AN INTRODUCTION TO TWISTORS

PAUL BAIRD

Abstract
We introduce the twistor correspondence in 4-dimensions via a classical formula of Whittaker for harmonic functions in 3-space. Throughout our discussion we compare the Riemannian with the semi-Riemannian case.

1. Some background

In very general terms, twistor theory consists of the association of a complex space (twistor space) to a real space in a natural and useful way. By useful, we mean that some important (second order differential) equations - e.g. the zero-rest-mass field equations, the harmonic map equation - can be solved in terms of holomorphic data in the twistor space. This kind of correspondence has its origins in classical algebraic geometry, with the Plücker-Klein correspondence which represents lines in a projective 3-space as points on a quadric in projective 5-space. The example of representing circles in 3-space as points on a complex quadric in complex projective 4-space (see Example (d) below) goes back to Laguerre.

Twistor theory in the context of space-time has been pioneered by Roger Penrose and others since the 1960s and is based on the association of a complex twistor space $\mathbf{C}P^3$ (or sometimes an unprojectivised twistor space C^4) to the space of light rays in space-time. The name derives from the Robinson congruence which is the natural realisation of a (non-null) twistor and is described in Section 7. Some appealing aspects of the theory are:

- twistor space becomes the basic space so that light rays are the fundamental objects from which space-time is derived;
- discrete quantities such as spin are reflected in the discrete values obtained by contour integration;
- elegance and simplicity!

Although from a mathematical point of view the correspondence is purely formal, from a physical point of view the different perspective becomes fundamental, particularly at the level of quantisation - see [29] for an excellent account of the motivation and aims of twistor theory.

At the same time as Penrose published his basic paper 'Twistor algebra' in 1967, Calabi described how minimal surfaces in the Euclidean sphere S^n can be described in terms of holomorphic curves in a complex space. This space is the bundle of almost Hermitian structures on an even-dimensional manifold and later, to people working in the areas of minimal surfaces and harmonic maps, became known as the *twistor bundle*. In favourable circumstances it is a complex manifold. For example, in the case of S^4, it is $\mathbf{C}P^3$, the same as the twistor space for Minkowski space-time. We will refer to the latter theory as the theory of *Riemannian twistors*, since they are derived from spaces having a Riemannian rather than a semi-Riemannian structure.

More recently the study of *harmonic morphisms* has exploited the twistor correspondence. As a special case a harmonic morphism is a mapping from a Riemannian manifold to a surface whose fibres form a conformal foliation by minimal submanifolds; they can be considered as the Riemannian analogue of null-solutions to Maxwell's equations in space-time.

In these lectures, we aim to give an introduction to twistor theory, sometimes using the Riemannian perspective to aid in understanding the space-time case, which will be the fundamental object of study. The author benefitted greatly from the summer schools on 'Quantum field theory and integrable systems' at Peyresq and would like to thank the organisers, Frédéric Helein and Joseph Kouneiher for their hospitality.

2. Examples

We give some examples of real spaces and their associated twistor spaces.

real space	object defining correspondence	twistor space
(a) R^3	lines	$T\mathbf{C}P^1$
(b) S^3	great circles	$\mathbf{C}P^1 \times \mathbf{C}P^1$
(c) H^3	geodesics	$\mathbf{C}P^1 \times \mathbf{C}P^1 \setminus \Delta$
(d) S^3	circles and points	$Q_3 \subset \mathbf{C}P^4$
(e) Minkowski space	light rays	$\mathcal{N}^5 \subset \mathbf{C}P^3$
(f) asd Einstein 4-mfld	almost Hermitian str.	$\mathcal{Z}^+$ twistor bundle

In all cases there is an *incidence relation* which describes 'incidence' between a point in the twistor space and a point of the real space belonging to the geometric object defining the correspondence.

(a) Each oriented line $\gamma(s)$ has a unique description:

$$\gamma(s) = \boldsymbol{c} + s\boldsymbol{u}$$

where $\boldsymbol{u} \in S^2$ is the direction of the line and $\boldsymbol{c}$ is the unique vector perpendicular to $\boldsymbol{u}$ lying on the line - the displacement vector. Then the pair $(\boldsymbol{u}, \boldsymbol{c}) \in TS^2 = \{(\boldsymbol{x}, \boldsymbol{y} \in \mathsf{R}^3 \times \mathsf{R}^3 : |\boldsymbol{x}| = 1, \langle \boldsymbol{x}, \boldsymbol{y} \rangle = 0\} \subset \mathsf{R}^3 \times \mathsf{R}^3$. To exhibit the complex properties of the twistor space TS^2 more usefully, we introduce a complex chart, for example by identifying $S^2 \setminus \{\text{point}\}$ with C by stereographic projection. So let $\sigma : S^2 \setminus \{(0,0,-1)\} \to \mathsf{C}$ be stereographic projection from the south pole defined by

$$\sigma(x) = \frac{x^1 + \mathrm{i}x^2}{1 + x^3}$$

Let $z = \sigma(\boldsymbol{u})$ and $w = \mathrm{d}\sigma_{\boldsymbol{u}}(\boldsymbol{c})$. Then the line γ is the solution set to the complex equation (see [22, 7]),

$$(x^1 + \mathrm{i}x^2) - 2\mathrm{i}zx^3 + z^2(x^1 - \mathrm{i}x^2) = 2w\,. \tag{1}$$

This equation is the incidence relation. We have the following correspondences:

Real space R^3	twistor space $T\mathbf{C}P^1$
line	point
change of orientation	real structure $\rho : T\mathbf{C}P^1 \to T\mathbf{C}P^1$ ($\rho^2 = \mathrm{id}$)
point x	$\mathbf{C}P^1(x) \subset T\mathbf{C}P^1$ (real section)

(b) The space of oriented great circles can be identified with the Grassmannian $G_2(\mathsf{R}^4)$ of oriented 2-planes passing through the origin in R^4. The identification $G_2(\mathsf{R}^4) \sim \mathbf{C}P^1 \times \mathbf{C}P^1$ is standard.

(c) Take the Poincaré model for hyperbolic 3-space. This is the open unit ball $B^3 \subset \mathsf{R}^3$ with metric $g^H = 4\mathrm{d}x^2/(1 - |x|^2)$ ($\mathrm{d}x^2 = \mathrm{d}{x_1}^2 + \mathrm{d}{x_2}^2 + \mathrm{d}{x_3}^2$). Then the geodesics are arcs of circles which hit the boundary $\partial B^3 = S^2$ at right-angles. If γ is an oriented geodesic then it is determined by its start point and its end point, both distinct points of S^2, so the space of all oriented geodesics is identified with $S^2 \times S^2 \setminus \Delta$ where Δ is the diagonal. It is natural to take the conjugate complex structure on the second factor.

In fact (a), (b) and (c) have a unified treatment [22], since the tangent space to the space of geodesics at a geodesic γ is given by the set of Jacobi fields along γ which are normal to γ. The complex structure is given by rotation through $\pi/2$ in the normal space to γ, that this gives another Jacobi field depends on having constant curvature.

(d) If we consider R^3 as identified with $S^3 \setminus \{\text{point}\}$ via stereographic projection and H^3 as identified with a hemi-sphere of S^3, then (a), (b) and (c) are also unified by considering the space of circles of S^3. This can be identified with the complex quadric $\mathcal{H} = \{[w_0, \ldots, w_4] \in \mathbf{C}P^4 : {w_0}^2 + \cdots {w_3}^2 - {w_4}^2 = 0\} \subset \mathbf{C}P^4$ as follows.

Consider the affine oriented 2-plane P^2,

$$w_0 y_0 + \cdots + w_3 y_3 + w_4 = 0 \tag{2}$$

in R^4. Let $S^3 = \{y \in \mathsf{R}^4 : |y| = 1\}$, then $P^2 \cap S^3$ is either a point or an oriented circle: if all $w_0, \ldots w_4$ are real, then (2) defines a tangent plane to S^3 with intersection a point, otherwise it is a circle.

In fact if we identify $S^3 \setminus \{\text{point}\} \to \mathsf{R}^3$ via stereographic projection and write $\xi = (\xi_1, \xi_2, \xi_3) = \frac{1}{w_0 + w_4}(w_1, w_2, w_3)$, then the equation of the circle in R^3 is given explicitly by

$$\xi^2 + 2\xi \cdot x + |x|^2 = 0 \qquad (\xi^2 = {\xi_1}^2 + {\xi_2}^2 + {\xi_3}^2)$$

Indeed, $\xi = -a + \mathrm{i}n$ where a is the centre of the circle and n is the normal to the plane containing the circle with $|n| = r$ the radius. The set of circles ξ satisfying a quadratic relation of the form

$$A\xi^2 + 2\xi \cdot b + C = 0 \qquad (b \in \mathsf{R}^3, A, C \in \mathsf{R})$$

are geodesics with respect to a space form of constant curvature $K = (AC - |b|^2)/C^2$ [5, 6].

(e) We intend to describe this in more detail below, but for the moment we give the following brief account.

Consider a space-like slice R^3 of Minkowski space. Then a light ray is determined by its origin $\in \mathsf{R}^3$ and its direction - a point in the celestial sphere S^2, i.e. it corresponds to a point of the unit tangent bundle $T^1\mathsf{R}^3 \cong \mathsf{R}^3 \times S^2$. It is better to compactify, so that a light ray now corresponds to a point of $T^1 S^3 \cong S^3 \times S^2$. Now let $\pi : \mathbf{C}P^3 \to S^4$ be the Hopf fibration given by

$$\pi([z_0, z_1, z_2, z_3]) = [z_0 + z_1\mathrm{j}, z_2 + z_3\mathrm{j}] \in \mathsf{H}P^1 \cong S^4 .$$

Here we use quaternion notation $x_0 + x_1\mathrm{i} + x_2\mathrm{j} + x_3\mathrm{k} \cong z + w\mathrm{j}$ $(z, w \in \mathsf{C})$ and $\mathsf{H}P^1$ is quaternionic projective space (we will be more explicit in Section 7). Let $S^3 \subset S^4$ be an equatorial sphere; then $\pi^{-1}(S^3) \cong T^1 S^3$ in a natural way. So we identify the space of light rays with a real 5-dimensional hypersurface in $\mathbf{C}P^3$ - the latter space is twistor space.

(f) Take $M^4 = S^4$. Identify $S^4 \setminus \{\text{point}\}$ with R^4 via stereographic projection and identify R^4 with C^2: $(x^0, x^1, x^2, x^3) \leftrightarrow (q^1 = x^0 + \mathrm{i}x^2, q^2 = x^2 + \mathrm{i}x^2)$. Then a positive almost Hermitian structure J at a point x, $J_x : T_x\mathsf{R}^4 \to T_x\mathsf{R}^4$ $({J_x}^2 = -\mathrm{id}, J_x$ isometry) is determined by its $\pm$i-eigenspaces in $T_x^{\mathsf{C}}\mathsf{R}^4 \cong \mathsf{C}^4$ (see Section 6 for more details); these are determined by specifying a number $\mu \in \mathbf{C}P^1$: explicitly

$$\frac{\partial}{\partial \overline{q_1}} - \mu \frac{\partial}{\partial q_2}, \quad \frac{\partial}{\partial \overline{q_2}} + \mu \frac{\partial}{\partial q_1}$$

span the $-$i-eigenspace. So the space of almost Hermitian structures is a $\mathbf{C}P^1$-bundle over S^4 - this is once more $\mathbf{C}P^3$.

Note that an immersed oriented surface $\iota : M^2 \hookrightarrow S^4$ determines a section $\sigma : M^2 \to \mathbf{C}P^3$. Indeed, for each $x \in M^2$, J_x is given by $J_x = (J_x^T, J_x^N)$ where J_x^T is rotation by $\pi/2$ in T_xM^2 and J_x^N is rotation by $\pi/2$ in the orthogonal compliment $(T_xM^2)^\perp$ in TS^4. Minimality of M^2 is reflected in properties of the complex curve $\sigma(M^2) \subset \mathbf{C}P^3$.

3. Whittaker's formula

UItimately we are interested in fields on space-time (solutions of some field equation - for example the wave equation) and their description as objects in the twistor space. As a first step we consider Laplace's equation in R^3 (a static solution to the wave equation); now the twistorial description is essentially a classical formula of Whittaker (1903).

The formula of Whittaker states that, up to a translation in space a (local) complex valued solution to Laplace's equation in R^3:

$$\frac{\partial^2\varphi}{\partial {x_1}^2} + \frac{\partial^2\varphi}{\partial {x_2}^2} + \frac{\partial^2\varphi}{\partial {x_3}^2} = 0$$

is given by an integral

$$\varphi(x) = \int_0^{2\pi} f(\theta, x_3 + \mathrm{i}x_1\cos\theta + \mathrm{i}x_2\sin\theta)\mathrm{d}\theta \tag{3}$$

where $f(z,w)$ is a complex analytic function in 2-variables (with singularities away from the path of integration). Before proving this formula let us give it a different interpretation.

Set $q = x_1 + \mathrm{i}x_2$, $u = x_3$ and $z(\theta) = \exp(\mathrm{i}\theta)$ to be the unit circle over which we take a contour integration. Then

$$q - 2\mathrm{i}zu + z^2\overline{q} = -2\mathrm{i}e^{\mathrm{i}\theta}(x_3 + \mathrm{i}x_1\cos\theta + \mathrm{i}x_2\sin\theta)\,,$$

so that we may equivalently write the integral (3) (up to a modification of f) as

$$\varphi(x) = \frac{1}{2\pi\mathrm{i}}\oint f(z, q - 2\mathrm{i}zu + z^2\overline{q})\mathrm{d}z\,. \tag{4}$$

¿From (1), we see that the 2nd argument $w = q - 2\mathrm{i}zu + z^2\overline{q}$, up to a factor of 2, is the incidence relation between a twistor (z,w) and the corresponding line in 3-space. It is therefore natural to view $f(z,w)$ as a function defined on a domain of twistor space $T\mathbf{C}P^1$.

Note that given a point $x \in \mathsf{R}^3$, the set of twistors incident with x (the set of lines passing through x) form a copy of $\mathbf{C}P^1 \subset T\mathbf{C}P^1$ which we write $\mathbf{C}P^1(x)$. We then consider the integration as taking place along a contour contained in $\mathbf{C}P^1(x)$. We therefore very loosely have the correspondence:

harmonic function on a domain of $\mathsf{R}^3 \leftrightarrow$

holomorphic function $f(z,w)$ on a domain of twistor space + choice of contour.

Proof of Whittaker's formula: We establish the formula (3). Now a solution φ to Laplace's equation $\Delta\varphi = 0$ is analytic. Let x_0 be a regular point for φ; by translation we may suppose that x_0 is the origin and we expand φ in a power series about the origin:

$$\varphi = \sum_I a_I x^I = a_0 + a_1x^1 + a_2x^2 + a_3x^3 + a_{11}(x^1)^2 + a_{12}x^1x^2 + \cdots$$

If we write this in homogeneous parts:

$$\varphi = Q_0 + Q_1 + Q_2 + \cdots$$

where Q_n is homogeneous of degree n, then it is easily seen that each Q_n is also harmonic. Now in 3 variables, there are $2n+1$ linearly independent harmonic homogeneous polynomials of degree n (see, for example [13]), e.g. $n=1: x,y,z$, $n=2: xy, yz, xz, x^2-y^2, y^2-z^2$. These can be generated as follows:

Consider the function of u, x^1, x^2, x^3, homogeneous of degree n in x, given by

$$(x^3 + \mathrm{i}x^1\cos u + \mathrm{i}x^2\sin u)^n = \sum_{k=0}^{n} g_k(x)\cos ku + \sum_{k=1}^{n} h_k(x)\sin ku\,.$$

Then $g_k = g_k(x^1,x^2,x^3)$ and $h_k = h_k(x^1,x^2,x^3)$ form $(2n+1)$ linearly independent harmonic functions of degree n. By the theory of Fourier series

$$g_k(x) = \frac{1}{\pi}\int_0^{2\pi}(x^3 + \mathrm{i}x^1\cos u + \mathrm{i}x^2\sin u)^n\cos ku\mathrm{d}u$$

$$h_k(x) = \frac{1}{\pi}\int_0^{2\pi}(x^3 + \mathrm{i}x^1\cos u + \mathrm{i}x^2\sin u)^n\sin ku\mathrm{d}u$$

which gives the required form.

Example Set $f(z,w) = z/w$. This function has simple poles at $z = \mathrm{i}(u \pm |x|)/\overline{q}$. Evaluate the contour integral (4) along a contour surrounding the pole $\mathrm{i}(u+|x|)/\overline{q}$, but not surrounding the other pole. To be more specific, take the contour $|z|=2$; then the above property is satisfied for $q \neq 0$ and $0 < 9|q|^2 - 16u^2$. Set $U = \{x \in \mathsf{R}^3 : q \neq 0, 0 < 9|q|^2 - 16u^2\}$. Then for $x \in U$, calculating the residue, the integral (4) gives the harmonic function

$$\varphi(x) = \frac{u+|x|}{2\overline{q}|x|}$$

well-defined off the x^3-axis $q=0$. Note that (4) only determines the harmonic function for $x \in U$, whereas the function clearly extends to $\mathsf{R}^3 \setminus \{x^3 - \text{axis}\}$.

If on the other hand we let $x \in V = \{x \in \mathsf{R}^3 : q \neq 0, 0 > 9|q|^2 - 16u^2\}$, the contour surrounds the other pole and we get a *different* harmonic function

$$\psi(x) = -\frac{u-|x|)}{2\overline{q}|x|}\,.$$

In order to describe the harmonic function φ in terms of twistor space we have to work a bit harder! We avoid discussion of twistor cohomology in these lectures, but to give a flavour of what occurs, we outline the procedure to determine a global solution. (We follow closely the description in [32] for the space-time case here.)

Take an appropriate open cover $\{U_i\}$ of twistor space $T\mathbf{C}P^1$. Note that for a given $x \in \mathsf{R}^3$, the integration takes place along a contour in the corresponding Riemann sphere $\mathbf{C}P^1(x) \subset T\mathbf{C}P^1$ (this is where $f(z,w)$ is defined!) Suppose that $U_1 \cap U_2 \supset \mathbf{C}P^1(x)$ and let $V_1 = U_1 \cap \mathbf{C}P^1(x)$, $V_2 = U_2 \cap \mathbf{C}P^1(x)$. Then we require the contour to lie in $V_1 \cap V_2$. Furthermore we require the twistor function f to be defined in a neighbourhood of this contour - in fact in $U_1 \cap U_2$ and we write it as f_{12}. More generally, with respect to the open cover $\{U_i\}$, we have a collection of twistor functions $\{f_{ij}\}$ defined on the intersections $U_i \cap U_j$. These must satisfy the cocycle conditions and define an element of cohomology. In the space-time context this is the basis of the Penrose transform (an integral transform) relating sheaf cohomology on twistor space and zero-rest-mass fields on space time (see [**?**, 37]). The twistor function is replaced by an element of the cohomology group and the field now becomes a function of an element of the cohomology group.

4. Static fields

We will consider a field as a section of a vector bundle $E \to M$ over a Riemannian or semi-Riemannian manifold satisfying some differential equations (the field equations). Following Penrose [28], we generate solutions of field equations which are the static analogue of the zero-rest-mass field equations in Minkowski space, from Whittaker's formula.

As a special case of (4), set

$$\varphi_r(x) = \frac{1}{2\pi \mathrm{i}} \oint z^r f(z, q - 2\mathrm{i}zu + z^2\overline{q})\mathrm{d}z\,. \tag{5}$$

Then we have the recurrence relations:

$$\begin{cases} \frac{\partial \varphi_r}{\partial u} &= -2\mathrm{i}\frac{\partial \varphi_{r+1}}{\partial q} \\ -2\mathrm{i}\frac{\partial \varphi_r}{\partial \overline{q}} &= \frac{\partial \varphi_{r+1}}{\partial u} \end{cases} \qquad r = 0, 1, 2, \dots \tag{6}$$

Now let s be a non-negative integer and set

$$\varphi_0 = \varphi_{00\ldots 0}, \quad \varphi_1 = \varphi_{10\ldots 0}, \quad \varphi_2 = \varphi_{11\ldots 0}, \ldots, \varphi_{2s} = \varphi_{11\ldots 1}$$

defining a tensor field $\varphi_{AB\ldots L}$, $A, B, \ldots, L = 0, 1$, which we insist is totally symmetric in the indices $A, B, \ldots, L$. We may naturally consider $\varphi_{AB\ldots L}(x)$ as a section of the 'spinor' bundle $\mathsf{C}^{2s} \times U \to U$ ($U \subset \mathsf{R}^3$). The indices $A, B, \ldots, L$ will be referred to as *spinor indices.*

Define the differential operators: $D_{00} = -\sqrt{2}\mathrm{i}\frac{\partial}{\partial q}$, $D_{01} = D_{10} = \frac{1}{\sqrt{2}}\frac{\partial}{\partial u}$, $D_{11} = -\sqrt{2}\mathrm{i}\frac{\partial}{\partial \overline{q}}$. These are naturally deduced from the analogues for Minkowski space which we discuss later (see [33]). Raise and lower indices using the skew forms $(\varepsilon^{AB}) = (\varepsilon_{AB}) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ on C^2. Thus $\xi^A = \varepsilon^{AB}\xi_B$, i.e. $\xi^0 = \xi_1$ and $\xi^1 = -\xi_0$ etc. Then (6) may be written in the convenient form

$$D^{AK}\varphi_{AB\cdots L} = 0\,. \tag{7}$$

Suppose the contour in (5) surrounds a kth-order pole of f. Let $\nu = \nu(x)$ be that pole, then, following Penrose once more,

$$\oint (z - \nu)^k z^r f(z, q - 2\mathrm{i}zu + z^2\overline{q})\mathrm{d}z = 0$$

so that, if $2s \geq k$,

$$\varphi_{r+k} - k\varphi_{r+k-1}\nu + \frac{1}{2}k(k-1)\varphi_{r+k-2}\nu^2 - \cdots + \varphi_r(-\nu)^k = 0 \tag{8}$$

for each $r = 0, \ldots, 2s - k$. Setting $\xi^0 = -\nu$, $\xi^1 = 1$, we can rewrite this as

$$\varphi_{AB\ldots L}\xi^A\xi^B \cdots \xi^D = 0$$

where there are $2s - k + 1$ indices $A, B, \ldots, D$. This is the condition for the spinor ξ^A to represent a $(2s - k + 1)$-fold principal direction [30, 31]. Then if $2s - k + 1 \geq 2$, the field $\varphi_{AB\ldots L}$ is called *algebraically special.* In particular, if $k = 1$ and the contour surrounds a simple pole, then all $2s$ principal directions coincide, so that

$$\varphi_{AB\cdots L} = \sigma\xi_{(A}\xi_B \cdots \xi_{L)}$$

for some function σ. When $s = 1$, equations (7) are the static analogue of Maxwell's equations.

5. Harmonic morphisms

A harmonic morphism on a domain of R^3 with values in C can be considered as an algebraically special time independent solution to Maxwell's equations ($s = 1$ in (7)). More generally, the equations determining a harmonic morphism provide a set of conditions which, in the context of Riemannian manifolds, specialize the harmonic map equation and in the context of semi-Riemannian manifolds, specialize the zero-rest-mass field equations. As such, their study may provide insight into problems such as that of defining twistors in arbitrarily curved spaces.

In the most general terms, a harmonic morphism is a map $\varphi : (M, g) \to (N, h)$ between manifolds M, N endowed with metrics g, h respectively, which preserves locally defined harmonic functions, i.e. if $f : V \to \mathsf{R}$ is a harmonic function defined on an open set $V \subset N$ with $\varphi^{-1}(V)$ non-empty, then the compostition $f \circ \varphi$ is harmonic on $\varphi^{-1}(V)$. Not that we do not impose any condition on the signatures of the metrics g and h and that 'harmonic' is defined with respect to a metric of any signature by the equation

$$\Delta^M f = g^{ij}\left(\frac{\partial^2 f}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial f}{\partial x^k}\right) = \frac{1}{\sqrt{|\det g_{ij}|}}\frac{\partial}{\partial x^i}\left(\sqrt{|\det g_{ij}|}g^{ij}\frac{\partial f}{\partial x^j}\right) .$$

In fact a harmonic morphism can be defined whenever the notion of harmonic function makes sense. For example, in a discrete setting, we may define a harmonic function on a graph as a function defined on the vertices which satisfies the averaging property (the average of its values on neighbouring vertices equals the value at the vertex) and so we may define harmonic morphisms between graphs [34, 35, 1]. For Riemannian manifolds we have the following characterization:

Theorem 5.1. ([20, 23, 14]) *Let $\varphi : (M, g) \to (N, h)$ be a smooth mapping between Riemannian manifolds, then φ is a harmonic morphism if and only if φ is both harmonic and semi-conformal.*

Here, with respect to local coordiantes (x^i) on M and (y^α) on N, harmonicity of φ is equivalent to the second order elliptic equation

$$\tau(\varphi)^\gamma = g^{ij}\left(\frac{\partial^2 \varphi^\gamma}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial \varphi^\gamma}{\partial x^k} + L^\gamma_{\alpha\beta}\frac{\partial \varphi^\alpha}{\partial x^i}\frac{\partial \varphi^\beta}{\partial x^j}\right) = 0 \tag{9}$$

where $\Gamma^k_{ij}, L^\gamma_{\alpha\beta}$ are the Christofell symbols on M, N respectively.

Semi-conformality is equivalent to the first order equation

$$g^{ij}\varphi^\alpha_i \varphi^\beta_j = \lambda^2 h^{\alpha\beta} \tag{10}$$

for some continuous function $\lambda : M \to \mathsf{R}(\geq 0)$ with λ^2 smooth. The regular fibres of a semi-conformal map form a *conformal foliation* (see [36]). Conversely, given a conformal foliation, on suitable open sets we can put a conformal structure on the leaf space such that the canonical projection onto the leaf space is semi-conformal, see [8].

With sufficient care, the same theorem holds when M and N have metrics of arbitrary signatures ([21, 12]) - now λ^2 should be replaced by a function $\Lambda : M \to \mathsf{R}$ which may take on negative values.

If $z : U \subset \mathsf{R}^m \to \mathsf{C}$, then (9) and (10) become

$$\sum \frac{\partial^2 z}{(\partial x^i)^2} = 0 \tag{11}$$

and

$$\sum \left(\frac{\partial z}{\partial x^i}\right)^2 = 0 \tag{12}$$

Consider the case $m = 3$. Set $\varphi_{AB} = D_{AB}z$. Then (12) is equivalent to the equation $\det \varphi_{AB} = 0$, which is equivalent to the decomposition of φ into an outer product of the form $\varphi_{AB} = \sigma\xi_A\xi_B$. Then equation (11) is equivalent to $D^{AK}\varphi_{AB} = 0$.

Conversely, suppose we are given a spinor field φ_{AB}. The equation $D^{AK}\varphi_{AB} = 0$ implies integrablility: $\varphi_{AB} = D_{AB}z$ for some function z. Harmonicity of z follows by summing over B and K and (12) is equivalent to $\varphi_{AB} = \sigma\xi_A\xi_B$. In summary, (11) and (12) are equivalent to the spinor equations

$$\left\{ \begin{array}{rcl} \varphi_{AB} & = & \sigma\xi_A\xi_B \\ D^{AK}\varphi_{AB} & = & 0 \end{array} \right.$$

i.e., φ is an algebraically special solution of the field equations (7). By (8), solutions have an explicit expression in terms of contour integrals given by $(\nu =)\ z = \varphi_1/\varphi_0$.

6. Twistor space of S^4

We describe the twistor bundle of positive almost Hermitian structures on S^4. Later we will relate this to the twistor space associated to Minkowski space.

In general, let M^{2m} be an oriented even dimensional manifold. For $x \in M^{2m}$, an *almost Hermitian structure at* x is an isometry $J_x : T_xM \to T_xM$ such that $J_x^2 = -\mathrm{id}$. Let $\{e_1, e_2, \ldots, e_{2m}\}$ be an orthonormal basis at x. Then the linear map determined by $J(e_1) = e_2, \ldots, J(e_{2m-1}) = e_{2m}$ defines an almost Hermitian structure at x. We say that J_x is *positive* if the basis $\{e_1, \ldots, e_{2m}\}$ is positive. We let $\Sigma_x^+ = \{J_x : J_x \text{ positive}\}$. Then $SO(2m)$ acts transitively on Σ_x^+ with isotropy subgroup $U(m)$ and we have an isomorphism $SO(2m)/U(m) \cong \Sigma_x^+$. If $m = 2$, the isomorphism $SO(4)/U(2) \cong \mathbf{C}P^1$ is realised explicitly by identifying $J \in \Sigma_x^+$ with $J(\partial/\partial x^0) \in S^2 \subset \mathsf{R}^3$, where $\mathsf{R}^3 = \{(x^0, \ldots, x^3) \in \mathsf{R}^4 : x^0 = 0\}$.

An almost Hermitian structure on $U \subset M^4$ defines a (tautological) section $\sigma_J : U \to (\Sigma^+, \mathcal{J}^\Sigma)$. Here $\mathcal{J}^\Sigma$ is the canonical almost complex structure on the twistor bundle (cf. [19]). It can be shown that σ_J is holomorphic with respect to J if and only if J is integrable ([19, 18]).

In the case $M^4 = S^4$, Σ^+ is identified with $\mathbf{C}P^3$ as follows. Let $\pi : \mathbf{C}P^3 \to S^4$ be the Hopf fibration defined in terms of complex homogeneous coordinates on the domain and quaternionic homogeneous coordinates on the codomain by $\pi([f, g, h, k]) = [f + g\mathrm{j}, h + k\mathrm{j}] \in \mathsf{H}P^1 \cong S^4$, i.e. identifying the quaternions H with R^4 and identifying $S^4 \setminus \{\infty\}$ with R^4 by stereographic projection, we have

$$S^4 = \mathsf{H} \cup \{\infty\} \ni q = \frac{(\overline{h} - k\mathrm{j})(f + g\mathrm{j})}{|h|^2 + |k|^2} \tag{13}$$

Equivalently, we have the relation

$$(f + g\mathrm{j}) - (h + k\mathrm{j})(q_1 + q_2\mathrm{j}) = 0 \tag{14}$$

where $q = q_1 + q_2\mathrm{j}, q_1, q_2 \in \mathsf{C}$.

Given a complex surface $\mathcal{S} \subset \mathbf{C}P^3$ locally parametrized holomorphically by $(z, w) \mapsto [f(z, w), \ldots]$, then we can locally solve (14) for $(z, w) = (z(q), w(q))$. Then (z, w) give complex coordinates with respect to the (integrable) Hermitian structure J determined by $\mathcal{S}$ via σ_J.

Rewrite equation (14) as the pair of complex equations

$$(h \quad k)\begin{pmatrix} q_1 & q_2 \\ -\overline{q_2} & \overline{q_1} \end{pmatrix} = (f \quad g)$$

This is a spinor equation; indeed, set $q_1 = x^0 + \mathrm{i}x^1, q_2 = x^2 + \mathrm{i}x^3$. We make the identification

$$(x^0, x^1, x^2, x^3) \leftrightarrow (x^{AB}) = \begin{pmatrix} x^0 + \mathrm{i}x^1 & x^2 + \mathrm{i}x^3 \\ -x^2 + \mathrm{i}x^3 & x^0 - \mathrm{i}x^1 \end{pmatrix}$$

Then (14) can be conveniently written

$$\xi_A x^{AB} = \eta^B \tag{15}$$

where $(\xi_A) = (h, k)$ and $(\eta^B) = (f, g)$. Note that $\det(x^{AB}) = |q_1|^2 + |q_2|^2$ is the Euclidean norm. Then for $(R, S) \in SU(2) \times SU(2)$, the mapping

$$(x^{AB}) \mapsto R(x^{AB})\overline{S}^{\mathrm{Tr}}$$

belongs to $SO(4)$ and realizes the double cover $SU(2) \times SU(2) \overset{2:1}{\to} SO(4)$.

Solutions $z = z(q), w = w(q)$ to (14) are automatically semi-conformal, being holomorphic maps into a surface. It is easy to calculate the Laplacian Δz with respect to the Euclidean metric and we find that $z = z(q)$ is harmonic (and so a harmonic morphism) if and only if

$$hk_w - kh_w = 0 \quad \Leftrightarrow h/k = \beta(z)$$

for some meromorphic function $\beta(z)$. Then $\beta(z) \in \mathbf{C}P^1$ is the section of the twistor bundle $\Sigma^+ = \mathsf{R}^4 \times \mathbf{C}P^1$ over R^4. It is constant along the fibres of z and we have an alternative proof of a Theorem of Wood [40]: *a submersive harmonic morphism* $z : U \subset \mathsf{R}^4 \to \mathsf{C}$ *is equivalent to an integrable Hermitian structure* J *on* U*. The fibres of* z *are the integrable surfaces of the distribution* $D_J = \{v \in TU : \nabla_v J = 0\}$. This result remains true on an anti-self-dual Einstein 4-manifold [40].

Choose inhomogeneous coordinates by setting $k \equiv 1$; then (14) becomes

$$\begin{aligned} \beta(z)q_1 - \overline{q_2} &= f(z, w) \\ \beta(z)q_2 + \overline{q_1} &= g(z, w)\,. \end{aligned}$$

Eliminating w leads to an equation $\psi(z, \beta(z)q_1 - \overline{q_2}, \beta(z)q_2 + \overline{q_1}) = 0$. Local solutions $z = z(q)$ give harmonic morphisms. In a similar way to the R^3 case, we consider ψ as a twistor function defined on a domain of $\mathbf{C}P^3$. Local harmonic (not necessarily semi-conformal) complex-valued functions are given by contour integrals:

$$\varphi(q_1, q_2) = \oint \psi(z, \beta(z)q_1 - \overline{q_2}, \beta(z)q_2 + \overline{q_1})\mathrm{d}z$$

It will be useful in what follows to describe how the unit tangent bundle to S^3 sits naturally in $\mathbf{C}P^3$.

Consider $S^3 \subset S^4$ as the equatorial sphere given in our quaternionic coordinates by the imaginary quaternions:

$$S^3 = \{q \in \mathsf{H} \cup \{\infty\} : \overline{q} = -q\}\,.$$

Let $\mathcal{N}^5 \subset \mathbf{C}P^3$ be the inverse image under the Hopf fibration π: $\mathcal{N}^5 = \pi^{-1}(S^3)$. Then $[f, g, h, k] \in \mathcal{N}^5$ if and only if $\mathrm{Re}\,\{\overline{h}f + k\overline{g} + (\overline{h}g - k\overline{f})\mathrm{j}\} = 0$ if and only if

$$\overline{h}f + k\overline{g} + h\overline{f} + \overline{k}g = 0\,. \tag{16}$$

Equation (16) is a real equation defining an inner product $Q = \overline{h}f + k\overline{g} + h\overline{f} + \overline{k}g$ of signature $(2, 2)$ on C^4. Then $\mathbf{C}P^3$ splits into 3 components $P\mathsf{T}^-, \mathcal{N}^5, P\mathsf{T}^+$, depending on whether Q is positive, zero or negative, respectively.

Identify the hyperbolic plane (upper-half-space model) with the hemisphere $\pi(P\mathsf{T}^+) \subset S^4$; explicitly:

$$H^4 = \{q = x^0 + x^1\mathrm{i} + x^2\mathrm{j} + x^3\mathrm{k} \in \mathsf{H} : x^0 > 0\}$$

with metric $\sum (\mathrm{d}x^i)^2/(x^0)^2$. Then if $\Phi : U \subset H^4 \to \mathsf{C}$ is a smooth function, the hyperbolic Laplacian is given by

$$\Delta^{H^4}\Phi = (x^0)^2 \Delta^{\mathsf{R}^4}\Phi - x^0 \frac{\partial \Phi}{\partial x^0}$$

Suppose that Φ extends in a C^1 fashion to $\partial H^4 \cong S^3$. It is easily checked that if $\Delta^{H^4}\Phi = 0$, then $\partial\Phi/\partial x^0 = 0$ at $x^0 = 0$. Suppose then that Φ is a harmonic morphism with respect to the hyperbolic metric and set $\varphi = \Phi|_{\partial H^4} : V \subset S^3 \to \mathsf{C}$. Then, since $\partial\Phi/\partial x^0 = 0$ at $x^0 = 0$, we have

$$\left(\frac{\partial \varphi}{\partial x^1}\right)^2 + \left(\frac{\partial \varphi}{\partial x^2}\right)^2 + \left(\frac{\partial \varphi}{\partial x^3}\right)^2 = 0 \tag{17}$$

and φ is semi-conformal. Conversely, any analytic solution to (17) on an open set $V \subset S^3$ extends uniquely to a complex valued harmonic morphism Φ on a domain of H^4 [10, 11].

7. Twistor space of Minkowski space

Consider (real, uncompactified) Minkowski space $M^4 = \{(t, x^1, x^2, x^3) \in \mathsf{R}^4\}$ with metric $\mathrm{d}t^2 - \mathrm{d}{x_1}^2 - \mathrm{d}{x_2}^2 - \mathrm{d}{x_3}^2$. We make the identification

$$(t, x^1, x^2, x^3) \leftrightarrow (x^{AA'}) = \begin{pmatrix} -\mathrm{i}(t - x^1) & x^2 + \mathrm{i}x^3 \\ -x^2 + \mathrm{i}x^3 & -\mathrm{i}(t + x^1) \end{pmatrix}$$

Then $(x^{AA'})$ is skew-Hermitian, i.e. $\overline{(x^{AA'})}^{\mathrm{Tr}} = -(x^{AA'})$ with $\det(x^{AA'}) = -t^2 + {x_1}^2 + {x_2}^2 + {x_3}^2$.

Remark 7.1. Our conventions here differ from the usual, when the matrix on the right-hand side is replaced by $\begin{pmatrix} t + x^1 & x^2 + \mathrm{i}x^3 \\ x^2 - \mathrm{i}x^3 & t - x^1 \end{pmatrix}$ which is now Hermitian. However, in order to facilitate the unification with the Riemannian case described in Section 8, we prefer to choose the above conventions and in particular to maintain the expression for the Hopf fibration given by (13).

For an element $R \in SL_2(\mathsf{C})$, $R(x^{AA'})\overline{R}^{\mathrm{Tr}}$ is also skew-Hermitian with the same determinant as $(x^{AA'})$ and the map $M^4 \to M^4$ given by

$$(x^{AA'}) \mapsto R(x^{AA'})\overline{R}^{\mathrm{Tr}}$$

belongs to the connected component of the identity $O_+^{\uparrow}(1,3)$ of the Lorentz group. This realizes the double cover

$$SL_2(\mathsf{C}) \stackrel{2:1}{\to} O_+^{\uparrow}(1,3)\,.$$

Given a null vector (light ray) $(t, x^1, x^2, x^3) \in M^4$, then $\det(x^{AA'}) = 0$ which implies that $(x^{AA'})$ decomposes as an outer product: $x^{AA'} = \mathrm{i}\xi^A\eta^{A'}$. The fact that $(x^{AA'})$ is also skew-Hermitian implies that $\eta^{A'} = \overline{\xi}^{A'}$, where $\overline{\xi}^{0'} = \overline{\xi^0}$ and $\overline{\xi}^{1'} = \overline{\xi^1}$. In fact, after appropriately choosing conventions, $[\xi^0, \xi^1] \in \mathbf{C}P^1 \sim S^2$ represents the direction of the corresponding light ray as a point of the celestial sphere.

Consider the equation

$$\xi_A x^{AA'} = \eta^{A'} \tag{18}$$

This is the Minkowski analogue of (15) and is the incidence relation between a point $[\eta^{A'}, \xi_A] \in \mathbf{C}P^3$ in twistor space and a point $(x^{AA'})$ in M^4. Observe that if $x^{AA'} = y^{AA'}$ is a solution to (18), then so is $x^{AA'} = y^{AA'} + \lambda\xi^A\overline{\xi}^{A'}$, $\lambda \in \mathsf{R}$ - this represents the null-geodesic passing through $y^{AA'}$ with direction $[\xi^A] \in \mathbf{C}P^1$.

There is a real solution (i.e. (t, x^1, x^2, x^3) is real) if and only if

$$\overline{\eta^{0'}}\xi_0 + \eta^{1'}\overline{\xi_1} + \eta^{0'}\overline{\xi_0} + \overline{\eta^{1'}}\xi_1 = 0$$

which is the condition (16), so that a null-geodesic in M^4 corresponds to a point of $\mathcal{N}^5 \subset \mathbf{C}P^3$. The converse does not quite hold: we need to compactify M^4 by adding in a null cone at infinity to obtain compactified Minkowski space $\overline{M}^4 \sim S^3 \times S^1$. This becomes clearer when we complexify in Section 8. We will call elements of $\mathbf{C}P^3$, *twistors* and elements of $\mathcal{N}^5$, *null-twistors*. A Robinson congruence and the name twistor derives from the interpretation of a non-null-twistor.

Let $[Z^\alpha] \in \mathbf{C}P^3$ be a non-null twistor and consider the linear subspace $\mathcal{S}$ it determines: $\mathcal{S} = \{[W_\alpha] \in \mathbf{C}P^3 : Z^0W_0 + Z^1W_1 + Z^2W_2 + Z^3W_3 = 0\}$ (this should be considered as a subspace of dual twistor space - but we leave this interpretation until Section 8). Then $\mathcal{S}$ intersects $\mathcal{N}^5$ in a 3-dimensional set which therefore determines a 3-parameter family of null-geodesics which fill out $\overline{M}^4$ - this is a *Robinson congruence* (cf. [27]). With the appropriate conventions, taking a space-like slice $S^3 \subset \overline{M}^4$, a choice of Robinson congruence has tangent vector field which projects to a vector field in S^3 which is tangent to the fibres of the Hopf fibration $S^3 \sim \mathsf{R}^3 \cup \{\infty\} \to S^2$. The twisting of these fibres gives rise to the name twistor.

Zero-rest-mass fields on M^4 can be defined in a similar way to the construction of harmonic sections of spinor bundles defined in Section 4. This follows Penrose [28] and Penrose-Ward [32].

Indeed, set $u = t + x^1, v = t - x^1, \zeta = x^2 + \mathrm{i}x^3$ and let f be a complex analytic function of 3 complex variables (the twistor function defined on a domain of twistor space). Now set

$$\varphi_r = \frac{1}{2\pi\mathrm{i}} \oint z^r f(z, u + z\overline{\zeta}, \zeta + zv)\mathrm{d}z$$

Then we have the recurrence relations:

$$\begin{cases} \frac{\partial\varphi_r}{\partial\overline{\zeta}} = \frac{\partial\varphi_{r+1}}{\partial u} \\ \frac{\partial\varphi_r}{\partial v} = \frac{\partial\varphi_{r+1}}{\partial\zeta} \qquad r = 0, 1, 2, \ldots \end{cases}$$

For a non-negative integer n, let $\varphi_{AB\cdots K}$ (with $2n$ spinor indices) be defined by

$$\varphi_0 = \varphi_{00\cdots0}, \quad \varphi_1 = \varphi_{00\cdots1}, \ldots, \varphi_{AB\cdots K} = \varphi_{(AB\cdots K)}$$

Then the recurrence relations above are equivalent to the spinor field equation

$$\nabla^{AA'}\varphi_{AB\cdots K} = 0\,,$$

where $\nabla_{AA'} = \partial/\partial x^{AA'}$.

Remark 7.2. It is more usual to define the twistor function as a function f on a domain of C^4 homogeneous of a certain degree. Then $f = f(Z^\alpha)$ where $Z^\alpha = (\omega^{A'}, \pi_A) = (\pi_A x^{AA'}, \pi_A)$ is an unprojectivised twistor; now the contour integral is replaced by an integral over a 2 real dimensional surface - the two descriptions are equivalent.

Massless free fields are now constructed as follows: let $\varphi_{AB\cdots L}, \psi_{A'B'\cdots L'}$ be symmetric spinor fields on space-time. Then we have the following interpretation:

(19) $$\text{spin } \frac{1}{2}n\hbar \quad \text{negative helicity:} \nabla^{AA'}\varphi_{AB\cdots L} = 0$$

(20) $$\text{spin } \frac{1}{2}n\hbar \quad \text{positive helicity:} \nabla^{AA'}\varphi_{A'B'\cdots L'} = 0$$

The first equation represents an anti-self-dual field, the second a self-dual field. For $n = 0$ (spin 0) we take the wave equation $\nabla_a\nabla^a\psi = 0$ as the corresponding field equation.

Example: $n = 1$: We define an anti-symmetric 2-tensor by

$$\begin{aligned} F_{ab} \leftrightarrow F_{AA'BB'} &= \varphi_{AB}\varepsilon_{A'B'} + \varepsilon_{AB}\psi_{A'B'} \\ &= F_{ab}^{(-)} + F_{ab}^{(+)} \end{aligned}$$

which is the decomposition into $-$ self-dual and $+$ self-dual parts, respectively. The field is real if $\overline{\varphi}_{A'B'} = \psi_{A'B'}$. Then equations (19) and (20) correspond to $\nabla_{[a}F_{bc]} = 0$ (i.e. $\mathrm{d}F = 0$) and $\nabla^a F_{ab} = 0$ (i.e. $d^*F = 0$) which are Maxwell's equations.

Example: $n = 2$: We set

$$K_{abcd} \leftrightarrow K_{AA'BB'CC'DD'} = \varphi_{ABCD}\varepsilon_{A'B'}\varepsilon_{C'D'} + \varepsilon_{AB}\varepsilon_{CD}\psi_{A'B'C'D'}$$

the right-hand-side being the decomposition into $-$ self-dual and $+$ self-dual parts, respectively. Now (19) and (20) correspond to the Bianchi identities: $\nabla_{[a}K_{bc]pq} = 0$, which are the linearized Einstein field equations.

Example (Dirac equation): Let $(\eta^{A'}, \xi_A)$ be a pair of spinor fields on space-time. Then the Dirac equation has an elegant expression as the pair of spinor equations:

$$\left\{ \begin{array}{rcl} \nabla^{AA'}\xi_A &=& -\frac{im}{\sqrt{2}}\eta^{A'} \\ \nabla_{AA'}\eta^{A'} &=& -\frac{im}{\sqrt{2}}\xi_A \end{array} \right.$$

where m represents mass. The pair of equations imply that $\nabla_{BA'}\nabla^{AA'}\xi_A = -\frac{m^2}{2}\xi_B$ which shows that in some sense the equations are the 'square root' of the Klein-Gordon equation.

8. Unification

Let F_{12} be the flag manifold consisting of pairs (S_1, S_2), where S_1, S_2 are 1- and 2-dimensional subspaces, respectively, of C^4 with $S_1 \subset S_2$. Then we have two projections $\pi_1 : \mathsf{F}_{12} \to \mathbf{C}P^3$ and $\pi_2 : \mathsf{F}_{12} \to G_2(\mathsf{C}^4)$ given by $\pi_i((S_1, S_2)) = S_i$ $(i = 1, 2)$. The pair (π_1, π_2) is called a *double fibration* and the manifold F_{12} is called the *correspondence space*. The large cell, diffeomorphic to C^4 in the Shubert cell decomposition of $G_2(\mathsf{C}^4)$, can be parametrized by (see [26])

$$\begin{pmatrix} z_1 & z_2 \\ -\widetilde{z}_2 & \widetilde{z}_1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad z_1, \widetilde{z}_1, z_2, \widetilde{z}_2 \in \mathsf{C} \tag{21}$$

In fact the two vectors $(z_1, -\widetilde{z}_2, 1, 0), (z_2, \widetilde{z}_1, 0, 1)$ define a basis for $S_2 \in G_2(\mathsf{C}^4)$. Then S^4 sits naturally in $G_2(\mathsf{C}^4)$ as the space defined by the equations $\overline{z_1} = \widetilde{z}_1$ and $\overline{z_2} = \widetilde{z}_2$, whereas compactified Minkowski sits naturally as the space $z_1 = -\overline{z_1}$, $\widetilde{z}_2 = -\overline{\widetilde{z}_2}$ and $\widetilde{z}_1 = -\overline{z_2}$ (see [25] for an amplified treatment as well as a description of ultra-hyperbolic space). Their intersection is diffeomorphic to S^3. With this picture is it natural to consider $G_2(\mathsf{C}^4)$ as the complexification of both compactified Minkowski space $\overline{M}^4$ and S^4.

Modulo conventions, there are other ways to see these embeddings. For example (cf. [2]), to embed S^4, consider the real structure defined on $\mathbf{C}P^3$ as follows. Identify H^2 with C^4 by setting $(p, q) \equiv (p_1 + p_2\mathrm{j}, q_1 + q_2\mathrm{j})$ $(p, q \in \mathsf{H})$. Let $\sigma : \mathsf{H}^2 \to \mathsf{H}^2$ be left multiplication by $-\mathrm{j}$, then this induces a conjugate linear mapping $\sigma : \mathsf{C}^4 \to \mathsf{C}^4$ given by $\sigma(p_1, p_2, q_1, q_2) = (\overline{p_2}, -\overline{p_1}, \overline{q_2}, -\overline{q_1})$. This in turn induces a conjugate linear mapping $\sigma : \mathbf{C}P^3 \to \mathbf{C}P^3$. There are no fixed points (real points), however, there are complex projective lines invariant under σ called *real lines*. These lines are precisely the fibres of the Hopf fibration $\pi : \mathbf{C}P^3 \to S^4$. Now $G_2(\mathsf{C}^4)$ parametrizes all the projective lines in $\mathbf{C}P^3$, so π induces a natural embedding $S^4 \to G_2(\mathsf{C}^4)$. On the other hand compactified Minkowski space is naturally parametrized by the ! projective lines lying in $\mathcal{N}^5$.

We have three objects which are determined by a complex surface $\mathcal{S}$ in $\mathbf{C}P^3$ and which are therefore essentially equivalent. These are (a) a Hermitian structure on a domain of S^4; (b) a conformal foliation by curves of a domain of S^3; (c) a shear-free ray congruence (SFR) on a domain of Minkowski space. Let us elaborate a bit further.

(a) Locally, a complex surface determines a section of the twistor bundle $\mathcal{Z}^+ \to S^4$ and defines an integrable Hermitian structure as described in Section 6.

(b) Suppose the surface $\mathcal{S}$ defines an integrable Hermitian structure on an open set $U \subset S^4$. We canonically embed S^3 in S^4 by, say, setting $x^0 = 0$ and we suppose that $V = U \cap S^3$ is non-empty. Put the hyperbolic metric on the hemisphere H^4 on one side of S^3, say $x^0 > 0$. Then the Hermitian structure determines a harmonic morphism $\Phi : U \cap H^4 \to \mathbf{C}P^1$; its extension to the boundary $\varphi : V \to \mathbf{C}P^1$ is a semi-conformal map. The fibres of a semi-conformal map form a conformal foliation by curves (Section 5).

(c) The complex surface $\mathcal{S}$ intersects $\mathcal{N}^5$ generically in a 3-dimensional set - locally this determines a 3-parameter family of light rays which fill out a domain of space-time - a shear-free-ray congruence. The term shear-free means the following. Take a space-like slice R^3 of M^4, say $t = 0$. Then projection along the t-axis of the tangent vector field to the congruence at $t = 0$ determines a vector field on a domain of R^3. This vector field is tangent to a conformal foliation by curves. It is the same foliation given by (b).

We can interpret all of the above in another way. The twistor $[\eta^B, \xi_A] \in \mathbf{C}P^3$ determines a 2-dimensional surface in $G_2(\mathsf{C}^4)$ via equation (15) (we use the notation of Section 6 and so the second index in not a primed one; also the x^{AB} are now independent complex variables). With respect to the chart given by (21), it is a plane, called an *α-plane*. We consider S^4 and $\overline{M}^4$ embedded in $G_2(\mathsf{C}^4)$ in the ways described above. Then the α-plane intersects S^4 in a point and gives the $(0,1)$-tangent space of the corresponding complex structure in the complexification $T^{\mathsf{C}}S^4$. The α-plane may or may not intersect $\overline{M}^4$, depending on whether the twistor is null or not. If it is null, then the intersection is a light ray. The surface $\mathcal{S}$ now corresponds to defining ξ_A and η^B as holomorphic functions of two complex variables (z, w) and so locally defines a two (com! plex) parameter holomorphically varying family of α-planes. This defines the objects (a) and (c) above. In broad terms, the two meet and are defined by a conformal foliation by curves (b). (For a conformal foliation by curves to extend to a Hermitian structure on a domain of H^4 we require an analyticity hypothesis, see [10, 11] for details.)

To conclude these lectures, we outline some ideas, first described in [3], Section VI, on a method to determine a geometric structure on a domain of S^4 in terms of twistor space and its dual.

To define dual twistor space, we first of all define the flag manifold F_{23} consisting of all pairs (S_2, S_3) where S_2 and S_3 are 2- and 3-dimensional subspaces, respectively, of C^4 with $S_2 \subset S_3$. Set P^* to be the projective space of complex 3-planes in C^4; then we can identify P^* with the space $(\mathbf{C}P^3)^*$ of 1-dimensional complex lines in the dual space $(\mathsf{C}^4)^*$. We have a similar double fibration $\mathsf{F}_{23} \to G_2(\mathsf{C}^4)$ and $\mathsf{F}_{23} \to (\mathbf{C}P^3)^*$. We call $(\mathbf{C}P^3)^*$ *dual projective twistor space.* It can also be interpreted as the *negative* twistor bundle $\mathsf{Z}^- \to S^4$ of almost Hermitian structures incompatible with the orientation.

In order to describe points of dual twistor space, we replace upper indices by lower ones and lower indices by upper ones, so a dual twistor has the form $[\mu_B, \omega^A]$. We have a dual equation $\omega^A = -\mu_B x^{AB}$ which determines a surface in $G_2(\mathsf{C}^4)$, called a *β-plane*. In fact, $G_2(\mathsf{C}^4)$ is a doubly ruled complex manifold, ruled by α- and β-planes. A β-plane also intersects S^4 in a point and corresponds to the $(0,1)$-tangent space of an almost Hermitian structure incompatible with the orientation at that point (see [3]).

Now suppose we have a complex surface $\mathcal{S} \subset \mathbf{C}P^3$ in twistor space. Then $\mathcal{S}$ determines a Hermitian structure on a domain $U \subset S^4$. If we now fix a metric g that is asd Einstein on U, then by the theorem of Wood (cf. Section 6), it determines a harmonic morphism $\varphi : (U, g) \to \mathbf{C}P^1$. Note that φ depends on g. Furthermore φ also determines an almost (not necessarily integrable) Hermitian structure that is *incompatible* with the orientation. Indeed, for $x \in U$, let $V_x = \ker \mathrm{d}\varphi_x$ and let H_x be the orthogonal compliment with repect to g. With respect to the canonical orientation on S^4, we let $J_x^V : V_x \to V_x$ be rotation by $+\pi/2$; similarly we let $J_x^H : H_x \to H_x$ be rotation by $+\pi/2$. Then φ determines two almost Hermitian structures $J_1 = (J^H, J^V)$ and $J_2 = (-J^H, J^V)$ (for some relations to other properties of 2-dimensional foliations in a 4-manifold, see [**?**]). One of these, say J_1, is integrable and corresponds to the section $\mathcal{S}$ of the twistor bundle. The other J_2 is not in general integrable, but it defines another section $\mathcal{T}$ of dual twistor space.

We can turn the above situation around: suppose we are given complex surfaces $\mathcal{S} \subset \mathbf{C}P^3$ and $\mathcal{T} \subset (\mathbf{C}P^3)^*$, not necessarily complex analytic. We suppose both determine sections over an open set $U \subset S^4$. Then at each $x \in U$, we have determined J_1 and J_2. What are the properties of $\mathcal{S}$ and $\mathcal{T}$ that ensure that (i) we have determined a vertical and horizontal distribution V and H; (ii) the vertical distribution V is integrable; (iii) the resulting foliation determines a harmonic morphism with respect to a metric g? With such a program we would have the intriguing prospect of determining a geometric structure g from surfaces $\mathcal{S} \subset \mathbf{C}P^3$ and $\mathcal{T} \subset (\mathbf{C}P^3)^*$.

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Département de Mathématiques, Université de Bretagne Occidentale 6 Avenue Le Gorgeu, B.P. 452, 29275 Brest Cedex, France.
Email: Paul.Baird@univ-brest.fr