Spectral theory
(Cours de M1, Université Paris Diderot)

Frédéric HÉLEIN *

12 décembre 2014

Table des matières

*Institut de Mathématiques de Jussieu–Paris Rive Gauche, UMR CNRS 7586 Université Paris Didero–Paris 7, helein@math.univ-paris-diderot.fr

Introduction

Spectral theory could be presented as an attempt to extend the well-known decomposition results in finite dimensional linear algebra (the diagonalization and triangularization of matrices) to analogous situations in infinite dimension. Following the usual terminology we call *operators* the linear maps between (generally speaking) infinite dimensional vector spaces. Most results and applications require to choose a topology on these vector spaces and this is one of the first deep difference with the finite dimensional theory. This makes the theory much more difficult and this is why we content ourself with the task of extending the diagonalization of normal operators to Hilbert spaces, the suitable generalization of the notion of Euclidean vector spaces. The range of applications of such a theory is however quite large.

A second toy model we could start from is the Fourier transform $\mathcal{F} : f \longmapsto \widehat{f}$ of functions on $\mathbb{R}^n$, defined by $\widehat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-i\xi\cdot x} f(x)dx$. The well-known relation $\widehat{\frac{\partial f}{\partial x^k}}(\xi) = i\xi_k \widehat{f}(\xi)$ can be interpreted by claiming that the operator $\widehat{f} \longmapsto \mathcal{F}\frac{\partial}{\partial x^k}\mathcal{F}^{-1}(\widehat{f})$ coincides with the multiplication operator :

$$[\xi \longmapsto \widehat{f}(\xi)] \longmapsto [\xi \longmapsto i\xi_k \widehat{f}(\xi)].$$

But the latter operator is just an infinite dimensional, continuous analogue of a diagonal operator. In particular the value $i\xi_k$ seems to play the role of an eigenvalue. However it is *not* an eigenvalue, as explained below. Moreover $\frac{\partial}{\partial x^k}$ can be understood as a normal operator (actually $\frac{1}{i}\frac{\partial}{\partial x^k}$ is self-adjoint) and $\mathcal{F}$ as a unitary transformation, provided we endow the space of functions with the L^2-Hermitian scalar product $\langle f, g\rangle := \int_{\mathbb{R}^n} \overline{f(x)} g(x)dx$. However a rigorous description of the Fourier transform as the unitary isomorphism of $L^2(\mathbb{R}^n, \mathbb{C})$ for the spectral decomposition of $\frac{\partial}{\partial x^k}$ leads to several difficulties, since the latter operator is not defined on the whole space $L^2(\mathbb{R}^n, \mathbb{C})$. We thus need a theory of *non bounded* operators to take into account such operators and more generally all normal operators which are built from differential operators, the most important class of operators for applications. We can see also why $i\xi_k$ is not a eigenvalue : it would be so, then the corresponding eigenspace would be a distribution supported by the hyperplane $\xi_k =$ Constant and hence could not be in $L^2(\mathbb{R}^n, \mathbb{C})$. Hence $i\xi_k$ is rather called a *spectral* value.

The spectral theory is also strongly connected to another theory : the *functional calculus*. The latter addresses the following question : given, say, a self-adjoint operator A, and a continuous function f defined on $\mathbb{R}$, can we make sense of $f(A)$ as an operator ? We note that, if for instance A is diagonalizable, then we can answer to the question by setting $f(A)u = f(\lambda)u$, for each eigenvalue λ and eigenvector u such that $Au = \lambda u$. This establishes hence a connection between the spectral theory and the functional calculus. But the relationship between both theories is in fact deeper and we will see that, conversely, if we are able to build a functional calculus of self-adjoint operators, we will able to perform a spectral decomposition of them. Actually this approach is the key for the general spectral theory.

Most ideas and tools of this theory were built in a short period by John Von Neumann and Marshall Stone approximatively in the same time when Werner Heisenberg, Max Born, Pascual Jordan, Wolfgang Pauli, Louis de Broglie, Erwin Schrödinger and Paul Dirac developped the theory of Quantum Mechanics in the years after the historical breakthrough of 1924–1925. This is not a coincidence, since the spectral theory and the functional calculus answer crucial questions of quantum mechanics. Heisenberg postulated that quantities which were measured in classical physics by *real numbers* should be replaced by *matrices*, which can be translated in the language of mathematicians by *operators*. This point of view was developped by Born, Heisenberg, Jordan and Dirac. It immediately led to the question of making sense of observable quantities which may be functions of other observable quantities, that is the functional calculus. But since Quantum Mechanics also postulated that the numbers which are observed in an experiment are eigenvalues, or more generally spectral values of the operators, a need for a spectral theory was also immediate. Of course both questions are also strongly related in Quantum Mechanics. This is the reason why many concepts in spectral theory have counterparts in physics.

Later on more abstract theories (which we will not present here) were developped starting from these ideas making sense of operators which does not act necessarily on a Hilbert space (the theory of Banach algebras and the theorem of Israel Gelfand and Mark Naimark in 1943) and the theorem of Israel Gelfand saying that a commutative C^*-algebra can identified with an algebra of continuous complex functions on a compact topological space. These ideas led rich developments, among which we find the theory of *noncommutative geometry* built by Alain Connes.

In this course we assume that the Reader is familiar with the basic notions of topology (norm, topology—open and closed sets) and of linear algebras (vector spaces).

1 Metric, Banach and Hilbert spaces

In this course we are mainly interested in *complex Hilbert spaces*, the natural framework where the spectral theory developped. We start by a few definitions.

1.1 Complete metric spaces, normed vector spaces, Banach spaces

Definition 1.1 *A metric space (X, d) is a set X endowed with a* distance *function*

$$\begin{array}{rccc} d: & X \times X & \longrightarrow & [0, +\infty) \\ & (x, y) & \longmapsto & d(x, y) \end{array}$$

which satisfies : (a) $\forall x, y, z \in X$, $d(x, z) \leq d(x, y) + d(y, z)$ and (b) $\forall x, y \in X$, $d(x, y) = 0 \Longleftrightarrow x = y$.

We recall that a sequence $(x_n)_{n\in\mathbb{N}}$ in X is a *Cauchy sequence* if

$$\forall\varepsilon > 0, \exists N \in \mathbb{N}, \text{ s.t. } \forall n, m \in \mathbb{N}, \quad n, m \geq N \Longrightarrow d(x_n, x_m) < \varepsilon.$$

We then say that (X, d) is *complete* if any Cauchy sequence with values in E converges in E.

Definition 1.2 *A complex* ***normed vector space*** *E is a complex vector space endowed with a function $N : E \longrightarrow [0, +\infty)$ which satisfies :*

(i) $\forall\lambda \in \mathbb{C}$, $\forall x \in E$, $N(\lambda x) = |\lambda|N(x)$ (N is positive homogeneous of degree 0*);*

*(ii) $\forall x, y \in E$, $N(x + y) \leq N(x) + N(y)$ (*triangular inequality*);*

(iii) $\forall x \in E$, $N(x) = 0$ iff $x = 0$.

The function N is then called a norm. We denote by (E, N) the vector space endowed with the norm N.

Note that we may define similarly a *real normed vector space* just by replacing $\mathbb{C}$ by $\mathbb{R}$ everywhere in the definition. Note also that Property (i) implies that $N(0) = 0$, hence Property (iii) could be replaced by $N(x) = 0 \Longrightarrow x = 0$.

Normed vector space are examples of metric spaces : here we just set $d(x, y) = N(x-y)$, $\forall x, y \in E$.

Recall that on a *finite dimensional* vector space E, any pair of norms N_1, N_2 are equivalent, i.e. $\exists C > 0$ s.t. $\forall x \in E$, $C^{-1}N_1(x) \leq N_2(x) \leq CN_1(x)$. Also a finite dimensional vector space is always complete. These properties are not true in general for infinite dimensional vector spaces.

Definition 1.3 *A normed vector space (E, N) is called a* ***Banach space*** *if it is complete.*

For instance, for $1 \leq p < +\infty$, the space $\ell^p(\mathbb{N}, \mathbb{C}) := \{a = (a_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}; \sum_{j=0}^{\infty} |a_j|^p < +\infty\}$ with the norm $N(a) := \|a\|_p := \left(\sum_{j=0}^{\infty} |a_j|^p\right)^{1/p}$ is a Banach space. Its subspace $\ell_{finite}(\mathbb{N}, \mathbb{C}) := \{a = (a_j)_{j\in\mathbb{N}} \in \mathbb{C}^{\mathbb{N}}; \exists J \in \mathbb{N}, \text{ s.t. } a_j = 0 \text{ if } j \geq J\}$ with the same norm is not complete. If $p \neq q$, $(\ell^p(\mathbb{N}, \mathbb{C}), \|\cdot\|_p)$ is not equivalent to $(\ell^q(\mathbb{N}, \mathbb{C}), \|\cdot\|_q)$.

Definition 1.4 (separable spaces) *A normed vector space is* ***separable*** *if it contains a countable dense subset.*

1.2 Hilbert spaces

Definition 1.5 (Real Hilbert spaces) *Let $\mathcal{H}$ be a real vector space.*

(i) A map

$$\begin{array}{rccc} \varphi : & \mathcal{H} \times \mathcal{H} & \longrightarrow & \mathbb{C} \\ & (u, v) & \longmapsto & \varphi(u, v) \end{array}$$

is a ***bilinear form*** *on $\mathcal{H}$ if it is* ***linear*** *in the first and in the second argument, i.e. $\forall\lambda, \mu \in C$, $\forall u, v, w \in \mathcal{H}$,*

$$\begin{aligned} \varphi(\lambda u + \mu v, w) &= \lambda\varphi(u, w) + \mu\varphi(v, w) \\ \varphi(u, \lambda v + \mu w) &= \lambda\varphi(u, v) + \mu\varphi(u, w). \end{aligned}$$

(ii) *A bilinear form* φ *on* $\mathcal{H}$ *is **symmetric** if* $\forall u, v \in \mathcal{H}$, $\varphi(v, u) = \varphi(u, v)$ *;*

(iii) *A symmetric bilinear form* φ *on* $\mathcal{H}$ *is **positive** if* $\forall u \in \mathcal{H}$, $\varphi(u, u) \geq 0$ *;*

(iv) *A positive symmetric bilinear form* φ *on* $\mathcal{H}$ *is **positive definite** if :*
$\forall u \in \mathcal{H}$, $\varphi(u, u) = 0 \Longrightarrow u = 0$.
We then say that $(\mathcal{H}, \varphi)$ *is **pre-Hilbertian space**. If so* $u \longmapsto \varphi(u, u)^{1/2}$ *defines a norm on* $\mathcal{H}$.

(v) *A pre-Hilbertian space* $(\mathcal{H}, \varphi)$ *is a **Hilbert space** if it is **complete**.*

This definition can be adapted to the complex case :

Definition 1.6 (Complex Hilbert spaces) *Let* $\mathcal{H}$ *be a complex vector space.*

(i) *A map*

$$\begin{array}{rccc} \varphi : & \mathcal{H} \times \mathcal{H} & \longrightarrow & \mathbb{C} \\ & (u, v) & \longmapsto & \varphi(u, v) \end{array}$$

*is a **sesquilinear form** on* $\mathcal{H}$ *if it is **antilinear** in the first argument and **linear** in the second argument, i.e.* $\forall \lambda, \mu \in C$, $\forall u, v, w \in \mathcal{H}$,

$$\begin{array}{rcl} \varphi(\lambda u + \mu v, w) & = & \overline{\lambda}\varphi(u, w) + \overline{\mu}\varphi(v, w) \\ \varphi(u, \lambda v + \mu w) & = & \lambda\varphi(u, v) + \mu\varphi(u, w). \end{array}$$

(ii) *A sesquilinear form* φ *on* $\mathcal{H}$ *is **Hermitian** if* $\forall u, v \in \mathcal{H}$, $\varphi(v, u) = \overline{\varphi(u, v)}$ *;*

(iii) *A Hermitian form* φ *on* $\mathcal{H}$ *is **positive** if* $\forall u \in \mathcal{H}$, $\varphi(u, u) \geq 0$ *;*

(iv) *A positive Hermitian form* φ *on* $\mathcal{H}$ *is **positive definite** if :*
$\forall u \in \mathcal{H}$, $\varphi(u, u) = 0 \Longrightarrow u = 0$.
We then say that $(\mathcal{H}, \varphi)$ *is **pre-Hilbertian space** and that* φ *is a* Hermitian scalar product. *If so* $u \longmapsto \varphi(u, u)^{1/2}$ *defines a norm on* $\mathcal{H}$.

(v) *A pre-Hilbertian space* $(\mathcal{H}, \varphi)$ *is a **Hilbert space** if it is **complete**.*

In the following we will often denote a Hermitian scalar product and its norm by :

$$\langle u, v\rangle = \varphi(u, v), \quad \|u\| = \langle u, u\rangle^{1/2}.$$

Remarks — **a) Caution !** Most Authors in mathematics use an opposite convention in (i) : a sesquilinear is then *linear* in its first argument and *antilinear* in its second argument. Our convention however is the same as in the book of Reed and Simon [3] and agrees with the general convention of physicists !
b) A Hermitian form satisfies the reality condition : $\forall u \in \mathcal{H}$, $\langle u, u\rangle \in \mathbb{R}$, as a straightforward consequence of (ii). The converse is true : if a sesquilinear form φ satisfies $\varphi(u, u) \in \mathbb{R}$, $\forall u \in \mathcal{H}$, then φ is Hermitian (see Lemma 10.1).

Proposition 1.1 *Let* $\mathcal{H}$ *be a **separable** Hilbert space. Then there exists a countable family* $(e_n)_{n\in\mathbb{N}}$ *of vectors such that* $\langle e_n, e_m\rangle = 1$ *if* $n = m$ *and* $= 0$ *if* $n \neq m$ *and such that* $Vec\{e_n; n \in \mathbb{N}\}$ *is dense in* $\mathcal{H}$.

If so $(e_n)_{n\in\mathbb{N}}$ *is called a* Hermitian orthogonal Hilbertian basis *of* $\mathcal{H}$. *Moreover for any* $x \in \mathcal{H}$, *the series* $\sum_{n\in\mathbb{N}}\langle e_n, x\rangle e_n$ *converges in* $\mathcal{H}$ *and its sum is equal to* x. *Lastly the Parseval identity* $\sum_{n\in\mathbb{N}} |\langle e_n, x\rangle|^2 = \|x\|^2$ *holds.*

1.3 Bounded operators between Banach spaces

Let X and Y be two Banach spaces. A linear map $T : X \longrightarrow Y$ is **bounded** if there exists a constant $C > 0$ s.t.

$$\forall x \in X, \quad \|Tx\|_Y \leq C\|x\|_X.$$

Then we set

$$\|T\| = \|T\|_{\mathcal{L}(X,Y)} = \sup_{x \in X; x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X}$$

and the preceding inequality holds with $C = \|T\|$ (the optimal constant).

The set of bounded operators from X to Y is denoted by $\mathcal{L}(X,Y)$. It is a vector space and $T \longmapsto \|T\|$ is a norm on it. The normed vector space $(\mathcal{L}(X,Y), \|\cdot\|)$ is a Banach space. Note that, since T is linear, T bounded iff $T : X \longrightarrow Y$ is continuous.

In the special case where $X = Y$, we set $\mathcal{L}(X) := \mathcal{L}(X,X)$.

An important case also is when $Y = \mathbb{C}$. Then $\mathcal{L}(X,\mathbb{C})$ is the set of continuous linear forms on X, i.e. the dual space of X : we denote it by X'.

To any $T \in \mathcal{L}(X,Y)$ we associate its adjoint operator defined by :

$$\begin{array}{rccc} T' : & Y' & \longrightarrow & X' \\ & \ell & \longmapsto & \ell \circ T \end{array}$$

By using the Hahn–Banach theorem one can then prove the following result :

Theorem 1.1 *If $T \in \mathcal{L}(X,Y)$ then T' is also bounded and moreover $\|T\|_{\mathcal{L}(X,Y)} = \|T'\|_{\mathcal{L}(Y',X')}$.*

2 Complex Hilbert spaces

In the following we assume that $(\mathcal{H}, \varphi)$ is a Hilbert space and we write $\varphi(x,y) = \langle x, y\rangle$. Then there is a natural map

$$\begin{array}{rccc} C : & \mathcal{H} & \longrightarrow & \mathcal{H}' \\ & y & \longmapsto & [x \longmapsto \langle y, x\rangle] \end{array}$$

Note that C is not linear but anti-linear, i.e.

$$C(\lambda x + \mu y) = \overline{\lambda} C(x) + \overline{\mu} C(y).$$

The Riesz theorem for Hilbert spaces states that C is one-to-one and onto [1], i.e. is a bijection. It is an anti-isomorphism.

1. In English *one-to-one* means *injective* and *onto* means *surjective*.

2.1 The Hilbertian adjoint of a bounded operator between Hilbert spaces

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two Hilbert spaces and let $C_1 : \mathcal{H}_1 \longrightarrow \mathcal{H}_1'$ and $C_2 : \mathcal{H}_2 \longrightarrow \mathcal{H}_2'$ be the corresponding Riesz anti-isomorphisms. Let $T \in \mathcal{L}(\mathcal{H}_1, \mathcal{H}_2)$ and consider its adjoint $T' \in \mathcal{L}(\mathcal{H}_2', \mathcal{H}_1')$.

Definition 1 — We define the Hilbertian adjoint of T to be the operator $T^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ s.t. the following diagramm is commutative :

$$\begin{array}{ccc} \mathcal{H}_2 & \underset{T^*}{\longrightarrow} & \mathcal{H}_1 \\ \downarrow C_2 & & \downarrow C_1 \\ \mathcal{H}_2' & \underset{T'}{\longrightarrow} & \mathcal{H}_1' \end{array}$$

This means that $C_1 \circ T^* = T' \circ C_2$ or $C_1 T^* = T' C_2$ for short. Note that since T' is bounded and $\|T'\| = \|T\|$, T^* is also bounded and $\|T^*\| = \|T\|$.

Let us analyze this definition by testing it :

$$\begin{array}{cl} & [C_1 T^* = T' C_2 \text{ in } \mathcal{L}(\mathcal{H}_2, \mathcal{H}_1')] \\ \iff & [\forall y \in \mathcal{H}_2, \quad C_1 T^* y = T' C_2 y \text{ in } \mathcal{H}_1'] \\ \iff & [\forall y \in \mathcal{H}_2, \forall x \in \mathcal{H}_1, \quad (C_1 T^* y)(x) = (T' C_2 y)(x) \text{ in } \mathbb{C}] \\ \iff & [\forall y \in \mathcal{H}_2, \forall x \in \mathcal{H}_1, \quad (C_1 T^* y)(x) = (C_2 y)(Tx) \text{ in } \mathbb{C}] \\ \iff & [\forall y \in \mathcal{H}_2, \forall x \in \mathcal{H}_1, \quad \langle T^* y, x \rangle_1 = \langle y, Tx \rangle_2 \text{ in } \mathbb{C}] \end{array}$$

Hence we arrive at the second (equivalent) definition of T^* :

Definition 2 — We define the Hilbertian adjoint of T to be the unique operator $T^* : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ s.t.

$$\forall y \in \mathcal{H}_2, \forall x \in \mathcal{H}_1, \quad \langle T^* y, x \rangle_1 = \langle y, Tx \rangle_2. \tag{1}$$

The existence and the uniqueness of T^* are garanted by the previous discussion. Note that the definition (1) is much more convenient for applications.

In the particular case where $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$ then the previous definition specializes to :

Definition 3 — The adjoint of an operator $T \in \mathcal{L}(\mathcal{H})$ is the unique operator $T^* \in \mathcal{L}(\mathcal{H})$ s.t.

$$\forall x, y \in \mathcal{H}, \quad \langle T^* y, x \rangle = \langle y, Tx \rangle.$$

Then :

Definition 4 — Let $T \in \mathcal{L}(\mathcal{H})$, then

- T is **self-adjoint** if $T^* = T$,
- T is **normal** if $TT^* = T^*T$, i.e. it commutes with its adjoint ;
- T is **unitary** if $TT^* = T^*T = 1_{\mathcal{H}}$, where $1_{\mathcal{H}}$ is the identity operator of $\mathcal{H}$, i.e. T is invertible and $T^{-1} = T^*$.

Note that any self-adjoint operator is normal, any unitary operator is normal.

Examples
a) The operator $T = -i\frac{d}{dx}$ acting on $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$. It is simply defined by $(Tf) = -i\frac{df}{dx}$. Heuristically it is self-adjoint for, if we assume that $f, g \in \mathcal{C}_c^\infty(\mathbb{R})$, then

$$\begin{aligned}\langle f, Tg\rangle_{L^2} &= \int_{\mathbb{R}} \overline{f}\left(-i\frac{dg}{dx}\right) dx = -i\int_{\mathbb{R}} \left[\frac{d}{dx}\left(\overline{f}g\right) - \frac{d\overline{f}}{dx}g\right] dx \\ &= 0 + \int_{\mathbb{R}} \overline{-i\frac{df}{dx}} g dx = \langle Tf, g\rangle_{L^2}.\end{aligned}$$

However there are serious difficulties with this operator : it is not true in general that, if $f \in L^2(\mathbb{R}, \mathbb{C})$ then its derivative in the sense of distribution $-i\frac{df}{dx}$ belongs to $L^2(\mathbb{R}, \mathbb{C})$, so that $-i\frac{d}{dx}$ is not defined! Hence the previous computation does not make sense in general. A correct treatment requires to define the concept of an *unbounded operator* T on a *domain* $D(T)$ which is a subspace fo $\mathcal{H}$. Also the understanding of the generalization of self-adjoint operators in this context is a delicate task that we postpone to the end of this course. Once these difficulties have been overcomed, then $-i\frac{d}{dx}$ will an unbounded self-adjoint operator.
b) For any $t \in \mathbb{R}$ we define the operator $U_t : f \longmapsto U_t f$ on $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$ by $(U_t f)(x) = f(x - t)$. Using the change of variable $y = x - t$, we see that $\forall f, g \in L^2(\mathbb{R}, \mathbb{C})$,

$$\langle f, U_t g\rangle_{L^2} = \int_{\mathbb{R}} \overline{f(x)} g(x-t) dx = \int_{\mathbb{R}} \overline{f(y+t)} g(t) dy = \langle U_{-t} f, g\rangle_{L^2}.$$

Hence $U_t^* = U_{-t}$. One also checks easily that U_t is invertible and $U_t^{-1} = U_{-t}$. Hence $U_t^{-1} = U_t^*$, i.e. U_t is unitary.

3 The spectrum of a bounded operator

In this section X is a complex Banach space. Let $T \in \mathcal{L}(X)$. The spectrum of T can be seen as the generalization in infinite dimension the notion of eigenvalues in finite dimension. We start by defining the complimentary set.
Definition 1 — Let $T \in \mathcal{L}(X)$. The *resolvent set* of T is the set

$$\rho(T) := \{\lambda \in \mathbb{C};\ \lambda - T \text{ is invertible }\}.$$

The *resolvent* is the map

$$\begin{array}{ccc}\rho(T) & \longrightarrow & \mathcal{L}(X) \\ \lambda & \longmapsto & R_\lambda(T) = (\lambda - T)^{-1}\end{array}$$

By $\lambda - T$ we mean $\lambda 1_X - T$, where 1_X is the identity operator of X. Note that, as a consequence of the closed graph theorem, if T is bounded and invertible, then its inverse is automatically bounded.

Definition 2 — Let $T \in \mathcal{L}(X)$. The *spectrum* of T is

$$Sp(T) := \mathbb{C} \setminus \rho(T) = \{\lambda \in \mathbb{C};\ \lambda - T \text{ is not invertible } \}.$$

There may be several reasons why the operator $\lambda - T$ be not invertible :

(i) $\mathrm{Ker}(\lambda - T) \neq \{0\}$, i.e. $\exists v \in X$ s.t. $v \neq 0$ and $Tv = \lambda v$, i.e. λ is an **eigenvalue**. The subset of $Sp(T)$ defined by

$$Sp_p(T) := \{\lambda \in \mathbb{C};\ \mathrm{Ker}(\lambda - T) \neq \{0\}\}$$

is composed of eigenvalues and is called the **punctual spectrum** of T ;

(ii) $\mathrm{Ker}(\lambda - T) = \{0\}$ but $\mathrm{Im}(\lambda - T) \neq X$. Then we may again consider two subcases :

(a) $\mathrm{Ker}(\lambda - T) = \{0\}$ and $\mathrm{Im}(\lambda - T)$ is *not dense* in X, we then say that λ is a **residual spectral value** of T. The subset

$$Sp_r(T) := \{\lambda \in \mathbb{C};\ \mathrm{Ker}(\lambda - T) = \{0\}, \overline{\mathrm{Im}(\lambda - T)} \neq X\}$$

is called the **residual spectrum** of T ;

(b) if none of the previous cases occur, i.e. if $\mathrm{Ker}(\lambda - T) = \{0\}$, $\mathrm{Im}(\lambda - T) \neq X$ but $\mathrm{Im}(\lambda - T)$ is **dense** in X, then we say that λ is a **continuous spectral value** of T. The subset

$$Sp_c(T) := \{\lambda \in \mathbb{C};\ \mathrm{Ker}(\lambda - T) = \{0\}, \mathrm{Im}(\lambda - T) \neq \overline{\mathrm{Im}(\lambda - T)} = X\}$$

is called the **continuous spectrum** of T.

All that gives us a complicated partition $Sp(T) = Sp_p(T) \cup Sp_r(T) \cup Sp_c(T)$. Fortunately when we will specialize ourself to *self-adjoint* operators later on, we will prove that the residual spectrum of these operators is empty, which will simplify the situation.

The following results are quite useful, although elementary. They also help to understand the continuous spectrum.

Lemma 3.1 *Let X be a Banach space and Y a normed vector space. Let $T \in \mathcal{L}(X, Y)$. Assume that there exists a constant $c > 0$ s.t.*

$$\forall x \in X, \quad \|Tx\|_Y \geq c\|x\|_X. \tag{2}$$

Then ImT is closed in Y.

Corollary 3.1 *Let X be a Banach space and Y a normed vector space. Let $T \in \mathcal{L}(X, Y)$. Then T is invertible iff the two following conditions are satisfied.*

*(i) $\exists c > 0$, $\forall x \in X$, $\|Tx\|_Y \geq c\|x\|_X$ (T is '*strongly one-to-one*');*

*(ii) ImT is dense in Y (T is '*weakly onto*').*

Proof of Lemma 3.1 — Let us assume (2). Let $(y_n)_n$ be a sequence with values in $\mathrm{Im}T$ and assume that this sequence converges to some $y \in Y$. We want to show that $y \in \mathrm{Im}T$. For any $n \in \mathbb{N}$, $y_n \in \mathrm{Im}T$ so there exists an $x_n \in X$ s.t. $Tx_n = y_n$. Thus, by (2), $\forall n, m \in \mathbb{N}$,

$$\|y_n - y_m\|_Y = \|T(x_n - x_m)\|_Y \geq c\|x_n - x_m\|_X.$$

But since it is convergent, $(y_n)_n$ is a Cauchy sequence in Y and the previous inequality implies that $(x_n)_n$ is also a Cauchy sequence in X. Since X is complete, this sequence converges to some $x \in X$. Now since T is continuous,

$$y = \lim_{n\to+\infty} y_n = \lim_{n\to+\infty} Tx_x = T \lim_{n\to+\infty} x_n = Tx$$

and so $y \in \mathrm{Im}T$. □

Proof of Corollary 3.1 — It is easy to prove that, if T is invertible, then (i) and (ii) are satisfied. Indeed by writing that T^{-1} is continuous, i.e. $\|T^{-1}y\|_X \leq \|T^{-1}\|\|y\|_Y$, we deduce easily (i) with $c = 1/\|T^{-1}\|$. And (ii) follows obviously.

Conversely if (i) holds then T is clearly one-to-one and, by Lemma 3.1, $\mathrm{Im}T$ is closed. Hence (ii) implies that $\mathrm{Im}T = \overline{\mathrm{Im}T} = Y$, i.e. that T is onto. □

As a consequence, let $T \in \mathcal{L}(X)$ and assume that $\lambda \in Sp_c(T)$. Then $\mathrm{Im}(\lambda - T)$ is dense in X, i.e. $\lambda - T$ satisfies Property (ii) of the previous lemma, however $\lambda - T$ is not invertible. So the only possibility is that $\lambda - T$ *does not* satisfy (i). This means that

$$\forall c > 0, \exists x \in X, \quad \|(\lambda - T)x\|_X < c\|x\|_X.$$

This property allows us to construct a sequence $(x_n)_n$ of X s.t. $\forall n \in \mathbb{N}$, $\|(\lambda - T)x_n\|_X < (1/n)\|x_n\|_X$. Hence the vectors x_n behave asymptotically as eigenvectors. But they are not eigenvectors, since the hypothesis $\lambda \in Sp_c(T)$ *excludes* that λ be an eigenvalue of T.

4 The spectrum of a bounded operator is compact and non-empty

Let $T \in \mathcal{L}(T)$. We start with the following result, which implies that $Sp(T)$ is *closed*.

Theorem 4.1 *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then*

(i) $\rho(T)$ is an open subset of $\mathbb{C}$;

(ii) $\lambda \longmapsto R_\lambda(T)$ is a holomorphic function from $\rho(T)$ to $\mathcal{L}(X)$;

(iii) $\forall \lambda, \mu \in \rho(T)$,

$$R_\lambda(T)R_\mu(T) = R_\mu(T)R_\lambda(T), \tag{3}$$

$$R_\lambda(T) - R_\mu(T) = -(\lambda - \mu)R_\lambda(T)R_\mu(T). \tag{4}$$

Proof — We will establish (i) and (ii) simultaneously by proving that, for any $\lambda_0 \in \rho(T)$, there exists a disk in $\mathbb{C}$ centered on λ_0 s.t. $\lambda \longmapsto \lambda - T$ admits an inverse for λ in this disk, which is complex analytic. Using the fact that $\lambda_0 - T$ is invertible, write

$$\begin{aligned}\lambda - T &= (\lambda - \lambda_0) + (\lambda_0 - T)\\ &= [(\lambda - \lambda_0)(\lambda_0 - T)^{-1} + 1]\,(\lambda_0 - T)\\ &= [1 - (\lambda_0 - \lambda)R_{\lambda_0}(T)]\,(\lambda_0 - T).\end{aligned}$$

Hence $\lambda - T$ is invertible iff $1 - (\lambda_0 - \lambda)R_{\lambda_0}(T)$ is invertible. This is possible if $(\lambda_0 - \lambda)R_{\lambda_0}(T)$ is sufficiently small, i.e. more precisely if

$$\|(\lambda_0 - \lambda)R_{\lambda_0}(T)\| < 1.$$

For then the series

$$\sum_{n=0}^{\infty} [(\lambda_0 - \lambda)R_{\lambda_0}(T)]^n$$

converges and its sum is equal to $[1 - (\lambda_0 - \lambda)R_{\lambda_0}(T)]^{-1}$. Hence, if $|\lambda - \lambda_0| < \|R_{\lambda_0}(T)\|^{-1}$, $\lambda - T$ is invertible and its inverse is

$$R_\lambda(T) = \sum_{n=0}^{\infty} R_{\lambda_0}(T)^{n+1}(\lambda_0 - \lambda)^n.$$

The proof of (iii) is left to the Reader. □

Corollary 4.1 *The spectrum of a bounded operator is* ***closed****.*

Lemma 4.1 *Let X be a Banach space and $T \in \mathcal{L}(X)$. Then*

$$\mathbb{C} \setminus \overline{B(0, \|T\|)} \subset \rho(T).$$

and $\forall \lambda \in \mathbb{C}$ s.t. $|\lambda| > \|T\|$,

$$R_\lambda(T) = \sum_{n=0}^{\infty} T^n \lambda^{-n-1}. \tag{5}$$

Proof — If $|\lambda| > \|T\|$, then $\|\lambda^{-1}T\| < 1$ and the series $\sum_{n=0}^{\infty}(\lambda^{-1}T)^n$ converges in $\mathcal{L}(X)$. Its sum is equal to

$$\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} = \left(1 - \lambda^{-1}T\right)^{-1} = \lambda(\lambda - T)^{-1}.$$

Hence $\lambda - T$ is invertible and its inverse is

$$R_\lambda(T) = (\lambda - T)^{-1} = \lambda^{-1}\sum_{n=0}^{\infty} \frac{T^n}{\lambda^n} = \sum_{n=0}^{\infty} \frac{T^n}{\lambda^{n+1}}.$$

Remark — By setting $z := \lambda^{-1}$ and defining $f_z(T) := R_{1/z}(T) = R_\lambda(T)$, we can translate the previous result by saying that the map $z \longmapsto f_z(T)$ is complex analytic on

$B(0, 1/\|T\|) \setminus \{0\}$ and is equal on this domain to $f_z(T) = \sum_{n=0}^{\infty} T^n z^{n+1}$. In particular we see that this map admits a holomorphic extension to $B(0, 1/\|T\|)$ by setting $f_0(T) = 0$. In other words $\lambda \longmapsto R_\lambda(T)$ can be extended holomorphically to $\left[\mathbb{C} \setminus \overline{B(0, \|T\|)}\right] \cup \{\infty\}$ and hence to $[\mathbb{C} \setminus Sp(T)] \cup \{\infty\} \simeq \mathbb{C}P \setminus Sp(T)$ by setting $R_\infty(T) = 0$. □

Corollary 4.2 *The spectrum of a bounded operator T is **bounded** and moreover*

$$Sp(T) \subset \overline{B(0, \|T\|)}.$$

To summarize : T is **compact**. It remains to answer the most stupid question...

Lemma 4.2 *Let $T \in \mathcal{L}(X)$, then its spectrum is non empty.*

Remark — This result is the analogue of d'Alembert's theorem on the existence of roots of a polynomial over $\mathbb{C}$. Indeed if X would be finite dimensional, we would simply argue that the characteristic polynomial $P_T(\lambda) := \det(\lambda - T)$ has at least one root in $\mathbb{C}$ (which is the statement of d'Alembert's theorem). In infinite dimension the determinant of $\lambda - T$ does not make sense (in general and without working hard), but the following proof follows the same lines as d'Alembert's theorem.

Proof of Lemma 4.2 — Argue by contradiction and assume that $Sp(T) = \emptyset$. Then $\rho(T) = \mathbb{C}$, i.e. $\lambda \longrightarrow R_\lambda(T)$ is an entire function (a holomorphic function defined on $\mathbb{C}$). But we have seen previously that $R_\lambda(T)$ tends to 0 when $|\lambda|$ tends to $+\infty$. Hence by applying Liouville's theorem, we deduce that $R_\lambda(T) \equiv 0$, a contradiction. □

Conclusion — The spectrum of any $T \in \mathcal{L}(X)$ is non empty and compact. Hence we may define

$$r(T) := \sup\{|\lambda|;\ \lambda \in Sp(T)\}, \quad \text{the } \textbf{spectral radius} \text{ of } T$$

and this supremum is achieved by some $\lambda \in Sp(T)$. Moreover we have

$$r(T) \leq \|T\|. \tag{6}$$

Note that in general $r(T) < \|T\|$. This occurs even if X is finite dimensional : for instance one may choose any *nilpotent* operator T acting on a finite dimension vector space. Then all eigenvalues of T are zero and hence $r(T) = 0$, but $\|T\| \neq 0$ in general.

4.1 The study of an example

We consider the *shift operator* $T \in \mathcal{L}(\ell^1)$ defined by :

$$\forall x = (x_1, x_2, \cdots) \in \ell^1, \quad T(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$$

and its adjoint [2] $T' \in \mathcal{L}(\ell^\infty)$, which satisfies :

$$\forall \alpha = (\alpha_1, \alpha_2, \cdots) \in \ell^\infty, \quad T'(\alpha_1, \alpha_2, \cdots) = (0, \alpha_1, \alpha_2, \cdots).$$

Note that $||T^n|| = ||(T')^n|| = 1$, $\forall n \in \mathbb{N}^*$, so that by Theorem 5.2 we have :

$$r(T) = \lim_{n\to\infty} ||T^n||^{1/n} = r(T') = \lim_{n\to\infty} ||(T')^n||^{1/n} = 1.$$

We shall prove the following properties :

spectrum	punctual spectrum	continuous spectrum	residual spectrum
$Sp(T) = \overline{B(0,1)}$	$Sp_p(T) = B(0,1)$	$Sp_c(T) = \partial B(0,1)$	$Sp_r(T) = \emptyset$
$Sp(T') = \overline{B(0,1)}$	$Sp_p(T') = \emptyset$	$Sp_c(T') = \emptyset$	$Sp_r(T') = \overline{B(0,1)}$

and, for that purpose, proceed by steps.

a) $Sp(T) \subset \overline{B(0,1)}$ and $Sp(T') \subset \overline{B(0,1)}$: this is a consequence of the fact that $r(T) = r(T') = 1$.
b) $Sp_p(T) \supset B(0,1)$: we need to show that, for any $\lambda \in B(0,1)$, $\lambda - T$ has a non vanishing kernel (i.e. a non trivial eigenspace for T with the eigenvalue λ). Note that it is straightforward for $\lambda = 0$, since we easily remark that $(1, 0, 0, \cdots)$ belongs to the kernel of T. The eigenvectors of T for $\lambda \neq 0$ are kind of perturbations of $(1, 0, 0, \cdots)$. Set

$$x_{[\lambda]} := (1, \lambda, \lambda^2, \lambda^3, \cdots).$$

Then $x_{[\lambda]} \in \ell^1$ ssi $|\lambda| < 1$. Moreover

$$T(x_{[\lambda]}) = (\lambda, \lambda^2, \lambda^3, \cdots) = \lambda(1, \lambda, \lambda^2, \cdots) = \lambda x_{[\lambda]}.$$

Thus $Sp_p(T) \supset B(0,1)$.
Corollary of a) and b) : we have

$$B(0,1) \subset Sp_p(T) \subset Sp(T) \subset \overline{B(0,1)},$$

but since we know that $Sp(T)$ is closed this implies :

$$Sp(T) = \overline{B(0,1)} = Sp(T'),$$

where we used Phillips' theorem for $Sp(T')$.
c) $Sp_p(T) = B(0,1)$: by the previous results it suffices to show that $\partial B(0,1) \cap Sp_p(T) = \emptyset$, i.e. that, for all λ s.t. $|\lambda| = 1$, $\lambda - T$ is one-to-one. Argue by contradiction and assume that there exists $\lambda \in \partial B(0,1)$ s.t. $\ker(\lambda - T) \neq \{0\}$. Let $x = (x_1, x_2, x_3, \cdots) \in \ker(\lambda - T)$

2. Recall that the dual space of ℓ^1 is ℓ^∞. However the dual space of ℓ^∞ is not ℓ^1 but a space which contains strictly ℓ^1 as a closed subspace. It means that the map $\ell^1 \longrightarrow (\ell^\infty)'$ which, to any sequence $x \in \ell^1$, associates $ev_x : \alpha \longmapsto \alpha(x)$ is an embedding which is not onto.

which is different from 0, then $(\lambda - T)(x) = 0$ holds iff the following system of equations holds

$$x_{n+1} = \lambda x_n, \quad \forall n \in \mathbb{N}^*.$$

We deduce by a straightforward recursion that :

$$x = x_1(1, \lambda, \lambda^2, \lambda^3, \cdots).$$

But we remark that, because of $|\lambda| = 1$, x cannot be in ℓ^1 unless $x_1 = 0$ — a contradiction.

d) $Sp_p(T') = \emptyset$: in order to show it, argue by contradiction and assume that there exists $\alpha = (\alpha_1, \alpha_2, \cdots) \in \ell^\infty$ and $\lambda \in \mathbb{C}$ s.t. $(\lambda - T')(\alpha) = 0$, which means that

$$\begin{array}{rcl} \lambda\alpha_1 & = & 0 \\ \lambda\alpha_2 & = & \alpha_1 \\ \lambda\alpha_3 & = & \alpha_2 \\ & \vdots & \end{array}$$

We then deduce that $(\alpha_1, \alpha_2, \cdots) = 0$ (argue by recursion : one needs to inspect separately the cases $\lambda = 0$ and $\lambda \neq 0$, but both cases are easy to treat). This leads to a contradiction.

e) $Sp_r(T') \supset B(0,1)$: recall that

$$Sp_r(T') = \{\lambda \in \mathbb{C} |\ \overline{\mathrm{Im}(\lambda - T')} \neq \ell^\infty\} \setminus Sp_p(T').$$

But since as seen previously $Sp_p(T') = \emptyset$, we deduce that $Sp_r(T') = \{\lambda \in \mathbb{C} |\ \overline{\mathrm{Im}(\lambda - T')} \neq \ell^\infty\}$. For all $\lambda \in B(0,1)$ we use the sequence $x_{[\lambda]} = (1, \lambda, \lambda^2, \lambda^3, \cdots) \in \ell^1$ as in b) and we set

$$f_{[\lambda]} := ev_{x_{[\lambda]}} : \alpha \longmapsto \alpha\left(x_{[\lambda]}\right).$$

Then, $\forall \alpha \in \ell^\infty$,

$$f_{[\lambda]}\left[(\lambda - T')(\alpha)\right] = \left[(\lambda - T')(\alpha)\right](x_{[\lambda]}) = \alpha\left[(\lambda - T)(x_{[\lambda]})\right] = \alpha(0) = 0,$$

i.e. $(\lambda - T')(\alpha) \in \mathrm{Ker} f_{[\lambda]}$. Hence $\mathrm{Im}(\lambda - T') \subset \mathrm{Ker} f_{[\lambda]}$, which implies that the closure of $\mathrm{Im}(\lambda - T')$ cannot be equal to ℓ^∞. Thus λ is a residual value.

f) In fact $Sp_r(T') = \overline{B(0,1)}$: because of the previous observations it suffices to show that any $\lambda \in \mathbb{C}$ s.t. $|\lambda| = 1$ is also in $Sp_r(T')$. Let λ be such a value. We start by computing a formal inverse of $\lambda - T'$: if $a \in \ell^\infty$ and if b is another sequence with values in $\mathbb{C}$, the equation $(\lambda - T')(b) = a$ reads

$$\left\{ \begin{array}{rcl} a_1 & = & \lambda b_1 \\ a_2 & = & \lambda b_2 - b_1 \\ & \vdots & \\ a_n & = & \lambda b_n - b_{n-1} \end{array} \right. \iff \left\{ \begin{array}{rcl} b_1 & = & \overline{\lambda} a_1 \\ b_2 & = & \overline{\lambda}(a_2 + b_1) \\ & \vdots & \\ b_n & = & \overline{\lambda}(a_n + b_{n-1}) \end{array} \right.$$

Hence this equation has the solution $b_n = \overline{\lambda} a_n + \overline{\lambda}^2 a_{n-1} + \cdots + \overline{\lambda}^n a_1$. We can already see that $\lambda - T'$ is not onto because, for $a = a_{[\overline{\lambda}]} := (1, \overline{\lambda}, \overline{\lambda}^2, \cdots) \in \ell^\infty$, the solution is $b_n = n\overline{\lambda}^n$ and this sequence cannot be in ℓ^∞ : thus $a_{[\overline{\lambda}]} \notin \mathrm{Im}(\lambda - T')$.

But we actually need a stronger result, i.e. that $\mathrm{Im}(\lambda - T')$ is not dense in ℓ^∞. For that purpose we show that $B(a_{[\overline{\lambda}]}, 1/2) \cap \mathrm{Im}(\lambda - T') = \emptyset$ in ℓ^∞. Let $a \in B(a_{[\overline{\lambda}]}, 1/2)$, we can write $a = a_{[\overline{\lambda}]} + \beta$, where $||\beta||_{\ell^\infty} < 1/2$. The solution in the space of complex valued series of the equation

$$(\lambda - T')b = a = a_{[\overline{\lambda}]} + \beta$$

is $b = (b_1, b_2, b_3, \cdots)$, with :

$$b_n = n\overline{\lambda}^n + \sum_{j=1}^{n} \overline{\lambda}^{n-j} \beta_j,$$

which implies that :

$$|b_n - n\overline{\lambda}^n| = \left| \sum_{j=1}^{n} \overline{\lambda}^{n-j} \beta_j \right| < \frac{n}{2}.$$

By using the triangular inequality $n = |n\overline{\lambda}^n| \leq |b_n - n\overline{\lambda}^n| + |b_n|$, we deduce that

$$|b_n| \geq n - |b_n - n\overline{\lambda}^n| > n - \frac{n}{2} = \frac{n}{2}.$$

Thus b is not in ℓ^∞.

g) Let us show that $Sp_r(T) = \emptyset$. We just need to show that, if $|\lambda| = 1$, then $\lambda \notin Sp_r(T)$. Assume the contrary : then there exists $\lambda \in \mathbb{C}$ s.t. $|\lambda| = 1$ and $\mathrm{Im}(\lambda - T)$ is not dense in ℓ^1. Then conclusion (ii) from Proposition 6.3, $\lambda \in Sp_p(T')$. However this cannot happen for we have seen at d) that $Sp_p(T') = \emptyset$.

h) The continuous spectra : according to the previous observations

$$Sp_c(T) = Sp(T) \setminus (Sp_p(T) \cup Sp_r(T)) = \overline{B(0,1)} \setminus (B(0,1) \cup \emptyset) = \partial B(0,1),$$

and

$$Sp_c(T') = Sp(T') \setminus (Sp_p(T') \cup Sp_r(T')) = \overline{B(0,1)} \setminus \left(\emptyset \cup \overline{B(0,1)}\right) = \emptyset.$$

5 An expression for the spectral radius

5.1 The spectral radius in a Banach space

We want to prove that for any bounded operator T acting on a Banach space $r(T) = \lim_{n\to\infty} \|T^n\|^{1/n}$. We will first prove that $r(T) = \overline{\lim}_{n\to\infty} \|T^n\|^{1/n}$ and then prove that the sequence $\|T^n\|^{1/n}$ actually converges.

We first translate the definition of $r(T)$ in terms of the resolvent set $\rho(T)$:

$$r(T) = \sup\{|\lambda|;\ \lambda \in Sp(T)\} = \inf\{r;\ [\lambda \longmapsto R_\lambda(T)] \text{ is holomorphic on } [\mathbb{C} \setminus \overline{B(0,r)}]\}.$$

Hence by setting $z = 1/\lambda$, $k = 1/r$ and $f_z(T) = R_\lambda(T)$, with $f_0(T) = 0$,

$$\frac{1}{r(T)} = \sup\{k;\ [z \longmapsto f_z(T)] \text{ is holomorphic on } B(0,k)]\}.$$

In other words $r(T)^{-1}$ is the radius of convergence of the analytic function $f_z(T)$.

Lemma 5.1 *Let $\sum_{n=0}^{\infty} a_n z^n$ be a series in the complex variable z with values in a complex Banach space X and denote by R its radius of convergence. Assume that $R > 0$. Then*

$$R = \underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}. \tag{7}$$

Proof — For $z \in B(0, R)$, we write $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$.
(a) We first prove that $R \leq \underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}$. For that it suffices to prove that :

$$\forall r > 0, \quad r < R, \quad \Longrightarrow \quad r \leq \underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}.$$

Indeed if $r < R$, then $z \longmapsto \phi(z)$ is analytic on $\overline{B(0,r)}$, hence $\forall z \in B(0, r)$,

$$\begin{aligned} \phi(z) &= \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{\phi(v)}{v - z} dv = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{\phi(v)}{v} \frac{dv}{1 - z/v} \\ &= \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{\phi(v)}{v} \sum_{n=0}^{\infty} \frac{z^n}{v^n} dv = \sum_{n=0}^{\infty} \frac{z^n}{2\pi i} \int_{\partial B(0,r)} \frac{\phi(v)}{v^{n+1}} dv. \end{aligned}$$

Hence

$$a_n = \frac{1}{2\pi i} \int_{\partial B(0,r)} \frac{\phi(v)}{v^{n+1}} dv.$$

Thus, by denoting $C := \sup_{v \in \partial B(0,r)} \|\phi(v)\| < +\infty$,

$$\|a_n\| \leq \frac{1}{2\pi} \int_0^{2\pi} \frac{C}{r^n} d\theta = \frac{C}{r^n},$$

Hence $\|a_n\|^{1/n} \leq C^{1/n} \frac{1}{r}$ or $r \leq C^{1/n} \|a_n\|^{-1/n}$ By letting $n \to +\infty$ we deduce that $r \leq \underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}$.
(b) Let us prove conversely that $\underline{\lim}_{n\to\infty} \|a_n\|^{-1/n} \leq R$. Let $z \in \mathbb{C}$ s.t. $|z| < \underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}$. Then

$$\overline{\lim}_{n\to\infty} \|a_n z^n\|^{1/n} = |z| \overline{\lim}_{n\to\infty} \|a_n\|^{1/n} = \frac{|z|}{\underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}} < 1.$$

Hence $\sum_{n=0}^{\infty} a_n z^n$ converges by the Cauchy criterion. Thus $|z| \leq R$. We have proved that $|z| < \underline{\lim}_{n\to\infty} \|a_n\|^{-1/n}$ implies that $|z| \leq R$, hence $\underline{\lim}_{n\to\infty} \|a_n\|^{-1/n} \leq R$. □

We are now able to prove the :

Theorem 5.1 *Let X be a complex Banach space and $T \in \mathcal{L}(X)$, then*

$$r(T) = \overline{\lim}_{n\to\infty} \|T^n\|^{1/n}. \tag{8}$$

Proof — We apply Lemma 5.1 with $\phi(z) = f_z(T)$ and $R = 1/r(T)$. But because of (5) we know that, on $\mathbb{C} \setminus \overline{B(0, \|T\|)}$, $f_z(T) = \sum_{n=0}^{\infty} z^{n+1} T^n$, thus $a_n = T^{n-1}$. Hence (7) gives us

$$\frac{1}{r(T)} = \underline{\lim}_{n\to\infty} \|T^{n-1}\|^{-1/n} = \underline{\lim}_{n\to\infty} \|T^n\|^{-1/n}.$$

So (8) follows. □
We next prove that the superior limit in (8) is actually a limit.

Theorem 5.2 *Let E be a complex Banach space and $T \in \mathcal{L}(E)$. Then the sequence $\left(||T^n||^{1/n}\right)_n$ converges and its limit is equal to $r(T)$.*

Proof — Because of Theorem 5.1 we just need to prove that the sequence $\left(||T^n||^{1/n}\right)_n$ converges. We start from the relation $T^{n+m} = T^n T^m$, which holds $\forall n, m \in \mathbb{N}$. It implies :

$$||T^{n+m}|| \leq ||T^n|| \, ||T^m||.$$

Hence by setting $a_n := \log||T^n||$, we deduce the following inequality

$$a_{n+m} \leq a_n + a_m$$

(sub-additivity). We will show that $\lim_{n\to\infty} \frac{a_n}{n}$ exists and is equal to $\beta := \inf_{m\in\mathbb{N}^*} \frac{a_m}{m} \in [-\infty, +\infty)$ (hence the limit of $||T^n||^{1/n}$ exists). Assume first that $\beta > -\infty$. Then

$$\forall \varepsilon > 0, \exists m \in \mathbb{N}^*, \quad \beta \leq \frac{a_m}{m} < \beta + \varepsilon.$$

Fix ε and hence $m \in \mathbb{N}^*$ s.t. the preceding inequality holds. For all $n > m$, let us write the Euclidean division of n by m :

$$\exists q \in \mathbb{N}, \exists r \in \mathbb{N} \quad \text{s.t. } 0 \leq r \leq m-1 \text{ and } n = qm + r.$$

Then

$$\frac{a_n}{n} = \frac{a_{qm+r}}{qm+r} \leq \frac{qa_m + a_r}{qm+r} \leq \frac{qa_m}{qm} + \frac{a_r}{n} = \frac{a_m}{m} + \frac{a_r}{n} < \beta + \varepsilon + \frac{a_r}{n}.$$

Let $C := \sup_{0\leq r\leq m-1} a_r < +\infty$. Then (still because ε and C are fixed), $\limsup_{n\to\infty} \frac{a_r}{n} \leq \lim_{n\to\infty} \frac{C}{n} = 0$. Thus

$$\limsup_{n\to\infty} \frac{a_n}{n} \leq \beta + \varepsilon.$$

Since ε is arbitrary we have in fact $\limsup_{n\to\infty} \frac{a_n}{n} \leq \beta$. But

$$\beta = \inf_{m\in\mathbb{N}^*} \frac{a_m}{m} \leq \liminf_{m\to\infty} \frac{a_m}{m} \leq \limsup_{n\to\infty} \frac{a_n}{n} \leq \beta,$$

which implies that all these quantities are equal and thus that $\frac{a_n}{n}$ converges to β.

In the case where the sequence $\frac{a_m}{m}$ is not bounded from below, i.e. if $\beta = -\infty$, then

$$\forall A > 0, \exists m \in \mathbb{N}^*, \quad \frac{a_m}{m} < -A$$

and, by the same reasoning as before, we get that, for $n > m$,

$$\frac{a_n}{n} \leq -A + \frac{a_r}{n}, \quad \text{where } 0 \leq r \leq m-1$$

and thus that $\limsup_{n\to\infty} \frac{a_n}{n} \leq -A$. Since A is arbitrary this implies that $\lim_{n\to\infty} \frac{a_n}{n} = -\infty = \beta$.

In whatever case we have showed that

$$\lim_{n\to\infty} ||T^n||^{1/n} = e^\beta \quad (\text{with } e^{-\infty} = 0).$$

□

5.2 Application : the spectral radius of a self-adjoint operator

We first show the following result.

Lemma 5.2 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ a bounded* ***self-adjoint*** *operator. Then*

$$||A|| = \sup_{||x||=1} |\langle Ax, x\rangle|, \tag{9}$$

Proof — Denote temporarily $[A] := \sup_{||x||=1} |\langle Ax, x\rangle|$. On the one hand the inequality $[A] \leq ||A||$ is a straightforward consequence of the Cauchy–Schwarz inequality $|\langle Ax, x\rangle| \leq ||Ax||\ ||x||$. On the other hand the reverse inequality $[A] \geq ||A||$ requires more work. For that purpose we first show the identity

$$\forall x, y \in \mathcal{H}, \quad 4\mathrm{Re}\langle x, Ay\rangle = \langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle. \tag{10}$$

Let $x, y \in \mathcal{H}$, by substituting

$$x = \frac{x+y}{2} + \frac{x-y}{2} \quad \text{and} \quad Ay = \frac{A(x+y)}{2} - \frac{A(x-y)}{2}$$

in $\langle x, Ay\rangle$ we get :

$$\begin{array}{rcl} 4\langle x, Ay\rangle & = & \langle x+y, A(x+y)\rangle + \langle x-y, A(x+y)\rangle \\ & & -\langle x+y, A(x-y)\rangle - \langle x-y, A(x-y)\rangle \\ & = & R + I, \end{array}$$

where

$$R := \langle x+y, A(x+y)\rangle - \langle x-y, A(x-y)\rangle$$

and

$$I := \langle x-y, A(x+y)\rangle - \langle x+y, A(x-y)\rangle.$$

We remark that

$$\langle x+y, A(x-y)\rangle = \overline{\langle A(x-y), x+y\rangle} = \overline{\langle A^*(x-y), x+y\rangle} = \overline{\langle x-y, A(x+y)\rangle}.$$

Thus we can write

$$I = \langle x-y, A(x+y)\rangle - \overline{\langle x-y, A(x+y)\rangle}.$$

In particular we see that R is real and I is imaginary. Hence $4\mathrm{Re}\langle x, Ay\rangle = R$ which gives us (10). Now we use this identity (and we use the definition of $[A]$) :

$$4\mathrm{Re}\langle x, Ay\rangle \leq [A]||x+y||^2 + [A]||x-y||^2 = 2[A](||x||^2 + ||y||^2).$$

This implies

$$\sup_{||x||,||y||\leq 1} 4\mathrm{Re}\langle x, Ay\rangle \leq \sup_{||x||,||y||\leq 1} 2[A](||x||^2 + ||y||^2) = 4[A].$$

But by Riesz' theorem, $||A|| = \sup_{||x||,||y||\leq 1} \mathrm{Re}\langle x, Ay\rangle$. Hence $||A|| \leq [A]$. □

Corollary 5.1 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then $r(A) = ||A||$.*

Proof — Let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. By applying (9) we obtain that

$$||A^2|| = ||A^*A|| = \sup_{||x||=1} |\langle A^*Ax, x\rangle| = \sup_{||x||=1} |\langle Ax, Ax\rangle| = \sup_{||x||=1} ||Ax||^2 = ||A||^2.$$

Since any integer power of A is self-adjoint (check it !), this result is also true for A^4 :

$$||A^4|| = ||(A^2)^2|| = ||A^2||^2 = (||A||^2)^2 = ||A||^4$$

and by recursion :

$$\forall n \in \mathbb{N}, \quad ||A^{2^n}|| = ||A||^{2^n}.$$

Thus by using Theorem 5.2

$$r(A) = \lim_{n\to\infty} ||A^n||^{1/n} = \lim_{n\to\infty} ||A^{2^n}||^{2^{-n}} = ||A||.$$

□

6 The spectrum of an operator and of its adjoint

6.1 Generalities

We start with a general result without giving its proof. We will instead show one corollary of it.

Theorem 6.1 (Phillips' theorem) *Let E be a complex Banach, $T \in \mathcal{L}(E)$ and $T' \in \mathcal{L}(E')$ be the adjoint of T, then*

$$Sp(T') = Sp(T) \quad \text{and} \quad \forall \lambda \in \rho(T), \quad R_\lambda(T') = R_\lambda(T)'.$$

In the following **we will denote by λ^* the complex conjugate of a complex number** $\lambda \in \mathbb{C}$, in order to avoid confusion with the topological closure of sets. In the particular case where $E = \mathcal{H}$ is a complex Hilbert space Phillips' Theorem has the following straightforward consequence :

Proposition 6.1 *Let $\mathcal{H}$ be a complex Hilbert space, $T \in \mathcal{L}(\mathcal{H})$ and $T^* \in \mathcal{L}(\mathcal{H})$ be the* Hilbertian adjoint *of T. Then*

$$Sp(T^*) = Sp(T)^* := \{\lambda^* |\ \lambda \in Sp(T)\} \quad \text{and} \quad \forall \lambda \in \rho(T), \quad R_{\lambda^*}(T^*) = R_\lambda(T)^*.$$

Proof of the proposition — We first observe that $\forall S, T \in \mathcal{L}(\mathcal{H})$,

$$(TS)^* = S^*T^*$$

and hence that, if T is invertible, then

$$\begin{aligned}(T^{-1})^*T^* = \left(T(T^{-1})\right)^* = Id^* = Id\\ T^*(T^{-1})^* = \left((T^{-1})T\right)^* = Id^* = Id,\end{aligned}$$

which means that T^* is also invertible and $(T^*)^{-1} = (T^{-1})^*$. Since $T \longmapsto T^*$ is an involution the converse is straightforward, i.e. : T *is invertible iff* T^* *is invertible.* Thus

$$\lambda \in \rho(T) \Longleftrightarrow (\lambda - T) \text{ is invertible} \Longleftrightarrow (\lambda^* - T^*) \text{ is invertible} \Longleftrightarrow \lambda^* \in \rho(T^*).$$

And so $\left((\lambda - T)^{-1}\right)^* = (\lambda^* - T^*)^{-1}$. □

6.2 Useful results in complex Hilbert spaces

Theorem 6.2 (orthogonal projection) *Let $\mathcal{H}$ be a complex Hilbert space and F be a closed vector subspace. Then for any y there exists an unique $x \in F$ such that*

$$\|x - y\| = \inf_{\xi \in F} \|\xi - y\|. \tag{11}$$

Moreover $x - y \perp F$.

Proof — The case where $y \in F$, for which $x = y$, is trivial. So we assume that $y \notin F$ (which means that necessarily $F \neq \mathcal{H}$). Let $d := \inf_{\xi \in F} \|\xi - y\|$. Note that $d > 0$ since otherwise this would mean that y belongs to $\overline{F} = F$. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence with values in F such that $\lim_{n\to+\infty} \|x_n - y\| = d$ and write $d_n := \|x_n - y\|$. We will first establish that $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence. Let $n, m \in \mathbb{N}$, using the identity $\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2$ with $a = \frac{1}{2}(x_n - y)$ and $b = \frac{1}{2}(x_m - y)$ we get

$$\left\|\frac{1}{2}(x_n + x_m) - y\right\|^2 + \left\|\frac{1}{2}(x_n - x_m)\right\|^2 = \frac{1}{2}\|x_n - y\|^2 + \frac{1}{2}\|x_m - y\|^2 = \frac{1}{2}(d_n^2 + d_m^2).$$

But since $\frac{1}{2}(x_n + x_m) \in F$ and because of the definition of d, we have $\left\|\frac{1}{2}(x_n + x_m) - y\right\| \geq d$. Hence $\frac{1}{2}(d_n^2 + d_m^2) - \frac{1}{4}\left\|(x_n - x_m)\right\|^2 \geq d^2$, which implies

$$2\left(d_n^2 + d_m^2 - 2d^2\right) \geq \|x_n - x_m\|^2. \tag{12}$$

But since $d_n, d_m \to d$, the l.h.s. of (12) tends to zero and hence $\|x_n - x_m\|$ tends also to zero. Hence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, thus it converges to some x. Since F is closed $x \in F$. Note also that if $x, x' \in F$ are two points such that $\|x - y\| = \|x' - y\| = d$, then by applying (12) with (x, x') in place of (x_n, x_m), we deduce that $0 \geq \|x - x'\|$, thus x is unique.

Lastly for any $\xi \in F$, the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $f(t) := \|x + t\xi - y\|^2 = \|\xi\|^2 t^2 + t(\langle x - y, \xi\rangle + \langle \xi, x - y\rangle) + \|x - y\|^2$ is a polynomial which achieves its minimum for $t = 0$, hence $f'(0) = 0$, which reads $\mathrm{Re}(\langle x - y, \xi\rangle) = 0$. The same reasoning with ξ replaced by $i\xi$ gives us $\mathrm{Im}(\langle x - y, \xi\rangle) = 0$. Hence $x - y \perp \xi$, $\forall \xi \in F$. □

In the following, for any $A \subset \mathcal{H}$ we pose $A^\perp := \{y \in \mathcal{H}; \forall a \in A, \langle a, y\rangle = 0\}$ and $\forall x \in \mathcal{H}$, we set $x^\perp := \{x\}^\perp$. Note that for any subset A, $A^\perp$ is a closed vector subspace of $\mathcal{H}$, since it is the intersection of the closed hyperplanes $a^\perp$, for $a \in A$.

Corollary 6.1 *Let $\mathcal{H}$ be a complex Hilbert space and $F \subset X$ be a vector subspace. Then*

$$\overline{F} = (F^\perp)^\perp. \tag{13}$$

Proof — The inclusion $\overline{F} \subset (F^\perp)^\perp$ follows from two facts : first $(F^\perp)^\perp$ is closed, second the obvious fact that $F \subset (F^\perp)^\perp$. This implies that $\overline{F} \subset \overline{(F^\perp)^\perp} = (F^\perp)^\perp$.

To conclude to (13), let's argue by contradiction and suppose that $\overline{F}$ is a strict subspace of $(F^\perp)^\perp$. Then there exists some $y \in (F^\perp)^\perp$ which does not belong to $\overline{F}$. By applying Theorem 6.2 we obtain some $x \in \overline{F}$ such that $y - x \in \overline{F}^\perp = F^\perp$. But since $x \in \overline{F} \subset (F^\perp)^\perp$ we have also $y - x \in (F^\perp)^\perp$. Both conclusions imply $y - x = 0$, a contradiction. □

Proposition 6.2 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Then*

$$KerT = (ImT^*)^\perp. \tag{14}$$

And as a straightforward consequence of Corollary 6.4

$$\overline{ImT^*} = (KerT)^\perp. \tag{15}$$

Proof — Let $x \in \mathcal{H}$, then

$$\begin{aligned} x \in \mathrm{Ker}T &\iff Tx = 0 \iff Tx \in \mathcal{H}^\perp \\ &\iff [\forall y \in \mathcal{H}, \quad \langle y, Tx\rangle = 0] \\ &\iff [\forall y \in \mathcal{H}, \quad \langle T^* y, x\rangle = 0] \\ &\iff x \in (\mathrm{Im}T^*)^\perp. \end{aligned}$$

Then (15) follows by taking the orthogonal subspaces of both sides of (14) and applying Corollary 6.1. □

Corollary 6.2 *If $\mathcal{H}$ is a complex Hilbert space, then*

$$\lambda \in Sp_p(T) \quad \textit{iff} \quad Im(\lambda^* - T^*) \textit{ is not dense in } \mathcal{H}. \tag{16}$$

Proof — Apply (15) with $\lambda - T$: this gives us $\overline{\mathrm{Im}(\lambda^* - T^*)} = (\mathrm{Ker}(\lambda - T))^\perp$. □

6.3 Generalizations in complex normed vector spaces

Theorem 6.2 does not make sense in an arbitrary Banach space. However we have the following substitute. First recall the following geometric version of the Hahn–Banach theorem theorem.

Theorem 6.3 (geometric Hahn–Banach, compact/closed) *Let E be a **real** normed vector space and F and K be two convex subsets of E. Assume that F is closed, K is compact and $F \cap K = \emptyset$. Then there exists a closed hyperplane H which separates F and K. This amounts to say : $\exists f \in E'$, $\exists \alpha \in \mathbb{R}$ s.t. $\forall x \in F$, $\forall y \in K$, $f(x) < \alpha < f(y)$.*

Corollary 6.3 *Let E be a **real** normed vector space, G be a vector subspace of E and $v \in E$. Assume that $v \notin \overline{G}$. Then $\exists \ell \in E'$, s.t. $\forall x \in G$, $\ell(x) = 0$ and $\ell(v) = 1$. As a bonus : $\overline{G} \oplus \mathbb{R}v$ is closed in E.*

Proof of Corollary 6.3 — We apply Theorem 6.3 with $F = \overline{G}$ and $K = \{v\}$. It gives us some $f \in E'$ and $\alpha \in \mathbb{R}$, s.t. $\forall x \in G$, $f(x) < \alpha < f(v)$. This forces that $\forall x \in G$, $f(x) = 0$. Indeed assume the contrary : then there exists some $a \in G$ s.t. $f(a) \neq 0$, but then we may find some $\lambda \in \mathbb{R}$ s.t. $\lambda f(a) = f(\lambda a) \geq \alpha$ and $\lambda a \in G$, a contradiction. Hence $f(v) > \alpha > 0$. By setting $\ell := f(v)^{-1} f$, we reach our conclusion.

Lastly $\overline{G} \oplus \mathbb{R}v$ is closed in E because it is the inverse image of the closed subspace $\overline{G}$ by the map $E \ni x \longmapsto x - \ell(x)v \in E$ which is linear and continuous. □

Theorem 6.4 (characterization of dense subspaces in the complex case) *Let E be a complex normed vector space and F be a vector subspace. Then*

$$\overline{F} \neq E \quad \Longleftrightarrow \quad \exists \Phi \in E' \text{ s.t. } \Phi \neq 0 \text{ and } F \subset \mathrm{Ker}\Phi.$$

Proof — (i) Assume that $\overline{F} \neq E$. Then there exists $v \in E$ s.t. $v \notin \overline{F}$. In the following we consider E as a *real vector space* and all vector subspaces (such as F) will also be considered as as *real vector subspaces* of E. We introduce the two vector subspaces

$$G_1 := \overline{F} \oplus \mathbb{R}iv \quad \text{and} \quad G_2 := \overline{F} \oplus \mathbb{R}v$$

and use Corollary 6.3 (the bonus of which says us that G_1 and G_2 are closed)

– to G_1 and $v \notin G_1$: we obtain $f_1 \in E'_{\mathbb{R}}$ s.t. $G_1 \subset \mathrm{Ker} f_1$ and $f_1(v) = 1$;

– to G_2 and $iv \notin G_2$: we obtain $f_2 \in E'_{\mathbb{R}}$ s.t. $G_2 \subset \mathrm{Ker} f_2$ and $f_2(iv) = 1$;

Hence the map $\varphi : E \longrightarrow \mathbb{C}$ defined by

$$\varphi(x) := f_1(x) + i f_2(x)$$

satisfies $F \subset G_1 \cap G_2 \subset \mathrm{Ker}\varphi$ and $\varphi((\lambda + i\mu)v) = \lambda + i\mu$, hence the restriction of φ on $G_1 + G_2 = \overline{F} \oplus \mathbb{C}v$ is $\mathbb{C}$-linear. Note that φ may not be $\mathbb{C}$-linear on E, but by setting

$$\Phi(x) := \frac{1}{2}\left[\varphi(x) - i\varphi(ix)\right],$$

we obtain a continuous complex linear form on E which agrees with φ on $G_1 + G_2$. In particular $F \subset \mathrm{Ker}\Phi$ and $\Phi(v) = 1$.

(ii) The converse is straightforward : if there exists some non vanishing $\Phi \in E'$ s.t. $F \subset \mathrm{Ker}\Phi$, we immediately get $\overline{F} \subset \mathrm{Ker}\Phi$ and hence $\overline{F} \neq E$. □

Let us introduce some notations. If X is a Banach space and $F \subset X$ is a vector subspace then

$$F^\perp := \{\alpha \in X'; \forall x \in F, \alpha(x) = 0\}$$

and, if A is a vector subspace of X',

$$A^\perp := \{x \in X; \forall \alpha \in A, \alpha(x) = 0\}.$$

Corollary 6.4 *Let X be a complex Banach space and $F \subset X$ be a vector subspace. Then*

$$\overline{F} = (F^{\perp})^{\perp}. \tag{17}$$

Proof — The inclusion $\overline{F} \subset (F^{\perp})^{\perp}$ follows from two facts : first $(F^{\perp})^{\perp} = \cap_{\alpha \in F^{\perp}} \mathrm{Ker}\alpha$ is closed (an intersection of closed hyperplanes), second the obvious fact that $F \subset (F^{\perp})^{\perp}$. This implies that $\overline{F} \subset \overline{(F^{\perp})^{\perp}} = (F^{\perp})^{\perp}$.

To conclude to (17), let's argue by contradiction and suppose that $\overline{F}$ is a strict subspace of $(F^{\perp})^{\perp}$. Then by applying Theorem 6.4 there exists a *non vanishing* $\alpha \in ((F^{\perp})^{\perp})'$ s.t. $\overline{F} \subset \mathrm{Ker}\alpha$. By using Hahn–Banach theorem and a reasoning similar to the proof of Theorem 6.4 we can extend α to $\alpha_X \in X'$. We still have $\overline{F} \subset \mathrm{Ker}\alpha_X$ and hence $\alpha_X \in \overline{F}^{\perp} = F^{\perp}$. Thus any $x \in (F^{\perp})^{\perp}$ must satisfy $\alpha_X(x) = 0$, meaning that $\alpha = \alpha_X|_{(F^{\perp})^{\perp}} = 0$, a contradiction. □

Similarly Proposition 6.2 can be generalized in Banach spaces :

Proposition 6.3 *Let E be a complex Banach space and $T \in \mathcal{L}(E)$. Then*

(i) If $\lambda \in Sp_p(T)$, then Im$(\lambda - T')$ is not dense in E' ;

(ii) If $\lambda \in \mathbb{C}$ is s.t. Im$(\lambda - T)$ is not dense in E, then $\lambda \in Sp_p(T')$;

(iii) Corollary : if E is reflexive, i.e. if $(E')' = E$, then $\lambda \in Sp_p(T)$ iff Im$(\lambda - T')$ is not dense in E'.

Proof — **Proof of (i)** : let $\lambda \in Sp_p(T)$, then $\exists x_0 \in E \setminus \{0\}$ s.t. $(\lambda - T)(x_0) = 0$. This implies in particular that $\forall \alpha \in E'$,

$$[(\lambda - T')(\alpha)](x_0) = \alpha\left((\lambda - T)(x_0)\right) = \alpha(0) = 0,$$

i.e. $(\lambda - T')(\alpha) \in x_0^{\perp} := \{\beta \in E' |\ \beta(x_0) = 0\}$. Hence $\mathrm{Im}(\lambda - T') \subset x_0^{\perp}$. This implies of course that $\mathrm{Im}(\lambda - T')$ cannot be dense in E', since $x_0^{\perp}$ is closed and is different from E'.

Proof of (ii) : let $\lambda \in \mathbb{C}$ s.t. $\mathrm{Im}(\lambda - T)$ is not dense in E. From Theorem 6.4, there exists $\alpha \in E' \setminus \{0\}$ s.t. $\mathrm{Im}(\lambda - T) \subset \mathrm{Ker}\alpha$. Thus,

$$\begin{aligned} \forall x \in E, \quad \alpha\left[(\lambda - T)(x)\right] = 0 &\iff \forall x \in E, \quad \left[(\lambda - T')(\alpha)\right](x) = 0 \\ &\iff (\lambda - T')(\alpha) = 0 \\ &\iff \alpha \in \mathrm{Ker}(\lambda - T'). \end{aligned}$$

Hence $\lambda \in Sp_p(T')$.

Proof of (iii) : the fact that $[\lambda \in Sp_p(T)$ implies $\mathrm{Im}(\lambda - T')$ is not dense in $E']$ has been proved in (i) ; the converse follows by applying (ii) to T' and by using the fact that $T'' = T$. □

6.4 Application to normal operators

Theorem 6.5 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Assume that T is normal, i.e. $TT^* = T^*T$, then its residual spectrum is empty, i.e. $Sp_r(T) = \emptyset$.*

Proof — We first prove that :

$$\text{if } TT^* = T^*T \quad \text{then} \quad \mathrm{Ker}T = \mathrm{Ker}T^*. \tag{18}$$

Indeed if $TT^* = T^*T$ for any $x \in \mathcal{H}$,

$$\begin{aligned} x \in \mathrm{Ker}T &\iff \|Tx\|^2 = 0 \\ &\iff 0 = \langle Tx, Tx\rangle = \langle T^*Tx, x\rangle = \langle TT^*x, x\rangle = \|T^*x\|^2 \\ &\iff x \in \mathrm{Ker}T^*. \end{aligned}$$

Applying (18) to $\lambda^* - T^*$, we get that :

$$\text{if } TT^* = T^*T \quad \text{then} \quad \mathrm{Ker}(\lambda^* - T^*) = \mathrm{Ker}(\lambda - T). \tag{19}$$

On the other hand if we apply (15) with T replaced by $\lambda^* - T^*$ we get

$$\overline{\mathrm{Im}(\lambda - T)} = \mathrm{Ker}(\lambda^* - T^*)^{\perp}. \tag{20}$$

Hence we deduce from (19) and (20) that, if $TT^* = T^*T$,

$$\overline{\mathrm{Im}(\lambda - T)} = \mathrm{Ker}(\lambda^* - T^*)^{\perp} = \mathrm{Ker}(\lambda - T)^{\perp}.$$

In particular if $TT^* = T^*T$

$$\lambda \in Sp_p(T) \quad \Longleftrightarrow \quad \mathrm{Ker}(\lambda - T) \neq \{0\} \quad \Longleftrightarrow \quad \mathrm{Im}(\lambda - T) \text{ is not dense in } \mathcal{H}$$

and thus $Sp_r(T) := \{\lambda \in \mathbb{C};\ \mathrm{Ker}(\lambda - T) = 0 \text{ and } \overline{\mathrm{Im}(\lambda - T)} \neq \mathcal{H}\} = \emptyset$. □

Remark — We can summarize the proof by the following identities :

$$\left\{\begin{array}{ccc} \mathrm{Ker}(\lambda - T) & = & \mathrm{Im}(\overline{\lambda} - T^*)^{\perp} \\ \| & & \\ \mathrm{Ker}(\overline{\lambda} - T^*) & = & \mathrm{Im}(\lambda - T)^{\perp} \end{array}\right. \Longrightarrow \left\{\begin{array}{ccc} \mathrm{Ker}(\lambda - T)^{\perp} & = & \overline{\mathrm{Im}(\overline{\lambda} - T^*)} \\ \| & & \\ \mathrm{Ker}(\overline{\lambda} - T^*)^{\perp} & = & \overline{\mathrm{Im}(\lambda - T)} \end{array}\right.$$

where the vertical identity is a consequence of $TT^* = T^*T$.

Conclusion : the spectrum of any normal operator is composed uniquely of eigenvalues and continuous spectral values.

Example 1 — Let $\mathcal{H}$ be a complex separable Hilbert space. This implies that $\mathcal{H}$ admits a Hermitian orthogonal Hilbertian basis $(e_n)_{n\in\mathbb{N}^*}$. We let $T : \mathcal{H} \longrightarrow \mathcal{H}$ be the linear operator s.t.

$$\forall n \in \mathbb{N}^*, \quad Te_n = \frac{1}{n}e_n.$$

Then T is obvious bounded, i.e. $T \in \mathcal{L}(\mathcal{H})$. It is also clear that $Sp_p(T) = \{\frac{1}{n};\ n \in \mathbb{N}^*\}$. Hence $Sp(T)$ contains at least $\{\frac{1}{n};\ n \in \mathbb{N}^*\}$. Are there other spectral values ? Certainly yes, since the spectrum should be closed and $\{\frac{1}{n};\ n \in \mathbb{N}^*\}$ is not closed. In fact its closure contains also 0, so 0 is a spectral value. It is not difficult to see that any λ which does not belong to $\{\frac{1}{n};\ n \in \mathbb{N}^*\} \cup \{0\}$ is in $\rho(T)$. Indeed for such a value, there exists $\varepsilon > 0$ s.t. $\forall n \in \mathbb{N}^*$, $|\lambda - \frac{1}{n}| > \varepsilon$ and hence we easily prove that

$$\forall x \in \mathcal{H}, \quad \|(\lambda - T)x\| \geq \varepsilon\|x\|. \tag{21}$$

Hence $\mathrm{Ker}(\lambda - T) = \{0\}$. Moreover thanks to (21) we can use Lemma 3.1 to deduce that $\mathrm{Im}(\lambda - T)$ is closed. But, since T is self-adjoint and hence normal, we deduce also from Corollary 6.2 that the image of $\lambda - T$ is dense in $\mathcal{H}$. Hence $\lambda - T$ is invertible. Thus $Sp(T)$

is also contained in $\{\frac{1}{n};\, n \in \mathbb{N}^*\} \cup \{0\}$, thus it coincides with this set. We deduce with the help of Theorem 6.5 that $Sp_c(T) = \{0\}$.

We actually check directly that ImT is dense in $\mathcal{H}$ but is not equal to $\mathcal{H}$. Indeed on the one hand $\mathcal{H}_{pol} := \{\sum_{n=1}^{N} x_n e_n;\, N \in \mathbb{N}^*, (x_1, \cdots, x_N) \in \mathbb{C}^N\}$ is dense in $\mathcal{H}$ and one checks easily that the image of T contains $\mathcal{H}_{pol}$, hence the image of T is dense in $\mathcal{H}$. On the other hand $y = \sum_{n=1}^{+\infty} \frac{e_n}{n}$ belongs to $\mathcal{H}$ but not to ImT since if there would be some $x = \sum_{n=1}^{+\infty} x_n e_n \in \mathcal{H}$ which would satisfy $Tx = y$, then we should have $x_n = 1$, $\forall n$, which makes impossible the condition $\sum_{n=1}^{+\infty} |x_n|^2 < +\infty$.

Example 2 — Let again $\mathcal{H}$ be a complex separable Hilbert space with a Hermitian orthogonal Hilbertian basis $(e_n)_{n\in\mathbb{N}^*}$. Let $\varphi : \mathbb{N}^* \longrightarrow [0,1] \cap \mathbb{Q}$ be a bijection and define $T \in \mathcal{L}(\mathcal{H})$ by

$$\forall n \in \mathbb{N}^*, \quad Te_n = \varphi(n) e_n.$$

An analysis similar to the one in the preceding example shows that $Sp_p(T) = [0,1] \cap \mathbb{Q}$, $Sp(T) = [0,1]$ and thus (since T is normal and hence $Sp_r(T) = \emptyset$) $Sp_c(T) = [0,1] \setminus \mathbb{Q}$. We can check directly the latter. Indeed if $\lambda \in [0,1] \setminus \mathbb{Q}$, then λ is not an eigenvalue of T and the image of $\lambda - T$ contains $\mathcal{H}_{pol}$ and hence is dense in $\mathcal{H}$. Lastly Im$(\lambda - T)$ is not equal to $\mathcal{H}$ because of the following. By using the density of $\mathbb{Q}$ in $\mathbb{R}$, we can find a sequence $\psi : \mathbb{N}^* \longrightarrow \mathbb{N}^*$ s.t. $\forall n \in \mathbb{N}^*$, $|\varphi \circ \psi(n) - \lambda| < \frac{1}{n}$ and then $y = \sum_{n=1}^{+\infty} [\lambda - \varphi \circ \psi(n)] e_{\psi(n)}$ belongs to $\mathcal{H}$ but not to Im$(\lambda - T)$.

6.5 The spectrum of a self-adjoint operator

We are now in position to prove the following.

Theorem 6.6 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Assume that T is self-adjoint, then*

(i) the spectrum of T is real, i.e. $Sp(T) \subset \mathbb{R}$;

(ii) if $\lambda, \mu \in Sp_p(T)$ and if $u, v \in \mathcal{H}$ are eigenvectors of T for the eigenvalues λ and μ respectively, i.e. $Tu = \lambda u$ and $Tv = \mu v$, then $\lambda \neq \mu$, implies $u \perp v$.

Proof — We first prove the following inequality. Assume that $\lambda, \mu \in \mathbb{R}$, then

$$\forall x \in \mathcal{H}, \quad \|(T - \lambda - i\mu)x\| \geq |\mu| \|x\|. \tag{22}$$

Let $x \in \mathcal{H}$, then, using the fact that $T^* = T$,

$$\begin{aligned} \|(T - \lambda - i\mu)x\|^2 &= \langle (T - \lambda - i\mu)x, (T - \lambda - i\mu)x \rangle \\ &= \langle (T - \lambda + i\mu)^* x, (T - \lambda - i\mu)x \rangle \\ &= \langle x, (T - \lambda + i\mu)(T - \lambda - i\mu)x \rangle = \langle x, [(T - \lambda)^2 + \mu^2]x \rangle \\ &= \langle x, (T - \lambda)^2 x \rangle + \mu^2 \|x\|^2. \end{aligned}$$

Observe that an application of this identity with $\mu = 0$ gives us also $\|(T - \lambda)x\|^2 = \langle x, (T - \lambda)^2 x \rangle$. Substituting this expression in the r.h.s. of our computation hence leads to

$$\|(T - \lambda - i\mu)x\|^2 = \|(T - \lambda)x\|^2 + \mu^2 \|x\|^2,$$

from which (22) follows.

Let us now prove (i) : let $\lambda, \mu \in \mathbb{R}$ such that $\mu \neq 0$, we need to prove that $T - \lambda - i\mu$ is invertible. Obviously (22) implies that $T - \lambda - i\mu$ is one-to-one [3]. Moreover by using Lemma 3.1 we also deduce from (22) that the image of $T - \lambda - i\mu$ is *closed*. Lastly by applying (22) for $T - \lambda + i\mu = (T - \lambda - i\mu)^*$, we also obtain that $\mathrm{Ker}(T - \lambda - i\mu)^* = \{0\}$. Hence by applying (15) to $(T - \lambda - i\mu)^*$ we deduce that the image of $T - \lambda - i\mu$ is *dense in* $\mathcal{H}$. Hence $\mathrm{Im}(\lambda - T - i\mu) = \mathcal{H}$, i.e. $T - \lambda - i\mu$ is onto [4]. This proves (i).

To prove (ii), let $\lambda, \mu \in Sp_p(T)$ and $u, v \in \mathcal{H}$ such that $Tu = \lambda u$ and $Tv = \mu v$. Then

$$\mu\langle v, u\rangle = \mu^*\langle v, u\rangle = \langle \mu v, u\rangle = \langle Tv, u\rangle = \langle v, Tu\rangle = \langle v, \lambda u\rangle = \lambda\langle v, u\rangle.$$

Hence $(\lambda - \mu)\langle v, u\rangle = 0$. It follows that, if $\lambda \neq \mu$, $\langle v, u\rangle = 0$. □

7 Compact operators

7.1 Definition and examples

In the following, if Y is a normed vector space, $a \in Y$ and $r \in (0, +\infty)$, we denote by $B_Y(a, r)$ the open ball of center a and of radius r in Y.

Definition 7.1 (relatively compact sets) *Let Y be a Banach space and $F \subset Y$. We say that F is* ***relatively compact*** *if one of the two following equivalent properties holds :*

(i) $\forall \varepsilon > 0$, there exists a finite number $n \in \mathbb{N}^*$ of points $y_1, \cdots, y_N \in Y$ such that $F \subset \cup_{1\leq i\leq N} B_Y(y_i, \varepsilon)$;

(ii) for any sequence $(u_n)_{n\in\mathbb{N}}$ with values in F, there exists a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ which converges in Y.

Definition 7.2 (compact operators) *Let X be a normed vector space and Y be a Banach space. A linear operator $T : X \longrightarrow Y$ is* ***compact*** *if $T(B_X(0,1))$ is relatively compact in Y.*

Examples (i) If Y is a finite dimensional vector space, a theorem of Riesz tells us that, for any $R > 0$, $B_Y(0, R)$ is relatively compact. Hence for any normed vector space X, any bounded linear operator T from X to Y is compact, for the image of $B_X(0,1)$ by T is contained in the ball $B_Y(0, \|T\|)$.
(ii) An operator T between two normed vector spaces X and Y is **of finite rank** if its image is a finite dimensional vector subspace of Y. Any bounded linear operator T of *finite rank* is compact, since $T(B_X(0,1))$ is contained in $\mathrm{Im}T \cap B_Y(0, \|T\|)$.

We denote by $\mathcal{K}(X, Y)$ the set of compact operators between two normed vector spaces X and Y. If $X = Y$, we set $\mathcal{K}(X) = \mathcal{K}(X, X)$. It is easy to check that $\mathcal{K}(X, Y)$ is a vector space which is contained in $\mathcal{L}(X, Y)$, since any relatively compact subset is bounded.

3. i.e. injective
4. i.e. surjective

Theorem 7.1 *The space of compact operator between two normed vector subspaces is closed in* $\mathcal{L}(X, Y)$

Proof — Let $(T_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{K}(X, Y)$ and assume that it converges to some operator T in $\mathcal{L}(X, Y)$. Let $\varepsilon > 0$, chose and fix $n \in \mathbb{N}$ such that $\|T - T_n\| < \varepsilon/2$. Then use the fact that T_n is compact : let $y_1, \cdots, y_N \in Y$ be a finite collection of points such that $T_n(B_X(0, 1)) \subset \cup_{i=1}^N B_Y(y_i, \varepsilon/2)$. Then we deduce easily that $T(B_X(0, 1)) \subset \cup_{i=1}^N B_Y(y_i, \varepsilon)$. □

The previous result allows us to enrich our list of examples of compact operators :
(iii) Any operator which is *the limit in* $\mathcal{L}(X, Y)$ *of a sequence of finite rank operators* is compact (since it is then the limit of a sequence of compact operators).

A natural question is : are there other examples ? this question turns out to be very difficult. We will see later on that if X is a Hilbert space, then there are no more examples of compact operators. However it took a long time before knowing whether this holds also in an arbitrary normed vector space, until one finds that in general there are compact operators which are not limit of a sequence of finite rank operators !

7.2 The weak topology and compact operators

Compact operators enjoy a remarkable property related with the notion of **weak convergence**.

Definition 7.3 *Let X be a normed vector space and X' its topological dual space. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X and $x \in X$. We say that $(x_n)_{n\in\mathbb{N}}$* ***converges weakly to*** *x and we write :*

$$x_n \rightharpoonup x \quad \textit{in } X, \textit{when } n \to +\infty$$

if,

$$\forall \alpha \in X', \quad \lim_{n\to+\infty} \alpha(x_n) = \alpha(x). \tag{23}$$

Remarks — **1)** One can easily check that, if a sequence converges weakly, then the weak limit is unique.
2) An important consequence of the Banach–Steinhaus theorem is that **any weakly convergent sequence is bounded**.
3) When X is a Hilbert space $\mathcal{H}$, we can replace condition (23) by

$$\forall y \in \mathcal{H}, \quad \lim_{n\to+\infty} \langle y, x_n\rangle = \langle y, x\rangle. \tag{24}$$

4) The standard notion of convergence in a normed space $(X, \|\cdot\|)$ reads

$$\lim_{n\to+\infty} \|x_n - x\| = 0.$$

When both notions are used it is safer to refer to the standard notion of convergence as being the **strong convergence**, to avoid confusion. If $(x_n)_{n\in\mathbb{N}}$ converges *strongly* to x, we write $x_n \to x \quad \text{in } X, \text{when } n \to +\infty$.

2) In a finite dimensional vector space both notions of convergence coincides. However there are different when $\dim X = +\infty$ as the names suggest :

If $(x_n)_{n\in\mathbb{N}}$ converges strongly to x, then $(x_n)_{n\in\mathbb{N}}$ converges weakly to x, because of the inequality $|\alpha(x_n) - \alpha(x)| \leq \|\alpha\|\|x_n - x\|$. But the converse is not true in general (see the Example below). In fact these notions of convergence are associated with different topologies : the strong topology is spanned by open balls of X, whereas the weak topology is associated with, roughly speaking, Cartesian products of finite codimensional closed vector subspaces with open balls in a (finite dimensional) supplementary subspace. In particular the open ball is not an open subset of the weak topology if X is infinite dimensional!

Example 7.1 *Let $\mathcal{H}$ be a Hilbert space and let $(x_n)_{n\in\mathbb{N}}$ an orthonormal family of vectors, i.e. $\forall n, m \in \mathbb{N}$, $\langle x_n, x_m\rangle$ is equal to 1 if $n = m$ and to 0 if $n \neq m$. Then $x_n \rightharpoonup 0$ as $n \to +\infty$, but x_n does not converge strongly. Indeed for any $N \in \mathbb{N}$ and any $y \in \mathcal{H}$, consider the finite sum $z_N := \sum_{n=0}^{N}\langle x_n, y\rangle x_n$ (i.e. the orthogonal projection of y on Vect$(x_0, \cdots, x_N)$). Then Pythagore's theorem says us that $\sum_{n=0}^{N} |\langle x_n, y\rangle|^2 = \|z_N\|^2 \leq \|y\|^2 < +\infty$. This implies that the series $\sum_{n=0}^{+\infty} |\langle x_n, y\rangle|^2$ converges[5]. Hence in particular the general term of this series converges to 0, i.e. $\lim_{n\to+\infty}\langle x_n, y\rangle = 0$. Since $y \in \mathcal{H}$ is arbitrary, this proves that $(x_n)_{n\in\mathbb{N}}$ **converges weakly to 0**.*

*Now let's assume by contradiction that $(x_n)_{n\in\mathbb{N}}$ converges strongly to some limit, say x. Then this limit must be the same as the limit in the weak convergence, i.e. $x = 0$. Thus we should have $\lim_{n\to+\infty} \|x_n\| = 0$, which is impossible, since $\|x_n\| = 1$, $\forall n \in \mathbb{N}$. Hence $(x_n)_{n\in\mathbb{N}}$ **does not converge strongly**.*

Recall that, according to Riesz' theorem, the unit ball of a normed vector space is relatively compact iff the dimension of the space is finite. Hence in particular the unit ball of an infinite dimensional normed vector space is not relatively compact. Thus the weak topology is extremely useful because of the following compactness result. First recall the :

Definition 7.4 *A normed vector space X is reflexive if the the canonical inclusion map $X \subset X''$ is a normed vector space isomorphism.*

Theorem 7.2 *Let X be a Banach space. Assume that X is **reflexive** and **separable**. Then the unit closed ball of X is compact for the weak topology. In particular any **bounded** sequence in X admits a weakly convergent subsequence.*

Corollary 7.1 *Let X be a Banach space. Assume that X is **reflexive**. Then any **bounded** sequence in X admits a weakly convergent subsequence.*

5. In fact this also implies that the Hilbertian series $\sum_{n+0}^{\infty}\langle x_n, y\rangle x_n$ converges in $\mathcal{H}$. Its sum is equal to the orthogonal projection of y on the closure of the vector space spanned by $(x_n)_{n\in\mathbb{N}}$

Proof of the Corollary — Let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence and consider $F := \text{Vect}\{u_n; n \in \mathbb{N}\}$ and its closure $\overline{F}$ in X (for the weak or the strong topology, this does not make any difference for convex subsets). Then $\overline{F}$ is separable by construction and reflexive. Hence we can apply Theorem 7.2 to $(u_n)_{n\in\mathbb{N}}$ in $\overline{F}$. □

The proof ot Theorem 7.2 is a consequence of the following series of important definitions and results :
1) Definition of the **weak ⋆ topology** : let X be a normed vector space and X' its dual space. Then we define on X' the weak ⋆ topology (also called the $\sigma(X', X)$ topology) which is in general even weaker than the weak topology. Without giving too much details a sequence $(a_n)_{n\in\mathbb{N}}$ in X' converges to $a \in X'$ in the weak ⋆ topology if, $\forall x \in X$, $a_n(x)$ converges to $a(x)$ when $n \to +\infty$.
2) The **Banach–Alaoglu–Bourbaki theorem** : let X be a normed vector space, then the unit closed ball of X' is compact for the weak ⋆ topology.
3) If X is a **reflexive** Banach space, then the weak topology and the weak ⋆ topology on X' coincide. In fact a deeper result provides an equivalence :
Kakutani theorem : let X be a Banach space. Then X is reflexive if and only if the closed unit ball $\overline{B}_{X'}(0,1)$ of X' is closed for the weak topology.
4) Theorem : Let X be Banach space and assume that X is **separable** (i.e. there exists a countable dense subset in X). Then the unit ball $B_{X'}(0,1)$ of X' is metrisable for the weak ⋆ topology (i.e. there exists a metric on $B_{X'}(0,1)$ which induces the same topology as the weak ⋆ topology).

Note that the Banach–Alaoglu–Bourbaki theorem and the Kakutani theorem provide only the Borel–Lebesgue compactness criterion (using covering by open subsets) in general but not the Bolzano–Weierstrass property. However the equivalence between the Borel–Lebesgue property and the Bolzano–Weierstrass property holds in a metrisable space.

Now we present the most important result concerning the weak convergence and compact operators.

Theorem 7.3 *Let X and Y be two Banach spaces and $T \in \mathcal{K}(X,Y)$. Then the image of any* ***weakly convergent sequence*** *by T is a* ***strongly convergent sequence****.*

Proof — Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in X such that $x_n \rightharpoonup x$ in X. We first remark that, since $T \in \mathcal{K}(X,Y) \subset \mathcal{L}(X,Y)$, $Tx_n \rightharpoonup Tx$ in Y. Indeed, for any $\beta \in Y'$, $\beta \circ T \in X'$, hence $(\beta \circ T)(x_n) \to (\beta \circ T)(x)$, which also reads $\beta(Tx_n) \to \beta(Tx)$.

Let us now prove that $Tx_n \to Tx$ (strongly) in Y and argue by contradiction, i.e. we assume that the sequence $(Tx_n)_{n\in\mathbb{N}}$ does not converge to Tx. Then we can extract a subsequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ and find some $\varepsilon > 0$ such that $Tx_{\varphi(n)} \notin B_Y(Tx, \varepsilon)$, $\forall n \in \mathbb{N}$. But we still have $x_{\varphi(n)} \rightharpoonup x$ and, in particular, the sequence $(x_{\varphi(n)})_{n\in\mathbb{N}}$ is bounded. So by using the compactness of T, we deduce that its image $\{Tx_{\varphi(n)};\ n \in \mathbb{N}\}$ is relatively compact. Hence we can extract a further subsequence $(x_{\psi(n)})_{n\in\mathbb{N}}$ such that $Tx_{\psi(n)} \to y$ (strongly) in Y. To summarize :

$$\begin{array}{rcl} \left[x_{\psi(n)} \rightharpoonup x, \text{ in } X\right] & \Longrightarrow & \left[Tx_{\psi(n)} \rightharpoonup Tx, \text{ in } Y\right] \\ \left[Tx_{\psi(n)} \to y, \text{ in } Y\right] & \Longrightarrow & \left[Tx_{\psi(n)} \rightharpoonup y, \text{ in } Y\right] \end{array}$$

thus, by the uniqueness of the weak limit, $y = Tx$. Hence we deduce that $Tx_{\psi(n)} \to Tx$ strongly in Y, which contradicts the inequality $\|Tx_{\psi(n)} - Tx\| \geq \varepsilon$, $\forall n \in \mathbb{N}$. □

Theorem 7.4 *Let $\mathcal{H}$ be a* ***separable*** *Hilbert space. Then the space of finite rank operators in $\mathcal{L}(\mathcal{H})$ is dense in $\mathcal{K}(\mathcal{H})$.*

Proof — Since $\mathcal{H}$ is separable, there exists a countable Hermitian orthogonal Hilbertian basis $(e_n)_{n\in\mathbb{N}^*}$. For any any $n \in \mathbb{N}^*$ we let E_n be the finite dimensional vector space spanned by $\{e_1, \cdots, e_n\}$ and $E_n^\perp$ the orthogonal subspace to E_n. We define the two associated orthogonal projections $P_n : \mathcal{H} \longrightarrow E_n$ and $P_n^\perp : \mathcal{H} \longrightarrow E_n^\perp$.

Now let $T \in \mathcal{K}(\mathcal{H})$. We want to construct a sequence $(T_n)_{n\in\mathbb{N}^*}$ of finite rank operators such that $\lim_{n\to+\infty} T_n = T$ in the $\mathcal{L}(\mathcal{H})$ topology. For that purpose we simply set $T_n := T \circ P_n$. Then first the image of T_n is $T(\text{Vect}\{e_1, \cdots, e_n\})$ and thus is finite dimensional. Second

$$\|T-T_n\| = \sup_{x\in\mathcal{H},\|x\|\leq 1} \|Tx-TP_nx\| = \sup_{x\in\mathcal{H},\|x\|\leq 1} \|TP_n^\perp x\| = \sup_{x_n\in E_n, x_n^\perp\in E_n^\perp, \|x_n\|^2+\|x_n^\perp\|^2\leq 1} \|Tx_n^\perp\|,$$

where in the last expression we have decomposed $x = x_n + x_n^\perp$ with $x_n = P_nx$ and $x_n^\perp = P_n^\perp x$. Obviously the last supremum is the same as the supremum over all $x_n^\perp \in E_n^\perp$ such that $\|x_n^\perp\| \leq 1$. Hence

$$\|T - T_n\| = \sup_{x\in E_n^\perp, \|x\|\leq 1} \|Tx\|.$$

Observe that the sequence $(\|T - T_n\|)_{n\in\mathbb{N}^*}$ is decreasing and positive, hence converges to some nonnegative real number λ. We need to show that $\lambda = 0$. Argue by contradiction and assume that $\lambda > 0$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}^*}$ such that $x_n \in E_n^\perp$, $\|x_n\| = 1$, and $\|Tx_n\| \geq \lambda/2$, $\forall n \in \mathbb{N}^*$. Then $x_n \rightharpoonup 0$ since, for any $y \in \mathcal{H}$, $|\langle y, x_n\rangle| = |\langle P_n^\perp y, x_n\rangle| \leq \|P_n^\perp y\|$ and $\lim_{n\to+\infty} \|P_n^\perp y\| = 0$ (use the same kind of argument as in Example 7.1). Thus Theorem 7.3 implies that $Tx_n \to 0$. But this contradicts the inequality $\|Tx_n\| \geq \lambda/2$. Hence $\lambda = 0$, which means that T_n converges to T. □

7.3 Towards the spectral decomposition of self-adjoint operators : the Fredholm alternative

Theorem 7.5 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{K}(\mathcal{H})$. Then*

(i) Ker$(1 - T)$ is finite dimensional;

(ii) Im$(1 - T)$ is closed.

Remark — Theorem 7.5 is a special case of a more general result which works in Banach spaces.

Proof — We first prove (i) : by Riesz' Theorem it suffices to prove that the unit ball of $\mathrm{Ker}(1-T)$ is relatively compact. Observe that $\forall x \in \mathrm{Ker}(1-T)$, $Tx = x$ and thus

$$B_{\mathcal{H}}(0,1) \cap \mathrm{Ker}(1-T) = T\left[B_{\mathcal{H}}(0,1) \cap \mathrm{Ker}(1-T)\right] \subset T\left(B_{\mathcal{H}}(0,1)\right),$$

which implies that $B_{\mathrm{Ker}(1-T)}(0,1)$ is relatively compact since T is compact.

We now turn to the proof of (ii), which requires more work. The starting point is to consider a sequence $(y_n)_{n\in\mathbb{N}}$ of points in $\mathrm{Im}(1-T)$ which converges to some y in $\mathcal{H}$. We then wish to prove that the limit y belongs to $\mathrm{Im}(1-T)$.

a) *Choosing a good sequence of pre-images* — For any $n \in \mathbb{N}$, there exists some $x_n \in \mathcal{H}$ such that $(1-T)x_n = y_n$. We would like to prove that the sequence $(x_n)_{n\in\mathbb{N}}$ converge to some limit in $\mathcal{H}$ (then we could conclude that y is the image by T of this limit). The problem is that x_n is not the unique solution of the equation $x - Tx = y_n$ in general (unless $\mathrm{Ker}(1-T) = \{0\}$), so that the sequence has $(x_n)_{n\in\mathbb{N}}$ no reason to converge in general. Consider the space of solutions

$$E_n := \{x_n \in \mathcal{H};\, x_n - Tx_n = y_n\},$$

an affine space parallel to $\xi \in \mathrm{Ker}(1-T)$. This space is in particular finite dimensional. Hence we can find an unique $u_n \in E_n$ which minimizes the continuous function $x \longmapsto \|x\|^2$ on E_n. Actually u_n is the projection of 0 on E_n and satisfies

$$u_n - Tu_n = y_n \tag{25}$$

and

$$u_n \perp \mathrm{Ker}(1-T). \tag{26}$$

b) *Proving that the sequence $(u_n)_{n\in\mathbb{N}}$ is bounded* — We argue by contradiction and assume that the sequence $(u_n)_{n\in\mathbb{N}}$ is not bounded. Thus there exists a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ such that $d_{\varphi(n)} := \mathrm{dist}(0, E_{\varphi(n)}) = \|u_{\varphi(n)}\| \to +\infty$. Set $v_n := u_n/\|u_n\|$. Then the sequence $(v_{\varphi(n)})_{n\in\mathbb{N}}$ is bounded, hence by Kakutani's theorem **??** we may extract a further subsequence $(u_{\psi(n)})_{n\in\mathbb{N}}$ such that

$$v_{\psi(n)} \rightharpoonup v \quad \text{in } \mathcal{H}, \tag{27}$$

which implies by Theorem 7.3 :

$$Tv_{\psi(n)} \to Tv \quad \text{in } \mathcal{H}, \tag{28}$$

But using (25) for $u_{\psi(n)}$ and dividing by $d_{\psi(n)}$ we obtain on the other hand

$$v_{\psi(n)} - Tv_{\psi(n)} = y_{\psi(n)}/d_{\psi(n)} \to 0 \quad \text{strongly in } \mathcal{H},$$

which implies by using (27) and (28) that

$$v - Tv = 0. \tag{29}$$

But also (26) implies that $v_{\psi(n)} \perp \mathrm{Ker}(1-T)$, which implies by (27) that

$$v \perp \mathrm{Ker}(1-T). \tag{30}$$

Hence (29) and (30) implies that $v \in \mathrm{Ker}(1-T) \cap \mathrm{Ker}(1-T)^{\perp}$, i.e. $v = 0$. However $(v_{\psi(n)})_{n\in\mathbb{N}}$ converges strongly because of the identity $v_{\psi(n)} = Tv_{\psi(n)} + y_{\psi(n)}/d_{\psi(n)}$ and because of (28). Thus since $\|v_{\psi(n)}\| = 1$, we must have $\|v\| = 1$, which contradicts $v = 0$.
c) *Conclusion* — Since we know that $(u_n)_{n\in\mathbb{N}}$ is bounded, by Kakutani's theorem **??**, we may extract a subsequence $(u_{\varphi(n)})_{n\in\mathbb{N}}$ such that $u_{\varphi(n)} \rightharpoonup u$ in $\mathcal{H}$. This implies $Tu_{\varphi(n)} \to Tu$ by Theorem 7.3. Hence by passing to the weak limit in the relation $u_{\varphi(n)} - Tu_{\varphi(n)} = y_{\varphi(n)}$, we deduce $u - Tu = y$, i.e. that $y \in \mathrm{Im}(1-T)$. □

Corollary 7.2 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{K}(\mathcal{H})$. Then if T is self-adjoint,*

$$Im(1-T) = Ker(1-T)^{\perp}. \tag{31}$$

Proof — Theorem 7.5, (ii) gives us that, for any compact operator T, $\mathrm{Im}(1-T) = \overline{\mathrm{Im}(1-T)}$. But by applying (15) to $1-T^*$, we deduce that $\overline{\mathrm{Im}(1-T)} = \mathrm{Ker}(1-T^*)^{\perp}$. Thus $\mathrm{Im}(1-T) = \mathrm{Ker}(1-T^*)^{\perp}$. If we further assume that $T^* = T$, this gives us (31). □

Note that a theorem of Schauder asserts that the adjoint operator of a compact operator is compact (its proof uses Ascoli's theorem).

We now derive consequences of Theorem 7.5 and Corollary 7.2 for compact self-adjoint operators.

Lemma 7.1 *Let $\mathcal{H}$ be a complex Hilbert space of* infinite dimension *and let $T \in \mathcal{K}(\mathcal{H})$ be* ***self-adjoint***. *Then*

(i) $0 \in Sp(T)$ *;*

(ii) $Sp(T) \setminus \{0\} = Sp_p(T) \setminus \{0\}$.

Proof — (i) Assume that $0 \notin Sp(T)$, then T is invertible and thus T^{-1} is a bounded operator. Hence

$$B_{\mathcal{H}}(0,1) = T\left[T^{-1}(B_{\mathcal{H}}(0,1))\right]$$

is relatively compact, which is impossible since $\dim\mathcal{H} = +\infty$.

(ii) Let $\lambda \in Sp(T) \setminus \{0\}$. Then we know by Theorem 6.6 that $\lambda \in \mathbb{R}$. Hence it suffices to study the case where $\lambda \in \mathbb{R} \setminus \{0\}$. But, $\forall \lambda \in \mathbb{R} \setminus \{0\}$, using also Corollary 7.2,

$$\mathrm{Im}(\lambda - T) = \mathrm{Im}\left(1 - \frac{T}{\lambda}\right) = \mathrm{Ker}\left(1 - \frac{T}{\lambda}\right)^{\perp} = \mathrm{Ker}\,(\lambda - T)^{\perp}.$$

Thus, for any $\lambda \in \mathbb{R} \setminus \{0\}$ which is not an eigenvalue (i.e. $\lambda \notin Sp_p(T)$), we have simultaneously $\mathrm{Ker}\,(\lambda - T) = \{0\}$ and $\mathrm{Im}(1-T) = \mathcal{H}$, so that $\lambda \in \rho(T)$. □

Lemma 7.2 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{K}(\mathcal{H})$ be* ***self-adjoint****. Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence with values in $Sp(T)\setminus\{0\}$ such that $\forall n,m \in \mathbb{N}$, $n \neq m \Longrightarrow \lambda_n \neq \lambda_m$ and $\lim_{n\to+\infty} \lambda_n = \lambda$. Then $\lambda = 0$.*
(In other words the only possible accumulation point of $Sp(T) \setminus \{0\}$ is 0.)

Proof — By Lemma 7.1 each λ_n is actually an eigenvalue of T, i.e. $\forall n \in \mathbb{N}$, $\exists e_n \in \mathcal{H}$ such that $Te_n = \lambda_n e_n$ and $\|e_n\| = 1$. Thus by Theorem 6.6, $\forall n, m \in \mathbb{N}$, $n \neq m \Longrightarrow \lambda_n \neq \lambda_m \Longrightarrow e_n \perp e_m$. Hence $e_n \rightharpoonup 0$ (see Example 7.1). Let's assume that $\lambda \neq 0$. Then $\exists N \in \mathbb{N}$, $\forall n \in \mathbb{N}$, $n \geq N \Longrightarrow |\lambda_n| \geq \lambda/2$. Hence $e_n/\lambda_n \rightharpoonup 0$ in $\mathcal{H}$. Since T is compact we deduce that $e_n = T(e_n/\lambda_n) \to 0$ in $\mathcal{H}$ (Theorem 7.3). But simultaneously $\|e_n\| = 1$, $\forall n \in \mathbb{N}$. Hence we get a contradiction and $\lambda = 0$. □

Theorem 7.6 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{K}(\mathcal{H})$ be* ***self-adjoint****. Then*

(i) either $Sp(T) = \{0\}$;

(ii) or $Sp(T) \setminus \{0\}$ is finite ;

(iii) or $Sp(T) \setminus \{0\}$ is the image of a sequence of real numbers which converges to 0.

Proof — Assume that neither (i) or (ii) occur. Then $Sp(T) \setminus \{0\}$ has an accumulation point (as an infinite bounded subset of $\mathbb{R}$). By Lemma 7.2 this accumulation point must be 0. Hence in particular :

$$\forall p \in \mathbb{N}, \quad Sp(T) \setminus B_{\mathbb{R}}(0, 1/p) \text{ is finite,}$$

since otherwise we could find an accumulation point in $\mathbb{R} \setminus B_{\mathbb{R}}(0, 1/p)$, i.e. away from 0. Since $Sp(T) \setminus \{0\} = \cup_{p\in\mathbb{N}^*} [Sp(T) \setminus B_{\mathbb{R}}(0, 1/p)]$, we can hence count all eigenvalues in $Sp(T) \setminus \{0\}$ by integers $n \in \mathbb{N}$ and label them λ_n. It is clear that λ_n converges to 0. □

Theorem 7.7 (Spectral decomposition of compact self-adjoint operators) *Let $\mathcal{H}$ be a complex separable Hilbert space and $T \in \mathcal{K}(\mathcal{H})$ be* ***self-adjoint****. Then $\mathcal{H}$ has a Hilbertian Hermitian orthogonal basis composed of eigenvectors of T.*

Proof — **1)** By the previous results there exists at most a countable sequence $(\lambda_n)_{n\in\mathcal{N}}$ of real nonvanishing eigenvalues of T which tends to 0 and such that $Sp(T) = \{0\} \cup \{\lambda_n;\, n \in \mathcal{N}\}$ ($\mathcal{N}$ being a subset of $\mathbb{N}$). Moreover, for any $n \in \mathcal{N}$, each eigenspace $E_n := \mathrm{Ker}(\lambda_n - T)$ is finite dimensional. Let $F := \mathrm{Vec}(E_1, E_2, \cdots) \subset \mathcal{H}$. For any $n \in \mathcal{N}$ we construct an orthonormal basis $\{v_{n1}, \cdots, v_{np_n}\}$ of E_n. Note that since $E_n \perp E_m$ if $n \neq m$, the union $\cup_{n\in\mathcal{N}}\{v_{n1}, \cdots, v_{np_n}\}$ is again a countable orthonormal family of vectors. Hence this gives us a Hilbertian Hermitian orthogonal basis of $\overline{F}$. We observe that F is obviously stable by T, since it is spanned by eigenvectors of T. The same holds for $\overline{F}$.

2) Consider $F^\perp = \overline{F}^\perp$. This subspace of $\mathcal{H}$ is also stable by T : $\forall y \in F^\perp$,

$$\forall x \in F, \quad \langle x, Ty\rangle = \langle Tx, y\rangle = 0,$$

since $Tx \in F$. Hence $Ty \in F^\perp$.

3) So let $T_0 := T|_{F^\perp} \in \mathcal{K}(F^\perp)$. This operator is also self-adjoint. Hence we can apply all previous results to T_0. We now observe that $Sp(T_0) = \{0\}$. Indeed any $\lambda \in Sp(T_0)$ which is different from 0 should be an eigenvalue of T as well, hence the corresponding eigenvectors should be in F. But it should also be in $F^\perp$, which is impossible. Hence $Sp(T_0) = \{0\}$. In particular $r(T_0) = 0$.

4) At this point we use the innocent looking result Corollary 5.1 to T_0 : it tells us that $\|T_0\| = r(T_0) = 0$. Hence $T_0 = 0$, which means that $F^\perp \subset \text{Ker}T$. In fact $F^\perp = \text{Ker}T$. In conclusion we have the decomposition $\mathcal{H} = \text{Ker}T \oplus \overline{F} = \text{Ker}T \oplus \overline{E_1 \oplus E_2 \oplus \cdots}$. The last observation is the rôle of the hypothesis that $\mathcal{H}$ is separable : it implies that Ker$T$ is separable and hence that this subspace has a Hilbertian Hermitian orthogonal basis which can be used to complete the Hilbertian basis of eigenvectors of $\overline{F}$. □

Remark — As it appears clearly in the proof of the previous theorem, the hypothesis that $\mathcal{H}$ be separable is not essential to obtain a spectral decomposition.

8 The spectral decomposition of the Laplace operator on a bounded domain

We now present an application of the results of the previous section. The *Laplacian* or *Laplace operator* acting on functions of n real variable is the second order differential operator

$$\Delta = \sum_{i=1}^{n} \frac{\partial^2}{(\partial x^i)^2} = \frac{\partial^2}{(\partial x^1)^2} + \cdots + \frac{\partial^2}{(\partial x^n)^2}.$$

Consider an open domain $\Omega \subset \mathbb{R}^n$. Our aim is to prove that, if Ω is bounded, any function $u : \Omega \longrightarrow \mathbb{R}$, satisfying regularity condition to be precised later can be decomposed in the form (and in a sense also to be precised later)

$$u = \sum_{n=0}^{+\infty} \widehat{u}_n e_n$$

where $(\widehat{u}_n)_{n\in\mathbb{N}}$ is a sequence of real numbers and each function e_n satisfies

$$\begin{cases} -\Delta e_n &= \lambda_n e_n \quad \text{on } \Omega; \\ e_n &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

A simple example is the case where $n = 1$ and $\Omega = (0, \pi)$. We then recover the Fourier decomposition with the functions $e_n(x) = \sqrt{2/\pi}\sin(nx)$ and the eigenvalues $\lambda_n = n^2$. The idea is to perform a spectral decomposition of the operator $-\Delta$. However a difficulty is that $-\Delta$ is not bounded. This difficulty can be overcomed by the use of the theory of non bounded self-adjoint operators that we shall see later on. An alternative approach is to work with the operator $(1 - \Delta)^{-1}$: in the following we define this operator and prove that it is compact and self-adjoint. An application of Theorem 7.7 will then gives us the decomposition result on $-\Delta$. *In this section Hilbert spaces are over real numbers.*

8.1 Lebesgue and Sobolev spaces

For $1 \leq p < +\infty$ and for any open subset $\Omega \subset \mathbb{R}^n$ we define

$$\mathcal{L}^p(\Omega) := \left\{ u : \Omega \to \mathbb{R};\ u \text{ is Lebesgue measurable and } \int_\Omega |u(x)|^p dx < +\infty \right\}.$$

The quantity

$$\|u\|_{\mathcal{L}^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}$$

is then a semi-norm on $\mathcal{L}^p(\Omega)$ which does not 'see' what happens on negligeable subsets, i.e. $\forall u, v \in \mathcal{L}^p(\Omega)$, $\|u - v\|_{\mathcal{L}^p(\Omega)} = 0 \Longleftrightarrow u = v$ a.e. This is why we consider the quotient set

$$L^p(\Omega) := \mathcal{L}^p(\Omega)/[u \sim v \text{ iff } u = v \text{ a.e.}].$$

Then for any class of functions $[u] \in L^p(\Omega)$ one can consistently define $\|[u]\|_{L^p(\Omega)} = \|u\|_{\mathcal{L}^p(\Omega)}$ and $(L^p(\Omega), \|\cdot\|_{L^p(\Omega)})$ is a Banach space (Lebesgue spaces). In the following we do the usual abuse of notation $u = [u]$.

Similarly one can also define the space of measurable essentially bounded functions

$$\mathcal{L}^\infty(\Omega) := \{u : \Omega \to \mathbb{R};\ \exists M > 0, \text{ such that } |u(x)| \leq M \text{ a.e.}\}.$$

And, as previously,

$$L^\infty(\Omega) := \mathcal{L}^\infty(\Omega)/[u \sim v \text{ iff } u = v \text{ a.e.}].$$

The space $L^\infty(\Omega)$ can be equipped with the norm

$$\|u\|_{L^\infty(\Omega)} := \inf\{M > 0, \text{ such that } |u(x)| \leq M \text{ a.e.}\}.$$

In the special case where $p = 2$, then the norm $\|u\|_{L^2(\Omega)}$ comes from the scalar product

$$\langle u, v \rangle_{L^2(\Omega)} := \int_\Omega u(x)v(x)dx.$$

We now define the *Sobolev* space

$$\begin{aligned} W^{1,p}(\Omega) \ &:= \ \{u \in L^p(\Omega);\ \exists u_1, \cdots, u_n \in L^p(\Omega), \forall i = 1, \cdots, n, \\ &\qquad \forall \varphi \in \mathcal{C}^\infty_c(\Omega), \int_\Omega u\frac{\partial \varphi}{\partial x^i} + u_i\varphi = 0\}. \end{aligned}$$

To understand the meaning of this definition, let us assume e.g. that Ω is bounded with a smooth boundary $\partial\Omega$. Then an application of Stokes' formula gives us : for any $u \in \mathcal{C}^1(\overline{\Omega})$, $\forall i = 1, \cdots, n$, $\forall \varphi \in \mathcal{C}^\infty_c(\Omega)$,

$$0 = \int_{\partial\Omega} u\varphi\ \nu^i d\sigma = \int_\Omega \frac{\partial(u\varphi)}{\partial x^i} dx = \int_\Omega \left(u\frac{\partial \varphi}{\partial x^i} + \frac{\partial u}{\partial x^i}\varphi \right) dx,$$

where $d\sigma$ is the Euclidean measure on $\partial\Omega$ and $\nu = (\nu^1, \cdots, \nu^n)$ is the exterior unit normal vector to $\partial\Omega$. Since our hypotheses also implies that $u \in L^p(\Omega)$ and $\frac{\partial u}{\partial x^i} \in L^p(\Omega)$, $\forall i$, we deduce that $u \in W^{1,p}(\Omega)$ with $u_i = \frac{\partial u}{\partial x^i}$, $\forall i$. Hence in this case the u_i's coincide with the partial derivatives of u. Reversing the perspective, we may interpret the functions u_i's in the definition of $W^{1,p}(\Omega)$ as playing the rôle of partial derivatives : the u_i's are actually called the *weak partial derivatives of* u or the *partial derivatives of* u *in the sense of distributions* and we may think of $W^{1,p}(\Omega)$ as being the subspace of $L^p(\Omega)$ with partial derivatives in the sense of distributions which are in $L^p(\Omega)$. For these reasons we usually write $\frac{\partial u}{\partial x^i}$ the weak partial derivatives of u, although these functions are not partial derivatives in the classical sense.

In the following we will be concerned with the case $p = 2$. We then denote

$$H^1(\Omega) := W^{1,2}(\Omega).$$

This space is then a **Hilbert space** when endowed with the Hilbertian scalar product

$$\langle u, v\rangle_{H^1(\Omega)} := \int_\Omega uv dx + \sum_{i=1}^n \int_\Omega \frac{\partial u}{\partial x^i}\frac{\partial v}{\partial x^i} dx$$

from which the $W^{1,2} = H^1$ topology derives.

8.2 Useful results

Smooth functions are dense in the Lebesgue spaces and Sobolev spaces. More precisely, if we denote by $\mathcal{C}^\infty_c(\overline{\Omega})$ the space of $\mathcal{C}^\infty$ functions with compact support on the closure $\overline{\Omega}$ of Ω, then :

Proposition 8.1 *Let $p \in [1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary open domain. Then*

– *$\forall u \in L^p(\Omega)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ with values in $\mathcal{C}^\infty_c(\overline{\Omega})$ such that*

$$\lim_{k\to+\infty} \|u - \varphi_k\|_{L^p(\Omega)} = 0.$$

– *$\forall u \in W^{1,p}(\Omega)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ with values in $\mathcal{C}^\infty_c(\overline{\Omega})$ such that*

$$\lim_{k\to+\infty} \|u - \varphi_k\|_{W^{1,p}(\Omega)} = 0.$$

Note that we do not need to assume that the domain Ω is bounded in this result (however in the case where Ω is bounded there is no difference between smooth functions on $\overline{\Omega}$ and compactly supported smooth function on $\overline{\Omega}$). Moreover the analogous result for $p = +\infty$ is definitively *false*.

Since for $1 \leq p < +\infty$ the L^p topology 'does not see' the values of a function on a negligeable subset, we can approximate a function in $L^p(\Omega)$ by a sequence of smooth functions with compact support in Ω (not $\overline{\Omega}$!), which in particular vanish on the boundary of Ω. This is however not the case in $W^{1,p}(\Omega)$. This phenomenon reflects the fact that the $W^{1,p}$ topology feels what happens on submanifolds of codimension 1.

Proposition 8.2 *Let $p \in [1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ be an arbitrary open domain. Then*

– *$\forall u \in L^p(\Omega)$, there exists a sequence $(\varphi_k)_{k\in\mathbb{N}}$ with values in $\mathcal{C}_c^\infty(\Omega)$ such that*

$$\lim_{k\to+\infty} \|u - \varphi_k\|_{L^p(\Omega)} = 0.$$

– ***However*** *$\mathcal{C}_c^\infty(\Omega)$ is not dense in $W^{1,p}(\Omega)$*

Let us define the closure of $\mathcal{C}_c^\infty(\Omega)$ in $W^{1,p}(\Omega)$:

$$W_0^{1,p}(\Omega) := \{u \in W^{1,p}(\Omega);\ \exists(\varphi_k)_{k\in\mathbb{N}} \in \mathcal{C}_c^\infty(\Omega)^{\mathbb{N}} \text{ s.t. } \|u - \varphi_k\|_{W^{1,p}(\Omega)} = 0\}.$$

For $p = 2$ we set $H_0^1(\Omega) = W_0^{1,2}(\Omega)$. We can think on $W_0^{1,p}(\Omega)$ as the (closed) subspace of $W^{1,p}(\Omega)$ of functions which vanish on $\partial\Omega$.

Another useful result is the following

Theorem 8.1 (Rellich–Kondrakov) *Let $p \in [1, +\infty)$ and $\Omega \subset \mathbb{R}^n$ be an open domain. Assume that Ω is **bounded**. Then the inclusion map*

$$\begin{array}{rccc} \iota_p : & W^{1,p}(\Omega) & \longrightarrow & L^p(\Omega) \\ & u & \longmapsto & u \end{array}$$

*is a **compact** operator.*

The proof of this result is based on the use of Ascoli's theorem.

8.3 An example : the space $H^1([a, b])$

In order to understand Proposition 8.1 and Proposition 8.2 let us consider $H^1([a, b])$, for some interval $[a, b] \subset \mathbb{R}$. We will see in particular that $H^1([a, b]) \subset \mathcal{C}^{0,1/2}([a, b])$, where $\mathcal{C}^{0,1/2}([a, b])$ is the space of Hölder continuous functions of exponent $1/2$ (a function $f : [a, b] \longrightarrow \mathbb{R}$ is so if there exists a constant $C > 0$ such that : $\forall x, y \in [a, b]$, $|f(x) - f(y)| \le C|x - y|^{1/2}$).

1) Observe that $\mathcal{C}^1([a, b]) \subset H^1([a, b])$. We will show that

$$\forall \chi \in \mathcal{C}^1([a, b]), \quad \|\chi\|_{L^\infty} \le \left(\frac{1}{\sqrt{b-a}} + \sqrt{b-a} \right) \|\chi\|_{H^1}. \tag{32}$$

Indeed consider some $\chi \in \mathcal{C}^1([a, b])$. First we remark that :

$$\exists x_0 \in [a, b], \quad |\chi(x_0)| \le \frac{\|\chi\|_{H^1}}{\sqrt{b-a}}, \tag{33}$$

as a consequence of the inequality $\int_a^b |\chi(x)|^2 dx \le \|\chi\|_{H^1}^2$. Second we show that :

$$\forall x_1, x_2 \in [a, b], \quad |\chi(x_2) - \chi(x_1)| \le \sqrt{|x_2 - x_1|}\|\chi\|_{H^1}. \tag{34}$$

Indeed by Cauchy–Schwarz :

$$|\chi(x_2) - \chi(x_1)| = \left| \int_{x_1}^{x_2} \chi'(x)dx \right| \leq \sqrt{\int_{x_1}^{x_2} |\chi'(x)|^2 dx} \sqrt{\int_{x_1}^{x_2} dx} \leq \|\chi\|_{H^1} \sqrt{|x_2 - x_1|}.$$

Lastly (32) follows easily by writing : $\forall x_1 \in [a, b]$, $|\chi(x_1)| \leq |\chi(x_1) - \chi(x_0)| + |\chi(x_0)|$ and by using (33), (34) and the inequality $\sqrt{|x_2 - x_1|} \leq \sqrt{b - a}$.

2) Next use the Proposition 8.1, i.e. the fact that $\mathcal{C}^\infty([a, b])$ is dense in $H^1([a, b])$. Let $u \in H^1([a, b])$ and consider a sequence $(\varphi_n)_{n\in\mathbb{N}}$ with values in $\mathcal{C}^\infty([a, b])$ which converges to u in $H^1([a, b])$. Then $(\varphi_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $H^1([a, b])$ and hence also in $L^\infty([a, b])$ because of (32). Since $L^\infty([a, b])$ is a Banach space, $(\varphi_n)_{n\in\mathbb{N}}$ converges uniformly to some function $\underline{u}$, which is necessarily continous on $[a, b]$. Note that since $(\varphi_n)_{n\in\mathbb{N}}$ converges to u in $H^1([a, b])$, there exists a subsequence $(\varphi_{\psi(n)})_{n\in\mathbb{N}}$ which converges a.e. to u. Hence $\underline{u} = u$ a.e. Thus we proved that any function $u \in H^1([a, b])$ coincides a.e. with a continuous function. In other words we can always choose a function representing it which is continuous. In the following we hence assume w.l.g. that u is continuous.

3) For any $x_1, x_2 \in [a, b]$, we have

$$|u(x_2) - u(x_1)| = \lim_{n\to+\infty} |\varphi_n(x_2) - \varphi_n(x_1)| \leq \|u\|_{H^1} \sqrt{|x_2 - x_1|}, \tag{35}$$

where we passed to the limit in the following consequence of (34) :

$$|\varphi_n(x_2) - \varphi_n(x_1)| \leq \|\varphi_n\|_{H^1} \sqrt{|x_2 - x_1|}.$$

Thus we see that u is 1/2-Hölder continuous with coefficient $\|u\|_{H^1}$.

4) In particular we can interpret the definition of $H_0^1([a, b])$ as the closure of $\mathcal{C}_c^\infty((a, b))$ in $H^1([a, b])$: any function $u \in H_0^1([a, b])$ can be approximated by a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of smooth functions which satisfy in particular that $\varphi_n(a) = \varphi_n(b) = 0$. From (35) we deduce that $|u(x)| \leq \|u\|_{H^1}\sqrt{x - a}$ and $|u(x)| \leq \|u\|_{H^1}\sqrt{b - x}$. This is why it makes sense to interpret $H_0^1([a, b])$ as the subspace of functions in $H^1([a, b])$ which vanish at a and b.

8.4 Weak solutions to Dirichlet problem

Let $f \in \mathcal{C}^0(\overline{\Omega})$. The Dirichlet problem on Ω consists in finding a map u (which we may momentaneously suppose to be in $\mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$) which is a solution to

$$-\Delta u + u = f \quad \text{on } \Omega, \tag{36}$$

with the so-called *Dirichlet boundary condition* (we recall that we denote by $\partial\Omega$ the boundary of Ω) :

$$u = 0 \quad \text{on } \partial\Omega. \tag{37}$$

Assume further that Ω is bounded and that $u \in \mathcal{C}^2(\overline{\Omega})$ is a solution of (36) and (37). Then $u \in H^1(\Omega)$ and $f \in L^2(\Omega)$. Moreover (36) is equivalent to claiming that $-\Delta u + u - f$ is zero in $L^2(\Omega)$, i.e. that $-\Delta u + u - f$ is orthogonal to $\mathcal{C}_c^\infty(\Omega)$ in $L^2(\Omega)$, since $\mathcal{C}_c^\infty(\Omega)$ is dense in $L^2(\Omega)$. This reads :

$$\forall \varphi \in \mathcal{C}_c^\infty(\Omega), \quad \int_\Omega (-\varphi \Delta u + \varphi u) dx = \int_\Omega \varphi f. \tag{38}$$

We now apply Stokes' formula to the vector field $X = \varphi \nabla u$ on Ω. Denoting by ν the outward normal vector to $\partial\Omega$ and by $d\sigma$ the Euclidean measure on $\partial\Omega$, this gives us

$$\int_\Omega \mathrm{div}(\varphi \nabla u) dx = \int_{\partial\Omega} \varphi(\nu \cdot \nabla u) d\sigma,$$

which vanishes because φ vanishes on $\partial\Omega$. But on the other hand $\mathrm{div}(\varphi \nabla u) = \nabla\varphi \cdot \nabla u + \varphi \mathrm{div}(\nabla u) = \nabla \varphi \cdot \nabla u + \varphi \Delta u$. Hence we deduce the identity

$$\int_\Omega (\nabla\varphi \cdot \nabla u + \varphi \Delta u) dx = 0.$$

Hence (38) reads

$$\forall \varphi \in \mathcal{C}_c^\infty(\Omega), \quad \int_\Omega (\nabla \varphi \cdot \nabla u + \varphi u) dx = \int_\Omega \varphi f dx,$$

or in more compact notations :

$$\forall \varphi \in \mathcal{C}_c^\infty(\Omega), \quad \langle \varphi, u \rangle_{H^1(\Omega)} = \langle \varphi, f \rangle_{L^2(\Omega)}. \tag{39}$$

On the other hand the boundary condition (37) admits also the concise formulation that

$$u \in H_0^1(\Omega). \tag{40}$$

Lastly since $\mathcal{C}_c^\infty(\Omega)$ is dense in $H_0^1(\Omega)$ and condition (39) obviously involves functionals of φ which are continuous in the $H^1(\Omega)$ topology, it is clear that (39) is equivalent to

$$\forall v \in H_0^1(\Omega), \quad \langle v, u \rangle_{H^1(\Omega)} = \langle v, f \rangle_{L^2(\Omega)}. \tag{41}$$

In conclusion it is possible to formulate the Dirichlet problem (36) and (37), if Ω is bounded and for functions $u \in \mathcal{C}^2(\overline{\Omega})$, by using uniquely notions of Sobolev space. In particular (41) and (40) make sense for any function $u \in H^1(\Omega)$ and even in the case where we only assume $f \in L^2(\Omega)$.

Definition 8.1 *A weak solution to (36) and (37) is a function $u \in H_0^1(\Omega)$ which satisfies (41).*

It is now easy to prove the existence and uniqueness of a weak solution to (36) and (37). Let $f \in L^2(\Omega)$, consider the linear form

$$\begin{array}{rccl} \alpha: & H_0^1(\Omega) & \longrightarrow & \mathbb{R} \\ & v & \longmapsto & \langle v, f\rangle_{L^2(\Omega)}. \end{array}$$

This form is continuous for

$$\langle v, f\rangle_{L^2(\Omega)} \leq \|v\|_{L^2(\Omega)}\|f\|_{L^2(\Omega)} \leq \|v\|_{H^1(\Omega)}\|f\|_{L^2(\Omega)},$$

and we remark that

$$\|\alpha\|_{(H^1(\Omega))'} \leq \|f\|_{L^2(\Omega)}. \tag{42}$$

Hence by Riesz' theorem there exists an unique $u \in H_0^1(\Omega)$ such that $\forall v \in H_0^1(\Omega)$, $\langle v, u\rangle_{H^1(\Omega)} = \alpha(v)$. But this property is exactly (41) !

We note that the unique solution to in $H_0^1(\Omega)$ of (41) is obtained by first mapping $f \in L^2(\Omega)$ to $C_{L^2}(f)$ by the Riesz isomorphism $C_{L^2} : L^2(\Omega) \longrightarrow (L^2(\Omega))'$, then mapping its image by $(\iota_2)' : (L^2(\Omega))' \longrightarrow (H_0^1(\Omega))'$ (the adjoint of the injection mapping $\iota_2 : H_0^1(\Omega) \longrightarrow L^2(\Omega)$) and then the image by the inverse of the Riesz isomorphism $C_{H_0^1} : H_0^1(\Omega) \longrightarrow (H_0^1(\Omega))'$. In other words the weak solution to the Dirichlet problem (36) is given by the linear operator

$$\begin{array}{rccc} R: & L^2(\Omega) & \longrightarrow & H_0^1(\Omega) \\ & f & \longmapsto & u \end{array}$$

which is given by $R := (C_{H_0^1})^{-1} \circ (\iota_2)' \circ C_{L^2}$. This operator is continuous and its norm satisfies

$$\|R\| \leq 1 \tag{43}$$

because of (42).

8.5 The spectral decomposition of $(1 - \Delta)^{-1}$

We now define $T := \iota_2 \circ R : L^2(\Omega) \longrightarrow L^2(\Omega)$, where $\iota_2 : H^1(\Omega) \longrightarrow L^2(\Omega)$ is the injection mapping[6]. We assume that Ω is bounded, then by Theorem 8.1 ι_2 is compact. Hence, since R is bounded, T is **compact**.

We now show that T is also **self-adjoint**. For that purpose recall the characterization of R by :

$$\forall f \in L^2(\Omega), \quad \forall w \in H_0^1(\Omega), \quad \langle w, Rf\rangle_{H^1} = \langle w, f\rangle_{L^2}. \tag{44}$$

Hence, $\forall f, g \in L^2(\Omega)$, by using two times (44) (the function playing the rôle of w in (44) being indicated by a bracket)

$$\langle f, Tg\rangle_{L^2} = \langle \underbrace{Tg}_{w}, f\rangle_{L^2} = \langle \underbrace{Tg}_{w}, Rf\rangle_{H^1} = \langle \underbrace{Tf}_{w}, Rg\rangle_{H^1} = \langle \underbrace{Tf}_{w}, g\rangle_{L^2}.$$

6. Hence $T = \iota_2 \circ (C_{H_0^1})^{-1} \circ (\iota_2)' \circ C_{L^2}$.

Thus $T^* = T$. Hence we can apply Theorem 7.7 to T : we deduce that there exists a countable Hilbertian Hermitian orthogonal basis $(e_n)_{n\in\mathbb{N}}$ of $L^2(\Omega)$ and a sequence of real numbers $(\mu_n)_{n\in\mathbb{N}}$ such that, $\forall n \in \mathbb{N}$, $Te_n = \mu_n e_n$ and $\lim_{n\to+\infty} \mu_n = 0$. Let us collect some observations about the sequence $(\mu_n)_{n\in\mathbb{N}}$:

– $\forall n \in \mathbb{N}$, $e_n \in H_0^1(\Omega)$ hence we can apply (44) with $w = f = e_n$:

$$\mu_n \langle e_n, e_n\rangle_{H^1} = \langle e_n, \mu_n e_n\rangle_{H^1} = \langle e_n, Te_n\rangle_{H^1} = \langle e_n, e_n\rangle_{L^2}.$$

Hence, because of the obvious inequality $0 < 1 = \langle e_n, e_n\rangle_{L^2} \leq \langle e_n, e_n\rangle_{H^1}$, we deduce that $0 < \mu_n \leq 1$. In particular 0 is not an eigenvalue of T ;

– Moreover $\mu_n < 1$, $\forall n \in \mathbb{N}$. We can argue by contradiction : assume that there exists some $n \in \mathbb{N}$ such that $\mu_n = 1$. Then in view of the preceding inequality we would have $\|e_n\|_{H^1} = \|e_n\|_{L^2}$, which is true iff $\|\nabla e_n\|_{L^2} = 0$. One can then show that this would implies that e_n agrees with a constant a.e., but since $e_n \in H_0^1(\Omega)$ (in particular its trace on $\partial\Omega$ vanishes) this constant should be zero, i.e. $e_n = 0$, which contradicts $\|e_n\|_{L^2} = 1$.

A further result that we will admit is that e_n is actually $\mathcal{C}^\infty$ on Ω. Hence the eigenvector equation $Te_n = \mu_n e_n$ is equivalent to the classical equation

$$(1-\Delta)^{-1} e_n = \mu_n e_n \quad \Longleftrightarrow \quad e_n = \mu_n e_n - \mu_n \Delta e_n \quad \Longleftrightarrow \quad -\Delta e_n = \lambda_n e_n$$

where we set $\lambda_n := 1/\mu_n - 1$. In particular we deduce that

$$0 < \lambda_n < +\infty \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n = +\infty.$$

Conclusion — Since $(e_n)_{n\in\mathbb{N}}$ is a Hilbertian Hermitian orthogonal basis we obtain that any function $f \in L^2(\Omega)$ can be decomposed through a *Parseval identity*

$$f = \sum_{n=0}^{\infty} \widehat{f}_n e_n, \tag{45}$$

where each e_n is a smooth function on Ω which vanishes on $\partial\Omega$ and which is a solution of $-\Delta e_n = \lambda_n e_n$ on Ω. In particular

$$-\Delta f = \sum_{n=0}^{\infty} \lambda_n \widehat{f}_n e_n$$

(this series does not converge in $L^2(\Omega)$ in general but in the larger Hilbert space $H^{-2}(\Omega)$, the dual space of $H^2(\Omega)$). This leads to a representation of, e.g., a solution $u : \mathbb{R}\times\Omega \longrightarrow \mathbb{R}$ to the *wave equation* on Ω :

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} - \Delta u = 0 & \text{on } \mathbb{R}\times\Omega \\ u = 0 & \text{on } \mathbb{R}\times\partial\Omega \text{ (Dirichlet boundary conditions)} \end{cases}$$

which has the general form

$$u(t,x) = \sum_{n=0}^{\infty} \left(a_n e^{i\sqrt{\lambda_n} t} + b_n e^{-i\sqrt{\lambda_n} t} \right) e_n(x).$$

We can model small vibrations of a membrane, the frontier of which is fixed, (e.g. a drum) by solutions to the wave equation with Dirichlet boundary conditions. Then each eigenvector e_n of $-\Delta$ corresponds to a vibrating mode, with frequency $2\pi/\sqrt{\lambda_n}$. The collection of frequencies (together with the corresponding amplitudes) characterizes the sound of the drum.

Example 8.1 (Fourier decomposition) *If $\Omega = (0, 2\pi)$, $-\Delta = -\frac{d^2}{(dx)^2}$, we obtain $e_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx)$ and $\lambda_n = n^2$. Then the decomposition $f = \sum_{n=1}^{\infty} \widehat{f}_n e_n$ corresponds to the Fourier decomposition.*

9 Hilbert–Schmidt and kernel operators

A special class of compact operators.

In the following we assume that $\mathcal{H}$ is a *complex separable Hilbert space*. Hence $\mathcal{H}$ has a Hilbertian Hermitian orthogonal basis $e = (e_n)_{n \in \mathbb{N}^*}$.

9.1 Hilbert–Schmidt operators

Definition 9.1 *An operator $T \in \mathcal{L}(\mathcal{H})$ is* ***Hilbert–Schmidt*** *if for any Hilbertian Hermitian orthogonal basis $e = (e_n)_{n \in \mathbb{N}^*}$ of $\mathcal{H}$,*

$$\sum_{n=1}^{\infty} \|Te_n\|^2 < +\infty. \tag{46}$$

We denote by $\mathcal{I}_2(\mathcal{H})$ the set of Hilbert–Schmidt operators of $\mathcal{H}$.

We first show that it suffices to check Condition (46) for one Hilbertian Hermitian orthogonal basis of $\mathcal{H}$ to prove that an operator is Hilbert–Schmidt.

Proposition 9.1 *For any $T \in \mathcal{L}(\mathcal{H})$ the quantity*

$$\|T\|_e^2 := \sum_{n=1}^{\infty} \|Te_n\|^2 \in [0, +\infty) \cup \{+\infty\}$$

does not depend on the Hilbertian Hermitian orthogonal basis $e = (e_n)_{n \in \mathbb{N}^*}$.

As a consequence we have thee

Definition 9.2 *If $T \in \mathcal{L}(\mathcal{H})$ is a **Hilbert–Schmidt** operator, we define its Hilbert–Schmidt norm to be*

$$\|T\|_2 := \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{1/2},$$

where $e = (e_n)_{n\in\mathbb{N}^*}$ is a Hilbertian Hermitian orthogonal basis of $\mathcal{H}$.

Proof of Propostion 9.1 — Let $e = (e_n)_{n\in\mathbb{N}^*}$ and $f = (f_p)_{p\in\mathbb{N}^*}$ be two Hilbertian Hermitian orthogonal bases of $\mathcal{H}$. Then using Parseval's identity two times we obtain

$$\begin{aligned}
\|T\|_e^2 &= \sum_{n=1}^{\infty} \|Te_n\|^2 = \sum_{n=1}^{\infty}\left(\sum_{p=1}^{\infty} |\langle f_p, Te_n\rangle|^2\right) \\
&= \sum_{n=1}^{\infty}\left(\sum_{p=1}^{\infty} |\langle T^* f_p, e_n\rangle|^2\right) = \sum_{p=1}^{\infty}\left(\sum_{n=1}^{\infty} |\langle T^* f_p, e_n\rangle|^2\right) \\
&= \sum_{p=1}^{\infty} \|T^* f_p\|^2 = \|T^*\|_f^2.
\end{aligned}$$

Hence $\|T\|_e^2$ does not depend on e. As a bonus we immediately get the following corollary. □

Corollary 9.1 *If $T \in \mathcal{L}(\mathcal{H})$ is Hilbert–Schmidt then its adjoint is also Hilbert–Schmidt and*

$$\|T^*\|_2 = \|T\|_2.$$

As announced Hilbert–Schmidt operators are a particular class of compact operator.

Theorem 9.1 *Let $T \in \mathcal{L}(\mathcal{H})$ be a Hilbert–Schmidt operator. Then*

$$\|T\|_{\mathcal{L}(\mathcal{H})} \leq \|T\|_2 \tag{47}$$

and T is a compact operator.

Proof — We first prove (47). Let $T \in \mathcal{L}(\mathcal{H})$ be a Hilbert–Schmidt operator, let $e = (e_n)_{n\in\mathbb{N}^*}$ be any Hilbertian Hermitian orthogonal bases of $\mathcal{H}$ and let $x \in \mathcal{H}$. By using Parseval identity,

$$\begin{aligned}
\|Tx\|^2 &= \sum_{n=1}^{\infty} |\langle e_n, Tx\rangle|^2 = \sum_{n=1}^{\infty} |\langle T^* e_n, x\rangle|^2 \\
&\leq \sum_{n=1}^{\infty} \|T^* e_n\|^2 \|x\|^2 = \|T^*\|_2^2 \|x\|^2 = \|T\|_2^2 \|x\|^2.
\end{aligned}$$

To prove that T is compact it suffices to prove that the sequence of finite rank operators $(T_n)_{n\in\mathbb{N}^*}$ defined by :

$$\forall x \in \mathcal{H}, \quad T_n x := \sum_{j=1}^{n} \langle e_j, Tx\rangle e_j$$

converges to T in the $\mathcal{L}(\mathcal{H})$ topology. For that purpose, observe that, since $\sum_{j=1}^{\infty} \|T^*e_j\|^2$ is a convergent series, $\forall \varepsilon > 0$, $\exists N \in \mathbb{N}^*$ such that, $\forall n \geq N$, $\sum_{j=n+1}^{\infty} \|T^*e_j\|^2 < \varepsilon^2$. Hence by a calculation which is similar to the previous one we have

$$\|(T - T_n)x\|^2 = \sum_{j=n+1}^{\infty} |\langle e_j, Tx\rangle|^2 \leq \sum_{j=n+1}^{\infty} \|T^*e_j\|^2\|x\|^2 \leq \varepsilon^2\|x\|^2.$$

Hence $\|T - T_n\| \leq \varepsilon$. □

Theorem 9.2 *The set $\mathcal{I}_2(\mathcal{H})$ is a vector subspace of $\mathcal{L}(\mathcal{H})$. Moreover it satisfies the following properties*

(i) $\forall T, S \in \mathcal{I}_2(\mathcal{H})$,

$$\sum_{n=1}^{\infty} |\langle Te_n, Se_n\rangle| \leq \|T\|_2\|S\|_2.$$

(ii) as a consequence the sesquilinear form

$$\langle T, S\rangle_2 := \sum_{n=1}^{\infty} \langle Te_n, Se_n\rangle$$

is well-defined, furthermore it does not depend on the choice of the Hilbertian Hermitian orthogonal basis $e = (e_n)_{n\in\mathbb{N}^*}$.

(iii) $(\mathcal{I}_2(\mathcal{H}), \langle\cdot,\cdot\rangle_2)$ *is a Hilbert space.*

Proof — For (i) we use Parseval identity :

$$\begin{aligned}
\sum_{n=1} \|(\lambda T + \mu S)e_n\|^2 &= \sum_{n,m=1}^{\infty} |\langle e_m, (\lambda T + \mu S)e_n\rangle|^2 \\
&\leq \sum_{n,m=1}^{\infty} |\lambda|^2|\langle e_m, Te_n\rangle|^2 + |\mu|^2|\langle e_m, Se_n\rangle|^2 = 2\left(|\lambda|^2\|T\|_2^2 + |\mu|^2\|S\|_2^2\right).
\end{aligned}$$

For (ii) we use first Cauchy–Schwarz in $\mathcal{H}$ and then in $\ell^2(\mathbb{N})$:

$$\begin{aligned}
\sum_{n=1}^{\infty} |\langle Te_n, Se_n\rangle| &\leq \sum_{n=1}^{\infty} \|Te_n\|\,\|Se_n\| \\
&\leq \left(\sum_{n=1}^{\infty} \|Te_n\|^2\right)^{1/2} \left(\sum_{n=1}^{\infty} \|Se_n\|^2\right)^{1/2} = \|T\|_2\|S\|_2.
\end{aligned}$$

(iii) follows from Proposition 9.1 and the following polarization formula :

$$\sum_{n=1}^{\infty} \langle Te_n, Se_n\rangle = \frac{1}{4}\left(\|T + S\|_e^2 - \|T - S\|_e^2 + i\|T + iS\|_e^2 - i\|T - iS\|_e^2\right).$$

(iv) is left to the Reader : the main point to check is that $(\mathcal{I}_2(\mathcal{H}), \langle\cdot,\cdot\rangle_2)$ is complete. For that purpose one first shows that it $(T_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, then, for any fixed $p \in \mathbb{N}^*$ the sequence $(T_ne_p)_{n\in\mathbb{N}^*}$ is Cauchy. One then concludes using a diagonal subsequence argument. □

9.2 Kernel operators

In the following (X,μ) denotes a measured space and $\mathcal{H} = L^2(X,\mu,\mathbb{C}) \simeq L^2(X,\mu)$. For any $K \in L^2(X\times X, \mu\otimes\mu, \mathbb{C})$ we define

$$\begin{array}{rccc} T_K : & \mathcal{H} & \longrightarrow & \mathcal{H} \\ & f & \longmapsto & T_K f \end{array}$$

where

$$\text{for } \mu\text{-a.e. } x \in X, \quad T_K f(x) = \int_X K(x,y)f(y)d\mu(y).$$

T_K is called a *kernel operator*. Any such operator is bounded as can be seen by using Fubini's theorem :

$$\text{for } \mu\text{-a.e. } x \in X, \quad [y \longmapsto K(x,y)] \in L^2(X,\mu)$$

and by using also Cauchy–Schwarz' inequality

$$\begin{aligned} |T_K f(x)|^2 &= \left(\int_X K(x,y)f(y)d\mu(y)\right)^2 \leq \int_X |K(x,y)|^2 d\mu(y) \int_X |f(y)|^2 d\mu(y) \\ &= \|f\|^2_{L^2(X,\mu)} \int_X |K(x,y)|^2 d\mu(y). \end{aligned}$$

Hence

$$\int_X |T_K(x)|^2 d\mu(x) \leq \|f\|^2_{L^2(X,\mu)} \int_X\int_X |K(x,y)|^2 d\mu\otimes d\mu(x,y),$$

i.e. $\|T_K f\|_{L^2(X,\mu)} \leq \|K\|_{L^2(X\times X,\mu\otimes\mu)}\|f\|_{L^2(X,\mu)}$. Hence we deduce that T_K is bounded and

$$\|T_K\|_{\mathcal{L}(\mathcal{H})} \leq \|T\|_{L^2(X\times X,\mu\otimes\mu)}. \tag{48}$$

Theorem 9.3 *Any kernel operator is Hilbert–Schmidt.*

Proof — Let $(e_n)_{n\in\mathbb{N}^*}$ be a Hilbertian Hermitian orthogonal basis of $\mathcal{H}$. Then for any $n \in \mathbb{N}^*$,

$$\text{for } \mu\text{-a.e. } x \in X, \quad T_K e_n(x) = \int_X K(x,y)e_n(y)d\mu(y),$$

hence for any $n, m \in \mathbb{N}^*$,

$$\begin{aligned} \langle e_m, T_K e_n\rangle &= \int_X \overline{e_m(x)} \int_X K(x,y)e_n(y)d\mu(y)d\mu(x) \\ &= \int_X\int_X K(x,y)\overline{e_m(x)}e_n(y)mu\otimes d\mu(x,y) \\ &= \langle e_m\otimes\overline{e_n}, K\rangle_{L^2(X\times X,\mu\otimes\mu)} \end{aligned}$$

But $(e_m\otimes\overline{e_n})_{n,m\in\mathbb{N}^*}$ is a Hilbertian Hermitian orthogonal basis of $L^2(X\times X,\mu\otimes\mu)$ (we admit it). Thus by using Parseval

$$\|T_K e_n\|^2_{L^2(X,\mu)} = \sum_{m=1}^{\infty} |\langle e_m, T_K e_n\rangle|^2 = \sum_{m=1}^{\infty} |\langle e_m\otimes\overline{e_n}, K\rangle_{L^2(X\times X,\mu\otimes\mu)}|^2.$$

By using again Parseval

$$\sum_{n=1}^{\infty} \|T_K e_n\|^2_{L^2(X,\mu)} = \sum_{n,m=1}^{\infty} |\langle e_m \otimes \overline{e_n}, K\rangle_{L^2(X\times X,\mu\otimes\mu)}|^2 = \|K\|^2_{L^2(X\times X,\mu\otimes\mu)}.$$

This proved more than the fact that T_K is Hilbert–Schmidt :

$$\|T_K\|_2 = \|K\|_{L^2(X\times X,\mu\otimes\mu)}, \tag{49}$$

i.e. the map

$$\begin{array}{ccc} L^2(X\times X,\mu\otimes\mu) & \longrightarrow & \mathcal{I}_2(\mathcal{H}) \\ K & \longmapsto & T_K \end{array} \tag{50}$$

is **an isometry**. □

Actually kernel operators with kernels in $L^2(X\times X,\mu\otimes\mu)$ are more or less the same thing as Hilbert–Schmidt operators, as shown by the following.

Theorem 9.4 *Let $\mathcal{H} = L^2(X,\mu,\mathbb{C}) \simeq L^2(X,\mu)$. Then any Hilbert–Schmidt operator on $\mathcal{H}$ is a kernel operator with a kernel in $L^2(X\times X,\mu\otimes\mu)$.*

Proof — Since the map in (50) is an isometry, it is one-to-one and its image is closed in $(\mathcal{I}_2(\mathcal{H}), \langle,\cdot,\cdot\rangle_2)$ (see Lemma 3.1). Thus it suffices to show that its image is dense in $(\mathcal{I}_2(\mathcal{H}), \langle,\cdot,\cdot\rangle_2)$ to prove that this map is an isomorphism. This can be done more or less as in the proof of Theorem 9.1 : take any $T \in \mathcal{I}_2(\mathcal{H})$ and define the sequence of operators $(T_n)_{n\in\mathbb{N}^*}$ by

$$\text{for } \mu\text{-a.e. } x \in X, \quad T_n f(x) = \sum_{j=1}^{n} \langle e_j, Tf\rangle_{\mathcal{H}} e_j(x).$$

Then one checks easily that, on the one hand, this sequence converges to T in $(\mathcal{I}_2(\mathcal{H}), \langle,\cdot,\cdot\rangle_2)$ and, on the other hand, that each T_n is a kernel operator with kernel K_n s.t.

$$\text{for } \mu\otimes\mu\text{-a.e. } (x,y) \in X\times X, \quad K_n(x,y) = \sum_{j=1}^{n} e_j(x)\overline{T^* e_j(y)}.$$

□

10 Interlude : real and positive operators

Lemma 10.1 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$. Assume that*

$$\forall x \in \mathcal{H}, \quad \langle x, Tx\rangle \in \mathbb{R}. \tag{51}$$

Then T is self-adjoint.

Proof — Take any $x, y \in \mathcal{H}$. From (51) we deduce that $\langle x+y, T(x+y)\rangle \in \mathbb{R}$ thus

$$\langle x, Ty\rangle + \langle y, Tx\rangle \in \mathbb{R} \quad \Longleftrightarrow \quad \mathrm{Im}\langle x, Ty\rangle = -\mathrm{Im}\langle y, Tx\rangle.$$

We deduce also that $\langle x+iy, T(x+iy)\rangle \in \mathbb{R}$ thus

$$i\langle x, Ty\rangle - i\langle y, Tx\rangle \in \mathbb{R} \quad \Longleftrightarrow \quad \mathrm{Re}\langle x, Ty\rangle = \mathrm{Re}\langle y, Tx\rangle.$$

Hence $\langle x, Ty\rangle = \overline{\langle y, Tx\rangle} = \langle Tx, y\rangle$. □

Definition 10.1 *Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$, then T is* ***positive*** *if*

$$\forall x \in \mathcal{H}, \quad \langle x, Tx\rangle \geq 0.$$

As a consequence of Lemma 10.1, any positive operator is self-adjoint.

10.1 Self-adjoint operator and physics

Lemma 10.1 and Theorem 6.6 are the reasons why self-adjoint operators are so important in mathematical physics and in particular in quantum physics.

Indeed classical mechanics was based on the definition of quantities such as the position of a particle, its energy, its momentum, etc. and on mathematical relations between these quantities which take into account the kinematic and express the laws of dynamics. But all the involved *observable* quantities were *real* numbers.

In quantum mechanics, the situation is the same, excepted that the observable should be represented by mathematical objects which are more complex than real numbers. Werner Heisenberg discovered these objects, Max Born and Pascual Jordan recognized that they are analogous to matrices and Paul Dirac called them q-numbers. The precise mathematical description was done by John Von Neumann : observable quantities are described by operators A acting on some Hilbert space $\mathcal{H}$.

The physical state of a particle, which was represented by its position and its velocity in classical mechanics, is now representated by a non vanishing vector $\varphi \in \mathcal{H}$. The relation of these data with experiments is now much more subtle as in classical mechanics. In particular if we know that a particle is, at some time, in a state $\varphi \in \mathcal{H}$, then we are not be able in general to predict precisely the result of an experimental measurement of the observable quantity associated to A. Indeed in general such experiments give randomly different results for the same φ. However the result of each measure will be in any case a spectral value of A. Moreover the random distribution of the possible results follows a probability law which we are able to predict. In particular the average will be :

$$\frac{\langle \varphi, A\varphi\rangle}{\langle \varphi, \varphi\rangle}.$$

Of course, we always observe real numbers! By Lemma 7.1 and Theorem 6.6 this forces A to be self-adjoint.

11 The spectral decomposition of self-adjoint bounded operators

11.1 Functional calculus for self-adjoint bounded operators

The question is to make sense of $f(A)$ for A a self-adjoint operator and f a continuous function of the real variable. This question has its origin in the paradigm of Quantum Mechanics where observable real numbers are replaced by self-ajdoint operators and, as a consequence, it is natural to consider continous functions of self-adjoint operators. The a priori unexpected gain is that this is the key to understand the spectral decomposition of bounded self-adjoint operators.

Theorem 11.1 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. Assume that $A^* = A$. Then there exists an unique operator*

$$\begin{array}{rccc} \Phi_A : & \mathcal{C}^0(SpA, \mathbb{C}) & \longrightarrow & \mathcal{L}(\mathcal{H}) \\ & f & \longmapsto & \Phi_A(f) \end{array}$$

such that the following properties hold.

a) $\forall f, g \in \mathcal{C}^0(SpA, \mathbb{C})$, $\forall \alpha, \beta \in \mathbb{R}$,

$$\begin{array}{l} \Phi_A(\alpha f + \beta g) = \alpha \Phi_A(f) + \beta \Phi_A(g), \\ \Phi_A(fg) = \Phi_A(f)\Phi_A(g), \quad \Phi_A(1) = 1_{\mathcal{H}}, \\ \Phi_A(\overline{f}) = \Phi_A(f)^*. \end{array}$$

The two first line means that Φ_A is an algebra homomorphism, with the last condition, we speak of a C^*-algebra homomorphism ;

b) *Φ_A is continuous and more precisely*

$$\|\Phi_A(f)\|_{\mathcal{L}(\mathcal{H})} = \|f\|_{\mathcal{C}^0(SpA,\mathbb{C})}. \tag{52}$$

c) *For $f = X|_{SpA}$ (the restriction of $X : \mathbb{R} \ni \lambda \longmapsto \lambda \in \mathbb{R}$ to SpA), we have $\Phi_A(X|_{SpA}) = A$.*

Moreover Φ_A satisfies the following conditions :

d) *For any eigenvalue λ of A, $\forall \psi \in \mathcal{H}$, $A\psi = \lambda\psi \Longrightarrow \Phi_A(f)\psi = f(\lambda)\psi$;*

e) *The spectrum of $\Phi_A(f)$ is equal to the image of SpA by f, i.e.*

$$Sp\,(\Phi_A(f)) = \{f(\lambda);\, \lambda \in SpA\};$$

f) *If $f \in \mathcal{C}^0(SpA, \mathbb{C})$ is positive, then $\Phi_A(f)$ is positive, i.e. $f \geq 0 \Longrightarrow \Phi_A(f) \geq 0$.*

In fact all these properties mean clearly that, for any continous function f, $\Phi_A(f)$ satisfies all reasonable conditions we could think about $f(A)$. Hence we will simply write

$$f(A) := \Phi_A(f)$$

in the following. Then all properties of Φ_A listed in Theorem 11.1 can be translated into

a) $\forall f, g \in \mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$, $\forall \alpha, \beta \in \mathbb{R}$,

$$\begin{aligned}&(\alpha f + \beta g)(A) = \alpha f(A) + \beta g(A),\\ &(fg)(A) = f(A)g(A), \quad 1(A) = 1_{\mathcal{H}},\\ &\overline{f}(A) = f(A)^*.\end{aligned}$$

b) $\|f(A)\|_{\mathcal{L}(\mathcal{H})} = \|f\|_{\mathcal{C}^0(\mathrm{Sp}A,\mathbb{C})}$;

c) $X|_{\mathrm{Sp}A}(A) = A$.

d) $A\psi = \lambda\psi \Longrightarrow f(A)\psi = f(\lambda)\psi$;

e) $\mathrm{Sp}\,(f(A)) = \{f(\lambda);\, \lambda \in \mathrm{Sp}A\}$;

f) $f \geq 0 \Longrightarrow f(A) \geq 0$.

Proof — We will first define $f(A)$ in the case where f is the restriction of a polynomial on $\mathrm{Sp}A$, which is an easy task. Then we will prove that this definition extends in an unique way to a continuous mapping Φ_A on the whole space $\mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$. This follows from two facts : first the Stone–Weierstrass theorem which implies that the subspace of polynomial functions on $\mathrm{Sp}A$ is dense in $\mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$, second the fact that the restriction of Φ_A on the subspace of polynomial functions is continuous. This latter task requires more work.

Step 1 — We let $\mathbb{C}[X]$ be the space of complex polynomials and we denote by $\mathbb{C}[X]|_{SpA}$ the space of maps from $\mathrm{Sp}A$ to $\mathbb{C}$ which are restrictions of complex polynomial maps. We endow $\mathbb{C}[X]|_{SpA}$ with the topology of $\mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$.

Assuming **a)**, **b)**, and **c)** we have no choice for defining the restriction of Φ_A on $\mathbb{C}[X]|_{SpA}$: rule $\Phi_A(1) = 1_{\mathcal{H}}$ in **a)** and condition **c)** gives us initial conditions. Then the rule $\Phi_A(fg) = \Phi_A(f)\Phi_A(g)$ implies by recursion that $\Phi_A(X^n|_{SpA}) = A^n$ and rule $\Phi_A(\alpha f + \beta g) = \alpha\Phi_A(f) + \beta\Phi_A(g)$ implies that $\Phi_A(P|_{SpA}) = P(A)$, for any $P \in \mathbb{C}[X]$.

Since $\mathrm{Sp}A$ is a compact subset of $\mathbb{R}$, Stone–Weierstrass theorem implies that $\mathbb{C}[X]|_{SpA}$ is dense in $\mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$. Hence we now need to prove the following weak version of **b)** :

b') $\forall P \in \mathbb{C}[X], \quad \|P(A)\|_{\mathcal{L}(\mathcal{H})} = \|P\|_{\mathcal{C}^0(\mathrm{Sp}A,\mathbb{C})}$,

which is itself a consequence of the following weak version of **e)** :

e') $\forall P \in \mathbb{C}[X], \quad \mathrm{Sp}\,(P(A)) = \{P(\lambda);\, \lambda \in \mathrm{Sp}A\}$.

Step 2 — Proof of **e')**. We first prove, for any $P \in \mathbb{C}[X]$, the inclusion $\mathrm{Sp}P(A) \supset \{P(\lambda);\, \lambda \in \mathrm{Sp}A\}$. Let $\lambda \in \mathrm{Sp}A$ and $P \in \mathbb{C}[X]$ (w.l.g. we assume P to be non constant, the case where P is constant being easy and left to the Reader). Consider the polynomial $S(X) = P(X) - P(\lambda)$. Obviously λ is a root of S, hence these exists a polynomial Q such that $P(X) - P(\lambda) = (X - \lambda)Q(X)$. Applying this identity to A we get

$$P(A) - P(\lambda) = (A - \lambda)Q(A).$$

Since $\lambda \in \mathrm{Sp}A$, $A - \lambda$ is not invertible, hence the preceding identity implies that $P(A) - P(\lambda)$ is not also invertible. Thus $P(\lambda) \in \mathrm{Sp}P(A)$.

We now prove the reverse inclusion $\mathrm{Sp}P(A) \subset \{P(\lambda);\, \lambda \in \mathrm{Sp}A\}$, for any $P \in \mathbb{C}[X]$. Again we assume that P is not constant and ask the Reader to check the easy case where

P is constant. Assume that $\mu \in \text{Sp}P(A)$ and consider the polynomial $S(X) = P(X) - \mu$. Then we can decompose $P(X) - \mu = S(X) = a(X - \lambda_1) \cdots (X - \lambda_n)$, where $a \in \mathbb{C} \setminus \{0\}$ and $\lambda_1, \cdots, \lambda_n$ are the roots of S. Applying this to A, we get

$$P(A) - \mu = a(A - \lambda_1) \cdots (A - \lambda_n).$$

Since $\mu \in \text{Sp}P(A)$, $P(A) - \mu$ is not invertible, which implies by the preceding identity that there exists at least some value λ_j such that $A - \lambda_j$ is not invertible, i.e. $\lambda_j \in \text{Sp}A$. On the other hand we have $S(\lambda_j) = 0$, from which we deduce that $\mu = P(\lambda_j)$. Hence $\mu \in \{P(\lambda);\ \lambda \in \text{Sp}A\}$.
Step 3 — Proof of **b')**. Let $P \in \mathbb{C}[X]$. Then

$$\begin{aligned} \|P(A)\|^2_{\mathcal{L}(\mathcal{H})} &= \sup_{\|x\| \le 1} \|P(A)x\|^2 = \sup_{\|x\| \le 1} \langle P(A)x, P(A)x \rangle \\ &= \sup_{\|x\| \le 1} \langle x, P(A)^* P(A)x \rangle = \|P(A)^* P(A)\|_{\mathcal{L}(\mathcal{H})}, \end{aligned}$$

where, in the last equality, we have used the fact that $P(A)^*P(A)$ is self-adjoint and Lemma 5.2. But using the fact hat $A^* = A$, we have $P(A)^* = \overline{P}(A)$, where, if $P(X) = \sum_{j=0}^n a_j X^j$, $\overline{P}(X) := \sum_{j=0}^n \overline{a_j} X^j$ or, equivalentely, $\forall \lambda \in \mathbb{C}$, $\overline{P}(\overline{\lambda}) = \overline{P(\lambda)}$. Hence

$$P(A)^* P(A) = \overline{P}(A) P(A) = |P|^2(A),$$

where $|P|^2 := \overline{P}P$. Note that $|P|^2$ satisfies the property : $|P|^2(\lambda) := |P(\lambda)|^2$, $\forall \lambda \in \mathbb{R}$. We now use the fact that $|P|^2(A)$ is self-adjoint and Corollary 5.1 :

$$\|P(A)\|^2_{\mathcal{L}(\mathcal{H})} = r(|P(A)|^2) = \sup_{\mu \in \text{Sp}|P|^2(A)} |\mu|,$$

but using **e')**, which implies $\text{Sp}\left(|P|^2(A)\right) = \{|P|^2(\lambda) = |P(\lambda)|^2;\ \lambda \in \text{Sp}A\}$, this gives us

$$\|P(A)\|^2 = \sup_{\lambda \in \text{Sp}A} |P(\lambda)|^2.$$

Hence

$$\|P(A)\|^2_{\mathcal{L}(\mathcal{H})} = \left(\sup_{\lambda \in \text{Sp}A} |P(\lambda)| \right)^2 = \|P|_{\text{Sp}A}\|^2_{\mathcal{C}^0}$$

and **b')** is proved.
Step 4 — Conclusion. We can now extend in an unique way Φ_A to $\mathcal{L}(\mathcal{H})$ in such a way that **b)** is satisfied. It is then easy to check **a)**. One needs also to check **d)**, **e)** and **f)**. All these properties are easy to check, excepted **e)**, which we won't prove. Let's just prove **f)** : let $f \in \mathcal{C}^0(\text{Sp}A, \mathbb{R})$ be a non negative function. Then there exists an unique continuous non negative function g on $\text{Sp}A$ such that $g^2 = f$. Hence, $\forall \psi \in \mathcal{H}$,

$$\langle \psi, f(A)\psi \rangle = \langle \psi, g^2(A)\psi \rangle = \langle \psi, g(A)^* g(A)\psi \rangle_= |g(A)\psi|^2 \ge 0,$$

i.e. $f(A)$ is non negative. □

11.2 The spectral measure and the Riesz–Markov theorem

Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. For any $\psi \in \mathcal{H}$, consider the map

$$\begin{array}{rccc} \ell_\psi : & \mathcal{C}^0(\mathrm{Sp}A, \mathbb{C}) & \longrightarrow & \mathbb{C} \\ & f & \longmapsto & \langle \psi, f(A)\psi \rangle \end{array}$$

then it follows from Theorem 11.1 that ℓ_ψ is a continous linear map on $\mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$. Moreover this map is nonnegative in the sense that

$$\forall f \in \mathcal{C}^0(\mathrm{Sp}A, \mathbb{R}), \quad f \geq 0 \quad \Longrightarrow \langle \psi, f(A)\psi \rangle \geq 0,$$

as a consequence of **e)**. Such a map is an example of a *Radon* measure.

Definition 11.1 (Radon measure) *If X is a (Borelian) subset of $\mathbb{R}^n$, a* ***nonnegative Radon measure on*** *X is a* ***nonnegative continous linear form*** *ℓ* ***on the space*** *$\mathcal{C}_c(X)$* ***of continuous functions with compact support in*** *X, where $\mathcal{C}_c(X)$ is endowed with the topology of uniform convergence on any compact subset of X. More precisely for any compact $K \subset X$, $\exists C_K > 0$ such that $\forall f \in \mathcal{C}_c(X)$, if the support of f is contained in K, then $|\ell(f)| \leq C_K \|f\|_{\mathcal{C}(K)}$.*
We denote by $\mathfrak{M}^+(X)$ the set of nonnegative Radon measures on $\mathbb{R}^n$.

Note that in the case where K is compact (e.g. if $K = \mathrm{Sp}A$) then a nonnegative linear form on $\mathcal{C}_c(K)$ is a nonnegative Radon measure if there exists a constant $C > 0$ such that $\forall f \in \mathcal{C}_c(X)$, $|\ell(f)| \leq C\|f\|_{\mathcal{C}(K)}$.

The Riesz–Markov Theorem says that any nonnegative Radon measure on X can be represented by using a nonnegative Borelian measure μ on X. In order to fully appreciate this result, it is worth to recall some notions of measure theory.

(i) given a set X, a **σ-algebra of** X is a collection $\mathcal{A}$ of subsets of X which contains $\emptyset$ and X, is stable by taking complements, finite intersections and countable unions (actually these properties imply also that Ω is stable by countable intersections) ;

(ii) a **nonnegative measure on** $(X, \mathcal{A})$ is a map $\mu : \mathcal{A} \longrightarrow [0, +\infty)$ such that, for all collection $(A_i)_{i\in I}$ of subsets in $\mathcal{A}$ which is at most countable, if $i \neq j \Longrightarrow A_i \cap A_j = \emptyset$, then $\mu\left(\cup_{i\in I} A_i\right) = \sum_{i\in I} \mu(A_i)$ (we then say that μ is σ-additive) ;

(iii) the **Borelian σ-algebra of** $\mathbb{R}^n$ (or on a compact subset $K \subset \mathbb{R}^n$) is the smallest σ-algebra which contains the open subsets of $\mathbb{R}^n$ (or K) ;

(iv) a **Borelian measure on** $\mathbb{R}^n$ (or on a compact subset $K \subset \mathbb{R}^n$) is a measure on the Borelian σ-algebra of $\mathbb{R}^n$ (or K).

Theorem 11.2 *Let X be a Borelian subset[7] of $\mathbb{R}^n$ and let ℓ be a nonnegative Radon measure on X, i.e. a nonnegative continuous linear form on $\mathcal{C}_c(X)$. Then there exists an unique nonnegative Borelian measure μ on X such that*

$$\forall f \in \mathcal{C}_c(X), \quad \ell(f) = \int_X f(x) d\mu(x). \tag{53}$$

7. Actually the definition of Borelian measure and the Riesz–Markov theorem extend to the case where X is a topological Hausdorff space.

Moreover μ is finite (i.e. $\mu(X) < +\infty$) and satisfies

$$\mu(A) = \sup\{\mu(K);\ K \text{ is a compact subset of } X \text{ s.t. } K \subset A\}, \tag{54}$$

and

$$\mu(A) = \inf\{\mu(U);\ U \text{ is an open subset of } X \text{ s.t. } U \supset A\}, \tag{55}$$

Conversely any finite nonnegative Borelian measure which satisfies (54) and (55) defines a Radon measure through (53).

When applied to ℓ_ψ on SpA, Theorem 11.2 gives us the existence of a Borelian measure μ_ψ such that

$$\forall f \in \mathcal{C}^0(\mathrm{Sp}A, \mathbb{C}), \quad \langle \psi, f(A)\psi \rangle = \int_{\mathrm{Sp}A} f(\lambda) d\mu_\psi(\lambda). \tag{56}$$

Definition 11.2 *The nonnegative Borelian measure μ_ψ on SpA defined by (56) and satisfying (54) and (55) is called the* ***spectral measure associated with*** *ψ.*

11.3 Cyclic vectors

Definition 11.3 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. A vector $\psi \in \mathcal{H}$ is* ***cyclic*** *for A if the vector subspace*

$$F_\psi := Vec\{A^n\psi;\ n \in \mathbb{N}\} = \{P(A)\psi;\ P \in \mathbb{C}[X]\}$$

is dense in $\mathcal{H}$.

Theorem 11.3 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Assume that there exists vector $\psi \in \mathcal{H}$ which is* ***cyclic*** *for A. Then there exists a positive Borelian measure μ on SpA with finite mass and an unitary map*

$$U : \mathcal{H} \longrightarrow L^2(SpA, \mu, \mathbb{C})$$

such that,

$$\forall f \in L^2(SpA, \mu, \mathbb{C}), \quad (UAU^{-1}f)(\lambda) = \lambda f(\lambda), \quad \text{for } \mu\text{-a.e. } \lambda \in SpA, \tag{57}$$

or equivalentely $UAU^{-1} = M_X$.

Remark — For any bounded μ-measurable function g on SpA we denote by

$$\begin{array}{rccc} M_g : & L^2(\mathrm{Sp}A, \mu, \mathbb{C}) & \longrightarrow & L^2(\mathrm{Sp}A, \mu, \mathbb{C}) \\ & f & \longmapsto & gf \end{array}$$

the operator of multiplication by g. Such operators play the role of 'diagonal matrices'. If $P \in \mathbb{C}[X]$ we denote by $P|_{\mathrm{Sp}A}$ or, abusing notations, simply by P the restriction of P to SpA. Here we meet the monomial $X : \lambda \longmapsto \lambda$.

Proof — We take $\mu = \mu_\psi$, the spectral measure associated with ψ and defined by (56). We start by defining [8]

$$\begin{array}{rccc} V: & \mathbb{C}[X]|_{\mathrm{Sp}A} & \longrightarrow & \mathcal{H} \\ & P & \longmapsto & P(A)\psi \end{array}$$

where $\mathbb{C}[X]|_{\mathrm{Sp}A} := \{P|_{\mathrm{Sp}A};\ P \in \mathbb{C}[X]\} \subset L^2(\mathrm{Sp}A, \mu, \mathbb{C})$. As we shall see later on V can be interpreted as the restriction to $\mathbb{C}[X]|_{\mathrm{Sp}A}$ of U^{-1}.

Observe that the image of V is F_ψ. We first show that V is an isometry from $(\mathbb{C}[X]|_{\mathrm{Sp}A}, L^2(\mathrm{Sp}A))$ to F_ψ. Indeed, $\forall P \in \mathbb{C}[X]$,

$$\begin{aligned} \|V(P)\|^2_{\mathcal{H}} &= \|P(A)\psi\|^2_{\mathcal{H}} = \langle \psi, P(A)^* P(A)\psi\rangle_{\mathcal{H}} \\ &= \langle \psi, \overline{P}(A)P(A)\psi\rangle_{\mathcal{H}} = \langle \psi, (\overline{P}P)(A)\psi\rangle_{\mathcal{H}} \\ &= \int_{\mathrm{Sp}A} (\overline{P}P)(\lambda) d\mu_\psi(\lambda), \end{aligned}$$

where we have used the definition (56) of μ_ψ. Hence

$$\|V(P)\|^2_{\mathcal{H}} = \int_{\mathrm{Sp}A} |P(\lambda)|^2 d\mu_\psi(\lambda) = \|P\|^2_{L^2(\mathrm{Sp}A)}.$$

Now since, by the Stone–Weierstrass theorem, $\mathbb{C}[X]|_{\mathrm{Sp}A}$ is dense in $\mathcal{C}^0(\mathrm{Sp}A, \mathbb{C})$, which is itself dense in $L^2(\mathrm{Sp}A, \mu, \mathbb{C})$, there exists an unique continous extension of V, that we will denote by $\overline{V}$, from $L^2(\mathrm{Sp}A, \mu, \mathbb{C})$ to $\mathcal{H}$ of V, i.e. $\overline{V}$ coincides with V on $\mathbb{C}[X]|_{\mathrm{Sp}A}$ and satisfies

$$\forall f \in L^2(\mathrm{Sp}A, \mu, \mathbb{C}), \quad \|\overline{V}(f)\|_{\mathcal{H}} = \|f\|_{L^2(\mathrm{Sp}A)}. \tag{58}$$

This implies in particular that $\overline{V}$ satisfies Hypothesis (i) in Corollary 3.1. But the image of $\overline{V}$ contains the image of V, i.e. F_ψ. Since ψ is cyclic the image of $\overline{V}$ is thus dense in $\mathcal{H}$, i.e. $\overline{V}$ fulfills Hypothesis (ii) in Corollary 3.1. Hence $\overline{V}$ is invertible. Let us denote by U its inverse. We deduce from (58) that U is an isometry.

It remains to prove (57). From the relations

$$\begin{aligned} V(P) &= P(A)\psi \\ V(XP) &= AP(A)\psi \end{aligned}$$

we deduce $V(XP) = AV(P)$ or $U^{-1}(XP) = AU^{-1}(P)$, $\forall P \in \mathbb{C}[X]$. Hence by density :

$$\forall f \in L^2(\mathrm{Sp}A, \mu, \mathbb{C}), \quad U^{-1}(M_X f) = AU^{-1}(f),$$

which gives us (57) by right composition with U. □

---

8. Note that we may identify $\mathbb{C}[X]|_{\mathrm{Sp}A}$ with the quotient of $\mathbb{C}[X]$ by the equivalence relation $P \sim Q \iff (P-Q)|_{\mathrm{Sp}A} = 0$. Hence the map V can be defined by first defining $\mathbb{C}[X] \ni P \longmapsto P(A) \in \mathcal{L}(\mathcal{H})$ and second by showing that this map is constant on the equivalence classes. But this follows by applying Property **e')** in the proof of Theorem 11.1 to the polynomial $|P-Q|^2$.

11.4 The meaning of the existence of a cyclic vector

Let us consider the previous construction in the case where the Hilbert space $\mathcal{H}$ has a finite dimension equal to n and let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then we know that we can diagonalize A in a Hermitian orthogonal basis, i.e. we can find such a basis $(e_1, \cdots, e_n)$ and n real eigenvalues $(\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$ such that, $\forall j = 1, \cdots, n$, $Ae_j = \lambda_j e_j$.

Consider some $\psi \in \mathcal{H}$ and let us see under which condition ψ is a cyclic vector for A. Because of the Cayley–Hamilton theorem we have $P_A(A) = 0$ where

$$P_A(X) := \det(X1_{\mathcal{H}} - A) = X^n - (\mathrm{tr}A)X^{n-1} + \cdots + (-1)^n \det A$$

is the characteristic polynomial of A. Hence $A^n = (\mathrm{tr}A)A^{n-1} + \cdots - (-1)^n \det A$ and

$$F_\psi = \mathrm{Vec}\{A^j\psi; j \in \mathbb{N}\} = \mathrm{Vec}(\psi, A\psi, \cdots, A^{n-1}\psi).$$

Thus ψ is cyclic, i.e. $F_\psi = \mathcal{H}$, iff $(\psi, A\psi, \cdots, A^{n-1}\psi)$ is a basis of $\mathcal{H}$. In a decomposition $\psi = \psi^1 e_1 + \cdots + \psi^n e_n$ we have $A^p\psi = \lambda_1^p \psi^1 e_1 + \cdots + \lambda_n^p \psi^n e_n$. Hence we see that the latter is equivalent to the condition

$$\begin{vmatrix} \psi^1 & \lambda_1\psi^1 & \cdots & \lambda_1^{n-1}\psi^1 \\ \vdots & \vdots & & \vdots \\ \psi^n & \lambda_n\psi^n & \cdots & \lambda_n^{n-1}\psi^n \end{vmatrix} = \psi^1 \cdots \psi^n \begin{vmatrix} 1 & \lambda_1 & \cdots & \lambda_1^{n-1} \\ \vdots & \vdots & & \vdots \\ 1 & \lambda_n & \cdots & \lambda_n^{n-1} \end{vmatrix} \neq 0.$$

We find a Vandermond determinant. Hence ψ is cyclic iff $\psi^1 \cdots \psi^n \neq 0$ and all eigenvalues $\lambda_1, \cdots, \lambda_n$ are pairwise disjoints. We conclude that if the eigenspaces of A are all one dimensional, then the Vandermond determinant does not vanish and there exist infinitely many cyclic vectors (it suffices that $\langle e_j, \psi\rangle \neq 0$, $\forall j = 1, \cdots, n$). However if there exists at least one multiple eigenvalue, then the Vandermond determinant vanishes and there is no cyclic vector.

In the case where all eigenvalues are pairwise disjoint the unitary operator U of Theorem 11.3 maps $\mathcal{H}$ to $L^2(\{\lambda_1, \cdots, \lambda_n\}, \mathbb{C})$ in such a way that $(Ue_j)(\lambda_k) = \delta_{jk}$ (0 if $j \neq k$, 1 if $j = k$). For a given cyclic vector $\psi = \psi^1 e_1 + \cdots + \psi^n e_n$ the corresponding measure is

$$\mu_\psi = |\psi^1|^2 \delta_{\lambda_1} + \cdots + |\psi^n|^2 \delta_{\lambda_n},$$

where δ_λ is the Dirac mass at λ. Hence by knowing the measure μ_ψ we can recover all components of ψ up to a diagonal $U(1)^n$ action on its components.

11.5 Spectral decomposition in the general case

In the case where there is no cyclic vector for A the idea is to use several vectors ψ_i such that the sum $\bigoplus_{i\in I} F_{\psi_i}$ is orthogonal and dense in $\mathcal{H}$. The construction of such a family of vectors $(\psi_i)_{i\in I}$ rests on the use of Zorn's theorem (itself equivalent to the axiom of choice). We first recall a definition :

Let $(\mathcal{D}, \leq)$ be a partially ordered set. We say that $(\mathcal{D}, \leq)$ is **inductive** if any totally ordered subset of $\mathcal{D}$ has a majorant.

Proposition 11.1 (Zorn's theorem) *Any inductive partially ordered set $(\mathcal{D}, \leq)$ has a maximal element, i.e. there exists some $\overline{D} \in \mathcal{D}$ such that : $\forall D \in \mathcal{D}$, if D is comparable to $\overline{D}$, then $D \leq \overline{D}$.*

We also need the following

Lemma 11.1 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Let $\phi, \psi \in \mathcal{H}$. Then*

$$\psi \perp F_\phi \quad \Longrightarrow \quad F_\psi \perp F_\phi.$$

Proof — The condition $\psi \perp F_\phi$ means : $\forall n \in \mathbb{N}$, $\langle A^n\phi, \psi\rangle = 0$. It implies

$$\forall n, m \in \mathbb{N}, \quad \langle A^n\phi, A^m\psi\rangle = \langle (A^*)^m A^n\phi, \psi\rangle = \langle A^{n+m}\phi, \psi\rangle = 0.$$

□

We deduce :

Proposition 11.2 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then there exists a collection $(\psi_i)_{i\in I}$ of vectors in $\mathcal{H}$ such that $\forall i, j \in I$, $i \neq j \Longrightarrow F_{\psi_i} \perp F_{\psi_j}$ and*

$$\bigoplus_{i\in I} F_{\psi_i} \text{ is dense in } \mathcal{H}.$$

Proof — We let $S_\mathcal{H} := \{\phi \in \mathcal{H}; \|\phi\| = 1\}$ (the unit sphere in $\mathcal{H}$) and we set

$$\mathcal{D} := \{D \subset S_\mathcal{H}; \forall \phi, \psi \in D, \phi \neq \psi \Longrightarrow F_\phi \perp F_\psi\}.$$

We observe that $(\mathcal{D}, \subset)$ is partially ordered and inductive. Indeed any totally ordered subset $\mathcal{D}_0$ of $\mathcal{D}$ has a majorant, which is $\cup_{D\subset\mathcal{D}_0} D$. By Zorn's theorem there exists a maximal element $\overline{D} \in \mathcal{D}$. We let

$$V := \bigoplus_{\phi\in\overline{D}} F_\phi.$$

We claim that V is dense in $\mathcal{H}$. Argue by contradiction : if not, there exists $\psi \in S_\mathcal{H}$ such that $\psi \perp V$. By Lemma 11.1 this implies that $F_\psi \perp F_\phi$, $\forall \phi \in \overline{D}$. Thus $\overline{D} \cup \{\psi\} \in \mathcal{D}$ contains strictly $\overline{D}$, which contradicts the fact that $\overline{D}$ is maximal. □

Remark — The set I may be infinite and even non countable. However if $\mathcal{H}$ is separable, I is at most countable.

Theorem 11.4 *Let $\mathcal{H}$ be a complex separable Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then there exists a finite or countable family $(\mu_n)_{n\in\mathcal{N}}$ of Borelian measures on SpA with finite masses and a unitary map*

$$U : \mathcal{H} \longrightarrow \text{the closure of } \bigoplus_{n\in\mathcal{N}} L^2(SpA, \mu_n, \mathbb{C})$$

such that, denoting by $f = (f_n)_{n\in\mathcal{N}}$ (where $\forall n \in \mathcal{N}$, $f_n \in L^2(SpA, \mu_n, \mathbb{C})$) the decomposition of any element f in the closure of $\bigoplus_{n\in\mathcal{N}} L^2(SpA, \mu_n, \mathbb{C})$

$$\forall f = (f_n)_{n\in\mathcal{N}}, \quad (UAU^{-1}f)_n = M_X f_n, \quad \forall n \in \mathcal{N}. \tag{59}$$

Proof — We use Proposition 11.2. Since $\mathcal{H}$ is separable, there exists some subset $\mathcal{N}$ which is at most countable (we may assume w.l.g. that $\mathcal{N}$ is a subset of $\mathbb{N}$) and a family $(\psi_n)_{n\in\mathcal{N}}$ of vectors in $S_{\mathcal{H}}$ such that $\forall n,m \in \mathcal{N}$, $n \neq m \Longrightarrow F_{\psi_n} \perp F_{\psi_m}$ and $\bigoplus_{n\in\mathcal{N}} F_{\psi_n}$ is dense in $\mathcal{H}$. Each subspace $\overline{F_{\psi_n}}$ is stable by A, hence, for any $n \in \mathcal{N}$, we may consider the restriction A_n of A to $\overline{F_{\psi_n}}$. Then for any $n \in \mathcal{N}$, ψ_n is obvious cyclic for A_n, thus we can apply Theorem 11.3 to A_n and ψ_n : there exists a Borelian measure μ_n with finite mass on SpA and a unitary map $U_n : \overline{F_{\psi_n}} \longrightarrow L^2(\text{Sp}A, \mu_n, \mathbb{C})$ such that $U_n A_n U_n^{-1} = M_X$ (acting on $L^2(\text{Sp}A, \mu_n, \mathbb{C})$). Collecting all these decompositions together we obtain U satisfying (59). □

We can represent differentely the closure of $\bigoplus_{n\in\mathcal{N}} L^2(\text{Sp}A, \mu_n, \mathbb{C})$ by considering the set $X := \mathcal{N} \times \text{Sp}A$ equipped with the measure $\nu = \sum_{n\in\mathcal{N}} \delta_n \otimes \mu_n$, i.e. such that, for any $h : \mathcal{N} \times \text{Sp}A \longrightarrow [0, +\infty)$,

$$\int_X h(n,\lambda) d\nu(n,\lambda) := \sum_{n\in\mathcal{N}} \int_{\text{Sp}A} h(n,\lambda) d\mu_n(\lambda).$$

Thus the closure of $\bigoplus_{n\in\mathcal{N}} L^2(\text{Sp}A, \mu_n, \mathbb{C})$ can be identified with $L^2(X, \nu, \mathbb{C})$, through the map $(h_n)_{n\in\mathcal{N}} \longmapsto [(x,\lambda) \longmapsto h(n,\lambda) = h_n(\lambda)]$. Then the unitary operator U in Theorem 11.4 maps $\mathcal{H}$ to $L^2(X, \nu, \mathbb{C})$ and (59) reads $UAU^{-1} = M_g$, where $\forall (n,\lambda) \in \mathcal{N} \times \text{Sp}A$, $g(n,\lambda) = \lambda$. Note that g takes values in Sp$A \subset \mathbb{R}$.

However the measured space (X, ν) has the defect that it has not a finite mass in general. This can be cured by considering the measure $\mu := \sum_{n\in\mathcal{N}} 2^{-n} \delta_n \otimes \mu_n$ on X. Then (X, μ) has a finite mass and

$$\begin{array}{rccl} T: & L^2(X,\nu,\mathbb{C}) & \longrightarrow & L^2(X,\mu,\mathbb{C}) \\ & f & \longmapsto & [Tf : (n,\lambda) \longmapsto \sqrt{2}^n f(n,\lambda)] \end{array}$$

is an isometry. Moreover if $\tilde{U} := TU : \mathcal{H} \longrightarrow L^2(X, \mu, \mathbb{C})$, we then have $\tilde{U} A \tilde{U}^{-1} = M_g$. Hence we have proved the following.

Theorem 11.5 *Let $\mathcal{H}$ be a complex separable Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then there exists a measured space (X, μ) with finite mass, a measurable bounded function $g : X \longrightarrow \mathbb{R}$ and a unitary map $U : \mathcal{H} \longrightarrow L^2(X, \mu, \mathbb{C})$ such that $UAU^{-1} = M_g$.*

11.6 The Borelian functional calculus

Up to now we have used the spectral measure μ_ψ associated to a vector ψ (or a family $(\psi_i)_{i\in I}$) as in Theorem 11.4. We now use this construction for all vectors $\psi \in \mathcal{H}$: this leads us to a generalization of the continuous functional calculus, the Borelian functional calculus. For any Borelian subset $X \subset \mathbb{R}^n$ denote by

$$\mathcal{B}(X, \mathbb{C})$$

the space of Borelian bounded functions from X to $\mathbb{C}$. We address the question : given a self-adjoint operator A and a bounded Borelian measurable function $g \in \mathcal{B}(\text{Sp}A, \mathbb{C})$, can we make sense of $g(A)$? The answer is *yes* and this will be achieved by defining $g(A)$ to be the solution of

$$\forall \psi \in \mathcal{H}, \quad \langle \psi, g(A)\psi \rangle = \int_{\mathrm{Sp}A} g(\lambda) d\mu_\psi(\lambda). \tag{60}$$

In the following, for all Borelian subset $X \subset \mathbb{R}^n$, we denote by $\mathfrak{M}^{\mathbb{C}}(X)$ the complex vector space of the Borelian measures [9] on X which are complex linear combinations of nonnegative Radon measures on X. In particular a complex Borelian measure $\mu \in \mathfrak{M}^{\mathbb{C}}(X)$ has a finite mass and satisfies (54) and (55) because of Theorem 11.2. (We will need these properties later on.)

Moreover for any $\mu \in \mathfrak{M}^{\mathbb{C}}(X, \mathbb{C})$ and $g \in \mathcal{B}(X, \mathbb{C})$, we use the more concise notation

$$\mu(g) := \int_X g(x) d\mu(x).$$

11.6.1 Analysis of the problem

Assume that we know a solution $g(A)$ to (60). Then we know all quantities $\langle \varphi, g(A)\psi \rangle$, for $\varphi, \psi \in \mathcal{H}$. Indeed from the identity

$$\begin{aligned} \langle \varphi, g(A)\psi \rangle &= \frac{1}{2} \left[\langle \varphi + \psi, g(A)(\varphi + \psi) \rangle - \langle \varphi, g(A)\varphi \rangle - \langle \psi, g(A)\psi \rangle \right] \\ &\quad + \frac{i}{2} \left[\langle i\varphi + \psi, g(A)(i\varphi + \psi) \rangle - \langle i\varphi, g(A)(i\varphi) \rangle - \langle \psi, g(A)\psi \rangle \right] \end{aligned}$$

we deduce that, since $g(A)$ is a solution of (60),

$$\langle \varphi, g(A)\psi \rangle = \frac{1}{2} \left(\mu_{\varphi+\psi}(g) - \mu_\varphi(g) - \mu_\psi(g) \right) + \frac{i}{2} \left(\mu_{i\varphi+\psi}(g) - \mu_{i\varphi}(g) - \mu_\psi(g) \right).$$

Thus by setting, $\forall \varphi, \psi \in \mathcal{H}$,

$$\mu_{(\varphi,\psi)} := \frac{1}{2} \left(\mu_{\varphi+\psi} - \mu_\varphi - \mu_\psi \right) + \frac{i}{2} \left(\mu_{i\varphi+\psi} - \mu_{i\varphi} - \mu_\psi \right), \tag{61}$$

we have

$$\forall \varphi, \psi \in \mathcal{H}, \quad \langle \varphi, g(A)\psi \rangle = \mu_{(\varphi,\psi)}(g) = \int_{\mathrm{Sp}A} g(\lambda) d\mu_{(\varphi,\psi)}(\lambda). \tag{62}$$

In particular all that implies easily that if a solution to (60) exists, then it is unique.

---

9. An alternative equivalent definition of $\mathfrak{M}^{\mathbb{C}}(X)$ is that it is the space of *complex Radon measures* : a complex linear form ℓ on $\mathcal{C}_0(X, \mathbb{C})$ is a **complex Radon measure** if it is continuous for the topology of uniform convergence on any compact, i.e. for any compact $K \subset X$, $\exists C_K > 0$ such that $\forall f \in \mathcal{C}_c(X, \mathbb{C})$, if $\mathrm{supp}(f) \subset K$, then $|\ell(f)| \leq C_K \|f\|_{\mathcal{C}^0}$. If furthermore $f \in \mathcal{C}_c(\mathbb{R}^n, \mathbb{R})$, $\ell(f) \in \mathbb{R}$, then ℓ is a real Radon measure ℓ on X. We denote by $\mathfrak{M}^{\mathbb{R}}(\mathbb{R}^n)$ the real space of real Radon measures

11.6.2 Synthesis

Recall that we were able to construct all measures μ_ψ, thanks to the continuous functional calculus and to the Riesz–Markov theorem, an hence all measures $\mu_{(\varphi,\psi)}$ by using (61). This gives us the map

$$\begin{array}{rccl} \mathfrak{m}_A: & \mathcal{H}\times\mathcal{H} & \longrightarrow & \mathfrak{M}^{\mathbb{C}}(\mathrm{Sp}A) \\ & (\varphi,\psi) & \longmapsto & \mathfrak{m}_A(\varphi,\psi) = \mu_{(\varphi,\psi)}. \end{array}$$

Hence we can construct all integrals $\int_{\mathrm{Sp}A} g(\lambda)d\mu_{(\varphi,\psi)}(\lambda)$, for $\varphi,\psi\in\mathcal{H}$ and $g\in\mathcal{B}(\mathrm{Sp}A,\mathbb{C})$. However we need that this collection of complex numbers satisfy some compatibility conditions in order to be sure that (62) be satisfied by some bounded linear operator $g(A)$. These conditions are :

$$\mathfrak{m}_A \quad \text{is sesquilinear,} \tag{63}$$

$$\forall g\in\mathcal{B}(\mathrm{Sp}A,\mathbb{C}),\quad \exists C_g>0 \text{ s.t.,}\quad \left|\int_{\mathrm{Sp}A} g(\lambda)d\mu_{(\varphi,\psi)}(\lambda)\right| \le C_g\|\varphi\|\,\|\psi\|. \tag{64}$$

The fact that (63) and (64) are necessary conditions for having a solution $g(A)$ to (62) is obvious (with $C_g = \|g(A)\|$). The fact that these conditions are sufficient follows from the Riesz–Fréchet theorem : for any $\varphi\in\mathcal{H}$, consider the map $L_\varphi:\mathcal{H}\longrightarrow\mathbb{C}$ defined by

$$\forall\psi\in\mathcal{H},\quad L_\varphi(\psi) := \int_{\mathrm{Sp}A} g(\lambda)d\mu_{(\varphi,\psi)}(\lambda).$$

Then L_φ is $\mathbb{C}$-linear because of (63) and bounded because of (64) with the estimate $\|L_\varphi\|\le C_g\|\varphi\|$. Hence, by the Riesz–Fréchet theorem, there exists an unique vector $\omega\in\mathcal{H}$ such that $L_\varphi(\psi) = \langle\omega,\psi\rangle$, $\forall\psi\in\mathcal{H}$. The map $\varphi\longmapsto\omega$ is linear thanks to (63) and bounded thanks to (64). Hence it is a bounded operator : call it $g(A)^*$, this hence defines $g(A)$, which is the solution of (62). Note that (64) implies also that

$$\|g(A)\|\le C_g. \tag{65}$$

11.6.3 Proving (63) and (64)

These properties will be proved by using the following result.

Lemma 11.2 *Let X be a Borelian subset of $\mathbb{R}^m$. Then for any function $g\in\mathcal{B}(X,\mathbb{C})$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ with values in $\mathcal{C}_c(X,\mathbb{C})$ such that*

$$\forall n\in\mathbb{N},\quad \sup_X|f_n|\le\sup_X|g|, \tag{66}$$

$$\forall\mu\in\mathfrak{M}^C(X,\mathbb{C}),\quad \lim_{n\to+\infty}\mu(f_n) = \mu(g). \tag{67}$$

For any sequence $(f_n)_{n\in\mathbb{N}}$ with values in $\mathcal{C}_c(X,\mathbb{C})$ we will write

$$f_n \xrightarrow[b]{\star} g \tag{68}$$

if (66) and (67) hold.

Remark — We will discuss about the proof of Lemma 11.2 later on. It is important to stress the fact that, as a consequence of the Riesz–Markov theorem 11.2 all measures $\mu_{(\varphi,\psi)}$ are in $\mathfrak{M}^C(\mathrm{Sp}A,\mathbb{C})$, so that we may apply (67) to these measures.

Proving (63) amounts to show that $\forall g \in \mathcal{B}(\mathrm{Sp}A,\mathbb{C})$, $(\varphi,\psi) \longmapsto \mathfrak{m}_A(\varphi,\psi)(g)$ is sesquilinear. Use Lemma 11.2 to get a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{C}_c(X,\mathbb{C})$ such that (67) holds. Then, $\forall n \in \mathbb{N}$, the identity $\mathfrak{m}_A(\varphi,\psi)(f_n) = \langle\varphi, f_n(A)\psi\rangle$ is valid and implies obviously that $\mathfrak{m}_A(\alpha\varphi_1+\beta\varphi_2,\psi)(f_n) = \overline{\alpha}\mathfrak{m}_A(\varphi_1,\psi)(f_n)+\overline{\beta}\mathfrak{m}_A(\varphi_2,\psi)(f_n)$, $\forall\alpha,\beta\in\mathbb{C}$ and $\mathfrak{m}_A(\psi,\varphi)(f_n) = \overline{\mathfrak{m}_A(\varphi,\psi)(f_n)}$. Passing to the limit when $n\to+\infty$ in these identities gives us (63).

Let us now prove (64). Take any function $g \in \mathcal{B}(\mathrm{Sp}A,\mathbb{C})$ and again a a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{C}_c(X,\mathbb{C})$ such that (66) and (67) hold. Then, $\forall\varphi,\psi\in\mathcal{H}$, $\forall n\in\mathbb{N}$,

$$|\mu_{(\varphi,\psi)}(f_n)| = |\langle\varphi, f_n(A)\psi\rangle| \le \|f_n(A)\|\|\varphi\|\|\psi\| = \sup_{\mathrm{Sp}A}|f_n|\|\varphi\|\|\psi\|,$$

where we have used (52). Using (66) this implies

$$|\mu_{(\varphi,\psi)}(f_n)| \le \sup_{\mathrm{Sp}A}|g|\|\varphi\|\|\psi\|.$$

Passing to the limit when $n\to+\infty$ and using (67) we obtain

$$|\mu_{(\varphi,\psi)}(g)| \le \sup_{\mathrm{Sp}A}|g|\|\varphi\|\|\psi\|.$$

Hence (64) holds with

$$C_g = \sup_{\mathrm{Sp}A}|g| = \|g\|_{\mathrm{sup}}. \tag{69}$$

11.6.4 Conclusion

Assuming that Lemma 11.2 has been proved, for any $g \in \mathcal{B}(\mathrm{Sp}A,\mathbb{C})$, (63) and (64) hold, hence there exists an unique bounded operator $g(A)\in\mathcal{L}(\mathcal{H})$ such that (60) holds. Moreover we deduce from (65) and (69) that

$$\|g(A)\| \le \sup_{\mathrm{Sp}A}|g| = \|g\|_{\mathrm{sup}}. \tag{70}$$

We have thus constructed an operator $g\longmapsto g(A)$. It satisfies the following properties.

Theorem 11.6 *Let $\mathcal{H}$ be a complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$. Assume that $A^* = A$. Then there exists an unique operator*

$$\begin{array}{ccc} \mathcal{B}(SpA, \mathbb{C}) & \longrightarrow & \mathcal{L}(\mathcal{H}) \\ g & \longmapsto & g(A) \end{array}$$

such that the following properties hold.

a) $\forall g, h \in \mathcal{B}(SpA, \mathbb{C})$, $\forall \alpha, \beta \in \mathbb{R}$,

$$\begin{array}{l} (\alpha g + \beta h)(A) = \alpha g(A) + \beta h(A), \\ (gh)(A) = g(A)h(A), \quad 1(A) = 1_{\mathcal{H}}, \\ \overline{g}(A) = g(A)^*. \end{array}$$

b) $\|g(A)\|_{\mathcal{L}(\mathcal{H})} \leq \|g\|_{sup}$ *;*

c) *if g is continuous $g(A)$ coincides with the operator defined in Theorem 11.1 ;*

d) *if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $\mathcal{B}(SpA, \mathbb{C})$ such that, $\forall \lambda \in SpA$, $\lim_{n\to+\infty} f_n(\lambda) = f(\lambda)$ and if $\|f_n\|_{sup}$ is bounded, then*[10]*, $\forall \varphi, \psi \in \mathcal{H}$,*

$$\lim_{n\to+\infty} \langle \varphi, f_n(A)\psi \rangle = \langle \varphi, f(A)\psi \rangle; \tag{71}$$

e) *if $A\psi = \lambda\psi$ then $g(A)\psi = g(\lambda)\psi$;*

f) *if $f \geq 0$ then $f(A) \geq 0$;*

g) *if $BA = AB$, then $g(A)B = Bg(A)$.*

Proof — The existence of the operator and property **b)** has already been proved in (70). The proof of **a)** is easy excepted for the product property. The latter one can be obtained by proving the property

$$\forall f, g \in \mathcal{B}(\mathrm{Sp}A, \mathbb{C}), \forall \varphi, \psi \in \mathcal{H}, \quad \langle \varphi, (fg)(A)\psi \rangle = \langle \varphi, f(A)g(A)\psi \rangle \tag{72}$$

in three steps with increasing generality.

(i) if $f, g \in \mathcal{C}(\mathrm{Sp}A, \mathbb{C})$, (72) is an obvious consequence of property **a)** in Theorem 11.1 ;

(ii) if $f \in \mathcal{B}(\mathrm{Sp}A, \mathbb{C})$ and $g \in \mathcal{C}(\mathrm{Sp}A, \mathbb{C})$. As a preliminary we prove the identity

$$\forall f, g \in \mathcal{B}(\mathrm{Sp}A, \mathbb{C}), \forall \varphi, \psi \in \mathcal{H}, \quad \langle \varphi, f(A)g(A)\psi \rangle = \overline{\mu_{(g(A)\psi,\varphi)}(\overline{f})}. \tag{73}$$

Indeed $\langle \varphi, f(A)g(A)\psi \rangle = \langle f(A)^*\varphi, g(A)\psi \rangle = \overline{\langle g(A)\psi, \overline{f}(A)\varphi \rangle} = \overline{\mu_{(g(A)\psi,\varphi)}(\overline{f})}$. By Lemma 11.2 there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{C}(\mathrm{Sp}A, \mathbb{C})$ such that $f_n \xrightarrow[b]{\star} f$, thus by (73)

$$\langle \varphi, f_n(A)g(A)\psi \rangle = \overline{\mu_{(g(A)\psi,\varphi)}(\overline{f_n})} \to \overline{\mu_{(g(A)\psi,\varphi)}(\overline{f})} = \langle \varphi, f(A)g(A)\psi \rangle.$$

10. In [3], it is claimed that $f_n(A)$ converges to $f(A)$ *strongly*, which is false, see the footnote in Theorem 11.3, **c)**.

Moreover we also deduce directly from $f_n \xrightarrow[b]{\star} f$ that $f_n g \xrightarrow[b]{\star} fg$, hence

$$\langle \varphi, (f_n g)(A)\psi \rangle = \mu_{(\varphi,\psi)}(f_n g) \to \mu_{(\varphi,\psi)}(fg) = \langle \varphi, (fg)(A)\psi \rangle.$$

Since by the previous step (72) holds for f_n and g, we deduce that f and g satisfies also (72) by passing to the limit.

(iii) if $f, g \in \mathcal{B}(\mathrm{Sp}A, \mathbb{C})$, we use again Lemma 11.2 to get a sequence $(g_n)_{n\in\mathbb{N}}$ with values in $\mathcal{C}(\mathrm{Sp}A, \mathbb{C})$ such that $g_n \xrightarrow[b]{\star} g$. We have on the one hand

$$\langle \varphi, f(A)g_n(A)\psi \rangle = \langle f(A)^*\varphi, g_n(A)\psi \rangle \longrightarrow \langle f(A)^*\varphi, g(A)\psi \rangle = \langle \varphi, f(A)g(A)\psi \rangle$$

and on the other hand, because $fg_n \xrightarrow[b]{\star} fg$,

$$\langle \varphi, (fg_n)(A)\psi \rangle = \mu_{(\varphi,\psi)}(fg_n) \to \mu_{(\varphi,\psi)}(fg) = \langle \varphi, (fg)(A)\psi \rangle.$$

By the previous step we can apply (72) to f and g_n and we can pass to the limit thanks to the previous observations : we then conclude that f and g satisfy (72).

Property **c)** is clear from the construction. Property **d)** follows from Lebesgue's dominated convergence which implies the convergence of $\int_{\mathrm{Sp}A} f_n(\lambda) d\mu_{(\varphi,\psi)}$ to $\int_{\mathrm{Sp}A} f(\lambda) d\mu_{(\varphi,\psi)}$. This translates as $\langle \varphi, f_n(A)\psi \rangle \to \langle \varphi, f(A)\psi \rangle$. We left properties **e)** and **f)** to the Reader.

Property **g)** is easy to prove for f being the restriction of a polynomial to $\mathrm{Sp}A$ and for f being continuous by using Theorem 11.1. Hence it remains to prove it for any f in $\mathcal{B}(\mathrm{Sp}A, \mathbb{C})$ as follows : this amounts indeed to prove that $\langle \varphi, Bf(A)\psi \rangle = \langle \varphi, f(A)B\psi \rangle$. First observe the identities

$$\langle \varphi, Bf(A)\psi \rangle = \langle B^*\varphi, f(A)\psi \rangle = \mu_{(B^*\varphi,\psi)}(f)$$

and

$$\langle \varphi, f(A)B\psi \rangle = \langle f(A)^*\varphi, B\psi \rangle = \overline{\langle B\psi, \overline{f}(A)\varphi \rangle} = \overline{\mu_{(B\psi,\varphi)}(\overline{f})}.$$

Take a sequence $(f_n)_{n\in\mathbb{N}}$ with values in $\mathcal{C}(\mathrm{Sp}A, \mathbb{C})$ such that $f_n \xrightarrow[b]{\star} f$. It satisfies $\langle \varphi, Bf_n(A)\psi \rangle = \langle \varphi, f_n(A)B\psi \rangle$, which translates as $\mu_{(B^*\varphi,\psi)}(f_n) = \overline{\mu_{(B\psi,\varphi)}(\overline{f_n})}$. But in this form, it is easy to pass to the limit by using the fact that $f_n \xrightarrow[b]{\star} f$. □

11.6.5 About the proof of Lemma 11.2

Two proofs can be given, the first one uses measure theory, the second one tools from functional analysis.

Sketch of proof 1 — We first prove the result for $g = 1_A$ being the indicator function of a Borelian subset $A \subset X$, i.e. the function defined by $1_A(x) = 1$ if $x \in A$ and $1_A(x) = 0$ if $x \in X \setminus A$. Any nonnegative measure $\mu \in \mathfrak{M}^+(X, \mathbb{C})$ satisfies the *inner regularity* property (54) and the *outer regularity* property (55). This implies that, $\forall \varepsilon > 0$, there exists a compact $K \subset X$ and an open subset $U \subset X$ such that $K \subset A \subset U$ and $\mu(A \setminus K) \leq \varepsilon/2$

and $\mu(U \setminus A) \leq \varepsilon/2$. Hence $\mu(U \setminus K) = \mu(U \setminus A) + \mu(A \setminus K) \leq \varepsilon$. Moreover one can construct a continuous function $f \in \mathcal{C}_c(X)$ which takes values in $[0,1]$ and such that $f = 1$ on K and $f = 0$ on $X \setminus U$. Then since $f - 1_A$ vanishes on $(X \setminus U) \cup K = X \setminus (U \setminus K)$, we have

$$|\mu(f) - \mu(1_A)| = \int_X |f(x) - 1_A(x)| d\mu(x) = \int_{U\setminus K} |f(x) - 1_A(x)| d\mu(x) \leq \mu(U \setminus K) \leq \varepsilon.$$

Simultanously we have also $\sup_X |f| = 1 \leq \sup_X 1_A$. Since any measure in $\mathfrak{M}^{\mathbb{C}}(X, \mathbb{C})$ is a complex linear combination of nonnegative measures in $\mathfrak{M}^+(X, \mathbb{C})$, a similar result follows for such measures. Hence by letting $\varepsilon = 1/n$, for $n \in \mathbb{N}^*$, we obtain a sequence $(f_n)_{n\in\mathbb{N}^*}$ satisfying (66) and (67) for $g = 1_A$.

In a second step one extends this result fo an arbitrary bounded Borelian function g on X, using the fact that such functions can be approached by finite linear combinations of indicator functions. □

Sketch of proof 2 — It rests on the following general result :
Lemma (see Proposition V.4.1 in [1]) *Let E be a Banach space, E' its dual space and E'' its bidual space. Let $\iota : E \longrightarrow E''$ be the canonical embedding, $\overline{B}_E$ be the unit closed ball in E and $\overline{B}_{E''}$ the unit closed ball in E''. Then $\iota(\overline{B}_E)$ is dense in $\overline{B}_{E''}$ for the topology $\sigma(E'', E')$.*

It is perhaps worth to recall some notions of functional analysis :

(i) if E and F is a pair of vector spaces in duality, i.e. endowed with a bilinear map $\beta : E \times F \longrightarrow \mathbb{C}$, then the topology $\sigma(E, F)$ is the coarser topology on E such that, $\forall y \in F$, the map $E \ni x \longmapsto \beta(x, y)$ is continuous. A basis of neighbourhoods of a point $x_0 \in E$ in this topology is the collection of finite intersections $\cap_{i=1}^n \{x \in E;\, |\beta(x - x_0, y_i)| \leq \varepsilon_i\}$, where $y_i \in F$ and $\varepsilon_i > 0$. The topology $\sigma(E'', E')$ is also called the weak star topology on E''.

(ii) a *locally convex* topological space is a vector space endowed with the topology induced by a family $(N_i)_{i\in I}$ of semi-norms. Simplest examples are normed vector spaces. But a vector space E endowed with the $\sigma(E, F)$ is also a locally convex vector space, the family of semi-norm being $N_y = |\beta(\cdot, y)|$, for $y \in F$. Hence $(E'', \sigma(E'', E'))$ is locally convex.

(iii) A geometric form of the Hahn–Banach theorem [11] holds in locally convex spaces : if A and B are two closed convex subset which are disjoint and if one of them is compact, say A, then there exists a hyperplane which separates strictly A and B.

The preceding lemma can be proved by contradiction : let β be the closure of $\iota(\overline{B}_E)$ in the $\sigma(E'', E')$ topology, then β is $\sigma(E'', E')$-closed and convex. Assume that there exists some $x'' \in \overline{B}_{E''}$ which not contained in β. Then by using the geometric Hahn–Banach theorem in the locally convex space $(E'', \sigma(E'', E'))$ one can find a $\sigma(E'', E')$-closed hyperplane $H \subset E''$ which separates strictly β and $A = \{x''\}$. Hence $\exists y' \in E'$,

11. Note that the geometric Hahn–Banach for separating open convex disjoint subsets holds in normed vector spaces but not in locally convex spaces in general.

$\exists r \in \mathbb{R}$ such that $\forall x \in \beta$, $y'(x) < r < y'(x'')$. Necessarily $r > 0$. By rescaling r and y' we can assume w.l.g. that $y'(x'') = 1$ so that $r \in (0,1)$. But we then have on the one hand : $\|y'\|_{E'} = \sup_{x\in B_E(0,1)} |y'(x)| \leq \sup_{x\in\beta} |y'(x)| \leq r < 1$ and, on the other hand, $1 = y'(x'') \leq \|y'\|_{E'}\|x''\|_{E''}$ which implies $\|y'\|_{E'} \geq 1$ since $\|x''\|_{E''} \leq 1$. Hence we reach a contradiction.

Now consider the case when $E = \mathcal{C}_c(X,\mathbb{C})$. Then E' can be identified with the space of complex Radon measures $\mathfrak{M}^{\mathbb{C}}(X,\mathbb{C})$. The space E'' then contains strictly $\mathcal{B}(X,\mathbb{C})$ as a subspace. By applying the previous result to this case, we obtain Lemma 11.2. □

11.7 Applications of the Borelian functional calculus

11.7.1 Measures with values in projections

A first application of the Borelian functional calculus is another point of view on the spectral decomposition of self-adjoint operators. To understand the idea, assume that $\mathcal{H}$ a finite dimensional space Hilbert space and let $A \in \mathcal{L}(\mathcal{H})$ be self-adjoint. Then there exists a finite collection of real eigenvalues $\lambda_1 < \lambda_2 < \cdots < \lambda_p$ (possibly with multiplicity) and a collection of corresponding eigenspaces $E_{\lambda_1}, \cdots, E_{\lambda_p} \subset \mathcal{H}$ which are pairwise orthogonal and such that $\mathcal{H} = E_{\lambda_1} \oplus \cdots \oplus E_{\lambda_p}$. A way to represent this decomposition is to introduce the orthogonal projections $P_{\{\lambda_1\}}, \cdots, P_{\{\lambda_p\}}$, where each operator $P_{\{\lambda_j\}} \in \mathcal{L}(\mathcal{H})$ is the orthogonal projection onto E_j. Then we have

$$A = \lambda_1 P_{\{\lambda_1\}} + \cdots + \lambda_p P_{\{\lambda_p\}}.$$

When passing to an infinite dimensional Hilbert space the difficulty is to adapt such a formula for a spectrum which can be any arbitrary compact subset of $\mathbb{R}$ and for spectral values which may not be eigenvalues. This is done by constructing a measure P on the σ-algebra of Borelian subset of SpA with values in orthogonal projections in $\mathcal{H}$. In the case of finite dimensional space, this measure associates to each subset $\{\lambda_{i_1}, \cdots, \lambda_{i_k}\} \subset \mathrm{Sp}A$ the orthogonal projection $P_{\{\lambda_{i_1},\cdots,\lambda_{i_k}\}}$ onto the subspace $E_{\{\lambda_{i_1}\}} \oplus \cdots \oplus E_{\{\lambda_{i_k}\}}$.

Now let $\mathcal{H}$ be an arbitrary complex Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be a self-adjoint operator. Let $\mathcal{A}_{\mathrm{Sp}A}$ be the set of all Borelian subsets of SpA. We apply Theorem 11.6 to A and to any indicator function 1_Ω, where $\Omega \in \mathcal{A}_{\mathrm{Sp}A}$: 1_Ω is the function which is equal to 1 on Ω and equal to 0 on $\mathrm{Sp}A \setminus \Omega$. Obviously $1_A \in \mathcal{B}(\mathrm{Sp}A,\mathbb{C})$. We then set

$$\forall \Omega \in \mathcal{A}_{\mathrm{Sp}A}, \quad P_\Omega := 1_\Omega(A) \in \mathcal{L}(\mathcal{H}).$$

Lemma 11.3 *For any Borelian subset $\Omega \subset SpA$, P_Ω is an orthogonal projection.*

Proof — From Theorem 11.6, **a)** we deduce that $(P_\Omega)^2 = 1_\Omega(A)1_\Omega(A) = (1_\Omega)^2(A) = 1_\Omega(A) = P_\Omega$, hence P_Ω is a projection. Moreover since P_Ω is self-adjoint, $\forall y \in \mathrm{Im}P_\Omega$, $\forall x \in \mathrm{Ker}P_\Omega$, $y = P_\Omega y$ and thus

$$\langle x, y\rangle = \langle x, P_\Omega y\rangle = \langle P_\Omega x, y\rangle = \langle 0, y\rangle = 0.$$

Hence the image of P_Ω is orthogonal to is kernel. □

Proposition 11.3 *The family* $(\Omega)_{\Omega\in\mathcal{A}_{SpA}}$ *satisfies the following properties.*

a) $\forall\Omega$, P_Ω *is an orthogonal projection ;*

b) $P_\emptyset = 0$, $P_{SpA} = 1_{\mathcal{H}}$ *;*

c) *If* $\Omega = \cap_{n\in\mathbb{N}^*}\Omega_n$ *and if* $n \neq m$ *implies* $\Omega_n \cap \Omega_m = \emptyset$, *then*[12] $\forall\varphi,\psi \in \mathcal{H}$,

$$\lim_{N\to+\infty}\sum_{n=1}^{N}\langle\varphi, P_{\Omega_n}\psi\rangle - \langle\varphi, P_\Omega\psi\rangle = 0$$

d) $P_{\Omega_1}P_{\Omega_2} = P_{\Omega_1\cap\Omega_2}$.

Proof — A straightforward consequence of the previous lemma and Theorem 11.6. □
We will give a name to the previous construction.

Definition 11.4 *A collection* $\{P_\Omega\}_{\Omega\in\mathcal{A}_{SpA}}$ *which satisfies Properties* ***a), b), c)*** *and* ***d)*** *in Proposition 11.3 is called a* ***Borelian measure with values in orthogonal projections*** *with compact support.*

Conversely :

Theorem 11.7 *Any Borelian measure* $\{P_\Omega\}_{\Omega\in\mathcal{A}_{SpA}}$ *with values in orthogonal projections with compact support defines an unique bounded self-adjoint operator in* $\mathcal{H}$ *such that* $P_\Omega = 1_\Omega(A)$, $\forall\Omega \in \mathcal{A}_{SpA}$.

Proof — Let us consider such a measure defined on a compact $K \subset \mathbb{R}$. For any $\varphi,\psi \in \mathcal{H}$, we define $\mu_{(\varphi,\psi)}$ by

$$\forall\Omega \in \mathcal{A}_{\mathrm{Sp}A},\quad \mu_{(\varphi,\psi)}(\Omega) := \langle\varphi, P_\Omega\psi\rangle.$$

Then one can checks easily that $\mu_{(\varphi,\psi)}$ is a complex Borelian measure with support in K. Moreover it is nonnegative if $\varphi = \psi$. Hence by repeating the arguments for the proof of Theorem 11.6 we can define for any $g \in \mathcal{B}(K,\mathbb{C})$ an operator $B_g \in \mathcal{L}(\mathcal{H})$ such that

$$\forall\varphi,\psi \in \mathcal{H},\quad \langle\varphi, B_g\psi\rangle = \int_K g(\lambda)d\mu_{(\varphi,\psi)}(\lambda) \tag{74}$$

Call the operator $A := B_{X|_K}$ corresponding to the particular function $g = X|_K$. Then one needs to check that the operator B_g in (74) is equal to $g(A)$, as defined by (60).

12. In [3] it is claimed that :

$$\lim_{N\to+\infty}\left\|\sum_{n=1}^{N}P_{\Omega_n} - P_\Omega\right\|_{\mathcal{L}(\mathcal{H})} = 0,$$

which is wrong : take for instance $\mathcal{H} = L^2([0,1])$, $A = M_X$ (the operator of multiplication by the function $\lambda \longmapsto \lambda$) and $\Omega_n = (\frac{1}{n+1}, \frac{1}{n}]$, $\forall n \in \mathbb{N}^*$. Then all Ω_n's are pairwise disjoint and $\cap_{n\in\mathbb{N}^*}\Omega_n = (0,1] = \Omega$. However $\left(\sum_{n=1}^{N}P_{\Omega_n}\right) - P_\Omega$ is equal to the projection operator $f \longmapsto 1_{(0,\frac{1}{N+1}]}f$, the norm of which is 1.

For that purpose we first approximate $X|_K$ by the sequence of functions $(\varphi_n)_{n\in\mathbb{N}^*}$ defined by : $\varphi_n(\lambda) = \frac{k}{n}$, $\forall k \in \mathbb{Z}$, $\forall \lambda \in [\frac{k}{n}, \frac{k+1}{n}) \cap K$. On the one hand we observe that $B_{\varphi_n} = \sum_{k\in\mathbb{Z}} \frac{k}{n} P_{[\frac{k}{n},\frac{k+1}{n})\cap K}$ and we deduce easily that $(B_{\varphi_n})^p = B_{(\varphi_n)^p}$, $\forall p \in \mathbb{N}$ and, more generally, $P(B_{\varphi_n}) = B_{P(\varphi_n)}$, $\forall P \in \mathbb{C}[X]$. On the other hand passing to the limit when $n \to +\infty$, we deduce $P(A) = B_{P|_K}$, $\forall P \in \mathbb{C}[X]$. Using density of polynomial functions in $\mathcal{C}^0(K,\mathbb{C})$ and Lemma 11.2, we can conclude that $B_g = g(A)$, $\forall g \in \mathcal{B}(K,\mathbb{C})$. □

It is useful to introduce suggestive notations (although there are nothing but notations) for writing the integral in (74). We write :

$$\forall \varphi, \psi \in \mathcal{H}, \quad \langle \varphi, g(A)\psi\rangle = \int_K g(\lambda) d\langle \varphi, P_\lambda \psi\rangle$$

and hence getting rid of φ and ψ (another notation)

$$g(A) = \int_K g(\lambda) dP_\lambda.$$

11.7.2 Commuting self-adjoint operators

We wish here to extend the functional calculus to a family of *commuting* operators. It is a standard result in linear algebra that a pair of commuting operators acting on a finite dimensional space which are diagonalizable can be diagonalizable simultaneously in the same basis. It is thus natural to expect a similar result for self-adjoint operators in general. This will achieved by first extending the functional calculus of commuting operators. Consider a pair $A, B \in \mathcal{L}(\mathcal{H})$ of self-adjoint operators and assume that

$$[A, B] = AB - BA = 0.$$

and let K be a compact subset of $\mathbb{R}$ which contains $\mathrm{Sp}A$ and $\mathrm{Sp}B$. Then by applying Theorem 11.6 to A and B we know that for any functions $f, g \in \mathcal{B}(K,\mathbb{C})$ we can define $f(A)$ and $g(B)$ and that these operations satisfy all properties listed in Theorem 11.6. In particular by applying **g)** we deduce that $[f(A), B] = 0$. But applying again Theorem 11.6, **g)** with $f(A)$ playing the role of B and B playing the role of A, we deduce that $[f(A), g(B)] = 0$.

If we apply this for the indicator functions $f = 1_U$ and $g = 1_V$, where $U, V \subset K$ are Borelian subsets of K, it gives us $[1_U(A), 1_V(B)] = 0$. We can thus define unambiguously the image of (A, B) by the function $1_{U\times V} = 1_U \otimes 1_V \in \mathcal{B}(K \times K,\mathbb{C})$ by $1_{U\times V}(A, B) := 1_U(A)1_V(B) = 1_V(B)1_U(A)$. More generally let $\mathcal{B}_{\text{finite}}(K^2,\mathbb{C})$ be the space of linear combinations of indicator functions of Cartesian products, i.e. of the form $F = \sum_{j=1}^k \alpha_j 1_{U_j\times V_j}$, where $U_1, \cdots, U_k$ and $V_1, \cdots, V_k$ are Borelian subsets of K and $\alpha_1, \cdots, \alpha_k \in \mathbb{C}$. Then we can define $F(A, B)$ for any $F \in \mathcal{B}_{\text{finite}}(K^2,\mathbb{C})$ by :

$$F(A, B) = \sum_{j=1}^k \alpha_j 1_{U_j\times V_j}(A, B) = \sum_{i=1}^k \alpha_j 1_{U_j}(A) 1_{V_j}(B).$$

One can then prove that, for any $\varphi, \psi \in \mathcal{H}$, there exists an unique Borelian measure $\mu_{(\varphi,\psi)}$ on K^2 satisfying (54) and (55) such that $\forall F \in \mathcal{B}_{\text{finite}}(K^2, \mathbb{C})$

$$\langle \varphi, F(A,B)\psi \rangle = \int_{K^2} F(\lambda_1, \lambda_2) d\mu_{(\varphi,\psi)}(\lambda_1, \lambda_2).$$

Moreover $\mu_{(\varphi,\psi)}$ depends in a sesquilinear way of (φ, ψ) and is nonnegative for $\varphi = \psi$. By using the same strategy as in the previous sections, one can then define $F(A, B)$ for any $F \in \mathcal{B}(K^2, \mathbb{C})$ such that the previous relation holds.

Then all results on the functional calculus and the spectral decomposition that we have seen can be generalized to a pair of commuting self-adjoint operators essentially by repeating the same arguments : Theorem 11.1 on the continuous functional calculus, Theorem 11.3 (by defining a cyclic vector ψ by the condition that $\mathrm{Vec}\{A^n B^m \psi;\, n, m \in \mathbb{N}\}$ be dense in $\mathcal{H}$) and Theorem 11.5 for the spectral decomposition of a pair of commuting self-adjoint operators, Theorem 11.6 on the Borelian functional calculus, Proposition 11.3.

Lastly all these results generalize also to any finite family of pairwise commuting operators.

11.7.3 Normal operators

As a preliminary we remark that *any* bounded operator $T \in \mathcal{L}(\mathcal{H})$ can be decomposed in an unique way in the form

$$T = A + iB,$$

where A and B are self-adjoint, so that, in particular $T^* = A - iB$. Indeed A and B are given by

$$A = \frac{1}{2}(T + T^*), \quad B = \frac{1}{2i}(T - T^*).$$

We call A and B respectively the real part and the imaginary part of T. In this decomposition the quantity $[T, T^*]$ reads

$$[T, T^*] = \frac{1}{4i}(A + iB)(A - iB) - \frac{1}{4i}(A - iB)(A + iB) = -\frac{1}{2}[A, B].$$

Hence *an operator is normal if and only if its real part and its imaginary part commute.* Hence by applying the results presented in the previous section on commuting self-adjoint operators, we can easily extend all the spectral theory to normal operators. An example of result is the analogue of Theorem 11.5 which follows.

Theorem 11.8 *Let $\mathcal{H}$ be a complex separable Hilbert space and $A \in \mathcal{L}(\mathcal{H})$ be normal. Then there exists a measured space (X, μ) with finite mass, a measurable bounded function $g : X \longrightarrow \mathbb{C}$ and a unitary map $U : \mathcal{H} \longrightarrow L^2(X, \mu, \mathbb{C})$ such that $UAU^{-1} = M_g$.*

The only difference with Theorem 11.5 is that here g may take complex instead of real values.

12 Non bounded operators

12.1 Basic definitions

We now wish to handle operators of the kind :
– $\widehat{x^j}$ on $L^2(\mathbb{R},\mathbb{C})$, for $j=1,2,3$, defined by $(\widehat{x^j}f)(x)=x^jf(x)$, for a.e. $x\in\mathbb{R}^3$;
– $\widehat{p_j}$ on $L^2(\mathbb{R},\mathbb{C})$, for $j=1,2,3$, defined by $(\widehat{p_j}f)(x)=\frac{\hbar}{i}\frac{\partial f}{\partial x^j}(x)$, for a.e. $x\in\mathbb{R}^3$;
– $\widehat{H}$ on $L^2(\mathbb{R}^3,\mathbb{C})$, defined by $(\widehat{H}f)(x)=-\frac{\hbar^2}{2m}\Delta f(x)$, for a.e. $x\in\mathbb{R}^3$.

These three operators play a fundamental role in quantum mechanics, they represent respectively the coordinates of the position ($\widehat{x}$), the momentum ($\widehat{p}$) and the energy or the Hamiltonian ($\widehat{H}$) of a particle. Here $\hbar$ is the Planck constant and m the mass of the particle.

These three operators have the same problem of not being defined on $L^2(\mathbb{R}^3,\mathbb{C})$. Consider the simplest situation of the operator $\widehat{x}$ on $L^2(\mathbb{R},\mathbb{C})$, defined by $(\widehat{x}f)(x)=xf(x)$, for a.e. $x\in\mathbb{R}$. There are indeed infinitely many functions $f\in L^2(\mathbb{R},\mathbb{C})$ such that $x\longmapsto xf(x)$ is not in $L^2(\mathbb{R},\mathbb{C})$, for instance $f(x)=\frac{1}{\sqrt{1+x^2}}$. So we need to choose a suitable vector subspace where this operator makes sense, e.g. $\{f\in L^2(\mathbb{R},\mathbb{C});\ \int_{\mathbb{R}}x^2|f(x)|^2dx<+\infty\}$. This vector subspace will be called the *domain* of the operator.

Definition 12.1 *Let $\mathcal{H}$ be a complex Hilbert space. A* ***non bounded operator on*** $\mathcal{H}$ *is a pair (D,T), where D is a vector subspace of $\mathcal{H}$ and T is a linear operator from D to $\mathcal{H}$.*

If the operator T is continuous on D with respect to the topology of $\mathcal{H}$, then there exists a continuous linear extension $\overline{T}$ of T on $\mathcal{H}$ (either because D is dense in $\mathcal{H}$, then the extension is unique, or in general by using the Banach–Steinhaus theorem, but the extension may be non unique). So in this case there is nothing new with respect to bounded operators. Hence in the following we will assume that the non bounded operators are not continuous on their domain with respect to the topology of $\mathcal{H}$.

We will also often abuse notation by writing $T\simeq(D,T)$. In that case we write $D(T)$ for the domain.

The key notion is to look at a non bounded operator through its graph.

Definition 12.2 *Let $\mathcal{H}$ be a complex Hilbert space and T be a non bounded operator on $\mathcal{H}$. Then its graph is the vector subspace of $\mathcal{H}\times\mathcal{H}$*

$$GrT:=\{(x,Tx);\ x\in D(T)\}.$$

The Cartesian product $\mathcal{H}\times\mathcal{H}$ is equipped with the standard product Hilbert structure and the product topology. It induces a pre-Hilbertian structure on GrT. There are two natural projection mappings $\pi_1,\pi_2:\mathcal{H}\times\mathcal{H}\longrightarrow\mathcal{H}$ on respectively the first and the second factor. Both restrictions of these projection maps to GrT are continuous.

The substitute for the notion of continuity is the notion of *closed graph*.

Definition 12.3 *Let $\mathcal{H}$ be a complex Hilbert space and T be a non bounded operator on $\mathcal{H}$. Then* *(i)* T *is **closed** if* GrT *is closed ;*

(ii) If T_1 *and* T_2 *are two non bounded operators on* $\mathcal{H}$ *then we say that* T_2 ***is an extension of*** T_2 *if* $D(T_1) \subset D(T_2)$ *and*

$$\forall x \in D(T_1), \quad T_1 x = T_2 x.$$

We then write $T_1 \subset T_2$ *;*

(iii) T *is **closable** if there exists an extension of* T *which is closed.*

Remark that $T_1 \subset T_2$ if and only if $\mathrm{Gr}T_1 \subset \mathrm{Gr}T_2$.

Proposition 12.1 *Let* T *be a non bounded operator on* $\mathcal{H}$*. Assume that* T *is closable. Then there exists an unique smallest extension* $\overline{T}$*, which is closed.*

This extension $\overline{T}$ *is called the **closure** of* T*.*

Proof — A naive idea is to contemplate the closure $\overline{\mathrm{Gr}T}$ of $\mathrm{Gr}T$ and to let $\overline{T}$ be the operator the graph of which is $\overline{\mathrm{Gr}T}$. Like many naive ideas it is the right one, but one has to be careful and to check that $\overline{\mathrm{Gr}T}$ is the graph of some operator ! One may think that this is an obvious point, but we will see later on an example that this is not always the case. As a preliminary the Reader is invited to check that :

A subspace $V \subset \mathcal{H} \times \mathcal{H}$ *is the graph of a non bounded operator if and only if the restriction of* π_1 *to* V *is one-to-one.*

Now let's use the hypothesis that T is closable : there exists at least a closed operator S which is an extension of T, i.e. $\mathrm{Gr}T \subset \mathrm{Gr}S$. Thus $\overline{\mathrm{Gr}T} \subset \overline{\mathrm{Gr}S} = \mathrm{Gr}S$. Since $\mathrm{Gr}S$ is a graph, the restriction of π_1 to it is one-to-one, hence the restriction of π_1 to $\overline{\mathrm{Gr}T}$. Hence $\overline{\mathrm{Gr}T}$ is the graph os some operator. Let's call it $\overline{T}$. This is a closed extension of T. Moreover any closed extension S of T should also be an extension of $\overline{T}$ (just read again the previous arguments). Hence $\overline{T}$ is the smallest closed extension. □

Example 12.1 *Let* $\mathcal{H}$ *be a complex separable Hilbert space and* $(e_n)_{n\in\mathbb{N}}$ *be a Hilbertian Hermitian orthogonal basis of* $\mathcal{H}$*. Set* $V := Vec\{e_n;\, n \in \mathbb{N}\}$ *(the set of finite linear combinations of vectors in* $\{e_n;\, n \in \mathbb{N}\}$*) and* $z \in \mathcal{H} \setminus V$*. Let* $D := V \oplus \mathbb{C}z$ *and let* T *be the non bounded operator with domain* D *and such that*

$$\forall x \in V, \forall \lambda \in \mathbb{C}, \quad T(x + \lambda z) = \lambda z.$$

Then $GrT = \{(x + \lambda z, \lambda z);\, x \in V, \lambda \in \mathbb{C}\}$ *and* $\overline{GrT} = \mathcal{H} \times \mathbb{C}z$*. Hence the restriction of* π_1 *to* $\overline{GrT}$ *is not one-to-one (its kernel is* $\{0\} \times \mathbb{C}z$*). Thus* T *cannot be closable because if so the previous Proposition would lead to a contradiction.*

Example 12.2 *Let* $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C})$*,* $D(T) = \mathcal{C}_c^\infty(\mathbb{R}, \mathbb{C})$ *(smooth functions with compact support in* $\mathbb{R}$*) and* $Tf = \frac{df}{dx}$*. Then* T *is closable,* $D(\overline{T}) = H^1(\mathbb{R}, \mathbb{C})$ *and* $\overline{T}f = \frac{df}{dx}$*.*

12.2 The adjoint of a non bounded operator

In the following all operators will be non bounded, unless the contrary is specified.

The definition of the adjoint of an operator T is a delicate task, but the basic idea is simple ; we would like to define it by the relation

$$\langle T^* x, y\rangle = \langle x, Ty\rangle. \tag{75}$$

However the problem is to decide the domain of the validity of such a formula : the right hand side makes sense a priori if $(x, y) \in \mathcal{H} \times D(T)$, whereas the left hand side makes sense a priori for all values of y in $\mathcal{H}$, but we expect that it may not be defined for any x, i.e. that *there is no reason a priori that T^* be bounded.* Hence we should find a domain $D(T)^*$ for T^*. Its definition will again be imposed by (75) : it says us that, if T^*x exists, then the linear form $y \longmapsto \langle x, Ty\rangle$ is bounded.

Definition 12.4 *Let $\mathcal{H}$ be a complex Hilbert space and T be an operator on $\mathcal{H}$. The adjoint domain is the space $D(T)^*$ of vectors $x \in \mathcal{H}$ such that the linear form*

$$\begin{array}{rccl} \ell_x : & D(T) & \longmapsto & \mathbb{C} \\ & y & \longmapsto & \langle x, Ty\rangle \end{array}$$

is bounded, i.e.

$$D(T)^* := \{x \in \mathcal{H}; \exists C > 0, \forall y \in D(T), |\langle x, Ty\rangle| \leq C\|y\|\}.$$

Hence for any $x \in D(T)^*$ we can extend the linear form ℓ_x to a linear continous form defined on $\mathcal{H}$: this extension can be found by using the Hahn–Banach theorem in the general case or simply by continuity if $D(T)$ is dense in $\mathcal{H}$. In the latter case the extension of ℓ_x is unique.

In both cases let's denote by $\overline{\ell_x}$ a continuous extension of ℓ_x. Then the Riesz–Fréchet theorem implies that there exists an unique element $T^*x \in \mathcal{H}$ such that $\forall y \in \mathcal{H}$, $\overline{\ell_x}(y) = \langle T^*x, y\rangle$. In particular this implies that $\forall y \in D(T)$, $\langle x, Ty\rangle = \langle T^*x, y\rangle$.

Definition 12.5 *Let $\mathcal{H}$ be a complex Hilbert space and T be an operator on $\mathcal{H}$. An adjoint of T is an operator T^* defined on $D(T)^*$ such that*

$$\forall x \in D(T)^*, \forall y \in D(T), \quad \langle x, Ty\rangle = \langle T^*x, y\rangle. \tag{76}$$

Note the important fact that, if $D(T)$ is **dense** in $\mathcal{H}$, then T^* is **unique**.

Proposition 12.2 *Let $\mathcal{H}$ be a complex Hilbert space and T and S non bounded operators. Then*

a) $T \subset S \Longrightarrow S^* \subset T^*$;

Moreover if the domain $D(T)$ of T is dense in $\mathcal{H}$, then the three following properties are true

b) T^* is closed ;

c) T is closable iff $D(T^*)$ is dense and, if so, $\overline{T} = T^{**}$;

d) if T is closable, then $(\overline{T})^* = T^*$.

Before giving the proof of this let us recall that very similar results hold for the notion of the orthogonal vector subspace $E^\perp$ to a given subspace $E \subset \mathcal{H}$.

a) $E \subset F \Longrightarrow F^\perp \subset E^\perp$;

b) $E^\perp$ is always closed ;

c) $\overline{E} = (E^\perp)^\perp$;

d) $(\overline{E})^\perp = E^\perp$.

This is not an accident. The idea of the proof of Proposition 12.2 is actually the following. We first prove that the graph of T^* is

$$\mathrm{Gr}T^* = \{(a,b) \in \mathcal{H} \times \mathcal{H};\ \forall (x,y) \in \mathrm{Gr}T, \langle b,x\rangle - \langle a,y\rangle = 0\}.$$

In other words $\mathrm{Gr}T^*$ is the orthogonal in $\mathcal{H} \times \mathcal{H}$ of $J\mathrm{Gr}T$, where $J : \mathcal{H} \times \mathcal{H} \longrightarrow V$ is defined by $J(a,b) = (b,-a)$. We will just prove that fact : $\forall (a,b) \in \mathcal{H} \times \mathcal{H}$,

$$\begin{array}{lcl} & \forall (x,y) \in \mathrm{Gr}T, & \langle b,x\rangle - \langle a,y\rangle = 0 \\ \Longleftrightarrow & \forall x \in D(T), & \langle b,x\rangle = \langle a,Tx\rangle \\ \Longleftrightarrow & \left\{ \begin{array}{ccc} D(T) & \longrightarrow & \mathbb{C} \\ x & \longmapsto & \langle a,Tx\rangle \end{array} \right. & \text{is continuous linear and coincides with } \langle b,\cdot\rangle \\ \Longleftrightarrow & & a \in D(T^*) \text{ and } b = T^*a. \end{array}$$

12.3 Self-adjoint and symmetric operators

Definition 12.6 *Let $\mathcal{H}$ be a complex Hilbert space and T be a non bounded operators. Then*

- *T is **symmetric** if $T \subset T^*$.*
- *T is **self-adjoint** if $T = T^*$.*

Remark — We can define a complex valued sesquilinear symplectic form ω on $\mathcal{H} \times \mathcal{H}$ by : $\omega((a,b),(x,y)) = \langle b,x\rangle - \langle a,y\rangle$. Then $\mathrm{Gr}T^*$ is the orthogonal in $\mathcal{H} \times \mathcal{H}$ of $\mathrm{Gr}T$ for ω. Moreover : (i) T is self-adjoint iff $\mathrm{Gr}T$ is Lagrangian for ω ; (ii) T is symmetric iff $\mathrm{Gr}T$ is isotropic for ω.

Proposition 12.3 *Let $\mathcal{H}$ be a complex Hilbert space and T be a non bounded operators. Assume that the domain $D(T)$ of T is dense, then*

a) if T is symmetric, T is closable ;

b) if T is symmetric, $T \subset T^{**} \subset T^*$;

c) if T is closed and symmetric, $T = T^{**} \subset T^*$;

d) if T is self-adjoint, $T = T^{**} = T^*$.

Proof — a) follows from the fact that T^* is always closed and from the definition of a symmetric operator ; b) follows from $T \subset T^* \Longrightarrow T^{**} \subset T^*$ and $T \subset \overline{T} = T^{**}$; c) is a consequence of b) and the fact that T is closed iff $T = \overline{T} = T^{**}$. □

Definition 12.7 *An operator is* ***essentially self-adjoint*** *if it is symmetric and if its closure is self-adjoint.*

The next result is the analogue of Theorem 6.6 for non bounded self-adjoint operators.

Theorem 12.1 *Let $\mathcal{H}$ be a complex Hilbert space and T be a non bounded self-adjoint operator on $\mathcal{H}$. Then*

a) $SpT \subset \mathbb{R}$;

b) Furthermore, $\forall \lambda, \mu \in \mathbb{R}$ such that $\mu \neq 0$, $(T - \lambda - i\mu)^{-1} \in \mathcal{L}(\mathcal{H})$ and the image of $(T - \lambda - i\mu)^{-1}$ is $D(T)$.

Proof — We first prove that

$$\forall x \in D(T), \quad \|(T - \lambda - i\mu)x\| \geq |\mu|\|x\|. \tag{77}$$

(the analogue of Identity (22)). For that purpose we first remark that $\forall x \in D(T)$, $\langle x, Tx\rangle \in \mathbb{R}$ because $\langle Tx, x\rangle = \langle x, Tx\rangle = \overline{\langle Tx, x\rangle}$. Hence from $\langle x, (T - \lambda - i\mu)x\rangle = \langle x, (T-\lambda)x\rangle - i\mu\|x\|^2$ we deduce that $\mathrm{Re}\langle x, (T-\lambda-i\mu)x\rangle = \langle x, (T-\lambda)x\rangle$ and $\mathrm{Im}\langle x, (T-\lambda - i\mu)x\rangle = -\mu\|x\|^2$. Hence

$$\forall x \in D(T), \quad |\mu|\|x\|^2 \leq |\langle x, (T - \lambda - i\mu)x\rangle| \leq \|x\|\|(T - \lambda - i\mu)x\|.$$

Thus (77) follows. This has the following consequences.
(i) If $\mu \neq 0$, $T - \lambda - i\mu$ is one-to-one.
(ii) If $\mu \neq 0$, $\mathrm{Im}(T - \lambda - i\mu)$ is dense in $\mathcal{H}$. For that set $S = T - \lambda - i\mu$ and let us prove that $\mathrm{Im}S$ is dense in $\mathcal{H}$ if and only if $\mathrm{Ker}S^* = \{0\}$. Indeed, for all $y \in \mathcal{H}$,

$$\begin{aligned} y \in (\mathrm{Im}S)^\perp &\Longleftrightarrow \forall x \in D(S), \langle y, Sx\rangle = 0 \\ &\Longleftrightarrow \left\{ \begin{array}{ccc} D(S) & \longrightarrow & \mathbb{C} \\ x & \longmapsto & \langle y, Sx\rangle \end{array} \right. \text{ exists, is continuous and vanishes} \\ &\Longleftrightarrow y \in D(S^*) \text{ and } S^*y = 0 \text{ (since } D(S) \text{ is dense in } \mathcal{H}) \\ &\Longleftrightarrow y \in \mathrm{Ker}S^*. \end{aligned}$$

Hence $\mathrm{Im}(T-\lambda-i\mu)$ is dense in $\mathcal{H}$ if and only if $\{0\} = \mathrm{Ker}(T-\lambda-i\mu)^* = \mathrm{Ker}(T-\lambda+i\mu)$. But this is a consequence of (77) (changing μ in $-\mu$) because $\mu \neq 0$.
(iii) If $\mu \neq 0$, $\mathrm{Im}(T-\lambda-i\mu)$ is closed. Let $(y_n)_{n\in\mathbb{N}}$ be a sequence with values in $\mathrm{Im}(T-\lambda-i\mu)$ which converges to some $y \in \mathcal{H}$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ with values in $D(T)$ such that $(T-\lambda-i\mu)x_n = y_n$, $\forall n \in \mathbb{N}$. The sequence $(y_n)_{n\in\mathbb{N}}$ is Cauchy, this implies that $(x_n)_{n\in\mathbb{N}}$ is also a Cauchy sequence because of (77). Hence $(x_n)_{n\in\mathbb{N}}$ converges to some $x \in \mathcal{H}$. Would T be continuous, then we could conclude directly pass to the limit in the relation $y_n = (T-\lambda-i\mu)x_n$ when $n \to +\infty$ and conclude $y = (T-\lambda-i\mu)x \in \mathrm{Im}(T-\lambda-i\mu)$

as in the proof of Lemma 3.1. We turn around the difficulty by writing the equation in the form

$$y_n = (T - \lambda - i\mu)x_n \Longleftrightarrow y_n + (\lambda + i\mu)x_n = Tx_n \Longleftrightarrow (x_n, y_n + (\lambda + i\mu)x_n) \in \mathrm{Gr}T.$$

Since $(x_n, y_n + (\lambda+i\mu)x_n)$ converges to $(x, y+(\lambda+i\mu)x)$ and $\mathrm{Gr}T$ is *closed*, a consequence of the fact that T is self-adjoint, we deduce the result.
In conclusion if $\mu \neq 0$, $T - \lambda - i\mu$ is a bijection between $D(T)$ and $\mathcal{H}$. By using (77) we deduce $\|(T - \lambda - i\mu)^{-1}y\| \leq |\mu|^{-1}\|y\|$. □

12.4 An example

Consider $\mathcal{H} = L^2([0,1], \mathbb{C})$ ($\simeq L^2(0,1)$ for short), and set
– $D(T) = H_0^1([0,1], \mathbb{C})$ ($\simeq H_0^1(0,1)$ for short) and
– $\forall f \in H_0^1(0,1)$, $Tf = i\frac{df}{dx}$.
Recall (see Chapter 8) that $H^1(0,1)$ can be defined to be the subspace of $L^2(0,1)$ of functions f which admit a weak derivative $\frac{df}{dt}$ in $L^2(0,1)$, i.e. a function characterized by $\int_0^1 (f\frac{d\varphi}{dt} + \frac{df}{dt}\varphi)dt = 0$, $\forall \varphi \in \mathcal{C}_c^\infty((0,1))$. Moreover $H_0^1(0,1)$ is the subspace of $H^1(0,1)$ of functions f such that $f(0) = f(1) = 0$.

Let us investigate the domain of T^* : it is the space of $f \in L^2(0,1)$ such that the map

$$\begin{array}{ccc} H_0^1(0,1) & \longrightarrow & \mathbb{C} \\ g & \longmapsto & \langle f, Tg \rangle \end{array}$$

can be extended continuously to $L^2(0,1)$ and, hence, can be identified with the map $g \longmapsto \langle T^*f, g\rangle$, for some $T^*f \in L^2(0,1)$. This condition reads : $\forall g \in H_0^1(0,1)$

$$\int_0^1 \overline{f}\left(i\frac{dg}{dt}\right)dt = \int_0^1 \overline{T^*f}g\,dt \quad \Longleftrightarrow \quad \int_0^1 \left(f\frac{d\overline{g}}{dt} - i(T^*f)\overline{g}\right)dt = 0.$$

This means that f has a weak derivative in $L^2(0,1)$, which is equal to $-iT^*f$. Thus $D(T^*)$ coincides with $H^1(0,1)$ and $\forall f \in D(T^*)$, $-iT^*f = \frac{df}{dt} \Longleftrightarrow T^*f = i\frac{df}{dt}$. Hence $T \subset T^*$, i.e. T is *symmetric*. Observe that T is *not self-adjoint*, since $H_0^1(0,1)$ is a strict subspace of $H^1(0,1)$ (more precisely of codimension 2).

Moreover T^* is neither self-adjoint, nor symmetric. This is a consequence of the fact that $\forall f, g \in D(T^*) = H^1(0,1)$,

$$\langle f, T^*g\rangle - \langle T^*f, g\rangle = \int_0^1 \overline{f}\left(i\frac{dg}{dt}\right) - \left(-i\frac{d\overline{f}}{dt}\right)g = i\int_0^1 \overline{f}\frac{dg}{dt} + \frac{d\overline{f}}{dt}g = \left[\overline{f}g\right]_0^1, \qquad (78)$$

a quantity which does not vanish in general. In fact (78) tells us that, for any $f \in H^1(0,1)$, the linear form $H^1(0,1) \ni g \longmapsto \langle f, T^*g\rangle \in \mathbb{C}$ coincides with $g \longmapsto \langle T^*f, g\rangle + [\overline{f}g]_0^1$, i.e. the sum of $g \longmapsto \langle T^*f, g\rangle$, which is continuous for the L^2-topology, and of $g \longmapsto [\overline{f}g]_0^1$, which is *not continuous* for the L^2-topology excepted if $f(0) = f(1) = 0$, i.e. if $f \in H_0^1(0,1)$.

Hence $D(T^{**}) = H^1_0(0,1)$ and we also deduce from (78) that $T^{**} = T$. In particular T^* is not symmetric.

There are however self-adjoint extensions of T *between* T and T^*. This means that there exist operators the graph of which contains $\mathrm{Gr}T$ and is contained in $\mathrm{Gr}T^*$ and which are self-adjoint. Note that the choices are relatively limited since, the codimension of $\mathrm{Gr}T$ in $\mathrm{Gr}T^*$ is equal to the codimension of $H^1_0(0,1)$ in $H^1(0,1)$, i.e. to 2. Moreover we know that neither T nor T^* are self-adjoint, so this means that we must look for extensions of T the domain of which has a codimension 1 in $H^1(0,1)$, i.e. a domain of equation $af(0) + bf(1) = 0$. By normalizing this amounts to define $H^1_{(\alpha)}(0,1) := \{f \in H^1(0,1);\ f(1) = \alpha f(0)\}$, for some $\alpha \in \mathbb{C} \cup \{\infty\}$. If we define $T_{(\alpha)}$ to be the operator of domain $H^1_{(\alpha)}(0,1)$ and such that $T_{(\alpha)}f = i\frac{df}{dt}$, then we deduce from (78) that $\forall f, g \in H^1_{(\alpha)}(0,1)$, $\langle f, T^*_{(\alpha)}g\rangle - \langle T^*_{(\alpha)}f, g\rangle = (|\alpha|^2 - 1)\overline{f(0)}g(0)$. This quantity vanishes if and only if $|\alpha| = 1$. Hence $T_{(\alpha)}$ is self-adjoint iff α belongs to the circle $\{e^{i\theta};\ \theta \in \mathbb{R}\}$. The spaces $H^1_{(\alpha)}(0,1)$ have nice interpretations : geometrically they represent sections of a complex line bundle over the circle $\mathbb{R}/\mathbb{Z}$.

12.5 The spectral decomposition of self-adjoint operators

A preliminary result follows by applying Theorem 12.1 for $\lambda + i\mu = \pm i$: it says us that $(T-i)^{-1}$ and $(T+i)^{-1}$ exist and are bounded operators. The following result implies that these operators are also normal.

Proposition 12.4 *Let $\mathcal{H}$ be a complex Hilbert space and T be a non bounded self-adjoint operator, such that $D(T)$ is dense in $\mathcal{H}$. Then*

(i) $(T-i)^{-1}$ and $(T+i)^{-1}$ commute ;

(ii) $(T-i)^{-1}$ is the adjoint of $(T+i)^{-1}$ and conversely.

Proof — First note that $(T \pm i)^{-1}$ is an isomorphism from $\mathcal{H}$ to $D(T)$ and also that $(T+i)^{-1}(T-i)^{-1}$ and $(T-i)^{-1}(T+i)^{-1}$ are isomorphism from $\mathcal{H}$ to $D(T^2)$. To prove (i), let $x \in \mathcal{H}$ and $y = (T+i)^{-1}(T-i)^{-1}x$ and $y' = (T-i)^{-1}(T+i)^{-1}x$ be in $D(T^2)$. Then obviously

$$(T-i)(T+i)y = x = (T+i)(T-i)y' \quad \Longrightarrow \quad y = y',$$

(because $(T+i)(T-i) = (T-i)(T+i)$) which means that $(T+i)^{-1}(T-i)^{-1}x = (T-i)^{-1}(T+i)^{-1}x$.
To prove (ii) let $y, y' \in \mathcal{H}$, set $x = (T+i)^{-1}y$ and $x' = (T-i)^{-1}y'$, so that $y = (T+i)x$ and $y' = (T-i)x'$. Then using the fact that the adjoint of $T-i$ is $T+i$, we have

$$\langle (T+i)^{-1}y, y'\rangle = \langle x, (T-i)x'\rangle = \langle (T+i)x, x'\rangle = \langle y, (T-i)^{-1}y'\rangle.$$

Hence $((T-i)^{-1})^* = (T+i)^{-1}$. □

We can hence apply Theorem 11.8 to $(T+i)^{-1}$. It gives us

There exists a finite mass measured space (X, μ) and a unitary map $U : \mathcal{H} \longrightarrow L^2(X, \mu, \mathbb{C})$ and there exists $g \in \mathcal{B}(X, \mathbb{C})$ such that

$$U(T+i)^{-1}U^{-1} = M_g. \tag{79}$$

Recall that M_g is the multiplication operator $\varphi \longmapsto g\varphi$ acting on $L^2(X, \mu, \mathbb{C})$. Since g is Borelian and bounded this operator is bounded.

We also observe that, since $(T+i)^{-1}$ is one to one, $\{x \in X;\ g(x) = 0\}$ is μ-negligeable, i.e. $g \neq 0$ μ-a.e. on X. We can thus define the μ-measurable function $f : X \longrightarrow \mathbb{C}$ by

$$\frac{1}{g} = f + i.$$

Moreover (79) implies that the image of M_g is equal to the image of $U(T+i)^{-1}$, i.e. to $UD(T)$. It is also equal to $\{\varphi \in L^2(X, \mu, \mathbb{C});\ f\varphi \in L^2(X, \mu, \mathbb{C})\}$. We define the non bounded operator M_f on $L^2(X, \mu, \mathbb{C})$, with the domain $D(M_f) := UD(T)$. Then both sides of (79) are bijections from $L^2(X, \mu, \mathbb{C})$ to $D(M_f)$. It is equivalent to

$$\begin{aligned}
& (T+i)^{-1}U^{-1} = U^{-1}M_g \\
\Longleftrightarrow\ & (T+i)^{-1} = U^{-1}M_gU \\
\Longleftrightarrow\ & T+i = (U^{-1}M_gU)^{-1} = U^{-1}M_g^{-1}U = U^{-1}M_{1/g}U \\
\Longleftrightarrow\ & T = U^{-1}M_fU.
\end{aligned}$$

We hence deduce the

Theorem 12.2 *Let $\mathcal{H}$ be a complex separable Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint such that $D(T)$ is dense in $\mathcal{H}$. Then there exists a measured space (X, μ) with finite mass, a measurable function $f : X \longrightarrow \mathbb{R}$ and a unitary map $U : \mathcal{H} \longrightarrow L^2(X, \mu, \mathbb{C})$ such that $UTU^{-1} = M_f$.*

Proof — Most of the statements have been proved. It remains to prove that f is real valued, a task left to the Reader. □

12.6 Borelian functional calculus for non bounded self-adjoint operators

One can also construct a Borelian functional calculus for non bounded self-adjoint operators. This can be done either by using the Borelian functional calculus for bounded normal operators in in a way similar to the method used previously, or by using the result of Theorem 12.2 as follows :

$$\forall h \in \mathcal{B}(\mathbb{R}, \mathbb{C}), \quad h(T) := U^{-1}M_{h\circ f}U,$$

i.e. $\forall \varphi \in UD(T)$,

$$(Uh(T)U^{-1}\varphi)(x) = h(f(x))\varphi(x), \quad \mu - \text{a.e.}$$

We then have the following.

Theorem 12.3 *Let $\mathcal{H}$ be a complex separable Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be self-adjoint such that $D(T)$ is dense in $\mathcal{H}$. Then the map*

$$\begin{array}{ccc} \mathcal{B}(\mathbb{R},\mathbb{C}) & \longrightarrow & \mathcal{L}(\mathcal{H}) \\ T & \longmapsto & h(T) \end{array}$$

satisfies the following properties

(i) it is a $*$-morphism;

(ii) $\forall h \in \mathcal{B}(\mathbb{R},\mathbb{C})$, $\|h(T)\|_{\mathcal{L}(\mathcal{H})} \leq \|h\|_{\mathcal{B}(\mathbb{R},\mathbb{C})}$;

(iii) if $(h_n)_{n\in\mathbb{N}}$ is a sequence with values in $\mathcal{B}(\mathbb{R},\mathbb{C})$ such that $\forall x \in \mathbb{R}$, $\lim_{n\to+\infty} h_n(x) = x$ and $|h_n(x)| \leq |x|$, $\forall x \in \mathbb{R}$, then, for any $u \in D(T)$, $\lim_{n\to+\infty} h_n(T)u = Tu$;

(iv) if $(h_n)_{n\in\mathbb{N}}$ is a sequence with values in $\mathcal{B}(\mathbb{R},\mathbb{C})$ such that $\forall x \in \mathbb{R}$, $\lim_{n\to+\infty} h_n(x) = h(x)$, $h \in \mathcal{B}(\mathbb{R},\mathbb{C})$ and $\|h_n\|_{\mathcal{B}(\mathbb{R},\mathbb{C})}$ is bounded then $\forall u, v \in \mathcal{H}$, $\lim_{n\to+\infty}\langle u, h_n(T)v\rangle = \langle u, h(T)v\rangle$;

(v) if $Tu = \lambda u$, then $h(T)u = h(\lambda)u$;

(vi) if $h \geq 0$, then $h(T) \geq 0$.

There are two important applications to this result :

1) for $h = 1_\Omega$, where Ω belongs to the σ-algebra $\mathcal{A}_\mathbb{R}$ of Borelian subsets of $\mathbb{R}$. We then set
$$P_\Omega := 1_\Omega(T)$$
and the collection $(P_\Omega)_{\Omega\in\mathcal{A}_\mathbb{R}}$ is a *Borelian measure with values in orthogonal projections.*

2) for the family of functions $(h_t)_{t\in\mathbb{R}}$, where $h_t(x) := e^{itx}$, $\forall t \in \mathbb{R}$. We can thus define $e^{itT} \in \mathcal{L}(\mathcal{H})$, for any $t \in \mathbb{R}$.

It is remarkable that this work for a non bounded operator, whereas an approach based on the standard formula
$$e^{tA} = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$
fails since this series does not make sense for a non bounded operator. Actually our definition of e^{itT} rests on the fact that T is *self-adjoint*.

Another consequence of Theorem 12.3 is the definition of spectral measures (by using the Riesz–Markov theorem).

Definition 12.8 *For any $\varphi \in \mathcal{H}$, there exists an unique nonnegative Borelian measure on $\mathbb{R}$ such that $\forall f \in \mathcal{B}(\mathbb{R},\mathbb{C})$,*

$$\langle \varphi, f(A)\varphi\rangle = \int_\mathbb{R} f(\lambda)d\mu_\varphi(\lambda).$$

Note that the mass of μ_φ is finite (since it is equal to $\|\varphi\|^2$).

Theorem 12.4 *Let A be a non bounded self-adjoint operator with dense domain and let $U(t) := e^{itA}$, $\forall t \in \mathbb{R}$. Then $\forall t \in \mathbb{R}$, $U(t) \in \mathcal{L}(\mathcal{H})$ and moreover*

(i) $\forall t, s \in \mathbb{R}$, *$U(t)$ is unitary and* $U(t)U(s) = U(t+s)$ *;*

(ii) $\forall \varphi \in \mathcal{H}$, $\lim_{t\to t_0} U(t)\varphi = U(t_0)\varphi$ *;*

(iii) $\forall \varphi \in D(A)$, $\lim_{t\to 0} \frac{U(t)\varphi-\varphi}{t} = iA\varphi$ *;*

(iv) $\forall \varphi \in \mathcal{H}$, *if* $\lim_{t\to 0} \frac{U(t)\varphi-\varphi}{t}$ *exists then* $\varphi \in D(A)$.

Proof — Property (i) is a consequence of Theorem 12.3 (i). For Property (ii) it suffices to prove it for t_0. For that purpose we compute

$$\begin{aligned}\|U(t)\varphi - \varphi\|^2 &= \|e^{itA}\varphi - \varphi\|^2 = \langle (e^{itA}-1)\varphi, (e^{itA}-1)\varphi\rangle \\ &= \langle \varphi, (e^{itA}-1)^*(e^{itA}-1)\varphi\rangle = \langle \varphi, (e^{-itA}-1)(e^{itA}-1)\varphi\rangle \\ &= \langle \varphi, g_t(A)\varphi\rangle = \int_{\mathbb{R}} g_t(\lambda) d\mu_\varphi(\lambda),\end{aligned}$$

where $g_t(\lambda) = (e^{-it\lambda}-1)(e^{it\lambda}-1) = |e^{-it\lambda}-1|^2$. Since the spectral measure μ_φ has a finite mass, $|g_t(\lambda)| \le 2$ and $\lim_{t\to 0} g_t(\lambda) = 0$, $\forall \lambda \in \mathbb{R}$, we conclude by using Lebesgue's dominated convergence theorem that $\|U(t)\varphi - \varphi\|^2$ tends to 0 when $t \to 0$.

The proof of (iii) follows the same lines. let $\varphi \in D(A)$. Note that the fact that $A\varphi \in \mathcal{H}$ implies that $\int_{\mathbb{R}} \lambda^2 d\mu_\varphi(\lambda) < +\infty$. We have

$$\left\|\frac{e^{itA}-1-itA}{t}\varphi\right\|^2 = \int_{\mathbb{R}} \left|\frac{e^{it\lambda}-1-it\lambda}{t}\right|^2 d\mu_\varphi(\lambda) = \int_{\mathbb{R}} h_t(\lambda) d\mu_\varphi(\lambda),$$

where $h_t(\lambda) := \left|\frac{e^{it\lambda}-1-it\lambda}{t}\right|^2$. Note that

$$\left|\frac{e^{it\lambda}-1-it\lambda}{t}\right| \le \left|\frac{e^{it\lambda}-1}{t}\right| + |\lambda| \le 2|\lambda|,$$

and $\lim_{t\to 0} h_t(\lambda) = 0$, $\forall \lambda \in \mathbb{R}$. Hence the result follows also by Lebesgue's theorem.

To prove (iv) set

$$D(B) := \{\psi \in \mathcal{H};\ \lim_{t\to 0} \frac{U(t)\psi - \psi}{t} \text{ exists } \}$$

and, if $\psi \in D(B)$,

$$iB\psi := \lim_{t\to 0} \frac{U(t)\psi - \psi}{t}.$$

Then one can check that B is a non bounded (linear) operator. Let us show that it is symmetric. For any $\psi \in D(B)$ we observe that

$$\begin{array}{ccc} D(B) & \longrightarrow & \mathbb{C} \\ \varphi & \longmapsto & \lim_{t\to 0} \left\langle \psi, \frac{U(t)\varphi-\varphi}{it}\right\rangle \end{array}$$

is bounded (for the topology on $D(B)$ induced by its embedding in $\mathcal{H}$), since its values at φ is equal to

$$\lim_{t\to 0}\left\langle \frac{U(-t)\psi-\psi}{-it},\varphi\right\rangle = \lim_{t\to 0}\left\langle \frac{U(t)\psi-\psi}{it},\varphi\right\rangle = \langle B\psi,\varphi\rangle.$$

Hence $\psi \in D(B^*)$ and $B^*\psi = B\psi$. Thus $B \subset B^*$. But observe that, because of (iii), $A \subset B$, which implies that $B^* \subset A^*$ so that, since $A = A^*$, $B^* \subset A$. Hence $A \subset B \subset B^* \subset A$, which implies $A = B$. □

Definition 12.9 *A family $(U(t))_{t\in\mathbb{R}}$ of operators acting on a complex Hilbert space which satisfies (i), (ii), (iii) and (iv) in Theorem 12.4 is called a* ***strongly continuous 1-paramater family of unitary operators****.*

Theorem 12.4 has a converse.

Theorem 12.5 (Stone theorem) *Let $(U(t))_{t\in\mathbb{R}}$ be a strongly continuous 1-paramater family of unitary operators. Then there exists a self-adjoint non bounded operator A such that $U(t) = e^{itA}$, $\forall t \in \mathbb{R}$.*

Références

[1] John B. Conway, *A course in functional analysis*, Springer–Verlag, New York, 1990

[2] John B. Conway, *A course in operator theory*, Graduate Studies in Mathematics, vol 21, American Mathematical Society.

[3] Michael Reed, Barry Simon, *Methods of mathematical physics, I : Functional analysis*, Academic Press.