Covariant Hamiltonian description of relativistic fields and the construction of observable quantities

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The question

- Find **first integrals** for solutions $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ of

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- Motivation: *the quantization of fields.* The space

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- Using each of these first integrals one defines *covariantly* an *observable* functional on $\mathcal{E}_W$

The linear Klein–Gordon equation

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- Fix $\varphi \in \mathcal{E}_0$ and $t \in \mathbb{R}$. We define a functional on the set of maps $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ by

$$I_t^\varphi[u]_t := \int_{\mathbb{R}^{n-1}} \left(u \frac{\partial \varphi}{\partial t} - \frac{\partial u}{\partial t} \varphi \right) |_{x^0=t}\ d\vec{x} = \int_t u \frac{\overleftrightarrow{\partial}}{\partial t} \varphi.$$

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- Claim: *if* $u \in \mathcal{E}_0$ *then* $I_t^\varphi[u]_t$ *is time independant* (this can be interpreted using Noether's theorem)

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- Claim: *if $u \in \mathcal{E}_0$ then $I_t^\varphi[u]_t$ is time independant* (this can be interpreted using Noether's theorem)
- However if $\varphi \in \mathcal{E}_0$ and $u \in \mathcal{E}_W$ with $W \neq 0$, then $I_t^\varphi[u]_t \neq I_s^\varphi[u]_s$ if $t \neq s$ in general.

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- Add a correction to $I_t^\varphi[u]_t$: $\mathcal{F}_{t,2}^\varphi[u]_t \otimes [u]_t :=$

$$-\int_t \int_t u(y_1)u(y_2)\overleftrightarrow{\partial_{y_1}}\overleftrightarrow{\partial_{y_2}} \int_0^t G(y_1-z)G(y_2-z) \int_0 G(z-x)\overleftrightarrow{\partial_x}\varphi(x),$$

where G is the solution of:
$\Box G + m^2 G = 0,\ G|_{x^0=0} = 0$ and $\frac{\partial G}{\partial x^0}|_{x^0=0} = \delta_0$.

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- For shortness we write:

$$\mathcal{F}_{t,2}^\varphi[u]_t \otimes [u]_t := -\int_t\int_t u_1 u_2 \overleftrightarrow{\partial_1}\overleftrightarrow{\partial_2}\int_0^t G_{1z}G_{2z}\int_0 G_{zx}\overleftrightarrow{\partial_x}\varphi$$

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- Then if we denote $\mathcal{F}^{\varphi}_{t,1}[u]_t := I^{\varphi}_t[u]_t$, the functional

$$\mathcal{F}^{\varphi}_{t,1}[u]_t + \mathcal{F}^{\varphi}_{t,2}[u]_t \otimes [u]_t$$

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- Idea: build a series

$$\mathcal{F}^{\varphi}_{t}[u]_t = \mathcal{F}^{\varphi}_{t,1}[u]_t^{\otimes 1} + \mathcal{F}^{\varphi}_{t,2}[u]_t^{\otimes 2} + \mathcal{F}^{\varphi}_{t,3}[u]_t^{\otimes 3} + \cdots$$

to obtain a conserved quantity.

The result by D. Harrivel (Annales IHP, 2006)

- For any $\varphi \in \mathcal{E}_0$, there exists a family (depending on t) of formal series $\mathcal{F}^{\varphi}_t[u]_t$ such that $\mathcal{F}^{\varphi}_0[u]_0 = \mathcal{F}^{\varphi}_{0,1}[u]_0$ (coincidence at time 0) and $\mathcal{F}^{\varphi}_t[u]_t = \mathcal{F}^{\varphi}_0[u]_0$ (conservation), i.e.

$$\forall t, \quad \sum_{p=1}^{\infty} \mathcal{F}^{\varphi}_{t,p}[u]_t^{\otimes p} = \mathcal{F}^{\varphi}_{0,1}[u]_0 = I^{\varphi}_0[u]_0,$$

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  for any solution u of $\Box u + m^2 u + u^2 = 0$.

- If $s > (n-1)/2$,

$$\varphi \in \mathcal{C}^0([0,T], H^{-s}(\mathbb{R}^{n-1})) \cap \mathcal{C}^1([0,T], H^{-s-1}(\mathbb{R}^{n-1})),$$
$$u \in \mathcal{C}^0([0,T], H^{s+2}(\mathbb{R}^{n-1})) \cap \mathcal{C}^1([0,T], H^{s+1}(\mathbb{R}^{n-1})) \text{ and}$$
$$T, \ ||u|| \text{ are small enough,}$$

  then the series $\mathcal{F}_t^\varphi[u]_t$ converges for $t \in [0,T]$.

Construction of the series

- Each term $\mathcal{F}^{\varphi}_{t,p}[u]_t^{\otimes p}$ is the sum over binary trees of multiple integrals built by using adapted Feynman rules from the tree.

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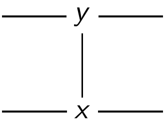

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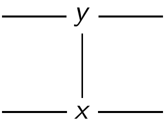
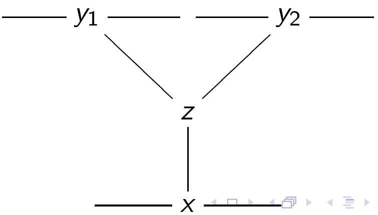


- and the second term is:

$$\mathcal{F}^{\varphi}_{t,2}[u]^{\otimes 2}_{t} = -$$



The general case

A classical analogue of Fock spaces

We consider a formal algebra $\mathcal{A}$ spanned by the symbols $(\phi_\triangleright(x), \phi_\triangleleft(y))_{x,y\in\mathbb{R}^n}$ and acting on two vector spaces $\mathbb{F}$ and $\mathbb{F}^*$ in duality, such that

- $\forall x, y \in \mathbb{R}^n$, $[\phi_\triangleleft(y), \phi_\triangleright(x)] = G(y - x)$;

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- $\forall x\in\mathbb{R}^n$, $\langle 0|\phi_\triangleright(x) = 0$.

A generating function...

- For any $|f\rangle \in \mathbb{F}$, $t \in \mathbb{R}$, set

$$\mathcal{F}_t[u]_t := \langle [u]_t | T\exp\left(-\int_0^t W(\phi_\triangleright(z), d\phi_\triangleright(z))\phi_\triangleleft(z) \right) |f\rangle,$$

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- and an example for $|f\rangle$ is:

$$|f\rangle = |f_\varphi\rangle := \int_{x^0=0} \phi_\triangleright(x) \overleftrightarrow{\partial_x} \varphi(x) d\vec{x}\ |0\rangle,$$

for some function φ. Then $\mathcal{F}_0[u]_0 = I_0^\varphi[u]_0$.

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Theorem (D. Harrivel, F.H., arXiv:0704.2674)

- If $u \in \mathcal{E}_W$, i.e. $\Box u + m^2 u + W(u, du) = 0$, then $\mathcal{F}_t[u]_t$ is *formally* a time independant functional of u.

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- If $s > (n-1)/2$,

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then the series $\mathcal{F}_t^{\varphi}[u]_t$ converges for $t \in [0,T]$ (and is really conserved).

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- If $\varphi|_{t=0} = 0$ and $\frac{\partial \varphi}{\partial t}|_{t=0} = \delta_{\vec{x}}$, then $I_0^{\varphi}[u] = u(0, \vec{x})$;

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  for $x = (0,\vec{x})$.

How to do it ?

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- Remark: then actually $\langle [u]_t|$ is the Dirac mass at the only map $\varphi_{[u]_t} \in \mathcal{E}_0$ which has the same Cauchy data as u at time t.

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- The 'creation' operators $\phi_\triangleright(x)$ and the 'annihilation' operators $\phi_\triangleleft(y)$ cannot be defined simultaneously;
  - building $\phi_\triangleright(x)$ requires at least that the fields in $\mathcal{E}_0$ should be continuous on $\mathbb{R}^n$
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- then the $\phi_\triangleright(x)$'s are well defined and the $\phi_\triangleleft(y)$'s are not defined but

$$\int_t u \overleftrightarrow{\partial} \phi_\triangleleft, \quad -\int_t (\Box u + m^2 u)\phi_\triangleleft, \quad \int_t W(\phi_\triangleright, d\phi_\triangleright)\phi_\triangleleft$$

are well defined.

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where $\mathbb{F}_r^{(k)}$ is the space of real analytic functionals on the ball of radius r in $\mathcal{E}_0$ which are bounded in a ‘$\mathcal{C}^k$ topology’;

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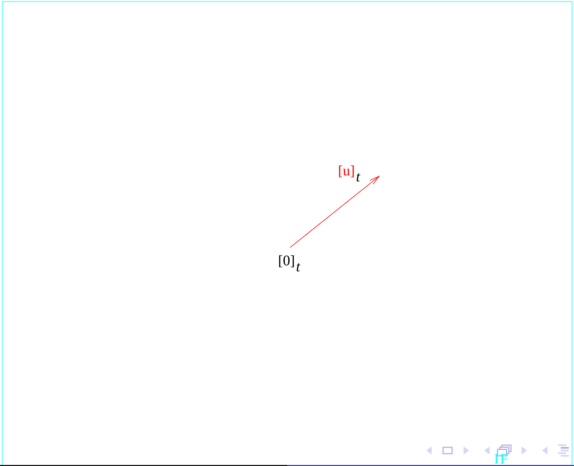
- As a consequence the (time ordered) exponential of an annihilation operator is bounded from $\mathbb{F}_r$ to some $\mathbb{F}_{e^{-tX}(r)}$, where $X(r) := ||\mathbb{D}||_{\mathbb{F}_r^{(1)}\to\mathbb{F}_r}\frac{d}{dr}$.

Structure of the proof I

$\langle 0|$

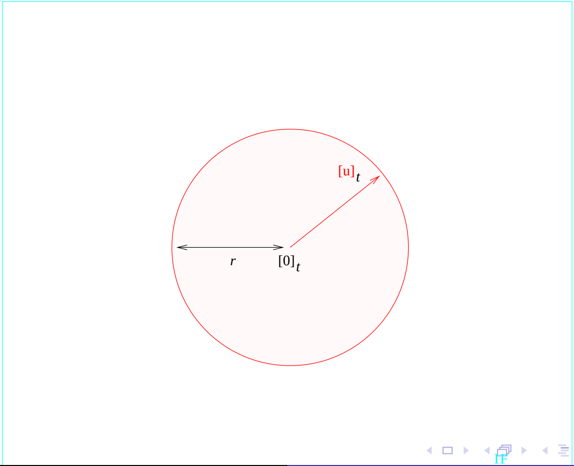
Structure of the proof II

$$\langle 0|\exp\left(\int_t u(y)\overleftrightarrow{\partial_y}\phi_\triangleleft(y)\right)$$



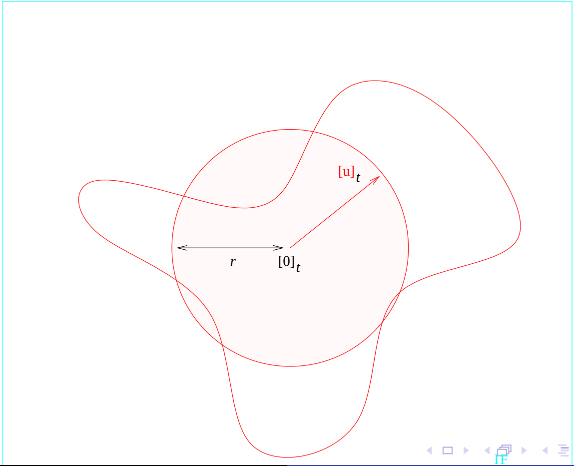
Structure of the proof III

$$\langle 0|\exp\left(\int_t u(y)\overleftrightarrow{\partial_y}\phi_\triangleleft(y)\right)$$



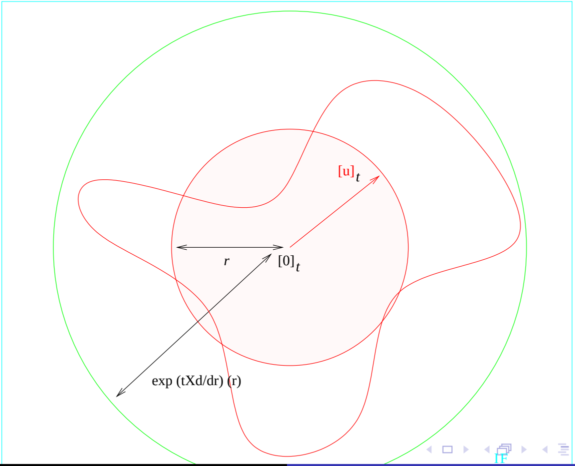
Structure of the proof IV

$$\langle 0|\exp\left(\int_t u(y)\overleftrightarrow{\partial_y}\phi_\triangleleft(y)\right) T\exp\left(-\int_0^t W(\phi_\triangleright(z), d\phi_\triangleright(z))\phi_\triangleleft(z)\right)$$



Structure of the proof V

$$\langle 0|\exp\left(\int_t u(y)\overleftrightarrow{\partial_y}\phi_\triangleleft(y)\right) T\exp\left(-\int_0^t W(\phi_\triangleright(z), d\phi_\triangleright(z))\phi_\triangleleft(z)\right)|f\rangle$$



Where does it come from ?

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- This can be done by using the **multisymplectic formalism**...

Multisymplectic manifolds

An *n-multisymplectic manifold* is a manifold $\mathcal{M}$ endowed with an $(n+1)$-form ω which satisfies:

- ω is *closed*, i.e. $d\omega = 0$

(Remark: in general the dimension of $\mathcal{M}$ can be anything.)

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- ω is *non degenerate*, i.e. for any tangent vector field ξ on $\mathcal{M}$, $\xi \,\lrcorner\, \omega = 0$ if and only if $\xi = 0$

(Remark: in general the dimension of $\mathcal{M}$ can be anything.)

An example is:

$\mathcal{M} = \Lambda^n T^*(\mathbb{R}^n \times \mathbb{R})$ with coordinates:

- $x = (x^0, x^1, \cdots, x^{n-1})$ on $\mathbb{R}^n$,

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- y on $\mathbb{R}$,
- let $\beta := dx^0 \wedge \cdots \wedge dx^{n-1}$ and $\beta_\mu := \frac{\partial}{\partial x^\mu} \lrcorner \beta$, then a basis of $\Lambda^n T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$ is $(\beta, dy \wedge \beta_0, \cdots, dy \wedge \beta_{n-1})$ and we denote the coordinates on $\Lambda^n T_{(x,y)}(\mathbb{R}^n \times \mathbb{R})$ in this basis by $(e, p^0, \cdots, p^{n-1})$

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- then the multisymplectic form reads

$$\omega = de \wedge \beta + dp^0 \wedge dy \wedge \beta_0 + \cdots + dp^{n-1} \wedge dy \wedge \beta_{n-1}.$$

The dynamics

A classical solution of the fields equation is pictured by an n-dimensional submanifold $\Gamma \subset \mathcal{M}$ which is a solution of the *Hamilton equations*:

- $$\Lambda^n T\Gamma \lrcorner \omega = 0 \text{ mod } d\mathcal{H},$$

  where $\mathcal{H} : \mathcal{M} \longrightarrow \mathbb{R}$ is a given function ('Hamiltonian'). We call a **Hamiltonian *n*-curve** any solution of this equation and we denote by $\mathcal{E}$ the set of all Hamiltonian n-curves.

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- For instance in $\mathcal{M} = \Lambda^n T^*(\mathbb{R}^n \times \mathbb{R})$ the Hamilton equations with

  $$\mathcal{H}(x, y, e, p) = e + \frac{1}{2}\eta_{\mu\nu}p^\mu p^\nu + \frac{1}{2}m^2 y^2 + V(u)$$

  correspond to $\Box u + m^2 u + V'(u) = 0$.

Observable functionals I

An observable functional is built from

- an *observable* $(n-1)$*-form* which is a $(n-1)$-form F on $\mathcal{M}$ such that there exists a tangent vector field ξ_F such that

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- Then one set

$$\begin{array}{rccc} \int_\Sigma F : & \mathcal{E} & \longrightarrow & \mathbb{R} \\ & \Gamma & \longmapsto & \int_{\Gamma \cap \Sigma} F \end{array}$$

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Furthermore we can define a 'Poisson' bracket between two $(n-1)$-forms F and G by:

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- is a Poisson bracket, which coincides with the standard Poisson bracket of fields theory.

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- (this relation expresses that the equation is invariant by ξ_F and the independance on Σ is then a version of Noether's theorem)

Back to the example

If $\mathcal{M} = \Lambda^n T^*(\mathbb{R}^n \times \mathbb{R})$ and $\mathcal{H} = e + \frac{1}{2}\eta_{\mu\nu}p^\mu p^\nu + \frac{1}{2}m^2 y^2 + V(u)$, there are **very few solutions** to the symmetry condition $\xi_F \lrcorner d\mathcal{H} = 0$ in general (only the ones corresponding to the Poincaré group)...

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- Then

$$(\text{for } \Sigma_0 := \{t = 0\}), \quad \int_{\Sigma_0} F_\varphi = \int_0 u\frac{\overleftrightarrow{\partial}}{\partial t}\varphi$$

Connection with our main result

(in progress)
If $V \neq 0$, we found the invariant functional:

$$\langle 0| \exp\left(\int_{\Sigma\cap\Gamma} F_\triangleleft\right) T\exp\left(-\int_{\Sigma_0}^{\Sigma} W_\triangleright \phi_\triangleleft\right) \int_{\Sigma_0\cap\Gamma_\triangleright} F_\varphi |0\rangle,$$

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- The corresponding 'symmetry' can be derived.

References on the use of series

Series expansions of solution to nonlinear ODE's:

- K.T. Chen (1957), Chen–Fliess (M. Fliess, 1981), W. Magnus (1954) (many applications to control theory)
- the Butcher series which explain the structure of Runge–Kutta methods: J.C. Butcher, E. Hairer and G. Wanner (1974). Relation with the Hopf algebra of D. Kreimer (1998) on the renormalization theory discovered by C. Brouder (2000).

Concerning nonlinear PDE's:

- known to physicists since J. Schwinger and R. Feynman
- few precise references: M. Dütsch, K. Fredenhagen (2003)
- first rigorous result (i.e. with a proof of convergence of the series) by D. Harrivel.

References on the multisymplectic formalism

Series expansions of solution to nonlinear ODE's:

- C. Carathéodory (1929), T. De Donder, H. Weyl (1935), T. Lepage (1936)
- P. Dedecker (1953)
- J. Kijowski, H. Goldschmidt–S. Sternberg, K. Gawędski (1972–73), J. Kijowski–W. Szczyrba (1976), J. Kijowski–W.M. Tulczyjew (1979), E. Binz–J. Śnyatycki–H. Fisher (1989)
- M.J. Gotay–J. Isenberg–J.E. Marsden–R. Montgomery–J. Śnyatycki–P.B. Yasskin (1998)
- I. V. Kanatchikov (1997-98), S. Hrabak (1999)
- F. Hélein–J. Kouneiher (2004)

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- Quantize...