Université Denis Diderot — **2010-2011**
Mathématiques M1 — **Spectral Theory**

Examen
Durée 3 heures

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1. Let $A, B, T \in \mathcal{B}(H)$ with $T = T^*$. We suppose that $AT = TB$ and $TA = BT$.
   (a) Let $g \in C(\mathbb{R})$ be an even function. Show that $g(T)$ commutes with A and B.
   (b) Let $f \in C(\mathbb{R})$ be an odd function. Show that $Af(T) = f(T)B$.

   Answer — We first show by recursion on n the property :

$$\mathcal{P}_n : \quad T^{2n}A = AT^{2n}, \quad T^{2n}B = BT^{2n}, \quad T^{2n+1}A = T^{2n+1}B, \quad T^{2n+1}B = AT^{2n+1}.$$

   For $n = 0$, $\mathcal{P}_0$ is a straightforward consequence of the hypotheses $AT = TB$ and $TA = BT$. Now assume that $\mathcal{P}_n$ has been proved. Then we write :

$$T^{2n+2}A = T^{2n+1}TA = T^{2n+1}BT = AT^{2n+1}T = AT^{2n+2}$$

   and we obtain by a similar computation that $T^{2n+2}B = AT^{2n+2}$. Similarly we obtain

$$T^{2n+3}A = T^{2n+2}TA = T^{2n+2}BT = BT^{2n+2}T = BT^{2n+3}$$

   and also $T^{2n+3}B = AT^{2n+3}$. Hence $\mathcal{P}_{n+1}$ holds. It follows that, for any odd polynomial P, we have $AP(T) = P(T)B$ and, for any even polynomial Q, $[A, Q(T)] = [B, Q(T)] = 0$. Now we use the fact that T is self-adjoint, so that we can define a continuous functional calculus $\mathcal{C}^0(\mathrm{Sp}T, \mathbb{C}) \ni f \longmapsto f(T) \in \mathcal{B}(H)$, which satisfies in particular $||f(T)||_{\mathcal{B}(H)} \leq ||f||_{\mathcal{C}^0(\mathrm{Sp}T)}$. Moreover by the Stone–Weierstrass theorem the set of restrictions of polynomial functions on $\mathrm{Sp}T$ is dense in $\mathcal{C}^0(\mathrm{Sp}T, \mathbb{C})$. Hence for any, say, even function $g \in \mathcal{C}^0(\mathrm{Sp}T, \mathbb{C})$, there exists a sequence of polynomials $(Q_n)_n$ s.t. $Q_n|_{\mathrm{Sp}T}$ converges to g. Then one proves easily that if we set : $\widehat{Q}_n(x) = \frac{1}{2}(Q_n(x) + Q_n(-x))$, then $\widehat{Q}_n$ is obviously even and $\widehat{Q}_n|_{\mathrm{Sp}T}$ converges to g also in $\mathcal{C}^0(\mathrm{Sp}T, \mathbb{C})$ and hence $\widehat{Q}_n(T)$ converges to $g(T)$ in $\mathcal{B}(H)$. Since $[\widehat{Q}_n(T), A] = [\widehat{Q}_n(T), B] = 0$, (a) follows. The proof of (b) is similar.

2. Let U be the bilateral shift on $\ell^2(\mathbb{Z})$ ie, $Ue_n = e_{n+1}$ where $(e_n)_{n\in\mathbb{Z}}$ is the canonical basis. We recall that the spectrum of U is the unit circle. Let $k \in \mathbb{Z}$. Show that e_k is a cyclic vector for U and compute the associated spectral measure.
   Answer — The vector e_k is cyclic iff the vector subspace spanned by $(U^n(U^*)^m e_k)_{n,m\in\mathbb{N}}$ is dense in $\ell^2(\mathbb{Z})$. Since $U^* = U^{-1}$ (this can be seen from the very definition of U or as a consequence of the fact that U is unitary because its spectrum is a subset of the unit circle), we have $U^n(U^*)^m = U^{n-m}$, so that this amounts to show that the vector subspace spanned by $(U^n e_k)_{n\in\mathbb{Z}}$ is dense in $\ell^2(\mathbb{Z})$. But this is clearly the fact, for $(e_{n+k})_{n\in\mathbb{Z}}$ being

a Hilbert basis of $\ell^2(\mathbb{Z})$.

The spectral measure $d\mu_{e_k}$ on the circle $S^1 \subset \mathbb{C}$ is defined by the relation

$$\forall f \in \mathcal{C}^0(S^1, \mathbb{C}), \quad \int_{S^1} f(e^{i\theta}) d\mu_{e_k}(\theta) = \langle e_k, f(U) e_k \rangle.$$

By using a Fourier series decomposition $f(e^{i\theta}) = \sum_{n\in\mathbb{Z}} \widehat{f}_n e^{in\theta}$ and the fact that $U^* = U^{-1}$, we obtain $f(U) = \sum_{n\in\mathbb{Z}} \widehat{f}_n U^n$. Hence

$$\langle e_k, f(U)e_k \rangle = \sum_{n\in\mathbb{Z}} \langle e_k, \widehat{f}_n U^n e_k \rangle = \sum_{n\in\mathbb{Z}} \widehat{f}_n \langle e_k, e_{n+k} \rangle = \widehat{f}_0 = \frac{1}{2\pi} \int_{S^1} f(e^{i\theta}) d\theta,$$

so that

$$d\mu_{e_k}(\theta) = \frac{1}{2\pi} d\theta.$$

3. Let $a = (a_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers and M_a be the following operator on $\ell^2(\mathbb{N})$

$$\mathcal{D}(M_a) = \{x = (x_n)_n \in \ell^2(\mathbb{N}) \,:\, \sum_n |a_n|^2 |x_n|^2 < \infty\}, \quad M_a x = (a_n x_n)_n.$$

(a) Show that M_a is densely defined and closed.

Answer — The fact that $\mathcal{D}(M_a)$ is dense in $\ell^2(\mathbb{N})$ is a consequence of the fact that $\mathcal{D}(M_a)$ contains all finite sequences and that finite sequences are dense in $\ell^2(\mathbb{N})$. We now need to show that $\mathrm{Gr} M_a$ is a closed subspace of $\ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$. Consider a sequence $(x^k, M_a x^k)_{k\in\mathbb{N}}$ with values in $\mathrm{Gr} M_a$ and assume that $(x^k, M_a x^k)$ converges to some $(x, y) \in \ell^2(\mathbb{N}) \times \ell^2(\mathbb{N})$, when $k \to +\infty$. Setting $x^k = (x_0^1, x_1^k, \cdots)$ and $x = (x_0, x_1, \cdots)$, this implies immediately that, for any fixed $n \in \mathbb{N}$, $\lim_{k\to\infty} x_n^k = x_n$ in $\mathbb{C}$ and hence that $\lim_{k\to\infty} a_n x_n^k = a_n x_n$ in $\mathbb{C}$. Hence since we also have $\lim_{k\to\infty} a_n x_n^k = y_n$, this implies that $a_n x_n = y_n$, $\forall n \in \mathbb{N}$. Hence, since $y \in \ell^2(\mathbb{N})$, $\sum_{n\in\mathbb{N}} |a_n|^2 |x_n|^2 < +\infty$ and we deduce that $x \in \mathcal{D}(M_a)$ and $y = M_a x$, i.e. $(x, y) \in \mathrm{Gr} M_a$. Hence $\mathrm{Gr} M_a$ is closed.

(b) Show that $\mathrm{sp}(M_a) = \overline{\{a_n \,:\, n \in \mathbb{N}\}}$.

Answer — We first observe that any value a_n is an eigenvalue of M_a for at least the eigenvector e_n. Hence $\{a_n \,:\, n \in \mathbb{N}\} \subset \mathrm{Sp}_p M_a \subset \mathrm{Sp} M_a$. Since we know from the course that the spectrum of any operator is closed, this implies that $\overline{\{a_n \,:\, n \in \mathbb{N}\}} \subset \mathrm{Sp} M_a$. Now let $b \in \mathbb{C} \setminus \overline{\{a_n \,:\, n \in \mathbb{N}\}}$. Then, since $\overline{\{a_n \,:\, n \in \mathbb{N}\}}$ is closed, there exists some $\varepsilon > 0$ s.t. $B(b, \varepsilon) \cap \{a_n \,:\, n \in \mathbb{N}\} = \emptyset$. We will then first show that $M_a - b$ is a bijection between $\mathcal{D}(M_a)$ and $\ell^2(\mathbb{N})$: given some $y \in \ell^2(\mathbb{N})$ we need to prove that there exists an unique $x \in \mathcal{D}(M_a)$ s.t. $(M_a - b)x = y$. If such an x would exist, it would be the unique solution of the equation $(a_n - b)x_n = y_n \Longleftrightarrow x_n = y_n/(a_n - b)$, $\forall n \in \mathbb{N}$. Lastly one needs to prove that $(b - M_a)^{-1} : \ell^2(\mathbb{N}) \longrightarrow \ell^2(\mathbb{N})$ is bounded, a consequence of :

$$||(b - M_a)^{-1} y||^2 = \sum_{n\in\mathbb{N}} \frac{|y_n|^2}{|b - a_n|^2} \leq \sum_{n\in\mathbb{N}} \frac{|y_n|^2}{\varepsilon^2} = \frac{||y||^2}{\varepsilon^2}.$$

Hence the resolvent set of M_a contains $\mathbb{C} \setminus \overline{\{a_n \,:\, n \in \mathbb{N}\}}$, which is equivalent to say that $\mathrm{Sp} M_a \subset \overline{\{a_n \,:\, n \in \mathbb{N}\}}$.

4. Let $a = (a_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers indexed by $\mathbb{Z}$ and T_a be the operator defined on $\ell^2(\mathbb{Z})$ by

$$\mathcal{D}(T_a) = \{x = (x_n)_{n\in\mathbb{Z}} \in \ell^2(\mathbb{Z}) \,:\, \sum_{n\in\mathbb{Z}} |a_n|^2 |x_{-n}|^2 < \infty\}, \quad T_a x = (a_n x_{-n})_{n\in\mathbb{Z}}.$$

(a) Show that T_a is densely defined and closed.
Answer — Use the same method as in question 3 – (a).

(b) Compute T_a^*.
Answer — We first compute its domain $\mathcal{D}(T_a^*)$: this is the set of $y \in \ell^2(\mathbb{Z})$ s.t. the linear form

$$\begin{array}{ccc} \mathcal{D}(T_a) & \longrightarrow & \mathbb{C} \\ x & \longmapsto & \langle y, T_a x\rangle \end{array}$$

admits a continuous extension on $\ell^2(\mathbb{Z})$ (which is then unique since $\mathcal{D}(T_a)$ is dense in $\ell^2(\mathbb{Z})$). By Riesz' theorem this property is equivalent to say that there exists some $z \in \ell^2(\mathbb{Z})$ s.t. $\langle y, T_a x\rangle = \langle z, x\rangle$, $\forall x \in \mathcal{D}(T_a)$. But

$$\langle y, T_a x\rangle = \sum_{n\in\mathbb{Z}} \overline{y_n} a_n x_{-n} = \sum_{n\in\mathbb{Z}} \overline{y_{-n}} a_{-n} x_n.$$

Hence such a z exists, its satisfies $z_n = \overline{a_{-n}} y_{-n}$. Thus $y \in \mathcal{D}(T_a^*)$ iff $\sum_{n\in\mathbb{Z}} |\overline{a_{-n}} y_{-n}|^2 < +\infty$, i.e.

$$\mathcal{D}(T_a^*) = \{y \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |a_n y_n|^2 < +\infty\} \quad \text{and} \quad \forall y \in \mathcal{D}(T_a^*), \quad T_a^* y = (\overline{a_{-n}} y_{-n})_{n\in\mathbb{Z}}.$$

(c) Find a necessary and sufficient condition on a for T_a to be normal.
Answer — For any $x \in \mathcal{D}(T_a T_a^*) := \{x \in \mathcal{D}(T_a^*) :\ T_a^* x \in \mathcal{D}(T_a)\}$,

$$T_a T_a^* x = T_a \left((\overline{a_{-n}} x_{-n})_n \right) = \left(a_n \left(\overline{a_n} x_n \right)_n \right) = \left(|a_n|^2 x_n \right)_n,$$

wheras for any $x \in \mathcal{D}(T_a^* T_a)$,

$$T_a^* T_a x = T_a^* \left((a_n x_{-n})_n \right) = \left(\overline{a_{-n}} \left(a_{-n} x_n \right)_n \right) = \left(|a_{-n}|^2 x_n \right)_n,$$

Whatever $\mathcal{D}(T_a T_a^*)$ and $\mathcal{D}(T_a^* T_a)$ are, they contain the space of finite sequences. Hence a necessary condition for T_a to be normal is that, for any finite sequence $x = (x_n)_n$, $|a_n|^2 x_n = |a_{-n}|^2 x_n$, $\forall n \in \mathbb{Z}$, which implies that :

$$|a_n| = |a_{-n}|, \quad \forall n \in \mathbb{Z}.$$

Conversely it is clear from the preceding computation that, if the above condition holds, then $\mathcal{D}(T_a T_a^*) = \mathcal{D}(T_a^* T_a)$ and $T_a T_a^* = T_a^* T_a$, i.e. T_a is normal.

(d) Find a necessary and sufficient condition on a for T_a to be bounded.
Answer — The operator T_a is bounded iff $\mathcal{D}(T_a) = \ell^2(\mathbb{Z})$ and there exists a constant $C \in [0, +\infty)$ s.t. $\forall x \in \ell^2(\mathbb{Z})$, $||T_a x|| \leq C||x||$. Testing this condition with $x = e_n$, for any $n \in \mathbb{Z}$ implies that $|a_n| \leq C$. Conversely, it is easy to check that the latter condition implies that T_a is bounded. Hence the necessary and sufficient condition is : the sequence $(a_n)_{n\in\mathbb{Z}}$ is bounded.

(e) Compute $\mathrm{sp}(T_a^* T_a)$ and $\mathrm{sp}(T_a T_a^*)$.
Answer — We have seen in question c) that $T_a^* T_a$ has the domain $\{x \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |a_{-n}|^4 |x_n|^2 < +\infty\}$ and is defined by $T_a^* T_a(x) = (|a_{-n}|^2 x_n)_n$. Hence by a reasoning similar to the question 3 – (b), we deduce that the spectrum of $T_a^* T_a$ is $\overline{\{|a_{-n}|^2 : n \in \mathbb{Z}\}} = \overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$. Similarly $T_a T_a^*$ has the domain $\{x \in \ell^2(\mathbb{Z}) : \sum_{n\in\mathbb{Z}} |a_n|^4 |x_n|^2 < +\infty\}$ and is defined by $T_a^* T_a(x) = (|a_n|^2 x_n)_n$. Hence the spectrum of $T_a T_a^*$ is also $\overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$.

(f) Find a necessary and sufficient condition on a for $T_a^* T_a$ (resp. $T_a T_a^*$) to be compact.
Answer — Assume that $T_a^* T_a$ is compact. Note also that this operator is self-adjoint. Then it follows from the course that the spectrum of $T_a^* T_a$ is equal to $\{0\} \cup \Lambda$, where Λ is a subset of $\mathbb{C} \setminus \{0\}$ which is at most countable and has no accumulation point, excepted may be 0. In particular, for any $r > 0$, $\mathrm{Sp}T_a^* T_a \cap (\mathbb{C} \setminus B(0, r^2))$ is finite. Moreover Λ is composed of eigenvalues λ associated with *finite* dimensional vector eigenspaces. However we have seen in the preceding question that $\mathrm{Sp}T_a^* T_a = \overline{\{|a_n|^2 : n \in \mathbb{Z}\}}$ and the dimension of the eigenspace corresponding to any value $\lambda \in \Lambda$ is the cardinal of $\{n \in \mathbb{Z} : |a_n|^2 = \lambda\}$. We hence deduce that the number of values $n \in \mathbb{Z}$ s.t. $|a_n|^2 \geq r^2$ is finite. In particular, if we set $N(r) := \sup\{|n| \in \mathbb{Z} : |a_n|^2 \geq r\}$, we have $\forall n \in \mathbb{Z}$ s.t. $|n| > N(r)$, $|a_n| < r$. Hence $\lim_{|n|\to\infty} a_n = 0$.
Conversely if $\lim_{|n|\to\infty} a_n = 0$, then we define for any $r > 0$ the operator K_r on $\ell^2(\mathbb{Z})$ by

$$K_r x = \sum_{n\in\mathbb{Z}; |a_n|^2 \geq r^2} |a_{-n}|^2 x_n e_n.$$

Then $||T_a^* T_a - K_r|| \leq r^2$, so that $\lim_{r\to 0} ||T_a^* T_a - K_r|| = 0$ and each K_r is a finite rank operator. Hence $T_a^* T_a$ is compact.
A similar reasonning shows that $T_a T_a^*$ is compact iff the same condition holds, i.e.

$$\lim_{|n|\to\infty} a_n = 0.$$

5. Let $\mathcal{H}$ be a complex separable Hilbert space. Let A be a non bounded self-adjoint operator such that its domain $\mathcal{D}(A)$ is dense in $\mathcal{H}$. We assume that A is positive, i.e. that $\forall \varphi \in \mathcal{D}(A)$, $\langle \varphi, A\varphi \rangle \geq 0$.

(a) recall the spectral decomposition theorem for A ;
Answer — There exists a topological space X endowed with a Borelian measure μ, an unitary operator $U : \mathcal{H} \longrightarrow L^2(X, \mu; \mathbb{C})$ and a real-valued Borelian [1] function $f : X \longrightarrow \mathbb{C}$ s.t., for any function $\varphi \in U(\mathcal{D}(A))$,

$$(UAU^{-1}\varphi)(x) = f(x)\varphi(x), \quad \text{for } \mu - \text{a.e. } x \in X.$$

In the following we will write $L^2(X) = L^2(X, \mu; \mathbb{C})$ for shortness. Note that the positivity condition translates here as $\forall \varphi \in U(\mathcal{D}(A))$, $\int_X f(x)|\varphi(x)|^2 d\mu(x) \geq 0$. Moreover since $\mathcal{D}(A)$ is dense in $\mathcal{H}$, $U(\mathcal{D}(A))$ is dense in $L^2(X)$ and hence the latter condition implies that f is real and $f \geq 0$, $\mu-$a.e.

1. not necessarily bounded

(b) Prove that, for any $t > 0$, one can define the operator e^{-tA} and that this operator is bounded on $\mathcal{H}$;

Answer — Since the function $x \longmapsto e^{-tf(x)}$ is Borelian and bounded, it suffices to define e^{-tA} by : $\forall \varphi \in L^2(X)$,

$$(Ue^{-tA}U^{-1}\varphi)(x) = e^{-tf(x)}\varphi(x), \quad \text{for } \mu - \text{a.e. } x \in X.$$

This operator is bounded and its norm is equal to the supremum of $x \longmapsto e^{-tf(x)}$ over X.

Beware : it is not possible to use the series $\sum_{n\geq 0}(-tA)^n/n!$, which does not converge since A is not bounded.

(c) Prove that $\forall t, s > 0$, $e^{-tA}e^{-sA} = e^{-(t+s)A}$;

Answer — This follows from the definition of e^{-tA} and the fact that $e^{-tf(x)}e^{-sf(x)} = e^{-(t+s)f(x)}$.

(d) Prove that $\forall t_0 > 0$, $\lim_{t>0, t\to 0} e^{-tA} = e^{-t_0 A}$;

Answer — We need to evaluate the limit of $||e^{-tA} - e^{-t_0 A}||_{\mathcal{H}}$, when $t \to t_0$. Without loss of generality we can assume that $t_0/2 < t < 3t_0/2$. For any $u \in \mathcal{H}$, let $\varphi := Uu \in L^2(X)$, so that $u = U^{-1}\varphi$. Then

$$||(e^{-tA} - e^{-t_0 A})u||_{\mathcal{H}}^2 = ||(e^{-tf} - e^{-t_0 f})\varphi||_{L^2}^2 = \int_X |e^{-tf(x)} - e^{-t_0 f(x)}|^2 |\varphi(x)|^2 d\mu(x).$$

By the mean value theorem, for any $x \in X$, there exists some τ between t_0 and t s.t. $|e^{-tf(x)} - e^{-t_0 f(x)}| = |-f(x)e^{-\tau f(x)}(t - t_0)| = |f(x)|e^{-t_0 f(x)/2}|t - t_0| \leq C|t - t_0|$, where $C := \sup_{z>0} z e^{-t_0 z/2}$ (actually $C = 2/et_0$). Hence

$$||(e^{-tA} - e^{-t_0 A})u||_{\mathcal{H}}^2 \leq C^2(t-t_0)^2 \int_X |\varphi(x)|^2 d\mu(x) = C^2(t-t_0)^2||\varphi||_{L^2}^2 = C^2(t-t_0)^2||u||_{\mathcal{H}}^2$$

and $||e^{-tA} - e^{-t_0 A}||_{\mathcal{H}} \leq C|t - t_0|$, which implies the result.

(e) Prove that $\forall u \in D(A)$,

$$\lim_{t>0, t\to 0} \frac{e^{-tA}u - u}{t} = -Au$$

Answer — For any $u \in \mathcal{D}(A)$ and $\varphi := Uu \in L^2(X)$, so that $u = U^{-1}\varphi$, we have, for $t > 0$:

$$\begin{aligned}
\left\| \left(\frac{e^{-tA} - 1}{t} + A \right) u \right\|_{\mathcal{H}}^2 &= \left\| \left(\frac{e^{-tf} - 1}{t} + f \right) \varphi \right\|_{L^2}^2 \\
&= \int_X \left| \frac{e^{-tf(x)} - 1}{t} + f(x) \right|^2 |\varphi(x)|^2 d\mu(x)
\end{aligned}$$

Using the mean value theorem, we know that, for any $x \in X$, there exists some $\tau \in (0, t)$ s.t.

$$\frac{e^{-tf(x)} - 1}{t} + f(x) = -f(x)e^{-\tau f(x)} + f(x) = f(x)(1 - e^{-\tau f(x)}),$$

and hence

$$\left\| \left(\frac{e^{-tA} - 1}{t} + A \right) u \right\|_{\mathcal{H}}^2 = \int_X |1 - e^{-\tau f(x)}|^2 f(x) |\varphi(x)|^2 d\mu(x).$$

The function $|1 - e^{-\tau f(x)}|^2 f(x)|\varphi(x)|^2$ is dominated by $f(x)|\varphi(x)|^2$, which is μ-integrable and it converges pointwise to 0 when t (and hence τ) converges to 0. Hence we can apply Lebesgue's dominated convergence theorem to conclude that $\frac{e^{-tA}u-u}{t} - Au$ converges to 0.

(f) We define $D(A^n)$ for all $n \in \mathbb{N}$ recursively by : $D(A^0) := \mathcal{H}$ and $\forall n \in \mathbb{N}$, $D(A^{n+1}) := \{u \in D(A^n) |\ A^n u \in D(A)\}$. Prove that $\forall t > 0$, $\forall n \in \mathbb{N}$, $\forall u \in \mathcal{H}$, $e^{-tA}u \in D(A^n)$.
Answer — This is just a consequence of the fact that, for any $n \in \mathbb{N}$, $x \longmapsto f(x)^n e^{-tf(x)}$ is bounded on X, since f is a nonnegative function.

Supplementary questions (an application) [2]

(g) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let A be the non bounded operator on $L^2(\Omega, \mathbb{C})$ with domain $D(A) = H^2(\Omega, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C})$ and such that $\forall u \in D(A)$, $Au = -\Delta u$. Show that A is symmetric.
Answer — We must show that $D(A) \subset D(A^*)$ and that A and A^* coincides on $D(A)$. In other words it suffices to check that, for any $u \in D(A)$, the linear form

$$\begin{array}{ccc} D(A) & \longrightarrow & \mathbb{C} \\ v & \longmapsto & \langle u, Av \rangle \end{array}$$

admits a continuous extension on $L^2(\Omega)$, which coincides with $v \longmapsto \langle Au, v \rangle$. But, since u and v belongs to $H^2(\Omega, \mathbb{C}) \cap H_0^1(\Omega, \mathbb{C})$, we have

$$\langle u, Av \rangle = -\int_\Omega \overline{u} \Delta v = -\int_\Omega v \Delta \overline{u} + \int_{\partial\Omega} v \frac{\partial \overline{u}}{\partial n} - \overline{u} \frac{\partial v}{\partial n} = -\int_\Omega v \Delta \overline{u} = \langle Au, v \rangle,$$

where we have used the fact that the boundary integral cancels because u and v vanish on $\partial\Omega$ ($u, v \in H_0^1(\Omega, \mathbb{C})$).

In the following we will admit that A is self-adjoint ;

(h) Prove that, for any $f \in L^2(\Omega, \mathbb{C})$, there exists [3] a $\mathcal{C}^1$ map

$$\begin{array}{cccc} u : & (0, +\infty) & \longrightarrow & D(A) \\ & t & \longmapsto & u(t, \cdot) \end{array}$$

such that :

2. One could use the following formula, valid for $f, g \in H^2(\Omega, \mathbb{C})$

$$\int_{\partial\Omega} f \frac{\partial g}{\partial n} - g \frac{\partial f}{\partial n} = \int_\Omega f \Delta g - g \Delta f,$$

where $\frac{\partial}{\partial n}$ denotes the derivatives with respect to the exterior normal vector to the boundary $\partial\Omega$.

3. Sorry, there were a slight mistake in the subject : the equation was $\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x)$ (and not $= -\Delta u(t, x)$).

– $u(0, \cdot) = f$;
– $\forall t > 0$, $u(t, \cdot) \in \mathcal{C}^\infty(\Omega)$ and $u(t, \cdot)$ vanishes on $\partial\Omega$;
– $\forall t > 0$, $\forall x \in \Omega$,

$$\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x).$$

Answer — We just take $u(t, x) = \left(e^{t\Delta} f\right)(x) = \left(e^{-tA} f\right)(x)$, $\forall t \geq 0$, $\forall x \in X$, with the convention that, for $t = 0$, $e^{-0 \cdot A} = 1$ is the identity map.

We first note that, $\forall t > 0$, $e^{t\Delta}$ maps $L^2(\Omega)$ to $\cap_{n \in \mathbb{N}} H_0^n(\Omega, \mathbb{C})$, because of the result of question (f) (where $H_0^n(\Omega, \mathbb{C})$ is the space of functions f in $L^2(\Omega, \mathbb{C})$ such that all derivatives of f of order less than or equal to n are in $L^2(\Omega, \mathbb{C})$ and such that f vanishes on $\partial\Omega$). But this space is contained in the space of $\mathcal{C}^\infty$ functions on Ω which vanish on $\partial\Omega$.

The map u is clearly continuous on $(0, +\infty)$, as a consequence of question (d). It is moreover continuous at $t = 0$: this can be proved by starting the same computation as in question (d), which gives

$$||e^{-tA} u - u||_{\mathcal{H}}^2 = \int_X |e^{-tf(x)} - 1|^2 |\varphi(x)|^2 d\mu(x)$$

and by using Lebesgue's dominated convergence theorem as in question (e). However u is not $\mathcal{C}^1$ at 0 : we can only prove that, for any $f \in D(A)$, the map $t \longmapsto e^{t\Delta} f$ is derivable at 0, with a derivative equal to Δf, as a straightforward consequence of question (e). Lastly u is continuously derivable on $(0, +\infty)$, since, we can write, for any $f \in L^2(\Omega, \mathbb{C})$ and for any $t > 0$:

$$\frac{u(t+s, \cdot) - u(t, \cdot)}{s} = \frac{e^{(t+s)\Delta} f - e^{t\Delta} f}{s} = \frac{e^{s\Delta}\left(e^{t\Delta} f\right) - e^{t\Delta}}{s} f$$

and we remark that $e^{t\Delta} f \in D(A)$ because of question (f). Hence we can apply question (e) with the function $e^{t\Delta} f$ to prove that this quotient has a limit when s goes to 0, which is equal to $\Delta\left(e^{t\Delta} f\right) = \Delta u(t, \cdot)$. Hence $\frac{\partial u}{\partial t}(t, x) = \Delta u(t, x)$, $\forall t > 0$, $\forall x \in X$. The fact that $t \longmapsto \frac{\partial u}{\partial t}(t, \cdot) = \Delta u(t, \cdot)$ is continuous can be seen by writing

$$\Delta u(t, \cdot) = \Delta\left(e^{t\Delta} f\right) = e^{(t-\tau)\Delta}\left(\Delta e^{\tau\Delta} f\right),$$

for some fixed τ such that $0 < \tau < t$: question (f) says us again that, for any $f \in L^2(\Omega, \mathbb{C})$, $\Delta e^{\tau\Delta} f$ belongs to $L^2(\Omega, \mathbb{C})$ and question (d) says us that its image by $e^{(t-\tau)\Delta}$ depends continuously on t.