)$ are automorphisms of $T_{\phi(z)}\mathcal{N}$, representing these by elements of G, they define a homomorphism $\mathfrak{R}(z) : S^1 \to G$, $\lambda \mapsto \mathfrak{R}_\lambda(z)$. By parallelity of the $\mathfrak{R}_\lambda(z)$ as z varies, these homomorphisms are all conjugate, so that the $\mathfrak{R}_\lambda$ define a map into the congugacy class of a circle subgroup $q : S^1 \to G$, $\lambda \mapsto q_\lambda$ with $q_{-1} = s_o$; this congugacy class is a flag manifold of the form G/C_q where C_q is the centralizer of q, and the $\mathfrak{R}_\lambda$ define a twistor lift into that manifold. Note that C_q is contained in K. Also a necessary condition for the existence of a circle subgroup q with $q_{-1} = s_o$ is that $\mathcal{N}$ be *inner*, i.e., s_o lies in the identity component of K. We thus obtain [69]: *Let $\phi : \mathcal{M} \to \mathcal{N}$ be a smooth map into an inner symmetric space $\mathcal{N} = G/K$ of compact type which is* full, *i.e., does not have image in a totally geodesic proper subspace of $\mathcal{N}$. Then ϕ is isotropic if and only if there is a flag manifold $Z = G/H$ with $H \subset K$ and a holomorphic superhorizontal map $\Phi : \mathcal{M} \to Z$ such that $\pi \circ \Phi = \phi$ where $\pi : G/H \to G/K$ is the natural projection.* In this setting, pluriharmonic maps into Lie groups G appear naturally by treating G as the symmetric space $G \times G/G$.

F. Burstall and M. A. Guest [34] take all this much further by showing that to every extended solution can be associated a homomorphism $q : \lambda \mapsto q_\lambda$ by flowing down the gradient lines of the energy of loops in G. The extended solution can be recovered from q by multiplication by a suitable holomorphic map into a loop group. The conditions (69) translate into conditions on the coefficients of the Fourier series of this map related to the eigenspace decomposition of $\mathrm{Ad}\, q_\lambda$. This leads to equations in the meromorphic parameters which can be solved by successive integrations leading to the theorem: *Every harmonic map $S^2 \to G$ arises from an extended solution which may be obtained explicitly by choosing a finite number of rational functions and then performing a finite number of algebraic operations and integrations.* They show how the work of Dorfmeister, Pedit and Wu [60] fits into this scheme, as well as Uhlenbeck's factorization.

Finite type solutions. An alternative way of finding harmonic maps into symmetric spaces, especially when the domain is a (2-)torus, is to integrate a pair of commuting Hamiltonian fields on the finite-dimensional subspace $\Omega^d\mathfrak{g} := \{\xi \in \Omega\mathfrak{g}|\ \xi_\lambda = \sum_{k=-d}^{d} \widehat{\xi}_k(1-\lambda^k)\}$ of the based[53] loop algebra $\Omega\mathfrak{g}$, for some $d \in \mathbb{N}^*$. Indeed the vector fields X_1 and X_2 defined on $\Omega^d\mathfrak{g}$ by $X_1(\xi) - iX_2(\xi) = 2[\xi, 2i(1-\lambda)\widehat{\xi}_d]$ *are tangent to* $\Omega^d\mathfrak{g}$ *and commute*. Thus we can integrate the Lax type equation $d\xi = [\xi, 2i(1-\lambda)\widehat{\xi}_d dz - 2i(1-\lambda^{-1})\widehat{\xi}_{-d}d\overline{z}]$, where $\xi : \mathbb{R}^2 \longrightarrow \Omega^d\mathfrak{g}$ (for a formulation of the harmonic map equations as a *Lax pair*, see the article by Wood in [73] or [53, 94]). Then, for any solution of this equation, the loop of 1-forms $A_\lambda := 2i(1-\lambda)\widehat{\xi}_d dz - 2i(1-\lambda^{-1})\widehat{\xi}_{-d}d\overline{z}$ satisfies the relation (70) and hence provides an extended harmonic map by integrating the relation $E_\lambda^*\omega = A_\lambda$; the resulting harmonic maps are said to be of *finite type*.

A nontrivial result is that, for all $n \in \mathbb{N}^*$, **all non-isotropic harmonic maps from the torus to S^n or $\mathbb{C}P^n$ are of finite type.** This was proved by N. Hitchin [122] for tori in S^3, by U. Pinkall and I. Sterling [169] for constant mean curvature tori in $\mathbb{R}^3$ and by Burstall, D. Ferus, Pedit and Pinkall [33] for non-conformal tori in rank one symmetric spaces ([122] and [33] propose a different approach, see [123] for a comparison). The case of conformal but non-isotropic tori in S^n or $\mathbb{C}P^n$ requires the notion of *primitive* maps introduced by Burstall [32], i.e. maps with values in a k-symmetric space fibred over the target. See [160, 161] for further developments. To each finite type harmonic map of a torus can be associated a compact Riemann surface called its *spectral curve*, together with some data on it called *spectral data*. This leads to a representation using techniques from algebraic geometry, done by A. Bobenko [19] for constant mean curvature tori and by I. McIntosh [154] for harmonic tori in complex projective spaces.

Harmonic maps from a higher genus surface $\mathcal{M}$. They can, in principle, be found by the DPW method by investigating harmonic maps on the universal cover of $\mathcal{M}$, but this is hard to implement. Another possible approach investigated by Y. Ohnita and S. Udagawa [160] is to look for (finite type) pluriharmonic maps on the Jacobian variety $J(\mathcal{M})$ of $\mathcal{M}$ and compose them with the Abel map $\mathcal{M} \longrightarrow J(\mathcal{M})$.

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