Master 2 – Mathématiques Fondamentales – 2012/2013
Université Pierre et Marie Curie

Cours « Analyse semi-classique »

Exercise 2.1 (Another proof of the Morse lemma)

Let $\varphi \in C^\infty(\Omega, \mathbb{R})$ with $x_0 \in \Omega$ a non-degenerate critical point : $\varphi'(x_0) = 0$, $\det \varphi''(0) \neq 0$. Suppose for simplicity that $x_0 = 0$ and $\varphi(x_0) = 0$.

a) Show that for x in some neighborhood of 0, $\varphi(x) = \frac{1}{2} \sum_{1 \leq i,j \leq n} q_{ij}(x)\, x_i\, x_j$, with $q_{ij} = q_{ji} \in C^\infty$ and $(q_{ij}(0))$ non-degenerate.

b) Show that after a linear change of variables, it is possible to suppose $q_{11}(0) \neq 0$. Then show that

$$\varphi(x) = \frac{1}{2}\Big(\varepsilon_1(\ell_1(x))^2 + \sum_{2 \leq i,j \leq n} \tilde{q}_{ij}(x)\, x_i\, x_j\Big)$$

with $\varepsilon_1 = \pm 1$, $\ell_1(x) = \sum_{1 \leq j \leq n} a_{1,j}(x)\, x_j$ and $a_{1,j}$, $\tilde{q}_{ij} \in C^\infty$.

c) Finally show that

$$\varphi(x) = \frac{1}{2} \sum_{i=1}^{n} \varepsilon_i \Big(\ell_i(x)\Big)^2$$

with $\ell_i(x) = \sum a_{ij}(x)\, x_j$, $\quad \varepsilon_i = \pm 1$, $\quad a_{ij} \in C^\infty$.

d) Show that $(x_1, \ldots, x_n) \mapsto (\ell_1(x), \ldots, \ell_n(x))$ is a diffeomorphism $U \to V$, U and V neighborhoods of 0.

e) Derive the Morse lemma.

Exercise 2.2

a) Do the following functions converge in $\mathcal{D}'(\mathbb{R})$ as $\lambda \to +\infty$?

$$\begin{aligned} u_\lambda(x) &= \lambda^N e^{-i\lambda x} \\ v_\lambda(x) &= \lambda^{\frac{1}{2}} e^{-i\lambda \frac{x^2}{2}} \\ w_\lambda(x) &= \lambda^{\frac{1}{2}} e^{+i\lambda \frac{x^2}{2}} \end{aligned}$$

b) Let $f \in C^\infty(\mathbb{R}, \mathbb{R})$ with $f'(x) \neq 0$, $\forall\, x \in \mathbb{R}$. Then same question for

$$\begin{aligned} u_\lambda(x) &= \lambda^N e^{-i\lambda f(x)} \\ v_\lambda(x) &= \lambda^{\frac{1}{2}} e^{-i\lambda \frac{(f(x))^2}{2}} \end{aligned}$$

Exercise 2.3

Let $\chi \in \mathcal{S}(\mathbb{R})$, $\chi(0) = 1$.

a) Show the existence of a limit as $\varepsilon \to 0$ of

$$\int e^{-i\lambda(y+\frac{y^3}{3})} \chi(\varepsilon y)\, dy\ , \quad \lambda \in \mathbb{R}\backslash\{0\}.$$

Show that the limit $I(\lambda)$ is a C^∞ function of λ.

Hint : choose a suitable differential operator L such that $L\Big(e^{-i\lambda(y+\frac{y^3}{3})}\Big) = e^{-i\lambda(y+\frac{y^3}{3})}$.

b) Show that for every N, $|I(\lambda)| \leq C_N |\lambda|^{-N}$ if $|\lambda| > 1$.

c) Show that $J(\lambda) = \lim\limits_{\varepsilon\to 0} \int e^{-i\lambda(y-\frac{y^3}{3})} \chi(\varepsilon y)\, dy$ does exist for $\lambda \in \mathbb{R}\backslash\{0\}$.

d) Find the asymptotics of $J(\lambda)$ as $\lambda \to \pm\infty$.

Exercise 2.4

The following problem appears in the method of steepest descent. Show that for $u \in C_0^\infty(\mathbb{R}^n)$, $\lambda \geq 1$:

$$\int e^{-\lambda x^2/2} u(x)\, dx = \sum_{k=0}^{N-1} \frac{(2\pi)^{\frac{n}{2}}}{k!\, \lambda^{k+\frac{n}{2}}} \left((\frac{1}{2}\Delta)^k u \right)(0) + S_N(u, \lambda),$$

where $|S_N(u, \lambda)| \leq C_{n,N}\, \lambda^{-N-\frac{n}{2}} \sum\limits_{|\alpha|=2N} \sup |\partial^\alpha u(x)|$.

Exercise 2.5 (Stirling's formula)

Let $F(\lambda) = \Gamma(\lambda + 1) = \int_0^{+\infty} e^{-t}\, t^\lambda\, dt$, $\lambda \geq 1$ $(F(n) = n!,\ n \in \mathbb{N})$.
One wants to find the asymptotics of $F(\lambda)$ as $\lambda \to +\infty$.

a) Rewrite the integral by means of the change of variable $t = \lambda(1 + s)$.

b) Use Exercise 2.4 and show

$$F(\lambda) = \left(\frac{\lambda}{e}\right)^\lambda \sqrt{2\pi\lambda}\, (1 + a_1\lambda^{-1} + a_2\lambda^{-2} + \ldots).$$

For $\lambda = n \in \mathbb{N}$ deduce Stirling's formula.

c) Calculate a_1 and a_2.