SuPeyrsymmetry:
An introduction to supersymmetry*

Frédéric Hélein
CMLA et Département de Mathématiques, ENS de Cachan,
61 avenue du Président Wilson, 94235 Cachan Cedex, France

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These notes are a short and informal introduction to the mathematical aspects of supersymmetry. It is largely inspired by the texts by D. Freed [5] and by P. Deligne and D. Freed [4] and by [6]. I also beneficiated from many discussions with D. Bennequin. The interested Reader can find complete expositions in [5], [4], [6], [7].
I first present a personal overview of the two first chapters of [5] in order to expound some simple examples of variational problems with supervariables. The important point here is that, for simplest examples, the supersymmetry can be seen both as a symmetry of the variational problem and as a generalized symmetry of superspace. I end with an original (I believe) presentation of the super Minkowski space and of the super Poincaré group.

1 Motivations

Superspaces or more generally supermanifolds are variants of the classical geometrical manifolds introduced by physicists. There are thought as kinds of manifolds W on which the ring of smooth functions $A := \mathcal{C}^\infty(W)$ is $\mathbb{Z}_2$ graded, i.e. can be decomposed as a vector space over $\mathbb{R}$ as the direct sum $A = A^0 \oplus A^1$, and is non commutative according to the following rule: if $f \in A^a$, $g \in A^b$, then $fg = (-1)^{ab} gf$. Each function in A^0 is called *even* and each function in A^1 is called *odd*. We say that $(A, +, \cdot)$ is a superalgebra. In most cases W is finite dimensional, which means that there exists a finite number of functions $\theta^1, \cdots, \theta^p \in A^1$ and a smooth manifold $|W|$ of dimension n such that $\mathcal{C}^\infty(W) = \mathcal{C}^\infty(|W|)[\theta^1, \cdots, \theta^p]$. Thus A is a modulus over $\mathcal{C}^\infty(|W|)$ and any function $F \in \mathcal{C}^\infty(W)$ can be written in an unique way as

$$F = \sum_{j=0}^{p} \sum_{1 \leq i_1 < \cdots < i_j \leq p} f_{i_1 \cdots i_j} \theta^{i_1} \cdots \theta^{i_j},$$

where each $f_{i_1 \cdots i_j}$ is a smooth function on $|W|$. The simplest example of such a superspace is $\mathbb{R}^{n|p}$ which could be characterized by the definition of $\mathcal{C}^\infty(\mathbb{R}^{n|p})$ to be $\mathcal{C}^\infty(\mathbb{R}^n)[\theta^1, \cdots, \theta^p]$.

In general such superspaces do not correspond to a physical situation. They were however introduced in theoretical physics for the following reasons: first supersymmetry is a hypothetical symmetry of a quantum field theory which would exchange Bosons with Fermions. Bust most quantum field descriptions of a particle are connected with a classical field description, i.e. a variational problem. And second it turns out that the supersymmetry can be read on this classical level as a symmetry which exchanges solutions of the Euler–Lagrange equations of two different variational problems. Lastly the nice point is that one can interpret this symmetry on solutions of partial differential equations as resulting from a generalized geometrical symmetry, provided that one works on supermanifolds.

A basic example is a superparticle evolving in a Riemannian manifold $\mathcal{M}$. A first (and relatively rough) description of this system is through a pair (x, ψ), where

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- x is a smooth map from $\mathbb{R}$ to $\mathcal{M}$
- ψ is a section of the pull-back bundle $x^*(\Pi T\mathcal{M})$, i.e. for all $t \in \mathbb{R}$, $\psi(t)$ belongs to $\Pi T_{x(t)}\mathcal{M}$. Here $\Pi T_{x(t)}\mathcal{M}$ is the space modelled on $T_{x(t)}\mathcal{M}$ with opposite parity, i.e. odd. This means that the set of smooth functions on $\Pi T_{x(t)}\mathcal{M}$ is $\mathbb{R}[\alpha^1, \cdots, \alpha^n] \simeq \Lambda^* T^*_{x(t)}\mathcal{M}$, where $(\alpha^1, \cdots, \alpha^n)$ is a basis of $T^*_{x(t)}\mathcal{M}$.

Here we have used the notation $\Pi T_{x(t)}\mathcal{M}$, where Π is supposed to be a kind of functor which reverses the parity (odd versus even) of a given space. This notation can be useful because of its concision but it may lead us to think that Π is a kind of morphism. It is so as long as we consider only $\mathbb{Z}_2$ graded vector spaces properties, but however Π is never a superalgebra morphism nor a geometrical morphism. For that reason we will replace the use of Π by more a precise description.

The Lagrangian for the superparticle is

$$\mathcal{L}[x,\psi] := \int_{\mathbb{R}} \left(\frac{1}{2}|\dot{x}|^2 + \frac{1}{2}\langle \psi, \nabla_{\dot{x}}\psi \rangle \right) dt,$$

where, if $\psi(t) = \psi^i(t)\frac{\partial}{\partial x^i}$, $\nabla_{\dot{x}}\psi(t) = \left(\frac{d\psi^i}{dt}(t) + \Gamma^i_{jk}(x(t))\dot{x}^j(t)\psi^k(t) \right) \frac{\partial}{\partial x^i}$ (the Γ^i_{jk}'s being Christoffel's symbols). We already see here that the ψ^i's should be anticommuting quantities: if not then we would have $\frac{1}{2}\langle \psi, \nabla_{\dot{x}}\psi \rangle = \frac{1}{4}\frac{d}{dt}\langle \psi, \psi \rangle$, so that the second term in the Lagrangian would be unuseful.

The Euler–Lagrange system of equations is

$$\left\{ \begin{array}{rcl} \nabla_{\dot{x}}\dot{x} & = & \frac{1}{2}R(\psi,\psi)\dot{x} \\ \nabla_{\dot{x}}\psi & = & 0. \end{array} \right.$$

Again $R(\psi,\psi)\dot{x} = R(x)_{ijk}{}^l \psi^i\psi^j\dot{x}^k\frac{\partial}{\partial x^l}$ does not vanish precisely because ψ^i and ψ^j anticommute. In the above presentation we have chosen to ignore some slight contradictions which arise here. For instance the first Euler–Lagrange equation reads $\ddot{x}^l + \Gamma^l_{jk}(x)\dot{x}^j\dot{x}^k = \frac{1}{2}R(x)_{ijk}{}^l\psi^i\psi^j\dot{x}^k$, which is not homogeneous (only modulo 2). Should we identify real numbers with bilinear functions of odd variables ? Certainly no, because in particular $\psi^i\psi^j$ is a nilpotent quantity. This means that we should understand better the meaning of the mathematical objects here.

The action $\mathcal{L}$ enjoys a particular symmetry — that we will call a *supersymmetry* soon — as follows: we apply to the field (x,ψ) the infinitesimal transformation

$$\left\{ \begin{array}{rcl} x & \longmapsto & x - \eta\psi \\ \psi & \longmapsto & \psi + \eta\dot{x}. \end{array} \right. \tag{1}$$

Here η has two functions: it plays the role of an infinitesimal parameter, but it is also an odd variable. Otherwise this transformation would exchange odd variables with even ones, which does make sense. So the "vector" $(-\eta\psi(t), \eta\dot{x}(t))$ can be interpreted as a vector tangent[1] to $\Pi T\mathcal{M}$ to $(x(t),\psi(t))$. Let us compute the effect of this infinitesimal transformation on the action $\mathcal{L}$: we need to compute $\eta \lrcorner \delta\mathcal{L}(x,\psi)$ such that $\mathcal{L}(x-\eta\psi, \psi+\eta\dot{x}) = \mathcal{L}(x,\psi) + \eta \lrcorner \delta\mathcal{L}(x,\psi)$. Observing that (1) implies that

$$\left\{ \begin{array}{rcl} \dot{x} & \longmapsto & \dot{x} - \eta\nabla_{\dot{x}}\psi \\ \nabla_{\dot{x}}\psi & \longmapsto & \nabla_{\dot{x}}\psi + \eta\nabla_{\dot{x}}\dot{x}, \end{array} \right.$$

we obtain that

$$\begin{array}{rcl} \eta \lrcorner \delta\mathcal{L}(x,\psi) & = & \displaystyle\int_{\mathbb{R}} \left(\langle \dot{x}, -\eta\nabla_{\dot{x}}\psi \rangle + \frac{1}{2}\langle \eta\dot{x}, \nabla_{\dot{x}}\psi \rangle - \frac{1}{2}\langle \psi, \eta\nabla_{\dot{x}}\dot{x} \rangle \right) dt \\ & = & -\displaystyle\frac{1}{2}\eta \int_{\mathbb{R}} \frac{d}{dt}\langle \psi, \dot{x} \rangle \, dt \\ & = & -\displaystyle\frac{1}{2}\eta \left[\langle \psi(+\infty), \dot{x}(+\infty) \rangle - \langle \psi(-\infty), \dot{x}(-\infty) \rangle \right]. \end{array}$$

[1] still with the restriction that it is homogeneous to a vector in $T_{(x(t),\psi(t))}\Pi T\mathcal{M}$ only mod 2.

This quantity does not vanish in general, but is exact. So we see that the transformation (1) will change infinitesimally a solution of the Euler–Lagrange equations into another one[2].

On the one hand we also have a kind of Noether theorem: let us introduce a smooth compactly supported function $\phi \in \mathcal{C}_0^\infty(\mathbb{R})$ and, instead of (1), apply to (x,ψ) the infinitesimal transformation

$$\left\{ \begin{array}{lcl} x & \longmapsto & x - \eta\phi\psi \\ \psi & \longmapsto & \psi + \eta\phi\dot{x}, \end{array} \right. \tag{2}$$

with

$$\left\{ \begin{array}{lcl} \dot{x} & \longmapsto & \dot{x} - \eta\left(\phi\nabla_{\dot{x}}\psi + \dot{\phi}\psi\right) \\ \nabla_{\dot{x}}\psi & \longmapsto & \nabla_{\dot{x}}\psi + \eta\left(\phi\nabla_{\dot{x}}\dot{x} + \dot{\phi}\dot{x}\right). \end{array} \right.$$

Then we have

$$\begin{aligned} (\phi\eta) \lrcorner \delta\mathcal{L}(x,\psi) &= \int_{\mathbb{R}} \phi\left(\langle \dot{x}, -\eta\nabla_{\dot{x}}\psi \rangle + \frac{1}{2}\langle \eta\dot{x}, \nabla_{\dot{x}}\psi\rangle - \frac{1}{2}\langle \psi, \eta\nabla_{\dot{x}}\dot{x}\rangle \right) dt \\ &\quad + \int_{\mathbb{R}} \left(\left\langle \dot{x}, -\eta\dot{\phi}\psi \right\rangle + \frac{1}{2}\left\langle \psi, -\eta\dot{\phi}\dot{x}\right\rangle \right) dt \\ &= -\frac{1}{2}\eta \int_{\mathbb{R}} \phi \frac{d}{dt}\langle \psi, \dot{x}\rangle\, dt + \frac{3}{2}\eta\int_{\mathbb{R}} \phi\frac{d}{dt}\langle\psi,\dot{x}\rangle\, dt \\ &= \eta\int_{\mathbb{R}} \phi\frac{d}{dt}\langle\psi,\dot{x}\rangle\, dt. \end{aligned}$$

In all these computations we have used that η, ψ^i and $\dot{\psi}^i$ anticommute. We hence conclude that the quantity $\langle\psi,\dot{x}\rangle$ is preserved, an indication that a kind of Noether's theorem should take place here[3] and hence that the symmetry which generates our fields transformation has a geometrical interpretation. Eventually this "Noether's charge" has an interesting physical meaning, since the corresponding quantum operator is nothing but the Dirac operator on $\mathcal{M}$ (see [1], [5]).

On the other hand we can interpret the transformation acting on the set of fields

$$Q := \eta\zeta : (x,\psi) \longmapsto (-\eta\psi, \eta\dot{x})$$

as a a kind of tangent vector field to the infinite dimensional set of fields (x,ψ):

$$Q(x,\psi) = \int_{\mathbb{R}} \eta\left(-\psi(t)\frac{\partial}{\partial x(t)} + \dot{x}(t)\frac{\partial}{\partial\psi(t)}\right) dt$$

In particular this operator respects the parity modulo 2 of the variables, i.e. is an even operator. It is interesting to choose two different anticommuting variables η_1 and η_2 and to compute the commutator of the "vector fields" $Q_1 := \eta_1\zeta$ and $Q_2 := \eta_2\zeta$. This can be done by computing the action of $Q_1 \circ Q_2$:

$$\begin{pmatrix} x \\ \psi \end{pmatrix} \overset{Q_2 = \eta_2\zeta}{\longmapsto} \begin{pmatrix} -\eta_2\psi \\ \eta_2\dot{x} \end{pmatrix} \overset{Q_1 = \eta_1\zeta}{\longmapsto} \begin{pmatrix} -\eta_2(\eta_1\dot{x}) \\ \eta_2(-\eta_1\nabla_{\dot{x}}\psi) \end{pmatrix} = \eta_1\eta_2 \begin{pmatrix} \dot{x} \\ \nabla_{\dot{x}}\psi \end{pmatrix},$$

and the action of $Q_2 \circ Q_1$ (vice-versa). Then by an argument which rests on the fact that the Levi–Civita connection ∇ is torsion-free, we can write that $[Q_1,Q_2](x,\psi) = Q_1(Q_2(x,\psi)) - Q_2(Q_1(x,\psi))$ and deduce that

$$[\eta_1\zeta, \eta_2\zeta](x,\psi) = [Q_1,Q_2](x,\psi) = 2\eta_1\eta_2\nabla_{\dot{x}}(x,\psi).$$

Factoring out by η_1 and η_2 (and taking into account the fact that the odd numbers η_i anticommute also with the odd operator ζ) we deduce that

$$[\zeta,\zeta](x,\psi) = -2\nabla_{\dot{x}}(x,\psi).$$

[2] The question whether this infinitesimal transformation can be exponentiated and if it generates a 1-parameter family of solutions is however quite obscure

[3] in a way similar to the fact that the time translation symmetry $t \longmapsto t + \varepsilon$ induces the fields transformation $(x,\psi) \longmapsto (x - \varepsilon\dot{x}, \psi - \varepsilon\nabla_{\dot{x}}\psi)$ which is related, through Noether's theorem, to the conservation of the energy $-\langle\dot{x},\nabla_{\dot{x}}\dot{x}\rangle - \langle\psi,\nabla_{\dot{x}}\psi\rangle$

Here the notation $[\zeta,\zeta]$ does not recover a commutator, but an anticommutator ($[\zeta,\zeta] = 2\zeta\zeta$). We conclude that the action of ζ on fields should be related somehow to an infinitesimal geometric symmetry τ_Q such that $[\tau_Q,\tau_Q]$ is 2 times the time translation generator $\frac{d}{dt}$.

As we shall see later this symmetry can be pictured geometrically indeed by viewing the multiplet (x,ψ) as the components of a single superfield $\Phi = x + \theta\psi$, from the supertime $\mathbb{R}^{1|1}$ on which the ring of functions is $\mathcal{C}^\infty(\mathbb{R})[\theta]$ to $\mathcal{M}$ (so θ is here an odd coordinate). Then the Lagrangian action can be written as $\mathcal{L}[\Phi] := \int\int_{\mathbb{R}^{1|1}} -\frac{1}{2}\left\langle D\Phi, \frac{\partial\Phi}{\partial t}\right\rangle dtd\theta$, where $D := \frac{\partial}{\partial\theta} - \theta\frac{\partial}{\partial t}$. And the transformation (1) results from an infinitesimal translation in $\mathbb{R}^{1|1}$ by the vector $\eta\tau_Q$, where $\tau_Q := \frac{\partial}{\partial\theta} + \theta\frac{\partial}{\partial t}$.

All this was a description of a nice intuitive framework designed by physicists. But it deserves precise definitions of the mathematical objects, which is a delicate task for which several approaches are possible. I will present next the framework expounded in [4] and [5], inspired by ideas in algebraic geometry, and try later to propose a picture of it which is more based on a geometrical and analytical intuition.

2 Superspaces

As expounded in the preceding section we can define a superspace W by means of characterizing its ring of smooth funtions $(\mathcal{C}^\infty(W),+,\cdot) = (\mathcal{C}^\infty(|W|)[\theta^1,\cdots,\theta^p],+,\cdot)$. This is quite satisfactory as long as we are interested in the properties of $\mathcal{C}^\infty(W)$ as an algebra of functions, but it presents two drawbacks: first it is hard to vizualize what are the "points" of W and second it leads to some paradoxes as soon as we try to understand $\mathcal{C}^\infty(W)$ as a set of maps between a mysterious space and $\mathbb{R}$. In the second point we mean that, if $F = \sum_{j=0}^p \sum_{1\leq i_1<\cdots<i_j\leq p} f_{i_1\cdots i_j}\theta^{i_1}\cdots\theta^{i_j}$ belongs to $\mathcal{C}^\infty(W,\mathbb{R})$, then its "values" should be real, or at least an even quantity. But here the coefficients $f_{i_1\cdots i_j}$ are real valued functions whereas the monomials $\theta^{i_1}\cdots\theta^{i_j}$ are more difficult to interpret and — worst — they must be odd if j is odd. Perhaps one should enlarge the ring of functions, allowing the "coefficients" $f_{i_1\cdots i_j}$ to be odd, when j is odd ? or something else ?

2.1 The morphism property

In order to analyze the above question in more precise terms, we need to understand $\mathcal{C}^\infty(W,\mathbb{R})$ not just as an algebra, but as an object which connects two other objects (the spaces W and $\mathbb{R}$). More generally it is reasonable to ask for some functorial properties, i.e. given two super spaces or supermanifolds $M^{m|k}$ and $\widetilde{M}^{n|l}$, to look for a consistent definition of a map $T : M^{m|k} \longrightarrow \widetilde{M}^{n|l}$. This can be done through a morphism of algebras

$$\begin{array}{rccc} T^*: & \left(\mathcal{C}^\infty(\widetilde{M}^{n|l},\mathbb{R}),+,.\right) & \longrightarrow & \left(\mathcal{C}^\infty(M^{m|k},\mathbb{R}),+,.\right) \\ & A & \longmapsto & T^*A, \end{array}$$

where we can think secretly that $T^*A = A\circ T$. The important thing is to check the morphism property, i.e. that

$$T^*(1) = 1, \quad \text{where 1 is the constant equal to one (the unit in algebraic terms)} \tag{3}$$

$$\forall\lambda,\mu\in\mathbb{R}, \forall A,B\in\mathcal{C}^\infty(\widetilde{M}^{n|l},\mathbb{R}), \quad T^*(\lambda A+\mu B) = \lambda T^*A + \mu T^*B \tag{4}$$

and

$$\forall A,B\in\mathcal{C}^\infty(\widetilde{M}^{n|l},\mathbb{R}), \quad T^*(AB) = (T^*A)(T^*B). \tag{5}$$

These conditions impose severe constraints [5]. In the following we analyze their consequences through some examples.

2.2 A list of examples

1. Maps $T : \mathbb{R}^{m|0} \longrightarrow \mathbb{R}^{0|l}$, from an **even** vector space to an **odd** vector space.
   This amounts to look at morphisms T^* from $\mathbb{R}[\theta^1,\cdots,\theta^l]$ to $\mathcal{C}^\infty(\mathbb{R}^m)$. Any function $F\in\mathbb{R}[\theta^1,\cdots,\theta^l]$ writes $F = \sum_{j=0}^l \sum_{1\leq i_1,\cdots,i_j\leq n} f_{i_1\cdots i_j}\theta^{i_1}\cdots\theta^{i_j}$, where the $f_{i_1\cdots i_j}$'s are real constants. Thus, because of (4), it suffices to characterize all pull-back images $T^*(\theta^{i_1}\cdots\theta^{i_j})$. If

$j = 0$, (3) implies that $T^*f = f$. For $j \geq 1$ we remark that $(\theta^{i_1} \cdots \theta^{i_j})^2 = 0$. Hence by (5) $\left(T^*(\theta^{i_1} \cdots \theta^{i_j})\right)^2 = T^*\left((\theta^{i_1} \cdots \theta^{i_j})^2\right) = 0$. We thus conclude that $T^*F = f = F(0)$, i.e. the

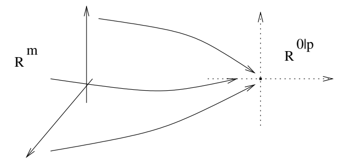

Figure 1: A map from $\mathbb{R}^m$ to $\mathbb{R}^{0|p}$

pull-back image of F by T is a constant function, whose value is equal to the coefficient f. This means that T looks like a constant function and its constant "value", if it would make sense, is 0; we can hence conclude that $T \equiv 0$.

2. Maps $T : \mathbb{R}^{0|1} \longrightarrow \mathcal{M}^{n|0} \simeq \mathcal{M}$, from an "**odd** line" to an (**even**) standard manifold.
   This the opposite situation. We look for morphisms $T^* : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathbb{R}[\theta]$. Any such map is characterized by two functionals $a, b : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ such that, $\forall f \in \mathcal{C}^\infty(\mathcal{M})$, $T^*f = a(f) + b(f)\theta$. The first condition (4) implies that a and b are linear functionals. Then condition (5) amounts to
$$a(fg) = a(f)a(g), \tag{6}$$
$$b(fg) = a(f)b(g) + b(f)a(g). \tag{7}$$
   The first relation (6) implies that there exists some point $m \in \mathcal{M}$ such that $a(f) = f(m)$, $\forall f$. Then the second one (7) reads $b(fg) = f(m)b(g) + b(f)g(m)$, which implies[4] that b is a derivation. So that there exists $\xi \in T_m\mathcal{M}$ such that $b(f) = df_m(\xi)$, $\forall f$. Hence

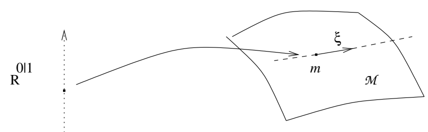

Figure 2: A map from $\mathbb{R}^{0|1}$ to $\mathcal{M}$

$$T^*f = f(m) + df_m(\xi)\theta, \quad \forall f \in \mathcal{C}^\infty(\mathcal{M}).$$

   Hence T is characterized by a point in the tangent bundle $T\mathcal{M}$.

3. Maps $T : \mathbb{R}^{0|2} \longrightarrow \mathcal{M} \simeq \mathcal{M}^{n|0}$, from an "**odd** plane" to an (**even**) standard smooth manifold.
   The analysis is similar but will lead to some pathological behaviour. We analyze morphisms $T^* : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathbb{R}[\theta^1, \theta^2]$: they are characterized by four linear functionals $a, b_1, b_2, c : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathbb{R}$ such that $\forall f \in \mathcal{C}^\infty(\mathcal{M})$, $T^*f = a(f) + b_1(f)\theta^1 + b_2(f)\theta^2 + c(f)\theta^1\theta^2$. Relation (5) for two functions $f, g \in \mathcal{C}^\infty(\mathcal{M})$ gives:
$$\begin{aligned} T^*(fg) &= a(fg) + b_1(fg)\theta^1 + b_2(fg)\theta^2 + c(fg)\theta^1\theta^2 \\ = T^*fT^*g &= \left(a(f) + b_1(f)\theta^1 + b_2(f)\theta^2 + c(f)\theta^1\theta^2\right)\left(a(g) + b_1(g)\theta^1 + b_2(g)\theta^2 + c(g)\theta^1\theta^2\right) \\ &= a(f)a(g) + (a(f)b_1(g) + b_1(f)a(g))\,\theta^1 + (a(f)b_2(g) + b_2(f)a(g))\,\theta^2 \\ &\quad + (a(f)c(g) + c(f)a(g) + b_1(f)b_2(g) - b_2(f)b_1(g))\,\theta^1\theta^2. \end{aligned}$$
   Again we deduce that there exists some point $m \in \mathcal{M}$ such that $a(f) = f(m)$ and there exist two vectors $\xi_1, \xi_2 \in T_m\mathcal{M}$ such that $b_1(f) = df_m(\xi_1)$ and $b_2(f) = df_m(\xi_2)$. We moreover obtain by identifying the $\theta^1\theta^2$ coefficients:
$$c(fg) = f(m)c(g) + c(f)g(m) + df_m(\xi_1)dg_m(\xi_2) - df_m(\xi_2)dg_m(\xi_1).$$

[4]assuming that T^* is continuous with respect to the $\mathcal{C}^1$ topology

¿From $fg = gf$ we deduce that the left hand side should be symmetric in f and g. However the right hand side is symmetric in f and g only if

$$(df \wedge dg)_m(\xi_1, \xi_2) = 0, \quad \forall f, g \in \mathcal{C}^\infty(\mathcal{M}).$$

This is possible only if ξ_1 and ξ_2 are linearly dependant, i.e. $\exists \xi \in T_m\mathcal{M}$, $\exists \lambda_1, \lambda_2 \in \mathbb{R}$, s.t. $\xi_1 = \lambda_1 \xi$ and $\xi_2 = \lambda_2 \xi$. If so we then conclude for c that $\exists \zeta \in T_m\mathcal{M}$ such that $c(f) = df_m(\zeta)$. Hence $T^*f = f(m) + (\lambda_1\theta^1 + \lambda_2\theta^2)df_m(\xi) + \theta^1\theta^2 df_m(\zeta)$. This conclusion does not look very natural: this enforces the relevance of the question asked at the end of the first paragraph of this Section, we come back to this point later.

2.3 A correction in the definition of maps $T : \mathbb{R}^{m|0} \longrightarrow \mathbb{R}^{0|l}$

The first example shows up some possible difficulties in applications. A first situation, as stressed out in [5], is if we are interested in the "pseudomechanics" [6] of a "particle" moving in an odd space $\mathbb{R}^{0|k}$. Such a mechanical problem is a candidate to be the classical description of fermions, usually described as quantum objects by a Dirac equation (even if in quantum field theory the Dirac equation needs to be quantized a second time). Indeed the classical motion of a "point" in $\mathbb{R}^{0|k}$ should be described by a map $\psi : \mathbb{R} \longrightarrow \mathbb{R}^{0|k}$, where the variable in the domain $\mathbb{R}$ is just the time t. But we have seen that if we define such a map according to the rules expounded in the subsection 2.2, it must be always constant and equal to 0. This reflects the fact that 0 is the only "classical" point in $\mathbb{R}^{0|k}$. The same difficulty occurs when we look at the superparticle model (x, ψ) expounded in the first section, in order to make sense of a non trivial section $t \longmapsto \psi(t) \in \Pi T_{x(t)}\mathcal{M}$.

As pointed out in [5] and [4], this can been cared by replacing the time line $\mathbb{R}$ by $\mathbb{R} \times B_L$, where B_L is a L dimensional odd space, such that $\mathcal{C}^\infty(B_L) = \mathbb{R}[\eta^1, \cdots, \eta^L]$. Then $\mathcal{C}^\infty(\mathbb{R} \times B_L) = \mathcal{C}^\infty(\mathbb{R})[\eta^1, \cdots, \eta^L]$. For example, if $L = 1$ the set of maps $T : \mathbb{R} \times B_1 \longrightarrow \mathbb{R}^{0|k}$ is described by the set of morphisms $T^* : \mathbb{R}[\psi^1, \cdots, \psi^k] \longrightarrow \mathcal{C}^\infty(\mathbb{R})[\eta]$. One then finds that for any such morphism there exists smooth functions $a^i \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$T^*\left(\sum_{j=0}^{\infty} \sum_{1 \le i_1, \cdots, i_j \le n} f_{i_1 \cdots i_j} \psi^{i_1} \cdots \psi^{i_j}\right) = f + \sum_{i=1}^{k} f_i a_i \eta, \quad \forall f_{i_1 \cdots i_j} \in \mathbb{R}.$$

We may interpret this result by the abuse of notations $\psi^i(t) \simeq \psi^i \circ T(t) = a^i(t)\eta$ (thinking on ψ^i as a coordinate on $\mathbb{R}^k$).

Coming back to the "pseudomechanics" of a "point" in an odd space, the equation of motion, in the absence of force, is

$$\frac{d\psi^i}{dt}(t) = 0.$$

And we just learned that we should consider this system of equations as a condition on a map $T : \mathbb{R} \times B_L \longrightarrow \mathbb{R}^k$, for some odd space B_L. This equation will the Euler–Lagrange equation for the action

$$\int_{\mathbb{R}} \left\langle \psi(t), \frac{d\psi}{dt}(t) \right\rangle dt.$$

But if we assume that $L = 1$, we encounter again a difficulty, since $\left\langle \psi(t), \frac{d\psi}{dt}(t) \right\rangle = \sum_i a^i(t) \frac{da^i}{dt}(t)\eta\eta = 0$, because η is odd. Thus we need actually here to suppose that $L \geq 2$, i.e. for example $\mathcal{C}^\infty(B_L) = \mathbb{R}[\eta^1, \eta^2]$. Then $\psi^i(t) = a^i(t)\eta^1 + b^i(t)\eta^2$, where a^i and b^i are smooth functions on $\mathbb{R}$ and $\left\langle \psi(t), \frac{d\psi}{dt}(t) \right\rangle = \sum_i \left(a^i(t)\frac{db^i}{dt}(t) - b^i(t)\frac{da^i}{dt}(t) \right) \eta^1\eta^2$.

This suggests that in general a map $T : M^{m|k} \longrightarrow \widetilde{M}^{n|l}$ should be defined through morphisms $T^* : \mathcal{C}^\infty(\widetilde{M}^{n|l}) \longrightarrow \mathcal{C}^\infty(M^{m|k} \times B_L) \simeq \mathcal{C}^\infty(M^{m|k}) \otimes \mathbb{R}[\eta^1, \cdots, \eta^L]$, for odd spaces B_L such that $\mathcal{C}^\infty(B_L) \simeq \mathbb{R}[\eta^1, \cdots, \eta^L]$. This is the point of view adopted in [4] and [5]. Equivalentely, as explained in the solution of the Fall Problem 2 in [9], a "map" $T : M^{m|k} \longrightarrow \widetilde{M}^{n|l}$ should be considered as a functor from the category of odd vector spaces B_L (or more generally superspaces) to the maps $T : M^{m|k} \times B_L \longrightarrow \widetilde{M}^{n|l}$, i.e. to morphisms $T^* : \mathcal{C}^\infty(\widetilde{M}^{n|l}) \longrightarrow \mathcal{C}^\infty(M^{m|k} \times B_L) \simeq$

$\mathcal{C}^\infty(M^{m|k}) \otimes \mathbb{R}[\eta^1, \cdots \eta^L]$.

Another approach would consist in replacing all rings of smooth functions $\mathcal{C}^\infty(W, \mathbb{R})$ over the field $\mathbb{R}$ by $\mathcal{C}^\infty(W, \mathbb{R}[\eta^1, \cdots, \eta^L])$ over the algebra $\mathbb{R}[\eta^1, \cdots, \eta^L]$. See for example [7].

2.4 A correction in the definition of maps from superspaces

Let us analyze now Examples 2 and 3. We are motivated by the superparticle (x, ψ) problem expounded in the first section. We will see in the next Section that the superspace formalism (i.e. by viewing the multiplet as the components of a single field $\Phi = x + \theta\psi$ from $\mathbb{R}^{1|1}$ to a Riemannian manifold $\mathcal{M}$) is much more convenient in order to picture and to compute the supersymmetries. However we see that if Φ is interpreted as in Examples 2 and 3, the components ψ^i are not odd as we would like, and moreover in case of a map $\Phi = x + \theta^1\psi_1 + \theta^2\psi_2 + \theta^1\theta^2 F$, from $\mathbb{R}^{n|2}$ into $\mathcal{M}$, ψ_1 and ψ_2 should be colinear. Again in order to care these contradictions one should either replace the field $\mathbb{R}$ by the algebra $\mathbb{R}[\eta^1, \cdots, \eta^L]$, or assume, in the spirit of [4] and [5], that a "geometrical map" $\Phi : \mathbb{R}^{n|p} \longrightarrow \mathcal{M}$ is actually a functor which to each finite dimensional odd space B_L associates the map $\Phi : \mathbb{R}^{n|p} \times B_L \longrightarrow \mathcal{M}$, i.e. the morphism $\Phi^* : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathcal{C}^\infty(\mathbb{R}^{n|p} \times B_L)$. We will see that this gives us the right framework for applications provided that we restrict ourself to maps Φ such that the morphism Φ^* is even, i.e. the image of $\mathcal{C}^\infty(\mathcal{M})$ by Φ^* is $\mathcal{C}^\infty(\mathbb{R}^{n|p} \times B_L)^0$, the even subspace of the super vector space $\mathcal{C}^\infty(\mathbb{R}^{n|p} \times B_L)$. This means that $\forall f \in \mathcal{C}^\infty(\mathcal{M})$, we can write

$$\Phi^* f = \sum_{j=0}^{p} \sum_{1 \leq i_1 < \cdots < i_j \leq p} \theta^{i_1} \cdots \theta^{i_j} a_{i_1 \cdots i_j}(f),$$

where each $f \longmapsto a_{i_1 \cdots i_j}(f)$ is a linear functional which takes values in $\mathcal{C}^\infty(\mathbb{R}^n \times B_L)^0$ if j is even and in $\mathcal{C}^\infty(\mathbb{R}^n \times B_L)^1$ if j is odd. Of course Φ^* should also satisfy conditions (3), (4) and (5).

So if we go back to Example 2., we now need to look at morphisms $T^* : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow (\mathbb{R}[\theta] \otimes \mathcal{C}^\infty(B_L))^0$. Setting $T^* f = A(f) + \theta B(f)$, where $A(f)$ is even and $B(f)$ is odd, we again find that (using that $A(f)$ and $A(g)$ commute with all other terms):

$$A(fg) = A(f)A(g) \quad \text{and} \quad B(fg) = A(f)B(g) + B(f)A(g).$$

Assume for example that $\mathcal{C}^\infty(B_L) = \mathbb{R}[\eta^1, \eta^2]$. Then $A(f) = a(f) + \eta^1\eta^2 a_{12}(f)$ and $B(f) = \eta^1 b_1(f) + \eta^2 b_2(f)$. The first of the above relations gives thus

$$a(fg) = a(f)a(g) \quad \text{and} \quad a_{12}(fg) = a(f)a_{12}(g) + a_{12}(f)a(g),$$

and hence there exists $m \in \mathcal{M}$ and $\zeta \in T_m\mathcal{M}$ such that $a(f) = f(m)$ and $a_{12}(f) = df_m(\zeta)$. The second relation gives

$$\eta^1 \left[a(f)b_1(g) + b_1(f)a(g)\right] + \eta^2 \left[a(f)b_2(g) + b_2(f)a(g)\right] = \eta^1 b_1(fg) + \eta^2 b_2(fg).$$

Hence $\exists \xi_1, \xi_2 \in T_m\mathcal{M}$ such that $b_k(f) = df_m(\xi_k)$. So

$$\begin{aligned} T^* f &= \left(f(m) + \eta^1\eta^2 df_m(\zeta)\right) + \theta \left(\eta^1 df_m(\xi_1) + \eta^2 df_m(\xi_2)\right) \\ &= f(m) + \eta^1\eta^2 df_m(\zeta) + \theta df_m \left(\eta^1\xi_1 + \eta^2\xi_2\right). \end{aligned}$$

If we add a time variable $t \in \mathbb{R}$ to $\mathbb{R}^{0|1}$ we deduce that, for example, a map Φ from $\mathbb{R}^{1|1} \times B_L$ to $\mathcal{M}$, where $\mathcal{C}^\infty(B_L) = \mathbb{R}[\eta^1, \eta^2]$ $(L = 2)$, can be written $\Phi = X + \theta\psi$, where now X is a kind of even map which, to each t associates a point $x(t)$ + an "infinitesimal displacement of $x(t)$" $\eta^1\eta^2\zeta(t)$ (here ζ is a section of $x^*T\mathcal{M}$) and $\psi(t) = \eta^1\xi_1(t) + \eta^2\xi_2(t)$, where ξ_1 and ξ_2 are also sections of $x^*T\mathcal{M}$.

Similarly we can come back to Example 3. and look at morphisms $T^* : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \left(\mathbb{R}[\theta^1, \theta^2] \otimes \mathcal{C}^\infty(B_L)\right)^0$. Assuming for example that $\mathcal{C}^\infty(B_L) = \mathbb{R}[\eta^1, \eta^2]$ one finds that $\exists m \in \mathcal{M}$, $\exists \zeta, \chi, \xi_{\alpha\beta}, \pi \in T_m\mathcal{M}$ (for $1 \leq \alpha, \beta \leq 2$) such that

$$\begin{aligned} T^* f = & \left(f(m) + \eta^1\eta^2 df_m(\zeta)\right) + \theta^1 df_m \left(\eta^1\xi_{11} + \eta^2\xi_{12}\right) + \theta^2 df_m \left(\eta^1\xi_{21} + \eta^2\xi_{22}\right) \\ & + \theta^1\theta^2 \left(df_m(\chi) + \eta^1\eta^2(Pf(m) + df_m(\pi))\right). \end{aligned}$$

Here P is the second order differential operator $P := Lie_\zeta \circ Lie_\chi - Lie_{\xi_{11}} \circ Lie_{\xi_{22}} + Lie_{\xi_{12}} \circ Lie_{\xi_{21}}$, where we have used arbitrary extensions of the vectors $\zeta, \chi, \xi_{\alpha\beta}, \pi \in T_m\mathcal{M}$ to vectors fields defined on a neighbourhood of m (for example we could choose the extensions in such a way that $[\zeta, \chi] = [\xi_{11}, \xi_{22}] = [\xi_{12}, \xi_{21}] = 0$). Clearly the operator P depends on the way we have extended the vectors fields. However two different choices of extensions lead to two operators P and P' such that $Pf(m) - P'f(m) = df_m(V)$, for some $V \in T_m\mathcal{M}$. Hence changing the extensions is just equivalent to change $\pi \in T_m\mathcal{M}$.

Lastly we can reinterpret the result that we obtained about $T^* : \mathcal{C}^\infty(\mathcal{M}) \longrightarrow \mathbb{R}[\theta^1, \theta^2] \otimes \mathcal{C}^\infty(B_2)$ by writing formally

$$\begin{aligned} T &= \left(m + \eta^1\eta^2\zeta\right) + \theta^1\left(\eta^1\xi_{11} + \eta^2\xi_{12}\right) + \theta^2\left(\eta^1\xi_{21} + \eta^2\xi_{22}\right) \\ &\quad + \theta^1\theta^2\left(\chi + \eta^1\eta^2\pi + (\eta^1\eta^2\zeta)\cdot\chi - (\eta^1\xi_{11} + \eta^2\xi_{22})\cdot(\eta^1\xi_{21} + \eta^2\xi_{21})\right) \\ &= \left(Id + \theta^1(\eta^1\xi_{11} + \eta^2\xi_{12})\right)\cdot\left(Id + \theta^1(\eta^1\xi_{11} + \eta^2\xi_{12})\right)\cdot\left(Id + \theta^1\theta^2(\chi + \eta^1\eta^2\pi)\right)\cdot\left(m + \eta^1\eta^2\zeta\right). \end{aligned}$$

2.5 Conclusion

More generally given a map $T : \mathbb{R}^{0|p} \longrightarrow \mathcal{M}$ we see that, if we let L increase and tend to ∞, T^*f will involve more and more derivatives of f, i.e. the map T can be understand as $T_0 + \sum_{j=1}^{p}\sum_{1\leq i_1<\cdots<i_j\leq p}\theta^{i_1}\cdots\theta^{i_j}T_{i_1\cdots i_j}$, where the image of T_0 is a point $m \in \mathcal{M}$ surrounded by a kind of formal neighbourhood of m (parametrized by even powers of η^i's) and each $T_{i_1\cdots i_j}$ for $j \geq 1$ is a vector in $T_m\mathcal{M}$ (even if j is so and odd if j is so) surrounded by formal deformations of it.

This picture is connected with other general frameworks for defining superspaces compatible with the morphism properties and with physical purposes, in which a smooth function on such a superspace should depend on formal deformations of points through Taylor expansions (even if the manifold is even !). Examples of such theories are proposed by de Witt or A. Rogers [8]. A brief exposition can be found in [6]. For example in this framework, one defines the Minkowski four dimensional superspace by first setting

$$M_L^{4|4} := B_L^0 \times B_L^0 \times B_L^0 \times B_L^0 \times B_L^1 \times B_L^1 \times B_L^1 \times B_L^1,$$

where B_L^0 (respectively B_L^1) is the even (resp. odd) subspace of B_L. Note that here the odd generators of two different factors B_L anticommute between themselves. One then specifies a point in $M_L^{4|4}$ by its four coordinates $(x^1, x^2, x^3, x^4, \theta^1, \theta^2, \theta^3, \theta^4)$. This set is fibered over $\mathbb{R}^4$ by the map $(x^1, x^2, x^3, x^4, \theta^1, \theta^2, \theta^3, \theta^4) \longmapsto (|x^1|, |x^2|, |x^3|, |x^4|)$, where, if $x = \sum_{j=1}^{L}\sum_{1\leq i_1<\cdots<i_j}\theta^{i_1}\cdots\theta^{i_j}x^{(j)}_{i_1\cdots i_j}$ belongs to B_L, $|x| := x^{(0)}$. ($|x|$ is also sometimes denoted by $\beta(x)$ and called the *body* of x.) Similarly one defines the *soul* of x to be $\sigma(x) := x - \beta(x)$. A topology can be defined on $M_L^{4|4}$ where each subset U is open if and only if $|U| := \{|x|)/x \in U\}$ is open. The superMinkowski space is then obtained by letting L tends to ∞. Then a superfunction of the variables (x^1, x^2, x^3, x^4) is by definition a function $z(f)$ such that there exists a $\mathcal{C}^\infty$ function f of the variables $(|x^1|, |x^2|, |x^3|, |x^4|)$ such that

$$z(f)[x^1, x^2, x^3, x^4] = \sum_{i_1\cdots i_4=1}^{L}\frac{1}{i_1!\cdots i_4!}\frac{\partial^{i_1+\cdots+i_4}f}{(\partial|x^1|)^{i_1}\cdots(\partial|x^4|)^{i_4}}\left(|x^1|, |x^2|, |x^3|, |x^4|\right)\sigma(x^1)^{i_1}\cdots\sigma(x^4)^{i_4}.$$

And an arbitrary superfunction of the variables $(|x^1|, |x^2|, |x^3|, |x^4|, \theta^1, \theta^2, \theta^3, \theta^4)$ is defined to be

$$F = \sum_{j=0}^{L}\sum_{1\leq i_1<\cdots<i_j\leq 4}\theta^{i_1}\cdots\theta^{i_j}z(F_{i_1\cdots i_j}),$$

where each $z(F_{i_1\cdots i_j})$ is superfunction of the variables (x^1, x^2, x^3, x^4) as defined above.

We see that all these points of view lead basically to same algebraic expressions. All follow from the purpose of a producing a mathematically correct definition of superspaces which allows us to add, multiply and pull-back functions in a consistent way. In particular we have seen that, in order to obtain *non trivial* functions (which are relevant for physicists), we have been forced to work with an arbitrary number of odd auxiliary variables. Eventually these frameworks lead to definitions of *supermanifolds*, *Lie supergroups*, etc., by gluing locally super vector spaces together with an appropriate notion of diffeomorphism (an adaptation of constructions from standard differential geometry). We shall not enter into these details here (see for example [8]).

3 The supertime

We come back to the variational problem on the variables (x, ψ) and its supersymmetry Q expounded in the first section. We now interpret this infinitesimal fields transformation as the effect of a generalized geometrical symmetry which takes place in a superspace. It is actually the simplest example of non trivial superspace, the "supertime" $M^{1|1}$. In an approximate description the set of smooth functions on $M^{1|1}$ is $\mathcal{C}^\infty(\mathbb{R})[\theta]$, where θ is an odd variable. But we now know what that means: actually we should consider functions depending on an arbitrary number of odd auxiliary variables for applications. For instance, following the approach in [5], a map $T \in \mathcal{C}^\infty(\mathbb{R})[\theta]$ is rather a functor which associates to any odd space B_L the morphism $T^* : \mathcal{C}^\infty(\mathbb{R}) \longrightarrow \mathcal{C}^\infty(\mathbb{R})[\theta]\otimes\mathcal{C}^\infty(B_L)$. In the following we shall "forget" to recall that we use an odd spaces B_L in order to "full" functions between superspaces, but this should be done. Functions on $M^{1|1}$ will be denoted by $f + \theta g$, where $f, g \in \mathcal{C}^\infty(\mathbb{R})$. The variable t can be understood as the time, the odd "variable" θ could be thought as a coordinate on an odd line. Beside the usual operator $\frac{\partial}{\partial t}$ acting on functions on $M^{1|1}$ by $\frac{\partial}{\partial t} \lrcorner\, d(f(t) + \theta g(t)) = \dot{f}(t) + \theta\dot{g}(t)$ (with the obvious geometrical interpretation), we consider two odd operators

$$D := \frac{\partial}{\partial \theta} - \theta\frac{\partial}{\partial t} \quad \text{and} \quad \tau_Q := \frac{\partial}{\partial \theta} + \theta\frac{\partial}{\partial t}.$$

They act on functions on $M^{1|1}$ through

$$D \lrcorner\, d(f(t) + \theta g(t)) = g(t) - \theta\dot{f}(t) \quad \text{and} \quad \tau_Q \lrcorner\, d(f(t) + \theta g(t)) = g(t) + \theta\dot{f}(t).$$

D (resp. τ_Q) can be considered as a left (resp. right) invariant odd vector field on $M^{1|1}$ (we shall see why later).

When acting on $\mathcal{C}^\infty(\mathbb{R})[\theta]$, operators $\frac{\partial}{\partial t}$, D and τ_Q satisfy the following relations:

$$D^2 = -\frac{\partial}{\partial t}, \quad \tau_Q^2 = \frac{\partial}{\partial t}, \quad D\tau_Q + \tau_Q D = 0,$$

$$D\frac{\partial}{\partial t} - \frac{\partial}{\partial t}D = \tau_Q\frac{\partial}{\partial t} - \frac{\partial}{\partial t}\tau_Q = 0.$$

For example

$$D^2(f(t) + \theta g(t)) = D \lrcorner\, (D \lrcorner\, (f(t) + \theta g(t))) = D \lrcorner\, \Big(g(t) - \theta\dot{f}(t)\Big) = -\dot{f}(t) - \theta\dot{g}(t).$$

These relations can be summarized together by introducing the supercommutator $[\cdot,\cdot]$ of two homogeneous operators A and B: if A has parity $|A|$ ($= 0$ if A is even, $= 1$ if A is odd) and B has parity $|B|$, then

$$[A, B] := A \circ B - (-1)^{|A||B|}B \circ A.$$

We thus have

$$[D, D] = -2\frac{\partial}{\partial t}, \quad [\tau_Q, \tau_Q] = 2\frac{\partial}{\partial t}, \quad [D, \tau_Q] = \left[D, \frac{\partial}{\partial t}\right] = \left[\tau_Q, \frac{\partial}{\partial t}\right] = 0.$$

This does not make the space "spanned" by $\frac{\partial}{\partial t}$, D and τ_Q a Lie algebra. It is a Lie superalgebra. One possible definition of such an object is that it is related to a genuine Lie algebra by the following construction. We consider again an odd space B_L and we consider the space spanned over $\mathcal{C}^\infty(B_L) = \mathbb{R}[\theta^1 \cdots \theta^L]$ by $\frac{\partial}{\partial t}$, D and τ_Q, $\mathfrak{g} := \mathcal{C}^\infty(B_L)[\frac{\partial}{\partial t}, D, \tau_Q]$, and we split into $\mathfrak{g}^0 \oplus \mathfrak{g}^1$, where $\mathfrak{g}^0$ is the subset of *even* combinations and $\mathfrak{g}^1$ is the subset of *odd* combinations. Then $\mathfrak{g}^0$ is a Lie algebra. Note that here it is important also to state that the variables η^j also anticommute with D and τ_Q (but commute with $\frac{\partial}{\partial t}$). This changes commutation relations and in particular all supercommutators are transmuted into commutators through the effect of the coefficients. For example, if η^1 and η^2 are odd, $\eta^1 D$ and $\eta^2\tau_Q$ are even and the anticommutation rule $[D,\tau_Q]=0$ implies the commutation

$$\left[\eta^1 D,\eta^2\tau_Q\right] = \eta^1 D\circ\eta^2\tau_Q - \eta^2\tau_Q\circ\eta^1 D = -\eta^1\eta^2 D\circ\tau_Q + \eta^2\eta^1\tau_Q\circ D = -2\eta^1\eta^2\left[D,\tau_Q\right] = 0.$$

It is then obvious that the induced bracket on $\mathfrak{g}^0$ is skew. A good exercise is to check that $\mathfrak{g}^0$ is a Lie algebra, and in particular that the Jacobi identity $[A,[B,C]]+[B,[C,A]]+[C,[A,B]]=0$ is true.

The action of the Lie superalgebra on $\mathcal{C}^\infty(\mathbb{R})[\theta]$ can also be read through the action of the Lie algebra $\mathfrak{g}^0$ on $\mathcal{C}^\infty(\mathbb{R})[\theta]\otimes\mathcal{C}^\infty(B_L)$ (where here θ anticommutes with all other odd variables).

Now we come back to the superparticle and we will present it differently. Instead of working on fields (x,ψ) we consider maps Φ from $M^{1|1}$ to $\mathcal{M}$. This relies on morphisms $\Phi^*:\mathcal{C}^\infty(\mathcal{M})\longrightarrow\mathcal{C}^\infty(\mathbb{R})[\theta]\otimes\mathcal{C}^\infty(B_L)$. One can check that according to the rule expounded there such a Φ^* should be such that there exists a even map $\mathbb{R}\ni t\longmapsto x(t)\in\mathcal{M}\otimes\mathcal{C}^\infty(B_L)^0$ and a odd section along x, $\mathbb{R}\ni t\longmapsto\psi(t)\in T_{x(t)}\mathcal{M}\otimes\mathcal{C}^\infty(B_L)^1$ (with some abuses of linguage), such that

$$\forall f\in\mathcal{C}^\infty(\mathcal{M}),\quad \Phi^* f = \left[t\longmapsto f(x(t)) + \theta df_{x(t)}(\psi(t))\right].$$

In more intuitive words, $\Phi(t)\simeq x(t)+\theta\psi(t)$. So we obtain the same data as for the superparticle of the preceding Section.

Recall that the vector field $Q=\eta\zeta$ was defined by

$$Q(x,\psi)(t) = (-\eta\psi(t),\eta\dot{x}(t)).$$

In the supertime framework, where $\Phi(t)\simeq x(t)+\theta\psi(t)$, this reads

$$Q\Phi(t) = -\eta\psi(t)+\theta\eta\dot{x}(t) = -\eta\left(\tau_Q \lrcorner\, d(x+\theta\psi)\right)(t).$$

Thus the action of Q on fields is related to infinitesimal translations of the fields along the constant vector field τ_Q:

$$Q\lrcorner\,\delta\Phi = \eta\zeta\lrcorner\,\delta\Phi = -\tau_q\lrcorner\, d\Phi. \tag{8}$$

Furthermore the action $\mathcal{L}[x,\psi]$ can be written in terms of the field Φ as follows. Consider:

$$\mathcal{L}[\Phi] := \int\int_{\mathbb{R}^{1|1}} -\frac{1}{2}\left\langle D\Phi,\frac{\partial\Phi}{\partial t}\right\rangle dt\,d\theta,$$

where the integration over the variable θ is defined according to Berezin (see [4], [7]) by: for any function $f+\theta g\in\mathcal{C}^\infty(\mathbb{R}^{1|1})$,

$$\int\int_{\mathbb{R}^{1|1}}(f+\theta g)dt\,d\theta := \int_{\mathbb{R}}\frac{\partial}{\partial\theta}\lrcorner\,(f+\theta g)dt = \int_{\mathbb{R}} g(t)\,dt.$$

Moreover through the substitution $\Phi = x+\theta\psi$ one finds

$$\begin{aligned}
-\frac{1}{2}\left\langle D\Phi,\frac{\partial\Phi}{\partial t}\right\rangle &= -\frac{1}{2}\left\langle\psi-\theta\dot{x},\dot{x}+\theta\nabla_{\dot{x}}\psi\right\rangle\\
&= -\frac{1}{2}\left\langle\psi,\dot{x}\right\rangle + \frac{\theta}{2}\left(|\dot{x}|^2+\langle\psi,\nabla_{\dot{x}}\psi\rangle\right).
\end{aligned}$$

Thus

$$\mathcal{L}[\Phi] = \frac{1}{2}\int_{\mathbb{R}}\left(|\dot{x}|^2+\langle\psi,\nabla_{\dot{x}}\psi\rangle\right)dt = \mathcal{L}[x,\psi].$$

Hence since $\mathcal{L}[\Phi]$ is defined only in terms of D and $\frac{\partial}{\partial t}$ and using the Berezin integration over $\mathbb{R}^{1|1}$ the invariance of $\mathcal{L}[\Phi]$ under the action of Q is then a consequence of (8) and

$$[D,\tau_Q] = \left[\frac{\partial}{\partial t},\tau_Q\right] = 0.$$

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