Lecture Notes on Real Analysis
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Proof. Let us first prove (1). For $\varphi, \chi_0 \in C_c^\infty(\mathbb{R})$ with $\chi_0 = 1$ near 0, we have

$$\varphi(x) = \underbrace{\sum_{0\le j<m} \frac{\varphi^{(j)}(0)}{j!} x^j}_{p_{\varphi,m}(x)} + \underbrace{\int_0^1 \frac{(1-t)^{m-1}}{(m-1)!} \varphi^{(m)}(tx) dt\, x^m}_{\psi_{m,\varphi}(x)}, \quad \psi_{m,\varphi} \in C^\infty(\mathbb{R}),$$

and thus, since $x^m u = 0$,

$$\langle u, \varphi\rangle = \overbrace{\langle x^m u, x^{-m}(1-\chi_0)\varphi\rangle}^{=0} + \langle u, \chi_0\varphi\rangle = \langle u, \chi_0 p_{m,\varphi}\rangle + \overbrace{\langle x^m u, \chi_0 \psi_{\varphi,m}\rangle}^{=0}$$
$$= \sum_{0\le j<m} \frac{\varphi^{(j)}(0)}{j!}\langle u, \chi_0\rangle = \sum_{0\le j<m} \langle c_j \delta_0^{(j)}, \varphi\rangle,$$

which the sought result. To obtain (2), for $\varphi \in C_c^\infty(\mathbb{R})$, and a given $v_0 \in \mathscr{D}'(\mathbb{R})$, we define, using the above notations,

$$\langle u, \varphi\rangle = \langle v_0, \chi_0 \psi_{m,\varphi}\rangle + \langle v_0, x^{-m}(1-\chi_0)\varphi\rangle.$$

This defines obviously a distribution on $\mathbb{R}$ and $\langle x^m u, \varphi\rangle = \langle u, x^m\varphi\rangle$; for the function $\phi(x) = x^m\varphi(x)$, we have $p_{\phi,m} = 0, x^m\psi_{m,\phi}(x) = x^m\varphi(x)$, so that the smooth functions $\psi_{m,\phi} = \varphi$,

$$\langle x^m u, \varphi\rangle = \langle v_0, \chi_0\varphi\rangle + \langle v_0, x^{-m}(1-\chi_0)x^m\varphi\rangle = \langle v_0, \varphi\rangle. \qquad \square$$

3.3 Distributions with compact support

3.3.1 Identification with $\mathscr{E}'$

Let Ω be an open subset of $\mathbb{R}^n$. We have already seen that the space $C^\infty(\Omega)$ (also denoted by $\mathscr{E}(\Omega)$) is a Fréchet space. Denoting by $\mathscr{E}'(\Omega)$ the topological dual of $\mathscr{E}(\Omega)$, we can consider $T \in \mathscr{E}'(\Omega)$ as a distribution $\tilde T$ on Ω by defining

$$\langle \tilde T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} \quad \text{(this makes sense since } \mathscr{D}(\Omega) \subset \mathscr{E}(\Omega)).$$

The linearity is obvious and the continuity of T as a linear form on the Fréchet space $\mathscr{E}(\Omega)$ implies that there exists $C > 0, N \in \mathbb{N}$, K compact subset of Ω such that

$$\forall \varphi \in \mathscr{E}(\Omega), \quad |\langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)}| \le C \sup_{|\alpha|\le N,\ x\in K} |(\partial_x^\alpha \varphi)(x)|.$$

This estimates also proves that $\tilde T$ belongs to $\mathscr{D}'(\Omega)$; moreover, it has compact support in the sense of the definition (3.1.8): we have $\langle \tilde T, \varphi\rangle = 0$ for $\varphi \in C_c^\infty(\Omega)$, $\operatorname{supp}\varphi \subset K^c$, so that $\tilde T_{|K^c} = 0$ and thus $\operatorname{supp}\tilde T \subset K$. The next theorem proves that we can identify the space $\mathscr{E}'(\Omega)$ with the distributions on Ω with compact support, denoted by $\mathscr{D}'_{\text{comp}}(\Omega)$.

Theorem 3.3.1. *Let Ω be an open subset of $\mathbb{R}^n$. The mapping $\iota : \mathscr{E}'(\Omega) \to \mathscr{D}'_{comp}(\Omega)$, defined as above by $\iota(T) = \tilde T$ is bijective.*

Proof. The mapping ι is linear and if $\iota(T) = 0$, we know that T vanishes on all functions of $\mathscr{D}(\Omega)$.

Lemma 3.3.2. *Let Ω be an open subset of $\mathbb{R}^n$. The space $\mathscr{D}(\Omega)$ is dense in $\mathscr{E}(\Omega)$.*

Proof of the lemma. We consider a sequence $(K_j)_{j\ge 1}$ of compact subsets of Ω such that the lemma 2.3.1 is satisfied. For each $j \ge 1$, we may use the lemma 3.1.3 to construct a function $\chi_j \in \mathscr{D}(\Omega)$ with $\chi_j = 1$ near K_j. For a given $\varphi \in \mathscr{E}(\Omega)$, the sequence $(\varphi\chi_j)_{j\ge 1}$ of functions in $\mathscr{D}(\Omega)$ converges in $\mathscr{E}(\Omega)$ to φ, thanks to the last property of the lemma 2.3.1, proving the lemma. □

Since T is continuous on $\mathscr{E}(\Omega)$, $\langle T, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \lim_j \langle T, \varphi\chi_j\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega),} = 0$ since T vanishes on $\mathscr{D}(\Omega)$. Let us consider now $T \in \mathscr{D}'_{\rm comp}(\Omega)$ with $\operatorname{supp} T = L$ (compact subset of Ω). Using the lemma 3.1.3, we consider $\chi_0 \in \mathscr{D}(\Omega)$ such that $\chi_0 = 1$ on a neighborhood of L. For $\varphi \in \mathscr{E}(\Omega)$, we define $S \in \mathscr{E}'(\Omega)$ by

$$\langle S, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \langle T, \chi_0\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} \quad (\text{note that } |\langle S, \varphi\rangle| \le C \sup_{|\alpha|\le N,\ x\in \operatorname{supp}\chi_0} |\partial_x^\alpha \varphi|),$$

We have $\iota(S) = T$ because

$$\langle \iota(S), \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle S, \varphi\rangle_{\mathscr{E}'(\Omega),\mathscr{E}(\Omega)} = \langle T, \chi_0\varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)} = \langle \chi_0 T, \varphi\rangle_{\mathscr{D}'(\Omega),\mathscr{D}(\Omega)},$$

and since for $\varphi \in \mathscr{D}(\Omega)$, the function $(1-\chi_0)\varphi$ vanishes on an open neighborhood V of L implying

$$\operatorname{supp}\big((1-\chi_0)\varphi\big) \subset V^c \subset L^c \Longrightarrow \langle T, (1-\chi_0)\varphi\rangle = 0,$$

so that $\iota(S) = \chi_0 T = \chi_0 T + \underbrace{(1-\chi_0)T}_{=0} = T$. The proof of the theorem is complete.

□

Remark 3.3.3. We can then identify $\mathscr{D}'_{\rm comp}(\Omega)$ with $\mathscr{E}'(\Omega)$, and we may note that for $T \in \mathscr{D}'_{\rm comp}(\Omega)$ with $\operatorname{supp} T = L$, T is of finite order N, and for all neighborhoods K of L, there exists $C > 0$ such that, for all $\varphi \in \mathscr{E}(\Omega)$,

$$|\langle T, \varphi\rangle| \le C \sup_{|\alpha|\le N,\ x\in K} |(\partial_x^\alpha \varphi)(x)|. \tag{3.3.1}$$

In general, it is not possible to take $K = L$ in the above estimate.

3.3.2 Distributions with support at a point

The next theorem characterizes the distributions supported in $\{0\}$.

Theorem 3.3.4. *Let Ω be an open subset of $\mathbb{R}^n$, $x_0 \in \Omega$ and let $u \in \mathscr{D}'(\Omega)$ such that $\operatorname{supp} u = \{x_0\}$. Then $u = \sum_{|\alpha|\le N} c_\alpha \delta_{x_0}^{(\alpha)}$, where the c_α are some constants.*

Proof. Let $\varphi \in C^\infty(\Omega)$; we have for $x \in V_0 \subset$ open neighborhood of x_0 (included in Ω), N_0 the order of u,

$$\varphi(x) = \sum_{|\alpha|\le N_0} \frac{(\partial_x^\alpha \varphi)(x_0)}{\alpha!}(x-x_0)^\alpha + \underbrace{\int_0^1 \frac{(1-\theta)^{N_0}}{N_0!}\varphi^{(N_0+1)}(x_0+\theta(x-x_0))d\theta(x-x_0)^{N_0+1}}_{\psi(x),\quad \psi\in C^\infty(V_0)},$$

and thus for $\chi_0 \in C_c^\infty(V_0), \chi_0 = 1$ near x_0,

$$\langle u, \varphi\rangle = \langle u, \chi_0\varphi\rangle = \sum_{|\alpha|\le N_0} \frac{(\partial_x^\alpha \varphi)(x_0)}{\alpha!}\langle u, \chi_0(x)(x-x_0)^\alpha\rangle + \langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1}\rangle. \tag{3.3.2}$$

We have also

$$|\langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1}\rangle| \le C_0 \sup_{|\alpha|\le N_0} |\partial_x^\alpha\big(\chi_0(x)\psi(x)(x-x_0)^{N_0+1}\big)|. \tag{3.3.3}$$

We can take $\chi_0(x) = \rho(\frac{x-x_0}{\epsilon})$, where $\rho \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball B_1, $\rho = 1$ in $\frac{1}{2}B_1$ and $\epsilon > 0$. We have then

$$\begin{aligned}\chi_0(x)\psi(x)(x-x_0)^{N_0+1} = \epsilon^{N_0+1}\rho(\frac{x-x_0}{\epsilon})\psi\big(x_0+\epsilon\frac{(x-x_0)}{\epsilon}\big)\frac{(x-x_0)^{N_0+1}}{\epsilon^{N_0+1}} \\ = \epsilon^{N_0+1}\rho_1(\frac{x-x_0}{\epsilon})\end{aligned}$$

with $\rho_1(t) = \rho(t)\psi(x_0+\epsilon t)t^{N_0+1}$, so that $\rho_1 \in C_c^\infty(\mathbb{R}^n)$ is supported in the unit ball B_1 has all its derivatives bounded independently of ϵ. From (3.3.3), we get for all $\epsilon > 0$,

$$|\langle u, \chi_0(x)\psi(x)(x-x_0)^{N_0+1}\rangle| \le C_0 \sup_{|\alpha|\le N_0} \epsilon^{N_0+1-|\alpha|}|(\partial_t^\alpha \rho_1)(\frac{x-x_0}{\epsilon})| \le C_1\epsilon,$$

which implies that the left-hand-side of (3.3.3) is zero. On the other hand, for $\chi_1 \in C_c^\infty(V_0), \chi_1 = 1$ near the support of χ_0, we have

$$\begin{aligned}\langle u, \chi_1(x)(x-x_0)^\alpha\rangle = \langle u, \underbrace{\chi_1(x)\chi_0(x)}_{=\chi_0(x)}(x-x_0)^\alpha\rangle + \langle u, \underbrace{\chi_1(x)(1-\chi_0(x))}_{\text{supported in } (\operatorname{supp} u)^c}(x-x_0)^\alpha\rangle \\ = \langle u, \chi_0(x)(x-x_0)^\alpha\rangle\end{aligned}$$

so that the latter does not depend on ε for ε small enough. The result of the theorem follows from (3.3.2). □

3.4 Tensor products

Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$ and $f \in C_c^\infty(X), g \in C_c^\infty(Y)$. The tensor product $f \otimes g$ is defined by $(f\otimes g)(x,y) = f(x)g(y)$ and belongs to $C_c^\infty(X \times Y)$. Now if $T \in \mathscr{D}'(X), S \in \mathscr{D}'(Y)$, we want to define a distribution $T \otimes S \in \mathscr{D}'(X \times Y)$ such that

$$\langle T \otimes S, f \otimes g\rangle = \langle T, f\rangle\langle S, g\rangle.$$

This triggers several questions: is such a construction possible? Is the definition above sufficient to determine unambiguously the distribution $T\otimes S$? We shall answer positively to these questions, but we first address a related question of derivation of an "integral" depending on a parameter.

3.4.1 Differentiation of a duality product

Theorem 3.4.1. *Let Ω be an open subset of $\mathbb{R}^n$, $u \in \mathscr{D}'(\Omega)$, U an open subset of $\mathbb{R}^m$ and $\phi \in C^\infty(\Omega \times U)$ such that*

$$\forall t \in U, \exists V_t \in \mathscr{V}_t, \exists K_t \text{ compact subset of } \Omega, \quad \forall s \in V_t, \quad \operatorname{supp} \phi(\cdot, s) \subset K_t. \tag{3.4.1}$$

Then the function f defined on U by $f(t) = \langle u, \phi(\cdot, t)\rangle$ makes sense and belongs to $C^\infty(U)$. Moreover we have for all $\alpha \in \mathbb{N}^m$, $(\partial_t^\alpha f)(t) = \langle u, (\partial_t^\alpha \phi)(\cdot, t)\rangle$.

Proof. The function f makes sense since for all $t \in U$, the function $\phi(\cdot, t)$ belongs to $C_c^\infty(\Omega)$. Let $t_0 \in U$ and B_0 be a closed ball with center t_0 and positive radius r_0 included in V_{t_0} given by (3.4.1). For $|h| \le r_0$, we have

$$f(t_0 + h) - f(t_0) = \langle u, \underbrace{\phi(\cdot, t_0 + h) - \phi(\cdot, t_0)}_{\text{supported in } K_{t_0},}\rangle$$

and using Taylor's formula with integral remainder, we get

$$f(t_0 + h) - f(t_0) = \langle u, (\partial_t \phi)(\cdot, t_0)\rangle h + \underbrace{\langle u, \int_0^1 (1 - \theta)\, \overbrace{\partial_s^2 \phi(\cdot, t_0 + \theta h)}^{\text{support in } K_{t_0}}\, d\theta\rangle h^2}_{r(t_0,h)}.$$

We have, since $K_{t_0} \times B_0$ is a compact subset of $\Omega \times U$,

$$|r(t_0, h)| \le |h|^2 C_0 \sup_{x \in K_{t_0}, |\alpha| \le N_0} \int_0^1 (1 - \theta) |(\partial_x^\alpha \partial_s^2 \phi)\underbrace{(x, t_0 + \theta_0 h)}_{\in K_{t_0} \times B_0}| d\theta \le C_1 |h|^2,$$

proving the differentiability of f on U along with $df(t) = \langle u, \partial_t \phi(\cdot, t)\rangle$. Inductively, we get that f is smooth and the result of the theorem. □

Corollary 3.4.2. *Let X, Y be open subsets of $\mathbb{R}^n, \mathbb{R}^m$, $\phi \in C^\infty(X \times Y)$ and $u \in \mathscr{D}'(X)$.*
(1) If ϕ is compactly supported in $X \times Y$, the function ψ defined by $\psi(y) = \langle u, \phi(\cdot, y)\rangle$ belongs to $C_c^\infty(Y)$.
(2) If $u \in \mathscr{E}'(X)$, the function ψ defined by $\psi(y) = \langle u, \phi(\cdot, y)\rangle$ belongs to $C^\infty(Y)$.

Proof. To prove (1), we need only to verify (3.4.1): we have indeed for all $y \in Y$

$$\operatorname{supp} \phi(\cdot, y) \subset proj_X(\operatorname{supp} \phi) \quad \text{which is a compact subset of } X,$$

which implies that $\psi \in C^\infty(Y)$; moreover the function $\phi(\cdot, y) = 0$ on the open subset of Y, $\big(proj_Y(\operatorname{supp} \phi)\big)^c$, and thus $\operatorname{supp} \psi \subset proj_Y(\operatorname{supp} \phi)$ which is a compact subset of Y. To obtain (2), we consider $\chi \in C_c^\infty(X)$ equal to 1 near the compact support of u. We have then $u = \chi u$ and consequently,

$$\langle u, \phi(\cdot, y)\rangle = \langle u, \phi(\cdot, y)\chi(\cdot)\rangle.$$

The function $\Phi(x, y) = \phi(x, y)\chi(x)$ is smooth on $X \times Y$ and $\operatorname{supp} \Phi(\cdot, y) \subset \operatorname{supp} \chi$ so that we can apply the theorem 3.4.1 whose assumptions are satisfied. □

3.4.2 Pull-back by the affine group

Let us now recall the definition of the affine group of $\mathbb{R}^n$: it is the group of mappings from $\mathbb{R}^n$ into itself of the form $x \mapsto Ax + t = \theta_{A,t}(x)$ where $A \in Gl(n, \mathbb{R})$($n \times n$ invertible matrices) and $t \in \mathbb{R}^n$. When A is the identity, $\Theta_{\mathrm{Id},t}$ is simply the translation of vector t; we have also $\theta_{A,t}^{-1} = \Theta_{A^{-1},-A^{-1}t}$. If u belongs to $L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Theta_{A,t}$ is in the affine group of $\mathbb{R}^n$, we can define the *pull-back* of u by the map Θ by the identity

$$\Theta_{A,t}^* u = u \circ \Theta_{A,t}, \quad \text{so that } (\Theta_{A,t}^* u)(x) = u(Ax + t). \tag{3.4.2}$$

As a result for $\varphi \in C_c^0(\mathbb{R}^n)$, we find

$$\langle \Theta_{A,t}^* u, \varphi\rangle = \int_{\mathbb{R}^n} u(Ax + t)\varphi(x)dx = \int_{\mathbb{R}^n} u(y)\varphi(A^{-1}y - A^{-1}t)|\det A|^{-1}dy. \tag{3.4.3}$$

We want to use that formula to define the pull-back of a distribution on $\mathbb{R}^n$ by an affine transformation.

Definition 3.4.3. *Let $A \in Gl(n, \mathbb{R}), t \in \mathbb{R}^n$, $\Theta_{A,t}$ the affine transformation defined above and let $u \in \mathscr{D}'(\mathbb{R}^n)$. We define the distribution $\Theta_{A,t}^* u$ by the identity*

$$\langle \Theta_{A,t}^* u, \varphi\rangle = \langle u, \varphi \circ \Theta_{A,t}^{-1}\rangle|\det A|^{-1}. \tag{3.4.4}$$

Remark 3.4.4. (1) Note that this defines a distribution on $\mathbb{R}^n$, since the mapping $\varphi \mapsto \varphi \circ \Theta_{A,t}^{-1}$ is an isomorphism of $\mathscr{D}(\mathbb{R}^n)$. Moreover, if $u \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the previous definition ensures that $\Theta_{A,t}^* u = u \circ \Theta_{A,t}$, thanks to the lemma 3.1.7.
(2) The mapping $u \mapsto \Theta_{A,t}^* u$ is sequentially continuous from $\mathscr{D}'(\mathbb{R}^n)$ into itself.
(3) A distribution u on $\mathbb{R}^n$ is even (resp. odd) if $\Theta_{-\mathrm{Id},0}^* u = u$ (resp. $-u$). Using the notation

$$\check{u} = \Theta_{-\mathrm{Id},0}^* u \quad (\text{for a function } u,\ \check{u}(x) = u(-x)), \tag{3.4.5}$$

u is even means $\check{u} = u$, odd means $\check{u} = -u$.

3.4.3 Homogeneous distributions

Definition 3.4.5. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution u is said to be homogeneous with degree λ if for all $t > 0$, $u(t\cdot) = t^\lambda u(\cdot)$ (here $u(t\cdot) = \theta^*_{t\,\mathrm{Id},0} u$).*

Proposition 3.4.6. *Let $u \in \mathscr{D}'(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$. The distribution u is homogeneous of degree λ if and only if the Euler equation is satisfied, namely*

$$\sum_{1\le j\le n} x_j \partial_{x_j} u = \lambda u. \tag{3.4.6}$$

Proof. A distribution u on $\mathbb{R}^n$ is homogeneous of degree λ means:

$$\forall \varphi \in C^\infty_c(\mathbb{R}^n), \forall t > 0, \qquad \langle u(y), \varphi(y/t)t^{-n}\rangle = t^\lambda \langle u(x), \varphi(x)\rangle,$$

which is equivalent to $\forall \varphi \in C^\infty_c(\mathbb{R}^n), \forall s > 0, \langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle$, also equivalent to

$$\forall \varphi \in C^\infty_c(\mathbb{R}^n), \qquad \frac{d}{ds}\big(\langle u(y), \varphi(sy)s^{n+\lambda}\rangle\big) = 0 \quad \text{on } s > 0. \tag{3.4.7}$$

Note that the differentiability property is due to the theorem 3.4.1 and that

$$\langle u(y), \varphi(sy)s^{n+\lambda}\rangle = \langle u(x), \varphi(x)\rangle \quad \text{ at } s = 1.$$

As a consequence, applying the theorem 3.4.1, we get that the homogeneity of degree λ of u is equivalent to

$$\forall s > 0, \quad \langle u(y), s^{n+\lambda-1}\big((n+\lambda)\varphi(sy) + \sum_{1\le j\le n}(\partial_j\varphi)(sy)sy_j\big)\rangle = 0,$$

also equivalent to $0 = \langle u(y), (n + \lambda + \sum_{1\le j\le n} y_j\partial_j)\big(\varphi(sy)\big)\rangle$ and by the definition of the differentiation of a distribution, it is equivalent to $(n+\lambda)u - \sum_{1\le j\le n} \partial_j(y_j u) = 0$, which is (3.4.6) by the Leibniz rule (3.2.14). ☐

Remark 3.4.7. (1) The Dirac mass at 0 in $\mathbb{R}^n$ is homogeneous of degree $-n$: we have for $t > 0$

$$\langle \delta_0(tx), \varphi(x)\rangle = \langle \delta_0(y), \varphi(y/t)t^{-n}\rangle = t^{-n}\varphi(0) = t^{-n}\langle \delta_0, \varphi\rangle.$$

(2) If T is an homogeneous distribution of degree λ, then $\partial_x^\alpha T$ is also homogeneous with degree $\lambda - |\alpha|$: taking the derivative of the Euler equation (3.4.6), we get

$$\partial_{x_k} u + \sum_{1\le j\le k} x_j \partial_{x_j}\partial_{x_k} u - \lambda \partial_{x_k} u = 0,$$

proving that $\partial_{x_k} u$ is homogeneous of degree $\lambda - 1$ and the result by iteration.
(3) It follows immediately from the definition (3.1.13) that the distribution $\mathrm{pv}(\frac{1}{x})$ is homogeneous of degree -1. The same is true for the distributions $\frac{1}{x \pm i0}$ as it is clear from (3.2.9)and (3.2.10).

(4) For $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > -1$ we define the $L^1_{\text{loc}}(\mathbb{R})$ functions

$$x_+^\lambda = \begin{cases} x^\lambda & \text{if } x > 0, \\ 0 & \text{if } x \le 0., \end{cases} \qquad \chi_+^\lambda = \frac{x_+^\lambda}{\Gamma(\lambda+1)}. \tag{3.4.8}$$

The distributions χ_+^λ and x_+^λ are homogeneous of degree λ and by an analytic continuation argument, we can prove that χ_+^λ may be defined for any $\lambda \in \mathbb{C}$, is an homogeneous distribution of degree λ and satisfies

$$\chi_+^\lambda = (\frac{d}{dx})^k(\chi_+^{\lambda+k}), \quad \chi_+^{-k} = \delta_0^{(k-1)}, \quad k \in \mathbb{N}^*.$$

Lemma 3.4.8. *Let $(u_j)_{1\le j\le m}$ be non-zero homogeneous distributions on $\mathbb{R}^n$ with distinct degrees $(\lambda_j)_{1\le j\le m}$ ($j \ne k$ implies $\lambda_j \ne \lambda_k$). Then they are independent in the complex vector space $\mathscr{D}'(\mathbb{R}^n)$.*

Proof. We assume that $m \ge 2$ and that there exists some complex numbers $(c_j)_{1\le j\le m}$ such that $\sum_{1\le j\le m} c_j u_j = 0$. Then applying the operator $\mathcal{E} = \sum_{1\le j\le m} x_j \partial_{x_j}$, we get for all $k \in \mathbb{N}$,

$$0 = \sum_{1\le j\le m} c_j \mathcal{E}^k(u_j) = \sum_{1\le j\le m} c_j \lambda_j^k u_j.$$

We consider now the Vandermonde matrix $m \times m$

$$V_m = \begin{pmatrix} 1 & 1 & \dots & 1 \\ \lambda_1 & \lambda_2 & \dots & \lambda_m \\ \dots & & & \\ \lambda_1^{m-1} & \lambda_2^{m-1} & \dots & \lambda_m^{m-1} \end{pmatrix}, \quad \det V_m = \prod_{1\le j<k\le m} (\lambda_k - \lambda_j) \ne 0.$$

We note that for $\varphi \in C_c^\infty(\mathbb{R}^n)$, and $X \in \mathbb{C}^m$ given by

$$X = \begin{pmatrix} c_1\langle u_1, \varphi\rangle \\ c_2\langle u_2, \varphi\rangle \\ \dots\dots\dots \\ c_m\langle u_m, \varphi\rangle \end{pmatrix},$$

we have $V_m X = 0$, so that $X = 0$, i.e. $\forall j, \forall \varphi \in C_c^\infty(\mathbb{R}^n), c_j\langle u_j, \varphi\rangle = 0$, i.e. $c_j u_j = 0$ and since u_j is not the zero distribution, we get the sought conclusion $c_j = 0$ for all j. □

3.4.4 Tensor products of distributions

We begin with a lemma.

Lemma 3.4.9. *Let $\phi \in C_c^\infty(]0,1[^n)$; one can find a sequence of functions in*

$$\operatorname{Vect}(\otimes^n C_c^\infty(]0,1[) \quad \textit{(the vector space generated by the tensor products)}$$

converging to ϕ in $C_c^\infty(]0,1[^n)$ in the sense of the definition 3.1.9.

Proof. We define for $k \in \mathbb{Z}^n$, $\hat\phi(k) = \int e^{-2i\pi x\cdot k}\phi(x)dx$, and we note that, with $\Delta = \sum_{1\le j\le n}\partial_{x_j}^2$, $m\in\mathbb{N}$,

$$\hat\phi(k) = (1+|k|^2)^{-m}\int (1-\frac{1}{4\pi^2}\Delta)^m\big(e^{-2i\pi x\cdot k}\big)\phi(x)dx$$
$$= (1+|k|^2)^{-m}\int e^{-2i\pi x\cdot k}\big((1-\frac{1}{4\pi^2}\Delta)^m\phi\big)(x)dx$$

so that
$$|\hat\phi(k)| \le (1+|k|^2)^{-m} C_m \max_{|\alpha|\le 2m}\|\partial_x^\alpha\phi\|_{L^\infty}. \tag{3.4.9}$$

As a result the series $\Phi(x) = \sum_{k\in\mathbb{Z}^n}\hat\phi(k)e^{2i\pi x\cdot k}$ converges and is a smooth function, periodic with periods $\mathbb{Z}^n$: we need only to check that $\sum_{k\in\mathbb{Z}^n}(1+|k|)^{-n-1} < +\infty$.[3] Moreover,

$$\text{for } x\in[0,1]^n, \qquad \Phi(x) = \phi(x). \tag{3.4.10}$$

We verify this first for $n=1$. We have in that case

$$\Phi(x) = \lim_{N\to+\infty}\int \sum_{|k|\le N} e^{2i\pi k(x-y)}\phi(y)dy,$$

and since
$$\sum_{|k|\le N} e^{2i\pi kt} = 1+2\operatorname{Re}\sum_{1\le k\le N} e^{2i\pi kt} = 1+2\operatorname{Re}\big(e^{2i\pi t}\frac{e^{2i\pi Nt}-1}{e^{2i\pi t}-1}\big)$$
$$= 1+2\operatorname{Re}\big(e^{i\pi(N+1)t}\frac{\sin(\pi N t)}{\sin(\pi t)}\big) = \frac{\sin(\pi t(2N+1))}{\sin(\pi t)},$$

we get that, since $\phi\in C_c^\infty(]0,1[)$, and for $x\in]0,1[$,

$$\Phi(x) = \lim_{N\to+\infty}\int \frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\phi(y)dy$$
$$= \lim_{N\to+\infty}\Big(\int_0^1 \frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\big(\phi(y)-\phi(x)\big)dy + \phi(x)\int_0^1\sum_{|k|\le N} e^{2i\pi k(x-y)}dy\Big)$$
$$= \phi(x),$$

because with $\psi\in C^\infty(\mathbb{R}^2)$, $\theta(s) = \frac{s}{\sin\pi s}$ (which is in $C^\infty(\mathbb{R}\backslash\pi\mathbb{Z}^*)$ and in particular on $]-1,+1[$), we have

$$\int_0^1 \frac{\sin\big(\pi(x-y)(2N+1)\big)}{\sin(\pi(x-y))}\big(\phi(y)-\phi(x)\big)dy$$
$$= \int_0^1 \sin\big(\pi(x-y)(2N+1)\big)\overbrace{\psi(x,y)\theta(x-y)}^{\substack{\text{smooth of } y \text{ on } [0,1]\\ \text{since } x\in]0,1[}}dy \underset{N\to+\infty}{\longrightarrow} 0,$$

[3] In fact, with $Q_k = k+(0,1)^n$ we have, replacing the Euclidean norm $|k|$ by the (equivalent) sup-norm $\|k\| = \max_{1\le j\le k}|k_j|$, we have for $x\in Q_k$, $k_j < x_j < k_j+1$ and thus

$$\|x\| = \max|x_j| \le 1+\|k\| \Longrightarrow 1+\|x\| \le 2+\|k\|$$

and $\sum_{k\in\mathbb{Z}^n}(2+\|k\|)^{-n-1} \le \int\sum_{k\in\mathbb{Z}^n}\mathbf{1}_{Q_k}(x)(1+\|x\|)^{-n-1}dx = \int(1+\|x\|)^{-n-1}dx < +\infty$.

since with $\omega \in C^\infty([0,1])$, we have $\int_0^1 \sin\bigl(\pi(x-y)(2N+1)\bigr)\omega(y)dy =$

$$\Bigl[\frac{\cos\bigl(\pi(x-y)(2N+1)\bigr)}{\pi(2N+1)}\omega(y)\Bigr]_{y=0}^{y=1} - \int_0^1 \frac{\cos\bigl(\pi(x-y)(2N+1)\bigr)}{\pi(2N+1)}\omega'(y)dy.$$

We have proven (3.4.10) for $n=1$ and $x \in]0,1[$. Since Φ, ϕ are both smooth on $[0,1]$ the equality holds as well for $x \in \{0,1\}$.

N.B. We could have used the Riemann-Lebesgue lemma (see e.g. the lemma 3.4.4 in [9]), but we have preferred a simple self-contained argument with an integration by parts since there was no shortage of regularity for the function ω.

To handle the case $n \geq 2$, we use an induction and in $n+1$ dimensions, we have for $\phi \in C_c^\infty(]0,1[^{n+1})$,

$$\forall x \in [0,1]^n, \quad \Phi(x, x_{n+1}) = \sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n} e^{2i\pi(x-y)\cdot k}\phi(y, x_{n+1})dy = \phi(x, x_{n+1}),$$

and thus $\forall x \in [0,1]^n, \forall x_{n+1} \in [0,1], \Phi(x, x_{n+1}) =$

$$\sum_{k \in \mathbb{Z}^n} \int_{(0,1)^n} e^{2i\pi(x-y)\cdot k}\Bigl(\sum_{k_{n+1} \in \mathbb{Z}} \int_0^1 e^{2i\pi(x_{n+1}-y_{n+1})k_{n+1}}\phi(y, y_{n+1})dy_{n+1}\Bigr)dy = \phi(x, x_{n+1}),$$

which is (3.4.10) since the series are uniformly converging. Since $\operatorname{supp}\phi \subset]0,1[^n$, there exists $\epsilon_0 > 0$ such that[4] $\operatorname{supp}\phi \subset [\epsilon_0, 1-\epsilon_0]^n$, and with $\chi \in C_c^\infty(]0,1[)$ equal to 1 on $[\epsilon_0, 1-\epsilon_0]$, we have

$$\chi(x_1)\dots\chi(x_n)\phi(x) = \phi(x) = \sum_{k \in \mathbb{Z}^n} e^{2i\pi x\cdot k}\hat{\phi}(k)\chi(x_1)\dots\chi(x_n). \tag{3.4.11}$$

The series is uniformly converging as well as all its derivatives, thanks to the fast decay of $\hat{\phi}(k)$ expressed by (3.4.9), and the functions

$$\sum_{|k| \leq N} e^{2i\pi x_1 k_1}\dots e^{2i\pi x_n k_n}\hat{\phi}(k)\chi(x_1)\dots\chi(x_n)$$

belong to $\operatorname{Vect}(\otimes^n C_c^\infty(]0,1[)$ with fixed compact support in $]0,1[^n$. The proof of the lemma is complete. □

As a consequence, we get the following result.

Proposition 3.4.10. *Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$. $\operatorname{Vect} C_c^\infty(X) \otimes C_c^\infty(Y)$ is dense in $C_c^\infty(X \times Y)$.*

[4]In fact, each projection $K_j = proj_j(\operatorname{supp}\phi)$ is a compact subset of $]0,1[$, thus $0 < \inf_{t \in K_j} t \leq \sup_{t \in K_j} t < 1$.

Proof. Let K be a compact subset of $X \times Y$. For each point $(x,y) \in K$, we can find some open bounded intervals $I_1, \dots, I_m, J_1, \dots, J_n$ of $\mathbb{R}$ such that

$$(x,y) \in Q = I_1 \times \cdots \times I_m \times J_1 \times \cdots \times J_n \subset X \times Y.$$

As a result, we can cover K with a finite number of open "cubes" $(Q_l)_{1 \le l \le N}$ included in $X \times Y$. Using a partition of unity given by the theorem 3.1.14, we can find $\psi_l \in C_c^\infty(Q_l)$ such that $\sum_{1 \le l \le N} \psi_l(x) = 1$ for $x \in V$ open such that $K \subset V \subset X \times Y$. For $\varphi \in C_c^\infty(X \times Y)$, $\operatorname{supp} \varphi = K$ compact subset of $X \times Y$, we have

$$\varphi = \sum_{1 \le l \le N} \varphi \psi_l, \quad \varphi \psi_l \in C_c^\infty(Q_l).$$

We can then apply the lemma 3.4.9 for each $\varphi \psi_l$ (rescaling the cube Q_l to $]0,1[^n$) to obtain the conclusion of the proposition. □

Theorem 3.4.11. *Let X be an open subset of $\mathbb{R}^m$, Y be an open subset of $\mathbb{R}^n$, and $u \in \mathscr{D}'(X), v \in \mathscr{D}'(Y)$. Then there exists a unique $w \in \mathscr{D}'(X \times Y)$ such that, $\forall \phi \in \mathscr{D}(X), \forall \psi \in \mathscr{D}(Y)$,*

$$\langle w, \phi \otimes \psi \rangle_{\mathscr{D}'(X \times Y), \mathscr{D}(X \times Y)} = \langle u, \phi \rangle_{\mathscr{D}'(X), \mathscr{D}(X)} \langle v, \psi \rangle_{\mathscr{D}'(Y), \mathscr{D}(Y)}, \tag{3.4.12}$$

where $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$. We shall denote w by $u \otimes v$ and call it the tensor product of u and v.

Proof. The uniqueness follows from the proposition 3.4.10. To find such a w, we define for $\Phi \in C_c^\infty(X \times Y)$, with obvious notations,

$$\langle w, \Phi \rangle = \big\langle v(y), \langle u(x), \Phi(x,y) \rangle \big\rangle. \tag{3.4.13}$$

As a matter of fact, thanks to the corollary 3.4.2 (1), the function $Y \ni y \mapsto \langle u(\cdot), \Phi(\cdot, y) \rangle$ belongs to $C_c^\infty(Y)$ so that (3.4.13) makes sense. Using the theorem 3.4.1, we obtain $\partial_y^\alpha \langle u(\cdot), \Phi(\cdot,y) \rangle = \langle u(\cdot), \partial_y^\alpha \Phi(\cdot, y) \rangle$. If $K = \operatorname{supp} \Phi$ (compact subset of $X \times Y$), both projections $proj_X K, proj_Y K$ are compact so that

$$|\langle u(\cdot), \partial_y^\alpha \Phi(\cdot, y) \rangle| \le C_1 \sup_{|\beta| \le N_1,\ x \in proj_X K} |(\partial_x^\beta \partial_y^\alpha \Phi)(x,y)|$$

and thus

$$|\big\langle v(y), \langle u(x), \Phi(x,y) \rangle \big\rangle| \le C_2 \sup_{\substack{|\alpha| \le N_2 \\ y \in proj_Y K}} |\partial_y^\alpha \langle u(\cdot), \Phi(\cdot, y) \rangle|$$
$$\le C_1 C_2 \sup_{\substack{|\beta| \le N_1, |\alpha| \le N_2 \\ (x,y) \in K}} |(\partial_x^\beta \partial_y^\alpha \Phi)(x,y)|,$$

implying that w is indeed a distribution on $X \times Y$. Since the formula (3.4.12) follows from (3.4.13), this concludes the proof of the theorem. □

Remark 3.4.12. (1) The uniqueness ensures that $w = u \otimes v$ is also defined by

$$\langle w, \Phi\rangle = \big\langle u(x), \langle v(y), \Phi(x,y)\rangle\big\rangle, \tag{3.4.14}$$

a formula for which (3.4.12) also holds.
(2) If $u \in L^1_{\text{loc}}(X), v \in L^1_{\text{loc}}(Y)$, then $u \otimes v$ belongs to $L^1_{\text{loc}}(X \times Y)$ and is defined by $u(x)v(y)$, thanks to the lemma 3.1.7 and to the proposition 3.4.10.
(3) For $u \in \mathscr{D}'(X), v \in \mathscr{D}'(Y)$, we have

$$\operatorname{supp}(u \otimes v) = \operatorname{supp} u \times \operatorname{supp} v. \tag{3.4.15}$$

In fact, if $\Phi \in C^\infty_c(X \times Y)$ with $\operatorname{supp}\Phi \subset X \times (\operatorname{supp} v)^c$ or with $\operatorname{supp}\Phi \subset (\operatorname{supp} u)^c \times Y$, it follows from (3.4.14) or (3.4.13) that $\langle u \otimes v, \Phi\rangle = 0$; this holds as well when

$$\operatorname{supp}\Phi \subset (\operatorname{supp} u \times \operatorname{supp} v)^c = \big((\operatorname{supp} u)^c \times Y\big) \cup \big(X \times (\operatorname{supp} v)^c\big),$$

since $\operatorname{supp}\Phi \subset \Omega_1 \cup \Omega_2$ with Ω_j open subset of $X \times Y$ and, thanks to the theorem 3.1.14, the compactly supported $\Phi = \Phi_1 + \Phi_2$, with $\operatorname{supp}\Phi_j \subset \Omega_j$ (it is also a direct consequence of the theorem 3.1.15 since $(u \otimes v)_{|\Omega_j} = 0$). We have proven that $\operatorname{supp}(u \otimes v) \subset \operatorname{supp} u \times \operatorname{supp} v$. Conversely, if $x_0 \in \operatorname{supp} u, y_0 \in \operatorname{supp} v$, and U, V are respective open neighborhoods of x_0, y_0 in X, Y, we can find $\phi_0 \in C^\infty_c(U), \psi_0 \in C^\infty_c(V)$ such that $\langle u, \phi_0\rangle \neq 0$ and $\langle v, \psi_0\rangle \neq 0$. As a result $\phi_0 \otimes \psi_0 \in C^\infty_c(U \times V)$ and $\langle u \otimes v, \phi_0 \otimes \psi_0\rangle = \langle u, \phi_0\rangle\langle v, \psi_0\rangle \neq 0$, so that $(u \otimes v)_{|U \times V}$ is not zero, proving that $(x_0, y_0) \in \operatorname{supp}(u \otimes v)$ and the sought result.
(4) With the notations of the previous theorem, we have obviously from the expression (3.4.13) and the theorem 3.4.1 that $\partial^\alpha_x \partial^\beta_y (u \otimes v) = (\partial^\alpha_x u) \otimes (\partial^\beta_y v)$.

Proposition 3.4.13. *Let $n \in \mathbb{N}^*$, U be an open subset of $\mathbb{R}^{n-1}$, I an interval of $\mathbb{R}$. Let $u \in \mathscr{D}'(U \times I)$ such that $\partial_{x_n} u = 0$. Then, there exists $v \in \mathscr{D}'(U)$ such that $u = v \otimes 1$. In other words, the differential equation $\partial_{x_n} u = 0$ has the only solutions $u(x', x_n) = v(x')$.*

Proof. From the remark 3.4.12 (3) above, the tensor products $v(x') \otimes 1$ are indeed solutions of $\partial_{x_n} u = 0$. Conversely the proposition is proven for $n = 1$ by the lemma 3.2.4. Let us assume $n \geq 2$; we consider $\chi_0 \in C^\infty_c(I)$ such that $\int \chi_0(t)dt = 1$ and we define $v \in \mathscr{D}'(U)$ by the identity

$$\langle v, \varphi\rangle_{\mathscr{D}'(U),\mathscr{D}(U)} = \langle u, \varphi \otimes \chi_0\rangle_{\mathscr{D}'(U \times I),\mathscr{D}(U \times I)}.$$

For $\varphi \in \mathscr{D}(U), \psi \in \mathscr{D}(I)$, we have with $J(\psi) = \int \psi(t)dt$,

$$\langle v \otimes 1, \varphi \otimes \psi\rangle = \langle u, \varphi \otimes \chi_0\rangle J(\psi).$$

From the proof of the lemma 3.2.4, we see that $\psi - \chi_0 J(\psi) = \theta'$ with $\theta \in C^\infty_c(I)$, and we get $\langle u, \varphi \otimes \big(\chi_0 J(\psi) - \psi\big)\rangle = \langle u, \partial_{x_n}(\varphi \otimes \theta)\rangle = 0$ so that $\langle v \otimes 1, \varphi \otimes \psi\rangle = \langle u, \varphi \otimes \psi\rangle$, which is the sought result. $\square$

3.5 Convolution

We want to define the convolution of two distributions on $\mathbb{R}^n$, provided one of them has compact support. Assuming first that $u \in L^1_{\text{comp}}(\mathbb{R}^n), v \in L^1_{\text{loc}}(\mathbb{R}^n), \phi \in C_c^\infty(\mathbb{R}^n)$ the integral

$$\iint u(x-y)v(y)\phi(x)dxdy = \iint u(x)v(y)\phi(x+y)dxdy, \tag{3.5.1}$$

makes sense since x and $x+y$ are moving in a compact set in the last integral (and so is y). This formula allows us to define

$$(u * v)(x) = \int u(x-y)v(y)dy = \int u(y)v(x-y)dy$$

and can naturally be extended to $u, v \in L^1(\mathbb{R}^n)$ so that $\|u*v\|_{L^1(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}\|v\|_{L^1(\mathbb{R}^n)}$, making $L^1(\mathbb{R}^n)$ a Banach algebra (without unit). The inequality of Young (see e.g. the Théorème 6.2.1 in [9]) is a non-trivial extension of that inequality. Anyhow, at the moment, we want to use the formula (3.5.1) for our general definition.

3.5.1 Convolution $\mathscr{E}'(\mathbb{R}^n) * \mathscr{D}'(\mathbb{R}^n)$

Definition 3.5.1. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We define the convolution $u * v$ by the following bracket of duality*

$$\langle u * v, \phi\rangle_{\mathscr{D}'(\mathbb{R}^n),\mathscr{D}(\mathbb{R}^n)} = \big\langle u(x), \langle v(y), \phi(x+y)\rangle\big\rangle = \big\langle v(y), \langle u(x), \phi(x+y)\rangle\big\rangle. \tag{3.5.2}$$

We note that the theorem 3.4.1 shows that the function $\mathbb{R}^n \ni x \mapsto \langle v(y), \phi(x+y)\rangle$ is C^∞ and thus that the first definition makes sense from the corollary 3.4.2 (2). To check the second equality above, we note that with $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the support of u, we have $\chi u = u$ and thus from the remark 3.4.12(1) and the formula (3.4.13),

$$\big\langle u(x), \langle v(y), \phi(x+y)\rangle\big\rangle = \big\langle u(x), \langle v(y), \chi(x)\phi(x+y)\rangle\big\rangle = \langle u(x)\otimes v(y), \chi(x)\phi(x+y)\rangle,$$

which is also equal to $\big\langle v(y), \langle u(x), \chi(x)\phi(x+y)\rangle\big\rangle = \big\langle v(y), \langle u(x), \phi(x+y)\rangle\big\rangle$. This proves as well that $u * v$ is a distribution on $\mathbb{R}^n$ since the mapping $C_c^\infty(\mathbb{R}^n) \ni \phi \mapsto \Phi \in C_c^\infty(\mathbb{R}^{2n})$, with $\Phi(x,y) = \phi(x+y)\chi(x)$ is continuous.

Remark 3.5.2. We note that whatever is $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the support of u, we have for $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$,

$$\langle u * v, \phi\rangle = \langle u(x)\otimes v(y), \chi(x)\phi(x+y)\rangle. \tag{3.5.3}$$

Proposition 3.5.3. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We have*

$$\operatorname{supp}(u * v) \subset \operatorname{supp} u + \operatorname{supp} v. \tag{3.5.4}$$

Proof. Note first that $\operatorname{supp} u + \operatorname{supp} v$ is a closed subset of $\mathbb{R}^n$ as the sum of a compact set and a closed set (exercise). Now if $\phi \in C_c^\infty(\mathbb{R}^n)$ with $\operatorname{supp} \phi \subset (\operatorname{supp} u + \operatorname{supp} v)^c$, then

$$\operatorname{supp}\big((x,y) \mapsto \phi(x+y)\big) \subset (\operatorname{supp} u \times \operatorname{supp} v)^c. \tag{3.5.5}$$

In fact, if $(x_0, y_0) \in \operatorname{supp} u \times \operatorname{supp} v$, then $x_0 + y_0 \in \operatorname{supp} u + \operatorname{supp} v \subset (\operatorname{supp} \phi)^c$, the latter being open so that there exists U open in $\mathscr{V}_0$ with $\phi(x_0 + U + y_0 + U) = 0$. As a consequence, the open set $(x_0 + U) \times (y_0 + U) \subset \big(\operatorname{supp}((x,y) \mapsto \phi(x+y))\big)^c$ and this implies $(x_0, y_0) \in \big(\operatorname{supp}((x,y) \mapsto \phi(x+y))\big)^c$ and proves (3.5.5), so that (3.5.3), (3.4.15) give the conclusion of the proposition. □

Remark 3.5.4. For u, v both in $\mathscr{E}'(\mathbb{R}^n)$, the formula (3.5.2) ensures that $u * v = v * u$.

3.5.2 Regularization

Proposition 3.5.5. *Let $u \in \mathscr{D}'(\mathbb{R}^n), \rho \in C_c^\infty(\mathbb{R}^n)$. Then $\rho * u$ belongs to $C^\infty(\mathbb{R}^n)$.*

Proof. We have from the definitions, with $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near $\operatorname{supp} \rho$, $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle \rho * u, \phi \rangle = \langle \rho(x) \otimes u(y), \chi(x)\phi(x+y) \rangle = \langle u(y), \langle \rho(x), \chi(x)\phi(x+y) \rangle \rangle, \tag{3.5.6}$$

and we note that $\langle \rho(x), \chi(x)\phi(x+y) \rangle = \int \rho(x)\phi(x+y)dx = \int \rho(x-y)\phi(x)dx$. As a result, we have

$$\langle \rho * u, \phi \rangle = \langle u(y), \int \underbrace{\rho(x-y)\phi(x)}_{\in C_c^\infty(\mathbb{R}^{2n})} dx \rangle = \int \phi(x) \langle u(y), \rho(x-y) \rangle dx$$

where the last equality is due to the theorem 3.4.1[5] which gives also that $\psi(x) = \langle u(y), \rho(x-y) \rangle$ is C^∞; we have proven $\rho * u = \psi$ and the result. We note also the formula following from (3.5.6)

$$\langle \rho * u, \phi \rangle = \langle u, \check{\rho} * \phi \rangle. \tag{3.5.7}$$

□

Lemma 3.5.6. *Let Ω be an open subset of $\mathbb{R}^n$ and $T \in \mathscr{D}'(\Omega)$. There exists a sequence $(\psi_j)_{j \geq 1}$ in $\mathscr{D}(\Omega)$ such that $\lim_j \psi_j = T$ in the weak-dual topology sense of the definition 3.1.16.*

Proof. We consider first a sequence $(K_j)_{j \geq 1}$ of compact subsets of Ω as in the lemma 2.3.1 and a sequence $(\chi_j)_{j \geq 1}$ such that $\chi_j \in C_c^\infty(\operatorname{int} K_{j+1}), \chi_j = 1$ near K_j (see the lemma 3.1.3). In the weak-dual topology sense, we have $\lim_j \chi_j T = T$: let $\varphi \in \mathscr{D}(\Omega)$, $K = \operatorname{supp} \varphi$. From the lemma 2.3.1, there exists j such that $\operatorname{supp} \varphi \subset K_j$ and thus $\varphi \chi_j = \varphi$, implying $\langle T\chi_j, \varphi \rangle = \langle T, \chi_j \varphi \rangle = \langle T, \varphi \rangle$. We can also consider the compactly supported distribution $\chi_j T$ and see it as a distribution on $\mathbb{R}^n$. We take now a function $\rho \in C_c^\infty(\mathbb{R}^n)$ such that $\int \rho(x)dx = 1$. According to the first example

[5] For $\Phi \in C_c^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, $u \in \mathscr{D}'(\mathbb{R}^n)$, $\langle 1 \otimes u, \Phi \rangle = \langle u(y), \int \Phi(x,y)dx \rangle = \int \langle u(y), \Phi(x,y) \rangle dx$.

in the section 3.1.3, we define ρ_ϵ (it tends to the Dirac mass at 0 in the weak-dual topology when $\epsilon \to 0_+$). For $\varphi \in \mathscr{D}(\Omega)$, using (3.5.7), we have

$$\langle \rho_\epsilon * (\chi_j T), \varphi\rangle = \langle \chi_j T, \check\rho_\epsilon * \varphi\rangle. \tag{3.5.8}$$

Considering now a decreasing sequence of positive numbers (ϵ_j) with limit 0 such that

$$\operatorname{supp}\chi_j + \epsilon_j \operatorname{supp}\rho \subset \operatorname{int}(K_{j+1}) \subset \Omega,$$

and we define $T_j = \rho_{\epsilon_j} * \chi_j T$. We have from the proposition 3.5.3 that $\operatorname{supp} T_j$ is compact included in Ω and also that $T_j \in C^\infty$ (proposition 3.5.5). Going back to (3.5.8), for a fixed φ, we can find j such that $\operatorname{supp}\varphi \subset K_{j-1}$ for $j \ge j_0$, implying that

$$\operatorname{supp}(\check\rho_{\epsilon_j} * \varphi) \subset K_{j-1} + \epsilon_j \operatorname{supp}\rho \subset \operatorname{supp}\chi_{j-1} + \epsilon_{j-1}\operatorname{supp}\rho \subset K_j,$$

implying that $\chi_j(\check\rho_{\epsilon_j} * \varphi) = \check\rho_{\epsilon_j} * \varphi$ and $\langle \rho_{\epsilon_j} * (\chi_j T), \varphi\rangle = \langle T, \check\rho_{\epsilon_j} * \varphi\rangle$. The result follows from the proposition 3.1.1 (implying $\lim_j(\check\rho_{\epsilon_j} * \varphi) = \varphi$ in $C_c^\infty(\Omega)$) and the (sequential) continuity of the distribution T. □

Proposition 3.5.7. *Let $u \in \mathscr{E}'(\mathbb{R}^n), v \in \mathscr{D}'(\mathbb{R}^n)$. We have*

$$\operatorname{singsupp}(u * v) \subset \operatorname{singsupp} u + \operatorname{singsupp} v. \tag{3.5.9}$$

Proof. We can choose $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 near the singsupp u, $\psi \in C^\infty$ equal to 1 near the singular support of v. We have from the proposition 3.5.5

$$u * v = (\chi u) * v + \underbrace{\overbrace{((1-\chi)u)}^{\in C_c^\infty(\mathbb{R}^n)} * v}_{\in C^\infty(\mathbb{R}^n)} \equiv (\chi u) * (\psi v) + \underbrace{\overbrace{(\chi u)}^{\in \mathscr{E}'(\mathbb{R}^n)} * \overbrace{((1-\psi)v)}^{\in C^\infty(\mathbb{R}^n)}}_{\in C^\infty(\mathbb{R}^n)} \mod C^\infty(\mathbb{R}^n)$$

and thus we get for all $\epsilon > 0$

$$\operatorname{singsupp}(u * v) \subset \operatorname{supp}\psi + \operatorname{supp}\psi \subset \operatorname{singsupp} u + \epsilon \bar B_1 + \operatorname{singsupp} v + \epsilon \bar B_1,$$

which gives the result. □

3.5.3 Convolution with a proper support condition

Looking at the formula (3.5.1), we see that we can extend it easily for $L^1_{\text{loc}}(\mathbb{R}^n)$ functions u, v so that the mapping

$$\operatorname{supp} u \times \operatorname{supp} v \ni (x, y) \mapsto x + y = \sigma(x, y) \in \mathbb{R}^n \tag{3.5.10}$$

is *proper*, i.e. such that $\sigma^{-1}(K)$ is compact for K compact subset of $\mathbb{R}^n$. In fact if $u, v \in L^1_{\text{loc}}(\mathbb{R}^n)$ are such that the map σ of (3.5.10) is proper, the function $u * v$ defined by

$$(u * v)(x) = \int u(x - y) v(y) dy$$

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