Université Denis Diderot — **2012-2013**
Mathématiques M1 — **Spectral Theory**

Corrigé de l'examen duu 16 janvier 2013
Durée 3 heures. Documents interdits.

---

1. Let $\mathcal{H}$ be complex Hilbert space, $A \in \mathcal{L}(\mathcal{H})$ and $P \in \mathbb{C}[X]$.
   (a) Recall the relation seen in the course which exists between $\mathrm{Sp}(P(A))$ and $\mathrm{Sp}A$ for a self-adjoint operator A. Show that the same result holds for an arbitrary bounded operator.
   Answer — The relation is
   $$\mathrm{Sp}(P(A)) = \{P(\lambda);\ \lambda \in \mathrm{Sp}A\}.$$
   It is proved by showing the inclusions $\mathrm{Sp}(P(A)) \subset \{P(\lambda);\ \lambda \in \mathrm{Sp}A\}$ and $\mathrm{Sp}(P(A)) \supset \{P(\lambda);\ \lambda \in \mathrm{Sp}A\}$. For the first inclusion one takes any $\mu \in \mathrm{Sp}(P(A))$, decomposes $P(X) - \mu = a(X - \lambda_1)\cdots(X - \lambda_n)$ and apply this identity by replacing X by A to conclude that $\mu = P(\lambda_j)$, for some $\lambda_j \in \mathrm{Sp}A$. For the second inclusion we take any $\lambda \in \mathrm{Sp}A$ and decompose the polynomial $P(X) - P(\lambda) = (X - \lambda)Q(X)$ and use the same reasoning to conclude that $P(\lambda) \in \mathrm{Sp}(P(A))$.
   In both reasoning the same kinds of arguments are used, which do not rely on the fact that A is self-adjoint. Hence the result actually extend to any bounded operator.
   (b) Prove that if $P(A) = 0$, then $\mathrm{Sp}A$ is contained in the set of roots of P.
   Answer — If $P(A) = 0$ we deduce from the previous question that $\{P(\lambda);\ \lambda \in \mathrm{Sp}A\} = \mathrm{Sp}(P(A)) = \{0\}$. Hence any λ in $\mathrm{Sp}A$ is contained in the set of roots of P.
2. Let $H = L^2(\mathbb{R}, \lambda)$ be the Hilbert space of complex-valued square integrable functions with respect to the Lebesgue measure. Let $f : \mathbb{R} \to \mathbb{C}$ be a measurable and λ-locally square integrable function on $\mathbb{R}$ (i.e. $\int_K |f|^2 d\lambda < \infty$ for all $K \subset \mathbb{R}$ compact). Define the unbounded operator M on H :
   $$\mathcal{D}(M) = \{\xi \in H\ :\ \int |f\xi|^2 d\lambda < \infty\} \quad (M\xi)(x) = f(x)\xi(x) \quad \xi \in \mathcal{D}(M),\ x \in \mathbb{R}.$$
   (a) Show that M is densely defined.
   Answer — It suffices to show that $\mathcal{D}(M)$ contains a dense subset of H, for instance the space $L^\infty_c(\mathbb{R}, \mathbb{C})$ of essentially bounded measurable functions with compact support. Indeed if $\xi \in L^\infty_c(\mathbb{R}, \mathbb{C})$, then $\mathrm{supp}\ \xi \subset K$, a compact subset of $\mathbb{R}$ and there exists some $M > 0$ s.t. $|\xi(x)| \leq M$, for a.e. $x \in \mathbb{R}$. Hence $\int |f\xi|^2 d\lambda \leq M^2 \int_K |f|^2 d\lambda < +\infty$, hence $\xi \in \mathcal{D}(M)$. Thus $L^\infty_c(\mathbb{R}, \mathbb{C}) \subset \mathcal{D}(M)$. The fact that $L^\infty_c(\mathbb{R}, \mathbb{C})$ is dense in H is a consequence of the fact that it contains the space $\mathcal{C}_c(\mathbb{R}, \mathbb{C})$ of continuous functions with compact support, which is known to be dense in H. Alternatively one may prove directly that $L^\infty_c(\mathbb{R}, \mathbb{C})$ is dense in H by proving that any function $\xi \in H$ is approximated by the sequence $(\xi_n)_n$ with values in $L^\infty_c(\mathbb{R}, \mathbb{C})$ defined by $\xi_n(x) := 1_{[-n,n]}(x) \inf(|\xi(x)|, n)\frac{\xi(x)}{n}$ and by using Lebesgue's dominated convergence theorem.

(b) Let T be the unbounded operator on H defined by

$$\mathcal{D}(T) = \mathcal{D}(M) \quad (T\xi)(x) = \overline{f(x)}\xi(x) \quad \xi \in \mathcal{D}(T),\ x \in \mathbb{R}.$$

Show that T is densely defined and $M = T^*$. Deduce that M is closed.
Answer — First $\mathcal{D}(T) = \mathcal{D}(M)$ is dense by the result of the previous question. The domain of T^* is the subspace of $\zeta \in H$ such that the linear form

$$\begin{array}{ccc} \mathcal{D}(T) & \longrightarrow & \mathbb{C} \\ \xi & \longmapsto & \int_{\mathbb{R}} \overline{\zeta}(x) f(x)\xi(x) d\lambda(x) \end{array}$$

admits a continuous linear extension to H. This linear form is uniquely defined because $\mathcal{D}(T)$ is dense in H. By the Riesz–Fréchet theorem, this property is equivalent to the fact that $[x \longmapsto \overline{f}(x)\zeta(x)]$ belongs to $L^2(\mathbb{R}, \mathbb{C})$. This latter condition is equivalent to $\zeta \in \mathcal{D}(M)$ and, if so, this linear map coincides with T. Hence $\mathcal{D}(T^*) = \mathcal{D}(T) = \mathcal{D}(M)$ and $T^* = M$. It follows that M is closed since it coincides with the adjoint operator of an operator defined on a dense domain.

(c) Compute M^*.
Answer — By changing $f \longmapsto \overline{f}$, we change T to M. Hence by applying the previous results, we deduce that $M^* = T$.

(d) Show that $\mathrm{sp}(M) = \mathrm{EssIm}(f)$ where

$$\mathrm{EssIm}(f) = \{\lambda \in \mathbb{C} : \forall \epsilon > 0\ \lambda(f^{-1}(B(\lambda, \epsilon))) > 0\}.$$

Answer — We first show that $\mathrm{EssIm}(f) \subset \mathrm{Sp}(M)$. Let $\lambda \in \mathrm{EssIm}(f)$. Then for any $n \in \mathbb{N}^*$, $\lambda[f^{-1}(B(\lambda, 1/n))] > 0$. Let A_n be a compact subset of $f^{-1}(B(\lambda, 1/n))$ such that $\lambda(A_n) > 0$. Set $\xi_n := 1_{A_n}/\sqrt{\lambda(A_n)}$, then $\int |\xi_n|^2 d\lambda = \lambda(A_n)^{-1} \int_{A_n} d\lambda = 1$. Moreover, $\forall x \in A_n$, $|f(x) - \lambda| < \frac{1}{n}$, so $f|_{A_n}$ is bounded by $|\lambda| + 1/n$. Hence $T\xi_n$, defined by $(T\xi_n)(x) = f(x)\xi_n(x)$, is in $L^2(\mathbb{R}, \mathbb{C})$, so $\xi_n \in \mathcal{D}(T)$. Now

$$\|(T - \lambda)\xi_n\|_H^2 = \int_{\mathbb{R}} |f(x) - \lambda|^2 |\xi_n(x)|^2 d\lambda \leq \frac{1}{n^2} \int_{A_n} |\xi_n(x)|^2 d\lambda = \frac{1}{n^2}.$$

Since $\|\xi_n\|_H^2 = 1$, this shows that $T - \lambda$ does not have a bounded inverse. Hence $\lambda \in \mathrm{Sp}(T)$.
Let us now show that $(\mathbb{C} \setminus \mathrm{EssIm}(f)) \subset (\mathbb{C} \setminus \mathrm{Sp}(T))$, i.e. that, for any λ which does not belong to $\mathrm{EssIm}(f)$, $T - \lambda$ is invertible. Indeed if $\lambda \notin \mathrm{EssIm}(f)$, then $\exists \varepsilon > 0$ such that $\lambda[f^{-1}(B(\lambda, \varepsilon))] = 0$, i.e. $\lambda(\{x; |f(x) - \lambda| < \varepsilon\}) = 0$. Hence, for a.e. $x \in \mathbb{R}$, $\frac{1}{|f(x)-\lambda|} \leq \frac{1}{\varepsilon}$. This proves that the operator $\xi \longmapsto \frac{\xi}{f-\lambda}$ is bounded, but this operator is the inverse of $T - \lambda$. Hence $\lambda \notin \mathrm{Sp}(T)$.

(e) Show that if f is continuous then $\mathrm{sp}(M) = \overline{f(\mathbb{R})}$.
Answer — Because of the previous question, it suffices to show that, if f is continuous, $\mathrm{EssIm}(f)) = \overline{f(\mathbb{R})}$. On the one hand, if $\lambda \in \mathrm{EssIm}(f))$, then, $\forall \varepsilon > 0$, $f^{-1}(B(\lambda, \varepsilon))$ has a non vanishing measure, thus is non empty, i.e. $f(\mathbb{R}) \cap B(\lambda, \varepsilon) \neq \emptyset$. Hence $\lambda \in \overline{f(\mathbb{R})}$ (note that, here, we do not need f to be continuous). On the other hand, if $\lambda \in \overline{f(\mathbb{R})}$, then $\forall \varepsilon > 0$, $B(\lambda, \varepsilon) \cap f(\mathbb{R}) \neq \emptyset$, i.e. $f^{-1}(B(\lambda, \varepsilon)) \neq \emptyset$. Since f is continuous $f^{-1}(B(\lambda, \varepsilon))$ is also open, hence it has a non vanishing measure.

(f) Let $A \subset \mathbb{R}$ be a Borel subset such that $\lambda(A) < \infty$, ν be the finite measure on $\mathbb{R}$ defined by $\nu(B) = \lambda(A \cap B)$ for all B Borel subset of $\mathbb{R}$. Observe that the integral with respect to ν is

$$\int g d\nu = \int_A g d\lambda.$$

Let $f_*(\nu)$ the measure image of ν by f i.e., $f_*(\nu)(B) = \nu(f^{-1}(B))$ for all Borel subset $B \subset \mathbb{C}$. Let $\xi = 1_A \in H$. Suppose that f is bounded and continuous. Show that M is bounded and the spectral measure μ_ξ of M associated to ξ is $\mu_\xi = f_*(\nu)$.

Answer — The fact that M is bounded follows from the fact that f is bounded, i.e. there exists $M > 0$ such that $|f(x)| \leq M$, $\forall x \in \mathbb{R}$. The spectral measure μ_ξ is defined by : $\forall \varphi \in \mathcal{C}(\mathbb{R}, \mathbb{C})$ (or for any Borelian function)

$$\int \varphi d\mu_\xi = \langle \xi, \varphi(M)\xi \rangle = \int_{\mathbb{R}} \overline{\xi(x)} \varphi(f(x))\xi(x) d\lambda = \int_A \varphi \circ f d\lambda = \int \varphi \circ f d\nu.$$

In the particular case where $\varphi = 1_B$ (where B is a Borelian subset of $\mathbb{C}$),

$$\int 1_B d\mu_\xi = \int 1_B \circ f d\nu = \int_{\mathbb{R}} 1_{f^{-1}(B)} d\nu = \nu(f^{-1}(B)) = f_*(\nu)(B)$$

and we deduce that $\mu_\xi(B) = \int 1_B d\mu_\xi = f_*(\nu)(B)$, for all Borelian subset B. This implies $\mu_\xi = f_*(\nu)$.

3. Let $\mathcal{H}$ be a complex Hilbert space and $T \in \mathcal{L}(\mathcal{H})$ be *self-adjoint*.

(a) Show that : $\forall n \in \mathbb{N}$, $||T^{2^n}|| = ||T||^{2^n}$.

Answer — We first show this result for $n = 1$, i.e. $||T^2|| = ||T||^2$. By using the fact that T^2 is self-adjoint and a result from the course, we have

$$\begin{aligned} ||T^2|| &= \sup_{||x|| \leq 1} |\langle x, T^2 x \rangle| = \sup_{||x|| \leq 1} |\langle T^* x, Tx \rangle| = \sup_{||x|| \leq 1} |\langle Tx, Tx \rangle| \\ &= \sup_{||x|| \leq 1} ||Tx||^2 = \left(\sup_{||x|| \leq 1} ||Tx|| \right)^2 = ||T||^2. \end{aligned}$$

Similarly, by using the fact that T^{2^n} is self-adjoint, we show that $||T^{2^n}|| = ||T^{2^{n-1}}||^2$. We conclude by recursion.

(b) Show by recursion that the following property $\mathcal{P}(n)$ is true for any $n \in \mathbb{N}$:

$$\mathcal{P}(n) : \quad \forall k \in \mathbb{N} \text{ such that } 0 \leq k \leq 2^n, \quad ||T^k|| = ||T||^k.$$

[Hint : for $2^n < k \leq 2^{n+1}$, consider $\ell := 2^{n+1} - k$ and $T^\ell T^k$.]

Answer — The property $\mathcal{P}(0)$ is straightforward ($||T^0|| = ||T||^0$ et $||T|| = ||T||$). Assume that we have shown $\mathcal{P}(n)$ let us prove $\mathcal{P}(n+1)$: it is enough to show that $||T^k|| = ||T||^k$ for $2^n < k \leq 2^{n+1}$. Let $\ell := 2^{n+1} - k$, then $0 \leq \ell < 2^n$. Hence by using the result of the previous question, we get

$$||T||^{2^{n+1}} = ||T^{2^{n+1}}|| = ||T^\ell T^k|| \leq ||T^\ell|| \, ||T^k||.$$

We can then use $\mathcal{P}(n)$ with ℓ, i.e. $||T^\ell|| = ||T||^\ell$. We hence obtain

$$||T||^{\ell+k} \leq ||T||^\ell ||T^k||$$

which gives by simplification $||T||^k \leq ||T^k||$. The reverse inequality $||T||^k \geq ||T^k||$ is actually true for any operator. We deduce $||T||^k = ||T^k||$.

Conclusion : we have shown that, for any **self-adjoint** operator $T \in \mathcal{L}(\mathcal{H})$,

$$\forall n \in \mathbb{N}, \quad ||T^n|| = ||T||^n.$$

4. Let $\mathcal{H}$ be a complex Hilbert space. The goal of this exercise is to show that there are no *bounded self-adjoint*[1] operators $P, Q \in \mathcal{L}(\mathcal{H})$ such that $[P, Q] = \lambda 1_{\mathcal{H}}$, where $1_{\mathcal{H}} \in \mathcal{L}(\mathcal{H})$ is the identity operator, λ is a complex constant[2] and $[P, Q] := PQ - QP$.

   (a) Assume that $\mathcal{H}$ has a finite dimension n. Show, by using a simple argument, that there are no operators $P, Q \in \mathcal{L}(\mathcal{H})$ such that $[P, Q] = 1_{\mathcal{H}}$.
   Answer — On the one hand the trace of $[P, Q]$ vanishes, on the other hand the trace of $1_{\mathcal{H}}$ is n.

   (b) In the following we assume that the dimension of $\mathcal{H}$ is infinite. Show that, for any pair of operators $P, Q \in \mathcal{L}(\mathcal{H})$, we have :

$$\forall n \in \mathbb{N}^*, \quad [P^n, Q] = \sum_{j=1}^{n} P^{n-j}[P, Q]P^{j-1}.$$

   Answer — We write that

$$\begin{aligned}
[P^n, Q] &= P^nQ - QP^n \\
&= (P^nQ - P^{n-1}QP) + (P^{n-1}QP - P^{n-2}QP^2) + \cdots + (PQP^{n-1} - QP^n) \\
&= \textstyle\sum_{j=1}^{n} P^{n-j}(PQ - QP)M^{j-1}.
\end{aligned}$$

   (c) We argue by contradiction and we assume that there exists operators $P, Q \in \mathcal{L}(\mathcal{H})$ such that $[P, Q] = 1_{\mathcal{H}}$. By using the result of the previous question and of the previous exercise, show that, for any $n \in \mathbb{N}^*$,

$$2||P||\ ||Q|| \geq n$$

   and conclude to a contradiction.
   Answer — *First method (assuming that P and Q are self-adjoint) :* Because of the previous question and the hypothesis $[P, Q] = 1_{\mathcal{H}}$, we have $[P^n, Q] = nP^{n-1}$. Hence, by using the result of Exercise 1,

$$n||P||^{n-1} = ||nP^{n-1}|| = ||[P^n, Q]|| \leq 2||P^n||\ ||Q|| = 2||P||^n||Q||,$$

   which gives us, by dividing by $||P||^{n-1}$, that $n \leq 2||P||\ ||Q||$. Since n can be chosen arbitrarily, this is impossible, since P and Q are bounded.
   Second method (without assuming that P and Q are selfadjoint) :

$$n\|P^{n-1}\| = \|[P, Q]\| = \|P^{n-1}PQ - QPP^{n-1}\| \leq \|P^{n-1}\|(\|P\|\,\|Q\| + \|Q\|\,\|P\|)$$

   implies $n \leq 2\|P\|\,\|Q\|$.

1. Sorry, this hypothesis was missing : the aim of this question was to use the result of the previous exercise on self-adjoint operators. We shall see, however, that the hypothesis that P and Q be self-adjoint is not essential.
2. The reason for this other modification of the hypothesis (we set $\lambda = 1$ in the original exam) is that, if P and Q are self-adjoint, then $[P, Q]$ is necessarily skew adjoint, i.e. satisfies $[P, Q]^* = -[P, Q]$, so that condition $[P, Q] = \lambda 1_{\mathcal{H}}$ makes sense iff λ is complex imaginary.

(d) Give an example of a complex Hilbert space $\mathcal{H}$ and two *non bounded* operators P and Q such that there exists a dense vector subspace $V \subset \mathcal{H}$ such that $\forall \varphi \in V$, $[P, Q]\varphi = \varphi$
Answer — We may choose $\mathcal{H} := L^2(\mathbb{R}, \mathbb{C})$, $\mathcal{D}(P) = H^1(\mathbb{R}, \mathbb{C})$, $P = i\frac{d}{dx}$, $\mathcal{D}(Q) = \{f \in L^2(\mathbb{R}, \mathbb{C});\ \int_{\mathbb{R}} |xf(x)|^2 dx < +\infty\}$ and Q defined by $(Qf)(x) = xf(x)$.

5. Let $\mathcal{H} := \ell^2(\mathbb{Z}, \mathbb{C}) \simeq \ell^2(\mathbb{Z})$. We denote by $(\epsilon_n)_{n\in\mathbb{Z}}$ the canonical Hilbertian Hermitian orthogonal basis of $\mathcal{H}$ (i.e. ϵ_n is the sequence which vanishes for all relative integer, excepted for n, for which it takes the value 1). We note $L, R \in \mathcal{L}(\mathcal{H})$ the operators defined by :

$$L\epsilon_n = \epsilon_{n-1} \quad \text{et} \quad R\epsilon_n = \epsilon_{n+1}, \quad \forall n \in \mathbb{Z}$$

and $A := L + R \in \mathcal{L}(\mathcal{H})$.

(a) Compute L^* and R^*. Deduce that A is self-adjoint.
Answer — For all $x = \sum_{n\in Z} x_n\epsilon_n$ and $y = \sum_{n\in Z} y_n\epsilon_n$, we have $Ly = \sum_{n\in Z} y_n\epsilon_{n-1} = \sum_{n\in Z} y_{n+1}\epsilon_n$ and thus :

$$\langle x, Ly\rangle = \sum_{n\in Z} x_n y_{n+1} = \sum_{n\in Z} x_{n-1} y_n = \langle Rx, y\rangle,$$

since $Rx = \sum_{n\in Z} x_n\epsilon_{n+1} = \sum_{n\in Z} x_{n-1}\epsilon_n$. Hence $L^* = R$. Similarly we obtain that $R^* = L$. We deduce then that A is self-adjoint.

(b) We note $U : \ell^2(\mathbb{Z}) \longrightarrow L^2(\mathbb{R}/\mathbb{Z})$ the unitary operator defined by : $(U\epsilon_n)(\theta) = e^{i2\pi n\theta}$, $\forall n \in \mathbb{Z}$ (Fourier series isomorphism). For all function $m \in L^\infty(\mathbb{R}/\mathbb{Z})$ we note $\widehat{m} \in \mathcal{L}\left(L^2(\mathbb{R}/\mathbb{Z})\right)$ the multiplication operator defined by :

$$\forall f \in L^2(\mathbb{R}/\mathbb{Z}), \quad (\widehat{m}f)(\theta) = m(\theta)f(\theta), \quad \text{p.p.}$$

Find ULU^{-1} and URU^{-1} and show that they coincide with multiplication operators by functions to be precised. Deduce UAU^{-1}.
Answer — For any function $f \in L^2(\mathbb{R}/\mathbb{Z})$ the Fourier series of which is $f(\theta) = \sum_{n\in Z} a_n e^{i2\pi n\theta}$, we have $U^{-1}f = \sum_{n\in Z} a_n\epsilon_n$ and thus $LU^{-1}f = \sum_{n\in Z} a_n\epsilon_{n-1}$, which implies :

$$ULU^{-1}f(\theta) = \sum_{n\in Z} a_n e^{i2\pi(n-1)\theta} = e^{-i2\pi\theta}\sum_{n\in Z} a_n e^{i2\pi n\theta} = e^{-i2\pi\theta}f(\theta).$$

Hence ULU^{-1} coincides with the multiplication the multiplication operator by the function $\theta \longmapsto e^{-i2\pi\theta}$. Similarly URU^{-1} coincides with the multiplication operator by the function $\theta \longmapsto e^{i2\pi\theta}$. We deduce that $UAU^{-1} = ULU^{-1} + URU^{-1}$ coincides with the multiplication operator by the function $\theta \longmapsto e^{i2\pi\theta} + e^{-i2\pi\theta} = 2\cos(2\pi\theta)$, i.e. $UAU^{-1} = 2\,\widehat{\cos(2\pi\theta)}$.

(c) Let $\psi \in \mathcal{H}$ be different of 0 and $g := U\psi$. We note $F_\psi := \{P(A)\psi |\ P \in \mathbb{C}[X]\} = \mathrm{Vect}_{\mathbb{C}}\{A^n\psi |\ n \in \mathbb{N}\}$.

Show that the UF_ψ, the image of F_ψ by U, is equal to : $\mathcal{P}_+ g := \{fg|\ f \in \mathcal{P}_+\}$, where $\mathcal{P}_+$ is the subspace of $L^2(\mathbb{R}/\mathbb{Z})$ of polynomials in $e^{i2\pi\theta}$ and $e^{-i2\pi\theta}$ which are *even* function of θ.
Answer — A consequence of the previous question is that, $\forall n \in \mathbb{N}$,

$$U(A^n\psi) = UA^nU^{-1}U\psi = (UAU^{-1})^n g = (2\cos(2\pi\theta))^n g,$$

hence, more generally, for any polynomial $P \in \mathbb{C}[X]$, $U(P(A)\psi) = P(2\cos(2\pi\theta))g$. We deduce that UF_ψ coincides with $\{P(2\cos(2\pi\theta))g|\ P \in \mathbb{C}[X]\}$. Hence it suffices to remark[3] that the set $\mathcal{Q} := \{P(2\cos(2\pi\theta))|\ P \in \mathbb{C}[X]\}$ coincides with $\mathcal{P}_+$
Lastly the closure of $\mathcal{P}_+$ dans $L^2(\mathbb{R}/\mathbb{Z})$ is the set of *even* functions of the variable θ.

(d) Let ψ and g be as in the preceding question. We define $h \in L^2(\mathbb{R}/\mathbb{Z})$ by :

$$h(\theta) = 2i\sin(2\pi\theta)\overline{g(-\theta)}.$$

Show that, $\forall f_1, f_2 \in \mathcal{P}_+$, $\langle f_1 g, f_2 h\rangle_{L^2} = 0$.
Answer — For any functions $f_1, f_2 \in \mathcal{P}_+$, we have :

$$\langle f_1 g, f_2 h\rangle_{L^2} = \int_{\mathbb{R}/\mathbb{Z}} \overline{f_1(\theta)g(\theta)} f_2(\theta) 2i\sin(2\pi\theta)\overline{g(-\theta)}d\theta,$$

which vanishes since the functions $\theta \longmapsto \overline{f_1(\theta)}f_2(\theta)$ and $\theta \longmapsto \overline{g(\theta)g(-\theta)}$ are even and the function $\theta \longmapsto 2i\sin(2\pi\theta)$ is odd, so that $\langle f_1 g, f_2 h\rangle_{L^2}$ is equal to the value of the integral of an odd function.

(e) Deduce from the preceding questions that A does not admit a cyclic vector.
Answer — For any $\psi \in \mathcal{H}\setminus\{0\}$, let $g := U\psi$ and $\chi := U^{-1}h \in \mathcal{H}$, where h is defined in terms of g as in the previous question. Then, because of Parseval's inequality, for any polynomials $P_1, P_2 \in \mathbb{C}[X]$,

$$\langle P_1(A)\psi, P_2(A)\chi\rangle = \langle UP_1(A)\psi, UP_2(A)\chi\rangle_{L^2} = \langle f_1 g, f_2 h\rangle_{L^2},$$

where $f_1, f_2 \in \mathcal{P}_+$ are such that $\widehat{f_1} = UP_1(A)U^{-1}$ and $\widehat{f_2} = UP_2(A)U^{-1}$ (recall that we denote by $\widehat{f}$ the multiplication operator by the function f). Hence, by the result of the previous question, $\langle P_1(A)\psi, P_2(A)\chi\rangle = 0$. This implies that $F_\psi \perp F_\chi$. Since $F_\chi \neq \{0\}$, we deduce that F_ψ is not dense in $\mathcal{H}$, i.e. that ψ is not cyclic for A.

3. Let us show it by recursion on N : the set $\mathcal{P}_N := \left\{\sum_{-N\le n\le N} a_n e^{i2\pi n\theta}|\ a_n \in \mathbb{C}, a_{-n} = a_n\right\}$ is equal to $\mathcal{Q}_N := \{P(2\cos(2\pi\theta))|\ \deg P \le N\}$. For $N = 0$, the result is straightforward. Assume that $\mathcal{P}_{N-1} = \mathcal{Q}_{N-1}$ let us show that $\mathcal{P}_N = \mathcal{Q}_N$. We use the identity :

$$(e^{i2\pi\theta} + e^{-i2\pi\theta})^N = \sum_{j=0}^{N} \frac{N!}{(N-j)!j!} e^{i2\pi(2j-N)\theta} = e^{i2\pi N\theta} + e^{-i2\pi N\theta} + \mathcal{R}_N(\theta), \tag{1}$$

where $\mathcal{R}_N(\theta) = \sum_{j=1}^{N-1} \frac{N!}{(N-j)!j!} e^{i2\pi(2j-N)\theta}$. It is clear that $\mathcal{R}_N \in \mathcal{P}_{N-1} = \mathcal{Q}_{N-1}$ (note in particular that $\mathcal{R}_N(\theta) = \mathcal{R}_N(-\theta)$). We deduce $e^{i2\pi N\theta} + e^{-i2\pi N\theta} \in \mathcal{Q}_N$ and $(e^{i2\pi\theta} + e^{-i2\pi\theta})^N \in \mathcal{P}_N$. Any element of $\mathcal{P}_N$ reads $a(e^{i2\pi N\theta} + e^{-i2\pi N\theta}) + \alpha(\theta)$, where $\alpha \in \mathcal{P}_{N-1}$, and is thus contained in $\mathcal{Q}_N$. Similarly any element of $\mathcal{Q}_N$ reads $b(e^{i2\pi\theta} + e^{-i2\pi\theta})^N + \beta(\theta)$, where $\beta \in \mathcal{Q}_{N-1}$, and is thus also contained in $\mathcal{P}_N$.