

COMPLEMENT TO: HAMILTONIAN STATIONARY TORI IN THE COMPLEX PROJECTIVE PLANE

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We present here an extended version of paragraphs 2.1. and 2.2 of [1].

0.1. *The space $\mathbb{C}P^n$*

The complex projective space $\mathbb{C}P^n$ can be identified with the quotient manifold $(\mathbb{C}^{n+1} \setminus \{0\})/\mathbb{C}^*$. We denote by $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}P^n$ the canonical projection (a.k.a. Hopf fibration): if $z = (z^1, \dots, z^{n+1}) \in \mathbb{C}^{n+1} \setminus \{0\}$, $\pi(z) = [z]$ is the equivalence class of z modulo \mathbb{C}^* (the complex punctured line spanned by z) and is also written in homogeneous coordinates $[z^1 : \dots : z^{n+1}]$.

The tangent bundle — If $[z] \in \mathbb{C}P^n$ a tangent vector $\xi \in T_{[z]}\mathbb{C}P^n$ can be represented by a \mathbb{C} -linear map $\ell : \mathbb{C}z \rightarrow \mathbb{C}^{n+1}$, $\lambda z \mapsto \ell(\lambda z)$ such that

$$\forall \lambda \in \mathbb{C}, \quad d\pi_{\lambda z}(\ell(\lambda z)) = \xi. \tag{0.1}$$

Note that ℓ is not unique and given some $\xi \in T_{[z]}\mathbb{C}P^n$ if ℓ and $\tilde{\ell}$ satisfy condition (0.1) with ξ , then there exists $k \in \mathbb{C}$ s.t. $\tilde{\ell}(\lambda z) = \ell(\lambda z) + k\lambda z$. However by using the standard Hermitian product $\langle \cdot, \cdot \rangle_{\mathbb{C}^{n+1}}$ on \mathbb{C}^{n+1} we can select an unique ℓ^0 which satisfies (0.1) and such that

$$\langle \ell^0(z), z \rangle_{\mathbb{C}^{n+1}} = 0. \tag{0.2}$$

Indeed it suffices to start from any ℓ satisfying (0.1) and to set $\ell^0(z) = \ell(z) + kz$. Then condition (0.2) holds if and only if $k = -\frac{\langle \ell(z), z \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2}$.

The Hermitian metric — We can define a Hermitian product on $\mathbb{C}P^n$ as follows. If $\xi_1, \xi_2 \in T_{[z]}\mathbb{C}P^n$ we consider the linear maps ℓ_1^0 and ℓ_2^0 satisfying (0.2) and such that $d\pi_z(\ell_1^0(z)) = \xi_1$ and $d\pi_z(\ell_2^0(z)) = \xi_2$. Then the Hermitian product of ξ_1 and ξ_2 is

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n} := \frac{\langle \ell_1^0(z), \ell_2^0(z) \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2},$$

a definition which is obviously invariant by transformations $z \mapsto \lambda z$. If we had started with linear mappings ℓ_1, ℓ_2 which lift respectively ξ_1 and ξ_2 according to (0.1) but without the condition (0.2) we could recover $\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n}$ by substitution of $\ell_a^0(z) = \ell_a(z) - \frac{\langle \ell_a(z), z \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2} z$ (for $a = 1, 2$) in the above definition. It gives

$$\langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n} = \frac{\langle \ell_1(z), \ell_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^2} - \frac{\langle \ell_1(z), z \rangle_{\mathbb{C}^{n+1}} \langle z, \ell_2(z) \rangle_{\mathbb{C}^{n+1}}}{|z|_{\mathbb{C}^{n+1}}^4}. \tag{0.3}$$

Note that the Hermitian metric $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n}$ on $\mathbb{C}P^n$ provides us with an Euclidean metric $\langle \cdot, \cdot \rangle_E$ and a symplectic form ω through[†] $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n} = \langle \cdot, \cdot \rangle_E - i\omega(\cdot, \cdot)$.

The horizontal distribution and the connection — For each $z \in \mathbb{C}^{n+1} \setminus \{0\}$ we let \mathcal{H}_z to be the complex n -subspace in \mathbb{C}^{n+1} which is Hermitian orthogonal to z (and hence to the fiber of π). Note that the definition of the Hermitian metric on $\mathbb{C}P^n$ is such that $\frac{1}{|z|_{\mathbb{C}^{n+1}}} d\pi_z : \mathcal{H}_z \rightarrow T_{[z]}\mathbb{C}P^n$ is an isometry between complex Hermitian spaces (and $d\pi_z$ allows us also to orient \mathcal{H}_z). We call the subbundle $\mathcal{H} := \cup_{z \in \mathbb{C}^{n+1} \setminus \{0\}} \mathcal{H}_z$ of $T\mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ the *horizontal distribution*. It defines in a natural way a connection $\nabla^{\text{Hopf}} \simeq \nabla^H$ on the Hopf bundle $\pi : \mathbb{C}^{n+1} \setminus \{0\} \rightarrow \mathbb{C}P^n$: if $f : \mathbb{C}P^n \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ is a section of this bundle its covariant derivative is

$$(\nabla_{\xi}^H f)_{[z]} := \frac{\langle df_{[z]}(\xi), f(z) \rangle_{\mathbb{C}^{n+1}}}{|f(z)|_{\mathbb{C}^{n+1}}^2} f(z).$$

In order to compute the curvature of this connection, let us choose two vector fields ξ_1 and ξ_2 on $\mathbb{C}P^n$ such that $[\xi_1, \xi_2] = 0$. Then using (0.3) one computes that

$$\nabla_{\xi_1}^H (\nabla_{\xi_2}^H f) - \nabla_{\xi_2}^H (\nabla_{\xi_1}^H f) = (\langle \xi_2, \xi_1 \rangle_{\mathbb{C}P^n} - \langle \xi_1, \xi_2 \rangle_{\mathbb{C}P^n}) f = 2i\omega(\xi_1, \xi_2)f.$$

Hence the curvature of this connection is proportional to the symplectic form ω . It has the following consequence:

PROPOSITION 0.1. *Let Ω be a simply connected open subset of \mathbb{R}^n and $u : \Omega \rightarrow \mathbb{C}P^n$ be a smooth Lagrangian immersion, i.e. such that $u^*\omega = 0$. Then there exists a lift*

$$\begin{array}{ccc} & & \mathbb{C}^{n+1} \\ & \nearrow \hat{u} & \downarrow \pi \\ \Omega & \xrightarrow{u} & \mathbb{C}P^n \end{array}$$

such that $(u^*\nabla^H)\hat{u} = 0$ (where $u^*\nabla^H$ is the pull-back by u of the connection ∇^H). Moreover this lift is unique modulo a multiplicative constant in \mathbb{C}^* , is isotropic (i.e. the pull-back by \hat{u} of the symplectic form on \mathbb{C}^{n+1} vanishes) and we can choose it in such a way that $|\hat{u}|_{\mathbb{C}^{n+1}} = 1$ everywhere (i.e. such that $\hat{u} : \Omega \rightarrow S^{2n+1}$).

Proof. The curvature of $u^*\nabla^H$ is the pull-back by u of the curvature of ∇^H , i.e. $2iu^*\omega$. But this 2-form vanishes because u is Lagrangian and thus $u^*\nabla^H$ is flat. Hence we can find a parallel section \hat{u} (unique up to multiplication by a non-zero constant). The condition $(u^*\nabla^H)\hat{u} = 0$ writes $\langle d\hat{u}, \hat{u} \rangle_{\mathbb{C}^{n+1}} = 0$, which implies $d|\hat{u}|_{\mathbb{C}^{n+1}}^2 = \langle d\hat{u}, \hat{u} \rangle_{\mathbb{C}^{n+1}} + \langle \hat{u}, d\hat{u} \rangle_{\mathbb{C}^{n+1}} = 0$. Hence $|\hat{u}|_{\mathbb{R}^{2n+2}}$ is constant and in particular we can choose \hat{u} to take values in S^{2n+1} . \square

Observe that in Proposition 0.1 the tangent n -subspace along \hat{u} at (x, y) is a sub-

[†]note the the sign convention may vary in the literature, e.g. some Authors use $\langle \cdot, \cdot \rangle_{\mathbb{C}P^n} = \langle \cdot, \cdot \rangle_E + i\omega(\cdot, \cdot)$.

space of $\mathcal{H}_{\widehat{u}(x,y)}$: we then say that \widehat{u} is a *Legendrian immersion*.

The Levi-Civita connection on $\mathbb{C}P^n$ — It can be expressed in terms of the trivial connection D of $(\mathbb{C}^{n+1}, \langle \cdot, \cdot \rangle_{\mathbb{R}^{2n+2}})$ as follows. Let $U \subset \mathbb{C}P^n$ be an open subset such that we can define a smooth local section $\sigma : U \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$ of the Hopf bundle. For any $m \in U$ we denote by $\widehat{m} := \sigma(m)$. Let X be a smooth tangent vector field on $\mathbb{C}P^n$ defined on U and, choosing some point $m \in U$, let $\xi \in T_m \mathbb{C}P^n$. In order to define the covariant derivative of X at m along ξ we let $\widehat{\xi} := d\sigma_m(\xi) \in T_{\widehat{m}} \mathbb{C}^{n+1}$ and $\widehat{X} := \sigma_* X$. Then we set

$$(\nabla_{\xi} X)(m) := d\pi_{\widehat{m}} \left((D_{\widehat{\xi}} \widehat{X})(m) \right) - \left(\frac{\langle \widehat{\xi}, \widehat{m} \rangle_{\mathbb{C}^{n+1}}}{|\widehat{m}|_{\mathbb{C}^{n+1}}^2} X(m) + \frac{\langle \widehat{X}(m), \widehat{m} \rangle_{\mathbb{C}^{n+1}}}{|\widehat{m}|_{\mathbb{C}^{n+1}}^2} \xi \right) \quad (0.4)$$

(Of course this definition does not depend on the choice of the local section σ .) One can then check easily that this connection respects the metric and is torsion-free and so it is the Levi-Civita connection on $\mathbb{C}P^n$.

0.2. The mean curvature vector of a Lagrangian submanifold in $\mathbb{C}P^n$

We consider here a simply connected Lagrangian submanifold Σ of $\mathbb{C}P^n$ and denote by $u : \Sigma \rightarrow \mathbb{C}P^n$ its immersion map. According to Proposition 0.1, we can construct a Legendrian lift $\widehat{u} : \Sigma \rightarrow \mathbb{C}^{n+1} \setminus \{0\}$. We also consider a smooth orthonormal moving frame along u , i.e. sections E_1, \dots, E_n of $u^* T\mathbb{C}P^n$ such that for all $x \in \Sigma$, $(E_1(x), \dots, E_n(x))$ is a direct orthonormal basis over \mathbb{R} of $T_{u(x)}\Sigma$ and, for all a , we let $e_a := \widehat{u}_* E_a$. Then for all $x \in \Sigma$, $(e_1(x), \dots, e_n(x))$ is a direct orthonormal basis over \mathbb{R} of $T_{\widehat{u}(x)}\widehat{u}(\Sigma)$. We next define the *mean curvature vector* \vec{H} of the immersion u which is the normal component of the trace of the second fundamental form. The main point here is that, since $T_{u(x)}\Sigma$ is Lagrangian, it is mapped by the complex structure[†] J of $T_{u(x)}\mathbb{C}P^n$ to the normal space $(T_{u(x)}\mathbb{C}P^n)^\perp$. And so $(JE_1(x), \dots, JE_n(x))$ is an orthonormal basis of the normal subspace to $T_{u(x)}\Sigma$ in $T_{u(x)}\mathbb{C}P^n$. Hence

$$\vec{H} := \frac{1}{n} \sum_{a,b=1}^n \langle \nabla_{E_a} E_a, JE_b \rangle_E JE_b. \quad (0.5)$$

Note that then, since $\langle e_a, \widehat{u} \rangle_{\mathbb{C}^{n+1}} = \langle e_b, \widehat{u} \rangle_{\mathbb{C}^{n+1}} = 0$, we have according to (0.4)

$$\nabla_{E_a} E_b = d\pi_{\widehat{u}(x)}(D_{E_a} e_b) = d\pi_{\widehat{u}(x)}(D_{e_a} e_b).$$

Hence

$$\vec{H} = \frac{1}{n} \sum_{a,b=1}^n \langle D_{e_a} e_a, ie_b \rangle_{\mathbb{R}^{2n+2}} JE_b. \quad (0.6)$$

The complex volume n -form along \widehat{u} — Let us consider on \mathbb{C}^{n+1} the complex volume $(n+1)$ -form $\Theta := dz^1 \wedge \dots \wedge dz^{n+1}$. Then we construct the complex volume n -form along \widehat{u} to be the section $\widehat{\theta}$ of $\widehat{u}^*(\Lambda^n \mathcal{H}^* \otimes \mathbb{C})$ by

$$\widehat{\theta}_x := (\widehat{u}(x) \lrcorner \Theta)|_{\mathcal{H}_{\widehat{u}(x)}}, \quad \forall x \in \Sigma,$$

[†]the complex structure J is the image of the canonical complex structure i on $\mathcal{H}_{\widehat{u}(x)}$ by $d\pi_{\widehat{u}(x)}$.

where \lrcorner is the interior product. Lastly we define the complex volume n -form θ along u to be the section of $u^*(\Lambda^n T^* \mathbb{C}P^n \otimes \mathbb{C})$ defined by $\theta := \widehat{u}^* \widehat{\theta}$. Using $\widehat{\theta}$ we define the *Lagrangian angle function* along \widehat{u} to be the function $\beta_{\widehat{u}} : \Sigma \rightarrow \mathbb{R}/2\pi\mathbb{Z}$ such that

$$e^{i\beta_{\widehat{u}}(x)} = \theta_x(E_1(x), \dots, E_n(x)) = \Theta(\widehat{u}(x), e_1(x), \dots, e_n(x)), \quad \forall x \in \Sigma.$$

It is not difficult to check that this definition is independent of the choice of the framing (E_1, \dots, E_n) . However all this construction depends on the choice of the Legendrian lift \widehat{u} : another choice \widehat{v} would lead to another complex volume n -form along \widehat{v} , and another Lagrangian angle function $\beta_{\widehat{v}}$, however $\beta_{\widehat{u}} - \beta_{\widehat{v}}$ is a constant in $\mathbb{R}/2\pi\mathbb{Z}$.

LEMMA 0.2. *The mean curvature vector of the Lagrangian surface Σ can be computed using the Lagrangian angle function $\beta_{\widehat{u}}$ through the relation*

$$\vec{H} = \frac{1}{n} J \nabla \beta_{\widehat{u}}. \quad (0.7)$$

Proof. We set here $\beta = \beta_{\widehat{u}}$. Relation (0.7) is equivalent to proving that

$$\forall V \in T_x \mathbb{C}P^n, \quad n \langle \vec{H}, JV \rangle_E = \langle \nabla \beta, V \rangle_E. \quad (0.8)$$

The left hand side of (0.8) is, by using (0.6) and the fact that J is an isometry, $\sum_{a,b=1}^n \langle D_{e_a} e_a, i e_b \rangle_{\mathbb{R}^{2n+2}} \langle E_b, V \rangle_E$. So we are led to the following computation, which uses first the fact that $\langle e_a, i e_b \rangle_{\mathbb{R}^{2n+2}} = 0$, second the fact that i is a complex structure, and third the fact that the Lie bracket $[e_a, e_b]$ is tangent to Σ :

$$\begin{aligned} \langle D_{e_a} e_a, i e_b \rangle_{\mathbb{R}^{2n+2}} &= - \langle e_a, i D_{e_a} e_b \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle i e_a, D_{e_a} e_b \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle i e_a, D_{e_b} e_a + [e_a, e_b] \rangle_{\mathbb{R}^{2n+2}} \\ &= \langle i e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}. \end{aligned}$$

So we have $n \langle \vec{H}, JV \rangle_E = \sum_{a,b=1}^n \langle i e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} \langle E_b, V \rangle_E$. Now the right hand side of (0.8) can be computed as follows.. We derivate both sides of the relation $e^{i\beta} = \Theta_{\widehat{u}}(\widehat{u}, e_1, \dots, e_n)$ with respect to e_b and use the decomposition of $D_{e_b} e_a$ in the \mathbb{R} -orthonormal basis $(\widehat{u}, e_1, \dots, e_n, i\widehat{u}, i e_1, \dots, i e_n)$:

$$\begin{aligned} i e^{i\beta} D_{e_b} \beta &= D_{e_b} e^{i\beta} \\ &= D_{e_b} (\Theta(\widehat{u}, e_1, \dots, e_n)) \\ &= \Theta(e_b, e_1, \dots, e_n) + \sum_{a=1}^n \Theta(\widehat{u}, e_1, \dots, D_{e_b} e_a, \dots, e_n) \\ &= \sum_{a=1}^n \Theta(\widehat{u}, e_1, \dots, e_a, \dots, e_n) (\langle e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} + i \langle i e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}) \\ &= i e^{i\beta} \sum_{a=1}^n \langle i e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}. \end{aligned}$$

Here we used in the last line the fact that $|e_a|_{\mathbb{R}^{2n+2}}^2 = 1 \implies \langle e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} = 0$. Hence we are left with $D_{e_b} \beta = \sum_{a=1}^n \langle i e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}}$, which implies that

$$\langle \nabla \beta, V \rangle_E = \sum_{b=1}^n D_{e_b} \beta \langle E_b, V \rangle_E = \sum_{a,b=1}^n \langle i e_a, D_{e_b} e_a \rangle_{\mathbb{R}^{2n+2}} \langle E_b, V \rangle_E.$$

So we have proved (0.8). \square

A *Hamiltonian stationary* Lagrangian submanifold Σ in $\mathbb{C}P^n$ is a Lagrangian submanifold which is a critical point of the n -volume functional \mathcal{A} under first variations which are *Hamiltonian vector fields* with compact support. This means that for any smooth function with compact support $h \in C_c^\infty(\mathbb{C}P^n, \mathbb{R})$, we have

$$\delta \mathcal{A}_{\xi_h}(\Sigma) := \int_{\Sigma} \left\langle \vec{H}, \xi_h \right\rangle_E d\text{vol} = 0,$$

where ξ_h is the Hamiltonian vector field of h , i.e. satisfies $\omega(\xi_h, \cdot) + dh = 0$ or $\xi_h = J\nabla h$. We also remark that if $f \in C_c^\infty(\Sigma, \mathbb{R})$, then there exist smooth extensions with compact support h of f , i.e. functions $h \in C_c^\infty(\mathbb{C}P^n, \mathbb{R})$ such that $h|_{\Sigma} = f$, and moreover the normal component of $(\xi_h)|_{\Sigma}$ does not depend on the choice of the extension h (it coincides actually with $J\nabla f$, where ∇ is here the gradient with respect to the induced metric on Σ). So we deduce from Lemma 0.2 that actually $\delta \mathcal{A}_{\xi_h}(\Sigma) = \frac{1}{n} \int_{\Sigma} \langle \nabla \beta, \nabla f \rangle_E d\text{vol}$. This implies the following.

COROLLARY 0.3. *Any Lagrangian submanifold Σ in $\mathbb{C}P^n$ is Hamiltonian stationary if and only if β is a harmonic function on Σ , i.e.*

$$\Delta_{\Sigma} \beta = 0.$$

The theory above extends to non simply connected surfaces Σ with the following restrictions. Let γ be a homotopically non trivial loop. The Legendrian lift of γ needs not close, so that in general its endpoints $p_1, p_2 \in S^{2n+1}$ are multiples of each other by a factor $e^{i\theta}$. The same holds for the Lagrangian angle: $\beta(p_2) \equiv \beta(p_1) + (n+1)\theta \pmod{2\pi}$ (since the tangent plane is also shifted by the Decktransformation $z \mapsto e^{i\theta}z$). In particular β is not always globally defined on surfaces in $\mathbb{C}P^n$ with non trivial topology, unless the Legendrian lift is globally defined in $S^{2n+1}/\mathbb{Z}_{n+1}$ (here \mathbb{Z}_{n+1} stands for the cubic roots of unity in $SU(n+1)$).

References

- [1] F. HÉLEIN, P. ROMON, *Hamiltonian stationary tori in the complex projective plane*, Proc. London Roy. Soc. (3) 90 (2005), 472–496.