Complement to: Willmore immersions and Loop groups

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The Euler–Lagrange equation of the Willmore functional

The derivation of the Euler–Lagrange equation of the Willmore functional was done in [1]. However I found this proof difficult to follow and I present here an alternative redaction.

Proof — Using the notations of [2] we know that we may write the Willmore functional as

$$\mathcal{W}(X) = \int_U -\omega_3^1 \wedge \omega_3^2 \equiv \int_U \Omega_X,$$

which is precisely minus the area covered by γ. We start from an immersion $X : U \longrightarrow S^3$, not necessarily conformal, and we consider some section e of the associated bundle $\mathcal{F}_X^{(\gamma)}$. We perform a variation of X induced by

$$e_0(t) = e_0 + t\lambda e_3, \tag{1}$$

where $\lambda : U \longrightarrow \mathbb{R}$ is some smooth map with compact support in U. We can construct for each t sufficiently small the associated conformal Gauss map $\gamma(t) = e_3(t)$, and the orthonormal frame $(e_1(t), e_2(t))$ for instance by the Gram-Schmitt orthonormalisation of $(\frac{\partial e_0}{\partial x}(t), \frac{\partial e_0}{\partial y}(t))$. By completing by an adequate vector $e_4(t)$ in $\mathcal{C}_+$, we construct a section $e(t)$ of $\mathcal{F}_{t,X}^{(\gamma)}$ for t sufficiently small. We now introduce the matrix of elements λ_j^i depending smoothly on z such that if the dot denotes the derivatives with respect to t at $t = 0$, we have

$$\dot{e}_j = e_i \lambda_j^i. \tag{2}$$

We have $B_{ac}\lambda_b^c + B_{bc}\lambda_a^c = 0$ and by (1)

$$\lambda_0^0 = \lambda_0^1 = \lambda_0^2 = 0\ ;\ \lambda_0^3 = \lambda. \tag{3}$$

We wish to compute the first variation of $\int_U \Omega_X$ by this deformation.

First step. Brutal computation of $\dot{\Omega}_X$: Using (2) we get that

$$\begin{aligned}
\dot{\omega}_3^1 &= \langle d\dot{e}_3, e_1\rangle + \langle de_3, \dot{e}_1\rangle \\
&= d\lambda_3^1 + \lambda_3^0\omega_0^1 + \lambda_3^2\omega_2^1 + \lambda_0^3\omega_1^0 + \lambda_1^2\omega_3^2 \\
\dot{\omega}_3^2 &= \langle d\dot{e}_3, e_2\rangle + \langle de_3, \dot{e}_2\rangle \\
&= d\lambda_3^2 + \lambda_3^0\omega_0^2 + \lambda_3^1\omega_1^2 + \lambda_0^3\omega_2^0 + \lambda_2^1\omega_3^1.
\end{aligned}$$

Thus

$$\begin{array}{rcl} -\dot{\Omega}_X & = & (d\lambda_3^1 \wedge \omega_3^2 + \lambda_3^2\omega_2^1 \wedge \omega_3^2 + \omega_3^1 \wedge d\lambda_3^2 + \lambda_3^1\omega_3^1 \wedge \omega_1^2) \\ & + & \lambda_0^3(\omega_1^0 \wedge \omega_3^2 + \omega_3^1 \wedge \omega_2^0) \\ & + & \lambda_3^0(\omega_0^1 \wedge \omega_3^2 + \omega_3^1 \wedge \omega_0^2). \end{array} \quad (4)$$

We then use (**??**) and the fact that $h_{11} + h_{22} = 0$ to cancel the last term in the right hand side of (4). Moreover using the structure equations

$$\begin{array}{rcl} d\omega_3^1 + \omega_0^1 \wedge \omega_3^0 + \omega_2^1 \wedge \omega_3^2 & = & 0 \\ d\omega_3^2 + \omega_0^2 \wedge \omega_3^0 + \omega_1^2 \wedge \omega_3^1 & = & 0 \end{array}$$

we can transform the first term in the right hand side of (4) as

$$d\lambda_3^1 \wedge \omega_3^2 - \lambda_3^2(d\omega_3^1 + \omega_0^1 \wedge \omega_3^0) + \omega_3^1 \wedge d\lambda_3^2 + \lambda_3^1(d\omega_3^2 + \omega_0^2 \wedge \omega_3^0)$$

$$= d(\lambda_3^1\omega_3^2 - \lambda_3^2\omega_3^1) + (\lambda_3^1\omega_0^2 - \lambda_3^2\omega_0^1) \wedge \omega_3^0.$$

Thus we obtain that

$$\begin{array}{rcl} -\dot{\Omega}_X & = & d(\lambda_3^1\omega_3^2 - \lambda_3^2\omega_3^1) + (\lambda_3^1\omega_0^2 - \lambda_3^2\omega_0^1) \wedge \omega_3^0 \\ & + & \lambda_0^3(\omega_1^0 \wedge \omega_3^2 + \omega_3^1 \wedge \omega_2^0). \end{array} \quad (5)$$

Second step. We compute the last term in the right hand side of (5). We set $\omega_3^0 = h_1\omega_0^1 + h_2\omega_0^2$, and we differentiate this expression. This gives

$$d\omega_3^0 = dh_1 \wedge \omega_0^1 + dh_2 \wedge \omega_0^2 + h_1 d\omega_0^1 + h_2 d\omega_0^2.$$

We develop this relation using the structure equations and (**??**) as

$$\begin{array}{rcl} (dh_1 + 2\omega_0^0 h_1 - \omega_1^2 h_2 - h_{11}\omega_1^0 - h_{12}\omega_2^0) \wedge \omega_0^2 & & \\ +(dh_2 + 2\omega_0^0 h_2 - \omega_2^1 h_1 - h_{21}\omega_1^0 - h_{22}\omega_2^0) \wedge \omega_0^1 & = & 0. \end{array} \quad (6)$$

And using Cartan lemma this implies that there exist smooth functions p_{11}, $p_{12} = p_{21}$ and p_{22} such that

$$\left\{ \begin{array}{rcll} dh_1 + 2\omega_0^0 h_1 & = & \omega_1^2 h_2 + h_{11}\omega_1^0 + h_{12}\omega_2^0 + p_{11}\omega_0^1 + p_{12}\omega_0^2 & (a) \\ dh_2 + 2\omega_0^0 h_2 & = & \omega_2^1 h_1 + h_{21}\omega_1^0 + h_{22}\omega_2^0 + p_{21}\omega_0^1 + p_{22}\omega_0^2 & (b) \end{array} \right. \quad (7)$$

And a computation of $(a) \wedge \omega_0^2 - (b) \wedge \omega_0^1$ gives

$$\begin{array}{rl} & dh_1 \wedge \omega_0^2 - dh_2 \wedge \omega_0^1 + 2\omega_0^0 \wedge (h_1\omega_0^2 - h_2\omega_0^1) \\ = & \omega_1^2 \wedge (h_2\omega_0^2 + h_1\omega_0^1) - (\omega_1^3 \wedge \omega_2^0 + \omega_1^0 \wedge \omega_2^3) + (p_{11} + p_{22})\omega_0^1 \wedge \omega_0^2. \end{array} \quad (8)$$

Relation (8) gives us an expression for $\omega_1^3 \wedge \omega_2^0 + \omega_1^0 \wedge \omega_2^3$, which we can again transform using the structure relations

$$\left\{ \begin{array}{rcl} d\omega_0^1 + \omega_0^1 \wedge \omega_0^0 + \omega_2^1 \wedge \omega_0^2 & = & 0 \\ d\omega_0^2 + \omega_0^2 \wedge \omega_0^0 + \omega_1^2 \wedge \omega_0^1 & = & 0 \end{array} \right. \quad (9)$$

as

$$\begin{aligned} \omega_3^1 \wedge \omega_2^0 + \omega_1^0 \wedge \omega_3^2 &= d(h_1\omega_0^2 - h_2\omega_0^1) + \omega_0^0 \wedge (h_1\omega_0^2 - h_2\omega_0^1) \\ &\quad -(p_{11}+p_{22})\omega_0^1 \wedge \omega_0^2. \end{aligned} \tag{10}$$

Third step. We inject the result of step 2 summarized in (10) in the formula for $\dot{\Omega}_X$ given by (5). It leads to

$$\begin{aligned} -\dot{\Omega}_X &= d(\lambda_3^1\omega_3^2 - \lambda_3^2\omega_3^1) + (\lambda_3^1\omega_0^2 - \lambda_3^2\omega_0^1) \wedge \omega_3^0 \\ &\quad +\lambda_0^3 d(h_1\omega_0^2 - h_2\omega_0^1) + \lambda_0^3\omega_0^0 \wedge (h_1\omega_0^2 - h_2\omega_0^1) \\ &\quad -\lambda_0^3(p_{11}+p_{22})\omega_0^1 \wedge \omega_0^2 \\ &= d(\lambda_3^1\omega_3^2 - \lambda_3^2\omega_3^1) + \lambda_0^3 d(h_1\omega_0^2 - h_2\omega_0^1) \\ &\quad +(\lambda_0^3\omega_0^0 - \lambda_3^1\omega_0^1 - \lambda_3^2\omega_0^2) \wedge (h_1\omega_0^2 - h_2\omega_0^1) \\ &\quad -\lambda_0^3(p_{11}+p_{22})\omega_0^1 \wedge \omega_0^2, \end{aligned} \tag{11}$$

where we used $\omega_3^0 = h_1\omega_0^1 + h_2\omega_0^2$.

Lastly we exploit a relation between the coefficients λ_j^i which we did not use before. Since we impose the constraint $\omega_0^3 = 0$, for small t, we have

$$0 = \langle d\dot{e}_0, e_3\rangle + \langle de_0, \dot{e}_3\rangle,$$

which leads to

$$d\lambda_0^3 = \lambda_0^3\omega_0^0 - \lambda_3^1\omega_0^1 - \lambda_3^2\omega_0^2. \tag{12}$$

We insert this relation in (11) to get

$$\begin{aligned} \dot{\Omega}_X &= \lambda_0^3(p_{11}+p_{22})\omega_0^1 \wedge \omega_0^2 \\ &\quad +d[\lambda_3^1\omega_3^2 - \lambda_3^2\omega_3^1 + \lambda_0^3(h_1\omega_0^2 - h_2\omega_0^1)]. \end{aligned} \tag{13}$$

Conclusion. We obtain that

$$\int_U \dot{\Omega}_X = \int_U \lambda(p_{11}+p_{22})\omega_0^1 \wedge \omega_0^2, \tag{14}$$

and this implies that any Willmore immersion satisfies the equation

$$p_{11} + p_{22} = 0. \tag{15}$$

[1] R. Bryant, *A duality theorem for Willmore surfaces*, Journal of Differential Geometry 20 (1984), 23–53.

[2] F. Hélein, *Willmore immersions and loop groups*, Journal of Differential Geometry, Vol. 50, n. 2 (1998), 331–388.