Lecture Notes on Real Analysis
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Chapter 3

Introduction to the Theory of Distributions

3.1 Test Functions and Distributions

3.1.1 Smooth compactly supported functions

Let Ω be an open subset of $\mathbb{R}^n$; we define $C_c^\infty(\Omega)$ as the vector space of complex-valued compactly supported functions defined on Ω. Even in the case $n = 1$ and $\Omega = \mathbb{R}$, it is not completely obvious that this space is not reduced to $\{0\}$. We leave to the reader as an exercise to check that the function

$$\rho_0(t) = \begin{cases} e^{-t^{-1}} & \text{if } t > 0, \\ 0 & \text{if } t \le 0, \end{cases} \tag{3.1.1}$$

is a C^∞ function on $\mathbb{R}$. Starting with ρ_0, we may define a function ρ on $\mathbb{R}^n$ by

$$\rho(x) = \rho_0(1 - \|x\|^2) \tag{3.1.2}$$

and we see right away that $\rho \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \rho = \bar{B}(0, 1)$. Here we have defined the support of ρ as the closure of the set $\{x \in \mathbb{R}^n, \rho(x) \neq 0\}$. Although that definition is fine when we deal with a continuous function, it will produce strange results if we want to define the support of a function in $L^1(\mathbb{R})$: for instance the characteristic function of $\mathbb{Q}$ is 0 a.e. and thus 0 as a function of $L^1(\mathbb{R})$, nevertheless the above set is $\mathbb{R}$. It is better to use the following definition, say for a function in $u \in L^1_{\text{loc}}(\Omega)$, Ω open subset of $\mathbb{R}^n$:

$$\operatorname{supp} u = \{x \in \Omega, \nexists U \text{open} \in \mathscr{V}_x, u_{|U} = 0\}, \quad (\operatorname{supp} u)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x, u_{|U} = 0\}. \tag{3.1.3}$$

The above definition makes sense for an L^1_{loc} function with $u_{|U} = 0$ meaning $u = 0$ a.e. in U. The smooth compactly supported functions are very useful as mollifiers, as shown by the next proposition.

Proposition 3.1.1. *Let $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)dx = 1$. For $\epsilon > 0$, we define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x\epsilon^{-1})$. Then, if $f \in C_c^m(\mathbb{R}^n)$, $\lim_{\epsilon\to 0_+} \phi_\epsilon * f = f$ (convergence in $C_c^m(\mathbb{R}^n)$) and if $f \in L^p(\mathbb{R}^n)$ with $1 \le p < +\infty$, $\lim_{\epsilon\to 0_+} \phi_\epsilon * f = f$ (convergence in $L^p(\mathbb{R}^n)$). In both cases the function $\phi_\epsilon * f$ is C^∞.*

Proof. We write

$$(\phi_\epsilon * f)(x) - f(x) = \int \phi_\epsilon(x-y) f(y) dy - f(x) = \int \phi(y)\big(f(x-\epsilon y) - f(x)\big) dy,$$

so that, if $\operatorname{supp}\phi \subset \bar{B}(0, R_0)$,

$$|(\phi_\epsilon * f)(x) - f(x)| \le \int |\phi(y)| dy \sup_{|x_1 - x_2| \le \epsilon R_0} |f(x_1) - f(x_2)|.$$

The function f is continuous and compactly supported, so is uniformly continuous on $\mathbb{R}^n$ (an easy consequence of the Heine theorem 1.5.10), thus

$$\lim_{\epsilon \to 0_+} \big(\sup_{x \in \mathbb{R}^n} |(\phi_\epsilon * f)(x) - f(x)|\big) = 0,$$

yielding the uniform convergence of $\phi_\epsilon * f$ towards f. If f is C_c^m, a simple differentiation under the integral sign (see e.g. the *Théorème 3.3.2.* in [9]) gives as well the uniform convergence of the derivatives, up to order m. The smoothness of $\phi_\epsilon * f$ for $\epsilon > 0$ is due to the same theorem when $f \in C_c^m(\mathbb{R}^n)$, since we have $(\phi_\epsilon * f)(x) = \int \phi_\epsilon(x-y) f(y) dy$.

Remark 3.1.2. *We have not defined a topology on the vector space $C_c^m(\mathbb{R}^n)$, but at the moment it will be enough for us to say that a sequence $(u_k)_{k\in\mathbb{N}}$ of functions in $C_c^m(\mathbb{R}^n)$ is converging if it converges in $C^m(\mathbb{R}^n)$ and if there exists a compact set K such that, for all $k \in \mathbb{N}$, $\operatorname{supp} u_k \subset K$.*

We note in particular that these conditions are satisfied by the "sequences" $(\phi_\epsilon * f)_{\epsilon > 0}$ since for $\epsilon \le 1$, $\operatorname{supp}(\phi_\epsilon * f) \subset \operatorname{supp} f + \operatorname{supp}\phi_\epsilon \subset \operatorname{supp} f + \operatorname{supp}\phi$.

Let us now take $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. With $\psi \in C_c^0(\mathbb{R}^n)$, we have

$$f * \phi_\epsilon - f = (f - \psi) * \phi_\epsilon + \psi * \phi_\epsilon - \psi + \psi - f,$$

so that

$$\begin{aligned}\|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} &\le (1 + \|\phi\|_{L^1}) \|f - \psi\|_{L^p(\mathbb{R}^n)} + \|\psi * \phi_\epsilon - \psi\|_{L^p(\mathbb{R}^n)} \\ &\le (1 + \|\phi\|_{L^1}) \|f - \psi\|_{L^p(\mathbb{R}^n)} + \underbrace{|\operatorname{supp}\phi + \epsilon|}_{\text{Lebesgue measure}}{}^{1/p} \|\psi * \phi_\epsilon - \psi\|_{L^\infty(\mathbb{R}^n)}.\end{aligned}$$

Since $\psi \in C_c^\infty(\mathbb{R}^n)$, the previous convergence argument implies the inequality

$$\limsup_{\epsilon \to 0_+} \|f * \phi_\epsilon - f\|_{L^p(\mathbb{R}^n)} \le (1 + \|\phi\|_{L^1}) \|f - \psi\|_{L^p(\mathbb{R}^n)}, \quad \text{for all } \psi \in C_c^\infty(\mathbb{R}^n).$$

The density of $C_c^\infty(\mathbb{R}^n)$ in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$ (see e.g. the *Théorème 3.4.1* in [9]) yields the result. For $\epsilon > 0, R > 0$, all the functions

$$\psi_{R,\epsilon}(y) = \sup_{|x| \le R} |(\partial_x^\alpha \phi_\epsilon)(x-y) f(y)|$$

belong to $L^1(\mathbb{R}^n_y)$ since

$$\int \psi_{R,\epsilon}(y) dy \le \|f\|_{L^p(\mathbb{R}^n)} \Big(\int \sup_{|x| \le R} |(\partial_x^\alpha \phi_\epsilon)(x-y)|^{p'} dy\Big)^{1/p'}, \quad \frac{1}{p} + \frac{1}{p'} = 1,$$

and $\operatorname{supp}\phi \subset \bar{B}_{R_0}$ gives that $|x - y| \le \epsilon R_0, |x| \le R$ imply $|y| \le \epsilon R_0 + R$, and the finiteness of the integral above, proving the smoothness of $\phi_\epsilon * f$ for $\epsilon > 0$. $\square$

N.B. The result of the proposition does not extend to the case $p = \infty$, since the uniform convergence of the continuous function $f * \phi_\epsilon$ would imply the continuity of the limit.

It will be also useful to use the compactly supported functions to construct some partitions of unity and, to begin with, to find C_c^∞ functions identically equal to 1 near a compact set.

Lemma 3.1.3. *Let Ω be an open subset of $\mathbb{R}^n$ and K be a compact subset of Ω. Then there exists a function $\varphi \in C_c^\infty(\Omega; [0,1])$ such that $\varphi = 1$ on a neighborhood of K.*

Proof. We claim that there exists $\epsilon_0 > 0$ such that $K + \epsilon_0 B_1 \subset \Omega$, ($B_1$ is the open unit ball). First we note that

$$d(K, \Omega^c) = \inf_{x \in K, y \in \Omega^c} |x - y| > 0, \tag{3.1.4}$$

otherwise, we could find sequences $(x_k)_{k\ge1}$ in K, $(y_k)_{k\ge1}$ in Ω^c such that $\lim_k |x_k - y_k| = 0$, and since K is compact, we may suppose that (x_k) converges with limit $x \in K$, implying $\Omega^c \ni \lim_k y_k = x$, which is impossible since $K \subset \Omega$. As a result, we have with $\epsilon_0 = d(K, \Omega^c)$

$$K + \epsilon_0 B_1 \subset \Omega,$$

otherwise, we could find $|t| < 1, x \in K$ such that $x + \epsilon_0 t = y \in \Omega^c$, implying $|x - y| < \epsilon_0 = d(K, \Omega^c)$, which is impossible. With the function ρ defined in 3.1.2, we define with $0 < \epsilon \le \frac{\epsilon_1}{2} < \frac{\epsilon_0}{4}$,

$$\varphi(x) = \int \mathbf{1}_{K+\epsilon_1 \bar{B}_1}(y) \rho\big((x-y)\epsilon^{-1}\big)\epsilon^{-n} dy \Big(\int \rho(t) dt\Big)^{-1}.$$

The function φ is C^∞ and such that

$$\operatorname{supp} \varphi \subset K + \epsilon_1 \bar{B}_1 + \epsilon \bar{B}_1 \subset K + \frac{3}{2}\epsilon_1 \bar{B}_1 \subset \underbrace{K + \frac{3}{4}\epsilon_0 \bar{B}_1}_{\text{compact}} \subset K + \epsilon_0 B_1 \subset \Omega.$$

Moreover $\varphi = 1$ on $K + \frac{\epsilon_1}{2}\bar{B}_1$ (which is a neighborhood of K), since if $x \in K + \frac{\epsilon_1}{2}\bar{B}_1$, we have, for y satisfying $|x - y| \le \epsilon$, that $y \in K + \frac{\epsilon_1}{2}\bar{B}_1 + \epsilon \bar{B}_1 \subset K + \epsilon_1 \bar{B}_1$. As a result, with $\tilde{\rho} = \rho\big(\int \rho(t) dt\big)^{-1}$, for $x \in K + \frac{\epsilon_1}{2}\bar{B}_1$, we have

$$1 = \int \tilde{\rho}((x-y)\epsilon^{-1})\epsilon^{-n} dy = \int \tilde{\rho}((x-y)\epsilon^{-1})\epsilon^{-n} \mathbf{1}_{K+\epsilon_1 \bar{B}_1}(y) dy = \varphi(x).$$

We note also that, since $\tilde{\rho} \ge 0$ with integral 1, $\mathbf{1}_L(y) \in [0,1]$, we have, for all $x \in \mathbb{R}^n$, $0 \le \varphi(x) \le 1$. The proof of the lemma is complete. $\square$

3.1.2 Distributions

Definition 3.1.4. *Let Ω be an open set of $\mathbb{R}^n$ and let $T : C_c^\infty(\Omega) \longrightarrow \mathbb{C}$ be a linear form with the following continuity property,*

$$\forall K \,\text{compact} \subset \Omega, \exists C_K > 0, \exists N_K \in \mathbb{N}, \forall \varphi \in C_K^\infty(\Omega),\ |\langle T, \varphi\rangle| \le C_K \sup_{\substack{|\alpha|\le N_K\\ x\in\mathbb{R}^n}} |(\partial_x^\alpha \varphi)(x)|, \tag{3.1.5}$$

where $C_K^\infty(\Omega) = \{\varphi \in C_c^\infty(\Omega), \operatorname{supp}\varphi \subset K\}$.

N.B. We shall use also the notation $\mathscr{D}(\Omega)$ for the space of test functions $C_c^\infty(\Omega)$ and $\mathscr{D}'(\Omega)$ for the space of distributions on Ω. We have not introduced a topology on $\mathscr{D}(\Omega)$ but we have defined a notion of converging sequence with the remark 3.1.2. It would have been certainly more elegant to start with the display of the natural topological structure on $\mathscr{D}(\Omega)$, at the (heavy) cost of having to deal with a non-metrizable locally convex topology defined by an uncountable family of semi-norms. The study of inductive limits of increasing sequences of Fréchet spaces is outlined in the appendix 3.7.2. Anyhow, one should think of $\mathscr{D}'(\Omega)$ as the topological dual of $\mathscr{D}(\Omega)$, a view supported by the next lemmas and remarks.

Remark 3.1.5. With $\mathscr{D}_K(\Omega) = C_K^\infty(\Omega)$, we have, using the sequence of compact sets $(K_j)_{j\ge1}$ of the lemma 2.3.1

$$\mathscr{D}(\Omega) = \cup_{j\ge1}\mathscr{D}_{K_j}(\Omega)$$

and it is not difficult to see that each $\mathscr{D}_{K_j}(\Omega)$ is a Fréchet space with the natural countable family of semi-norms given by $p_{K_j,m}(u) = \sup_{\substack{|\alpha|\le m\\ x\in K_j}} |(\partial_x^\alpha u)(x)|$. If we want to use the countable family $p_{K_j,m}$, we end-up with the topology on the Fréchet space $C^\infty(\Omega)$ as described in the subsection 2.3.3; the actual topology on $\mathscr{D}(\Omega)$ is finer and it is important to understand that, with ρ defined in (3.1.2) (say with $n = 1$), the sequence $(u_k)_{k\in\mathbb{N}}$, given by

$$u_k(x) = \rho(x - k)$$

does converge to 0 in the Fréchet space $C^\infty(\mathbb{R})$ but is *not* convergent in $C_c^\infty(\mathbb{R})$, since the second condition of the remark 3.1.2 is not satisfied: there is no compact subset K of $\mathbb{R}$ such that $\forall k \in \mathbb{N}, \operatorname{supp} u_k \subset K$.

Remark 3.1.6. Note that a linear form T on $C_c^\infty(\Omega)$ is a distribution if and only if, for all compact subsets K of Ω, its restriction to the Fréchet space $\mathscr{D}_K(\Omega)$ is continuous.

A L^1_{loc} function is a distribution: for Ω open subset of $\mathbb{R}^n$, for $f \in L^1_{\text{loc}}(\Omega)$, we define for $\varphi \in \mathscr{D}(\Omega)$

$$\langle T, \varphi\rangle = \int f(x)\varphi(x)dx \Longrightarrow |\langle T, \varphi\rangle| \le \|\varphi\|_{L^\infty(\mathbb{R}^n)} \int_{\operatorname{supp}\varphi} |f(x)|dx, \tag{3.1.6}$$

so that (3.1.5) is satisfied with $C_K = \int_K |f(x)|dx, N_K = 0$. Moreover the canonical mapping from $L^1_{\text{loc}}(\Omega)$ into $\mathscr{D}'(\Omega)$ is injective, as shown by the next lemma.

Lemma 3.1.7. *Let Ω be an open subset of $\mathbb{R}^n$, $f \in L^1_{loc}(\Omega)$ such that, for all $\varphi \in \mathscr{D}(\Omega)$, $\int f(x)\varphi(x)dx = 0$. Then we have $f = 0$.*

Proof. Let K be a compact subset of Ω and $\chi \in \mathscr{D}(\Omega)$ equal to 1 on a neighborhood of K as in the lemma 3.1.3. With ϕ as in the proposition 3.1.1, we get that $\lim_{\epsilon\to 0_+} \phi_\epsilon * (\chi f) = \chi f$ in $L^1(\mathbb{R}^n)$. We have

$$\big(\phi_\epsilon * (\chi f)\big)(x) = \int f(y) \underbrace{\chi(y)\phi\big((x-y)\epsilon^{-1}\big)\epsilon^{-n}}_{=\varphi_x(y)} dy, \quad \operatorname{supp}\varphi_x \subset K, \varphi_x \in \mathscr{D}(\Omega),$$

and from the assumption of the lemma, we obtain $\big(\phi_\epsilon * (\chi f)\big)(x) = 0$ for all x, implying $\chi f = 0$ from the convergence result; the conclusion follows. □

We note that it makes sense to restrict a distribution $T \in \mathscr{D}'(\Omega)$ to an open subset $U \subset \Omega$: just define

$$\langle T_{|U}, \varphi\rangle_{\mathscr{D}'(U),\mathscr{D}(U)} = \langle T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}, \tag{3.1.7}$$

and $T_{|U}$ is obviously a distribution on U. With this in mind, we can define the support of a distribution exactly as in (3.1.8).

Definition 3.1.8. *Let Ω be an open subset of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. We define the support of T as*

$$\operatorname{supp} T = \{x \in \Omega, \forall U \, open \in \mathscr{V}_x,\ T_{|U} \neq 0\}. \tag{3.1.8}$$

We define the C^∞ singular support of T as

$$\operatorname{singsupp} T = \{x \in \Omega, \forall U \, open \in \mathscr{V}_x,\ T_{|U} \notin C^\infty(U)\}. \tag{3.1.9}$$

Note that the support and the singular support are closed subset of Ω since their complements in Ω are open: we have

$$(\operatorname{supp} T)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x,\ T_{|U} = 0\}, \tag{3.1.10}$$

$$(\operatorname{singsupp} T)^c = \{x \in \Omega, \exists U \text{open} \in \mathscr{V}_x,\ T_{|U} \in C^\infty(U)\}. \tag{3.1.11}$$

A simple consequence of that definition is that, for $T \in \mathscr{D}'(\Omega)$, $\varphi \in \mathscr{D}(\Omega)$,

$$\operatorname{supp}\varphi \subset (\operatorname{supp} T)^c \Longrightarrow \langle T, \varphi\rangle = 0. \tag{3.1.12}$$

3.1.3 First examples of distributions

The Dirac mass

We define for $\varphi \in C^0_c(\mathbb{R}^n)$, $\langle \delta_0, \varphi\rangle = \varphi(0)$; the property (3.1.5) is satisfied with $C_K = 1, N_K = 0$. We have $\operatorname{supp}\delta_0 = \{0\}$. From this, the Dirac mass cannot be an L^1_{loc} function, otherwise, since it is 0 a.e., it would be 0. Let ϕ, ϵ as in the proposition 3.1.1: then we have from that proposition

$$\lim_{\epsilon\to 0_+} \int \phi_\epsilon(x)\varphi(x)dx = \varphi(0),$$

so that the Dirac mass appears as the weak limit of $\epsilon^{-n}\phi(x\epsilon^{-1})$.

The simple layer

We consider in $\mathbb{R}^n$ the hypersurface $\Sigma = \{(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}, x_n = f(x')\}$, where $f \in C^1(\mathbb{R}^{n-1})$. We define for $\varphi \in C_c^0(\mathbb{R}^n)$,

$$\langle \delta_\Sigma, \varphi \rangle = \int_{\mathbb{R}^{n-1}} \varphi\big(x', f(x')\big)\big(1 + |\nabla f(x')|^2\big)^{1/2} dx'.$$

The property (3.1.5) is satisfied with $C_K = area(\Sigma \cap K), N_K = 0, \operatorname{supp} \delta_\Sigma = \Sigma$, and since Σ has Lebesgue measure 0 in $\mathbb{R}^n$, the simple layer potential cannot be an L^1_{loc} function.

The principal value of $1/x$

We define for $\varphi \in C_c^1(\mathbb{R})$,

$$\langle \operatorname{pv} \frac{1}{x}, \varphi \rangle = \lim_{\epsilon \to 0_+} \int_{|x| \ge \epsilon} \frac{\varphi(x)}{x} dx. \tag{3.1.13}$$

Let us check that this limit exists. We have for parity reasons,

$$\begin{aligned}\int_{|x| \ge \epsilon} \frac{\varphi(x)}{x} dx &= \int_\epsilon^{+\infty} \big(\varphi(x) - \varphi(-x)\big) \frac{dx}{x} \\ &= \big[\ln x\big(\varphi(x) - \varphi(-x)\big)\big]_{x=\epsilon}^{x=+\infty} - \int_\epsilon^{+\infty} \big(\varphi'(x) + \varphi'(-x)\big) \ln x dx\end{aligned}$$

and thus, using that $\lim_{\epsilon \to 0_+} \epsilon \ln \epsilon = 0$, $\ln |x| \in L^1_{\text{loc}}(\mathbb{R})$, we get

$$\langle \operatorname{pv} \frac{1}{x}, \varphi \rangle = -\int_0^{+\infty} \big(\varphi'(x) + \varphi'(-x)\big) \ln x dx = -\int_{\mathbb{R}} \varphi'(x)(\ln |x|) dx,$$

yielding $|\langle \operatorname{pv} \frac{1}{x}, \varphi \rangle| \le \int_{\operatorname{supp} \varphi'} |\ln |x|| dx \|\varphi'\|_{L^\infty}$.

3.1.4 Continuity properties

Definition 3.1.9. *Let Ω be an open subset of $\mathbb{R}^n$ and let $(\varphi_j)_{j \ge 1}$ be a sequence of functions in $C_c^\infty(\Omega)$. We shall say that $\lim_j \varphi_j = 0$ in $C_c^\infty(\Omega)$ when the two following conditions are satisfied:*
(1) there exists a compact set $K \subset \Omega$, such that $\forall j \ge 1, \operatorname{supp} \varphi_j \subset K$,
(2) $\lim_j \varphi_j = 0$ in the Fréchet space $C_K^\infty(\Omega)$, i.e. $\forall \alpha \in \mathbb{N}^n$, $\lim_j \big(\sup_{x \in K} |(\partial_x^\alpha \varphi_j)(x)|\big) = 0$.

Proposition 3.1.10. *Let Ω be an open subset of $\mathbb{R}^n$ and T be a linear form defined on $C_c^\infty(\Omega)$. The linear form T is a distribution on Ω if and only if it is sequentially continuous.*

Proof. Assuming $|\langle T, \varphi \rangle| \le C_K \max_{|\alpha| \le N_K} \|\partial_x^\alpha \varphi\|_{L^\infty}$ for all $\varphi \in C_K^\infty(\Omega)$ and all K compact $\subset \Omega$ implies readily the sequential continuity. Conversely, if T does not satisfy (3.1.5), we have

$$\exists K_0 \text{compact} \subset \Omega, \forall k \ge 1, \forall N \in \mathbb{N}, \exists \varphi_{k,N} \in C_{K_0}^\infty(\Omega), |\langle T, \varphi_{k,N} \rangle| > k \max_{|\alpha| \le N} \|\partial_x^\alpha \varphi_{k,N}\|_{L^\infty}.$$

From the strict inequality, we infer that the function $\varphi_{k,N}$ is not identically 0, and we may define

$$\psi_k = \frac{\varphi_{k,k}}{k \max_{|\alpha|\le k} \|\partial_x^\alpha \varphi_{k,k}\|_{L^\infty}}, \quad \text{so that } |\langle T, \psi_k\rangle| > 1.$$

But the sequence $(\psi_k)_{k\ge1}$ converges to 0 since $\operatorname{supp}\psi_k \subset K_0$ and for $|\beta| \le k$, $\|\partial_x^\beta \psi_k\|_{L^\infty} \le 1/k$, implying for each multi-index β that $\lim_k \|\partial_x^\beta \psi_k\|_{L^\infty} = 0$. The sequential continuity is violated since $|\langle T, \psi_k\rangle| > 1$ and the converse is proven. □

Definition 3.1.11. *Let Ω be an open subset of $\mathbb{R}^n$, $T \in \mathscr{D}'(\Omega)$ and $N \in \mathbb{N}$. The distribution T will be said of finite order N if*

$$\exists N \in \mathbb{N}, \forall K \text{ compact} \subset \Omega, \exists C_K > 0, \forall \varphi \in C_K^\infty(\Omega), |\langle T, \varphi\rangle| \le C_K \sup_{\substack{|\alpha|\le N \\ x\in\mathbb{R}^n}} |(\partial_x^\alpha \varphi)(x)|. \tag{3.1.14}$$

The vector space of distributions of order N on Ω will be denoted by ${\mathscr{D}'}^N(\Omega)$. The vector space ${\mathscr{D}'}^0(\Omega)$ is called the space of Radon measures on Ω.

Proposition 3.1.12. *Let Ω be an open subset of $\mathbb{R}^n$ and $m \in \mathbb{N}$. The vector space ${\mathscr{D}'}^m(\Omega)$ is equal to the sequentially continuous[1] linear forms on $C_c^m(\Omega)$: if $T \in {\mathscr{D}'}^m(\Omega)$, it can be extended to a sequentially continuous linear form on $C_c^m(\Omega)$. If T is a sequentially continuous linear form on $C_c^m(\Omega)$, then $T \in {\mathscr{D}'}^m(\Omega)$.*

Proof. Let us first consider $T \in {\mathscr{D}'}^m(\Omega), \varphi \in C_c^m(\Omega)$. Applying the proposition 3.1.1, we find a sequence $(\varphi_k)_{k\ge1}$ in $C_c^\infty(\Omega)$, converging in $C_c^m(\Omega)$ with limit φ. Since we may assume that all the functions φ_k and φ are supported in a fixed compact subset K of Ω, we have, according to the estimate (3.1.14),

$$|\langle T, \varphi_k - \varphi_l\rangle| \le C \max_{|\alpha|\le m} \|\partial_x^\alpha(\varphi_k - \varphi_l)\|_{L^\infty} = Cp(\varphi_k - \varphi_l),$$

where p is the norm in the Banach space $C_K^m(\Omega)$. Since the sequence $(\varphi_k)_{k\ge1}$ converges in $C_K^m(\Omega)$, we get that the sequence $(\langle T, \varphi_k\rangle)_{k\ge1}$ is a Cauchy sequence in $\mathbb{C}$, thus converges; moreover, if for some compact subset L of Ω, $(\psi_k)_{k\ge1}$ is another sequence of $C_L^m(\Omega)$ converging to φ, we have

$$|\langle T, \psi_k - \varphi_k\rangle| \le C' \max_{|\alpha|\le m} \|\partial_x^\alpha(\varphi_k - \psi_k)\|_{L^\infty} = C'p(\varphi_k - \psi_k) \le C'p(\varphi_k - \varphi) + C'p(\varphi - \psi_k)$$

and $\lim_k \langle T, \psi_k - \varphi_k\rangle = 0$ so that, we can extend the linear form to $C_c^m(\Omega)$ by defining $\langle T, \varphi\rangle = \lim_k \langle T, \varphi_k\rangle$. We get also immediately that (3.1.14) holds with $N = m$ and $C_K^\infty(\Omega)$ replaced by $C_K^m(\Omega)$, so that T is obviously sequentially continuous.

Let us now consider a sequentially continuous linear form T on $C_c^m(\Omega)$; reproducing the proof of the proposition 3.1.10, we get that the estimate (3.1.14) holds with $N = m$, proving that $T \in {\mathscr{D}'}^m(\Omega)$. The proof of the proposition is complete. □

Remark 3.1.13. We have already proven directly that functions in $L^1_{\text{loc}}(\Omega)$(see (3.1.6)), the Dirac mass and a simple layer (see the section 3.1.3) are distributions of order 0. It is an exercise left to the reader to prove that the distribution $\operatorname{pv}\frac{1}{x}$ defined in (3.1.13) is of order 1 and not of order 0.

[1] The convergence of a sequence in $C_c^m(\Omega)$ is analogous to the convergence given in the definition 3.1.9, except that (2) is required in the Banach space $C_K^m(\Omega)$, i.e. $|\alpha| \le m$.

3.1.5 Partitions of unity and localization

Theorem 3.1.14 (Partition of unity). *Let Ω be an open subset of $\mathbb{R}^n$, K a compact subset of Ω and $\Omega_1, \dots, \Omega_m$ open subsets of Ω such that $K \subset \Omega_1 \cup \dots \cup \Omega_m$. Then for $1 \le j \le m$, there exists $\psi_j \in C_c^\infty(\Omega_j; [0,1])$ and V open such that*

$$\Omega \supset V \supset K, \ \forall x \in V, \sum_{1 \le j \le m} \psi_j(x) = 1,$$

and for all $x \in \Omega$, $\sum_{1 \le j \le m} \psi_j(x) \in [0,1]$.

Proof. The case $m = 1$ of the theorem is proven in the lemma 3.1.3. We consider now $m > 1$ and we note that, since $x \in K$ implies $x \in$ one of the Ω_j,

$$K \subset \cup_{x \in K} B(x, r_x), \quad \bar{B}(x, r_x) \subset \text{one of the } \Omega_j, \quad r_x > 0.$$

From the compactness of K, we get that $K \subset \cup_{1 \le l \le N} B(x_l, r_{x_l})$ and we may assume that

$$\begin{aligned}
&\bar{B}(x_l, r_{x_l}) \subset \Omega_1, \quad \text{for } 1 \le l \le N_1, \\
&\bar{B}(x_l, r_{x_l}) \subset \Omega_2, \quad \text{for } N_1 < l \le N_2, \\
&\dots\dots\dots\dots\dots \\
&\bar{B}(x_l, r_{x_l}) \subset \Omega_m, \quad \text{for } N_{m-1} < l \le N_m = N.
\end{aligned}$$

We define then the compact sets

$$K_1 = \cup_{1 \le l \le N_1} \bar{B}(x_l, r_{x_l}), \quad \dots \quad , K_m = \cup_{N_{m-1} < l \le N_m} \bar{B}(x_l, r_{x_l}),$$

and we have $K \subset \cup_{1 \le j \le m} K_j$, and for each j, $K_j \subset \Omega_j$. Using the lemma 3.1.3, we find $\varphi_j \in C_c^\infty(\Omega_j; [0,1])$ such that $\varphi_j = 1$ on a neighborhood $V_j (\subset \Omega_j)$ of K_j. We define then

$$\begin{aligned}
&\psi_1 = \varphi_1, \\
&\psi_2 = \varphi_2(1 - \varphi_1), \\
&\dots\dots \\
&\psi_j = \varphi_j(1 - \varphi_1) \dots (1 - \varphi_{j-1}),
\end{aligned}$$

so that $\psi_j \in C_c^\infty(\Omega_j; [0,1])$ and we have

$$\sum_{1 \le j \le m} \psi_j = \sum_{1 \le j \le m} \varphi_j \Big(\prod_{1 \le k < j} (1 - \varphi_k) \Big) = 1 - \prod_{1 \le k \le m} (1 - \varphi_k), \tag{3.1.15}$$

since the formula (second equality above) is true for $m = 1$ and inductively,

$$\begin{aligned}
\sum_{1 \le j \le m+1} \varphi_j \Big(\prod_{1 \le k < j} (1 - \varphi_k) \Big) &= 1 - \prod_{1 \le k \le m} (1 - \varphi_k) + \varphi_{m+1} \prod_{1 \le k \le m} (1 - \varphi_k) \\
&= 1 - (1 - \varphi_{m+1}) \prod_{1 \le k \le m} (1 - \varphi_k) = 1 - \prod_{1 \le k \le m+1} (1 - \varphi_k).
\end{aligned}$$

We have thus for $x \in \cup_{1 \le j \le m} V_j$ (which is a neighborhood of K in Ω), using (3.1.15) and $\varphi_j = 1$ on V_j, $\sum_{1 \le j \le m} \psi_j(x) = 1$. On the other hand, (3.1.15) and φ_j valued in $[0,1]$ show that $\sum_{1 \le j \le m} \psi_j(x) \in [0,1]$ for all x. The proof is complete. □

Theorem 3.1.15. *Let Ω be an open set of $\mathbb{R}^n$ and $(\Omega_j)_{j\in J}$ be an open covering of Ω: each Ω_j is open and $\cup_{j\in J}\Omega_j = \Omega$. Let us assume that for each $j \in J$, we are given $T_j \in \mathscr{D}'(\Omega_j)$ in such a way that*

$$T_{j|\Omega_j\cap\Omega_k} = T_{k|\Omega_j\cap\Omega_k}. \tag{3.1.16}$$

Then there exists a unique $T \in \mathscr{D}'(\Omega)$ such that for all $j \in J$, $T_{|\Omega_j} = T_j$.

Proof. Uniqueness: if T, S are such distributions, we get that $(T-S)_{|\Omega_j} = 0$, so that for all $j \in J$, $\Omega_j \subset (\operatorname{supp}(T-S))^c$ and thus $\Omega = \cup_{j\in J}\Omega_j \subset (\operatorname{supp}(T-S))^c$, i.e. $T - S = 0$.
Existence: let $\varphi \in \mathscr{D}(\Omega)$ and let us consider the compact set $K = \operatorname{supp}\varphi$. We have $K \subset \cup_{j\in M}\Omega_j$ with M a finite subset of J. Using the theorem on partitions of unity, we find some function $\psi_j \in C_c^\infty(\Omega_j)$ for $j \in M$ such that $\sum_{j\in M}\psi_j = 1$ on a neighborhood of K. As a consequence, we have $\varphi = \sum_{j\in M}\psi_j\varphi$ and we define

$$\langle T, \varphi\rangle = \sum_{j\in M}\langle T_j, \psi_j\varphi\rangle.$$

The required estimates (3.1.5) are easily checked, but the linearity and the independence with respect to the decomposition deserve some attention. Assume that we have $\varphi = \sum_{k\in N}\phi_k\varphi$, where N is a finite subset of J and $\phi_k \in C_c^\infty(\Omega_k)$: we have

$$\sum_{k\in N}\langle T_k, \phi_k\varphi\rangle = \sum_{j\in M, k\in N}\langle T_k, \phi_k\psi_j\varphi\rangle \underbrace{=}_{\text{from (3.1.16)}} \sum_{j\in M, k\in N}\langle T_j, \phi_k\psi_j\varphi\rangle = \sum_{j\in M}\langle T_j, \psi_j\varphi\rangle,$$

proving that T is defined independently of the decomposition. The linearity follows at once. The proof is complete. □

3.1.6 Weak convergence of distributions

We have not defined a topology on the space of test functions $\mathscr{D}(\Omega)$, although we gave the definition of convergence of a sequence (see the definition 3.1.9); we shall need also a simple notion of weak-dual convergence of a sequence of distributions, which is the $\sigma(\mathscr{D}', \mathscr{D})$ convergence.

Definition 3.1.16. *Let Ω be an open set of $\mathbb{R}^n$, $(T_j)_{j\ge1}$ be a sequence of $\mathscr{D}'(\Omega)$ and $T \in \mathscr{D}'(\Omega)$. We shall say that $\lim_j T_j = T$ in the weak-dual topology if*

$$\forall \varphi \in \mathscr{D}(\Omega), \quad \lim_j\langle T_j, \varphi\rangle = \langle T, \varphi\rangle. \tag{3.1.17}$$

Remark 3.1.17. We have already seen (see the section 3.1.3) that for $\rho \in C_c^\infty(\mathbb{R}^n)$, $\epsilon > 0$, $\rho_\epsilon(x) = \epsilon^{-n}\rho(x\epsilon^{-1})$, $\lim_{\epsilon\to0_+}\rho_\epsilon = \delta_0\int\rho(t)dt$. Moreover, on $\mathscr{D}'(\mathbb{R})$, we have with $T_\lambda(x) = e^{i\lambda x}$, $\lim_{\lambda\to+\infty}T_\lambda = 0$ since for $\varphi \in \mathscr{D}(\mathbb{R})$,

$$\int_{\mathbb{R}} e^{i\lambda x}\varphi(x)dx = (i\lambda)^{-1}\int_{\mathbb{R}}\frac{d}{dx}(e^{i\lambda x})\varphi(x)dx = -(i\lambda)^{-1}\int_{\mathbb{R}} e^{i\lambda x}\varphi'(x)dx.$$

Theorem 3.1.18. *Let Ω be an open set of $\mathbb{R}^n$, $(T_j)_{j\ge 1}$ be a sequence of $\mathscr{D}'(\Omega)$ such that, for all $\varphi \in \mathscr{D}(\Omega)$, the (numerical) sequence $(\langle T_j, \varphi\rangle)_{j\ge 1}$ converges. Defining the linear form T on $\mathscr{D}(\Omega)$, by $\langle T, \varphi\rangle = \lim_j \langle T_j, \varphi\rangle$, we obtain that T belongs to $\mathscr{D}'(\Omega)$.*

Proof. This is an important consequence of the Banach-Steinhaus theorem 2.1.8; let us consider a compact subset K of Ω. Then defining $T_{j,K}$ as the restriction of T_j to the Fréchet space $\mathscr{D}_K(\Omega)$, we see that the assumptions of the corollary 2.1.8 are satisfied since $T_{j,K}$ belongs to the topological dual of $\mathscr{D}_K(\Omega)$, according to the remark 3.1.6. As a consequence the restriction of T to $\mathscr{D}_K(\Omega)$ belongs to the topological dual of $\mathscr{D}_K(\Omega)$ and from the same remark 3.1.6, it gives that $T \in \mathscr{D}'(\Omega)$. □

N.B. The reader may note that we have used $E = \mathscr{D}(\Omega) = \cup_{j\in\mathbb{N}} \mathscr{D}_{K_j}(\Omega) = \cup_j E_j$, and that our definition of the topological dual of E as linear forms T on E such that, for all j, $T_{|E_j} \in$ the topological dual of the Fréchet space E_j. This structure allows us to use the Banach-Steinhaus theorem, although we have not defined a topology on E; this observation is a good introduction to the more abstract setting of LF spaces, the so-called inductive limits of Fréchet spaces.

3.2 Differentiation of distributions, multiplication by C^∞ functions

3.2.1 Differentiation

Definition 3.2.1. *Let Ω be an open set of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. We define the distributions $\partial_{x_j} T$ and for a multi-index $\alpha \in \mathbb{N}^n$ (see (2.3.6)), $\partial_x^\alpha T$ by*

$$\langle \partial_{x_j} T, \varphi\rangle = -\langle T, \partial_{x_j}\varphi\rangle, \quad \langle \partial_x^\alpha T, \varphi\rangle = (-1)^{|\alpha|}\langle T, \partial_x^\alpha \varphi\rangle. \tag{3.2.1}$$

We note that $\partial_x^\alpha T$ is indeed a distribution on Ω, since the mappings $\varphi \mapsto \partial_x^\alpha \varphi$ are continuous on each Fréchet space $\mathscr{D}_K(\Omega)$.

Remark 3.2.2. If $\lim_j T_j = T$ in the weak-dual topology of $\mathscr{D}'(\Omega)$, then, for all multi-indices α, $\lim_j \partial_x^\alpha T_j = \partial_x^\alpha T$ (in the weak-dual topology): we have, for each $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_x^\alpha T_j, \varphi\rangle = (-1)^{|\alpha|}\langle T_j, \partial_x^\alpha \varphi\rangle \underset{j\to+\infty}{\longrightarrow} (-1)^{|\alpha|}\langle T, \partial_x^\alpha \varphi\rangle = \langle \partial_x^\alpha T, \varphi\rangle.$$

Remark 3.2.3. If $u \in C^1(\Omega)$, its derivative $\partial_{x_j} u$ as a distribution coincides with the distribution defined by the continuous function $\partial u/\partial x_j$: for $\varphi \in \mathscr{D}(\Omega)$,

$$\langle \partial_{x_j} u, \varphi\rangle = -\langle u, \partial_{x_j}\varphi\rangle = -\int u(x)\frac{\partial \varphi}{\partial x_j}(x)dx = \int \frac{\partial u}{\partial x_j}(x)\varphi(x)dx = \langle \frac{\partial u}{\partial x_j}, \varphi\rangle.$$

Also, if $u, v \in C^0(\Omega)$ are such that $\partial_{x_1} u = v$ in $\mathscr{D}'(\Omega)$, then the function u admits v as a partial derivative with respect to x_1. To prove this, we may assume that u, v are both compactly supported in Ω: in fact it is enough to prove that for $\chi \in C_c^\infty(\Omega)$

identically equal to 1 near a point x_0, the function χu (compactly supported) has a partial derivative with respect to x_1 which is $\chi v + u\partial_{x_1}\chi$ (compactly supported) and we know that in $\mathscr{D}'(\Omega)$ we have

$$\langle \partial_{x_1}(\chi u), \varphi\rangle = -\langle u, \chi\partial_{x_1}\varphi\rangle = -\langle u, \partial_{x_1}(\chi\varphi)\rangle + \langle u, \varphi\partial_{x_1}\chi\rangle = \langle \partial_{x_1}u, \chi\varphi\rangle + \langle u\partial_{x_1}\chi, \varphi\rangle$$

which implies a particular case of Leibniz' formula $\partial_{x_1}(\chi u) = \chi\partial_{x_1}u + u\partial_{x_1}\chi = \chi v + u\partial_{x_1}\chi$. Assuming then that u, v are compactly supported, we have from the proposition 3.1.1, $u = \lim_\epsilon (u * \phi_\epsilon)$ in $C_c^0(\Omega)$ and the functions $u * \phi_\epsilon \in C_c^\infty(\Omega)$. Also we have, with the ordinary differentiation,

$$(\partial_{x_1}(u*\phi_\epsilon))(x) = \int u(y)(\partial_{x_1}\phi_\epsilon)(x-y)dy = \langle u(\cdot), -\partial_{y_1}\big(\phi_\epsilon(x-\cdot)\big)\rangle = \int v(y)\phi_\epsilon(x-y)dy,$$

and $\lim_\epsilon (v * \phi_\epsilon) = v$ in $C_c^0(\Omega)$. As a result the sequences $(u * \phi_\epsilon), (\partial_{x_1}(u * \phi_\epsilon))$ are both uniformly converging sequences of (compactly supported) continuous functions with respective limits u, v, and this implies that the continuous function u has v as a partial derivative with respect to x_1.

3.2.2 Examples

Defining the Heaviside function H as $\mathbf{1}_{\mathbb{R}_+}$, we get

$$H' = \delta_0 \tag{3.2.2}$$

since for $\varphi \in \mathscr{D}(\mathbb{R})$, we have $\langle H', \varphi\rangle = -\langle H, \varphi'\rangle = -\int_0^{+\infty} \varphi'(t)dt = \varphi(0)$. Still in one dimension, we have

$$\langle \delta_0^{(k)}, \varphi\rangle = (-1)^k \varphi^{(k)}(0), \tag{3.2.3}$$

since it is true for $k = 0$ and inductively $\langle \delta_0^{(k+1)}, \varphi\rangle = -\langle \delta_0^{(k)}, \varphi'\rangle = -(-1)^k \varphi'^{(k)}(0) = (-1)^{k+1}\varphi^{(k+1)}(0)$. Looking at the definition (3.1.13), we see that we have proven

$$\operatorname{pv}\big(\frac{1}{x}\big) = \frac{d}{dx}(\ln|x|), \qquad \text{(distribution derivative)}. \tag{3.2.4}$$

Let f be a finitely-piecewise C^1 function defined on $\mathbb{R}$: it means that there is an increasing finite sequence of real numbers $(a_n)_{1\le n\le N}$, so that f is C^1 on all closed intervals $[a_n, a_{n+1}]$ for $1 \le n < N$ and on $]-\infty, a_1]$ and $[a_N, +\infty[$. In particular, the function f has a left-limit $f(a_n^-)$ and a right-limit $f(a_n^+)$ which may be different. Let us compute the distribution derivative of f; for $\varphi \in \mathscr{D}(\mathbb{R})$, since f is locally integrable, we have, setting $a_0 = -\infty, a_{N+1} = +\infty$,

$$\begin{aligned}
\langle f', \varphi\rangle = -\langle f, \varphi'\rangle &= -\int_{\mathbb{R}} f(x)\varphi'(x)dx = -\sum_{0\le n\le N}\int_{a_n}^{a_{n+1}} f(x)\varphi'(x)dx \\
&= \sum_{0\le n\le N}\int_{a_n}^{a_{n+1}} \frac{df}{dx}(x)\varphi(x)dx + \sum_{0\le n\le N}\big(f(a_n^+)\varphi(a_n) - f(a_{n+1}^-)\varphi(a_{n+1})\big) \\
&= \int \varphi(x)\big(\sum_{0\le n\le N}\frac{df}{dx}(x)\mathbf{1}_{[a_n,a_{n+1}]}(x)\big) + \sum_{1\le n\le N} f(a_n^+)\varphi(a_n) - \sum_{1\le n\le N} f(a_n^-)\varphi(a_n),
\end{aligned}$$

so that we have obtained the so-called *formula of jumps*

$$f' = \sum_{0\le n\le N} \frac{df}{dx}\mathbf{1}_{[a_n,a_{n+1}]} + \sum_{1\le n\le N} \big(f(a_n^+) - f(a_n^-)\big)\delta_{a_n}, \tag{3.2.5}$$

where δ_{a_n} is the Dirac mass at a_n, defined by $\langle \delta_{a_n}, \varphi\rangle = \varphi(a_n)$.

We consider now the following determination of the logarithm given for $z \in \mathbb{C}\backslash\mathbb{R}_-$ by

$$\operatorname{Log} z = \oint_{[1,z]} \frac{d\xi}{\xi}, \tag{3.2.6}$$

which makes sense since $\mathbb{C}\backslash\mathbb{R}_-$ is star-shaped with respect to 1, i.e. the segment $[1,z] \subset \mathbb{C}\backslash\mathbb{R}_-$ for $z \in \mathbb{C}\backslash\mathbb{R}_-$. Since the function Log coincides with ln on $\mathbb{R}_+^*$ and is holomorphic on $\mathbb{C}\backslash\mathbb{R}_-$, we get by analytic continuation that

$$e^{\operatorname{Log} z} = z, \quad \text{for } z \in \mathbb{C}\backslash\mathbb{R}_-. \tag{3.2.7}$$

Also by analytic continuation, we have for $|\operatorname{Im} z| < \pi, \quad \operatorname{Log}(e^z) = z$. We want now to study the distributions on $\mathbb{R}$,

$$u_y(x) = \operatorname{Log}(x+iy), \quad \text{where } y \ne 0 \text{ is a real parameter.}$$

We leave as an exercise for the reader to prove that

$$\lim_{y\to 0_\pm} \operatorname{Log}(x+iy) = \ln|x| \pm i\pi\big(1 - H(x)\big), \tag{3.2.8}$$

where the limits are taken in the sense of the definition 3.1.16; also the reader can check

$$\frac{1}{x \pm i0} = \operatorname{pv}\big(\frac{1}{x}\big) \mp i\pi\delta_0, \tag{3.2.9}$$

where we have defined

$$\langle \frac{1}{x \pm i0}, \varphi\rangle = \lim_{\epsilon\to 0_+} \int \frac{\varphi(x)}{x \pm i\epsilon} dx \tag{3.2.10}$$

(part of the exercise is to prove that these limits exist for $\varphi \in \mathscr{D}(\mathbb{R})$). We conclude that section of examples with a more general lemma on a simple ODE.

Lemma 3.2.4. *Let I be an open interval of $\mathbb{R}$. The solutions in $\mathscr{D}'(I)$ of $u' = 0$ are the constants. The solutions in $\mathscr{D}'(I)$ of $u' = f$ make a one-dimensional affine subspace of $\mathscr{D}'(I)$.*

Proof. We assume first that $f = 0$; if u is a constant, then it is of course a solution. Conversely, let us assume that $u \in \mathscr{D}'(I)$ satisfies $u' = 0$. Let $\chi_0 \in C_c^\infty(I)$ such that $\int_{\mathbb{R}} \chi_0(x)dx = 1$; then we have for any $\varphi \in C_c^\infty(I)$, with $J(\varphi) = \int_{\mathbb{R}} \varphi(x)dx$, $\psi(x) = \int_{-\infty}^x \big(\varphi(t) - J(\varphi)\chi_0(t)\big)dt$, noting that ψ belongs[2] to $C_c^\infty(I)$,

$$\langle u, \varphi - J(\varphi)\chi_0\rangle = \langle u, \psi'\rangle = -\langle u', \psi\rangle = 0,$$

[2]The function ψ is obviously smooth and if φ, χ_0 are both supported in $\{a \le x \le b\}, a, b \in I$, so is ψ, thanks to the condition $\int \chi_0 = 1$.

which gives $\langle u, \varphi\rangle = J(\varphi)\langle u, \chi_0\rangle$, i.e. $u = \langle u, \chi_0\rangle$ proving that u is indeed a constant. We have proven that the solutions $u \in \mathscr{D}'(I)$ of $u' = 0$ are simply the constants. If $f \in \mathscr{D}'(I)$, we need only to construct a solution v_0 of $v_0' = f$ and then use the previous result to obtain that the set of solutions of $u' = f$ is $v_0 + \mathbb{R}$. Let us construct such a solution v_0. For $\varphi \in \mathscr{D}(I)$, we define with the same ψ as above,

$$\langle v_0, \varphi\rangle = -\langle f, \psi\rangle. \tag{3.2.11}$$

It is a distribution since for $\operatorname{supp}\varphi$ compact $\subset I$, we define (the compact set) $K_1 = \operatorname{supp}\varphi \cup \operatorname{supp}\chi_0$, and we have

$$|\langle v_0, \varphi\rangle| = |\langle f, \psi\rangle| \le C_{K_1} \max_{0\le j\le N_{K_1}} \|\psi^{(j)}\|_{L^\infty} \le C \max_{0\le j\le (N_{K_1}-1)_+} \|\varphi^{(j)}\|_{L^\infty}.$$

Moreover the formula (3.2.11) implies the sought result

$$\langle v_0', \varphi\rangle = -\langle v_0, \varphi'\rangle = \langle f, \psi_{\varphi'}\rangle = \langle f, \varphi\rangle,$$

since $\psi_{\varphi'}(x) = \int_{-\infty}^x \big(\varphi'(t) - J(\varphi')\chi_0(t)\big)dt = \varphi(x)$ because $J(\varphi') = 0$. The proof of the lemma is complete. □

3.2.3 Product by smooth functions

We define now the product of a C^∞ (resp. C^N) function by a distribution (resp. of order N).

Definition 3.2.5. *Let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. For $f \in C^\infty(\Omega)$, we define the product $f \cdot u$ as the distribution defined by*

$$\langle f\cdot u, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle u, f\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)}. \tag{3.2.12}$$

If u is of order N and $f \in C^N(\Omega)$, we define the product $f\cdot u$ as the distribution of order N defined by

$$\langle f\cdot u, \varphi\rangle_{\mathscr{D}'^N(\Omega),C_c^N(\Omega)} = \langle u, f\varphi\rangle_{\mathscr{D}'^N(\Omega),C_c^N(\Omega)}. \tag{3.2.13}$$

Remark 3.2.6. Since the multiplication by a $C^\infty(\Omega)$ (resp. $C^N(\Omega)$) function is a continuous linear operator from $C_c^\infty(\Omega)$ (resp. $C_c^N(\Omega)$) into itself, we get that the above formulas actually define the products as distributions on Ω with the right order (see the proposition 3.1.12). Also the product defined in the second part coincides with the first definition whenever $f \in C_c^\infty(\Omega)$ and if $u \in L^1_{\text{loc}}(\Omega), f \in C^0(\Omega)$, the usual product fu coincides with the $f\cdot u$ defined here, thanks to the lemma 3.1.7.

The next theorem is providing an extension to the classical Leibniz' formula for the derivatives of a product.

Theorem 3.2.7. *Let Ω be an open set of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, $f \in C^\infty(\Omega)$ and $\alpha \in \mathbb{N}^n$ be a multi-index (see (2.3.6)). Then we have*

$$\frac{\partial_x^\alpha(fu)}{\alpha!} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \frac{\partial_x^\beta(f)}{\beta!}\frac{\partial_x^\gamma(u)}{\gamma!}. \tag{3.2.14}$$

Proof. We get immediately by induction on $|\alpha|$ the formula

$$\frac{\partial_x^\alpha(fu)}{\alpha!} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\partial_x^\beta(f)}{\beta!}\frac{\partial_x^\gamma(u)}{\gamma!}, \quad \text{with } \sigma_{\beta,\gamma}\in\mathbb{R}_+.$$

To find the $\sigma_{\beta,\gamma}$, we choose $f(x)=e^{x\cdot\xi}, u(x)=e^{x\cdot\eta}$, with $\xi,\eta\in\mathbb{R}^n$. We find then for all $\xi,\eta\in\mathbb{R}^n$, the identity

$$\frac{(\xi+\eta)^\alpha}{\alpha!} = \frac{\partial_x^\alpha(e^{x\cdot(\xi+\eta)})}{\alpha!}_{|x=0} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\partial_x^\beta(e^{x\cdot\xi})}{\beta!}\frac{\partial_x^\gamma(e^{x\cdot\eta})}{\gamma!}_{|x=0} = \sum_{\substack{\beta,\gamma\in\mathbb{N}^n\\ \beta+\gamma=\alpha}} \sigma_{\beta,\gamma}\frac{\xi^\beta}{\beta!}\frac{\eta^\gamma}{\gamma!},$$

and the formula (2.3.7) shows that for β,γ such that $\beta+\gamma=\alpha$

$$\sigma_{\beta,\gamma} = \partial_\xi^\beta\partial_\eta^\gamma\Big(\frac{(\xi+\eta)^\alpha}{\alpha!}\Big)_{|\xi=\eta=0} = 1,$$

completing the proof of the theorem. □

Examples. Let f be a continuous function on $\mathbb{R}$ and δ_0 be the Dirac mass at 0. The product $f\cdot\delta_0$ is equal to $f(0)\delta_0$: since δ_0 is a distribution of order 0, we can multiply it by a continuous function and if $\varphi\in C_c^0(\mathbb{R})$, we have

$$\langle f\cdot\delta_0,\varphi\rangle = \langle\delta_0, f\varphi\rangle = f(0)\varphi(0) = \langle f(0)\delta_0,\varphi\rangle \Longrightarrow f\cdot\delta_0 = f(0)\delta_0. \tag{3.2.15}$$

On the other hand if $f\in C^1(\mathbb{R})$ we have

$$f\cdot\delta_0' = f(0)\delta_0' - f'(0)\delta_0, \tag{3.2.16}$$

since the Leibniz' formula (3.2.14) gives $f(0)\delta_0' = (f\cdot\delta_0)' = f'\cdot\delta_0 + f\cdot\delta_0' = f'(0)\delta_0 + f\cdot\delta_0'$. In particular $x\delta_0' = -\delta_0$.

3.2.4 Division of distribution on $\mathbb{R}$ by x^m

We want now to address the question of division of a function (or a distribution) by a polynomial; a typical example is the division of 1 by the linear function x expressed by the identity

$$x\,\mathrm{pv}(1/x) = 1 \tag{3.2.17}$$

which is an immediate consequence of (3.1.13). We note also from the previous examples that, for any constant c, we have $x\big(\mathrm{pv}(1/x)+c\delta_0\big)=1$. The next theorem shows that $T=\mathrm{pv}(1/x)+c\delta_0$ are the only distributions solutions of the equation $xT=1$.

Theorem 3.2.8. *Let $m\ge 1$ be an integer.*
(1) If $u\in\mathscr{D}'(\mathbb{R})$ is such that $x^m u=0$, then $u=\sum_{0\le j<m} c_j\delta_0^{(j)}$.
(2) Let $v\in\mathscr{D}'(\mathbb{R})$; there exists $u\in\mathscr{D}'(\mathbb{R})$ such that $v=x^m u$.

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