Mean curvature flow and self-similar submanifolds

Henri Anciaux

Introduction

This short survey is a completed version of an informal talk given in Peyresq at the meeting of the *mathematical physics group* in June 2002. The first three parts provide some basics about geometric flows. The last one briefly describes original results which are stated and proved in details in [Anc].

1 The mean curvature flow

Let us consider a smooth, compact submanifold Σ of $\mathbb{R}^n$ (however, there is no major difficulty to replace the ambient space by a Riemannian manifold). The mean curvature vector H is a fundamental feature attached to Σ which, roughly speaking, measures how the area of Σ (which we will denote by $\mathcal{A}$) varies when it is deformed. This property of H may be used as its definition:

Proposition and Definition 1 *There exists a unique vectorfield H tangent to Σ such that for any family of diffeomorphisms $(\Phi_t)_{t\geq 0}$ satisfying $\Phi_0 = Id$ and whose derivative at $t=0$ is denoted by V, the following holds:*

$$\mathcal{A}(\Phi_t(\Sigma)) = \mathcal{A}(\Sigma) - \int_\Sigma \langle V, H\rangle + o(t);$$

H will be called mean curvature vector.

This formula suggests to flow a submanifold by mean curvature: if we want to deform a submanifold by a diffeomorphism in such a way that the area decrease mostly and with the constraint that the L^2-norm of the derivative is some fixed constant, we find by Cauchy-Schwarz inequality that this derivative should be H (up to a multplicative constant that be may chosen to be 1). This is exactly the meaning of the *mean curvature equation*

$$\frac{\partial X}{\partial t} = H(X).$$

The decay of area of such a flow is then the L^2-norm of H.

In the context of symplectic geometry, the mean curvature vector has another property: if Σ is Lagrangian (*i.e.*, the standard symplectic form vanishes on Σ), then H is of Lagrangian type, which means that the mean curvature flow evolves within the class of Lagrangian submanifolds.

From the viewpoint of analysis, the mean curvature equation is of parabolic type, and by standard results, we know that for initial data having some regularity (C^0), existence and uniqueness of the flow holds for "a short time." By these words we mean for $t \in [0, T)$, where the non negative constant T does not admit any universal upper bound. Moreover, the flow is regularizing, *i.e.*, the evolved surface becomes instantaneously C^∞.

More interesting and more difficult is the long time behaviour of the evolution: we do not have general answers to this, except in some particular cases:

Theorem 1 *If Σ is the boundary of a convex, compact set in $\mathbb{R}^n, n \geq 3$, or a compact, embedded curve in $\mathbb{R}^2$, then it shrinks in finite time to a point. Moreover, its shape tends to be spherical as it shrinks.*

This is due to Grayson ([Gr]) for curves and to Huisken ([Hu]) in higher dimension.

Beyond theses cases, the general picture is that the flow generally develops singularities, and a first question is how these singularities look like.

Another question arises as follows: it has been a powerful tool to define weak notions of MCF (levels-set flows, viscosity solutions) which allow in particular to get very general existence theorems, even after formation of singularity. Then we are led to ask ourselves whether the good properties of the smooth flow still hold. Current studies ([AIV]) on the subject tend to answer "no": we expect that past a singularity, several distinct evolutions may occur, and that the Lagrangian property could be lost. In the last section, we shall give a kind of illustration of this phenomenon.

2 Self-similar flows

It is natural to seek for simple solutions of the MCF assuming that the flow preserves the shape of the evolving surface. Then the problem is reduced to look for a surface satisfying some curvature property. Analytically speaking, this amounts to plug a particular ansatz in the parabolic PDE describing the flow in order to split it into an ODE in the time variable (which is trivial) and an elliptic PDE in the space variables.

The first instance is self-shrinking, that is, a flow which is a homothetie with ratio tending to zero. Its importance comes from the fact that near a singularity the MCF is asymptotically self-shrinking.

The case of self-shrinking planar curves has been studied by U. Abresch and J. Langer in 1986 (cf [AbLa]). They show that the only embedded self-shrinking curve is the circle, and that most of them are not even properly embedded (*i.e.*, their image is dense in subset of $\mathbb{R}^2$). The other ones turn to be very beautiful rosettes. They are indexed by two relatively prime numbers p and q subject to the the following condition: $1/2 < p/q < \sqrt{2}/2$. These numbers have a geometric meaning: p is the winding number of the curve, and q is the number of petals of the rosette. On Figure 1 and 2 are two examples.

Self-shrinking is not the only example of self-similarity. We may look for self-expanding flows, that is flows which are a homothetie with ratio tending to infinity. This may seem somewhat paradoxical that the MCF evolution —which locally decreases area— should be an expansion; however such surfaces do exist, but are of course never compact. The Figure 3 shows an example of a self-expanding planar curve.

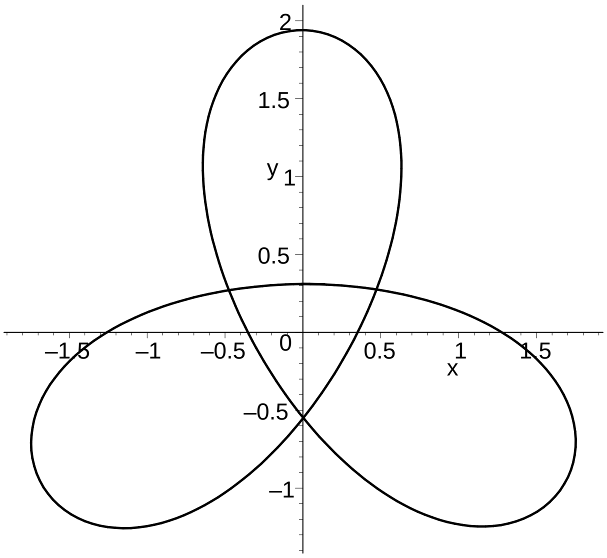

Figure 1: An Abresch and Langer curve with $p = 2, q = 3$

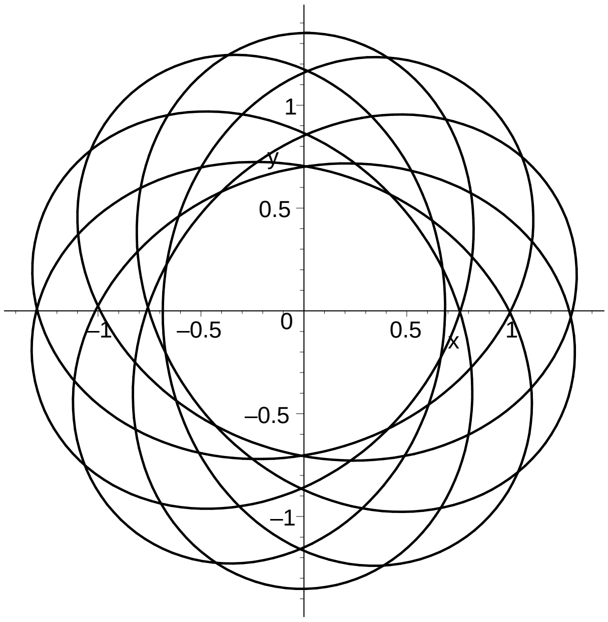

Figure 2: A Abresch and Langer curve with $p = 7, q = 10$

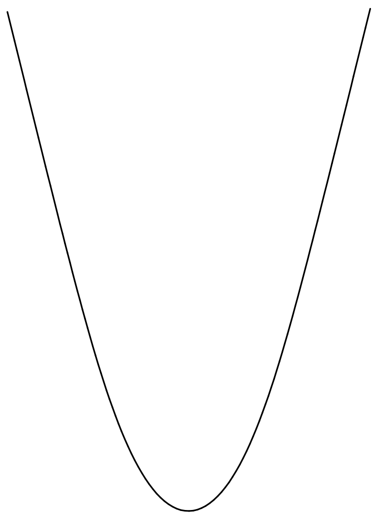

Figure 3: A self-expanding curve

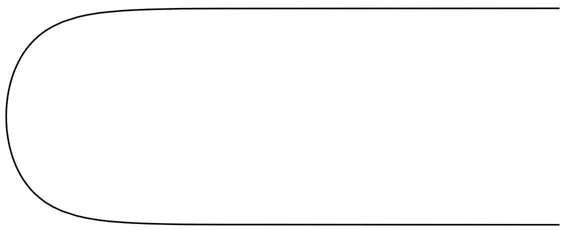

Figure 4: The Gream Reaper

One may also ask the flow to be a solid motion, for example a translation; this yields to the "Gream Reaper" (Fig. 4); finally, there is the "Yin-Yang" curve, which has the property of being self-rotated under the MCF (Fig. 5).

Beyond the case of planar curves, very little is known about self-similar flows. Angenent has proved the existence of self-similar tori in $\mathbb{R}^3$ ([Ang]) and Chopp has given numerical evidences for exitence of a variety of other self-shrinkers ([Ch]). In [AIV] a flow is described which starts being self-shrinking (say for $t < 0$), then becomes a cone at $t = 0$, and then pursues its evolution by self-expanding. In the last section we shall provide examples of Lagrangian self-shrinkers and self-expanders.

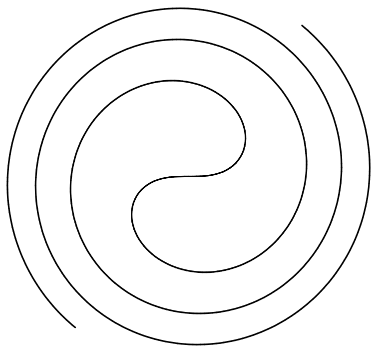

Figure 5: The Yin-Yang curve

3 Further geometric flows

It turns out that the MCF is just one example of geometric flow. Indeed, there is a general philosophy to define geometric flows: given an elliptic equation which is of variational form, one can consider the flow whose velocity is the gradient of the associated functional. In particular, stationary states of the flow are exactly solutions of the elliptic equation. For example, the MCF is the flow associated to the minimal surface equation, because the gradient of the area functional is the mean curvature vector, as expressed by the formula we began with.

Another very interesting example is the surface tension flow, which models the motion of the interface of two phases towards equilibrium. In this case, the interface tends to have least area with the constraint that the volume filled by each of the two phases remain constant. It can be proved that the stationary states are just constant mean curvature surfaces. This is actually the old famous *isoperimetric problem*: minimizing the area of a surface, some volume constraint given.

We can observe this phenomenon by dropping a few oil on a plate containing water: after a while, the connected components of oil become round disks, that is the interface between water and oil is a union of circles, which are the only closed curves with constant curvature.

In addition the surface tension flow illustrates well the similarity between the isoperimetric problem and the Oh's conjecture; the first one leads to the study of CMC hypersurfaces and the Oh's conjecture (that is minimizing the area of a Lagrangian submanifold among Hamiltonian deformations) yields the notion of Hamiltonian stationarity. These two variational problems have many common features, and are equal in dimension 1 (planar curves). One more similarity appears in our context: in both cases the associated flow is the surface tension flow.

There is another geometric flow whose study has drawn consequences in theoretical physics, that is the inverse mean curvature flow. Using it, Huisken and Ilmanen were able to proove the famous Penrose inequality (cf [HuIl]).

Many geometric flows are mentionned in the following array, where $\vec{V}$ stands for the normal velocity of the evolving surface:

	Geometric flow	Stationary solutions
Extrinsic flows	MCF $\vec{V} = \vec{H}(X)$	Minimal surfaces $\vec{H} = 0$
	Lagrangian MCF	Special Lagrangian surfaces
	Surface tension flow, [EMS] $\vec{V} = \Delta \vec{H}(X)$	CMC hypersurfaces $H = Const$
	$\vec{V} = \Delta \vec{H}(X) = J\nabla\Delta\beta$	Hamiltonian stationary Lagrangian submanifolds $\Delta\beta = 0$ [Oh]
	Willmore flow [KuSc] $\vec{V} = \delta\mathcal{W}$	Willmore surfaces [Wi] $\delta\mathcal{W} = 0$
	Inverse mean curvature flow [HuIl] $\|V\| = 1/\|H\|$	No stationary states
Intrinsic flows	Ricci flow $g_t^{ij} = -2R_{ij}$	Einstein metrics
	Yamabe flow	Conformal metrics with constant scalar curvature (Yamabe problem)

4 Self-similar equivariant submanifolds in $\mathbb{R}^{2n}$

In this section, we shall briefly describe further examples of self-similar submanifolds, which are a kind of generalisation of the Abresch and Langer curves. Let us consider $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, with coordinates $(x^1 + iy^1, \ldots, x^n + iy^n)$. This identification with a complex space provides a standard symplectic structure: $\omega = \sum_{j=1}^n dx^j \wedge dy^j$. We now consider n-dimensional submanifolds of $\mathbb{R}^{2n}$ taking the following form:

$$\Sigma = \{(\gamma\sigma_1, \ldots, \gamma\sigma_n)\},$$

where $(\sigma_1, \ldots, \sigma_n) \in S^{n-1} \subset \mathbb{R}^n$ and γ is some planar curve: $\gamma \subset \mathbb{C} \simeq \mathbb{R}^2$. A short computation shows that whatever γ is, Σ is Lagrangian for ω.

Then we show that Σ is self-similar, if and only if γ satisfies de following ODE:

$$k = \langle \gamma, \vec{N} \rangle \left(\frac{n-1}{|\gamma|^2} - \lambda \right), \tag{1}$$

where k denotes the curvature of γ and $\vec{N}$ its normal unit vector. If the constant λ is positive (resp. negative), Σ is self-shrinking (resp. self-expanding). Of course, the case of vanishing λ corresponds to a minimal submanifold.

It turns out that Equation (1) can be solved almost explicitly. Indeed, if we parametrize the curve $\gamma = re^{i\phi}$ by arclength and denote by θ the angle made by the tangent with any fixed axis, this equation may written as a the following system:

$$\begin{cases} \dot{r} = \cos(\theta - \phi) \\ \dot{\phi} = \frac{1}{r}\sin(\theta - \phi) \\ \dot{\theta} = (\lambda r - \frac{n-1}{r})\sin(\theta - \phi) \end{cases}$$

Then, introducing the new variable $\alpha := \theta - \phi$, we can reduce this system to:

$$\begin{cases} \dot{r} = \cos\alpha \\ \dot{\alpha} = (\lambda r - \frac{n}{r})\sin\alpha \end{cases}$$

This system admits the following first integral: $E := r^n \exp(-\lambda r^2/2)\sin\alpha$, which allows to draw the phase portrait, distinguishing the case $\lambda > 0$ in which the trajectories are bounded, and the case $\lambda \leq 0$, in which they are not (see Fig. 7).

It is not difficult now to deduce the main properties of the solutions of (1). We shall distinguish three cases according to the sign of λ.

4.1 The minimal case

This case is the easiest one because we get an explicit parametrization of solutions, that is

$$\phi = -\frac{1}{n}\alpha = -\frac{1}{n}\arcsin\frac{E}{r^n},$$

up to a constant of integration.

If $n = 1$, we get a straight line and in larger dimensions, we recognize the *Lagrangian catenoids* which were first identifed by Harvey and Lawson in [HaLa] and more widely described by Castro and Urbano in [CaUr]. The angle ϕ has range $(-\pi/n, 0)$, which corresponds to the fact that the catenoid is asymptotic to two Lagrangian hyperplanes with a constant angle π/n. For a more complete description of these submanifolds, see [CaUr].

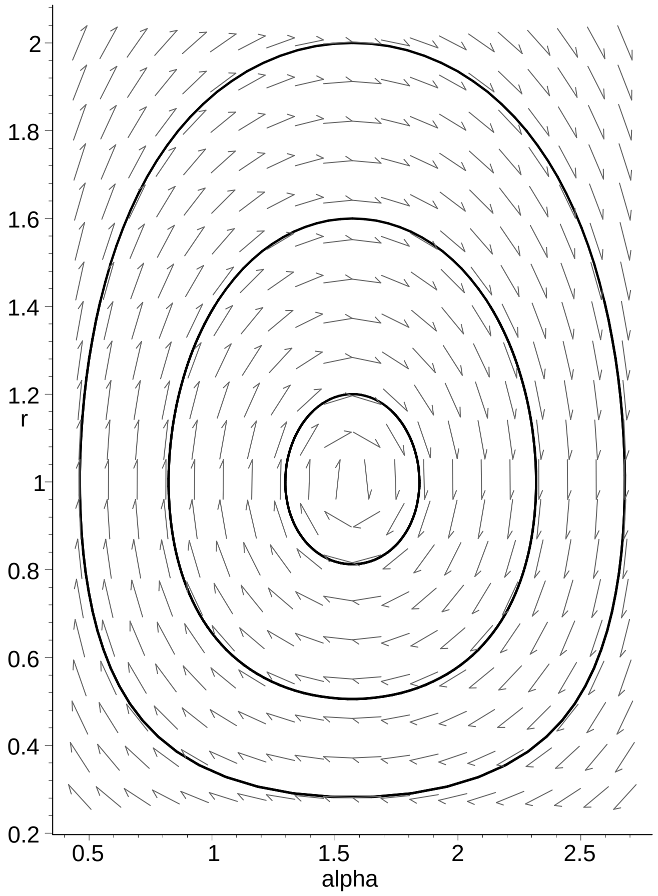

Figure 6: Phase portrait, $\lambda > 0$

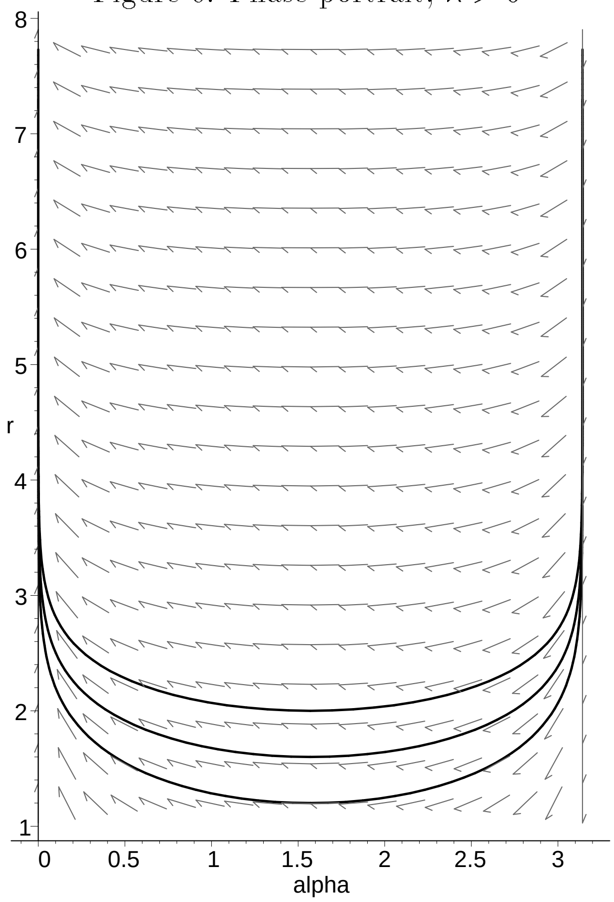

Figure 7: Phase portrait, $\lambda \leq 0$

4.2 The self-expanding case

The situation is quite similar to the previous case. We find that the curve γ is convex and asymptotic to two straight lines with an angle which ranges $(0, \pi/n)$. Therefore the corresponding submanifold is asymptotic to two Lagrangian hyperplanes making a constant angle in the same interval.

In the case of dimension 2, these surfaces yield to the following interesting observation: given two Lagrangian planes P and Q making a constant angle $\beta \in (0, \pi/2)$ between them, there exists a self-expanding surface Σ asymptotic to $P \cup -Q$ (the minus sign here means that Q is taken with opposite orientation as P's). This means that there exists a MCF starting from $P \cup -Q$, which is self-expanding: at time $t > 0$, the evolved surface is $\sqrt{t}\Sigma$.

Now, we notice that $P \cup Q$ is not only Lagrangian for the symplectic form $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$, but also for $\omega_1 = dx_1 \wedge dy_2 + dx_2 \wedge dy_1$. Therefore, it is also Lagrangian for any linear combination of ω and ω_1.

A consequence of this fact is that we can construct for each of these different symplectic forms a self-expanding surface, by the same construction as above, but in different coordinates, and each of these surfaces will be asymptotic to $P \cup -Q$. In other words, for each of these symplectic forms, we get a Lagrangian self-expanding evolution starting from $P \cup -Q$.

The conclusion is that we have exhibited a (singular) data, the intersection of two Lagrangian planes, admitting an infinity of distinct evolutions. Moreover, for a fixed symplectic structure, only one of these evolutions remains Lagrangian (however, as we have seen, the Lagrangian property is always conserved in some sense).

4.3 The self-shrinking case

Without loss of generality, we may fix $\lambda = n$. System (2) has as equilibrium points $(r = 1, \alpha = \frac{\pi}{2}[\pi])$. Geometrically, they correspond to the round circle of radius 1 which is solution of (1). These equilibrium points are local extrema of the energy E, so integrals curves are closed curves around them, excepted the vertical lines $\{\alpha = 0[\pi]\}$.

In order to describe completly the solutions of (1), we need to compute on each integral curve of (2) the number $\int \dot{\phi}$: this represents how much the curve γ winds around the origin. Actually, the curve will be closed if and only if this number is fraction times π.

A computation shows that $\int \dot{\phi}$ ranges the interval $[\pi/2n, \pi\sqrt{2/n}]$, so for any pair of integer numbers p and q such that $1/2n < p/q < \sqrt{2/n}$, there exists a closed curve satisfying Equation (1). The winding number of this curve is p, and q is the number of of patterns (or petals) that are periodically repeated. In particular, if $n = 1$, one recovers the Abresch and Langer curves. On Figures 8,9 and 10 are other instances.

Finally, we observe that except in the case of dimension 1, some of the curves we obtain are embedded, and so are the corresponding Lagrangian submanifolds. Indeed, it is necessary that the winding number p of γ be one, otherwise γ will have self-intersections. Then there are two slightly different cases: if q is odd Σ is an embedding of $S^{n-1} \times S^1$; if q is even, the parametrisation from $S^{n-1} \times S^1$ is a 2-covering (because of the central symmetry of γ), so the image is an embedding of $S^{n-1} \times S^1/\{0,1\}$. For example, in dimension two there is an embedded Lagrangian

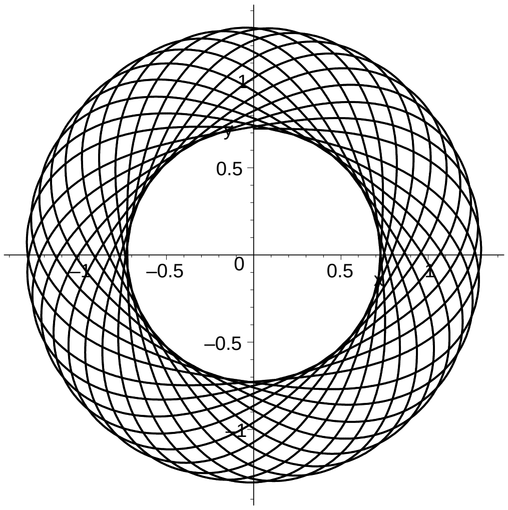

Figure 8: $n = 2, p = 17, q = 35$

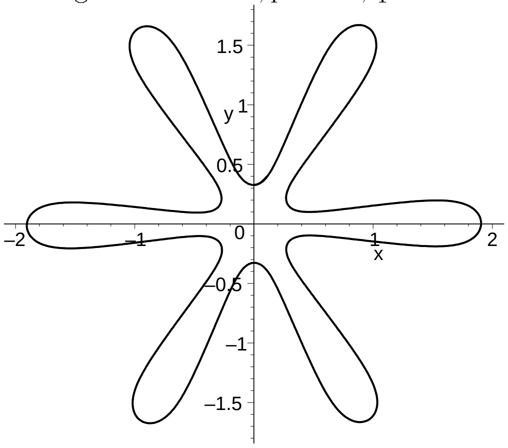

Figure 9: $n = 5, p = 1, q = 6$

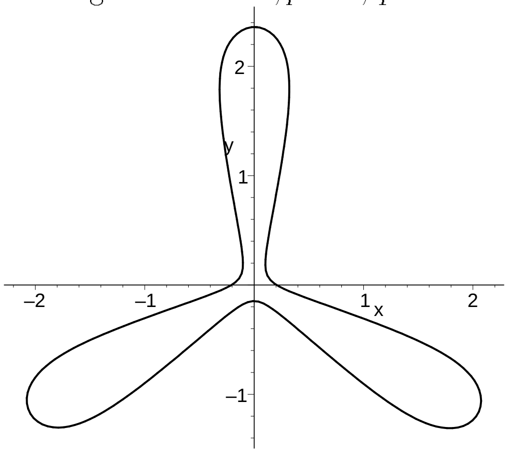

Figure 10: $n = 2, p = 1, q = 3$

self-shrinking torus (with $(p, q) = (1, 3)$). We notice also that the range condition on p/q prevents sharply the cases $(p, q) = (1, 2)$ and $(p, q) = (1, 4)$ which would be examples of Lagrangian Klein bottles. The interest of this last remark comes from a conjecture claiming that there is no Lagrangian embedding of the Klein bottle.

References

[AbLa] U. Abresch, J. Langer, *The normalized curve shortening flow and homothetic solutions,* J. of differential geometry, **23**(1986)175–196

[Anc] H. Anciaux, *Self-similar equivariant submanifolds in* $\mathbb{R}^{2n}$ in preparation

[Ang] S. Angenent, *Shrinking donuts,* in *Nonlinear diffusion reaction equations & their equilibrium, States 3,* editor N.G. Lloyd, Birkaüser, Boston, 1992.

[AIV] S. Angenent, T. Ilmanen, J.J.L. Velázquez, *Fattening from smooth initial data in mean curvature flow,* in preparation.

[CaUr] I. Castro, F. Urbano, *On a Minimal Lagrangian Submanifold of* $\mathbb{C}^n$ *Foliated by Spheres,* Mich. Math. J., **46**(1999),71–82

[Ch] D. L. Chopp *Numerical Computations of Self-Similar Solutions for Mean Curvature Flow,* Exper. Math. **3**(1994)1–16

[EMS] J. Escher, U. Mayer, G. Simonett, *The surface tension flow for immersed hypersurfaces,* SIAM J. Math. Anal. **29**(1998),1419–1433

[Gr] M. Grayson *The heat equation shrinks embedded plane curves to round circles* Journal of Differential Geometry, **26**(1987),285

[HaLa] R. Harvey, H. B. Lawson, *Calibrated geometries,* Acta Mathematica, **148**(1982),47–157

[Hu] G. Huisken *Flow by Mean Curvature of Convex Surfaces into Spheres* Journal of Differential Geometry, **20**(1984),237–266

[HuIl] G. Huisken and T. Ilmanen *The inverse mean curvature flow and the Riemannian Penrose inequality* available to `http://www.math.ethz.ch/~ilmanen/papers/hp.ps`

[Il] Tom Ilmanen, lectures on the mean curvature flow, `http://www.math.ethz.ch/~ilmanen/papers/notes.ps`

[KuSc] E. Kuwert and R. Schatlze, *The Willmore flow with small initial energy* Journal of Differential Geometry, **57**(1998),1–22

[Oh] Y. G. Oh, *Second variation and stabilities of minimal Lagrangian submanifolds in Kähler manifolds,* Invent. Math. **101**(1990),501–519

[Wi] T. J. Willmore, *Riemannian geometry,* Oxford Sciences Publications