# A Weierstrass representation for Willmore surfaces *

Frédéric Hélein

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## 1 Willmore surfaces

Willmore surfaces are immersed surfaces in the threedimensional Euclidean space $\mathbb{R}^3$ which are critical points of the functional

$$\mathcal{W}(\mathcal{S}) := \int_{\mathcal{S}} H^2 dA,$$

where $H := (k_1 + k_2)/2$ is the mean curvature of the surface $\mathcal{S}$, $dA$ is the area form of $\mathcal{S}$ induced by the immersion on $\mathcal{S}$ in $\mathbb{R}^3$. This functional and the study of its critical points were proposed by T. Willmore in 1960 [Wi]. Here we assume for simplicity that all surfaces are oriented (which is always true locally). If the surface has no boundary, a variant of $\mathcal{W}$ is

$$\tilde{\mathcal{W}}(\mathcal{S}) := \int_{\mathcal{S}} \frac{1}{4}(k_1 - k_2)^2 dA,$$

and because $\mathcal{W}(\mathcal{S}) - \tilde{\mathcal{W}}(\mathcal{S}) = \int_{\mathcal{S}} K dA = 4\pi(1-g)$, where $K = k_1 k_2$ is the Gauss curvature of $\mathcal{S}$ and $g$ is the genus of $\mathcal{S}$, both functionals have the same critical points, called *Willmore surfaces*. A Willmore surface is a solution of the fourth order partial differential equation

$$\Delta H + 2H(H^2 - K) = 0,$$

where the Laplacian $\Delta$ is constructed using the first fundamental form of the immersion.

The subject is actually older, since it appears in the book of W. Blaschke in 1929 [Bl] with the appellation of *conformal minimal surfaces*. This terminology is justified by the invariance of that problem under the action of the conformal group of $\mathbb{R}^3 \cup \{\infty\}$ - the Moebius group - for $\frac{1}{4}(k_1 - k_2)^2 dA$ being locally conformally invariant. In some sense, this problem is the analog in conformal geometry of the problem of minimal surfaces in Euclidean geometry.

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As a consequence, we can also view Willmore surfaces in the three sphere $\mathbb{S}^3$ or in the hyperbolic space $\mathbb{H}^3$, locally conformally equivalent to $\mathbb{R}^3$. They are critical points of

$$\int_{\mathcal{S}} (H^2 + c) dA,$$

where $H$ is the mean curvature of the surface $\mathcal{S}$ in $\mathbb{S}^3$ or $\mathbb{H}^3$ and $c = 1$ for $\mathbb{S}^3$ and $c = -1$ for $\mathbb{H}^3$. Through sterographic projections of $\mathbb{S}^3$ or $\mathbb{H}^3$ into $\mathbb{R}^3$, these surfaces are mapped into Willmore surfaces of $\mathbb{R}^3$. Thus these problems are locally the same.

As a problem in conformal geometry, the Euclidean notion of Gauss map (the map which associates to each point $m \in \mathcal{S}$ the unit normal vector in $\mathbb{S}^2$) is not that relevant. Instead we can define the notion of *conformal Gauss map*. We assume that $\mathcal{S}$ is oriented and we denote by $\mathcal{Q}$ the set of all oriented spheres and oriented planes in the affine space $\mathbb{R}^3$ (through the stereographic projection, this set is actually the same as the set of all oriented spheres in $\mathbb{S}^3$). To each point $m \in \mathcal{S}$, we associate the unique oriented sphere or plane $\mathbb{S}^2_{\gamma(m)}$ in $\mathcal{Q}$ which is tangent to $\mathcal{S}$ at $m$ and which has the same mean curvature as $\mathcal{S}$ at $m$ (it means that its radius is $1/H$). Then it is easy to check that the conformal Gauss map is equivariant with respect to the Moebius group acting on $\mathbb{R}^3 \cup \{\infty\}$ and on $\mathcal{Q}$. A theorem in [Bl], but rediscovered by R. Bryant in 1984 [Br 1] asserts that $\mathcal{S}$ is a Willmore surface if and only if its conformal Gauss map is a minimal branched immersion (i.e. a weakly conformal harmonic map into $\mathcal{Q}$). The meaning of this characterisation will be precised later.

Another construction is the following. If the immersion $\mathcal{S}$ is of class $\mathcal{C}^3$, for any point $m$ in $\mathcal{S}$ which is not *umbilic* (i.e. such that $k_1 \neq k_2$), there exists a unique point $\hat{m}$ in $\mathbb{R}^3 \cup \{\infty\}$ such that, if $T$ is any conformal transformation of $\mathbb{R}^3 \cup \{\infty\}$ which maps $\hat{m}$ at infinity, then the mean curvature and the first derivatives of the mean curvature of $T(\mathcal{S})$ at $T(m)$ vanish. This construction is also equivariant under the action of the Moebius group. The map $\mathcal{D}$ which associates $\hat{m}$ to $m$ is called the conformal transform by R. Bryant. This map is of class $\mathcal{C}^{k-3}$ if $\mathcal{S}$ is of class $\mathcal{C}^k$, outside the umbilic points. But $\mathcal{D}$ may not be defined at the umbilic points, even if the surface is smooth. We call $\mathcal{U}$ the set of umbilic points. The image of $\mathcal{S} \setminus \mathcal{U}$ by $\mathcal{D}$ is another surface, called conformal dual. The importance of that construction is illustrated by these two properties: if $\mathcal{D}$ is well-defined (i.e. outside $\mathcal{U}$), then $\mathcal{D} \circ \mathcal{D}$ is the identity map, and if $\mathcal{S}$ is a Willmore surface, then $\mathcal{D}(\mathcal{S} \setminus \mathcal{U})$ is also a Willmore surface. This was proved by R. Bryant in [Br 1].

## 2 Examples

Natural questions about these surfaces are: is there a Willmore surface for all genus? Are there surfaces minimizing the Willmore energy functional in each genus class? The more simple examples of Willmore surfaces are the spheres

$$\mathbb{S}^2_{a,r} := \{x \in \mathbb{R}^3 / |x-a| = r\}.$$

These are the only surfaces which are totally umbilic, i.e. $k_1 = k_2$ everywhere. Moreover they minimize the Willmore functional among surfaces of genus 0 since for all $\mathcal{S} \in \mathcal{D}$ of genus 0,

$$\mathcal{W}(\mathcal{S}) = \tilde{\mathcal{W}}(\mathcal{S}) + 4\pi(1-g) \geq 4\pi = \mathcal{W}(\mathbb{S}^2_{a,r}).$$

Actually all Willmore surfaces of genus 0 have been characterized using a Weierstrass type representation by R. Bryant ([Br 1] and [Br 2]) (they are basically the same thing as minimal surfaces of $\mathbb{R}^3$ satisfying some condition at infinity).

The next question is to understand the genus 1 case, i.e. Willmore tori. In 1965, T. Willmore showed that the torus of revolution spanned by the rotation of a circle of radius 1 around an straight line, such that the center of the rotating circle is at a distance $\sqrt{2}$ from the line is a Willmore surface and conjectured that this torus minimizes the Willmore functional among all tori. This conjecture is still unsolved despite some partial (positive) answer obtained by P. Li and S.T. Yau in [LY] or S. Montiel and A. Ros in [MR]. Recently, L. Simon proved that the minimum of that functional is achieved among tori [Si], but it is unknown whether or not it coincides with Willmore's candidate.

The invariance by the Moebius group implies that we need to enlarge the set of tori proposed by Willmore as candidates to be minimizing by adding all the images of these tori under conformal transformations.

Other instances of Willmore surfaces are all minimal surfaces in $\mathbb{R}^3$, $\mathbb{S}^3$ or $\mathbb{H}^3$. It means that all minimal surfaces of $\mathbb{R}^3$ are Willmore surfaces and that the image by any conformal local diffeomorphism, say from $\mathbb{S}^3$ or $\mathbb{H}^3$ into $\mathbb{R}^3$, of a minimal surface of $\mathbb{S}^3$ or $\mathbb{H}^3$ is a Willmore surface. For instance the Willmore torus corresponds to the minimal Clifford torus in $\mathbb{S}^3$. Also all the family of minimal surfaces in $\mathbb{S}^3$ constructed by H. B. Lawson [L] provides examples of Willmore surfaces of arbitrary genus (they are also good candidates to be Willmore minimizers, as conjectured by R. Kusner).

Lastly, more sophisticated examples of Willmore surfaces were constructed in [Pi], [FP], [BaBo] using ideas from the theory of completely integrable systems.

## 3 Translation in the Minkowski geometry

All that can be translated in the framework of the Minkowski space $\mathbb{R}^{4,1}$. Points in $\mathbb{R}^3 \cup \{\infty\} \simeq \mathbb{S}^3$ are identified with half lines contained in the light cone of the Minkowski space. For that purpose, we let $\mathcal{C}^+$ be the set of future pointing lightvectors - the half of the light cone - and an equivalence relation in $\mathcal{C}^+$ as follows: $l_1 \simeq l_2$ if and only if $l_1 = rl_2$, for some positive $r$. The quotient set $\mathcal{C}^+/\simeq$ may be identified with the sphere $\mathbb{S}^3$ and we denote $P : \mathcal{C}^+ \longrightarrow \mathbb{S}^3$ the canonical projection. Then we can check that the action of the identity component of the Lorentz group $SO(4,1)$ acting on $\mathbb{R}^{4,1}$ is mapped by $P$ into an action on $\mathbb{S}^3$ and that this action coincides with the action of the Moebius group. Second the set $\mathcal{Q}$ may also be identified with the Minkowski sphere $\mathbb{S}^{3,1} := \{y \in \mathbb{R}^{4,1}/|y|^2 = 1\}$ (a de Sitter space) by the following correspondance: to each vector $y$ in $\mathbb{S}^{3,1}$, we associate its orthogonal subspace in $\mathbb{R}^{4,1}$, $y^\perp$. This 4-dimensional subspace intersects $\mathcal{C}^+$ along a 3-dimensional subcone and the image of this cone $y^\perp \cap \mathcal{C}^+$ by $P$ is a 2-dimensional sphere in $\mathbb{S}^3$. Also the action of the group of conformal transformations of $\mathbb{S}^3$ coincides with the action of the Lorentz group on $\mathbb{S}^{3,1}$.

Thanks to that identification, the set $\mathcal{Q}$ possesses now a canonical differential structure and a pseudo-Riemannian metric and the notion of weakly conformal and harmonic map into $\mathcal{Q}$ makes sense. We should then precise that the conformal Gauss map $\gamma$ of a Willmore surface is a space-like branched minimal immersion, but with some possible accidents: at umbilic points, either the rank of the differential of the conformal Gauss map is 0 - the rank 0 case - either the rank of the differential of the conformal Gauss map is 1 and the conformal Gauss map moves along a lightline at this point - the rank 1 case.

One strategy could be to reduce the study of Willmore surfaces to the study of space-like branched minimal immersions into $\mathbb{S}^{3,1}$ with the possible accidents at umbilic points as described previously. The difficulty is then the "inverse problem": constructing the Willmore immersion starting from its conformal Gauss map. R. Bryant describes geometrically how it can be done: at the points where the conformal Gauss map is a spacelike immersion, the vectorial subspace of $T_\gamma\mathbb{S}^{3,1}$ orthogonal to the tangent plane to the image of $\gamma$ has a signature $(1,1)$ and in particular contains two light-lines. One of this light-line represents a Willmore immersion through the projection $P$, the other its conformal dual - another Willmore immersion. (Hence Willmore immersions have to be considered as pairs in conformal duality.) This nice picture degenerates for umbilic points. Then, in the rank 1 case, the image of $d\gamma$ is a light-line, the normal subspace to this light-line is then a 4-dimensional subspace which is the direct sum of that tangent light-line with a spacelike 3-dimensional space. This tangent light-line represents a point in $\mathbb{R}^3$ and we see that the pair of Willmore surfaces in conformal duality coincides at this point. In the rank 0 case, the value of the corresponding pair of Willmore immersions in conformal duality is not defined.

Notice that a classification of the possible behaviours has been done by R. Bryant. Beside the very particular case of round spheres which are totally umbilic Willmore surfaces, the umbilic locus is a closed set of dimension at most 1. M. Babich and A. Bobenko constructed a Willmore torus with a line of umbilic points, showing that Bryant's result is optimal [BaBo]. We shall see below an alternative description of conformal Willmore immersions which avoids these difficulties with umbilic points.

## 4 Harmonic maps into symmetric manifolds

Many results have been obtained recently concerning the study of harmonic maps defined on a surface, with values into a compact Lie group or a symmetric manifold: N. Hitchin for maps into $\mathbb{S}^3$ [Hi], K. Uhlenberck for maps into $U(n)$ [U], F. Burstall, D. Ferus, F. Pedit and U. Pinkall for maps between tori and homogeneous spaces [BFPP]. My approach is inspired by a more recent work due to J. Dorfmeister, F. Pedit and H.Y. Wu: a Weierstrass representation for any harmonic map into a manifold of the form $\mathfrak{G}/\mathfrak{K}$, where $\mathfrak{G}$ is a compact Lie group and $\mathfrak{K}$ is a Lie subgroup of $\mathfrak{H}$ [DPW]. They assume furthermore that there exists an automorphism $\tau$ of $\mathfrak{G}$, such that $\tau^k = 1\!\!1$, for some $k \in \mathbb{N}$ and such that $\mathfrak{K}$ is the subgroup of $\mathfrak{K}$ invariant by $\tau$. (An example is $\mathbb{S}^n = SO(n+1)/SO(n)$.) To explain the method, let us assume for simplicity that $k = 2$.

The first step is to remark that the differential of $\tau$ at $1\!\!1$ is a linear mapping $Ad_\tau$ of $\mathfrak{g}$, the Lie algebra of $\mathfrak{G}$, and that $Ad_\tau$ can be diagonalised with eigenvalues 1 and -1. We decompose $\mathfrak{g}$ as the direct sum

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$

where $\mathfrak{k}$ is the Lie algebra of $\mathfrak{K}$ (and the 1-eigenspace of $Ad_\tau$) and $\mathfrak{p}$ is the (-1)-eigenspace of $Ad_\tau$.

Now we consider a map $u : \Omega \longrightarrow \mathfrak{G}/\mathfrak{K}$ and any lift $F : \Omega \longrightarrow \mathfrak{G}$ of $u$, i.e. such that the composition of $F$ with the projection mapping from $\mathfrak{G}/\mathfrak{K}$ into $\mathfrak{G}$ is $u$. The Maurer-Cartan form of $F$, $\omega := F^{-1}.dF$, is a 1-form on $\Omega$ with coefficients in $\mathfrak{g}$. Automatically, such a 1-form is a solution of the equation

$$d\omega + \omega \wedge \omega = 0.$$

According to the above splitting of $\mathfrak{g}$, we decompose $\omega$ as

$$\omega = \omega_0 + \omega_1,$$

where the coefficients of $\omega_0$ are in $\mathfrak{k}$ and the coefficients of $\omega_1$ are in $\mathfrak{p}$. We consider a family of deformations of $\omega$ given by

$$\omega_\lambda = \lambda^{-1}\omega_1' + \omega_0 + \lambda\omega_1'',$$

for any $\lambda \in \mathbb{C}^\star$. Here $\omega_1' = \omega_1(\frac{\partial}{\partial z})dz$ and $\omega_1'' = \omega_1(\frac{\partial}{\partial \overline{z}})d\overline{z}$ are respectively the (1,0) and the (0,1) parts of $\omega_1$. Notice that in general $\omega_\lambda$ is a 1-form with coefficients in $\mathfrak{g} \otimes \mathbb{C}$ (the coefficients are in $\mathfrak{g}$ if and only if $|\lambda| = 1$). Then we have the following

**Theorem 1** *u is harmonic if and only if*

$$d\omega_\lambda + \omega_\lambda \wedge \omega_\lambda = 0,$$

*for all* $\lambda \in \mathbb{C}^\star$.

If $\Omega$ is simply connected, this equation is the necessary and sufficient condition for the existence of some map $F_\lambda$ form $\Omega$ such that

$$dF_\lambda = F_\lambda . \omega_\lambda,$$

for any $\lambda \in \mathbb{C}^\star$. Moreover $F_\lambda$ is unique if we assume a further initial condition $F_\lambda(z_0) = 1\!\!1$, for some $z_0 \in \Omega$.

Thus, if $\Omega$ is simply connected, the study of harmonic maps into $\mathfrak{G}/\mathfrak{K}$ is equivalent to the study of families of maps $F_\lambda$ such that $dF_\lambda = F_\lambda . \omega_\lambda$, for $\omega_\lambda = \lambda^{-1}\omega_1' + \omega_0 + \lambda\omega_1''$ defined as above. Since $F_\lambda$ is a complex analytic function of $\lambda$ on $\mathbb{C}^\star$, it is enough to restrict ourself to the case where $\lambda$ belongs to the circle $\mathbb{S}^1 \subset \mathbb{C}$.

An equivalent point of view is to define the set of maps from the circle (loops) $\mathbb{S}^1$ into $\mathfrak{G}$,

$$L\mathfrak{G} := \{g_\circ : \mathbb{S}^1 \to \mathfrak{G}\},$$

where we will denote the value of $g_\circ$ at $\lambda \in \mathbb{S}^1$ by $g_\lambda$. This set of loops is endowed with the law of multiplication

$$g_\circ . g_\circ' \text{ has the value } g_\lambda . g_\lambda' \text{ at } \lambda,$$

which gives to $L\mathfrak{G}$ an infinite dimensional Lie group structure. Thus, $L\mathfrak{G}$ is called a *loop group.* Its Lie algebra is, in a natural way

$$L\mathfrak{g} := \{\xi_\circ : \mathbb{S}^1 \to \mathfrak{g}\},$$

with the Lie bracket $[\xi_\circ, \xi_\circ'] : \lambda \longmapsto \xi_\lambda \xi_\lambda' - \xi_\lambda' \xi_\lambda$.

Here we shall consider maps $F_\circ$ from $\Omega$ into $L\mathfrak{G}$, such that $dF_\lambda = F_\lambda . \omega_\lambda$ and $F_\lambda(z_0) = 1\!\!1$ for some point $z_0 \in \Omega$. We call $F_\circ$ an *extended harmonic lift.*

J. Dorfmeister, F. Pedit and H.Y. Wu realised that extended harmonic lifts can been constructed by (i) integrating holomorphic datas in the framework of loop groups and (ii) performing some nonlinear decomposition which is the analog in loop groups of an Iwasawa decomposition (like the decompostion, for any $M \in GL(n, \mathbb{R})$, $M = R.T$, where $R \in O(n)$ and $T$ is an lower triangular matrix with positive diagonal elements). To describe this algorithm, one needs to define further special objects.

### Iwasawa decomposition

We let $B$ be a (solvable) subgroup of $\mathfrak{G}^{\mathbb{C}}$ such that $\forall g \in \mathfrak{G}^{\mathbb{C}}$, $\exists! a \in \mathfrak{G}$, $\exists! b \in B$, such that $g = a.b$. Moreover the mapping $\begin{array}{ccc} \mathfrak{G} \times B & \longrightarrow & \mathfrak{G}^{\mathbb{C}} \\ (a,b) & \longmapsto & a.b \end{array}$ is a diffeomorphism. We write "$\mathfrak{G}^{\mathbb{C}} = \mathfrak{G}.B$" for meaning this property, called *Iwasawa decomposition* (see [Hel]). An example is $\mathfrak{G} = U(n)$, $\mathfrak{G}^{\mathbb{C}} = GL(n,\mathbb{C})$ and

$$B = \left\{ \begin{pmatrix} b_1^1 & 0 & \dots & 0 \\ b_1^2 & b_2^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_1^n & b_2^n & \dots & b_n^n \end{pmatrix} / b_i^i \in ]0,\infty[, b_j^i \in \mathbb{C} \text{ if } i > j, b_j^i = 0 \text{ if } i < j \right\}.$$

Remark that in this case, the decomposition $g = a.b$ is done by the Schmidt orthonormalisation algorithm of a system of $n$ independant vectors of $\mathbb{C}^n$ whose components are the elements of $g$.

### Special loop groups

We define also

$$\begin{gathered}
L\mathfrak{G}^{\mathbb{C}} = \{g_\circ : \mathbb{S}^1 \longrightarrow \mathfrak{G}^{\mathbb{C}}\}, \\
L\mathfrak{G}^{\mathbb{C}}_\tau = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / g_{-\lambda} = \tau(g_\lambda)\}, \\
L\mathfrak{G}_\tau = L\mathfrak{G}^{\mathbb{C}}_\tau \cap L\mathfrak{G}, \\
L^+\mathfrak{G}^{\mathbb{C}}_\tau = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / \lambda \longmapsto g_\lambda \text{ extends holomorphically inside the disk } |\lambda| < 1\}, \\
L^+_B\mathfrak{G}^{\mathbb{C}}_\tau = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / \lambda \longmapsto g_\lambda \text{ extends holomorphically inside the disk } |\lambda| < 1 \\
\text{and } g_0 \in B\}, \\
L^-\mathfrak{G}^{\mathbb{C}}_\tau = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / \lambda \longmapsto g_{\lambda^{-1}} \text{ extends holomorphically inside the disk } |\lambda| < 1\}, \\
L^-_\star\mathfrak{G}^{\mathbb{C}}_\tau = \{g_\circ \in L\mathfrak{G}^{\mathbb{C}} / \lambda \longmapsto g_{\lambda^{-1}} \text{ extends holomorphically inside the disk } |\lambda| < 1 \\
\text{and } g_\infty = \mathbb{1}\}.
\end{gathered}$$

Similarly we define their respective Lie algebra

$$\begin{gathered}
L\mathfrak{g}^{\mathbb{C}} = L\mathfrak{g} \otimes \mathbb{C}, \\
L\mathfrak{g}^{\mathbb{C}}_\tau, \\
L\mathfrak{g}_\tau, \\
L^+\mathfrak{g}^{\mathbb{C}}_\tau, \\
L^+_b\mathfrak{g}^{\mathbb{C}}_\tau, \text{ where } b \text{ is the Lie algebra of } B, \\
L^-\mathfrak{g}^{\mathbb{C}}_\tau, \\
L^-_\star\mathfrak{g}^{\mathbb{C}}_\tau.
\end{gathered}$$

### Playing with these loop groups

Coming back to our original problem, a short inspection reveals also that $\omega_\lambda = \lambda^{-1}\omega_1' + \omega_0 + \lambda\omega_1''$ can be viewed as a 1-form with coefficients in $L\mathfrak{g}_\tau$. Thus we shall consider an extended harmonic lift $F_\circ$ as a map into $L\mathfrak{G}_\tau$. Working a little more, we can characterize all extended harmonic lifts $F_\circ$ as being all mappings into $L\mathfrak{G}_\tau$ such that the Fourier expansion of $F_\lambda^{-1}.dF_\lambda$ in function of $\lambda$ has the form

$$F_\lambda^{-1}.dF_\lambda = \sum_{k=-1}^{\infty} \hat{\omega}_k \lambda^k,$$

where $\hat{\omega}_{-1}$ is a (1,0)-form, i.e. $\hat{\omega}_{-1}(\frac{\partial}{\partial \bar{z}}) = 0$.

Now, this framework together with decomposition results for loop groups, as shown in the book of A. Pressley and G. Segal [PS] allow to establish a correspondance between such extended harmonic lifts and holomorphic or meromorphic datas.

### From holomorphic datas to extended harmonic lifts

It is obtained from an infinite dimensional analog of the Iwasawa decomposition

$$\begin{pmatrix} f_1^1 & f_2^1 & \dots & f_n^1 \\ f_1^2 & f_2^2 & \dots & f_n^2 \\ \vdots & \vdots & & \vdots \\ f_1^n & f_2^n & \dots & f_n^n \end{pmatrix} = \begin{pmatrix} e_1^1 & e_2^1 & \dots & e_n^1 \\ e_1^2 & e_2^2 & \dots & e_n^2 \\ \vdots & \vdots & & \vdots \\ e_1^n & e_2^n & \dots & e_n^n \end{pmatrix} .. \begin{pmatrix} b_1^1 & 0 & \dots & 0 \\ b_1^2 & b_2^2 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ b_1^n & b_2^n & \dots & b_n^n \end{pmatrix},$$

where $f \in GL(n, \mathbb{C})$, $e \in U(n)$ and $b \in B$. This is the following

**Theorem 2 (PS)** *, [DPW] For any $g_\circ \in L\mathfrak{G}_\tau^{\mathbb{C}}$, $\exists ! \phi_\circ \in L\mathfrak{G}_\tau$, $\exists ! b_\circ \in L_B^+\mathfrak{G}_\tau^{\mathbb{C}}$, such that $g_\circ = \phi_\circ . b_\circ$. Moreover the mapping*

$$\begin{array}{ccc} L\mathfrak{G}_\tau \times L_B^+\mathfrak{G}_\tau^{\mathbb{C}} & \longrightarrow & L\mathfrak{G}_\tau^{\mathbb{C}} \\ (\phi_\circ, b_\circ) & \longmapsto & \phi_\circ . b_\circ \end{array}$$

*is a diffeomorphism. We write "$L\mathfrak{G}_\tau^{\mathbb{C}} = L\mathfrak{G}_\tau . L_B^+\mathfrak{G}_\tau^{\mathbb{C}}$" for meaning this property.*

This result is the key for the following recipe, producing extended harmonic lifts.

- Take a closed holomorphic 1-form with values into $\lambda^{-1}L^+\mathfrak{g}^{\mathbb{C}} \cap L\mathfrak{g}_\tau^{\mathbb{C}}$, of the form

$$\mu_\lambda = \sum_{k=-1}^{\infty} \hat{\mu}_k \lambda^k = \sum_{k=-1}^{\infty} \xi_k \lambda^k dz.$$

  $\mu_\lambda$ is closed means that $d\mu_\lambda = 0$ or $\frac{\partial \xi_k}{\partial \bar{z}} = 0$.

- Integrate this 1-form. Assuming that $\Omega$ is simply connected, we construct thus a map $g_\circ : \Omega \longrightarrow L\mathfrak{G}_\tau^{\mathbb{C}}$ such that $g_\circ(z_0) = \mathbb{1}$ and $dg_\circ = g_\circ.\mu_\circ$.
- Decompose $g_\circ$ into a "real part" and a remaining "imaginary part". This step uses Theorem 2 as follows: fix $z \in \Omega$ and then apply Theorem 2 to $g_\circ(z)$. $\exists! \phi_\circ(z) \in L\mathfrak{G}_\tau$, $\exists! b_\circ(z) \in L_B^+\mathfrak{G}_\tau^{\mathbb{C}}$, such that

$$g_\circ(z) = \phi_\circ(z).b_\circ(z).$$

- Then one can check that $\phi_\circ$ is automatically an extended harmonic lift.

**From extended harmonic lifts to meromorphic datas**

It is possible to prove a converse to the above algorithm, namely:

**Theorem 3** *Let $F_\circ : \Omega \longrightarrow L\mathfrak{G}_\tau$ be an extended harmonic lift. Then there exists a map $g_\circ$ from $\Omega$ to $L\mathfrak{G}_\tau^{\mathbb{C}}$ and a map $b_\circ$ from $\Omega$ to $L_B^+\mathfrak{G}_\tau^{\mathbb{C}}$ such that*

$$\forall z \in \Omega, g_\circ(z) = F_\circ(z).b_\circ(z)$$

*and*

$$g_\lambda^{-1}.dg_\lambda = \sum_{k=-1}^{\infty} \hat{\mu}_k \lambda^k = \sum_{k=-1}^{\infty} \xi_k \lambda^k dz$$

*is a holomorphic 1-form, called the* holomorphic potential.

In the above Theorem, $g_\circ$ and $b_\circ$ are not unique. One of the main result in [DPW] is that one could choose an unique $g_\circ$ such that

$$g_\lambda^{-1}.dg_\lambda = \hat{\mu}_{-1}\lambda^{-1} = \xi_{-1}\lambda^{-1}dz,$$

where $\xi_{-1}$ is meromorphic. This *meromorphic potential* is obtained by using the following decomposition theorem.

**Lemma 1 (PS)** *, [DPW] There exists an open subset $\mathcal{C}$ of $L\mathfrak{G}_\tau^{\mathbb{C}}$, called* big cell*, such that "$\mathcal{C} = L_\star^-\mathfrak{G}_\tau^{\mathbb{C}}.L^+\mathfrak{G}_\tau^{\mathbb{C}}$", meaning that the map*

$$\begin{array}{ccc} L_\star^-\mathfrak{G}_\tau^{\mathbb{C}} \times L^+\mathfrak{G}_\tau^{\mathbb{C}} & \longrightarrow & L\mathfrak{G}_\tau^{\mathbb{C}} \\ (g_\circ^-, g_\circ^+) & \longmapsto & g_\circ^-.g_\circ^+ \end{array}$$

*is a diffeomorphism into its image $\mathcal{C}$.*

The meromorphic potential is constructed as follows.

**Theorem 4 (DPW)** *Take some extended harmonic lift $F_\circ$.*

- *Outside isolated points in* $\Omega$, $F_\circ(z) \in \mathcal{C}$*. Thus* $\exists! g_\circ^-(z) \in L_\star^-\mathfrak{G}_\tau^\mathbb{C}$, $\exists! g_\circ^+(z) \in L^+\mathfrak{G}_\tau^\mathbb{C}$,

$$F_\circ(z) = g_\circ^-(z).g_\circ^+(z).$$

- $z \longmapsto g_\circ^-(z)$ *is a meromorphic map from* $\Omega$ *to* $L_\star^-\mathfrak{G}_\tau^\mathbb{C}$.
- $\left(g_\circ^-\right)^{-1}.dg_\circ^- := \mu_\circ$ *has the form*

$$\mu_\lambda = \xi_{-1}\lambda^{-1}dz$$

*and* $\xi_{-1} : \Omega \longmapsto \mathfrak{p} \otimes \mathbb{C}$ *is meromorphic.*

# 5 Willmore surfaces and loop groups

The question now is: is it possible to adapt the previous results to the study of Willmore surfaces, in particular, by exploiting the fact that the conformal Gauss map of a Willmore immersion is a conformal harmonic map into the de Sitter space $\mathbb{S}^{3,1}$? This manifold is diffeomorphic to the quotient of $SO(4,1)$ by $SO(3,1)$ (here we fix any space-like vector $v \in \mathbb{R}^{4,1}$ and we identify $SO(3,1)$ with the subgroup of $SO(4,1)$ which leaves $v$ invariant). One difficulty is that $SO(4,1)$ is not compact and thus some of the results in [PS] which were used in [DPW] are not known (in particular the decomposition $L\mathfrak{G}^\mathbb{C} = L\mathfrak{G}.L_B^+\mathfrak{G}^\mathbb{C}$). Another difficulty is that describing Willmore immersions by their conformal Gauss maps has some defects and degenerates at umbilic points as evocated previously.

The first difficulty will force our result to be local (i.e. we can construct only neighbourhoods of Willmore surfaces using holomorphic datas). I want to describe here how we can eliminate the second difficulty by avoiding the use of the conformal Gauss map.

An intermediate step is the following alternative description.

## 5.1 Harmonic maps into a Grassmannian

As a preliminary, we let $Gr_3(\mathbb{R}^{4,1})$ be the Grassmannian of all oriented 3-dimensional spacelike subspaces of $\mathbb{R}^{4,1}$. Let us be more precise. We set

$$B = \begin{pmatrix} 0 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and

$$SO(4,1) := \{g \in GL(5,\mathbb{R}) / \ {}^t g.B.g = B\}.$$

The connected component of $SO(4,1)$ is

$$SO_0(4,1) := \{g \in SO(4,1) / \ \det g = 1, g_0^0 > 0\}.$$

We let

$$S = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix},$$

and we define the automorphism of $SO(4,1)$

$$\tau(g) = S^{-1}.g.S.$$

Then $\mathfrak{K} := \{g \in SO_0(4,1)/\tau(g) = g\}$ is isomorphic to $SO(1,1) \times SO(3)$. Moreover, $Gr_3(\mathbb{R}^{4,1})$ is diffeomorphic to $SO_0(4,1)/\mathfrak{K}$.

Outside the umbilic set $\mathcal{U}$ of a surface $\mathcal{S}$ are defined two points: $m \in \mathcal{S}$ and its conformal dual $\hat{m}$. These points span two light-lines in $\mathbb{R}^{4,1}$ and the orthogonal subspace to these two lines is an oriented 3-dimensional spacelike subspace, an element of $Gr_3(\mathbb{R}^{4,1})$. Hence we get a map $Z$ from $\mathcal{S} \setminus \mathcal{U}$ into $Gr_3(\mathbb{R}^{4,1})$. We proved in [Hé 1] that $\mathcal{S} \setminus \mathcal{U}$ is a Willmore surface if and only if this map $Z$ is harmonic.

To formulate this property, let us consider two parametrization mappings $u, \hat{u} : \Omega \longrightarrow \mathbb{S}^3$ such that $u$ parametrizes one surface without umbilic points $\mathcal{S}$, $\hat{u}$ parametrizes its conformal dual surface $\hat{\mathcal{S}}$ and for any $z \in \Omega$, $u(z)$ and $\hat{u}(z)$ are in conformal duality. We can choose $u$ and $\hat{u}$ to be conformal maps. These two maps have the same conformal Gauss map $\gamma : \Omega \longrightarrow \mathbb{S}^{3,1}$. Then, as in [Br 1], we can construct a moving frame $e = (e_0, e_1, e_2, e_3, e_4)$ in $\mathbb{R}^{4,1}$ such that $\langle e_a, e_b \rangle = B_{ab}$, defined above. And $e_0(z)$ and $e_4(z)$ span half light-lines which represent respectively $u(z)$ and $\hat{u}(z)$ and $e_3(z) = \gamma(z)$.

Then $Z(z)$ is the 3-dimensional subspace spanned by $(e_1(z), e_2(z), e_3(z))$. If we define the Maurer-Cartan form of the moving frame $e(z)$, $\omega = (\omega_a^b)$, such that

$$de_a = e_b \omega_a^b,$$

then it satifies the structure equation $d\omega + \omega \wedge \omega = 0$. We decompose $\omega = \omega_0 + \omega_1$ where the coefficients of $\omega_0$ are in

$$\mathfrak{k} = \left\{ \begin{pmatrix} \xi_0^0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \xi_2^1 & -\xi_1^3 & 0 \\ 0 & -\xi_2^1 & 0 & \xi_3^2 & 0 \\ 0 & \xi_1^3 & -\xi_3^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\xi_0^0 \end{pmatrix} \in so(4,1) \right\},$$

the Lie algebra of $\mathfrak{K}$. And the coefficients of $\omega_1$ are in

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0 & \xi_1^0 & \xi_2^0 & \xi_3^0 & 0 \\ \xi_0^1 & 0 & 0 & 0 & \xi_1^0 \\ \xi_0^2 & 0 & 0 & 0 & \xi_2^0 \\ \xi_0^3 & 0 & 0 & 0 & \xi_3^0 \\ 0 & \xi_0^1 & \xi_0^2 & \xi_0^3 & 0 \end{pmatrix} \in so(4,1) \right\}$$

(remark that $so(4,1) = \mathfrak{k} \oplus \mathfrak{p}$). Notice that, since $\gamma = e_3$ is the conformal Gauss map of $u$ and $\hat{u}$, $\omega_0^3 = \omega_3^0 = 0$ and

$$\omega_1 = \begin{pmatrix} 0 & \omega_1^0 & \omega_2^0 & 0 & 0 \\ \omega_0^1 & 0 & 0 & 0 & \omega_1^0 \\ \omega_0^2 & 0 & 0 & 0 & \omega_2^0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \omega_0^1 & \omega_0^2 & 0 & 0 \end{pmatrix}.$$

We further decompose $\omega_1 = \omega_1' + \omega_1''$, where $\omega_1' = \omega_1(\frac{\partial}{\partial z})dz$ and $\omega_1'' = \omega_1(\frac{\partial}{\partial \overline{z}})d\overline{z}$. Then the 3 following properties are equivalent

1. $u$ and $\hat{u}$ are conformal Willmore immersions
2. $Z$ is harmonic
3. $\forall \lambda \in \mathbb{C}^\star$,
$$d\omega_\lambda = \omega_\lambda \wedge \omega_\lambda = 0,$$
where $\omega_\lambda = \lambda^{-1}\omega_1' + \omega_0 + \lambda\omega_1''$.

The disadvantage of that formulation is, as before, that this still does not work at umbilic points of $u$. But there exists a way to deform $Z$ into a map $Z'$ with values in $Gr_3(\mathbb{R}^{4,1})$ defined even at umbilic points of $u$.

### 5.2 Roughly harmonic maps

We replace the pair of maps $u$ and $\hat{u}$ in conformal duality by a pair $u$, $v$, where $v$ is now any map into $\mathbb{S}^3$ such that $\forall z \in \Omega, v(z) \in \mathbb{S}^2_\gamma \setminus \{u(z)\}$. Given $u$, there is a large freedom in the choice of $v$: for any $z$, since $\mathbb{S}^2_\gamma \setminus \{u(z)\}$ is equivalent through the stereographic projection to $\mathbb{R}^2$, the set of all possible values of $v(z)$ is diffeomorphic to $\mathbb{R}^2$. We say that the additive gauge group $\mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ acts transitively on the set of all such $(u,v)$'s.

We define a map $Z : \Omega \longrightarrow Gr_3(\mathbb{R}^{4,1})$ which associates to each $z$ the 3-dimensional space-like subspace orthogonal to the two light-lines spanned by $u(z)$ and $v(z)$. The point is that we can characterize conformal Willmore immersions by using such a construction.

As before we consider a framing $e = (e_0, e_1, e_2, e_3, e_4)$, with $e_0$, $e_4$ and $e_3$ describing $u$, $v$ and $\gamma$ respectively. And $Z$ is spanned by $(e_1, e_2, e_3)$. This map is not anymore harmonic in general. Let $\omega$ be the Maurer-Cartan form of this moving frame and let us decompose $\omega = \omega_0 + \omega_1$, according to $so(4,1) = \mathfrak{k} \oplus \mathfrak{p}$ as before. We now decompose $\omega_1 = \omega_1' + \omega_1''$ in a slightly different way, namely

$$\omega_1' = \omega_1(\frac{\partial}{\partial z})dz + i \begin{pmatrix} 0 & \phi & -\star\phi & 0 & 0 \\ 0 & 0 & 0 & 0 & \phi \\ 0 & 0 & 0 & 0 & -\star\phi \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} ; \omega_1'' = \overline{\omega_1'},$$

where $\phi = \frac{1}{2}(\omega_2^0 - \star\omega_1^0)$ and the Hodge operator $\star$ is defined by $\star\omega_0^1 = \omega_0^2$ and $\star\omega_0^2 = -\omega_0^1$. Here $\omega_1'$ is not a (1,0)-form in general. Lastly we denote $\omega_\lambda = \lambda^{-1}\omega_1' + \omega_0 + \lambda\omega_1''$.

**Theorem 5 (Hé 1)** *$u$ is a conformal Willmore immersion if and only if* $\forall \lambda \in \mathbb{C}^\star$,

$$d\omega_\lambda + \omega_\lambda \wedge \omega_\lambda = 0.$$

We say that $Z$ is *roughly harmonic*. We remark that there are infinitely ways for constructing $Z$ and hence for representing a conformal Willmore immersion. This is due to the gauge action of $\mathcal{C}^\infty(\Omega, \mathbb{R}^2)$ on the set of all roughly harmonic maps representing the same conformal Willmore immersion. Some of the $Z$'s, corresponding to certain gauge choices, are such that $\phi = 0$ (*harmonic gauge*) and then $\omega_1'$ is really a (1,0)-form and $Z$ is a harmonic map. For any given point in $\Omega$, it is always possible to construct in a neighbourhood of that point a harmonic gauge (and then, $Z$ is harmonic on this neighbourhood). This is done by solving a complex Ricatti equation of the type $\frac{\partial f}{\partial \overline{z}} = a(z,\overline{z}) + b(z,\overline{z})f + c(z,\overline{z})f^2$. But since Ricatti equations have no global smooth solutions in general, there is no way to construct a global harmonic gauge in general. Notice that a particular harmonic gauge is when $\phi = 0$ and $\omega_3^0 = 0$: it corresponds to the case $v = \hat{u}$ and this gauge exists outside umbilic points.

# 6 Results

We shall use the characterisation of conformal Willmore immersions in terms of roughly harmonic maps $Z$ and their *extended harmonic roughly lifts*. For that purpose we define

$LSO(4,1)_\tau = \{g_\circ \in LSO(4,1)/\tau(g_\lambda) = g_{-\lambda}\}$ as in section 4. Extended roughly harmonic lifts are constructed using Theorem 5: it is a map $F_\circ : \Omega \longrightarrow LSO(4,1)_\tau$ which is a solution of $F_\circ(z_0) = \mathbb{1}$ and $dF_\circ = F_\circ.\omega_\circ$, where $\omega_0$ is as in Theorem 5.

Now it is possible to apply part of the programm of [DPW] for extended roughly harmonic lifts. Notice first that Lemma 1 still holds for $\mathfrak{G} = SO(4,1)$, just because $SO(4,1)^{\mathbb{C}} = SO(5)^{\mathbb{C}}$, a complexification of a compact Lie group. Here is an analog of Theorem 4.

**Theorem 6 (Hé 1)** *Take an extended roughly harmonic lift $F_\circ$ (associated to a conformal Willmore immersion $u$), then*

- *Outside isolated points in $\Omega$, $F_\circ \in \mathcal{C}$. Thus $\exists! g_\circ^-(z) \in L_\star^- SO(4,1)_\tau^{\mathbb{C}}$, $\exists! g_\circ^+(z) \in L^+SO(4,1)_\tau^{\mathbb{C}}$,*

$$F_\circ(z) = g_\circ^-(z).g_\circ^+(z).$$

- *$(g_\circ^-)^{-1}.dg_\circ^- := \mu_\circ$ has the form*

$$\mu_\lambda = \hat{\mu}_{-1}\lambda^{-1}$$

*and*

$$\hat{\mu}_{-1} = \begin{pmatrix} 0 & m^1 & m^2 & m^3 & 0 \\ l^1 & 0 & 0 & 0 & m^1 \\ l^2 & 0 & 0 & 0 & m^2 \\ l^3 & 0 & 0 & 0 & m^3 \\ 0 & l^1 & l^2 & l^3 & 0 \end{pmatrix} dz + \gamma \begin{pmatrix} 0 & l^1 & l^2 & l^3 & 0 \\ 0 & 0 & 0 & 0 & l^1 \\ 0 & 0 & 0 & 0 & l^2 \\ 0 & 0 & 0 & 0 & l^3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} d\overline{z},$$

*where $l =^t (l^1, l^2, l^3)$, $m =^t (m^1, m^2, m^3) \in \mathbb{C}^3$ are such that $(l^1)^2 + (l^2)^2 + (l^3)^2 = l^1m^1 + l^2m^2 + l^3m^3 = 0$ and $d\hat{\mu}_{-1} = 0$.*

Notice that we do not get here a meromorphic potential but rather a potential determinaed by a closed 1-form $(ldz, mdz + \gamma l d\overline{z})$.

But if we are given some conformal Willmore immersion $u$, we have many roughly harmonic maps associated to $u$, which differ by the action of a gauge group $\mathcal{C}^\infty(\Omega, \mathbb{R}^2)$. Let $Z_1$, $Z_2$ be two such roughly harmonic maps and let $F_{\circ,1}$ and $F_{\circ,2}$ be their extended roughly harmonic lifts respectively. Then the potential $\mu_{\circ,1}$ for $F_{\circ,1}$ and the potential $\mu_{\circ,1}$ for $F_{\circ,2}$ are related by

$$\mu_{\circ,2} = \mu_{\circ,1} + d\left[\delta\begin{pmatrix} 0 & l^1 & l^2 & l^3 & 0 \\ 0 & 0 & 0 & 0 & l^1 \\ 0 & 0 & 0 & 0 & l^2 \\ 0 & 0 & 0 & 0 & l^3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}\right],$$

where $\delta : \Omega \longrightarrow \mathbb{C}$.

Hence the potential for a given conformal Willmore immersion is rather an equivalence class of closed forms.

Now, for a choice of $Z$ corresponding to a harmonic gauge, $Z$ is really a harmonic map into $Gr_3(\mathbb{R}^{4,1})$. The corresponding potential is then a meromorphic form

$$\mu_\lambda = \lambda^{-1}\begin{pmatrix} 0 & m^1 & m^2 & m^3 & 0 \\ l^1 & 0 & 0 & 0 & m^1 \\ l^2 & 0 & 0 & 0 & m^2 \\ l^3 & 0 & 0 & 0 & m^3 \\ 0 & l^1 & l^2 & l^3 & 0 \end{pmatrix} dz,$$

such that $(l^1)^2 + (l^2)^2 + (l^3)^2 = l^1m^1 + l^2m^2 + l^3m^3 = 0$, $\frac{\partial l}{\partial \bar{z}} = \frac{\partial m}{\partial \bar{z}} = 0$.

Among the harmonic gauges, one corresponds to choose $v = \hat{u}$, working outside the umbilic points. Then the corresponding potential is

$$\mu_\lambda = \lambda^{-1}\begin{pmatrix} 0 & \nu l^1 & \nu l^2 & \nu l^3 & 0 \\ l^1 & 0 & 0 & 0 & \nu l^1 \\ l^2 & 0 & 0 & 0 & \nu l^2 \\ l^3 & 0 & 0 & 0 & \nu l^3 \\ 0 & l^1 & l^2 & l^3 & 0 \end{pmatrix} dz,$$

such that $(l^1)^2 + (l^2)^2 + (l^3)^2 = 0$, $\frac{\partial l}{\partial \bar{z}} = \frac{\partial \nu l}{\partial \bar{z}} = 0$.

Lastly it is possible to identify the special cases where $\nu$ is a real constant. Namely

- if $\nu < 0$, the surface is a minimal surface of $\mathbb{S}^3$.
- if $\nu = 0$, the surface is a minimal surface of $\mathbb{R}^3$.
- if $\nu > 0$, the surface is a minimal surface of $\mathbb{H}^3$.

Moreover, in the case where $\nu = 0$, $ldz$ is exactly the classical Weierstrass data for the corresponding minimal surface in $\mathbb{R}^3$.

It should be interesting to obtain a geometric characterisation of Willmore surfaces such that $\nu$ is a complex, non real constant.

Conversely, we can obtain a partial converse of these result, namely constructing conformal Willmore immersions from potentials, analogously to the algorithm of [DPW] using Theorem 2. However I was not able to obtain a global version of these results, but only a local one, since we do not know whether the analog of Theorem 2 works for non compact Lie groups such as $SO(4,1)$. This question is the subject of the work in [K].

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F.H., CMLA, ENS de Cachan, 61, avenue du Président Wilson, 94235 Cachan Cedex, France
helein@cmla.ens-cachan.fr