Constant mean curvature surfaces, harmonic maps and integrable systems

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February 5, 2001

Notes written by Roger Moser after a Nachdiplomsvorlesung at the Eidgenössische Technische Hochschule Zürich in Spring 1999

Preface

One of the most striking development of the last decades in the study of minimal surfaces, constant mean surfaces and harmonic maps is the discovery that many classical problems in differential geometry - including these examples - are actually integrable systems.

This theory grew up mainly after the important discovery of the properties of the Korteweg-de Vries equation in the sixties. After C. Gardner, J. Greene, M. Kruskal et R. Miura [44] showed that this equation could be solved using the inverse scattering method and P. Lax [62] reinterpreted this method by his famous equation, many other deep observations have been made during the seventies, mainly by the Russian and the Japanese schools. In particular this theory was shown to be strongly connected with methods from algebraic geometry (S. Novikov, V.B. Matveev, I.M. Krichever...), loop techniques (M. Adler, B. Kostant, W.W. Symes, M.J. Ablowitz...) and Grassmannian manifolds in Hilbert spaces (M. Sato...). Approximatively during the same period, the twistor theory of R. Penrose, built independentely, was applied successfully by R. Penrose and R.S. Ward for constructing self-dual Yang-Mills connections and four-dimensional self-dual manifolds using complex geometry methods. Then in the eighties it became clear that all these methods share the same roots and that other instances of integrable systems should exist, in particular in differential geometry. This led K. Uhlenbeck [82] to describe harmonic maps on a two-sphere, with values in $\mathbb{C}P^n$ using families of curvature free connections depending on a complex "spectral" parameter. At the same period N. Hitchin [54] investigated finite type tori into $SU(2)$ starting from similar methods. Such formulations were already proposed in the seventies by K. Pohlmeyer [69], V.E. Zhakarov - A.V. Mikhailov [90] and V.E. Zhakarov - A.B. Shabat [91]. A catalysor of these developments was the construction by H. Wente in 1984 of an immersed constant mean curvature torus in $\mathbb{R}^3$, which had the effect of removing an old inhibition due to the fact that people believed that such tori should not exist. Many results then followed quickly and we have now a very rich and fruitful theory for constructing constant mean curvature surfaces and harmonic maps of surfaces with values into symmetric manifolds using integrable systems methods.

All that seems apparentely a new theory, but many features of the "completely integrable behaviour" of constant mean surfaces have been guessed by geometers of the nineteenth century. Namely the existence of associated families of such surfaces by O. Bonnet, the study of special surfaces with planar curvature lines by A. Enneper and his students and the various Bäcklund transformations discovered and studied by A.V. Bäcklund, L. Bianchi, S. Lie, G. Darboux, E. Goursat and J. Clairin (see [71] for details and references).

This Monograph is intended to give an introduction to this old and new theory from the point of view of differential geometry. For that reason, it has seemed more natural for me to introduce the existence of families of curvature free connections for harmonic maps starting from the associated family of constant mean surfaces. Note however that this is not the historical way the theory actually developed and contemporary people were initially more inspired by the example of the KdV equation and the twistor theory. We also presented here some basic exposition of the twistor theory for harmonic maps, which was initiated to my knowledge by E. Calabi. Indeed this theory was an important stimulation for the integrable system theory, in particular in the work of K. Uhlenbeck, and shares some similarities with the integrable systems theory (for a complete exposition, see [23]). I also made an effort to present the beautiful result of U. Pinkall and I. Sterling concerning constant mean curvature tori (Chapter 9) in the framework of loop groups theory, in order to show how this result connects with the rest of theory, which uses loop groups.

The present text is just an introduction and is far from being complete. We may recommand as parallel lectures the books [46] and [43] plus of course... reading the cited papers.

These Notes come from a lecture that I gave at the Eidgenössische Technische Hochschule Zürich during Spring 1999. Most parts of the text were written and typed by R. Moser. I wish here to thank the ETH Zürich and more particularly Prof. M. Struwe for his hospitality and to thank R. Moser for his very nice work.

I also want to thank Joseph Dorfmeister and Pascal Romon for their valuable comments on this text.

1 Introduction: Surfaces with prescribed mean curvature

Curvature

For a curve Γ in a plane, and for any point on this curve, there exists a circle (or a straight line) which is the best approximation at this point of the curve up to third order. The inverse of its radius $k = \frac{1}{R}$ (or $k = 0$ if the best approximation is a straight line) is called the curvature of the curve at the given point.

For a surface Σ in $\mathbb{R}^3$ and a point $m \in \Sigma$, we consider the 1-parameter family of affine planes which contain the straight line passing through m and being normal to Σ at m. Each of these planes locally intersects the surface along a planar curve containing m. We may label these planes by choosing one, say P_0, and for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ naming by P_θ the image of P_0 by a rotation of angle θ around the normal line to Σ at m. Then the curvature $k(\theta)$ of $P_\theta \cap \Sigma$ at m is of the form $\frac{k_1+k_2}{2} + \frac{k_1-k_2}{2}\cos(2(\theta-\theta_0))$, where k_1 and k_2 are two numbers, called the principal curvatures of Σ at m. They are the extremal curvatures, achieved at two extremal positions of P_θ which are orthogonal (the principal directions). The average $H = \frac{1}{2}(k_1 + k_2)$ of k_1 and k_2 is called the mean curvature of Σ at m, their product $K = k_1 k_2$ the Gauss curvature.

If $k_1 = k_2$, i. e. if the curvature $k(\theta)$ is a constant, then the point m is called umbilic.

Experiments with soap

We are going to describe some experiments, carried out by J. Plateau, which may give some physical motivation for what is to follow.

Imagine a piece of wire, bent in the shape of a closed curve Γ, that is dipped into a solution of water and soap, such that it becomes spanned with a soap film. This film will take the form of a surface Σ_0 with boundary $\partial\Sigma_0 = \Gamma$, having least area among all surfaces with the same boundary. Computing the Euler-Lagrange equation of this variation problem, we see that Σ_0 satisfies

$$H = 0, \tag{1}$$

where H is the mean curvature of Σ_0 as described above. Such a surface is called a *minimal surface.*

Suppose now that there is some device that allows to have different pressure on either side of the soap film in question. Then instead of (1), the equation

$$H = C \tag{2}$$

will hold for Σ_0, where C is a constant different from 0 (depending on the difference of the pressures applied). In this case we call Σ_0 a *constant mean curvature* (CMC) *surface.*

First and second fundamental form

Let Σ be a piece of a surface, embedded in $\mathbb{R}^3$, which is diffeomorphic to the unit disk $D^2 = \{z = x + iy \in \mathbb{C} \colon\ |z| < 1\}$. Let $X \colon D^2 \to \mathbb{R}^3$ be a parametrization of Σ. Then we can define the following.

Definition 1.1 *The first fundamental form of an embedding $X\colon D^2 \to \mathbb{R}^3$ is the quadratic form given by the matrix*

$$\mathrm{I} = \begin{pmatrix} \left|\frac{\partial X}{\partial x}\right|^2 & \left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y} \right\rangle \\ \left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y} \right\rangle & \left|\frac{\partial X}{\partial y}\right|^2 \end{pmatrix},$$

which depends on $z \in D^2$.

For $\xi = (\xi_1, \xi_2) \in \mathbb{R}^2$, we compute ${}^t\xi\,\mathrm{I}\,\xi = |dX(\xi)|^2$. Now fix an orientation of Σ by assuming that $(\frac{\partial X}{\partial x}, \frac{\partial X}{\partial y})$ is an oriented basis of $T_{X(z)}\Sigma$. Consider

$$u = \frac{\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}}{\left|\frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}\right|},$$

where $\times$ denotes the vector product in $\mathbb{R}^3$. The map $u\colon D^2 \to S^2$ thus given is called the Gauss map of X.

Definition 1.2 *The second fundamental form of X is the quadratic form given by*

$$\mathrm{II} = \begin{pmatrix} \left\langle \frac{\partial^2 X}{\partial x^2}, u \right\rangle & \left\langle \frac{\partial^2 X}{\partial x \partial y}, u \right\rangle \\ \left\langle \frac{\partial^2 X}{\partial x \partial y}, u \right\rangle & \left\langle \frac{\partial^2 X}{\partial y^2}, u \right\rangle \end{pmatrix}.$$

With the notions of the first and second fundamental form, we are able to define the principal curvatures of a surface in a more convenient way than before.

Definition 1.3 *The principal curvatures k_1, k_2 are the eigenvalues of* II *with respect to* I*, i. e. the solutions of*

$$\det(\mathrm{II} - \lambda \mathrm{I}) = 0.$$

By a multiplication of $\mathrm{II} - \lambda\mathrm{I}$ with I^{-1} from the right it is easily verified that the mean curvature and the Gauss curvature can be computed from I and II in the following way:

$$H = \frac{1}{2}\mathrm{tr}(\mathrm{II} \cdot \mathrm{I}^{-1}), \quad K = \det(\mathrm{II})/\det(\mathrm{I}). \tag{3}$$

Conformal coordinates

To simplify things, we will use conformal coordinates. Choose Σ like above. The following is a well-known result.

Theorem 1.1 *There exists an embedding map $X\colon D^2 \to \mathbb{R}^3$, such that*

i) $X(D^2) = \Sigma$, and

ii) X is conformal, i. e.

$$\left|\frac{\partial X}{\partial x}\right|^2 - \left|\frac{\partial X}{\partial y}\right|^2 = \left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y} \right\rangle = 0. \tag{4}$$

Note that (4) means that the vectors $\frac{\partial X}{\partial x}$ and $\frac{\partial X}{\partial y}$ have at each point the same length and are perpendicular to each other. This implies that there is a function $\omega\colon D^2 \to \mathbb{R}$, such that

$$\frac{\partial X}{\partial x} = e^{\omega} e_1, \quad \frac{\partial X}{\partial y} = e^{\omega} e_2,$$

and (e_1, e_2) establish an orthonormal basis of $T_{X(z)}\Sigma$ for each $z \in D^2$. Thus we may write

$$\mathrm{I} = e^{2\omega} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathrm{II} = e^{2\omega} \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix}.$$

Then k_1 and k_2 are the solutions of

$$\det \left(\begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) = 0,$$

i. e. the eigenvalues of the matrix

$$\begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix}.$$

Consequently,

$$H = \frac{h_{11} + h_{22}}{2}, \quad K = \det \begin{pmatrix} h_{11} & h_{21} \\ h_{12} & h_{22} \end{pmatrix}.$$

We see that, for conformal X,

$$2H = h_{11} + h_{22} = e^{-2\omega} \langle \Delta X, u \rangle,$$

or

$$\langle \Delta X, u \rangle = 2e^{2\omega} H.$$

Moreover, we have the following.

Lemma 1.1 *If X is conformal, then*

$$\Delta X \perp \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}.$$

This can easily be verified by differentiating (4). The details are left to the reader. It allows, however, the following conclusions.

Corollary 1.1 *The map X satisfies*

$$\Delta X = \langle \Delta X, u \rangle u = 2He^{2\omega} u = 2H \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y},$$

as soon as it is conformal.

Apply this to the equation (2) in the form

$$H = C = H_0.$$

We see that it is equivalent to

$$\Delta X = 2H_0 \frac{\partial X}{\partial x} \times \frac{\partial X}{\partial y}.$$

In particular this implies that minimal surfaces are always images of harmonic and conformal parameterizations. This leads to the problem of finding maps $X\colon D^2 \to \mathbb{R}^3$, such that

- X is conformal, i. e.

$$0 = \left|\frac{\partial X}{\partial x}\right|^2 - \left|\frac{\partial X}{\partial y}\right|^2 - 2i\left\langle \frac{\partial X}{\partial x}, \frac{\partial X}{\partial y}\right\rangle,$$

- $\Delta X = 0$.

Weierstrass representation

Using the notation of complex differentiation,

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right), \quad \frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \quad \Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}},$$

we find that our problem can be expressed in the form

$$\begin{cases} \left(\dfrac{\partial X}{\partial z}\right)^2 & = & 0, \\ \dfrac{\partial}{\partial \bar{z}}\left(\dfrac{\partial X}{\partial z}\right) & = & 0. \end{cases}$$

Write $f = 2\frac{\partial X}{\partial z} : D^2 \to \mathbb{C}^3$. We want to solve

$$f^2 = 0, \quad \frac{\partial f}{\partial \bar{z}} = 0.$$

This can be done explicitly by the representation

$$f = \begin{pmatrix} \frac{i}{2}(w^2 - 1) \\ \frac{1}{2}(w^2 + 1) \\ iw \end{pmatrix} h,$$

where $w, h : D^2 \to \mathbb{C}$ are holomorphic functions. (The function w might be meromorphic.) Eventually, this gives us Weierstrass representation of a solution of our problem:

$$X(z) = \mathrm{Re}\left[\int_{z_0}^{z} f(\zeta)\, d\zeta\right],$$

where z_0 is an arbitrarily chosen point in D^2.

Completely integrable systems

The Weierstrass representation for minimal surfaces seems to be a miracle, since it describes all solutions of a nonlinear geometrical problem by a very simple algebraic construction. We shall see in these lectures more sophisticated miracles, occuring in various geometrical situations. People called them *completely integrable systems*. Classically, they are nonlinear equations on which mathematicians and physicists discovered unusual properties. Some of these properties are

- **existence of solitons.** This terminology comes from evolution problems and the most famous example is the Korteweg-de Vries (or KdV) equation $\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$, modelling water waves in a flat shallow channel. The story started in 1844 with the experimental observation of a solitary wave travelling along such a channel along a very long distance by J.S. Russel [73]. A model equation was derived in 1895 by D.J. Korteweg and G. de Vries [59]. Solitons are solutions of nonlinear partial differential equations which are localised in space, whose profile is not dispersed after a long period of time and which resist to interactions with other solitons [1]. Thus it is a smooth field, solution of some partial differential equation, which behaves like a particle.

- **Bäcklund transformations.** A baby example is the following: start from a harmonic function f of two real variables x and y. We may write the Laplace equation for f as $d\left(-\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy\right) = 0$, which implies that, if we work on a simply connected domain, there exists a function g such that $-\frac{\partial f}{\partial y}dx + \frac{\partial f}{\partial x}dy = dg$. Then g is another harmonic function, namely the conjugate function of f (i.e. $f+ig$ is a holomorphic function of $x+iy$). Such transformations, producing a solution of some partial differential equation starting from another solution work also in nonlinear situations.

- *by a nonlinear change of variable, the problem reduces to solving linear equations.* Weierstrass representation does it obviously, reducing the minimal surface equation to the Cauchy-Riemann system. We shall meet the same situation, but involving a much more complicated "change of variable".

- **a Hamiltonian structure**. Solutions are spanned by the Hamiltonian flows of commuting functions on a symplectic (or Poisson) manifold. This is described by Liouville's theorem.

- **Infinitely many symmetries**. Infinite dimensional Lie groups acts on the set of solutions of these systems.

The theory of completely integrable systems is not a pragmatic point of view, in the sense that it does not really provide a method (like the analytic approach based on functional analysis) where, starting from some qualitative intuition, one works (often hardly) to construct the tools one needs for proving (or disproving) what one believe to be true (existence, regularity,...). Indeed working on completely integrable systems is rather based on a contemplation of some very exceptional equations which hide a Platonic structure: although these equations do not look trivial a priori, we shall discover that they are elementary, once we understand how they are encoded in the language of symplectic geometry, Lie groups and algebraic geometry. It will turn out that this contemplation is fruitful and leads to many results.

[1] for the KdV equation, the simplest solution involving the soliton behaviour is the one-soliton $u(x,t) = 2a^2\text{sech}^2(a(x-4a^2t))$, for $a > 0$

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