JOURNAL OF MATHEMATICAL PHYSICS **49**, 023506 (2008)

A representation formula for maps on supermanifolds

Frédéric Hélein[a)]
Institut de Mathématiques de Jussieu, UMR 7586, Université Denis Diderot-Paris 7, Case 7012, 2 place Jussieu, 75251 Paris Cedex 5, France

(Received 17 September 2007; accepted 11 January 2008; published online 12 February 2008)

We analyze the notion of morphisms of rings of superfunctions which is the basic concept underlying the definition of supermanifolds as ringed spaces (i.e., following Berezin, Leites, Manin, etc.). We establish a representation formula for all (pull-back) morphisms from the algebra of functions on an ordinary manifolds to the superalgebra of functions on an open subset of a superspace. We then derive two consequences of this result. The first one is that we can integrate the data associated with a morphism in order to get a (nonunique) map defined on an ordinary space (and uniqueness can be achieved by restriction to a scheme). The second one is a simple and intuitive recipe to compute pull-back images of a function on a manifold $\mathcal{M}$ by a map from a superspace to $\mathcal{M}$. © *2008 American Institute of Physics.* [DOI: 10.1063/1.2840464]

I. INTRODUCTION

The theory of supermanifolds, first proposed by Salam and Strathdee[16] as a geometrical framework for understanding the supersymmetry, is now well understood mathematically and can be formulated in roughly two different ways: either by defining a notion of superdifferential structure with "supernumbers" which generalizes the differential structure of $\mathbb{R}^p$ and by gluing together these local models to build a supermanifold. This is the approach proposed by Dewitt[5] and Rogers.[14,10] Alternatively, one can define supermanifolds as ringed spaces, i.e., objects on which the algebra (or the sheaf) of functions is actually a superalgebra (or a sheaf of superalgebras). This point of view was adopted by Berezin,[4] Leĭtes,[11] and Manin[12] and was recently further developed[13] by Deligne and Morgan,[7] Freed,[9] and Varadarajan.[17] The first approach is influenced by differential geometry, whereas the second one is inspired by algebraic geometry. Of course, all these points of view are strongly related, but they may lead to some subtle differences (see Batchelor,[3] Bartocci *et al.*,[2] and Bahraini[1]). For a synthetic overview and a comparison of the various existing theories, see Ref. 15.

The starting point of this paper was to understand some implications of the theory of supermanifolds according to the second point of view,[4,11,12,8,9,17] i.e., the one inspired by algebraic geometry. The basic question is to understand $\mathbb{R}^{p|q}$, the space with p ordinary (bosonic) coordinates and q odd (fermionic) coordinates. There is no direct definition nor picture of such a space beside the fact that the algebra of functions on $\mathbb{R}^{p|q}$ should be isomorphic to $\mathcal{C}^\infty(\mathbb{R}^p)[\eta^1,\dots,\eta^q]$, i.e., the algebra over $\mathcal{C}^\infty(\mathbb{R}^p)$ spanned by q generators $\eta^1,\dots,\eta^q$ which satisfy the anticommutation relations $\eta^i\eta^j+\eta^j\eta^i=0$. Hence, $\mathcal{C}^\infty(\mathbb{R}^p)[\eta^1,\dots,\eta^q]$ is isomorphic to the set of sections of the flat vector bundle over $\mathbb{R}^p$ whose fiber is the exterior algebra $\Lambda^*\mathbb{R}^q$. To experiment further $\mathbb{R}^{p|q}$, we define what should be maps from open subsets of $\mathbb{R}^{p|q}$ to ordinary manifolds. We adopt the provisional definition of an *open subset* of Ω of $\mathbb{R}^{p|q}$ to be a space on which the algebra of functions is isomorphic to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\dots,\eta^q]$, where $|\Omega|$ is an open subset of $\mathbb{R}^p$. So let us choose such an open set Ω and a smooth ordinary manifold $\mathcal{N}$ and analyze what should be maps ϕ from

[a)]Electronic mail: helein@math.jussieu.fr.

Ω to $\mathcal{N}$. Again, there is no direct definition of such an object except that by the chain rule it should define a ring morphism ϕ^* from the ring $\mathcal{C}^\infty(\mathcal{N})$ of smooth functions on $\mathcal{N}$ to the ring $\mathcal{C}^\infty(|\Omega|)\times[\eta^1,\dots,\eta^q]$. The morphism property means that

$$\phi^*(\lambda f+\mu g)=\lambda\phi^* f+\mu\phi^* g,\quad \forall\,\lambda,\mu\in\mathbb{R},\quad \forall\, f,g\in\mathcal{C}^\infty(\mathcal{N}) \tag{1}$$

and

$$\phi^*(fg)=(\phi^* f)(\phi^* g),\quad \forall\, f,g\in\mathcal{C}^\infty(\mathcal{N}). \tag{2}$$

We restrict ourself to *even* morphisms, which means here that we impose to $\phi^* f$ to be in the even part $\mathcal{C}^\infty(|\Omega|)[\eta^1,\dots,\eta^q]_0$ of $\mathcal{C}^\infty(|\Omega|)[\eta^1,\dots,\eta^q]$.

In the Sec. II, we prove our main result (Theorem 1.1) which shows that, for any even morphism ϕ^*, there exists a smooth map φ from $|\Omega|$ to $\mathcal{N}$ and a family of vector fields $(\Xi_x)_{x\in|\Omega|}$ depending on $x\in|\Omega|$ and tangent to $\mathcal{N}$ and with coefficients in the commutative subalgebra $\mathbb{R}[\eta^1,\dots,\eta^q]_0$ such that

$$\phi^* f=(1\times\varphi)^*(e^{\Xi}f),\quad \forall\, f\in\mathcal{C}^\infty(\mathcal{N}). \tag{3}$$

One may interpret the term e^{Ξ} as an analog with odd variables of the standard Taylor series representation,

$$g(x)=\sum_{k=0}^{\infty}\frac{\partial^k g}{(\partial x)^k}(x_0)\frac{(x-x_0)^k}{k!}=(e^{\sum_{i=1}^n(x^i-x_0^i)(\partial/\partial x^i)}g)(x_0)$$

for a function g which is analytic in a neighbourhood of x_0. We also show that the vector field Ξ (which is not unique) can be build as a combination of commuting vector fieds. Then, the rest of this paper is devoted to the consequences of this result.

The second section explores in details the structure behind relation (3). First, we exploit the fact that one can assume that the vector fields which compose Ξ commute, so that one can integrate them locally. This gives us an alternative description of morphisms. Eventually, this study leads us to a factorization result for all even morphisms as follows. First, let us denote by $\Lambda_+^{2*}\mathbb{R}^q$ the subspace of all even elements of the exterior algebra $\Lambda^*\mathbb{R}^q$ of positive degree (i.e., $\Lambda_+^{2*}\mathbb{R}^q\simeq\mathbb{R}^{2^{q-1}-1}$). We construct an ideal $\mathcal{I}^q(|\Omega|)$ of the algebra $\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$ in such a way that if we consider the quotient algebra $\mathcal{A}^q(|\Omega|):=\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)/\mathcal{I}^q(|\Omega|)$, then there exists a canonical isomorphism $T_\Omega^*:\mathcal{A}^q(|\Omega|)\to\mathcal{C}^\infty(\Omega)$. By following the theory of scheme of Grothendieck, we associate to $\mathcal{A}^q(|\Omega|)$ its spectrum Spec $\mathcal{A}^q(|\Omega|)$, a kind of geometric object embedded in $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$. Then, for any even morphism ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\dots,\eta^q]$, there exists a smooth map Φ from $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$ to $\mathcal{N}$, such that

$$\phi^*=T_\Omega^*\circ\Phi_{|\star}^*,$$

where $\forall f\in\mathcal{C}^\infty(\mathcal{N})$, $\Phi_{|\star}^* f=f\circ\Phi \bmod \mathcal{I}^q(|\Omega|)$. So by dualizing we can think of the map $\Phi_{|\star}:\text{Spec}\,\mathcal{A}^q(|\Omega|)\to\mathcal{N}$ as the restriction of Φ to Spec $\mathcal{A}^q(|\Omega|)$. Hence, we obtain an interpretation of a map on a supermanifold as a function defined on an (almost) ordinary space. This reminds somehow the theory developped by Vladimirov and Volovich[18] who represent a map on a superspace as a function depending on many auxiliary ordinary variables satisfying a system of so-called "Cauchy–Riemann-type equations." However, their description in terms of ordinary functions satisfying first order equations differs from our point of view.

The last section is devoted to applications of our results for understanding the use of supermanifolds by physicists. First, we explain briefly how one can reduced the study of maps between two supermanifolds to the study of maps from a supermanifold to an ordinary one by using charts. Second, we recall why it is necessary to incorporate the notion of the functor of point (as illustrated in this framework in Refs. 7, 9, and 19) in the definition of a map ϕ between supermanifolds in terms of ring morphisms. Then, we address the simple question of computing the pull-back image $\phi^* f$ of a map f on an ordinary manifold $\mathcal{N}$ by a map ϕ from an open subset of $\mathbb{R}^{p|q}$ to $\mathcal{N}$. For instance, consider a superfield $\phi=\varphi+\theta^1\psi_1+\theta^2\psi_2+\theta^1\theta^2 F$ from $\mathbb{R}^{3|2}$ [with coordinates $(x^1,x^2,t,\theta^1,\theta^2)$] to $\mathbb{R}$ and look at the Berezin integral,

$$I := \int_{\mathbb{R}^{3|2}} d^3x d^2\theta \phi^* f,$$

where $f:\mathbb{R}\to\mathbb{R}$ is a smooth function. Such a quantity arises for instance in the action $\int_{\mathbb{R}^{3|2}} d^3x d^2\theta\left(\frac{1}{4}\epsilon^{ab}D_a\phi D_b\phi+\phi^* f\right)$ and then f plays the role of a superpotential. Following Berezin's rules, the integral I is equal to the integral over $\mathbb{R}^3$ of the coefficient of $\theta^1\theta^2$ in the development of $\phi^* f$, which is actually

$$\phi^* f = f\circ\varphi + \theta^1(f'\circ\varphi)\psi_1 + \theta^2(f'\circ\varphi)\psi_2 + \theta^1\theta^2[(f'\circ\varphi)F-(f''\circ\varphi)\psi_1\psi_2], \tag{4}$$

so that $I=\int_{\mathbb{R}^3} d^3x[(f'\circ\varphi)F-(f''\circ\varphi)\psi_1\psi_2]$. The development (4) is well known and can be obtained by several approaches. For instance in Ref. 6 or 9, one computes the coefficient of $\theta^1\theta^2$ in the development of $\phi^* f$ by the rule $\iota^*\left(-\frac{1}{2}(D_1D_2-D_2D_1)\phi^* f\right)$, where D_1 and D_2 are derivatives with respect to θ^1 and θ^2, respectively, and ι is the canonical embedding $\mathbb{R}^3\hookrightarrow\mathbb{R}^{3|2}$. Here, we propose a recipe which, I find, is simple, intuitive, but mathematically safe for performing this computation (this recipe is of course equivalent to the already existing rules!). It consists roughly in the following: we reinterpret the relation $\phi=\varphi+\theta^1\psi_1+\theta^2\psi_2+\theta^1\theta^2F$ as

$$\phi^* = \varphi^* e^{\theta^1\psi_1+\theta^2\psi_2+\theta^1\theta^2F} = \varphi^*(1+\theta^1\psi_1)(1+\theta^2\psi_2)(1+\theta^1\theta^2F), \tag{5}$$

where

- ψ_1, ψ_2, and F are first order differential operators which acts on the right, i.e., for instance, $\forall f\in\mathcal{C}^\infty(\mathbb{R})$, $\psi_a f=df(\psi_a)=f'\psi_a$, and so $\varphi^*\psi_a f=(f'\circ\varphi)\psi_a$;
- ψ_1, ψ_2, and F are $\mathbb{Z}_2$-graded in such a way that ϕ^* is even, i.e., since θ^1 and θ^2 are odd, ψ_1 and ψ_2 are odd and F is even; and
- all the symbols θ^1, θ^2, ψ_1, ψ_2, and F supercommute.

Let us use the supercommutation rules to develop (5), we obtain

$$\phi^* f = \varphi^* f + \theta^1\varphi^*\psi_1 f + \theta^2\varphi^*\psi_2 f + \theta^1\theta^2\varphi^* F f - \theta^1\theta^2\varphi^*\psi_1\psi_2 f, \quad \forall f\in\mathcal{C}^\infty(\mathbb{R}).$$

Then, we let the first order differential operators act and this gives us exactly (4).

All these rules are expounded in details in the Sec. IV of this paper. Their justification is precisely based on the results of Sec. II.

II. EVEN MAPS FROM $\mathbb{R}^{p|q}$ TO A MANIFOLD $\mathcal{N}$

Our first task will be to study even morphisms ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\ldots,\eta^q]$, i.e., maps between these two superalgebras which satisfy (1) and (2). Let us first precise the sense of *even*. If $A=A_0\oplus A_1$ and $B=B_0\oplus B_1$ are two $\mathbb{Z}_2$-graded rings with unity, a ring morphism $\phi:B\to A$ is said to be *even* if it respects the grading, i.e., $\forall b\in B_\alpha$, $\phi(b)\in A_\alpha$ for $\alpha=0,1$. In the case at hand, $B=\mathcal{C}^\infty(\mathcal{N})$ is purely even, i.e., $B_1=\{0\}$, and so ϕ^* is even if and only if it maps $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(|\Omega|)[\eta^1,\ldots,\eta^q]_0$, the even part of $\mathcal{C}^\infty(|\Omega|)[\eta^1,\ldots,\eta^q]$. We then say that ϕ is an *even* map from Ω to $\mathcal{N}$. In the following, we shall denote the rings $\mathcal{C}^\infty(|\Omega|)[\eta^1,\ldots,\eta^q]$ and $\mathcal{C}^\infty(|\Omega|)[\eta^1,\ldots,\eta^q]_0$ by, respectively, $\mathcal{C}^\infty(\Omega)$ and $\mathcal{C}^\infty(\Omega)_0$ and we shall denote by $\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathcal{C}^\infty(\Omega)_0)$ the set of even morphisms from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\Omega)$.

We observe that because of hypothesis (1), any such morphism is given by a finite family $(a_{i_1\ldots i_{2k}})$ of linear functionals on $\mathcal{C}^\infty(\mathcal{N})$ with values in $\mathcal{C}^\infty(|\Omega|)$, where $(i_1\cdots i_{2k})\in[\![1,q]\!]^{2k}$ and $0\leqslant k\leqslant[q/2]$ ($[q/2]$ is the integer part of $q/2$), by the relation

$$\phi^* f = \sum_{k=0}^{[q/2]} \sum_{1\leq i_1<\cdots<i_k\leq q} a_{i_1\cdots i_{2k}}(f)\eta^{i_1}\cdots\eta^{i_{2k}} = a_\emptyset(f) + \sum_{1\leq i_1<i_2\leq q} a_{i_1 i_2}(f)\eta^{i_1}\eta^{i_2} + \cdots .$$

Here, we will assume that the $a_{i_1\cdots i_{2k}}$'s are skew symmetric in $(i_1\cdots i_{2k})$. At this point, it is useful to introduce the following notations: For any positive integer k, we let $\mathbb{I}^q(k) := \{(i_1\cdots i_k) \in [\![1,q]\!]^k | i_1<\cdots<i_k\}$, we denote by $I=(i_1\cdots i_k)$ an element of $\mathbb{I}^q(k)$ and we then write $\eta^I := \eta^{i_1}\ldots\eta^{i_k}$. It will be also useful to use the convention $\mathbb{I}^q(0)=\{\emptyset\}$. We let $\mathbb{I}^q := \cup_{k=0}^q \mathbb{I}^q(k)$, $\mathbb{I}_0^q := \cup_{k=0}^{[q/2]}\mathbb{I}^q(2k)$, $\mathbb{I}_1^q := \cup_{k=0}^{[(q-1)/2]}\mathbb{I}^q(2k+1)$, and $\mathbb{I}_2^q := \cup_{k=1}^{[q/2]}\mathbb{I}^q(2k)$. Hence, the preceding relation can be written as

$$\phi^* f = \sum_{k=0}^{[q/2]} \sum_{I\in\mathbb{I}^q(2k)} a_I(f)\eta^I = \sum_{I\in\mathbb{I}_0^q} a_I(f)\eta^I \tag{6}$$

or

$$(\phi^* f)(x) = \sum_{I\in\mathbb{I}_0^q} a_I(f)(x)\eta^I, \qquad \forall\, x\in|\Omega|.$$

A. Construction of morphisms

We start by providing a construction of morphisms satisfying (1) and (2). We note $\pi:|\Omega|\times\mathcal{N}\to\mathcal{N}$ the canonical projection map and consider the vector bundle $\pi^*T\mathcal{N}$: the fiber over each point $(x,q)\in|\Omega|\times\mathcal{N}$ is the tangent space $T_q\mathcal{N}$. For any $I\in\mathbb{I}_2^q$, we choose a smooth section ξ_I of $\pi^*T\mathcal{N}$ over $|\Omega|\times\mathcal{N}$ and we consider the $\mathbb{R}[\eta^1,\ldots,\eta^q]_0$-valued vector field,

$$\Xi := \sum_{I\in\mathbb{I}_2^q} \xi_I \eta^I.$$

Alternatively, Ξ can be seen as a smooth family $(\Xi_x)_{x\in|\Omega|}$ of smooth tangent vector fields on $\mathcal{N}$ with coefficients in $\mathbb{R}[\eta^1,\ldots,\eta^q]_0$. So each Ξ_x defines a first order differential operator which acts on the algebra $\mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\ldots,\eta^q]_0$, i.e., the set of smooth functions on $\mathcal{N}$ with values in $\mathbb{R}[\eta^1,\ldots,\eta^q]_0$, by the relation

$$\Xi_x f = \sum_{I\in\mathbb{I}_2^q} ((\xi_I)_x\cdot f)\eta^I, \qquad \forall\, f\in\mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\ldots,\eta^q]_0.$$

Here, we do not need to worry about the position of η^I since it is an even monomial. We now define (letting $\Xi^0=1$)

$$e^\Xi := \sum_{n=0}^{\infty} \frac{\Xi^n}{n!} = \sum_{n=0}^{[q/2]} \frac{\Xi^n}{n!},$$

which can be considered again as a smooth family parametrized by $x\in|\Omega|$ of differential operators of order at most $[q/2]$ acting on $\mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\ldots,\eta^q]_0$. Now, we choose a smooth map $\varphi:|\Omega|\to\mathcal{N}$ and we consider the map

$$1\times\varphi: |\Omega| \to |\Omega|\times\mathcal{N}$$

$$x \mapsto (x,\varphi(x)),$$

which parametrizes the graph of φ. Lastly, we construct the following linear operator on $\mathcal{C}^\infty(\mathcal{N})\subset\mathcal{C}^\infty(\mathcal{N})\otimes\mathbb{R}[\eta^1,\ldots,\eta^q]_0$:

$$\mathcal{C}^\infty(\mathcal{N}) \ni f \mapsto (1 \times \varphi)^*(e^{\Xi} f) \in \mathcal{C}^\infty(\Omega),$$

where

$$(1 \times \varphi)^*(e^{\Xi} f)(x) := (e^{\Xi_x} f)(\varphi(x)) = \sum_{n=0}^{[q/2]} \left(\frac{(\Xi)_x^n}{n!} f \right)(\varphi(x)), \quad \forall\, x \in |\Omega|.$$

We observe that actually, for any $x \in |\Omega|$, we only need to define Ξ_x on a neighborhood of $\varphi(x)$ in $\mathcal{N}$, i.e., it suffices to define the section Ξ on a neighborhood of the graph of φ in $|\Omega| \times \mathcal{N}$ [or even on their Taylor expansion in q at order $[q/2]$ around $\varphi(x)$].

Lemma 1.1: The map $f \mapsto (1 \times \varphi)^*(e^{\Xi} f)$ is a morphism from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\Omega)_0$, i.e., satisfies assumptions (1) and (2).

Proof: Property (1) is obvious, so we just need to prove (2). We first remark that for any $x \in |\Omega|$, Ξ_x satisfies the Leibniz rule,

$$\Xi_x(fg) = (\Xi_x f)g + f(\Xi_x g), \quad \forall\, f,g \in \mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \ldots, \eta^q]_0,$$

which immediately implies by recursion that

$$\Xi_x^n(fg) = \sum_{j=1}^{n} \frac{n!}{(n-j)!\,j!} (\Xi_x^{n-j} f)(\Xi_x^j g), \quad \forall\, f,g \in \mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \ldots, \eta^q]_0, \quad \forall\, n \in \mathbb{N}. \tag{7}$$

We deduce easily that

$$e^{\Xi_x}(fg) = (e^{\Xi_x} f)(e^{\Xi_x} g), \quad \forall\, x \in |\Omega|, \quad \forall\, f,g \in \mathcal{C}^\infty(\mathcal{N}) \otimes \mathbb{R}[\eta^1, \ldots, \eta^q]_0, \tag{8}$$

by developping both sides and using (7). Now, relation (8) is true, in particular, for functions $f, g \in \mathcal{C}^\infty(\mathcal{N})$ and if we evaluate this identity at the point $\varphi(x) \in \mathcal{N}$ we immediately conclude that $f \mapsto (1 \times \varphi)^*(e^{\Xi} f)$ satisfies (2). ■

The following result says that actually all morphisms are of the previous type.

Theorem 1.1: *Let $\phi^*: \mathcal{C}^\infty(\mathcal{N}) \to \mathcal{C}^\infty(\Omega)_0$ be a morphism. Then there exists a smooth map $\varphi: |\Omega| \to \mathcal{N}$ and a smooth family $(\xi_I)_{I \in \mathbb{I}_2^q}$ of sections of $\pi^* T\mathcal{N}$ defined on a neighborhood of the graph of φ in $|\Omega| \times \mathcal{N}$, such that if $\Xi := \sum_{I \in \mathbb{I}_2^q} \xi_I \eta^I$, then*

$$\phi^* f = (1 \times \varphi)^*(e^{\Xi} f), \quad \forall\, f \in \mathcal{C}^\infty(\mathcal{N}). \tag{9}$$

Proof: Let $\phi^*: \mathcal{C}^\infty(\mathcal{N}) \to \mathcal{C}^\infty(\Omega)_0$ which satisfies (1) and (2). We denote by a_I the functionals involved in identity (6). We also introduce the following notation: for any $N \in \mathbb{N}$, $\mathcal{O}(\eta^{(N)})$ will represent a quantity of the form

$$\mathcal{O}(\eta^{(N)}) = \sum_{n=N}^{\infty} \sum_{I \in \mathbb{I}^q(n)} c_I \eta^I,$$

where the coefficients c_I's may be real constants or functions. The result will follow by proving by recursion on $n \in \mathbb{N}^*$ the following property:

(P_n): There exists a smooth map $\varphi: |\Omega| \to \mathcal{N}$ and there exists a family of vector fields $(\xi_I)_I$, where $I \in \mathbb{I}^q(2k)$ and $1 \leq k \leq n$, defined on a neighborhood of the graph of φ in $|\Omega| \times \mathcal{N}$, such that if

$$\Xi_n := \sum_{k=1}^{n} \sum_{I \in \mathbb{I}^q(2k)} \xi_I \eta^I,$$

then

$$\phi^* f = (1 \times \varphi)^*(e^{\Xi_n} f) + \mathcal{O}(\eta^{(2n+1)}), \quad \forall\, f \in \mathcal{C}^\infty(\mathcal{N}).$$

Proof of (P_1)*:* We start from relation (2) and we expand both sides by using (6): we first obtain by identifying the terms of degree 0 in the η^j's,

$$a_\emptyset(fg)(x) = (a_\emptyset(f)(x))(a_\emptyset(g)(x)), \quad \forall\, x \in |\Omega|, \quad \forall\, f,g \in \mathcal{C}^\infty(\mathcal{N}),$$

which implies that for any $x \in |\Omega|$, there exists some value $\varphi(x) \in \mathcal{N}$ such that

$$a_\emptyset(f)(x) = f(\varphi(x)), \quad \forall\, x \in |\Omega|, \quad \forall\, f \in \mathcal{C}^\infty(\mathcal{N}).$$

In other words, there exists a function $\varphi : |\Omega| \to \mathcal{N}$ such that $a_\emptyset(f) = f \circ \varphi$. Since $a_\emptyset(f)$ must be $\mathcal{C}^\infty$ for any smooth f, this implies that $\varphi \in \mathcal{C}^\infty(|\Omega|, \mathcal{N})$. The relations between the terms of degree 2 in (2) are

$$\begin{aligned} a_I(fg)(x) &= (a_I(f)(x))(a_\emptyset(g)(x)) + (a_\emptyset(f)(x))(a_I(g)(x)) \\ &= (a_I(f)(x))g(\varphi(x)) + f(\varphi(x))(a_I(g)(x)), \quad \forall\, x \in |\Omega|, \quad \forall\, f,g \in \mathcal{C}^\infty(\mathcal{N}), \quad \forall\, I \in \mathbb{I}^q(2), \end{aligned}$$

which implies that for any $x \in |\Omega|$, each $a_I(\cdot)(x)$ is a derivation acting on $\mathcal{C}^\infty(\mathcal{N})$, with support $\{\varphi(x)\}$, i.e., $\forall I \in \mathbb{I}^q(2)$ there exist tangent vectors $(\xi_I)_x \in T_{\varphi(x)}\mathcal{N}$ such that

$$a_I(f)(x) = ((\xi_I)_x \cdot f)(\varphi(x)), \quad \forall\, f \in \mathcal{C}^\infty(\mathcal{N}).$$

Since $a_I(f)$ must be smooth for any $f \in \mathcal{C}^\infty(\mathcal{N})$, the vectors $(\xi_I)_x$ should depend smoothly on x, i.e., $x \mapsto (\xi_I)_x$ is a smooth section of $\varphi^* T\mathcal{N}$. It is then possible (see the Proposition 1.1 below) to extend it to a smooth section of $\pi^* T\mathcal{N}$ on a neighborhood of the graph of φ. If we now set $(\Xi_1)_x := \sum_{I \in \mathbb{I}^q(2)} (\xi_I)_x \eta^I$, we have on the one hand,

$$e^{(\Xi_1)_x} f = f + \sum_{I \in \mathbb{I}^q(2)} ((\xi_I)_x \cdot f)\, \eta^I + \mathcal{O}(\eta^{(3)}), \quad \forall\, x \in |\Omega|, \quad \forall\, f \in \mathcal{N},$$

and on the other hand,

$$(\phi^* f)(x) = f(\varphi(x)) + \sum_{I \in \mathbb{I}^q(2)} ((\xi_I)_x \cdot f)(\varphi(x))\, \eta^I + \mathcal{O}(\eta^{(3)}), \quad \forall\, x \in |\Omega|,$$

from which (P_1) follows.

Proof of $(P_n) \Rightarrow (P_{n+1})$: We assume (P_n) so that a map $\varphi \in \mathcal{C}^\infty(|\Omega|, \mathcal{N})$ and a vector field Ξ_n have been constructed. Let us denote by b_I the linear forms on $\mathcal{C}^\infty(\mathcal{N})$ such that

$$(1 \times \varphi)^*(e^{\Xi_n} f) = \sum_{k=0}^{[q/2]} \sum_{I \in \mathbb{I}^q(2k)} b_I(f)\, \eta^I. \tag{10}$$

Then, property (P_n) is equivalent to

$$a_I = b_I, \quad \forall\, k \in [\![0,n]\!], \quad \forall\, I \in \mathbb{I}^q(2k). \tag{11}$$

We use Lemma 1.1: it says us that $f \mapsto (1 \times \varphi)^*(e^{\Xi_n} f)$ is a morphism; hence, $(1 \times \varphi)^*(e^{\Xi_n}(fg)) = [(1 \times \varphi)^*(e^{\Xi_n} f)][(1 \times \varphi)^*(e^{\Xi_n} g)]$, so by using (10),

$$\sum_{k=0}^{n+1} \sum_{I \in \mathbb{I}^q(2k)} b_I(fg)\, \eta^I = \sum_{k=0}^{n+1} \sum_{j=0}^{k} \sum_{J \in \mathbb{I}^q(2k-2j), K \in \mathbb{I}^q(2j)} b_J(f) b_K(g)\, \eta^J \eta^K + \mathcal{O}(\eta^{(2n+3)}). \tag{12}$$

However, the morphism property (2) for ϕ^* implies also

$$\sum_{k=0}^{n+1}\sum_{I\in\mathbb{I}^q(2k)} a_I(fg)\eta^I = \sum_{k=0}^{n+1}\sum_{j=0}^{k}\sum_{J\in\mathbb{I}^q(2k-2j),K\in\mathbb{I}^q(2j)} a_J(f)a_K(g)\eta^J\eta^K + \mathcal{O}(\eta^{(2n+3)}). \tag{13}$$

We now substract (12) to (13) and use (11): it gives us

$$\sum_{I\in\mathbb{I}^q(2n+2)} (a_I(fg)-b_I(fg))\eta^I = \sum_{I\in\mathbb{I}^q(2n+2)} [(a_I(f)-b_I(f))a_\varnothing(g)+a_\varnothing(f)(a_I(g)-b_I(g))]\eta^I.$$

Hence, if we denote $\delta a_I := a_I - b_I$, we obtain that

$$\delta a_I(fg) = \delta a_I(f)(g\circ\varphi) + (f\circ\varphi)\delta a_I(g), \quad \forall\, I\in\mathbb{I}^q(2n+2).$$

By the same reasoning as in the proof of (P_1), we conclude that $\forall I\in\mathbb{I}^q(2n+2)$, there exist smooth sections ξ_I of $\pi^*T\mathcal{N}$ defined on a neighborhood of the graph of φ, such that

$$\delta a_I(f)(x) = ((\xi_I)_x\cdot f)(\varphi(x)), \quad \forall\, x\in|\Omega|, \quad \forall\, I\in\mathbb{I}^q(2n+2).$$

Now let us define

$$\Xi_{n+1} := \Xi_n + \sum_{I\in\mathbb{I}^q(2n+2)} \xi_I\eta^I.$$

Then, it turns out that

$$e^{\Xi_{n+1}}f = \sum_{k=0}^{n+1}\frac{\left(\Xi_n + \sum_{I\in\mathbb{I}^q(2k+2)}\xi_I\eta^I\right)^k}{k!}f + \mathcal{O}(\eta^{(2n+3)}) = \sum_{k=0}^{n+1}\frac{\Xi_n^k}{k!}f + \sum_{I\in\mathbb{I}^q(2n+2)}\xi_I\cdot f\eta^I + \mathcal{O}(\eta^{(2n+3)})$$

$$= e^{\Xi_n}f + \sum_{I\in\mathbb{I}^M(2n+2)}\xi_I\cdot f\eta^I + \mathcal{O}(\eta^{(2n+3)}),$$

so that

$$(1\times\varphi)^*(e^{\Xi_{n+1}}f) = \phi^*f + \mathcal{O}(\eta^{(2n+3)}).$$

Hence, we deduce (P_{n+1}). ■

Proposition 1.1: In the preceding result, it is possible to construct smoothly the vector fields ξ_I's in such a way that

$$[(\xi_I)_x,(\xi_J)_x] = 0, \quad \forall\, x\in|\Omega|, \quad \forall\, I,J\in\mathbb{I}_2^q.$$

Proof: Recall that in the previous proof, in order to build Ξ_{n+1} out of Ξ_n, we introduced, for each $I\in\mathbb{I}^q(2n+2)$, an unique smooth section $x\mapsto(\xi_I)_x$ of $\varphi^*T\mathcal{N}$. We will explain here how to extend each such vector fields defined along the graph of φ to a neighborhood of the graph of φ in $|\Omega|\times\mathcal{N}$ in order to achieve the claim in the proposition. For that purpose, we prove that for some set

$$\mathcal{V} := \{(x,\xi,q)\in\varphi^*T\mathcal{N}\times\mathcal{N} | x\in|\Omega|, \xi\in T_{\varphi(x)}\mathcal{N}, q\in V_{\varphi(x)}\},$$

where each $V_{\varphi(x)}$ is a neighborhood of $\varphi(x)$ in $\mathcal{N}$, there exists a smooth map

$$\mathcal{V}\to T\mathcal{N},$$

$$(x,\xi,q)\mapsto(q,\mathbb{V}(x,\xi,q)),$$

such that $\forall(x,\xi)\in\varphi^*T\mathcal{N}$, $\mathbb{V}(x,\xi,\varphi(x))=\xi$ and $\forall x\in|\Omega|$ fixed, $\forall\xi,\zeta\in T_{\varphi(x)}\mathcal{N}$, $[\mathbb{V}(x,\xi,\cdot),\mathbb{V}(x,\zeta,\cdot)]=0$, i.e., the vector fields $q\mapsto\mathbb{V}(x,\xi,q)$ and $q\mapsto\mathbb{V}(x,\zeta,q)$ commute on $V_{\varphi(x)}$.

Then, the proposition will follows by extending each vector $(\xi_I)_x \in T_{\varphi(x)}\mathcal{N}$ on $V_{\varphi(x)}$ by $q \mapsto \mathbb{V}(x,(\xi_I)_x,q)$.

The construction is the following. Let $(U_a)_{a\in A}$ be a covering of $\mathcal{N}$ by open subsets, let $(\chi_a)_{a\in A}$ be a partition of unity, and let $(y_a)_{a\in A}$ be a family of charts associated with this covering. For any $x\in|\Omega|$, let $A_x := \{a\in A \,|\, \varphi(x)\in U_a\}$. For any $a\in A_x$ and for any linear isomorphism $\ell : T_{\varphi(x)}\mathcal{N} \to \mathbb{R}^n$, where $n=\dim\mathcal{N}$, let $R_{x,\ell,a}$ be the unique linear automorphism of $\mathbb{R}^n$ such that

$$R_{x,\ell,a}\circ dy_{a|\varphi(x)} = \ell.$$

We then set

$$y_{x,\ell}(q) := \sum_{a\in A_x} \chi_a(q) R_{x,\ell,a}\circ y_a(q), \quad \forall\, q\in\mathcal{N}.$$

We observe that $dy_{x,\ell|\varphi(x)}=\ell$ and hence, by the inverse mapping theorem, there exists an open neighborhood $V_{\varphi(x)}$ of $\varphi(x)$ in $\mathcal{N}$ such that the restriction of $y_{x,\ell}$ to $V_{\varphi(x)}$ is a diffeomorphism. We then define

$$\mathbb{V}(x,\xi,q) := (dy_{x,\ell|q})^{-1}(\ell(\xi)), \quad \forall\, q\in V_{\varphi(x)}.$$

Because of the obvious relation $y_{x,u\circ\ell}=u\circ y_{x,\ell}$ for all linear automorphism u of $\mathbb{R}^n$, it is clear that the definition of $\mathbb{V}(x,\xi,q)$ does not depend on ℓ (for the same reason $V_{\varphi(x)}$ is also independant of ℓ). Moreover, $q\mapsto\mathbb{V}(x,\xi,q)$ is simply a vector field which is a linear combination with constant coefficients of the vector fields $(\partial/\partial y^i_{x,\ell})_{i=1,\ldots,n}$ so that the property $[\mathbb{V}(x,\xi,\cdot),\mathbb{V}(x,\zeta,\cdot)]=0$ follows. Note also that these vector fields are of course not canonical since they obviously depend on the charts. ■

Remark 1.1: If we assume furthermore that the image of φ is contained in an open subset U of $\mathcal{N}$ such that there exists a local chart $y=(y^1,\ldots,y^n):U\to\mathbb{R}^n$, then it is possible to choose all the vector fields ξ_I such that

$$(\xi_I)_x\cdot(\xi_J)_x\cdot y = 0, \quad \forall\, x\in|\Omega|, \quad \forall\, I,J\in\mathbb{I}_2^q. \tag{14}$$

Indeed in this case the proof of Proposition 1.1 is much simpler since we do not need to use a partition of unity in order to build $\mathbb{V}$. We just set $\mathcal{V}:=\{(x,\xi,q)\in\varphi^*T\mathcal{N}\times\mathcal{N} \,|\, x\in|\Omega|,\xi\in T_{\varphi(x)}\mathcal{N}, q\in U\}$ and define $\mathbb{V}$ by $\mathbb{V}(x,\xi,q):=(dy_{|q})^{-1}\circ dy_{|\varphi(x)}(\xi)$. Then, for each $(x,\xi)\in\varphi^*T\mathcal{N}$ fixed, the vector field $q\mapsto\mathbb{V}(x,\xi,q)$ has constant coordinates in the variables y^α. Hence, (14) follows.

Remark 1.2: We can write an alternative formula for e^{Ξ} by developping this exponential: in each term of the form $(\Sigma_I\xi_I\eta^I)^n$, we can see that each monomial which appears contains at most one time any operator ξ_I, so we obtain

$$e^{\Xi} = \sum_{I\in\mathbb{I}_0^q} \eta^I \left(\sum_{n\geqslant 0} \frac{1}{n!} \sum_{I_1,\ldots,I_n\in\mathbb{I}_0^q} \epsilon_I^{I_1\cdots I_n} \xi_{I_1}\cdots\xi_{I_n} \right), \tag{15}$$

with the convention that the $\mathbb{I}_0^q(0)=\emptyset$ contribution is the identity. Here, we have introduced the notation $\epsilon_I^{I_1\cdots I_n}$: first all the $\epsilon_\emptyset^{I_1\cdots I_n}$'s vanish except for $\epsilon_\emptyset^\emptyset=1$, so that $e^{\Xi}=1 \bmod[\eta^1,\ldots,\eta^q]$. Second, for $k\geqslant 1$, if $I_1=(i_{1,1},\ldots,i_{1,2k_1}),\ldots, I_n=(i_{n,1},\ldots,i_{n,2k_n})$, and $I=(i_1,\ldots,i_{2k})$ we write that $I_1\sqcup\cdots\sqcup I_n=I$ if and only if $k_1+\cdots+k_n=k$, $\{i_{1,1},\ldots,i_{1,2k_1},\ldots,i_{n,1},\ldots,i_{n,2k_n}\}=\{i_1,\ldots,i_{2k}\}$ and $\forall j$, $I_j\neq\emptyset$ (i.e., $k_j>0$). Then,

- *if $I_1\sqcup\cdots\sqcup I_n\neq I$, $\epsilon_I^{I_1\cdots I_n}=0$;* and
- *if $I_1\sqcup\cdots\sqcup I_n=I$, $\epsilon_I^{I_1\cdots I_n}$ is the signature of the permutation $(i_{1,1},\ldots,i_{1,2k_1},\ldots,i_{n,1},\ldots,i_{n,k_{2n}})\mapsto(i_1,\ldots,i_{2k})$.*

The preceding expression of e^{Ξ} can be recovered by another way: since all the operators $\eta^I\xi_I$ commute, we have[1]

$$e^{\Xi} = e^{\sum_{I\in\mathbb{I}_2^q}\eta^I\xi_I} = \prod_{I\in\mathbb{I}_2^q} e^{\eta^I\xi_I},$$

which gives also the same result by a straightforward development.

III. A FACTORIZATION OF THE MORPHISM ϕ^*

A. Integrating the vector fields ξ_I's

In the same spirit as a tangent vector at a point q to a manifold $\mathcal{N}$ can be seen as the time derivative of a smooth curve which reaches q, we can describe the η^I-components of the morphism ϕ^* as higher order approximations of a smooth map from some vector space with values in $\mathcal{N}$. Indeed, let $\phi^*\in \mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathcal{C}^\infty(\Omega)_0)$: then by the preceding result ϕ^* is characterized by a map $\varphi\in\mathcal{C}^\infty(|\Omega|,\mathcal{N})$ and $2^{q-1}-1$ vector fields[1] ξ_I tangent to $\mathcal{N}$ defined on a neighborhood of the graph of φ in $|\Omega|\times\mathcal{N}$. By Proposition 1.1, these vector fields can moreover be chosen so that they pairwise commute when $x\in|\Omega|$ is fixed. So, for any $x\in|\Omega|$, we can integrate simultaneously all vector fields $(\xi_I)_x$ in order to construct a map,

$$\Phi(x,\cdot):U_x(\Lambda_+^{2*}\mathbb{R}^q)\to\mathcal{N},$$

where $\Lambda_+^{2*}\mathbb{R}^q\simeq\mathbb{R}^{2^{q-1}-1}$ is the subspace of even elements of positive degree of the exterior algebra $\Lambda^*\mathbb{R}^q$ and $U_x(\Lambda_+^{2*}\mathbb{R}^q)$ is a neighborhood of 0 in $\Lambda_+^{2*}\mathbb{R}^q$, such that

$$\Phi(x,0)=\varphi(x) \tag{16}$$

and denoting by $(\mathfrak{s}^I)_{I\in\mathbb{I}_2^q}$ the linear coordinates on $\Lambda_+^{2*}\mathbb{R}^q$,

$$\frac{\partial\Phi}{\partial\mathfrak{s}^I}(x,\mathfrak{s})=\xi_I(\Phi(x,\mathfrak{s})),\qquad\forall\,\mathfrak{s}\in U_x(\Lambda_+^{2*}\mathbb{R}^q),\qquad\forall\, I\in\mathbb{I}_2^q. \tag{17}$$

We hence obtain a map Φ from a neighborhood of $|\Omega|\times\{0\}$ in $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$ to $\mathcal{N}$. By using a cutoff function argument, we can extend this map to an application $\Phi:|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q\mapsto\mathcal{N}$. Lastly, we introduce the $\mathbb{R}[\eta^1,\ldots,\eta^q]$-valued vector field on $|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$,

$$\vartheta\coloneqq\sum_{I\in\mathbb{I}_2^q}\eta^I\frac{\partial}{\partial\mathfrak{s}^i},$$

so that by (17) $\Phi_*\vartheta=\Xi=\sum_{I\in\mathbb{I}_2^q}\eta^I\xi_I$. Then, relation (9) implies

$$\phi^*f(x)=(e^{\vartheta}(f\circ\Phi))(x,0),\qquad\forall\, f\in\mathcal{C}^\infty(\mathcal{N}),\qquad\forall\, x\in|\Omega|,$$

or by letting $\iota:|\Omega|\to|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q$, $x\mapsto(x,0)$ to be the canonical injection,

$$\phi^*f=\iota^*(e^{\vartheta}(f\circ\Phi)). \tag{18}$$

Alternatively by using (15), we have

$$\phi^*f(x)=\sum_{I\in\mathbb{I}_0^q}\eta^I\left(\sum_{k\geqslant 0}\frac{1}{k!}\sum_{I_1,\ldots,I_k\in\mathbb{I}_0^q}\epsilon_I^{I_1\cdots I_k}\frac{\partial^k(f\circ\Phi)}{\partial\mathfrak{s}^{I_1}\cdots\partial\mathfrak{s}^{I_k}}(x,0)\right),\qquad\forall\, x\in|\Omega|. \tag{19}$$

It is useful to introduce the differential operators $\mathcal{D}_\varnothing\coloneqq 1$ and

[1]Note that card $\mathbb{I}^q(2k)=q!/(q-2k)!(2k)!$ and $\sum_{k=0}^{[q/2]}q!/(q-2k)!(2k)!=2^{q-1}$.

$$\mathcal{D}_I := \sum_{k\geqslant 0}\frac{1}{k!}\sum_{I_1,\ldots,I_k\in\mathbb{I}_0^q}\epsilon_I^{I_1\cdots I_k}\frac{\partial^k}{\partial\mathfrak{s}^{I_1}\cdots\partial\mathfrak{s}^{I_k}},$$

so that $\phi^*f(x)=\Sigma_{I\in\mathbb{I}_0^q}\eta^I\mathcal{D}_I(f\circ\Phi)(x,0)$. Conversely to any map smooth map $\Phi\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q,\mathcal{N})$, we can associate a unique morphism $\phi^*\in\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathbb{R}[\eta^1,\ldots,\eta^q]_0)$ defined by (18) or (19). This defines an application

$$\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q,\mathcal{N})\to\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}),\mathcal{C}^\infty(\Omega)_0),$$

$$\Phi\mapsto\Phi_{|\circ}^*,$$

where $\Phi_{|\circ}^*f=\iota^*(e^\vartheta(f\circ\Phi))$, $\forall f\in\mathcal{C}^\infty(\mathcal{N})$. It is clear from the previous discussion that this application is onto. It is, however, certainly not injective since $\Phi_{|\circ}^*$ depends only on the $[q/2]$th order Taylor expansion of Φ at 0. This will be precised in the following.

B. Expressions using local coordinates on the target manifold

Assume that we have local coordinates on $\mathcal{N}$: we let U to be an open subset of $\mathcal{N}$ and we consider a chart $y=(y^1,\ldots,y^n):U\to V\subset\mathbb{R}^n$. Then, any function $f:U\to\mathbb{R}$ can be represented by an unique function $F:V\to\mathbb{R}$ such that $f=F\circ y$. For any $y_0\in V\subset\mathbb{R}^n$, let $P_{F,y_0}^{[q/2]}$ be the $[q/2]$th order Taylor expansion of F at y_0 and let $R_{F,y_0}^{[q/2]}$ be the rest, so that we have the decomposition $F(y)=P_{F,y_0}^{[q/2]}(y)+R_{F,y_0}^{[q/2]}(y)$. The expressions for $P_{F,y_0}^{[q/2]}$ and $R_{F,y_0}^{[q/2]}$ are

$$P_{F,y_0}^{[q/2]}(y)=\sum_{r\in\mathbb{N}^n,|r|\leqslant[q/2]}\frac{\partial^rF}{(\partial y)^r}(y_0)\frac{(y-y_0)^r}{r!},\quad\forall\ y\in\mathbb{R}^n$$

and

$$R_{F,y_0}^{[q/2]}(y)=\sum_{r\in\mathbb{N}^n,|r|=[q/2]+1}(y-y_0)^rR_{F,y_0,r}(y),\quad\forall\ y\in V,$$

where if $r=(r_1,\ldots,r_n)\in\mathbb{N}^n$, $|r|:=r_1+\cdots+r_n$, $(y)^r:=(y^1)^{r_1}\cdots(y^n)^{r_n}$, and $\partial^rF/(\partial y)^r:=\partial^{|r|}F/(\partial y^1)^{r_1}\cdots(\partial y^n)^{r_n}$, assuming that V is star-shaped around y_0,

$$R_{F,y_0,r}(y):=\frac{[q/2]+1}{r!}\int_0^1(1-t)^{[q/2]}\frac{\partial^rF}{(\partial y)^r}(y_0+t(y-y_0))dt.$$

Proposition 2.1: Let $y:\mathcal{N}\supset U\to V\in\mathbb{R}^n$ be a local chart and $\phi^*:\mathcal{C}^\infty(U)\to\mathcal{C}^\infty(\Omega)_0$ be a morphism. For any $f\in\mathcal{C}^\infty(U)$, let $F\in\mathcal{C}^\infty(V)$ such that $f=F\circ y$. Then,

$$(\phi^*f)(x_0)=\sum_{r\in\mathbb{N}^n,|r|\leqslant[q/2]}\frac{\partial^rF}{(\partial y)^r}(y_0)\frac{(\phi^*y-y_0)^r}{r!},\quad\forall\ x_0\in|\Omega|,\tag{20}$$

where y_0 is the unique point in $\mathbb{R}^n$ such that $y\circ\phi(x_0)-y_0$ has nilpotent components.

Proof: The morphism property implies that

$$\phi^*(F\circ y)=\phi^*(P_{F,y_0}^{[q/2]}(y))+\sum_{r\in\mathbb{N}^n,|r|=[q/2]+1}\phi^*((y-y_0)^r)\phi^*(R_{F,y_0,r}\circ y).\tag{21}$$

However, still by using the morphism property, we have $\phi^*(P(y))=P(\phi^*y)$ for any polynomial P in n real variables. Hence,

$$\phi^* f = \phi^*(F \circ y) = P^{[q/2]}_{F,y_0}(\phi^* y) + \sum_{r \in \mathbb{N}^n, |r| = [q/2]+1} (\phi^* y - y_0)^r \phi^*(R_{F,y_0,r} \circ y).$$

In particular, when we evaluate this last identity at the point x_0, we get (20) because $(\phi^* y - y_0)^r(x_0) = 0$ for $|r| = [q/2]+1$. ■

Now let $\Phi : |\Omega| \times \Lambda_+^{2*}\mathbb{R}^q \to U \subset \mathcal{N}$, then we have the diagram

$$\begin{array}{ccccc} |\Omega| \times \Lambda_+^{2*}\mathbb{R}^q & \xrightarrow{\Phi} & \mathcal{N} \supset U & \xrightarrow{f} & \mathbb{R} \\ & \searrow y \circ \Phi & \downarrow y & F \nearrow & \\ & & \mathbb{R}^n \supset V & & \end{array}$$

Corollary 2.1: Let $y : U \to \mathbb{R}^n$ be a local chart on $\mathcal{N}$ and let $\Phi, \tilde{\Phi} \in \mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q, \mathcal{N})$ such that $\iota^*\Phi = \iota^*\tilde{\Phi} =: \varphi$. Then,

$$\Phi^*_{|\circ} = \tilde{\Phi}^*_{|\circ}, \tag{22}$$

if and only if

$$\mathcal{D}_I(y^\alpha \circ \Phi)(x,0) = \mathcal{D}_I(y^\alpha \circ \tilde{\Phi})(x,0), \quad \forall\, \alpha, \quad \forall\, I \in \mathbb{I}_0^q, \quad \forall\, x \in |\Omega|. \tag{23}$$

Proof: Since $\Phi^*_{|\circ} f = \iota^* \Sigma_{I \in \mathbb{I}_0^q} \eta^I \mathcal{D}_I(f \circ \Phi)$ condition (23) just means that $\Phi^*_{|\circ} y^\alpha = \tilde{\Phi}^*_{|\circ} y^\alpha$, $\forall \alpha$, and hence is a trivial consequence of (22). Conversely if (23) is true then we recover (22) by applying (20) for $\phi^* = \Phi^*_{|\circ}$ and $\phi^* = \tilde{\Phi}^*_{|\circ}$ and with $y_0 = \varphi(x_0)$. ■

It is natural to define the following equivalence relation in $\mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q, \mathcal{N})$: for any $\Phi, \tilde{\Phi} \in \mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q, \mathcal{N})$,

$$\Phi \sim \tilde{\Phi} \Leftrightarrow \Phi^*_{|\circ} = \tilde{\Phi}^*_{|\circ}.$$

Then, clearly morphisms in $\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}), \mathcal{C}^\infty(\Omega)_0)$ are in one to one correspondence with equivalence classes in $\mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q, \mathcal{N})/\sim$. This gives us a direct geometric picture (which we shall discuss below) of a map $\phi : \mathbb{R}^{p|q} \supset \Omega \to \mathcal{N}$ (thought as dual to a morphism ϕ^* in $\mathrm{Mor}(\mathcal{C}^\infty(\mathcal{N}), \mathcal{C}^\infty(\Omega)_0)$): it can be identified with a class of maps in $\mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q, \mathcal{N})/\sim$, i.e., a map of $|\Omega|$ into $\mathcal{N}$ surrounded by a family of infinitesimal deformations inside $\mathcal{N}$.

C. The chain rule for the operators $\mathcal{D}_I$

We exploit relation (20) again but we use a different expression for the Taylor polynomial,

$$P^{[q/2]}_{F,y_0}(y) = \sum_{k=0}^{[q/2]} \frac{1}{k!} \sum_{\alpha_1,\ldots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_k}}(y_0)(y^{\alpha_1} - y_0^{\alpha_1}) \cdots (y^{\alpha_k} - y_0^{\alpha_k}).$$

Hence, by (20)

$$\Phi^*_{|\circ} f(x_0) = \sum_{k=0}^{[q/2]} \frac{1}{k!} \sum_{\alpha_1,\ldots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1} \cdots \partial y^{\alpha_k}}(y_0) \prod_{\ell=1}^{k} (\Phi^*_{|\circ} y^{\alpha_\ell} - y_0^{\alpha_\ell}). \tag{24}$$

However, since

$$\Phi^*_{|\circ} y^\alpha(x_0) - y_0^\alpha = \sum_{I \in \mathbb{I}_2^q} \eta^I \mathcal{D}_I(y^\alpha \circ \Phi)(x_0,0),$$

we deduce by substitution,

$$\Phi^*_{|\circ}f(x_0) = F(y_0) + \sum_{k=1}^{[q/2]} \frac{1}{k!} \sum_{I;I_1,\ldots,I_k \in \mathbb{I}_0^q} \eta^I \epsilon_I^{I_1\cdots I_k} \sum_{\alpha_1,\ldots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1}\cdots\partial y^{\alpha_k}}(y_0) \prod_{\ell=1}^{k} \mathcal{D}_{I_\ell}(y^{\alpha_\ell}\circ\Phi)(x_0,0).$$

However, on the other hand, we have

$$\Phi^*_{|\circ}f(x_0) = f\circ\varphi(x_0) + \sum_{I\in\mathbb{I}_2^q} \eta^I \mathcal{D}_I(f\circ\Phi)(x_0,0) = F(y_0) + \sum_{I\in\mathbb{I}_2^q} \eta^I \mathcal{D}_I(F\circ y\circ\Phi)(x_0,0).$$

These two relations give us by an identification an expression for each $\mathcal{D}_I(F\circ y\circ\Phi)(x_0,0)$ in terms of $\mathcal{D}_I(y^\alpha\circ\Phi)(x_0,0)$. By setting $Y^\alpha := y^\alpha\circ\Phi$, it can be formulated as follows.

Proposition 2.2: For any map $Y\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q,\mathbb{R}^n)$, for any $x_0\in|\Omega|$, for any open neighborhood V of $y_0 := Y(x_0,0)$ in $\mathbb{R}^n$, and for any map $F\in\mathcal{C}^\infty(V)$, we have

$$\mathcal{D}_I(F\circ Y)(x_0,0) = \sum_{k\geq 0} \frac{1}{k!} \sum_{I_1,\ldots,I_k\in\mathbb{I}_0^q} \epsilon_I^{I_1\cdots I_k} \sum_{\alpha_1,\ldots,\alpha_k=1}^{n} \frac{\partial^k F}{\partial y^{\alpha_1}\cdots\partial y^{\alpha_k}}(y_0) \prod_{\ell=1}^{k} \mathcal{D}_{I_\ell} Y^{\alpha_\ell}(x_0,0), \quad \forall\, I\in\mathbb{I}_2^q. \tag{25}$$

1. An application

We use a specialization of identity (25) by choosing $\mathbb{R}^n = \Lambda_+^{2*}\mathbb{R}^q$ and by substituting to Y a smooth map $S:\Lambda_+^{2*}\mathbb{R}^q \to \Lambda_+^{2*}\mathbb{R}^q$ such that $S(0)=0$. We hence get

$$\mathcal{D}_I(F\circ S)(0) = \sum_{p\geq 0} \frac{1}{p!} \sum_{I_1,\ldots,I_p\in\mathbb{I}_0^q} \epsilon_I^{I_1\cdots I_p} \sum_{J_1,\ldots,J_p\in\mathbb{I}_0^q} \frac{\partial^p F}{\partial\mathfrak{s}^{J_1}\cdots\partial\mathfrak{s}^{J_p}}(0)(\mathcal{D}_{I_1}S^{J_1}\cdots\mathcal{D}_{I_p}S^{J_p})(0).$$

In the special case where $\mathcal{D}_I S^J(0) = \delta_I^J$, this simplifies to

$$\mathcal{D}_I(F\circ S)(0) = \sum_{p\geq 0} \frac{1}{p!} \sum_{I_1,\ldots,I_p\in\mathbb{I}_0^q} \epsilon_I^{I_1\cdots I_p} \frac{\partial^p F}{\partial\mathfrak{s}^{I_1}\cdots\partial\mathfrak{s}^{I_p}}(0) = \mathcal{D}_I F(0). \tag{26}$$

We conclude that if $S:\Lambda_+^{2*}\mathbb{R}^q \to \Lambda_+^{2*}\mathbb{R}^q$ is a smooth diffeomorphism such that $S(0)=0$ and $\mathcal{D}_I S^J(0) = \delta_I^J$, then $V \sim V\circ S$. Hence, if we define

$$\mathcal{T}_q := \{\text{diffeomorphisms } S:\Lambda_+^{2*}\mathbb{R}^q \to \Lambda_+^{2*}\mathbb{R}^q \mid S(0)=0, \mathcal{D}_I S^J(0) = \delta_I^J\},$$

then we remark that $\mathcal{T}_q$ is a group for the composition law [another consequence of (26)] and we see that the morphism $\Phi^*_{|\circ}$ is characterized by the behavior of Φ modulo the action of $\mathcal{T}_q$ hence by duality we can identify a map $T:\mathbb{R}^{0|q}\to\mathcal{N}$ with a class of maps from $\Lambda_+^{2*}\mathbb{R}^q$ to $\mathcal{N}$ modulo the action of $\mathcal{T}_q$ on $\Lambda_+^{2*}\mathbb{R}^q$.

D. Leibniz identities for the operators $\mathcal{D}_I$

The operators $\mathcal{D}_I$ satisfy nice Leibniz-type identities.

Proposition 2.3: For any pair of functions $a,b\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q)$ and for any $I\in\mathbb{I}_0^q$,

$$\mathcal{D}_I(ab) = \sum_{I_1,I_2\in\mathbb{I}_0^q} \epsilon_I^{I_1I_2}(\mathcal{D}_{I_1}a)(\mathcal{D}_{I_2}b), \tag{27}$$

where in the summation we allow $(I_1,I_2)=(\emptyset,I)$ or $(I,\emptyset)$.

Proof: By applying relation (8) for $\vartheta := \Sigma_I \eta^I(\partial/\partial\mathfrak{s}^I)$, we obtain

$$e^\vartheta(ab) = (e^\vartheta a)(e^\vartheta b), \quad \forall\, a, \quad b\in\mathcal{C}^\infty(|\Omega|\times\Lambda_+^{2*}\mathbb{R}^q). \tag{28}$$

By using $e^\vartheta a = \Sigma_{I\in\mathbb{I}_0^q}\eta^I(\mathcal{D}_I a)$ to develop this relation, we obtain (27). ■

A straightforward consequence of Proposition 2.3 is that the set

$$\mathcal{I}^q(|\Omega|) := \{f \in \mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q) | \; \forall \, I \in \mathbb{I}_0^q, \iota^*(\mathcal{D}_I f) = 0\}$$

is an ideal of the commutative algebra $(\mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q), +, \cdot)$. Hence, the quotient $\mathcal{A}^q(|\Omega|) := \mathcal{C}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q)/\mathcal{I}^q(|\Omega|)$ is an algebra over $\mathbb{R}$. We will recover that this algebra is isomorphic to $\mathcal{C}^\infty(|\Omega|)[\eta^1, \ldots, \eta^q]_0$. First, we may also write $\mathcal{A}^q(|\Omega|) \simeq \mathcal{C}_{pol}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q)/\mathcal{I}_{pol}^q(|\Omega|)$, where $\mathcal{C}_{pol}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q)$ is the subalgebra of smooth functions on $|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q$ which have a polynomial dependence in the variables $\mathfrak{s}^I$ and $\mathcal{I}_{pol}^q(|\Omega|) = \mathcal{C}_{pol}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q) \cap \mathcal{I}^q(|\Omega|)$. Any function $f \in \mathcal{C}_{pol}^\infty(|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q)$ can be written as

$$f(x,\mathfrak{s}) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{I_1,\ldots,I_n \in \mathbb{I}_0^q} \frac{\partial^n f}{\partial \mathfrak{s}^{I_1} \cdots \partial \mathfrak{s}^{I_n}}(x,0)\mathfrak{s}^{I_1} \cdots \mathfrak{s}^{I_n}.$$

Now $f \in \mathcal{I}_{pol}^q(|\Omega|)$ if and only if

$$\frac{\partial f}{\partial \mathfrak{s}^I}(x,0) = -\sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I_1,\ldots,I_n \in \mathbb{I}_0^q} \epsilon_I^{I_1 \cdots I_n} \frac{\partial^n f}{\partial \mathfrak{s}^{I_1} \cdots \partial \mathfrak{s}^{I_n}}(x,0), \quad \forall \, I \in \mathbb{I}_2^q, \quad \forall \, x \in |\Omega|.$$

Hence, for such a function,

$$f(x,\mathfrak{s}) = \sum_{n=2}^{\infty} \frac{1}{n!} \sum_{I_1,\ldots,I_n \in \mathbb{I}_0^q} \frac{\partial^n f}{\partial \mathfrak{s}^{I_1} \cdots \partial \mathfrak{s}^{I_n}}(x,0)\left[\mathfrak{s}^{I_1} \cdots \mathfrak{s}^{I_n} - \sum_{I \in \mathbb{I}_0^q} \epsilon_I^{I_1 \cdots I_n} \mathfrak{s}^I\right].$$

So $\mathcal{I}_{pol}^q(|\Omega|)$ is the ideal spanned by the family

$$\left(\mathfrak{s}^{I_1} \cdots \mathfrak{s}^{I_n} - \sum_{I \in \mathbb{I}_0^q} \epsilon_I^{I_1 \cdots I_n} \mathfrak{s}^I\right)_{n \geq 2, I_1,\ldots,I_n \in \mathbb{I}_0^q}.$$

Hence, it is clear that the linear application from $\mathrm{Span}_{\mathcal{C}^\infty(|\Omega|)}(\mathfrak{s}^I)$ to $\mathrm{Span}_{\mathcal{C}^\infty(|\Omega|)}(\eta^I)$ which maps $\mathfrak{s}^I$ to η^I can be extended in an unique way into an algebra *isomorphism* from $\mathcal{A}^q(|\Omega|)$ to $\mathcal{C}^\infty(|\Omega|) \otimes \mathbb{R}[\eta^1, \ldots, \eta^q]_0$. Moreover, this isomorphism is nothing but

$$\iota^* \circ e^{\vartheta}: \mathcal{A}^q(|\Omega|) \to \mathcal{C}^\infty(|\Omega|) \otimes \mathbb{R}[\eta^1, \ldots, \eta^q]_0,$$

$$f \mapsto \iota^* \circ (e^{\vartheta} f).$$

E. An alternative description using schemes

Let us start by assuming that $p=0$ for simpliclity. The "geometry" of $\mathbb{R}^{0|q}$ appears to be related with another "geometric" object living in a neighbourhood of 0 in $\Lambda_+^{2*}\mathbb{R}^q$ and such that the ring of functions on it is isomorphic to the algebra $\mathcal{A}^q := \mathcal{A}^q(\{0\})$ that we just constructed. It turns out that this object can be described accurately by using Grothendieck's theory of schemes. We refer to Ref. 8 for a complete and comprehensive presentation of this theory and recall here only notions which may be relevant for us. To any commutative ring R, we can associate an (affine) scheme which is called the *spectrum* of R and is denoted by $\mathrm{Spec}\, R$. It consists in three data: a set of points, a topology (the Zariski topology), and a sheaf of regular functions on it. The set of points is simply the set of prime ideals of R. In the case at hand where $R = \mathcal{A}^q$, the prime ideals are of the form[2]

[2]Here if $a_1, \ldots, a_p \in R$, we denote by $(a_1, \ldots, a_p)$ the ideal $\{a_1 f_1 + \cdots + a_p f_p | f_1, \ldots, f_p \in R\}$.

$$\mathfrak{A} = \left(\sum_{I \in \mathbb{I}_2^q} \alpha_{1,I}\mathfrak{s}^I, \ldots, \sum_{I \in \mathbb{I}_2^q} \alpha_{p,I}\mathfrak{s}^I \right),$$

where $p \in \mathbb{N}$ and the $\alpha_{j,I}$ are real parameters so that, $\forall f,g \in R$, if $fg \in \mathfrak{A}$ then either $f \in \mathfrak{A}$ or $g \in \mathfrak{A}$. The "point" which corresponds to such an ideal is the "generic point" living in the vector subspace defined by $\Sigma_{I\in\mathbb{I}_2^q}\alpha_{1,I}\mathfrak{s}^I = \cdots = \Sigma_{I\in\mathbb{I}_2^q}\alpha_{p,I}\mathfrak{s}^I = 0$. Note that by dualizing the canonical ring morphism $\mathcal{C}^\infty_{pol}(\Lambda_+^{2*}\mathbb{R}^q) \to \mathcal{A}^q$, we can view Spec $\mathcal{A}^q$ as embedded in $\Lambda_+^{2*}\mathbb{R}^q$.

Example 2.1: For all $q \in \mathbb{N}$, set $\mathcal{A}^q_{(2)} := \{\Sigma_{1\leqslant i \leqslant j < q}\alpha_{ij}\mathfrak{s}^{ij}\}$. Then, for any $1 \leqslant p \leqslant (q(q-1)/2)$ if $f_1, \ldots, f_p$ are p linearly independants vectors of $\mathcal{A}^q_{(2)}$, then $(f_1, \ldots, f_p)$ is a prime ideal of $\mathcal{A}^q$ (and for $p=(q(q-1)/2)$ it is the maximal ideal, see below). For $q \leqslant 4$, there are no other prime ideals. However, for $q \geqslant 5$, other instances of prime ideal exist like $(\mathfrak{s}^{1234}+\mathfrak{s}^{15})$ for $q=5$.

So in general the concept of a point of a scheme is different from the usual one, except if the point is a maximal ideal. For $R=\mathcal{A}^q$, there is only one maximal ideal[3] which is $(\mathfrak{s}^I)_{I\in\mathbb{I}^q(2)}$: it corresponds to the point $0 \in \Lambda_+^{2*}\mathbb{R}^q$. This point is also the unique closed point for the Zariski topology, all the other ones are open[4]. For $p \geqslant 1$, similarly we can associate to any open subset Ω of $\mathbb{R}^{p|q}$ the scheme associated with $\mathcal{A}^q(|\Omega|)$, and we can picture its spectrum Spec $\mathcal{A}^q(|\Omega|)$ as an object embedded in $|\Omega| \times \Lambda_+^{2*}\mathbb{R}^q$. Then, we can interpret our results as follows: first for any morphism $\phi^*: \mathcal{C}^\infty(\mathcal{N}) \to \mathcal{C}^\infty(\Omega)_0$, we have found that there exists a family of maps $\Phi: |\Omega| \times \Lambda_+^{2*}\mathbb{R}^q \to \mathcal{N}$ (a class of maps modulo $\sim$) such that $\Phi^*_{|\circ} = \phi^*$. We can simply denote by $\Phi_{|\circ} = \phi$ this relation. Second through the algebra isomorphism $\iota^* \circ e^{\vartheta}: \mathcal{A}^q(|\Omega|) \to \mathcal{C}^\infty(\Omega)_0$ constructed in the previous section, we can decompose $\Phi^*_{|\circ} = \iota^* \circ e^{\vartheta} \circ \Phi^*_{|\star}$, where

$$(\Phi^*_{|\star}f)(x,\mathfrak{s}) := \sum_{I\in\mathbb{I}_0^q} \mathfrak{s}^I\mathcal{D}_I(f\circ\Phi)(x,0) = \left(e^{\Sigma_{I\in\mathbb{I}_2^q}\mathfrak{s}^I(\partial/\partial\mathfrak{s}^I)} f\circ\Phi\right)(x,0) = (f\circ\Phi)(x,\mathfrak{s}) \bmod \mathcal{I}^q(|\Omega|).$$

Hence, $\Phi_{|\star}$ can be thought as a restriction of Φ to Spec $\mathcal{A}^q(|\Omega|)$. Moreover, if we denote by T_Ω the isomorphism from Spec $\mathcal{C}^\infty(\Omega)_0$ to Spec$\mathcal{A}^q(|\Omega|)$ which is dual of $\iota^*\circ e^{\vartheta}$, we can dualize the relation $\Phi^*_{|\circ} = \iota^*\circ e^{\vartheta}\circ\Phi^*_{|\star}$ as $\phi = \Phi_\circ = \Phi_{|\star}\circ T_\Omega$. All that can be summarized in the following diagrams:

$$\begin{array}{ccccccc} \Omega & & & \mathcal{C}^\infty(\Omega)_0 & & \\ T_\Omega\downarrow & \searrow \phi = \Phi_{|\circ} & & \iota^*\circ e^{\vartheta}\uparrow & \nwarrow \phi^* = \Phi^*_{|\circ} & \\ \text{Spec } \mathcal{A}^q(|\Omega|) & \underset{\Phi_{|\star}}{\rightarrow} & \mathcal{N} & \mathcal{A}^q(|\Omega|) & \underset{\Phi^*_{|\star}}{\leftarrow} & \mathcal{C}^\infty(\mathcal{N}) \end{array}.$$

IV. SUPERMANIFOLDS

The previous and provisional definition of $\mathbb{R}^{p|q}$ can be recast in the more sophisticated language of ringed space, then functions on such superspaces can be seen as sections of sheaves of superalgebras. Let us recall the definition of a supermanifold according to Ref. 11, 12, 7, and 17. First, one defines the space $\mathbb{R}^{p|q}$ to be the topological space $\mathbb{R}^p$ endowed with the sheaf of real superalgebras $\mathcal{O}_{\mathbb{R}^{p|q}}$ whose sections are smooth functions on open subsets of $\mathbb{R}^p$, with values in $\mathbb{R}[\theta^1, \ldots, \theta^q]$, where $\theta^1, \ldots, \theta^q$ are odd variables. So for any open subset $|\Omega|$ of $\mathbb{R}^p$, the superal-

[3]Rings with an unique maximal ideal are *called local rings*.

[4]Then, R can be interpreted as the ring of functions on the points of Spec R: to each prime ideal $\mathfrak{A}$ of R we associate the *residue field* $R/\mathfrak{A}$ and each $f \in R$ has an image $[f \bmod \mathfrak{A}]$ in $R/\mathfrak{A}$ through the canonical projection, so each $f \in R$ is identified with the "map" f: Spec $R \to$ residue fields and $\mathfrak{A} \mapsto [f \bmod \mathfrak{A}]$. Here, we can interpret $[f \bmod \mathfrak{A}]$ as being isomorphic to the set of functions on the zero set of all functions contained in $\mathfrak{A}$. A more refined description of functions on Spec R is given by the construction of a sheaf $\mathcal{O}_{\text{Spec } R}$ on the topological space Spec R such that the ring of global sections of $\mathcal{O}_{\text{Spec } R}$ is R (see Ref. 8).

gebra $\Gamma(|\Omega|,\mathcal{O}_\Omega)$ of sections of $\mathcal{O}_{\mathbb{R}^{p|q}}$ over $|\Omega|$ is spanned over $\mathcal{C}^\infty(\Omega)$ by $\theta^1,\ldots,\theta^q$: $f=\Sigma_{I\in\mathbb{I}^q} f_I\theta^I$, $\forall f\in\Gamma(|\Omega|,\mathcal{O}_\Omega)$, where $f_I\in\mathcal{C}^\infty(|\Omega|)$, $\forall I\in\mathbb{I}^q$. The open subsets of $\mathcal{M}$ are then the objects $\Omega=(|\Omega|,\mathcal{O}_\Omega)$, where $|\Omega|$ is an open subset of $\mathbb{R}^p$. If Ω and Ω' are two such open subsets then a *morphism* $\varphi:\Omega\to\Omega'$ is given by a continuous map $|\varphi|:|\Omega|\to|\Omega'|$ and an even morphism φ^* of sheaves of superalbegras from $|\varphi|^*\mathcal{O}_{\Omega'}$ to $\mathcal{O}_\Omega$ (this implies in particular that $|\varphi|$ should be smooth).[5] If furthermore $|\varphi|$ is a *homeomorphism* and φ^* is an isomorphism of sheaves, we then say that φ is an *isomorphism*.

A supermanifold $\mathcal{M}$ of dimension $p|q$ is a topological space $|\mathcal{M}|$ endowed with a sheaf $\mathcal{O}_\mathcal{M}$ of real superalgebras which is *locally isomorphic* to $\mathbb{R}^{p|q}$. An *open subset* U of $\mathcal{M}$ is an open subset $|U|$ of $|\mathcal{M}|$ endowed with the sheaf of superalgebras $\mathcal{O}_U$ which is the restriction of $\mathcal{O}_\mathcal{M}$ over $|U|$. By saying *locally isomorphic*, we mean that for any point $m\in|\mathcal{M}|$ there is an open subset U of $\mathcal{M}$ such that $m\in|U|$, an open subset V of $\mathbb{R}^{p|q}$ and a isomorphism of sheaves X from U to V. There is, however, a difference with $\mathbb{R}^{p|q}$: the sheaf $\mathcal{O}_{|U|}$ of smooth real valued functions on $|U|$ is not embedded in a canonical way in $\mathcal{O}_U$.[6] However, it may be identified with $\mathcal{O}_U/\mathcal{J}$, where $\mathcal{J}$ is the nilpotent ideal $(\theta^1,\ldots,\theta^q)$.[7] Then, the isomorphism $X:U\to V$ plays the role of a local chart and the pull-back image of the canonical coordinates $x^1,\ldots,x^p,\theta^1,\ldots,\theta^q$ by X are the analogs of local coordinates.

A. Maps from an open subset of $\mathbb{R}^{p|q}$ to a supermanifold

Let $\mathcal{N}$ be a supermanifold of dimension $n|m$, U be an open subset of $\mathcal{N}$ and $Y:U\to V\subset\mathbb{R}^{n|m}$ be a local chart (i.e., a sheaf isomorphism). Let $y^1,\ldots,y^n,\psi^1,\ldots,\psi^m$ be the canonical coordinates on $\mathbb{R}^{n|m}$. By abusing notations, we write also $y^\alpha:\simeq Y^*y^\alpha$ and $\psi^j:\simeq Y^*\psi^j$. Then, any section f of $\mathcal{O}_\mathcal{N}$ over U decomposes as

$$f=\sum_{J\in\mathbb{I}^m} F_J(y^1,\ldots,y^n)\psi^J,$$

where $F_J\in\mathcal{C}^\infty(|V|)$, $\forall J\in\mathbb{I}^m$ and $\psi^J:=\psi^{j_1}\ldots\psi^{j_k}$, $\forall J=(j_1,\ldots,j_k)$.

Now, let Ω be an open subset of $\mathbb{R}^{p|q}$ and ϕ be a map from Ω to U, i.e., by dualizing an even morphism ϕ^* of superalgebra from $\mathcal{C}^\infty(U)$ to $\mathcal{C}^\infty(\Omega)$. Then, the morphism property of ϕ^* implies that

$$\phi^*f=\sum_{J\in\mathbb{I}^m}\phi^*(F_J\circ(y^1,\ldots,y^n))\chi^J,$$

where $\chi^j:=\phi^*\psi^j$, $\forall j\in[\![1,m]\!]$, $\chi^J:=\chi^{j_1}\cdots\chi^{j_k}$, $\forall J=(j_1,\cdots,j_k)$, and each $\phi^*(F_J\circ(y^1,\cdots,y^n))$ can be expressed in terms of $(\phi^*y^1,\cdots,\phi^*y^n)$ by using Proposition 2.1. Hence, *ϕ^*f can be computed as soon as we know $(\phi^*y^1,\ldots,\phi^*y^n)$ and $(\phi^*\psi^1,\ldots,\phi^*\psi^m)$*. This generalizes Proposition 2.1.

B. The use of the functor of point

When we study supersymmetric differential equations, a brutal application of the previous definitions suffers from incoherences. These are largely discussed in Ref. 9 An example is the superspace formulation of supergeodesics on an Euclidean sphere S^n. Let us view S^n as a submanifold of $\mathbb{R}^{n+1}$ and we consider the "supertime" $\mathbb{R}^{1|1}$ with coordinates t,θ. Then, we look at maps $\phi:\mathbb{R}^{1|1}\to S^n$ (i.e., morphisms ϕ^* from $\mathcal{C}^\infty(S^n)$ to $\mathcal{C}^\infty(\mathbb{R}^{1|1})$) which are solutions of

[5]Then, when restricted to the subsheaf $\mathcal{O}_{|\Omega'|}$ of smooth functions on $|\Omega'|$, φ^* corresponds to the usual pull-back operation on functions by $|\varphi|$.

[6]That is, by dualizing there is no canonical fibration $\mathcal{M}\to|\mathcal{M}|$.

[7]That is, by dualizing the projection map $\mathcal{O}_\mathcal{M}\to\mathcal{O}_\mathcal{M}/\mathcal{J}$, there is a canonical embbeding $|\mathcal{M}|\hookrightarrow\mathcal{M}$.

$$D\frac{\partial\phi}{\partial t}+\left\langle D\phi,\frac{\partial\phi}{\partial t}\right\rangle\phi=0,$$

where $D:=(\partial/\partial\theta)-\theta(\partial/\partial t)$. This means that the image of any coordinate function y^α on $\mathbb{R}^{n+1}\supset S^n$ by $D(\partial\phi^*/\partial t)+\langle D\phi^*,\partial\phi^*/\partial t\rangle\phi^*$ vanishes. Set $\phi^*y=\varphi+\theta\psi$, where $\varphi\in\mathcal{C}^\infty(\mathbb{R},S^n)$ and ψ is a section of φ^*TS^n. A first problem is that ψ should be *odd*: this is the usual requirement made by physicists and in our context, it is imposed by the fact that ϕ^* should be an even morphism because θ is odd. This could be cared by introducing a further (dumb) odd variable, say, η, and by letting $\psi=\eta v$, where v is an ordinary section of φ^*TS^n. However, then the next problem is that the preceding equation is equivalent to the system,

$$\frac{\partial^2\varphi}{(\partial t)^2}+\left|\frac{\partial\varphi}{\partial t}\right|^2\varphi=-\left\langle\psi,\frac{\partial\varphi}{\partial t}\right\rangle\psi\quad\text{and}\quad\frac{\partial\psi}{\partial t}+\left\langle\psi,\frac{\partial\varphi}{\partial t}\right\rangle\varphi=0.$$

We see that the right hand side of the first equation contains two times ψ, hence $\eta\eta$, which vanishes. So we should build ψ out of a linear combination of at least two dumb odd variables, say, η^1 and η^2. However, then we see that φ cannot be an ordinary map into S^n, still because of the first equation. Note that all these difficulties are absent in the differential geometric point of view used in Refs. 5 and 14 for defining supermanifolds.

An alternative solution is proposed in Refs. 7 and 19 (see also Ref. 17): it relies on Grothendieck's notion of *functor of points* in algebraic geometry. We will adopt that point of view in the following. For any $L\in\mathbb{N}$, we set $B:=\mathbb{R}^{0|L}$. The starting point is to see a map ϕ from a supermanifold $\mathcal{M}$ of dimension $p|q$ into a supermanifold $\mathcal{N}$ of dimension $n|m$ as *a functor from* $\mathcal{C}^\infty(B)$ *to even morphisms* $\phi^*:\mathcal{C}^\infty(\mathcal{N})\to\mathcal{C}^\infty(\mathcal{M}\times B)$. So we need to understand morphisms ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\mathcal{M}\times B)$: from a technical point of view, nothing is new and it suffices to apply all the previous results. For simplicity, we restrict ourself to the case where the target manifold $\mathcal{N}$ is an ordinary manifold and the source domain Ω is an open subset of $\mathbb{R}^{p|q}$.

C. Our final representation of a map from an open subset of $\mathbb{R}^{p|q}$ to an ordinary manifold

It is convenient to note $(x^1,\ldots,x^p)$ and $(\theta^1,\ldots,\theta^q)$, respectively, the even and the odd local coordinates on Ω and $(\eta^1,\ldots,\eta^L)$ the odd coordinates on B. Hence, for any open subset Ω of $\mathcal{M}$, $\mathcal{C}^\infty(\Omega\times B)\simeq\mathcal{C}^\infty(|\Omega|)[\theta^1,\ldots,\theta^q,\eta^1,\ldots,\eta^L]$. Furthermore, we note $\mathbb{A}^q(0)=\{\emptyset\}$ and for any $k\in\mathbb{N}^*$, $\mathbb{A}^q(k):=\{(a_1\cdots a_k)\in[\![1,q]\!]^k|a_1<\cdots<a_k\}$. We denote by $A=(a_1\cdots a_k)$ an element of $\mathbb{A}^q(k)$ and we then write $\theta^A:=\theta^{a_1}\cdots\theta^{a_k}$. We let $\mathbb{A}^q:=\cup_{k=0}^q\mathbb{A}^q(k)$, $\mathbb{A}_0^q:=\cup_{k=0}^{[q/2]}\mathbb{A}^q(2k)$, $\mathbb{A}_1^q:=\cup_{k=0}^{[(q-1)/2]}\mathbb{A}^q(2k+1)$, $\mathbb{A}_2^q:=\cup_{k=1}^{[q/2]}\mathbb{A}^q(2k)$, and $\mathbb{A}_+^q:=\mathbb{A}_1^q\cup\mathbb{A}_2^q$. Lastly, we set $\mathbb{AI}:=\{AI|A\in\mathbb{A}^q,I\in\mathbb{I}^L\}$ and defining the length of AI to be the some of the lengthes of A and I, we define similarly $\mathbb{AI}(j)$, $\mathbb{AI}_0$, $\mathbb{AI}_1$, and $\mathbb{AI}_2$. Hence any (even) function $f\in\mathcal{C}^\infty(\Omega\times B)$ (where Ω is an open subset of $\mathbb{R}^{p|q}$) can be decomposed as $f=\Sigma_{AI\in\mathbb{AI}_0}\theta^A\eta^I f_{AI}$, where $f_{AI}\in\mathcal{C}^\infty(|\Omega|)$, $\forall AI\in\mathbb{AI}_0$.

Then, Theorem 1.1 implies that for any morphism ϕ^* from $\mathcal{C}^\infty(\mathcal{N})$ to $\mathcal{C}^\infty(\Omega\times B)$, there exists a smooth map $\varphi\in\mathcal{C}^\infty(|\Omega|,\mathcal{N})$ and a smooth family $(\xi_{AI})_{AI\in\mathbb{AI}_2}$ of sections of $\pi^*T\mathcal{N}$ defined on a neighborhood of the graph of φ in $|\Omega|\times\mathcal{N}$ such that if $\Xi:=\Sigma_{AI\in\mathbb{AI}_2}\xi_{AI}\theta^A\eta^I$, then $\forall f\in\mathcal{C}^\infty(\mathcal{N})$, $\phi^*f=(1\times f)^*(e^\Xi f)$. Moreover, thanks to Proposition 1.1, the vector fields $(\xi_{AI})_{AI\in\mathbb{AI}_2}$ can be chosen in order to commute pairwise. We decompose Ξ as

$$\Xi=\sum_{A\in\mathbb{A}^q}\theta^A\Xi_A=\Xi_\emptyset+\sum_{a\in\mathbb{A}^q(1)}\theta^a\Xi_a+\sum_{(a_1,a_2)\in\mathbb{A}^q(2)}\theta^{a_1}\theta^{a_2}\Xi_{a_1a_2}+\cdots,$$

where $\Xi_A=\Sigma_{I\in\mathbb{I}_1^L}\xi_{AI}\eta^I$, $\forall A\in\mathbb{A}_1^q$ and $\Xi_A=\Sigma_{I\in\mathbb{I}_2^L}\xi_{AI}\eta^I$, $\forall A\in\mathbb{A}_0^q$. In particular, $\Xi_\emptyset=\Sigma_{I\in\mathbb{I}_2^L}\xi_{\emptyset I}\eta^I$ and we see that Ξ_A is odd if A is odd and is even if A is even. Then, the relations $[\xi_{AI},\xi_{A'I'}]=0$ implies that the vector fields Ξ_A *supercommute* pairwise, i.e.,

$$\Xi_A \Xi_{A'} - (-1)^{kk'} \Xi_{A'} \Xi_A = 0, \quad \forall A \in \mathbb{A}^q(k), \quad \forall A' \in \mathbb{A}^q(k').$$

This is equivalent to the fact that $[\theta^A \Xi_A, \theta^{A'} \Xi_{A'}]=0$, $\forall A, A' \in \mathbb{A}^q$. This last commutation relation implies that

$$e^{\Xi} = e^{\Sigma_{A \in \mathbb{A}^q} \theta^A \Xi_A} = e^{\Xi_\varnothing} \prod_{A \in \mathbb{A}_+^q} e^{\theta^A \Xi_A},$$

Hence,

$$\phi^* f = (1 \times \varphi)^* \left(e^{\Xi_\varnothing} \prod_{A \in \mathbb{A}_+^q} e^{\theta^A \Xi_A} f \right), \quad \forall f \in \mathcal{C}^\infty(\mathcal{N}). \tag{29}$$

Alternatively, one can integrate these vector fields as in the second section of this paper. Let us denote by $(\mathfrak{s}^{AI})_{AI \in \mathbb{AI}_2}$ the coordinates on $\Lambda_+^{2*}\mathbb{R}^{q+L}$ and

$$\vartheta := \sum_{AI \in \mathbb{AI}_2} \theta^A \eta^I \frac{\partial}{\partial \mathfrak{s}^{AI}} = \sum_{A \in \mathbb{A}^q} \theta^A \vartheta_A,$$

where $\vartheta_A := \Sigma_{I \in \mathbb{I}_1^L} \eta^I (\partial / \partial \mathfrak{s}^{AI})$, $\forall A \in \mathbb{A}_1^q$ and $\vartheta_A = \Sigma_{I \in \mathbb{I}_2^L} \eta^I (\partial / \partial \mathfrak{s}^{AI})$, $\forall A \in \mathbb{A}_0^q$. Then, there exists a smooth map Φ from a neighborhood of $|\Omega| \times \{0\}$ in $|\Omega| \times \Lambda_+^{2*}\mathbb{R}^{q+L}$ to $\mathcal{N}$ such that

$$\phi^* f = \iota^* \left(e^{\vartheta_\varnothing} \prod_{A \in \mathbb{A}_+^q} e^{\theta^A \vartheta_A} (f \circ \Phi) \right), \quad \forall f \in \mathcal{C}^\infty(\mathcal{N}). \tag{30}$$

D. Forgetting the ugly notations

We now propose some abuses and adaptations of notation to lighten all this description. However, we try to keep the important property that each Ξ_A is vector field[8] defined along the graph of φ (even if it has coefficients in a Grassmann algebra). First of all, we simply write $\varphi^* : \simeq (1 \times \varphi)^*$. Second, the operator $e^{\Xi_\varnothing}$ has no direct geometrical signification and his presence there is only necessary to "thicken" φ^*, so that we can absorb it by a redefinition of φ^*,

$$\varphi^* : \simeq \varphi^* e^{\Xi_\varnothing} : \simeq (1 \times \varphi)^* e^{\Xi_\varnothing}.$$

We can hence rewrite (29) as

$$\phi^* f = \varphi^* \left(\prod_{A \in \mathbb{A}_+^q} e^{\theta^A \Xi_A} \right) f, \quad \forall f \in \mathcal{C}^\infty(\mathcal{N}). \tag{31}$$

For example, if $q=2$, we have (keeping in mind the fact that Ξ_1 and Ξ_2 are odd, whereas Ξ_{12} is even)

$$\phi^* f = \varphi^* (1 + \theta^1 \Xi_1)(1 + \theta^2 \Xi_2)(1 + \theta^1 \theta^2 \Xi_{12}) f = \varphi^* (1 + \theta^1 \Xi_1 + \theta^2 \Xi_2 + \theta^1 \theta^2 (\Xi_{12} - \Xi_1 \Xi_2)) f. \tag{32}$$

Similarly, relation (30) can be written as

$$\phi^* f = \left(\prod_{A \in \mathbb{A}_+^q} e^{\theta^A \vartheta_A} \right) \Phi^* f, \quad \forall f \in \mathcal{C}^\infty(\mathcal{N}). \tag{33}$$

[8] That is, a *first order* differential operator.

1. Use of a local chart on the target manifold

The use of relation (31) is particularly convenient if we assume that the image of $\phi:\Omega\to\mathcal{N}$ is contained in an open subset $U\subset\mathcal{N}$ on which there is a chart $y:U\to\mathbb{R}^n$. Indeed, Remark 1.1 tells us that we can choose the vector fields (ξ_I) in such a way that $\xi_I\xi_J y=0$ [see (14)]. This implies that $\Xi_A\Xi_{A'}y=0$, $\forall A,A'\in\mathbb{A}^q$. Now, what physicists denote "ϕ" or "$(\phi^\alpha)_\alpha$" is just $\phi^* y$ or $(\phi^* y^\alpha)_\alpha$ and then when they write the decomposition

$$\phi=\varphi+\sum_{A\in\mathbb{A}^q}\theta^A\psi_A, \tag{34}$$

it implies by using (31) that

$$\varphi+\sum_{A\in\mathbb{A}^q_+}\theta^A\psi_A=\phi^* y=\varphi^*\left(\prod_{A\in\mathbb{A}^q_+}e^{\theta^A\Xi_A}\right)y.$$

However, since $\Xi_A\Xi_{A'}y=0$ the development of the right hand side of this identity is particularly simple. We deduce

$$\varphi+\sum_{A\in\mathbb{A}^q_+}\theta^A\psi_A=\varphi^* y+\sum_{A\in\mathbb{A}^q_+}\theta^A\varphi^*\Xi_A y.$$

Hence, $\psi_A=\varphi^*\Xi_A y$, $\forall A\in\mathbb{A}^q_+$. Our last abuse of notation is to let $\psi_A\simeq\Xi_A$. So we reinterpret (34) as

$$\phi^*=\varphi^*\prod_{A\in\mathbb{A}^q_+}e^{\theta^A\psi_A},$$

where the rules to manipulate such an expression are

- each ψ_A acts as a first order differential operator to its right and
- two different ψ_A, $\psi_{A'}$ supercommute pairwise and with the θ^A's.

An example of application

Assume that we find in the physics literature a map ϕ from $\mathbb{R}^{p|2}$ to $\mathbb{R}$ which has the expression

$$\phi=\varphi+\theta^1\psi_1+\theta^2\psi_2+\theta^1\theta^2 F, \tag{35}$$

and we want to compute $\phi^* f\simeq f\circ\phi$, where $f\in\mathcal{C}^\infty(\mathbb{R})$. Then, we reinterpret (35) as

$$\phi^*=\varphi^*(1+\theta^1\psi_1)(1+\theta^2\psi_2)(1+\theta^1\theta^2 F).$$

Then,

$$\begin{aligned}\phi^* f&=\varphi^*(1+\theta^1\psi_1)(1+\theta^2\psi_2)(1+\theta^1\theta^2F)f=\varphi^* f+\theta^1\varphi^*\psi_1 f+\theta^2\varphi^*\psi_2 f+\theta^1\theta^2\varphi^* Ff-\theta^1\theta^2\varphi^*\psi_1\psi_2 f\\&=f\circ\varphi+\theta^1(f'\circ\varphi)\psi_1+\theta^2(f'\circ\varphi)\psi_2+\theta^1\theta^2[(f'\circ\varphi)F-(f''\circ\varphi)\psi_1\psi_2].\end{aligned}$$

[1] Bahraini, A., "Supersymétrie et géométrie complexe," Ph.D. thesis, Université Paris 7, 2004.
[2] Bartocci, C., Bruzzo, U., and Hernández-Ruipérez, D., *The Geometry of Supermanifolds* (Kluwer Academic, Dordrecht, 1991).
[3] Batchelor, M., "Two Aapproaches to Supermanifolds," Trans. Am. Math. Soc. **258**, 257–270 (1980).
[4] Berezin, F. A., *Introduction to Superanalysis* (Reidel, Dordrecht, 1987).
[5] Dewitt, B., *Supermanifolds* (Cambridge University Press, Cambridge, 1984).
[6] Deligne, P., and Freed, D., *Quantum Fields and Strings: A Course for Mathematicians* (Ref. 13).
[7] Deligne, P., and Morgan, J., *Quantum Fields and Strings: A Course for Mathematicians*, (Ref. 13), pp. 41–96.
[8] Eisenbud, D., and Harris, J., *The Geometry of Schemes* (Springer, Berlin, 2000).
[9] Freed, D., *Five Lectures on Supersymmetry* (AMS, Providence, 1999).
[10] Freund, P. G. O., *Introduction to Supersymmetry*, Cambridge Monographs on Mathematical Physics (Cambridge Uni-

versity Press, Cambridge, 1986).
[11] Leĭtes, D. A., "Introduction to the theory of supermanifolds," Usp. Mat. Nauk **35**, 3–57 (1980) [Russ. Math. Surveys **35**, 1–64 (1980)].
[12] Manin, Y. I., *Gauge Field Theory and Complex Geometry*, Grundlehren der Mathematischen Wissenschaften Vol. 289, (Springer-Verlag, Berlin, 1988).
[13] *Quantum Fields and Strings: A Course for Mathematicians*, edited by Deligne, P., Etingof, P., Freed, D., Jeffrey, L., Kazhdan, D., Morgan, J., Morrison, D., and Witten, E. (AMS, Providence, 1999), Vol. 1.
[14] Rogers, A., "A global theory of supermanifolds," J. Math. Phys. **21**, 1352–65 (1980); "Super Lies groups: global topology and local structures," *ibid.* **22**, 939–45 (1981).
[15] Rogers, A., *Supermanifolds: Theory and Applications* (World Scientific, Singapore, 2007).
[16] Salam, A., and Strathdee, J., "Super-gauge transformations," Nucl. Phys. B **76**, 477–482 (1974).
[17] Varadarajan, V. S., *Supersymmetry for Mathematicians: An Introduction*, Courant Lecture Notes (AMS, Providence, 2004).
[18] Vladimirov, V. S., and Volovich, I. V., "Superanalysis. I. differential calculus," Teor. Mat. Fiz. **59**, 3–27 (1984).
[19] Witten, E., *Quantum Fields and Strings: A Course for Mathematicians* (Ref. 13); Deligne, P., Freed, D., Jeffrey, L., and Wu, S., *ibid.* (Ref. 13).