Lecture Notes on Real Analysis
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is also $L^1_{\rm loc}(\mathbb{R}^n)$, since for K compact subset of $\mathbb{R}^n$, we have

$$\iint |u(x-y)||v(y)|\mathbf{1}_K(x)dydx = \iint |u(x)||v(y)|\mathbf{1}_K(x+y)dxdy$$
$$= \iint_{\sigma^{-1}(K)} |u(x)||v(y)|dxdy < \infty, \quad \text{since } \sigma^{-1}(K) \text{ is compact in } \mathbb{R}^{2n}.$$

We can extend as well the convolution product of distributions u, v, provided σ in (3.5.10) is proper. Before doing so, we prove a simple lemma.

Lemma 3.5.8. *Let $F_1, \dots, F_m$ be closed subsets of $\mathbb{R}^n$ such that the mapping $\sigma : F_1 \times \dots \times F_m \to \mathbb{R}^n$, defined by $\sigma(x_1, \dots, x_m) = x_1 + \dots + x_m$ is proper. Defining for $\epsilon > 0$, $F_{j,\epsilon} = \{x \in \mathbb{R}^n, |x - F_j| \le \epsilon\}$, the mapping $\sigma_\epsilon : F_{1,\epsilon} \times \dots \times F_{m,\epsilon} \to \mathbb{R}^n$, defined by $\sigma_\epsilon(x_1, \dots, x_m) = x_1 + \dots + x_m$ is also proper.*

Proof. We note first that $F_{j,\epsilon} = F_j + \epsilon \bar{B}_1$ ($\bar{B}_1$ is the closed unit ball of $\mathbb{R}^n$) is closed as the sum of a compact and a closed set. Let K be compact subset of $\mathbb{R}^n$; if $(x_1, \dots, x_m) \in \sigma_\epsilon^{-1}(K)$, then there exists $y_j \in F_j, t_j \in \mathbb{R}^n, |t_j| \le \epsilon$, such that $x_j = y_j + t_j$, $\sum_{1\le j\le m}(y_j + t_j) \in K$ and thus $\sum_{1\le j\le m} y_j \in K + m\epsilon\bar{B}_1$, so that $(y_j)_{1\le j\le m} \in \sigma^{-1}(K + m\epsilon\bar{B}_1)$, a compact subset of $\prod F_j$. As a consequence, $(x_j)_{1\le j\le m} \in \sigma^{-1}(K + m\epsilon\bar{B}_1) + \epsilon\bar{B}_{1,nm}$ ($\bar{B}_{1,nm}$ is the closed unit ball of $\mathbb{R}^{nm}$), which is compact. As a result, $\sigma_\epsilon^{-1}(K)$ is compact as a closed subset of $\prod F_{j,\epsilon}$ (σ_ϵ is continuous) included in a compact set. □

Definition 3.5.9. *Let $u_1, \dots, u_m \in \mathscr{D}'(\mathbb{R}^n)$ such that the mapping σ*

$$\prod_{1\le j\le m} \operatorname{supp} u_j \ni (x_j)_{1\le j\le m} \mapsto \sum_{1\le j\le m} x_j \in \mathbb{R}^n \quad \textit{is proper.} \tag{3.5.11}$$

For $\epsilon > 0$, we take $\chi_{j,\epsilon} \in C^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \chi_{j,\epsilon} \subset \operatorname{supp} u_j + \epsilon\bar{B}_1$ and $\operatorname{supp} \chi_{j,\epsilon}$ is 1 on a neighborhood of $\operatorname{supp} u_j$. We define then

$$\langle u_1 * \dots * u_m, \phi\rangle_{\mathscr{D}'(\mathbb{R}^n),\mathscr{D}(\mathbb{R}^n)} = \langle u_1 \otimes \dots \otimes u_m, \tilde{\phi}\rangle_{\mathscr{D}'(\mathbb{R}^{nm}),\mathscr{D}(\mathbb{R}^{nm})} \tag{3.5.12}$$

with $\tilde{\phi}(x_1, \dots, x_m) = \prod_{1\le j\le m} \chi_{j,\epsilon}(x_j)\phi(\sum_{1\le j\le m} x_j)$: we note that $\tilde{\phi}$ is in $\mathscr{D}(\mathbb{R}^{nm})$ since

$$\operatorname{supp} \tilde{\phi} \subset \{(x_j)_{1\le j\le m} \in \prod_{1\le j\le m} \operatorname{supp} \chi_{j,\epsilon} \textit{ with } \sigma((x_j)) \in \operatorname{supp} \phi\}$$

which is compact from the previous lemma and (3.5.11).

It is also easy to prove that this definition does not depend on the choices of the functions $\chi_{j,\epsilon}$ having the properties listed above and that this definition coincides with the definition of convolution in the previous section. In particular, we can prove the associativity of the convolution using the identity (3.5.12), *provided the condition (3.5.11) is satisfied.* As a counterexample we can take $u_1 = 1, u_2 = \delta_0', u_3 = H$ and we have since $1 * \delta_0' = 0, \delta_0' * H = \delta_0$,

$$(u_1 * u_2) * u_3 = 0, \quad u_1 * (u_2 * u_3) = 1 * \delta_0 = 1.$$

Naturally the hypothesis (3.5.11) is violated here since the mapping σ defined on $\mathbb{R}\times\{0\}\times\mathbb{R}_+$ is not proper: $\sigma^{-1}(\{0\})\supset\{(-N,0,N)\}_{N\in\mathbb{N}}$. The assumption (3.5.11) is satisfied for $m=2$ if $\operatorname{supp} u_1$ is compact and also for distributions on $\mathbb{R}$ with support in $\mathbb{R}_+$. We get also that

$$\forall u\in\mathscr{D}'(\mathbb{R}^n),\quad u*\delta=u,\quad \text{since } \langle u(x_1)\otimes\delta(x_2),\phi(x_1+x_2)\rangle=\langle u,\phi\rangle. \tag{3.5.13}$$

and for $u,v\in\mathscr{D}'(\mathbb{R}^n)$ such that (3.5.11) holds

$$\partial_x^\alpha(u*v)=(\partial_x^\alpha u)*v=u*(\partial_x^\alpha v), \tag{3.5.14}$$

since $\langle\partial_x^\alpha(u*v),\phi\rangle=(-1)^{|\alpha|}\langle u*v,\partial_x^\alpha\phi\rangle=(-1)^{|\alpha|}\langle u(x)\otimes v(y),(\partial^\alpha\phi)(x+y)\rangle=\langle(\partial_x^\alpha u)(x)\otimes v(y),\phi(x+y)\rangle$ and putting inside the brackets the cut-off functions χ_ϵ does not change the outcome of the computation.

3.6 Some fundamental solutions

3.6.1 Definitions

Definition 3.6.1. *We consider a constant coefficients differential operator*

$$P=P(D)=\sum_{|\alpha|\le m}a_\alpha D_x^\alpha,\quad \textit{where } a_\alpha\in\mathbb{C}, D_x^\alpha=\frac{1}{(2i\pi)^{|\alpha|}}\partial_x^\alpha. \tag{3.6.1}$$

A distribution $E\in\mathscr{D}'(\mathbb{R}^n)$ is called a fundamental solution of P when $PE=\delta_0$.

We note that if $f\in\mathscr{E}'(\mathbb{R}^n)$ and E is a fundamental solution of P, we have from (3.5.14), (3.5.13),

$$P(E*f)=PE*f=\delta_0*f=f,$$

which allows to find a solution of the Partial Differential Equation (PDE for short) $P(D)u=f$, at least when f is a compactly supported distribution.

Examples. We have on the real line already proven (see (3.2.2)) that $\frac{dH}{dt}=\delta_0$, so that the Heaviside function is a fundamental solution of d/dt (note that from the lemma 3.2.4, the other fundamental solutions are $C+H(t)$). This also implies that

$$\partial_{x_1}\big(H(x_1)\otimes\delta_0(x_2)\otimes\cdots\otimes\delta_0(x_n)\big)=\delta_0(x),\quad \text{(the Dirac mass at 0 in } \mathbb{R}^n).$$

Let $N\in\mathbb{N}$. With x_+^λ defined in (3.4.8), we get, since $\partial_{x_1}^{N+1}(x_{1,+}^{N+1})=H(x_1)(N+1)!$, that

$$(\partial_{x_1}\dots\partial_{x_n})^{N+2}\Big(\prod_{1\le j\le n}\Big(\frac{x_{j,+}^{N+1}}{(N+1)!}\Big)\Big)=\delta_0(x).$$

The last example has the following interesting consequence.

Proposition 3.6.2. *Let $u\in\mathscr{D}'(\mathbb{R}^n)$ and Ω a bounded open set. Then $u_{|\Omega}$ is a derivative of finite order of a continuous function.*

Proof. We consider for $\chi \in C_c^\infty(\mathbb{R}^n)$ equal to 1 on Ω the distribution $\chi u \in \mathscr{E}'(\mathbb{R}^n)$ whose restriction to Ω coincides with $u_{|\Omega}$. The distribution χu has finite order N (see the remark 3.3.3). We have with $E(x) = \prod_{1\le j\le n} \frac{x_{j,+}^{N+1}}{(N+1)!}$

$$\chi u = \chi u * \delta_0 = (\partial_{x_1} \dots \partial_{x_n})^{N+2}\big(\chi u * E\big). \tag{3.6.2}$$

Since the function E is C^N with Nth derivatives (Lipschitz) continuous, we may consider the function ψ defined by

$$\psi(x) = \langle \chi(y)u(y), E(x-y)\rangle.$$

Since χu is compactly supported with order N, we have with K compact subset of $\mathbb{R}^n$,

$$|\psi(x+h) - \psi(x)| \le C \sup_{|\alpha|\le N, y\in K} |\partial_y^\alpha\big(E(x+h-y) - E(x-y)\big)|.$$

Since the function E is C^N with Nth derivatives Lipschitz continuous, we find that ψ is Lipschitz continuous. We have from the definitions, with $\phi \in C_c^\infty(\mathbb{R}^n)$,

$$\langle E * \chi u, \phi\rangle = \langle E(x) \otimes (\chi u)(y), \phi(x+y)\rangle = \langle (\chi u)(y), \langle E(x), \phi(x+y)\rangle\rangle,$$

and we note that $\langle E(x), \phi(x+y)\rangle = \int E(x-y)\phi(x)dx$. As a result, we have

$$\langle E * \chi u, \phi\rangle = \langle u(y), \int \underbrace{\chi(y)E(x-y)\phi(x)}_{\in C_c^N(\mathbb{R}^{2n})} dx\rangle = \int \phi(x)\langle (\chi u)(y), E(x-y)\rangle dx$$

where the last equality is due to the theorem 3.4.1[6] and gives also that $\psi = \chi u * E$. The result follows from the continuity of ψ and (3.6.2). □

3.6.2 The Laplace and Cauchy-Riemann equations

We define the Laplace operator Δ in $\mathbb{R}^n$ as

$$\Delta = \sum_{1\le j\le n} \partial_{x_j}^2. \tag{3.6.3}$$

In one dimension, we have from (3.2.2) that $\frac{d^2}{dt^2}(t_+) = \delta_0$ and for $n \ge 2$ the following result describes the fundamental solutions of the Laplace operator. In $\mathbb{R}^2_{x,y}$, we define the operator $\bar\partial$ (a.k.a. the Cauchy-Riemann operator) by

$$\bar\partial = \frac{1}{2}(\partial_x + i\partial_y). \tag{3.6.4}$$

Theorem 3.6.3. *We have $\Delta E = \delta_0$ with $\|\cdot\|$ standing for the Euclidean norm,*

$$E(x) = \frac{1}{2\pi}\ln\|x\|, \quad \text{for } n = 2, \tag{3.6.5}$$

$$E(x) = \|x\|^{2-n}\frac{1}{(2-n)|S^{n-1}|}, \quad \text{for } n \ge 3, \text{ with } |S^{n-1}| = \frac{2\pi^{n/2}}{\Gamma(n/2)}, \tag{3.6.6}$$

$$\bar\partial\Big(\frac{1}{\pi z}\Big) = \delta_0, \quad \text{with } z = x+iy \text{ (equality in } \mathscr{D}'(\mathbb{R}^2_{x,y})). \tag{3.6.7}$$

[6] For $\Phi \in C_c^N(\mathbb{R}^n \times \mathbb{R}^n)$, $v \in \mathscr{D}'(\mathbb{R}^n)$, order$(v) \le N$ $\langle 1 \otimes v, \Phi\rangle = \langle v(y), \int \Phi(x,y)dx\rangle = \int \langle v(y), \Phi(x,y)\rangle dx$.

Proof. We start with $n \ge 3$, noting that the function $\|x\|^{2-n}$ is L^1_{loc} and homogeneous with degree $2-n$, so that $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ (see the remark 3.4.7 (2)). Moreover, the function $\|x\|^{2-n} = f(r^2), r^2 = \|x\|^2, f(t) = t_+^{1-\frac{n}{2}}$ is smooth outside 0 and we can compute there

$$\Delta(f(r^2)) = \sum_j \partial_j(f'(r^2)2x_j) = \sum_j f''(r^2)4x_j^2 + 2nf'(r^2) = 4r^2 f''(r^2) + 2nf'(r^2),$$

so that with $t = r^2$,

$$\Delta(f(r^2)) = 4t(1-\frac{n}{2})(-\frac{n}{2})t^{-\frac{n}{2}-1} + 2n(1-\frac{n}{2})t^{-\frac{n}{2}} = t^{-\frac{n}{2}}(1-\frac{n}{2})(-2n+2n) = 0.$$

As a result, $\Delta\|x\|^{2-n}$ is homogeneous with degree $-n$ and supported in $\{0\}$. From the theorem 3.3.4, we obtain that

$$\underbrace{\Delta\|x\|^{2-n} = c\delta_0}_{\substack{\text{homogeneous}\\ \text{degree } -n}} + \sum_{1\le j\le m} \underbrace{\sum_{|\alpha|=j} c_{j,\alpha}\delta_0^{(\alpha)}}_{\substack{\text{homogeneous}\\ \text{degree } -n-j}}.$$

The lemma 3.4.8 implies that for $1 \le j \le m$, $0 = \sum_{|\alpha|=j} c_{j,\alpha}\delta_0^{(\alpha)}$ and $\Delta\|x\|^{2-n} = c\delta_0$. It remains to determine the constant c. We calculate, using the previous formulas for the computation of $\Delta(f(r^2))$, here with $f(t) = e^{-\pi t}$,

$$\begin{aligned} c = \langle \Delta\|x\|^{2-n}, e^{-\pi\|x\|^2}\rangle &= \int \|x\|^{2-n} e^{-\pi\|x\|^2}\big(4\|x\|^2\pi^2 - 2n\pi\big)dx \\ &= |S^{n-1}| \int_0^{+\infty} r^{2-n+n-1} e^{-\pi r^2}(4\pi^2 r^2 - 2n\pi)dr \\ &= |S^{n-1}|\Big(\frac{1}{2\pi}[e^{-\pi r^2}(4\pi^2r^2 - 2n\pi)]^0_{+\infty} + \frac{1}{2\pi}\int_0^{+\infty} e^{-\pi r^2} 8\pi^2 r dr\Big) \\ &= |S^{n-1}|(-n+2), \end{aligned}$$

giving (3.6.6). For the convenience of the reader, we calculate explicitely $|S^{n-1}|$. We have indeed

$$\begin{aligned} 1 &= \int_{\mathbb{R}^n} e^{-\pi\|x\|^2} dx = |S^{n-1}| \int_0^{+\infty} r^{n-1} e^{-\pi r^2} dr \\ &\underbrace{=}_{r=t^{1/2}\pi^{-1/2}} |S^{n-1}|\pi^{(1-n)/2} \int_0^{+\infty} t^{\frac{n-1}{2}} e^{-t}\frac{1}{2}t^{-1/2} dt \pi^{-1/2} = |S^{n-1}|\pi^{-n/2}2^{-1}\Gamma(n/2). \end{aligned}$$

Turning now our attention to the Cauchy-Riemann equation, we see that $1/z$ is also $L^1_{\text{loc}}(\mathbb{R}^2)$, homogeneous of degree -1, and satisfies $\bar\partial(z^{-1}) = 0$ on the complement of $\{0\}$, so that the same reasoning as above shows that

$$\bar\partial(\pi^{-1}z^{-1}) = c\delta_0.$$

To check the value of c, we write $c = \langle \bar\partial(\pi^{-1}z^{-1}), e^{-\pi z\bar z}\rangle = \int_{\mathbb{R}^2} e^{-\pi z\bar z}\pi^{-1}z^{-1}\pi z dx dy = 1$, which gives (3.6.7). We are left with the Laplace equation in two dimensions and we note that with $\frac{\partial}{\partial z} = \frac{1}{2}(\partial_x - i\partial_y)$, $\frac{\partial}{\partial \bar z} = \frac{1}{2}(\partial_x + i\partial_y)$, we have in two dimensions

$$\Delta = 4\frac{\partial}{\partial z}\frac{\partial}{\partial \bar z} = 4\frac{\partial}{\partial \bar z}\frac{\partial}{\partial z}. \tag{3.6.8}$$

Solving the equation $4\frac{\partial E}{\partial z} = \frac{1}{\pi z}$ leads us to try $E = \frac{1}{2\pi}\ln|z|$ and we check directly[7] that $\frac{\partial}{\partial z}\big(\ln(z\bar{z})\big) = z^{-1}$

$$\Delta(\frac{1}{2\pi}\ln|z|) = \pi^{-1}2^{-2}4\frac{\partial}{\partial \bar{z}}\frac{\partial}{\partial z}\big(\ln(z\bar{z})\big) = \pi^{-1}\frac{\partial}{\partial \bar{z}}\big(z^{-1}\big) = \delta_0. \qquad \square$$

3.6.3 Hypoellipticity

Definition 3.6.4. *Let P be a linear operator of type (3.6.1). We shall say that P is hypoelliptic when for all open subsets Ω of $\mathbb{R}^n$ and all $u \in \mathscr{D}'(\Omega)$, we have*

$$\operatorname{singsupp} u = \operatorname{singsupp} Pu. \tag{3.6.9}$$

It is obvious that $\operatorname{singsupp} Pu \subset \operatorname{singsupp} u$, so the hypoellipticity means that $\operatorname{singsupp} u \subset \operatorname{singsupp} Pu$, which is a very interesting piece of information since we can then determine the singularities of our (unknown) solution u, which are located at the same place as the singularities of the source f, which is known when we try to solve the equation $Pu = f$.

Theorem 3.6.5. *Let P be a linear operator of type (3.6.1) such that P has a fundamental solution E satisfying*

$$\operatorname{singsupp} E = \{0\}. \tag{3.6.10}$$

Then P is hypoelliptic. In particular the Laplace and the Cauchy-Riemann operators are hypoelliptic.

N.B. The condition (3.6.10) appears as an iff condition for the hypoellipticity of the operator P since it is also a consequence of the hypoellipticity property.

Proof. Assume that (3.6.10) holds, let Ω be an open subset of $\mathbb{R}^n$ and $u \in \mathscr{D}'(\Omega)$. We consider $f = Pu \in \mathscr{D}'(\Omega)$, $x_0 \notin \operatorname{singsupp} f$, $\chi_0 \in C_c^\infty(\Omega)$, $\chi_0 = 1$ near x_0. We have from the proposition 3.5.5 that

$$\chi u = \chi u * PE = (P\chi u) * E = ([P,\chi]u) * E + \underbrace{\overbrace{(\chi f)}^{\in C_c^\infty(\mathbb{R}^n)} * E}_{\in C^\infty(\mathbb{R}^n)}$$

and thus, using the the proposition 3.5.7 for singular supports, we get

$$\operatorname{singsupp}(\chi u) \subset \operatorname{singsupp}([P,\chi]u) + \operatorname{singsupp} E = \operatorname{singsupp}([P,\chi]u) \subset \operatorname{supp}(u\nabla\chi),$$

and since χ is identically 1 near x_0, we get that $x_0 \notin \operatorname{supp}(u\nabla\chi)$, implying $x_0 \notin \operatorname{singsupp}(\chi u)$, proving that $x_0 \notin \operatorname{singsupp} u$ and the result. $\square$

[7] Noting that $\ln(x^2+y^2)$ and its first derivatives are $L^1_{\text{loc}}(\mathbb{R}^2)$, we have for $\varphi \in C_c^\infty(\mathbb{R}^2)$, $\langle \frac{\partial}{\partial z}(\ln|z|^2), \varphi\rangle =$

$$\frac{1}{2}\iint_{\mathbb{R}^2}(-\partial_x\varphi + i\partial_y\varphi)\ln(x^2+y^2)dxdy = \iint \varphi(x,y)(xr^{-2} - iyr^{-2})dxdy = \iint (x+iy)^{-1}\varphi(x,y)dxdy.$$

3.7 Appendix

3.7.1 The Gamma function

The gamma function Γ is a meromorphic function on $\mathbb{C}$ given for $\operatorname{Re} z > 0$ by the formula

$$\Gamma(z) = \int_0^{+\infty} e^{-t} t^{z-1} dt. \tag{3.7.1}$$

For $n \in \mathbb{N}$, we have $\Gamma(n+1) = n!$; another interesting value is $\Gamma(1/2) = \sqrt{\pi}$. The functional equation

$$\Gamma(z+1) = z\Gamma(z) \tag{3.7.2}$$

is easy to prove for $\operatorname{Re} z > 0$ and can be used to extend the Γ function into a meromorphic function with simple poles at $-\mathbb{N}$ and $\operatorname{Res}(\Gamma, -k) = \frac{(-1)^k}{k!}$. For instance, for $-1 < \operatorname{Re} z \le 0$ with $z \ne 0$ we define

$$\Gamma(z) = \frac{\Gamma(z+1)}{z}, \qquad \text{where we can use (3.7.1) to define } \Gamma(z+1).$$

More generally for $k \in \mathbb{N}$, $-1-k < \operatorname{Re} z \le -k$, $z \ne -k$, we can define

$$\Gamma(z) = \frac{\Gamma(z+k+1)}{z(z+1)\dots(z+k)}.$$

There are manifold references on the Gamma function. One of the most comprehensive is certainly the chapter VII of the Bourbaki volume *Fonctions de variable réelle* [2].

3.7.2 *LF* spaces

3.7.3 The Schwartz kernel theorem

3.7.4 Coordinate transformations and pullbacks

Chapter 4

Introduction to Fourier Analysis

4.1 Fourier Transform of tempered distributions

4.1.1 The Fourier transformation on $\mathscr{S}(\mathbb{R}^n)$

Let $n \geq 1$ be an integer. The Schwartz space $\mathscr{S}(\mathbb{R}^n)$ is defined in the section 2.3.5, is a Fréchet space, as the space of C^∞ functions u from $\mathbb{R}^n$ to $\mathbb{C}$ such that, for all multi-indices[1] $\alpha, \beta \in \mathbb{N}^n$,

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial_x^\beta u(x)| < +\infty.$$

A simple example of such a function is $e^{-|x|^2}$, ($|x|$ is the Euclidean norm of x) and more generally if A is a symmetric positive definite $n \times n$ matrix the function

$$v_A(x) = e^{-\pi \langle Ax, x \rangle}$$

belongs to the Schwartz class.

Definition 4.1.1. *For $u \in \mathscr{S}(\mathbb{R}^n)$, we define its Fourier transform $\hat{u}$ as*

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-2i\pi x \cdot \xi} u(x) dx. \tag{4.1.1}$$

Lemma 4.1.2. *The Fourier transform sends continuously $\mathscr{S}(\mathbb{R}^n)$ into itself.*

Proof. Just notice that $\xi^\alpha \partial_\xi^\beta \hat{u}(\xi) = \int e^{-2i\pi x \xi} \partial_x^\alpha (x^\beta u)(x) dx (2i\pi)^{|\beta| - |\alpha|} (-1)^{|\beta|}$. □

Lemma 4.1.3. *For a symmetric positive definite $n \times n$ matrix A, we have*

$$\widehat{v_A}(\xi) = (\det A)^{-1/2} e^{-\pi \langle A^{-1} \xi, \xi \rangle}. \tag{4.1.2}$$

[1] Here we use the multi-index notation: for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$ we define

$$x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}, \quad \partial_x^\alpha = \partial_{x_1}^{\alpha_1} \ldots \partial_{x_n}^{\alpha_n}, \quad |\alpha| = \sum_{1 \leq j \leq n} \alpha_j.$$

Proof. In fact, diagonalizing the symmetric matrix A, it is enough to prove the one-dimensional version of (4.1.2), i.e. to check

$$\int e^{-2i\pi x\xi} e^{-\pi x^2} dx = \int e^{-\pi(x+i\xi)^2} dx e^{-\pi\xi^2} = e^{-\pi\xi^2},$$

where the second equality is obtained by taking the ξ-derivative of $\int e^{-\pi(x+i\xi)^2} dx$: we have indeed

$$\frac{d}{d\xi}\Big(\int e^{-\pi(x+i\xi)^2} dx\Big) = \int e^{-\pi(x+i\xi)^2}(-2i\pi)(x+i\xi)dx = -i\int \frac{d}{dx}\big(e^{-\pi(x+i\xi)^2}\big)dx = 0.$$

For $a>0$, we obtain $\int_{\mathbb{R}} e^{-2i\pi x\xi} e^{-\pi a x^2} dx = a^{-1/2} e^{-\pi a^{-1}\xi^2}$, which is the sought result in one dimension. If $n \ge 2$, and A is a positive definite symmetric matrix, there exists an orthogonal $n\times n$ matrix P (i.e. ${}^tPP = \mathrm{Id}$) such that

$$D = {}^tPAP, \quad D = \mathrm{diag}(\lambda_1, \dots, \lambda_n), \text{ all } \lambda_j > 0.$$

As a consequence, we have, since $|\det P| = 1$,

$$\begin{aligned}
\int_{\mathbb{R}^n} e^{-2i\pi x\cdot\xi} e^{-\pi\langle Ax,x\rangle} dx &= \int_{\mathbb{R}^n} e^{-2i\pi(Py)\cdot\xi} e^{-\pi\langle APy,Py\rangle} dy = \int_{\mathbb{R}^n} e^{-2i\pi y\cdot({}^tP\xi)} e^{-\pi\langle Dy,y\rangle} dy \\
(\text{with } \eta = {}^tP\xi) \quad &= \prod_{1\le j\le n} \int_{\mathbb{R}} e^{-2i\pi y_j\eta_j} e^{-\pi\lambda_j y_j^2} dy_j = \prod_{1\le j\le n} \lambda_j^{-1/2} e^{-\pi\lambda_j^{-1}\eta_j^2} \\
= (\det A)^{-1/2} e^{-\pi\langle D^{-1}\eta,\eta\rangle} &= (\det A)^{-1/2} e^{-\pi\langle {}^tPA^{-1}P\ {}^tP\xi, {}^tP\xi\rangle} = (\det A)^{-1/2} e^{-\pi\langle A^{-1}\xi,\xi\rangle}. \qquad \square
\end{aligned}$$

Proposition 4.1.4. *The Fourier transformation is an isomorphism of the Schwartz class and for $u \in \mathscr{S}(\mathbb{R}^n)$, we have*

$$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi. \tag{4.1.3}$$

Proof. Using (4.1.2) we calculate for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\epsilon > 0$, dealing with absolutely converging integrals,

$$\begin{aligned}
u_\epsilon(x) &= \int e^{2i\pi x\xi} \hat{u}(\xi) e^{-\pi\epsilon^2|\xi|^2} d\xi \\
&= \iint e^{2i\pi x\xi} e^{-\pi\epsilon^2|\xi|^2} u(y) e^{-2i\pi y\xi} dy d\xi \\
&= \int u(y) e^{-\pi\epsilon^{-2}|x-y|^2} \epsilon^{-n} dy \\
&= \int \underbrace{\big(u(x+\epsilon y) - u(x)\big)}_{\text{with absolute value} \le \epsilon|y|\|u'\|_{L^\infty}} e^{-\pi|y|^2} dy + u(x).
\end{aligned}$$

Taking the limit when ϵ goes to zero, we get the Fourier inversion formula

$$u(x) = \int e^{2i\pi x\xi} \hat{u}(\xi) d\xi. \tag{4.1.4}$$

We have also proven for $u \in \mathscr{S}(\mathbb{R}^n)$ and $\check{u}(x) = u(-x)$

$$u = \check{\hat{\hat{u}}}. \tag{4.1.5}$$

Since $u \mapsto \hat{u}$ and $u \mapsto \check{u}$ are continuous homomorphisms of $\mathscr{S}(\mathbb{R}^n)$, this completes the proof of the proposition. □

Proposition 4.1.5. *Using the notation*

$$D_{x_j} = \frac{1}{2i\pi}\frac{\partial}{\partial x_j}, \quad D_x^\alpha = \prod_{j=1}^n D_{x_j}^{\alpha_j} \quad \textit{with } \alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}^n, \tag{4.1.6}$$

we have, for $u \in \mathscr{S}(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \qquad (D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha u(x)}(\xi) \tag{4.1.7}$$

Proof. We have for $u \in \mathscr{S}(\mathbb{R}^n)$, $\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx$ and thus

$$(D_\xi^\alpha \hat{u})(\xi) = (-1)^{|\alpha|} \int e^{-2i\pi x\cdot\xi} x^\alpha u(x)dx,$$
$$\xi^\alpha \hat{u}(\xi) = \int (-2i\pi)^{-|\alpha|}\partial_x^\alpha\big(e^{-2i\pi x\cdot\xi}\big)u(x)dx = \int e^{-2i\pi x\cdot\xi}(2i\pi)^{-|\alpha|}(\partial_x^\alpha u)(x)dx,$$

proving both formulas. □

N.B. The normalization factor $\frac{1}{2i\pi}$ leads to a simplification in the formulas (4.1.7), but the most important aspect of these formulas is certainly that the Fourier transformation exchanges the operation of derivation against the operation of multiplication. For instance if $P(D)$ is given by a formula (3.6.1), we have

$$\widehat{Pu}(\xi) = \sum_{|\alpha|\le m} a_\alpha \xi^\alpha \hat{u}(\xi) = P(\xi)\hat{u}(\xi).$$

Remark 4.1.6. We have the following continuous inclusions[2]

$$\mathscr{D}(\mathbb{R}^n) \hookrightarrow \mathscr{S}(\mathbb{R}^n) \hookrightarrow \mathscr{E}(\mathbb{R}^n), \tag{4.1.8}$$

triggering the (continuous) inclusions of topological duals,

$$\mathscr{E}'(\mathbb{R}^n) \hookrightarrow \mathscr{S}'(\mathbb{R}^n) \hookrightarrow \mathscr{D}'(\mathbb{R}^n). \tag{4.1.9}$$

The space $\mathscr{S}'(\mathbb{R}^n)$ is the topological dual of the Fréchet space $\mathscr{S}(\mathbb{R}^n)$ and is called the space of *tempered distributions on* $\mathbb{R}^n$. We shall sometimes omit the "$\mathbb{R}^n$" in $\mathscr{S}(\mathbb{R}^n), \mathscr{S}'(\mathbb{R}^n)$, at least when it is clear that the dimension is fixed equal to n.

The Fourier transformation can be extended to $\mathscr{S}'(\mathbb{R}^n)$.

[2] The first inclusion is certainly sequentially continuous according to the definition 3.1.9 and the second is an inclusion of Fréchet spaces: for each semi-norm p on $\mathscr{E}(\mathbb{R}^n)$, there exists a semi-norm q on $\mathscr{S}(\mathbb{R}^n)$ such that for all $u \in \mathscr{S}(\mathbb{R}^n)$, $p(u) \le q(u)$.

4.1.2 The Fourier transformation on $\mathscr{S}'(\mathbb{R}^n)$

Definition 4.1.7. *Let T be a tempered distribution ; the Fourier transform $\hat{T}$ of T is the tempered distibution defined by the formula*

$$\langle \hat{T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}. \tag{4.1.10}$$

The linear form $\hat{T}$ is obviously a tempered distribution since the Fourier transformation is continuous on $\mathscr{S}$. Thanks to the lemma 3.1.7, if $T \in \mathscr{S}$, the present definition of $\hat{T}$ and (4.1.1) coincide.

Note that for $T, \varphi \in \mathscr{S}$, we have $\langle \hat{T}, \varphi\rangle = \iint T(x)e^{-2i\pi x\cdot\xi}\varphi(\xi)dxd\xi = \langle T, \hat{\varphi}\rangle$. This definition gives that

$$\widehat{\delta_0} = 1, \tag{4.1.11}$$

since $\langle \widehat{\delta_0}, \varphi\rangle = \langle \delta_0, \widehat{\varphi}\rangle = \widehat{\varphi}(0) = \int \varphi(x)dx = \langle 1, \varphi\rangle$.

Theorem 4.1.8. *The Fourier transformation is an isomorphism of $\mathscr{S}'(\mathbb{R}^n)$. Let T be a tempered distribution. Then we have*[3]

$$T = \check{\hat{\hat{T}}}. \tag{4.1.12}$$

With obvious notations, we have the following extensions of (4.1.7),

$$\widehat{D_x^\alpha T}(\xi) = \xi^\alpha \hat{T}(\xi), \qquad (D_\xi^\alpha \hat{T})(\xi) = (-1)^{|\alpha|}\widehat{x^\alpha T(x)}(\xi). \tag{4.1.13}$$

Proof. Using the notation $(\check{\varphi})(x) = \varphi(-x)$ for $\varphi \in \mathscr{S}$, we define $\check{S}$ for $S \in \mathscr{S}'$ by (see the remark 3.4.4), $\langle \check{S}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle S, \check{\varphi}\rangle_{\mathscr{S}',\mathscr{S}}$ and we obtain for $T \in \mathscr{S}'$

$$\langle \check{\hat{\hat{T}}}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{\hat{T}}, \check{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \hat{\check{\varphi}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \hat{\hat{\check{\varphi}}}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \varphi\rangle_{\mathscr{S}',\mathscr{S}},$$

where the last equality is due to the fact that $\varphi \mapsto \check{\varphi}$ commutes[4] with the Fourier transform and (4.1.4) means $\hat{\hat{\check{\varphi}}} = \varphi$, a formula also proven true on $\mathscr{S}'$ by the previous line of equality. The formula (4.1.7) is true as well for $T \in \mathscr{S}'$ since, with $\varphi \in \mathscr{S}$ and $\varphi_\alpha(\xi) = \xi^\alpha\varphi(\xi)$, we have

$$\langle \widehat{D^\alpha T}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, (-1)^{|\alpha|}D^\alpha\hat{\varphi}\rangle_{\mathscr{S}',\mathscr{S}} = \langle T, \widehat{\varphi_\alpha}\rangle_{\mathscr{S}',\mathscr{S}} = \langle \hat{T}, \varphi_\alpha\rangle_{\mathscr{S}',\mathscr{S}},$$

and the other part is proven the same way. □

The following lemma will be useful.

Lemma 4.1.9. *Let $T \in \mathscr{S}'(\mathbb{R}^n)$ be a homogeneous distribution of degree m. Then its Fourier transform is a homogeneous distribution of degree $-m-n$*

Proof. We check

$$(\xi\cdot D_\xi)\hat{T} = -\xi\cdot\widehat{xT} = -(\widehat{D_x\cdot xT}) = -\frac{n}{2i\pi}\hat{T} - \frac{1}{2i\pi}(\widehat{x\cdot\partial_x T}) = -\frac{(n+m)}{2i\pi}\hat{T},$$

so that the Euler equation (3.4.6) $\xi\dot{\partial}_\xi\hat{T} = -(n+m)\hat{T}$ is satisfied. □

[3] According to the remark 3.4.4, $\check{T}$ is the distribution defined by $\langle \check{T}, \varphi\rangle = \langle T, \check{\varphi}\rangle$ and if $T \in \mathscr{S}'$, $\check{T}$ is also a tempered distribution since $\varphi \mapsto \check{\varphi}$ is an involutive isomorphism of $\mathscr{S}$.

[4] If $\varphi \in \mathscr{S}$, we have $\widehat{\check{\varphi}}(\xi) = \int e^{-2i\pi x\cdot\xi}\varphi(-x)dx = \int e^{2i\pi x\cdot\xi}\varphi(x)dx = \hat{\varphi}(-\xi) = \check{\hat{\varphi}}(\xi)$.

4.1.3 The Fourier transformation on $L^1(\mathbb{R}^n)$

Theorem 4.1.10. *The Fourier transformation is linear continuous from $L^1(\mathbb{R}^n)$ into $L^\infty(\mathbb{R}^n)$ and for $u \in L^1(\mathbb{R}^n)$, we have*

$$\hat{u}(\xi) = \int e^{-2i\pi x\cdot\xi} u(x)dx, \quad \|\hat{u}\|_{L^\infty(\mathbb{R}^n)} \le \|u\|_{L^1(\mathbb{R}^n)}. \tag{4.1.14}$$

Proof. The formula (4.1.1) can be used to define directly the Fourier transform of a function in $L^1(\mathbb{R}^n)$ and this gives an $L^\infty(\mathbb{R}^n)$ function which coincides with the Fourier transform: for a test function $\varphi \in \mathscr{S}(\mathbb{R}^n)$, and $u \in L^1(\mathbb{R}^n)$, we have by the definition (4.1.10) above and the Fubini theorem

$$\langle \hat{u}, \varphi\rangle_{\mathscr{S}',\mathscr{S}} = \int u(x)\hat{\varphi}(x)dx = \iint u(x)\varphi(\xi)e^{-2i\pi x\cdot\xi}dxd\xi = \int \widetilde{u}(\xi)\varphi(\xi)d\xi$$

with $\widetilde{u}(\xi) = \int e^{-2i\pi x\cdot\xi}u(x)dx$ which is thus the Fourier transform of u. □

4.1.4 The Fourier transformation on $L^2(\mathbb{R}^n)$

We refer the reader to the section 5.3 in Chapter 5.

4.1.5 Some standard examples of Fourier transform

Let us consider the Heaviside function defined on $\mathbb{R}$ by $H(x) = 1$ for $x > 0$, $H(x) = 0$ for $x \le 0$; it is obviously a tempered distribution, so that we can compute its Fourier transform. With the notation of this section, we have, with δ_0 the Dirac mass at 0, $\check{H}(x) = H(-x)$,

$$\widehat{H} + \widehat{\check{H}} = \hat{1} = \delta_0, \quad \widehat{H} - \widehat{\check{H}} = \widehat{\operatorname{sign}}, \qquad \frac{1}{i\pi} = \frac{1}{2i\pi}2\widehat{\delta_0}(\xi) = \widehat{D\operatorname{sign}}(\xi) = \xi\widehat{\operatorname{sign}}\xi$$

so that $\xi\big(\widehat{\operatorname{sign}}\xi - \frac{1}{i\pi}pv(1/\xi)\big) = 0$ and from the theorem 3.2.8, we get

$$\widehat{\operatorname{sign}}\xi - \frac{1}{i\pi}pv(1/\xi) = c\delta_0,$$

with $c = 0$ since the lhs is odd. We obtain

$$\widehat{\operatorname{sign}}(\xi) = \frac{1}{i\pi}pv\frac{1}{\xi}, \tag{4.1.15}$$

$$\widehat{pv(\frac{1}{\pi x})} = -i\operatorname{sign}\xi, \tag{4.1.16}$$

$$\hat{H} = \frac{\delta_0}{2} + \frac{1}{2i\pi}pv(\frac{1}{\xi}) = \frac{1}{(x-i0)}\frac{1}{2i\pi}. \tag{4.1.17}$$

Let us consider now for $0 < \alpha < n$ the $L^1_{\text{loc}}(\mathbb{R}^n)$ function $u_\alpha(x) = |x|^{\alpha-n}$ ($|x|$ is the Euclidean norm of x); since u_α is also bounded for $|x| \ge 1$, it is a tempered distribution. Let us calculate its Fourier transform v_α. Since u_α is homogeneous of degree $\alpha - n$, we get from the lemma 4.1.9 that v_α is a homogeneous distribution of degree $-\alpha$. On the other hand, if $S \in O(\mathbb{R}^n)$ (the orthogonal group), we have in the distribution sense (see the definition 3.4.3), since u_α is a radial function,

$$v_\alpha(S\xi) = v_\alpha(\xi). \tag{4.1.18}$$

The distribution $|\xi|^\alpha v_\alpha(\xi)$ is homogeneous of degree 0 on $\mathbb{R}^n\backslash\{0\}$ and is also "radial", i.e. satisfies (4.1.18). Moreover on $\mathbb{R}^n\backslash\{0\}$, the distribution v_α is a C^1 function which coincides with

$$\int e^{-2i\pi x\cdot\xi}\chi_0(x)|x|^{\alpha-n}dx + |\xi|^{-2N}\int e^{-2i\pi x\cdot\xi}|D_x|^{2N}\big(\chi_1(x)|x|^{\alpha-n}\big)dx,$$

where $\chi_0 \in C_c^\infty(\mathbb{R}^n)$ is 1 near 0 and $\chi_1 = 1-\chi_0$, $N \in \mathbb{N}, \alpha+1 < 2N$. As a result $|\xi|^\alpha v_\alpha(\xi) = c_\alpha$ on $\mathbb{R}^n\backslash\{0\}$ and the distribution on $\mathbb{R}^n$ (note that $\alpha < n$)

$$T = v_\alpha(\xi) - c_\alpha|\xi|^{-\alpha}$$

is supported in $\{0\}$ and homogeneous (on $\mathbb{R}^n$) with degree $-\alpha$. From the theorem 3.3.4 and the lemma 3.4.8, the condition $0 < \alpha < n$ gives $v_\alpha = c_\alpha|\xi|^{-\alpha}$. To find c_α, we compute

$$\int |x|^{\alpha-n}e^{-\pi x^2}dx = \langle u_\alpha, e^{-\pi x^2}\rangle = c_\alpha\int |\xi|^{-\alpha}e^{-\pi\xi^2}d\xi$$

which yields

$$2^{-1}\Gamma(\frac{\alpha}{2})\pi^{-\frac{\alpha}{2}} = \int_0^{+\infty} r^{\alpha-1}e^{-\pi r^2}dr = c_\alpha\int_0^{+\infty} r^{n-\alpha-1}e^{-\pi r^2}dr = c_\alpha 2^{-1}\Gamma(\frac{n-\alpha}{2})\pi^{-\frac{n-\alpha}{2}}.$$

We have proven the following lemma.

Lemma 4.1.11. *Let $n \in \mathbb{N}^*$ and $\alpha \in]0,n[$. The function $u_\alpha(x) = |x|^{\alpha-n}$ is $L^1_{loc}(\mathbb{R}^n)$ and also a temperate distribution on $\mathbb{R}^n$. Its Fourier transform v_α is also $L^1_{loc}(\mathbb{R}^n)$ and given by*

$$v_\alpha(\xi) = |\xi|^{-\alpha}\pi^{\frac{n}{2}-\alpha}\frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{n-\alpha}{2})}.$$

4.2 The Poisson summation formula

4.2.1 Wave packets

We define for $x \in \mathbb{R}^n$, $(y,\eta) \in \mathbb{R}^n \times \mathbb{R}^n$

$$\varphi_{y,\eta}(x) = 2^{n/4}e^{-\pi(x-y)^2}e^{2i\pi(x-y)\cdot\eta} = 2^{n/4}e^{-\pi(x-y-i\eta)^2}e^{-\pi\eta^2}, \tag{4.2.1}$$

$$\text{where for } \zeta = (\zeta_1,\dots,\zeta_n) \in \mathbb{C}^n, \quad \zeta^2 = \sum_{1\le j\le n}\zeta_j^2. \tag{4.2.2}$$

We note that the function $\varphi_{y,\eta}$ is in $\mathcal{S}(\mathbb{R}^n)$ and with L^2 norm 1. In fact, $\varphi_{y,\eta}$ appears as a *phase translation* of a normalized Gaussian. The following lemma introduces the wave packets transform as a Gabor wavelet.

Lemma 4.2.1. *Let u be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$. We define*

$$(Wu)(y,\eta) = (u, \varphi_{y,\eta})_{L^2(\mathbb{R}^n)} = 2^{n/4} \int u(x) e^{-\pi(x-y)^2} e^{-2i\pi(x-y)\cdot\eta} dx \tag{4.2.3}$$

$$= 2^{n/4} \int u(x) e^{-\pi(y-i\eta-x)^2} dx e^{-\pi\eta^2}. \tag{4.2.4}$$

For $u \in L^2(\mathbb{R}^n)$, the function Tu defined by

$$(Tu)(y+i\eta) = e^{\pi\eta^2} Wu(y,-\eta) = 2^{n/4} \int u(x) e^{-\pi(y+i\eta-x)^2} dx \tag{4.2.5}$$

is an entire function. The mapping $u \mapsto Wu$ is continuous from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}(\mathbb{R}^{2n})$ and isometric from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^{2n})$. Moreover, we have the reconstruction formula

$$u(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} Wu(y,\eta)\varphi_{y,\eta}(x) dy d\eta. \tag{4.2.6}$$

Proof. For u in $\mathcal{S}(\mathbb{R}^n)$, we have

$$Wu(y,\eta) = e^{2i\pi y\eta} \widehat{\Omega}^1(\eta, y)$$

where $\widehat{\Omega}^1$ is the Fourier transform with respect to the first variable of the $\mathcal{S}(\mathbb{R}^{2n})$ function $\Omega(x,y) = u(x)e^{-\pi(x-y)^2} 2^{n/4}$. Thus the function Wu belongs to $\mathcal{S}(\mathbb{R}^{2n})$. It makes sense to compute

$$2^{-n/2}(Wu, Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+} \int u(x_1)\overline{u}(x_2) e^{-\pi[(x_1-y)^2+(x_2-y)^2+2i(x_1-x_2)\eta+\epsilon^2\eta^2]} dy d\eta dx_1 dx_2. \tag{4.2.7}$$

Now the last integral on $\mathbb{R}^{4n}$ converges absolutely and we can use the Fubini theorem. Integrating with respect to η involves the Fourier transform of a Gaussian function and we get $\epsilon^{-n} e^{-\pi\epsilon^{-2}(x_1-x_2)^2}$. Since

$$2(x_1-y)^2 + 2(x_2-y)^2 = (x_1+x_2-2y)^2 + (x_1-x_2)^2,$$

integrating with respect to y yields a factor $2^{-n/2}$. We are left with

$$(Wu, Wu)_{L^2(\mathbb{R}^{2n})} = \lim_{\epsilon\to 0_+} \int u(x_1)\, \overline{u}(x_2) e^{-\pi(x_1-x_2)^2/2} \epsilon^{-n} e^{-\pi\epsilon^{-2}(x_1-x_2)^2} dx_1 dx_2. \tag{4.2.8}$$

Changing the variables, the integral is

$$\lim_{\epsilon\to 0_+} \int u(s+\epsilon t/2)\, \overline{u}(s-\epsilon t/2) e^{-\pi\epsilon^2 t^2/2} e^{-\pi t^2} dt ds = \|u\|^2_{L^2(\mathbb{R}^n)}$$

by Lebesgue's dominated convergence theorem: the triangle inequality and the estimate $|u(x)| \le C(1+|x|)^{-n-1}$ imply, with $v = u/C$,

$$\begin{aligned} |v(s+\epsilon t/2)\, \overline{v}(s-\epsilon t/2)| &\le (1+|s+\epsilon t/2|)^{-n-1}(1+|s+\epsilon t/2|)^{-n-1} \\ &\le (1+|s+\epsilon t/2|+|s-\epsilon t/2|)^{-n-1} \\ &\le (1+2|s|)^{-n-1}. \end{aligned}$$

Eventually, this proves that

$$\|Wu\|^2_{L^2(\mathbb{R}^{2n})} = \|u\|^2_{L^2(\mathbb{R}^n)} \tag{4.2.9}$$

i.e.

$$W : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^{2n}) \quad \text{with} \quad W^*W = \mathrm{id}_{L^2(\mathbb{R}^n)}. \tag{4.2.10}$$

Noticing first that $\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$ belongs to $L^2(\mathbb{R}^n)$ (with a norm smaller than $\|Wu\|_{L^1(\mathbb{R}^{2n})}$) and applying Fubini's theorem, we get from the polarization of (4.2.9) for $u, v \in \mathcal{S}(\mathbb{R}^n)$,

$$\begin{aligned}(u,v)_{L^2(\mathbb{R}^n)} &= (Wu, Wv)_{L^2(\mathbb{R}^{2n})} \\ &= \iint Wu(y,\eta)(\varphi_{y,\eta}, v)_{L^2(\mathbb{R}^n)}dyd\eta \\ &= (\iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta, v)_{L^2(\mathbb{R}^n)},\end{aligned}$$

yielding the result of the lemma $u = \iint Wu(y,\eta)\varphi_{y,\eta}dyd\eta$. □

4.2.2 Poisson's formula

The following lemma is in fact the Poisson summation formula for Gaussian functions in one dimension.

Lemma 4.2.2. *For all complex numbers z, the following series are absolutely converging and*

$$\sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}} e^{-\pi m^2} e^{2i\pi m z}. \tag{4.2.11}$$

Proof. We set $\omega(z) = \sum_{m\in\mathbb{Z}} e^{-\pi(z+m)^2}$. The function ω is entire and 1-periodic since for all $m \in \mathbb{Z}$, $z \mapsto e^{-\pi(z+m)^2}$ is entire and for $R > 0$

$$\sup_{|z|\le R} |e^{-\pi(z+m)^2}| \le \sup_{|z|\le R} |e^{-\pi z^2}| e^{-\pi m^2} e^{2\pi|m|R} \in l^1(\mathbb{Z}).$$

Consequently, for $z \in \mathbb{R}$, we obtain, expanding ω in Fourier series[5],

$$\omega(z) = \sum_{k\in\mathbb{Z}} e^{2i\pi kz} \int_0^1 \omega(x) e^{-2i\pi kx} dx.$$

[5] Note that we use this expansion only for a C^∞ 1-periodic function. The proof is simple and requires only to compute $1 + 2\,\mathrm{Re}\sum_{1\le k\le N} e^{2i\pi kx} = \frac{\sin\pi(2N+1)x}{\sin\pi x}$. Then one has to show that for a smooth 1-periodic function ω such that $\omega(0) = 0$,

$$\lim_{\lambda\to+\infty} \int_0^1 \frac{\sin\lambda x}{\sin\pi x}\omega(x)dx = 0,$$

which is obvious since for a smooth ν (here we take $\nu(x) = \omega(x)/\sin\pi x$), $|\int_0^1 \nu(x)\sin\lambda x dx| = O(\lambda^{-1})$ by integration by parts.

We also check, using Fubini's theorem on $L^1(0,1)\times l^1(\mathbb{Z})$

$$\begin{aligned}\int_0^1 \omega(x)e^{-2i\pi kx}dx &= \sum_{m\in\mathbb{Z}}\int_0^1 e^{-\pi(x+m)^2}e^{-2i\pi kx}dx\\ &= \sum_{m\in\mathbb{Z}}\int_m^{m+1} e^{-\pi t^2}e^{-2i\pi kt}dt\\ &= \int_{\mathbb{R}} e^{-\pi t^2}e^{-2i\pi kt} = e^{-\pi k^2}.\end{aligned}$$

So the lemma is proven for real z and since both sides are entire functions, we conclude by analytic continuation. □

It is now straightforward to get the n-th dimensional version of the previous lemma: for all $z\in\mathbb{C}^n$, using the notation (4.2.2), we have

$$\sum_{m\in\mathbb{Z}^n} e^{-\pi(z+m)^2} = \sum_{m\in\mathbb{Z}^n} e^{-\pi m^2}e^{2i\pi m\cdot z}. \tag{4.2.12}$$

Theorem 4.2.3 (The Poisson summation formula). *Let n be a positive integer and u be a function in $\mathcal{S}(\mathbb{R}^n)$. Then we have*

$$\sum_{k\in\mathbb{Z}^n} u(k) = \sum_{k\in\mathbb{Z}^n} \hat{u}(k), \tag{4.2.13}$$

where $\hat{u}$ stands for the Fourier transform of u. In other words the tempered distribution $D_0=\sum_{k\in\mathbb{Z}^n}\delta_k$ is such that $\widehat{D_0}=D_0$.

Proof. We write, according to (4.2.6) and to Fubini's theorem

$$\begin{aligned}\textstyle\sum_{k\in\mathbb{Z}^n} u(k) &= \sum_{k\in\mathbb{Z}^n}\iint Wu(y,\eta)\varphi_{y,\eta}(k)dyd\eta\\ &= \iint Wu(y,\eta)\sum_{k\in\mathbb{Z}^n}\varphi_{y,\eta}(k)dyd\eta.\end{aligned}$$

Now, (4.2.12), (4.2.1) give $\sum_{k\in\mathbb{Z}^n}\varphi_{y,\eta}(k)=\sum_{k\in\mathbb{Z}^n}\widehat{\varphi}_{y,\eta}(k)$, so that (4.2.6) and Fubini's theorem imply the result. □

4.3 Fourier transformation and convolution

4.3.1 Fourier transformation on $\mathscr{E}'(\mathbb{R}^n)$

Theorem 4.3.1. *Let $u\in\mathscr{E}'(\mathbb{R}^n)$. Then $\hat{u}$ is an entire function on $\mathbb{C}^n$.*

Proof. We have for $\varphi\in\mathscr{D}(\mathbb{R}^n)$, according to the definition (3.4.14),

$$\begin{aligned}\langle\hat{u},\varphi\rangle = \langle u,\hat{\varphi}\rangle = \langle u(x),\int e^{-2i\pi x\cdot\xi}\varphi(\xi)d\xi\rangle &= \langle u(x)\otimes\varphi(\xi), e^{-2i\pi x\cdot\xi}\rangle_{\mathscr{E}'(\mathbb{R}^{2n}),\mathscr{E}(\mathbb{R}^{2n})}\\ &= \langle\varphi(\xi),\underbrace{\langle u(x),e^{-2i\pi x\cdot\xi}\rangle}_{\tilde{u}(\xi)}\rangle,\end{aligned}$$

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