Université Denis Diderot — **2014-2015**
Mathématiques M1 — **Spectral Theory**

Corrigé de l'examen du 5 janvier 2015
Durée 3 heures

For the following we recall that kernel linear operators with a kernel in L^2 are compact.

1. Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers and M_a be the following operator on $\ell^2(\mathbb{N})$

$$\mathcal{D}(M_a) = \{x = (x_n)_n \in \ell^2(\mathbb{N}) \, : \, \sum_n |a_n|^2|x_n|^2 < \infty\}, \quad M_a x = (a_n x_n)_n.$$

(a) Show that M_a is densely defined and closed.
Answer — The fact that $\mathcal{D}(M_a)$ is dense in $\ell^2(\mathbb{N})$ is a consequence of the fact that $\mathcal{D}(M_a)$ contains all finite sequences and that finite sequences are dense in $\ell^2(\mathbb{N})$. We now need to show that $\mathrm{Gr}M_a$ is a closed subspace of $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$. Consider a sequence $(x^k, M_a x^k)_{k\in\mathbb{N}}$ with values in $\mathrm{Gr}M_a$ and assume that $(x^k, M_a x^k)$ converges to some $(x, y) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$, when $k \to +\infty$. Setting $x^k = (x^k_0, x^k_1, \cdots)$ and $x = (x_0, x_1, \cdots)$, this implies immediately that, for any fixed $n \in \mathbb{N}$, $\lim_{k\to\infty} x^k_n = x_n$ in $\mathbb{C}$ and hence that $\lim_{k\to\infty} a_n x^k_n = a_n x_n$ in $\mathbb{C}$. Hence since we also have $\lim_{k\to\infty} a_n x^k_n = y_n$, this implies that $a_n x_n = y_n$, $\forall n \in \mathbb{N}$. Hence, since $y \in \ell^2(\mathbb{N})$, $\sum_{n\in\mathbb{N}} |a_n|^2|x_n|^2 < +\infty$ and we deduce that $x \in \mathcal{D}(M_a)$ and $y = M_a x$, i.e. $(x, y) \in \mathrm{Gr}M_a$. Hence $\mathrm{Gr}M_a$ is closed.

(b) Show that $\mathrm{sp}(M_a) = \overline{\{a_n \, : \, n \in \mathbb{N}\}}$.
Answer — We first observe that any value a_n is an eigenvalue of M_a for at least the eigenvector e_n. Hence $\{a_n \, : \, n \in \mathbb{N}\} \subset \mathrm{Sp}_p M_a \subset \mathrm{Sp}M_a$. Since we know from the course that the spectrum of any operator is closed, this implies that $\overline{\{a_n \, : \, n \in \mathbb{N}\}} \subset \mathrm{Sp}M_a$. Now let $b \in \mathbb{C} \setminus \overline{\{a_n \, : \, n \in \mathbb{N}\}}$. Then, since $\overline{\{a_n \, : \, n \in \mathbb{N}\}}$ is closed, there exists some $\varepsilon > 0$ s.t. $B(b, \varepsilon) \cap \{a_n \, : \, n \in \mathbb{N}\} = \emptyset$. We will then first show that $M_a - b$ is a bijection between $\mathcal{D}(M_a)$ and $\ell^2(\mathbb{N})$: given some $y \in \ell^2(\mathbb{N})$ we need to prove that there exists an unique $x \in \mathcal{D}(M_a)$ s.t. $(M_a - b)x = y$. If such an x would exist, it would be the unique solution of the equation $(a_n - b)x_n = y_n \Longleftrightarrow x_n = y_n/(a_n - b)$, $\forall n \in \mathbb{N}$. Lastly one needs to prove that $(b - M_a)^{-1} : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ is bounded, a consequence of :

$$||(b - M_a)^{-1}y||^2 = \sum_{n\in\mathbb{N}} \frac{|y_n|^2}{|b - a_n|^2} \leq \sum_{n\in\mathbb{N}} \frac{|y_n|^2}{\varepsilon^2} = \frac{||y||^2}{\varepsilon^2}.$$

Hence the resolvent set of M_a contains $\mathbb{C} \setminus \overline{\{a_n \, : \, n \in \mathbb{N}\}}$, which is equivalent to say that $\mathrm{Sp}M_a \subset \overline{\{a_n \, : \, n \in \mathbb{N}\}}$

2. Let $a = (a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers indexed by $\mathbb{Z}$ and T_a be the operator defined on $\ell^2(\mathbb{Z})$ by

$$\mathcal{D}(T_a) = \{x = (x_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \,:\, \sum_{n\in\mathbb{Z}} |a_n|^2|x_{-n}|^2 < \infty\}, \quad T_a x = (a_n x_{-n})_{n\in\mathbb{Z}}.$$

(a) Show that T_a is densely defined and closed.
Answer — Use the same method as in question 3 – (a).

(b) Compute T_a^*.
Answer — We first compute its domain $\mathcal{D}(T_a^*)$: this is the set of $y \in \ell^2(\mathbb{Z})$ s.t. the linear form

$$\begin{array}{ccc} \mathcal{D}(T_a) & \longrightarrow & \mathbb{C} \\ x & \longmapsto & \langle y, T_a x\rangle \end{array}$$

admits a continuous extension on $\ell^2(\mathbb{Z})$ (which is then unique since $\mathcal{D}(T_a)$ is dense in $\ell^2(\mathbb{Z})$). By Riesz' theorem this property is equivalent to say that there exists some $z \in \ell^2(\mathbb{Z})$ s.t. $\langle y, T_a x\rangle = \langle z, x\rangle$, $\forall x \in \mathcal{D}(T_a)$. But

$$\langle y, T_a x\rangle = \sum_{n\in\mathbb{Z}} \overline{y_n} a_n x_{-n} = \sum_{n\in\mathbb{Z}} \overline{y_{-n}} a_{-n} x_n.$$

Hence such a z exists, its satisfies $z_n = \overline{a_{-n}} y_{-n}$. Thus $y \in \mathcal{D}(T_a^*)$ iff $\sum_{n\in\mathbb{Z}} |\overline{a_{-n}} y_{-n}|^2 < +\infty$, i.e.

$$\mathcal{D}(T_a^*) = \{y \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |a_n y_n|^2 < +\infty\} \quad \text{and} \quad \forall y \in \mathcal{D}(T_a^*), \quad T_a^* y = (\overline{a_{-n}} y_{-n})_{n\in\mathbb{Z}}.$$

(c) Find a necessary and sufficient condition on a for T_a to be normal.
Answer — For any $x \in \mathcal{D}(T_a T_a^*) := \{x \in \mathcal{D}(T_a^*) : \ T_a^* x \in \mathcal{D}(T_a)\}$,

$$T_a T_a^* x = T_a\left((\overline{a_{-n}} x_{-n})_n\right) = \left(a_n\, (\overline{a_n} x_n)_n\right) = \left(|a_n|^2 x_n\right)_n,$$

wheras for any $x \in \mathcal{D}(T_a^* T_a)$,

$$T_a^* T_a x = T_a^*\left((a_n x_{-n})_n\right) = \left(\overline{a_{-n}}\, (a_{-n} x_n)_n\right) = \left(|a_{-n}|^2 x_n\right)_n,$$

Whatever $\mathcal{D}(T_a T_a^*)$ and $\mathcal{D}(T_a^* T_a)$ are, they contain the space of finite sequences. Hence a necessary condition for T_a to be normal is that, for any finite sequence $x = (x_n)_n$, $|a_n|^2 x_n = |a_{-n}|^2 x_n$, $\forall n \in \mathbb{Z}$, which implies that :

$$|a_n| = |a_{-n}|, \quad \forall n \in \mathbb{Z}.$$

Conversely it is clear from the preceding computation that, if the above condition holds, then $\mathcal{D}(T_a T_a^*) = \mathcal{D}(T_a^* T_a)$ and $T_a T_a^* = T_a^* T_a$, i.e. T_a is normal.

(d) Find a necessary and sufficient condition on a for T_a to be bounded.
Answer — The operator T_a is bounded iff $\mathcal{D}(T_a) = \ell^2(\mathbb{Z})$ and there exists a constant $C \in [0, +\infty)$ s.t. $\forall x \in \ell^2(\mathbb{Z})$, $||T_a x|| \le C||x||$. Testing this condition with $x = e_n$, for any $n \in \mathbb{Z}$ implies that $|a_n| \le C$. Conversely, it is easy to check that the latter condition implies that T_a is bounded. Hence the necessary and sufficient condition is : the sequence $(a_n)_{n\in\mathbb{Z}}$ is bounded.

(e) Compute $\mathrm{sp}(T_a^*T_a)$ and $\mathrm{sp}(T_aT_a^*)$.
Answer — We have seen in question c) that $T_a^*T_a$ has the domain $\{x \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |a_{-n}|^4|x_n|^2 < +\infty\}$ and is defined by $T_a^*T_a(x) = (|a_{-n}|^2x_n)_n$. Hence by a reasoning similar to the question 1 – (b), we deduce that the spectrum of $T_a^*T_a$ is $\overline{\{|a_{-n}|^2 : n \in \mathbb{Z}\}} = \overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$. Similarly $T_aT_a^*$ has the domain $\{x \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |a_n|^4|x_n|^2 < +\infty\}$ and is defined by $T_a^*T_a(x) = (|a_n|^2x_n)_n$. Hence the spectrum of $T_aT_a^*$ is also $\overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$.

(f) Find a necessary and sufficient condition on a for $T_a^*T_a$ (resp. $T_aT_a^*$) to be compact.
Answer — Assume that $T_a^*T_a$ is compact. Note also that this operator is self-adjoint. Then it follows from the course that the spectrum of $T_a^*T_a$ is equal to $\{0\} \cup \Lambda$, where Λ is a subset of $\mathbb{C} \setminus \{0\}$ which is at most countable and has no accumulation point, excepted may be 0. In particular, for any $r > 0$, $\mathrm{Sp}T_a^*T_a \cap (\mathbb{C} \setminus B(0, r^2))$ is finite. Moreover Λ is composed of eigenvalues λ associated with *finite* dimensional vector eigenspaces. However we have seen in the preceding question that $\mathrm{Sp}T_a^*T_a = \overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$ and the dimension of the eigenspace corresponding to any value $\lambda \in \Lambda$ is the cardinal of $\{n \in \mathbb{Z} : |a_n|^2 = \lambda\}$. We hence deduce that the number of values $n \in \mathbb{Z}$ s.t. $|a_n|^2 \geq r^2$ is finite. In particular, if we set $N(r) := \sup\{|n| \in \mathbb{Z} : |a_n|^2 \geq r\}$, we have $\forall n \in \mathbb{Z}$ s.t. $|n| > N(r)$, $|a_n| < r$. Hence $\lim_{|n|\to\infty} a_n = 0$.
Conversely if $\lim_{|n|\to\infty} a_n = 0$, then we define for any $r > 0$ the operator K_r on $\ell^2(\mathbb{Z})$ by

$$K_r x = \sum_{n\in\mathbb{Z};|a_n|^2\geq r^2} |a_{-n}|^2 x_n e_n.$$

Then $||T_a^*T_a - K_r|| \leq r^2$, so that $\lim_{r\to 0} ||T_a^*T_a - K_r|| = 0$ and each K_r is a finite rank operator. Hence $T_a^*T_a$ is compact.
A similar reasonning shows that $T_aT_a^*$ is compact iff the same condition holds, i.e.

$$\lim_{|n|\to\infty} a_n = 0.$$

3. In the following $\alpha \in (0, +\infty)$. The space $L^2(\mathbb{R}, \mathbb{C})$ is endowed with the Hermitian product $\langle\cdot,\cdot\rangle_{L^2}$ defined by $\langle f, g\rangle_{L^2} := \int_\mathbb{R} \overline{f(x)}g(x)dx$, $\forall f, g \in L^2(\mathbb{R}, \mathbb{C})$ and we set $\|f\|_{L^2} = \langle f, f\rangle_{L^2}^{1/2}$. The space $H^1(\mathbb{R}, \mathbb{C})$ is endowed with the Hermitian product $\langle\cdot,\cdot\rangle_\alpha$ defined by : $\langle f, g\rangle_\alpha := \int_\mathbb{R} (\overline{f'(x)}g'(x)) + \alpha^2\overline{f(x)}g(x))dx$, $\forall f, g \in H^1(\mathbb{R}, \mathbb{C})$ and we set $\|f\|_\alpha = \langle f, f\rangle_\alpha^{1/2}$.

(a) For any $f \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$ and we define

$$(R_\alpha f)(x) = e^{\alpha x}\int_x^{+\infty} e^{-\alpha y}f(y)dy, \quad (L_\alpha f)(x) = e^{-\alpha x}\int_{-\infty}^x e^{\alpha y}f(y)dy.$$

Compute $(R_\alpha f)'$ in function of $R_\alpha f$ and of f ; compute $(L_\alpha f)'$ in function of $L_\alpha f$ and of f.
Answer — $(R_\alpha f)' = \alpha R_\alpha f - f$, $(L_\alpha f)' = -\alpha L_\alpha f + f$.

(b) Let $f \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$, $R > 0$ such that $\mathrm{supp}(f) \subset [-R, R]$ and $\|f\|_\infty = \sup_x |f(x)|$. Show that $|R_\alpha f(x)| \leq Ce^{\alpha x}$ and $|L_\alpha f(x)| \leq Ce^{-\alpha x}$, where you can express C in terms of R and $\|f\|_\infty$. Study the support of $R_\alpha f$ and $L_\alpha f$. Deduce that $R_\alpha f, L_\alpha f \in L^2(\mathbb{R}, \mathbb{C})$.

Answer — $|R_\alpha f(x)| \leq 2\|f\|_\infty \frac{\sinh \alpha R}{\alpha} e^{\alpha x}$ and $|L_\alpha f(x)| \leq 2\|f\|_\infty \frac{\sinh \alpha R}{\alpha} e^{-\alpha x}$. Moreover $\operatorname{supp}(R_\alpha f) \subset (-\infty, R]$ and $\operatorname{supp}(L_\alpha f) \subset [-R, +\infty)$. Hence

$$\|R_\alpha f\|_{L^2}^2, \|L_\alpha f\|_{L^2}^2 \leq \|f\|_\infty^2 \frac{2\sinh^2 \alpha R}{\alpha^{3/2}} e^{2\alpha R}.$$

(c) For any $f \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$, we define $(\Sigma_\alpha f)(x) = \frac{1}{2\alpha}(R_\alpha + L_\alpha)$. Show that $(-\frac{d^2}{dx^2} + \alpha^2)\Sigma_\alpha f = f$ and that $\Sigma_\alpha f \in H^1(\mathbb{R}, \mathbb{C})$.
Answer — Compute, observe that $(\Sigma_\alpha f)' = \frac{1}{2}(R_\alpha f - L_\alpha f)$ and use the previous question.

(d) Show that the injection map $j : H^1(\mathbb{R}, \mathbb{C}) \longrightarrow L^2(\mathbb{R}, \mathbb{C})$, $f \longmapsto f$ is continuous.
Answer — $\|f\|_{L^2} \leq \frac{1}{\alpha}\|f\|_\alpha$.

(e) Show that, for any $f \in L^2(\mathbb{R}, \mathbb{C})$, there exists a unique $u \in H^1(\mathbb{R}, \mathbb{C})$ such that :

$$\forall \varphi \in H^1(\mathbb{R}, \mathbb{C}), \quad \langle u, \varphi \rangle_\alpha = \langle f, \varphi \rangle_{L^2}.$$

Prove that this defines a linear bounded operator $S_\alpha : L^2(\mathbb{R}, \mathbb{C}) \longrightarrow H^1(\mathbb{R}, \mathbb{C})$ such that $S_\alpha f = u$.
Answer — Let $\mathcal{L}_{L^2} : L^2(\mathbb{R}, \mathbb{C}) \longrightarrow (L^2(\mathbb{R}, \mathbb{C}))'$ be the Riesz anti-isomorphism, i.e. such that $\forall g \in L^2(\mathbb{R}, \mathbb{C})$, $(\mathcal{L}_{L^2} f)(g) = \langle f, g \rangle_{L^2}$. Similarly define the Riesz anti-isomorphism $\mathcal{L}_\alpha : H^1(\mathbb{R}, \mathbb{C}) \longrightarrow (H^1(\mathbb{R}, \mathbb{C}))'$ by $(\mathcal{L}_\alpha f)(g) = \langle f, g \rangle_\alpha$, $\forall f, g \in H^1(\mathbb{R}, \mathbb{C})$. Then $S_\alpha = \mathcal{L}_\alpha^* \circ j^* \circ \mathcal{L}_{L^2}$.
In the following we make a frequent use of the property : $\forall \varphi \in H^1(\mathbb{R}, \mathbb{C})$, $\langle S_\alpha f, \varphi \rangle_\alpha = \langle f, \varphi \rangle_{L^2}$.

(f) Abusing notations, we also denote by S_α the operator $L^2(\mathbb{R}, \mathbb{C}) \longrightarrow L^2(\mathbb{R}, \mathbb{C})$, $f \longmapsto S_\alpha f$. Show that S_α is self-adjoint. What can we say about the spectrum of S_α ?
Answer — Actually S_α is nonnegative because of $\langle S_\alpha f, \varphi \rangle_\alpha = \langle f, \varphi \rangle_{L^2}(\mathbb{R}, \mathbb{C})$, $\forall f \in L^2, \forall \varphi \in H^1(\mathbb{R}, \mathbb{C})$: indeed, by letting $\varphi = S_\alpha f$, this implies $\langle f, S_\alpha f \rangle_{L^2} = \langle S_\alpha f, S_\alpha f \rangle_\alpha = \|S_\alpha f\|_\alpha^2 \geq 0$. Hence S_α is self-adjoint and its spectrum is contained in $[0, +\infty)$ and is composed only of eigenvalues and of continuous spectral values (i.e. does not contain residual spectral values).

(g) Show that, $\forall f \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$,

$$\forall \varphi \in H^1(\mathbb{R}, \mathbb{C}), \quad \langle \Sigma_\alpha f, \varphi \rangle_\alpha = \langle f, \varphi \rangle_{L^2}$$

and deduce that the restriction of S_α on $\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$ coincides with Σ_α.
Answer — For any $f \in \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$, set $u = \Sigma_\alpha f$. Then, by the results of question (c), $u \in H^1(\mathbb{R}, \mathbb{C})$ and $-u'' + \alpha^2 u = f$. Hence

$$\forall \varphi \in H^1(\mathbb{R}, \mathbb{C}), \quad \int_{\mathbb{R}} \overline{\varphi}(x)(-u''(x) + \alpha^2 u(x))dx = \int_{\mathbb{R}} \overline{\varphi}(x) f(x) dx$$

which is equivalent to :

$$\forall \varphi \in H^1(\mathbb{R}, \mathbb{C}), \quad \int_{\mathbb{R}} (\overline{\varphi'}(x) u'(x) + \alpha^2 \overline{\varphi}(x) u(x))dx = \int_{\mathbb{R}} \overline{\varphi}(x) f(x) dx,$$

i.e. the required result. The fact that $S_\alpha|_{\mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})} = \Sigma_\alpha$ follows from the uniqueness of the solution of this problem shown in question (e).

(h) We let $G_\alpha \in \mathcal{C}^0(\mathbb{R}^2,\mathbb{R})$ be defined by

$$G_\alpha(x,y) = \frac{1}{2\alpha} e^{-\alpha|x-y|}.$$

Find an explicit expression for Σ_α in terms of G_α. Does G_α belong to $L^2(\mathbb{R}^2,\mathbb{R})$?
Answer — For any $f \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C})$,

$$(\Sigma_\alpha f)(x) = \int_{\mathbb{R}} G_\alpha(x,y) f(y) dy$$

because of the result of Question (c). We observe that G_α does not belong to $L^2(\mathbb{R}^2,\mathbb{R})$. Indeed

$$\int_{\mathbb{R}}\int_{\mathbb{R}} |G_\alpha(x,y)|^2 dxdy = \frac{1}{4\alpha^2}\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy e^{-2\alpha|x-y|}$$

which gives by the change of variable $z = x-y$

$$= \frac{1}{4\alpha^2}\int_{\mathbb{R}} dx \int_{\mathbb{R}} dz e^{-2\alpha|z|} = \frac{1}{4\alpha^2}\int_{\mathbb{R}} dx \frac{dz}{\alpha} = +\infty.$$

(i) We consider a smooth function $U \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{R})$ with compact support $\text{supp}U \subset [-1,1]$ and nonnegative values ($U(x) \geq 0$, $\forall x \in \mathbb{R}$). We define

$$T_\alpha := U^{1/2} S_\alpha U^{1/2}$$

where $U^{1/2} : f \longmapsto U^{1/2} f$ is the operator of multiplication by $U^{1/2}$. Show that $\forall f \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C})$,

$$(T_\alpha f)(x) = \int_{\mathbb{R}} K_\alpha(x,y) f(y) dy, \tag{1}$$

where $K_\alpha \in \mathcal{C}^0(\mathbb{R}^2,\mathbb{R})$.
In the following we will denote by $\mathcal{K}_\alpha$ the operator acting on $\mathcal{C}_c^\infty(\mathbb{R},\mathbb{C})$ defined by $(\mathcal{K}_\alpha f)(x) = \int_{\mathbb{R}} K_\alpha(x,y) f(y) dy$.
Answer — $K_\alpha(x,y) = U(x)^{1/2}(x)\frac{e^{-\alpha|x-y|}}{2\alpha}U(x)^{1/2}(y) = U(x)^{1/2}(x) G_\alpha(x,y) U(x)^{1/2}(y)$.

(j) Show that $K_\alpha \in L^2(\mathbb{R}^2,\mathbb{R})$. Deduce that the operator $\mathcal{K}_\alpha$ has an unique continuous extension from $L^2(\mathbb{R},\mathbb{C})$ to itself. Show that Relation (1) is actually true for any $f \in L^2(\mathbb{R},\mathbb{C})$.
Answer — We compute

$$\|K_\alpha\|_{L^2}^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} U(x)U(y)\frac{e^{-2\alpha|x-y|}}{4\alpha^2} dxdy \leq \frac{1}{4\alpha^2}\int_{\mathbb{R}}\int_{\mathbb{R}} U(x)U(y) dxdy = \frac{\|U\|_{L^1}^2}{4\alpha^2}.$$

Hence $\|K_\alpha\|_{L^2} \leq \|U\|_{L^1}/2\alpha$. This implies in particular that, $\forall f,g \in \mathcal{C}_c^\infty(\mathbb{R},\mathbb{C})$,

$$\begin{aligned}
|\langle f, \mathcal{K}_\alpha g\rangle_{L^2}| &= \left| \int_{\mathbb{R}}\int_{\mathbb{R}} \overline{f(x)} K_\alpha(x,y) g(y) dxdy \right| \\
&\leq \left(\int_{\mathbb{R}^2} |f(x)|^2 |g(y)|^2 dxdy\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |K_\alpha(x,y)|^2 dxdy\right)^{\frac{1}{2}} \\
&= \|K_\alpha\|_{L^2} \|f\|_{L^2} \|g\|_{L^2}.
\end{aligned}$$

Hence $\|\mathcal{K}_\alpha f\|_{L^2} \leq \|K_\alpha\|_{L^2}\|f\|_{L^2} \leq (\|U\|_{L^1}/2\alpha)\|f\|_{L^2}$. This implies that $\mathcal{K}_\alpha$ admits an unique continuous extension from $L^2(\mathbb{R},\mathbb{C})$ to itself.

But T_α is also bounded. Since both operators are continuous and coincide on a dense subspace (i.e. $\mathcal{C}_c^\infty(\mathbb{R},\mathbb{C})$), they coincide : $T_\alpha = \mathcal{K}_\alpha$.

(k) Show that T_α is compact.
Answer — T_α is a kernel operator, hence is Hilbert–Schmidt. Hence it is compact.

(l) Show that T_α is nonnegative and self-adjoint. What can we say about the spectrum of T_α ?
Answer — For any $f \in L^2(\mathbb{R},\mathbb{C})$, by setting $g = U^{1/2}f$ in the following,

$$\langle f, T_\alpha f\rangle_{L^2} = \langle f, U^{1/2}S_\alpha U^{1/2}f\rangle_{L^2} = \langle U^{1/2}f, S_\alpha U^{1/2}f\rangle_{L^2} = \langle g, S_\alpha g\rangle_{L^2}$$

hence, by using the definition of S_α,

$$\langle f, T_\alpha f\rangle_{L^2} = \langle S_\alpha g, S_\alpha g\rangle_\alpha = \|S_\alpha g\|_\alpha^2 \geq 0.$$

This implies that T_α is nonnegative and hence in particular self-adjoint. Hence, since T_α is also compact, its spectrum is equal to $\{0\} \cup \{\lambda_n; n \in \mathcal{N}\}$, where $\mathcal{N} \subset \mathbb{N}$ and $(\lambda_n)_{n\in\mathcal{N}}$ is a sequence of positive real numbers which, either is finite, or tends to 0 as $n \to +\infty$, if $\mathcal{N}$ is infinite.

(m) Show the *Birman–Schwinger principle* : $\mathrm{Ker}(1 - T_\alpha) \neq \{0\}$ if and only if $-\alpha^2$ is an eigenvalue of the operator $-\frac{d^2}{dx^2} - U$, i.e. $\exists \varphi \in H^1(\mathbb{R},\mathbb{C}) \cap \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$ such that $-\varphi'' - U\varphi = -\alpha^2\varphi$ in the weak sense [1].
Answer — First of all observe that, $\forall\varphi \in H^1(\mathbb{R},\mathbb{C})$, $-\varphi'' - U\varphi = -\alpha^2\varphi$ in a weak sense iff $\forall\chi \in L^2(\mathbb{R},\mathbb{C})$, $\langle\varphi,\chi\rangle_\alpha = \langle U\varphi,\chi\rangle_{L^2}$.
Now let $f \in L^2(\mathbb{R},\mathbb{C})$ such that $f \neq 0$ and $T_\alpha f = f$, i.e. $f = \sqrt{U}S_\alpha\sqrt{U}f$. Then $\sqrt{U}f = US_\alpha\sqrt{U}f$. Hence, by setting $\varphi := S_\alpha\sqrt{U}f \in H^1(\mathbb{R},\mathbb{C})$,

$$\forall\chi \in L^2(\mathbb{R},\mathbb{C}), \quad \langle\varphi,\chi\rangle_\alpha = \langle S_\alpha\sqrt{U}f,\chi\rangle_\alpha = \langle\sqrt{U}f,\chi\rangle_{L^2} = \langle US_\alpha\sqrt{U}f,\chi\rangle_{L^2} = \langle U\varphi,\chi\rangle_{L^2}.$$

Thus φ is a weak solution of $-\varphi'' - U\varphi = -\alpha^2\varphi$. Conversely let $\varphi \neq 0$ be a solution in $\mathcal{C}^\infty(\mathbb{R},\mathbb{C}) \cap H^1(\mathbb{R},\mathbb{C})$ of $-\varphi'' - U\varphi = -\alpha^2\varphi$. Then $f := \sqrt{U}\varphi \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C}) \cap H^1(\mathbb{R},\mathbb{C})$ is a solution of $T_\alpha f = f$.

(n) Show that any negative eigenvalue λ of $-\frac{d^2}{dx^2} - U$ belongs to $[-\|U\|_{L^1}^2/4, 0)$.
Answer — Set $\lambda = -\alpha^2$ with $\alpha > 0$. Then λ is a eigenvalue of $-\frac{d^2}{dx^2} - U$ iff 1 is an eigenvalue of T_α. If so, by using the result of Question (j),

$$1 \leq r(T_\alpha) \leq \|T_\alpha\| \leq \|U\|_{L^1}/2\alpha,$$

which implies that $\alpha \leq \|U\|_{L^1}/2$. Hence the result.

1. We admit the following regularity result : for any function $V \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$, any weak solution $v \in H^1(\mathbb{R},\mathbb{C})$ of the equation $-v'' + Vv = 0$ is smooth, i.e. $v \in \mathcal{C}^\infty(\mathbb{R},\mathbb{C})$.