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Harmonic diffeomorphisms with rotational symmetry

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Let $\mathcal{N}$ be a Riemannian manifold with boundary of class C^1 which is diffeomorphic to the unit open ball $B^n = \{x \in \mathbb{R}^n \,|\, |x| < 1\}$, where $n \geqq 3$. And let φ be a diffeomorphism of class C^1 between B^n and $\mathcal{N}$. B^n is equipped with the canonical Euclidean metric c. As in [CH] and [H] we can use the chart φ^{-1} in order to represent $\mathcal{N}$. So we need only to study the identity map (which we note Id) from (B^n, c) into (B^n, h) where h is the pull-back image of the metric on $\mathcal{N}$ by φ.

We will say that the map φ is rotationally symmetric or SO(n)-equivariant if the pull-back image of the metric on $\mathcal{N}$ by φ can be represented with two continuous maps $h_{//}$ and $h_\perp$ from $[0, 1]$ into $(0, +\infty)$ by the formula

$$h_{ij}(x) = h_{//}(r) x^i x^j r^{-2} + h_\perp(r)[\delta_{ij} - x^i x^j r^{-2}] \tag{1}$$

where $r = |x|$, δ_{ij} is the Kronecker symbol, and because of the continuity of h in 0,

$$h_{//}(0) = h_\perp(0). \tag{2}$$

We consider the space $H^1 * (B^n, B^n)$ of the maps in Sobolev space H^1 from B^n to B^n which agree with Id on the boundary of B^n. The energy functional is defined as follows: for any map u in $H^1 * (B^n, B^n)$ we denote by u^i_α the partial derivative $\partial u^i / \partial x^\alpha$, and we define the energy of u by

$$E(u) = \int_\Omega e(u)\,dx = (1/2) \int_\Omega \sum_{\alpha,i,j} h_{ij}[u(x)] u^i_\alpha u^j_\alpha \,dx. \tag{3}$$

A map $u: (B^n, c) \to (B^n, h)$ is called weakly harmonic if it is a critical point of the energy functional, moreover if this map is regular it will be called harmonic (see [EL]). In the case where φ or equivalentely Id $: (B^n, c) \to (B^n, h)$ is weakly harmonic then the following system of equations is verified in the distribution sense on B^n (see [CH])

$$\frac{\partial}{\partial x^\gamma}\left[\sum_i h_{ii}(x)\right] = \sum_i \frac{\partial}{\partial x^i}[h_{i\gamma}(x) + h_{\gamma i}(x)] \quad \text{for} \quad \gamma = 1, \ldots, n. \tag{4}$$

If φ is rotationally symmetric, φ is weakly harmonic if and only if we have in the distribution sense (see [CG], here the dash denotes the derivative with respect to r):

$$(n-1)[rh'_{\perp}+2h_{\perp}] = rh'_{//}+2(n-1)h_{//}. \tag{5}$$

Finally Id is minimizing if for any map u in $H^1*(B^n, B^n)$, $E(u) \geqq E(\mathrm{Id})$.

We want to know if harmonic diffeomorphisms as above are minimizing. In [CH] it was shown that any harmonic diffeomorphism between Riemannian manifolds of dimension 2 are minimizing and some examples of minimizing diffeomorphisms were given in higher dimensions. In [H] we gave sufficient conditions which ensures that some harmonic diffeomorphisms in dimension 3 are minimizing. Particularly this condition is true for a rotationally symmetric map on B^3. In the present paper we show the following

Theorem. *Let n be an integer not smaller than 3. Let $\mathrm{Id}:(B^n,\delta_{ij}) \to (B^n,h_{ij})$ be a harmonic diffeomorphism of class C^1. If this map is $\mathrm{SO}(n)$-equivariant then $\mathrm{Id}:(B^n,\delta_{ij}) \to (B^n,h_{ij})$ is minimizing.*

Proof. As in [L] and in [H] the proof is based on a comparison with a null Lagrangian. Let u be a map in $H^1*(B^n, B^n)$, and let us show that $E(u) \geqq E(\mathrm{Id})$. Thank a result of density in [CH], Appendix A, it suffices to work with a map u of class C^1 on B^n. First we will show that for any x in B^n, $e(u)(x)$ is not smaller than $L(u(x), \nabla u(x))$ and that equality holds when $u = \mathrm{Id}$ where L is a Lagrangian, and second we will show that L is a null Lagrangian, i.e. that the integral over B^n of $L(u(x), \nabla u(x))$ depends only on the trace of u on B^n.

First step. Let x be in $B^n\setminus\{0\}$, there exists an orthonormal basis of $\mathbb{R}^n$ in which $u(x)$ has coordinates $|u(x)|(1,0,\ldots,0)$. We start by working in this basis. Then

- $h_{ij}(u) = 0$ if $i \neq j$,
- $h_{ii}(u) = h_{\perp}(|u|)$ if $i \neq 1$,
- $h_{11}(u) = h_{//}(|u|)$.

So

$$\begin{aligned}
e(u)(x) &= (1/2)\sum_{\alpha}\Big[h_{//}(|u|)|u^1_\alpha|^2 + \sum_{2\leqq i\leqq n} h_{\perp}(|u|)|u^i_\alpha|^2\Big] \\
&= (1/2)\sum_{\alpha}\Big[\sum_{2\leqq i\leqq n}\frac{h_{//}(|u|)}{n-1}(|u^1_\alpha|^2+|u^i_\alpha|^2) \\
&\qquad + \sum_{2\leqq i<j\leqq n}\frac{(n-1)h_{\perp}(|u|)-h_{//}(|u|)}{(n-1)(n-2)}(|u^i_\alpha|^2+|u^j_\alpha|^2)\Big].
\end{aligned}$$

We use the following assumption the proof of which is given in the lemma at the end of this paper

$$\forall y\in B^n, (n-1)h_{\perp}(|y|) - h_{//}(|y|) \geqq 0, \tag{6}$$

then,

$$e(u)(x) \geqq \sum_{2\leqq i\leqq n} \frac{h_{//}(|u|)}{n-1} \begin{vmatrix} u_1^1 & u_i^1 \\ u_1^i & u_i^i \end{vmatrix} + \sum_{2\leqq i<j\leqq n} \frac{(n-1)h_\perp(|u|) - h_{//}(|u|)}{(n-1)(n-2)} \begin{vmatrix} u_i^i & u_j^i \\ u_i^j & u_j^j \end{vmatrix}.$$

Let $L_{//}(|u|) = \dfrac{h_{//}(|u|)}{n-1}$, $L_\perp(|u|) = \dfrac{(n-1)h_\perp(|u|) - h_{//}(|u|)}{(n-1)(n-2)}$, and $U_{ab}^{ij} = \begin{vmatrix} u_a^i & u_b^i \\ u_a^j & u_b^j \end{vmatrix}$. We obtain

$$e(u)(x) \geqq L_{//}(|u|) \sum_{1\leqq i<j\leqq n} U_{ij}^{ij} + [L_\perp(|u|) - L_{//}(|u|)] \sum_{2\leqq i<j\leqq n} U_{ij}^{ij},$$

or

$$e(u)(x) \geqq L_{//}(|u|) \sum_{1\leqq i<j\leqq n} U_{ij}^{ij} + [L_\perp(|u|) - L_{//}(|u|)] \sum_{1\leqq i<j\leqq n} \sum_{1\leqq a<b\leqq n} \Pi_{ij}^{ab}(u) U_{ab}^{ij}, \tag{7}$$

where $\Pi_{ij}^{ab}(u) = \begin{vmatrix} P_i^a & P_j^a \\ P_i^b & P_j^b \end{vmatrix}(u)$, and $P_i^a(u) = \delta_i^a - u^i(x)u^a(x)|u(x)|^{-2}$.

Last inequality (7) does not vary under a change of orthonormal coordinates, hence is true in any system of Euclidean coordinates. Moreover it is obvious that if $u = \mathrm{Id}$ equality holds in (7).

Second step. We prove that the right hand term in (7) that we will denote L is a null Lagrangian. We pose

$$L(u, \nabla u) = \sum_{1\leqq i<j\leqq n} \sum_{1\leqq a<b\leqq n} C_{ij}^{ab}(u) U_{ab}^{ij}$$

where of course $C_{ij}^{ab}(u) = L_{//}(|u|)\delta_{ij}^{ab} + [L_\perp(|u|) - L_{//}(|u|)]\Pi_{ij}^{ab}(u)$, and $\delta_{ij}^{ab} = \begin{vmatrix} \delta_i^a & \delta_j^a \\ \delta_i^b & \delta_j^b \end{vmatrix}$.

Let us introduce the exterior n-form ψ on $\{(x,y) \in B^n \times B^n\}$ defined by

$$\psi = \sum_{1\leqq i<j\leqq n} \sum_{1\leqq a<b\leqq n} C_{ij}^{ab}(y) dx^1 \wedge \ldots \wedge dx^{a-1} \wedge dy^i \wedge dx^{a+1} \wedge \ldots \wedge dx^{b-1} \wedge dy^j \wedge dx^{b+1} \wedge \ldots \wedge dx^n.$$

We remark that if $\Gamma = \{(x,y) \in B^n \times B^n \mid y = u(x)\}$ is the graph of u then

$$\int_{B^n} L(u, \nabla u)dx = \int_\Gamma \psi, \tag{8}$$

so it suffices to prove that ψ is closed i.e. $d\psi = 0$ to get our conclusion. But the condition $d\psi = 0$ is equivalent to

(9)
$$\frac{\partial}{\partial y^i} C^{ab}_{jk} + \frac{\partial}{\partial y^j} C^{ab}_{ki} + \frac{\partial}{\partial y^k} C^{ab}_{ij} = 0.$$

Coefficients Π^{ab}_{ij} are characterized by

(10)
$$\begin{cases} \Pi^{ab}_{ij} = -\Pi^{ba}_{ij} = -\Pi^{ab}_{ji} = \Pi^{ba}_{ji}, & \\ \text{if } \{a,b\} \cap \{i,j\} = \emptyset, & \Pi^{ab}_{ij} = 0, \\ \text{if } \{a,b\} \cap \{i,j\} = \{a\}, & \Pi^{ab}_{aj} = -\dfrac{u^b u^j}{|u|^2}, \\ \text{if } \{a,b\} \cap \{i,j\} = \{a,b\}, & \Pi^{ab}_{ab} = 1 - \dfrac{(u^a)^2 + (u^b)^2}{|u|^2}. \end{cases}$$

Let us verify that (9) is always true. There are three cases

(a) $\{a,b\} \cap \{i,j,k\} = \emptyset$, then (9) is obvious.

(b) $\{a,b\} \cap \{i,j,k\} = \{a\} = \{i\}$. Let us note $f(y) = \dfrac{L_\perp(|y|) - L_{//}(|y|)}{|y|^2}$. Then (9) is equivalent to

$$\frac{\partial}{\partial y^j}[f(y) y^b y^k] - \frac{\partial}{\partial y^k}[f(y) y^b y^j] = 0,$$

which is always true.

(c) $\{a,b\} \cap \{i,j,k\} = \{a,b\}$, and $(a,b) = (i,j)$, then (9) is equivalent to

$$\frac{\partial}{\partial y^k}[L_\perp(|y|) - f(y)[(y^a)^2 + (y^b)^2]] + \frac{\partial}{\partial y^a}[f(y) y^a y^k] + \frac{\partial}{\partial y^b}[f(y) y^b y^k] = 0$$
$$\Leftrightarrow\ y^k L'_\perp(|y|) - [(y^a)^2 + (y^b)^2] y^k f'(y) + 2y^k |y| f(y) + y^k (y^a)^2 f'(y) + y^k (y^b)^2 f'(y) = 0$$
$$\Leftrightarrow\ |y| L'_\perp(|y|) + 2L_\perp(|y|) = 2L_{//}(|y|)$$
$$\Leftrightarrow\ (n-1)[|y| h'_\perp + 2h_\perp] = |y| h'_{//} + 2(n-1) h_{//},$$

which is precisely (5).

To terminate our proof we must show the following lemma:

Lemma. *If* $\mathrm{Id}: (B^n, \delta_{ij}) \to (B^n, h_{ij})$ *is harmonic and* $\mathrm{SO}(n)$*-equivariant then* (6) *is verified.*

Proof. This follows immediatly from (2) and from

$$|y|[(n-1)h_\perp(|y|) - h_{//}(|y|)]' = 2(n-2) h_{//}(|y|) \geqq 0. \qquad \text{Q.E.D.}$$

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