THE COMPLEX WKB METHOD FOR DIFFERENCE EQUATIONS AND AIRY FUNCTIONS

ALEXANDER FEDOTOV AND FRÉDÉRIC KLOPP

Abstract. We consider the difference Schrödinger equation $\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0$ where z is a complex variable, $h>0$ is a parameter, and v is an analytic function. As $h\to 0$ analytic solutions to this equation have a standard quasiclassical behavior near the points where $v(z)\neq\pm 2$. We study analytic solutions near the points z_0 satisfying $v(z_0)=\pm 2$ and $v'(z_0)\neq 0$. For the finite difference equation, these points are the natural analogues of the simple turning points defined for the differential equation $-\psi''(z)+v(z)\psi(z)=0$. In an h-independent neighborhood of such a point, we derive uniform asymptotic expansions for analytic solutions to the difference equation.

1. Introduction, Preliminaries, and Main Results

1.1. **The problem.** We study analytic solutions to the difference Schrödinger equation

$$\psi(z+h)+\psi(z-h)+v(z)\psi(z)=0 \tag{1.1}$$

where z is a complex variable, h is a positive parameter and v is an analytic function. We describe their asymptotics as $h\to 0$.
Note that the parameter h is a standard quasiclassical parameter. Indeed, formally, $\psi(z+h)=\sum_{l=0}^{\infty}\frac{h^l}{l!}\frac{d^l\psi}{dz^l}(z)=e^{h\frac{d}{dz}}\psi(z)$, and h can be regarded as a small parameter in front of the derivative.
One encounters difference equations in the complex plane in many fields of mathematics and physics. For example, they arise when studying an electron in a crystal submitted to a constant magnetic field (e.g., [17]), wave scattering by wedges (e.g., [1]) and one-dimensional quasi-periodic differential Schrödinger equations with two frequencies (e.g., [10]). The quasiclassical case corresponds respectively to the cases of a small magnetic field, of a thin wedge and the case where one frequency is small with respect to another.
The quasiclassical theory of difference equations in the complex plane can also be useful to study orthogonal polynomials, see section 1.7.
The quasiclassical asymptotics of analytic solutions to ordinary differential equations in the complex plane are described by the well-known complex WKB method (see, e.g., [21, 7]). The complex WKB method for difference equations was developed in [3, 13, 15].
The present paper is devoted to uniform asymptotic formulas describing analytic solutions to (1.1) in h-independent complex neighborhoods of simple turning points (see sections 1.2.1 and 1.3.3). The results of this paper were partially announced in [12].

The idea to study the asymptotics of solutions to a difference equation in a complex neighborhood of a turning point appears to be very natural. One can say

Key words and phrases. Difference Schrödinger equation, complex WKB method, Airy functions.

The work was supported by CNRS, France, and the Russian Foundation for Basic Research under the grant No 17-51-150008.

that this idea and the techniques developed to get the asymptotics are the main analytic innovations of the paper.

In the next sections, we first recall some basic definitions and statements of the complex WKB method for difference equations. Next, we introduce a few objects needed to formulate our results that we then state.

We assume that v is analytic on a disk $U \subset \mathbb{C}$.

Below, a neighborhood is a δ-neighborhood, in particular, a neighborhood of a point is an open disk with the center at this point.

1.2. **A very short introduction to the complex WKB method.** Here, following [13, 15], we briefly describe basic definitions and results of the complex WKB method for difference equations.

1.2.1. *The complex momentum.* The main analytic object of the complex WKB method is the *complex momentum* p. For (1.1) it is defined by the formula

$$2\cos p + v(z) = 0. \tag{1.2}$$

It is a multivalued analytic function on U. At its branching points $\cos p(z) \in \{\pm 1\}$, thus, $v(z) \in \{\pm 2\}$.

In analogy with the glossary of the complex WKB method for differential equations, the points where $v(z) \in \{\pm 2\}$ are called *turning points.*

A set $D \subset U$ is *regular* if $v(z) \neq \pm 2$ in D.

1.2.2. *The main theorem of the complex WKB method.* As in the case of differential equations, one of the main geometric notions of the complex WKB method is the *canonical domain.* In this paper we do not use it directly, and the reader needs to keep in mind only that the canonical domains are regular, simply connected domains independent of h, and that one has the following two theorems.

Theorem 1.1. *Any regular point belongs to a canonical domain.*

The proof of this statement repeats the proof of Lemma 5.2 from [11].

Theorem 1.2. *Let $K \subset U$ be a canonical domain, let $z_0 \in K$, and let p be a branch of the complex momentum analytic in K. Then there exist solutions $\psi_\pm$ to* (1.1) *analytic in K and such that as $h \to 0$*

$$\psi_\pm(z) = \frac{1}{\sqrt{\sin(p(z))}}\, e^{\pm \frac{i}{h}\int_{z_0}^{z} p(z)\,dz + o(1)}. \quad z \in K. \tag{1.3}$$

This asymptotic representation is locally uniform in K.

In [13] this theorem was proved for v analytic in bounded domains, and in [15] it was proved in the case where v is a trigonometric polynomial.

Note that, by definition, at a turning point of p, one has $\sin p(z) = 0$. Thus, representation (1.3) cannot be valid in a neighborhood of a turning point.

Remark 1.1. For the differential equation $-h^2\psi''(z) + v(z)\psi(z) = 0$, formula (1.3) has to be replaced with (see, e.g. [21]) $\psi_\pm(z) = \frac{1}{\sqrt{p(z)}}\, e^{\pm \frac{i}{h}\int_{z_0}^{z} p(z)\,dz + o(1)}$, where the complex momentum is defined by the relation $p^2 + v(z) = 0$, i.e., as for (1.1), by the symbol of the equation.

1.3. **The complex momentum and the conformal mapping ζ.** Here, we discuss properties of the complex momentum that we use throughout this paper. These properties easily follow from the definition of p.

1.3.1. *Analytic branches of the complex momentum.* Let p_0 be a branch of the complex momentum analytic in a regular simply connected domain D. Then, an analytic function $\tilde{p}: \ D \to \mathbb{C}$ is a branch of the complex momentum if and only if there exists $s \in \{\pm 1\}$ and $n \in \mathbb{Z}$ such that

$$\tilde{p}(z) = sp_0(z) + 2\pi n, \quad \forall z \in D. \tag{1.4}$$

1.3.2. *The values of p at turning points.* We note that $z_0 \in U$ is a turning point if and only if $p(z_0) = 0 \bmod \pi$. A simple transformation of the equation shows that it suffices to consider the case where $p(z_0) = 0 \bmod 2\pi$. Indeed, for ψ, a solution to (1.1), we set $\phi(z) = e^{i\pi z/h}\psi(z)$. Then, ϕ satisfies equation

$$\phi(z+h) + \phi(z-h) - v(z)\phi(z) = 0. \tag{1.5}$$

The complex momenta for equations (1.1) and (1.5) differ by $\pi \bmod 2\pi$, and z_0 is a turning point for (1.1) if and only if it is a turning point for (1.5).

1.3.3. *The complex momentum near a turning point.* Let z_0 be a turning point. If $v'(z_0) \neq 0$, we call the turning point z_0 *simple.* In this case, the complex momentum is analytic in $\tau = \sqrt{z - z_0}$ in a neighborhood of 0, and as $\tau \to 0$ any of its analytic branches admits a representation of the form

$$p(z) = p(z_0) + k_1\tau + O(\tau^2), \quad \tau = \sqrt{z - z_0}, \qquad k_1 \neq 0. \tag{1.6}$$

1.3.4. *Our assumptions.* From now on, we assume that

- in the disk U, there exists a single turning point, namely its center z_0, and it is simple;
- $p(z_0) = 0 \bmod 2\pi$.

1.3.5. *The function ζ.* The function ζ we describe here plays an important role in the asymptotic analysis of (1.1) near turning points.
We cut U from z_0 to a point of its boundary along a simple curve and denote the thus obtained domain by U'. In U', we fix an analytic branch p of the complex momentum.

We have $p(z_0) = 2\pi n, \ n \in \mathbb{Z}$. Clearly, $p(z) - 2\pi n$ also is a branch of the complex momentum analytic in U'. So, we can and do assume that $p(z_0) = 0$.

Let us fix in U' an analytic branch ζ of the function

$$z \mapsto \left(\frac{3}{2i}\int_{z_0}^{z} p(z)\,dz\right)^{\frac{2}{3}}. \tag{1.7}$$

This branch is actually analytic in U. One has $\zeta(z_0) = 0$, and $\zeta'(z_0) \neq 0$.

Remark 1.2. There are three different analytic branches of function (1.7): they equal $e^{4\pi ij/3}\zeta$, $j \in \mathbb{Z}_3 = \mathbb{Z}/3\mathbb{Z}$. The set of these branches is independent of the curve along which we cut U to get U' and on the precise choice of the branch p.

We note that the definition of ζ implies that it satisfies one of the two equations

$$\sqrt{\zeta(z)}\zeta'(z) = \pm ip(z), \qquad z \in U. \tag{1.8}$$

Possibly reducing U somewhat, we can and do assume that

- ζ is a bi-analytic bijection of U onto its image.

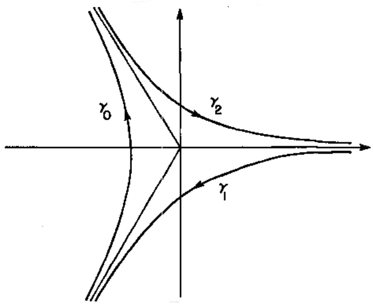

FIGURE 1. Integration paths

1.4. **Basic facts on Airy functions.** The equation

$$w''(\zeta) = \zeta w(\zeta), \qquad \zeta \in \mathbb{C}, \tag{1.9}$$

is the Airy equation. Its solutions are *Airy functions.*
Let $(\gamma_j)_{j\in\mathbb{Z}_3}$ be the curves shown in Fig. 1 borrowed from [22]; γ_0 is asymptotic to the half-lines $e^{\pm 2i\pi/3}\mathbb{R}_+$, $\mathbb{R}_+ = [0, +\infty) \subset \mathbb{R}$; for $j \in \mathbb{Z}_3$, rotating γ_0 around 0 by $2j\pi/3$, one obtains γ_j. The functions defined by the formulas

$$w_j(\zeta) = \int_{\gamma_j} e^{-\left(\frac{s^3}{3} - \zeta s\right)}\, ds, \quad j \in \mathbb{Z}_3, \qquad \zeta \in \mathbb{C}, \tag{1.10}$$

are three Airy functions related to the standard Airy function Ai by the formulas (see, e.g., [22])

$$w_j(\zeta) = 2\pi i \omega^j \mathrm{Ai}(\omega^j \zeta), \quad \omega = e^{2\pi i/3}, \qquad \zeta \in \mathbb{C}. \tag{1.11}$$

Assume that $|\arg z| < 2\pi/3$. As $|z| \to \infty$ one has

$$\mathrm{Ai}\,(z) = \frac{\exp\left(-\frac{2}{3}z^{3/2} + o(1)\right)}{2\sqrt{\pi}\; z^{1/4}}, \ \mathrm{Ai}\,(-z) = \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4} + o(1)\right)}{\sqrt{\pi}\; z^{1/4}}\,(1+o(1)) \tag{1.12}$$

where we use the analytic branches of $z \to z^{3/2}$ and $z \to z^{1/4}$ that are positive for $z > 0$ (see [18], pp. 116, 118 and 392).

1.5. **Notations.** The letter C denotes various positive constants independent of z and h.
For two functions f and g defined on a domain $D \subset \mathbb{C}$, we write that $g(z) = O(f(z))$ in D if $|g(z)| \le C|f(z)|$ for all $z \in D$.

1.6. **Solutions in a complex neighborhood of a branch point.**

1.6.1. *Asymptotic solutions.* First, let us describe asymptotic solutions to (1.1). Therefore, we introduce several objects. For a function f defined on U, we set

$$[H(f)](z) := f(z+h) + f(z-h) + v(z)f(z) \quad \text{if} \quad \{z-h, z, z+h\} \subset U. \tag{1.13}$$

We let

$$g(z) := \frac{sinh\left(\sqrt{\zeta(z)}\zeta'(z)\right)}{\sqrt{\zeta(z)}}, \qquad z \in U, \tag{1.14}$$

where the determination of the square roots in the denominator and the numerator are the same (the definition of g is independent of its choice). The function g is analytic in U. Possibly reducing U somewhat, we can and do assume that

- g does not vanish in U.

We further define

$$A_0(z) := \frac{1}{\sqrt{g(z)}}. \tag{1.15}$$

The function A_0 is analytic in U.
One has

Theorem 1.3. *There exist functions $(A_l)_{l\in\mathbb{N}\cup\{0\}}$ and $(B_l)_{l\in\mathbb{N}}$, (A_0 being defined by (1.15)), all analytic on U and such that, for any $L\in\mathbb{N}\cup\{0\}$ the following holds. Let w be one of the Airy functions w_j, $j\in\mathbb{Z}_3$. If we define*

$$w_h(z) = w\left(\zeta(z)/h^{\frac{2}{3}}\right), \qquad w_h'(z) = w'\left(\zeta(z)/h^{\frac{2}{3}}\right). \tag{1.16}$$

and

$$W(z) = h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{L} h^l A_l(z) + h^{\frac{2}{3}} w_h'(z) \sum_{l=1}^{L} h^l B_l(z), \tag{1.17}$$

then one has

$$H(W) = O\left(h^{L+2+\frac{1}{3}} w_h\right) + O\left(h^{L+2+\frac{2}{3}} w_h'\right). \tag{1.18}$$

We call the formal expression

$$h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{\infty} h^l A_l(z) + h^{\frac{2}{3}} w_h'(z) \sum_{l=1}^{\infty} h^l B_l(z) \tag{1.19}$$

an *asymptotic solution* to (1.1).

Theorem 1.3 is proved in section 3, where we describe, inter alia, a way to compute the coefficients $(A_l)_l$ and $(B_l)_l$.

Let us comment on the results of Theorem 1.3. First, we note that, for the differential equation $-\psi''(z)+v(z)\psi(z)=0$, in a neighborhood of a simple turning point (a point where $v(z)=0$ and $v'(z)\neq 0$), there are asymptotic solutions of the form (1.19) (with different coefficients $(A_l)_l$, $(B_l)_l$ and function ζ).

To justify the Ansatz (1.19) for the difference equation, one has to derive asymptotic formulas of the form

$$w_h(z\pm h) = f(z)\, w_h(z) \pm h^{\frac{1}{3}}\, g(z)\, w_h'(z) + \dots, \tag{1.20}$$

where $f(z) = \cosh(\sqrt{\zeta(z)}\zeta'(z))$ and the dots denote smaller order terms. If one tries to prove this formula using Taylor expansions for the left hand side, one has to handle an infinite number of infinite subsequences of terms of the same order. So, an effective resummation of these sequences is required. As we see in this paper, to derive formulas analogous to (1.20), instead of resummation of Taylor series, it is very natural to use tools from complex analysis.

Formula (1.20) imply that

$$H\left(w_h\right)(z) = \left(2\cosh(\sqrt{\zeta(z)}\zeta'(z)) + v(z)\right) w_h(z) + \dots. \tag{1.21}$$

In view of (1.2) and (1.8), the leading term in the right-hand side of (1.21) is zero.

Finally, we note that if $h^{-\frac{2}{3}}|\zeta(z)|$ is large, then $w_h(z)$ and w_h' in (1.17) can be replaced by their asymptotic representations. As a result, in view of (1.12), the leading term in (1.17) turns into a linear combination of the leading terms from (1.3).

1.6.2. *Solutions with standard asymptotic behavior.* Our main result is

Theorem 1.4. *Let $L \in \mathbb{N}$, and let W be one of the functions constructed in Theorem 1.3 for the order L. Then there exists an h-independent neighborhood $\overset{\circ}{U} \subset U$ of z_0 such that, for sufficiently small h, there exists ψ, a solution to equation* (1.1) *that is analytic in $\overset{\circ}{U}$ and admits there the asymptotic representation*

$$\psi(z) = W(z) + O(w_h \, h^{L+1+\frac{1}{3}}) + O(w_h' \, h^{L+1+\frac{2}{3}}) \tag{1.22}$$

where w_h and w_h' are defined in (1.16).

Theorem 1.4 is proved in sections 5 and 6.

Let us briefly explain the idea of the proof of Theorem 1.4. First, in section 5, using the approximate solutions constructed in Theorem 1.3, we construct a parametrix R, i.e., an operator such that, for suitable functions f, one has $HRf = f + Df$, where H is defined in (1.13) and D is a small operator. The operator D is a singular integral operator. We estimate its norm using natural geometric objects of the complex WKB method. This allows us to prove Theorem 1.4 on some special subdomains of U. In section 6, we study the thus constructed solutions on larger domains and complete the proof of Theorem 1.4.

To complete this short description, let us underline that, as equation (1.1) is non-local in z, the ideas of analysis of (1.1) are different from those used to study the analogous differential equation.

1.7. **Related results.** The WKB asymptotics of solutions of difference equations on $\mathbb{Z}$ with "slowly varying" coefficients have been studied since the end of 1960-s. In [20] and [19], the authors essentially studied equations of the form

$$Y_{k+1} = M(hk)Y_k, \quad k \in \mathbb{Z}, \tag{1.23}$$

with a small positive h and an $(n \times n)$-matrix valued function M defined on $\mathbb{R}$. We note that if

$$Y(x+h) = M(x)Y(x), \quad x \in \mathbb{R}, \tag{1.24}$$

then, the sequence $(Y_k)_{k\in\mathbb{Z}} = (Y(kh))_{k\in\mathbb{Z}}$ satisfies (1.23). We note also that equation (1.1) restricted to $\mathbb{R}$ is equivalent to (1.24) with $M(x) = \begin{pmatrix} -v(x) & -1 \\ 1 & 0 \end{pmatrix}$, and that a turning point for equation (1.1) is a point x where the eigenvalues of the matrix $M(x)$ coincide.

The short note [20] is essentially devoted to the case where all the eigenvalues of the matrix M in (1.23) are distinct. In [19] the author constructed asymptotic solutions to (1.23) in a small (depending on h) neighborhood of a point where two eigenvalues of $M(x)$ become equal.

In [5] the authors considered difference equations of the form

$$\sum_{j=I}^{J} a_j(hk, h)\, y_{k+j} = 0, \quad k \in \mathbb{Z}.$$

We note that this class includes the difference Schrödinger equations

$$y_{k+1} + y_{k-1} + v(hk)y_k = 0, \quad k \in \mathbb{Z}.$$

The authors described the asymptotics of solutions to this equations for hk being in a small (as $h \to 0$) neighborhood of a point where $v(x) \in \{\pm 1\}$.

We mention also three (series of) papers motivated by problems originating in the theory of orthogonal polynomials.

First, there is a series of papers by J.S. Geronimo and co-authors, see, e.g. [16] and references therein, devoted to uniform asymptotic formulas for solutions to the

equation $a_{k+1}\psi_{k+1}+b_k\psi_k+a_k\psi_{k-1}=\lambda\psi_k,\ \ k\in\mathbb{Z}$, where λ is the spectral parameter, and the coefficients a_k are positive and b_k are real.
Also we mention papers by R.Wong and coauthors, see e.g., [23], who also studied solutions to three terms recurrence relations with real coefficients for large values of the integer variable.
Finally, we mention paper [6] where the authors studied WKB asymptotics of solutions to a difference equation using the Maslov canonical operator.
There are more papers devoted to the subject. The reader can find more references in the papers that we mentioned above.

To the best of our knowledge, the present paper is the first where one rigorously obtains uniform asymptotics of analytic solutions to a difference equation on $\mathbb{C}$ in an h-independent neighborhood of a turning point.

2. The space of solutions to equation (1.1)

The observations that we discuss now are well-known in the theory of difference equations and are easily proved.
Let $c\in\mathbb{R}$ and $I=\{z\in U:\operatorname{Im}z=c\}$. We assume that the length of the segment I is sufficiently large (with respect to h) and discuss the set S of solutions to equation (1.1) on I.
Let $\{f,g\}\subset S$. The expression

$$(f(z),g(z))=f(z+h)g(z)-f(z)g(z+h),\quad z,\,z+h\in I,\tag{2.1}$$

is called *the Wronskian* of f and g. It is h-periodic in z.
If the Wronskian of two solutions does not vanish, they form a basis in S, i.e, $\psi\in S$ if and only if

$$\psi(z)=a(z)f(z)+b(z)g(z),\quad z,\,z+h\in I,\tag{2.2}$$

where a and b are h-periodic complex coefficients. One has

$$a(z)=\frac{(\psi(z),\,g(z))}{(f(z),\,g(z))},\quad b(z)=\frac{(f(z),\,\psi\,(z))}{(f(z),\,g(z))}.\tag{2.3}$$

3. Asymptotic solutions: the proof of Theorem 1.3

3.1. **The proof of Theorem 1.3 up to two propositions.** First, we formulate two statements needed to construct asymptotic solutions to (1.1). Below we use the notations introduced in (1.16).

Proposition 3.1. *Let A be analytic in U. Let $N\in\mathbb{N}$. If $\{z-h,z,z+h\}\subset U$,*

$$\begin{aligned}H\left(A\,h^{\frac13}w_h\right)=h^{\frac13}w_h\,\sum_{l=2}^{N}h^l a_l+O\left(h^{N+1+\frac13}w_h\right)\\+h^{\frac23}w_h'\,\sum_{l=1}^{N}h^l b_l+O\left(h^{N+1+\frac23}w_h'\right)\end{aligned}\tag{3.1}$$

as $h\to0$. All the coefficients $(a_l)_{l\geq2}$ and $(b_l)_{l\geq1}$ are analytic in U, independent of the choice of w in (1.16)*, and*

$$b_1=b_1[A]=Ag\frac{d}{dz}\log\left(A^2g\right).\tag{3.2}$$

Proposition 3.2. *Let B be analytic in U, and let $N \in \mathbb{N}$. If $\{z-h, z, z+h\} \subset U$,*

$$H\left(B\, h^{\frac{2}{3}} w_h'\right) = h^{\frac{1}{3}} w_h \sum_{l=1}^{N} h^l c_l + O\left(h^{N+1+\frac{1}{3}} w_h\right) + h^{\frac{2}{3}} w_h' \sum_{l=2}^{N} h^l d_l + O\left(h^{N+1+\frac{2}{3}} w_h'\right) \tag{3.3}$$

as $h \to 0$. All the coefficients $(c_l)_{l\geq 1}$ and $(d_l)_{l\geq 2}$ are analytic in U, independent of the choice of w in (1.16)*, and*

$$c_1 = c_1[B] = \zeta B g \frac{d}{dz} \log(\zeta B^2 g). \tag{3.4}$$

Before proving Propositions 3.1 and 3.2, we use them to prove Theorem 1.3. The proof is done by induction on the order L. For $L = 0$, one has $W = A_0 h^{\frac{1}{3}} w_h$. In view of (1.15) and (3.2), the coefficient b_1 corresponding to $A = A_0$ is equal to 0. So, the statement of Theorem 1.3 for $L = 0$ immediately follows from (3.1) with $N = 1$.

Now, we assume that Theorem 1.3 is proved up to the order $L = L_0 - 1$, $L_0 \in \mathbb{N}$. Let us prove it for $L = L_0$. We set

$$W(z) = h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{L_0} h^l A_l(z) + h^{\frac{2}{3}} w_h'(z) \sum_{l=1}^{L_0} h^l B_l(z), \tag{3.5}$$

where $(A_l)_{l<L_0}$ and $(B_l)_{l<L_0}$ are chosen as in the case $L = L_0 - 1$, A_{L_0} and B_{L_0} still having to be chosen. By the induction hypothesis

$$H\left(h^{\frac{1}{3}} w_h \sum_{l=0}^{L_0-1} h^l A_l + h^{\frac{2}{3}} w_h' \sum_{l=1}^{L_0-1} h^l B_l\right) = O\left(h^{L_0+1+\frac{1}{3}} w_h\right) + O\left(h^{L_0+1+\frac{2}{3}} w_h'\right). \tag{3.6}$$

In view of Propositions 3.2 and 3.1 this implies that

$$H\left(h^{\frac{1}{3}} w_h \sum_{l=0}^{L_0-1} h^l A_l + h^{\frac{2}{3}} w_h' \sum_{l=1}^{L_0-1} h^l B_l\right) = a h^{L_0+1+\frac{1}{3}} w_h + b h^{L_0+1+\frac{2}{3}} w_h' + O\left(h^{L_0+2+\frac{1}{3}} w_h\right) + O\left(h^{L_0+2+\frac{2}{3}} w_h'\right),$$

where a and b are analytic functions in U. On the other hand, using (3.1) and (3.3) with $N = 1$, we get

$$H(A_{L_0} h^{L_0+\frac{1}{3}} w_h) = h^{L_0+1+\frac{2}{3}} w_h'\, b_1[A_{L_0}] + O(h^{L_0+2+\frac{1}{3}} w_h) + O(h^{L_0+2+\frac{2}{3}} w_h'),$$
$$H(B_{L_0} h^{L_0+\frac{2}{3}} w_h') = h^{L_0+1+\frac{1}{3}} w_h\, c_1[B_{L_0}] + O(h^{L_0+2+\frac{1}{3}} w_h) + O(h^{L_0+2+\frac{2}{3}} w_h').$$

Therefore,

$$HW = h^{L_0+1}\left(h^{\frac{1}{3}} w_h\,(a + c_1[B_{L_0}]) + h^{\frac{2}{3}} w_h'\,(b + b_1[A_{L_0}])\right) + O(h^{L_0+2+\frac{1}{3}} w_h) + O(h^{L_0+2+\frac{2}{3}} w_h').$$

So, to prove Theorem 1.3, it suffices to choose A_{L_0} and B_{L_0} so that

$$a + c_1[B_{L_0}] = 0, \quad \text{and} \quad b + b_1[A_{L_0}] = 0.$$

In view of (3.2) and (3.4), these relations are equivalent to the equations

$$\zeta B_{L_0} g \frac{d}{dz} \log\left(\zeta B_{L_0}^2 g\right) = -a, \quad \text{and} \quad A_{L_0} g \frac{d}{dz} \log\left(A_{L_0}^2 g\right) = -b. \tag{3.7}$$

One constructs solutions to these equations by the formulas

$$A_{L_0}(z) = -\frac{1}{2\sqrt{g(z)}}\int_{z_0}^{z}\frac{b\,dz}{\sqrt{g}}, \quad \text{and} \quad B_{L_0}(z) = -\frac{1}{2\sqrt{\zeta(z)g(z)}}\int_{z_0}^{z}\frac{a\,dz}{\sqrt{\zeta g}}. \tag{3.8}$$

As

- g and ζ are analytic in U,
- g does not vanish in U,
- ζ vanishes in U only at z_0 where it has a simple zero,

the coefficients A_{L_0} and B_{L_0} are analytic in U. This completes the proof of Theorem 1.3. ☐

Remark 3.1. The function B_{L_0} constructed by (3.8) is the only solution to the first equation in (3.7) that is analytic in U. The function A_{L_0} constructed by (3.8) is unique up to a solution to the homogeneous equation $A_{L_0} g \frac{d}{dz}\log\left(A_{L_0}^2 g\right) = 0$ that is proportional to A_0 given by (1.15).

3.2. **The proof of Proposition 3.1.** Consider $(w_j)_{j\in\mathbb{Z}_3}$ the three solutions to the Airy equation (1.9) defined by (1.10). Let w be w_j for some $j \in \mathbb{Z}_3$ and let γ be the corresponding integration path γ_j in (1.10).
Note that

$$h^{\frac{1}{3}} w(h^{-\frac{2}{3}}\zeta) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta\right)}\,dt, \qquad h^{\frac{2}{3}} w'(h^{-\frac{2}{3}}\zeta) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta\right)}\,t\,dt. \tag{3.9}$$

Below, we use the notations from (1.16). Let $K \subset U$ is a closed disk centered at z_0 and independent of h. Below, we assume that $z \in K$ and h is sufficiently small. The proof of the asymptotics of $H(Ah^{\frac{1}{3}}w)$ in K as $h \to 0$ is broken into several steps.
1. In view of (3.9), we get

$$H\left(Ah^{\frac{1}{3}}w\right) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} F_0(t,z,h)\,dt, \tag{3.10}$$

$$F_0(t,z,h) = A(z+h)e^{\frac{t}{h}(\zeta(z+h)-\zeta(z))} + A(z-h)e^{\frac{t}{h}(\zeta(z-h)-\zeta(z))} + v(z)A(z). \tag{3.11}$$

Note that $(t,z,h) \mapsto F_0(t,z,h)$ is analytic in $\mathbb{C}\times K\times V$, where V is a sufficiently small neighborhood of zero.
2. To get the asymptotics of the integral in (3.10), we apply the well-known method described in detail in section 4 of chapter VII of [22]. First, we represent $F_0(t,z,h)$ in the form

$$F_0(t,z,h) = a_0(z,h) + b_0(z,h)t + (t^2-\zeta(z))f_0(t,z,h) \tag{3.12}$$

with

$$a_0(z,h) = \frac{1}{2}\left(F_0(\sqrt{\zeta(z)},z,h) + F_0(-\sqrt{\zeta(z)},z,h)\right), \tag{3.13}$$

$$b_0(z,h) = \frac{1}{2\sqrt{\zeta(z)}}\left(F_0(\sqrt{\zeta(z)},z,h) - F_0(-\sqrt{\zeta(z)},z,h)\right), \tag{3.14}$$

where, in (3.13) and (3.14), we use one and the same branch of $\sqrt{\zeta(z)}$.
Both a_0 and b_0 are analytic in $(z,h)\in K\times V$ (we remove the removable singularities at $z=0$). With a_0 and b_0 so chosen, it is easily seen that the function f_0 is analytic in $(t,z,h)\in\mathbb{C}\times K\times V$.

3. Substituting (3.12) into (3.10) and integrating by parts, we get

$$\begin{aligned} H\left(Ah^{\frac{1}{3}}w_h\right) &= a_0 h^{\frac{1}{3}} w_h + b_0 h^{\frac{2}{3}} w_h' + \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)}(t^2 - \zeta(z)) f_0\, dt \\ &= a_0 h^{\frac{1}{3}} w_h + b_0 h^{\frac{2}{3}} w_h' + h\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_1(t,z,h)\, dt. \end{aligned} \tag{3.15}$$

where $F_1(t,z,h) = \frac{\partial f_0}{\partial t}(t,z,h)$.

4. Now, we transform the last integral in (3.15), the one containing F_1, in the same way as we transformed the integral with F_0 from (3.10).

For a fixed positive integer N, we repeat this procedure inductively $N+2$ times. Reasoning as above, one proves that

$$H\left(A\, h^{\frac{1}{3}} w_h\right) = h^{\frac{1}{3}} w_h \sum_{l=0}^{N+1} h^l a_l(z,h) + h^{\frac{2}{3}} w_h' \sum_{l=0}^{N+1} h^l b_l(z,h) + h^{N+2} I_{N+2}, \tag{3.16}$$

where, for $l \in \mathbb{N} \cup \{0\}$, we have defined

$$I_l = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} F_l(t,z,h)\, dt. \tag{3.17}$$

As when $l = 0$, the coefficients a_l and b_l are expressed in terms of F_l by

$$\begin{aligned} a_l(z,h) &= \frac{1}{2}\left(F_l(\sqrt{\zeta(z)}, z, h) + F_l(-\sqrt{\zeta(z)}, z, h)\right), \\ b_l(z,h) &= \frac{1}{2\sqrt{\zeta(z)}}\left(F_l(\sqrt{\zeta(z)}, z, h) - F_l(-\sqrt{\zeta(z)}, z, h)\right), \end{aligned}$$

and the function f_l is defined by the relation

$$F_l(t,z,h) = a_l(z,h) + b_l(z,h)t + (t^2 - \zeta(z)) f_l(t,z,h) \tag{3.18}$$

Finally, for $l \geq 1$, one has $F_l(t,z,h) = \frac{\partial f_{l-1}}{\partial t}(t,z,h)$.

For $l \in \mathbb{N} \cup \{0\}$, the coefficients a_l and b_l are analytic in $(z,h) \in K \times V$.

5. To estimate the integrals I_l, one has to estimate the functions F_l. Below, the constants C are independent on z, h and t. The symbol $O(\cdot)$ is subsequently used for estimates uniform in z, t and h.

Let us assume that that$(t,z,h) \in \mathbb{C} \times K \times V$ and show that there exists a constant $C_0 > 0$ such that, for any $l \in \mathbb{N}$, one has

$$F_l(t,z,h) = O\left(e^{C_0|t|}\right), \tag{3.19}$$

where the implicit constant in (3.19) depends only on the index l.

This estimate is obvious for F_0. Let us assume that it is proved for some $l = l_0$ and prove it for $l = l_0 + 1$.

Clearly, $\zeta(z)$ is bounded on K. In view of the definitions of $(a_l)_{l\geq 0}$, $(b_l)_{l\geq 0}$ and the induction hypothesis, we have $a_{l_0}(z,h) = O(1)$ and $b_{l_0}(z,h) = O(1)$. These observations, the definition of f_l (3.18) and the induction hypothesis imply that there exists $R > 0$ independent of h such that, for all $|t| \geq R$, $f_{l_0}(t,z,h) = O(e^{C_0|t|})$. By the maximum principle, this implies that f_{l_0} satisfies this estimate for all $t \in \mathbb{C}$. Now the Cauchy estimates for the derivatives of the analytic functions imply (3.19) for $l = l_0 + 1$.

6. Let us prove that

$$I_l = O\left(h^{\frac{1}{3}} w_h\right) + O\left(h^{\frac{1}{3}} w_h'\right). \tag{3.20}$$

Therefore, one essentially has to repeat the reasoning made in Section 4, Chapter VII of [22]. So, we omit some details.

If $Z = h^{-\frac{2}{3}}\zeta(z)$ is bounded by a constant, setting $T = h^{-\frac{1}{3}}t$, we change variable in (3.17). In view of step 5, we get

$$I_l = h^{\frac{1}{3}} \int_\gamma e^{-\left(\frac{T^3}{3} - TZ\right)} O(e^{C_0 h^{\frac{1}{3}}|T|})\, dT = O(h^{\frac{1}{3}}),$$

and this leads to (3.20) as w and w' have no common zero.
If $Z = h^{-\frac{2}{3}}\zeta(z)$ is large, we estimate the integral I_l using the method of steepest descent. In view of the fifth step, we have

$$I_l = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} O(e^{C_0|t|})\, dt.$$

We deform the integration path to a path of steepest descent for $e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)}$ exactly as when computing the asymptotics of the Airy function w, i.e., the asymptotics of the integral $\int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)}\, dt$. The saddle points $\pm\sqrt{\zeta(z)}$ are uniformly bounded when $z \in K$. Let $r > 0$ be sufficiently large for the saddle points to be inside the disk of radius r centered at 0. We compute the asymptotics of the integral over $\gamma \cap \{|t| \leq r\}$ directly by means of the method of steepest descents and, comparing the answer with the asymptotics of the Airy function $w(Z)$ as $Z \to \infty$, we find that this integral is bounded by $O(h^{\frac{1}{3}} w_h) + O(h^{\frac{2}{3}} w_h')$. The integral over the remaining part of γ quickly tends to 0 as $h \to 0$: actually, it is exponentially small with respect to $O(h^{\frac{1}{3}} w_h) + O(h^{\frac{2}{3}} w_h')$. This yields (3.20).
7. Formula (3.16) and estimate (3.20) lead to the representation

$$\begin{aligned} H\left(A\, h^{\frac{1}{3}} w_h\right)(z) &= h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{N} h^l a_l(z,h) + h^{\frac{2}{3}} w_h'(z) \sum_{l=0}^{N} h^l b_l(z,h) \\ &\quad + O\left(h^{N+1+\frac{1}{3}} w_h(z)\right) + O\left(h^{N+1+\frac{2}{3}} w_h'(z)\right), \end{aligned} \tag{3.21}$$

The coefficients $(a_l)_{l\in\mathbb{N}\cup\{0\}}$ and $(b_l)_{l\in\mathbb{N}\cup\{0\}}$ being analytic in h, we can approximate them by Taylor polynomials. This yields

$$\begin{aligned} H\left(A\, h^{\frac{1}{3}} w_h\right)(z) &= h^{\frac{1}{3}} w_h(z) \sum_{l=0}^{N} h^l a_l(z) + h^{\frac{2}{3}} w_h'(z) \sum_{l=0}^{N} h^l b_l(z) \\ &\quad + O\left(h^{N+1+\frac{1}{3}} w_h(z)\right) + O\left(h^{N+1+\frac{2}{3}} w_h'(z)\right), \end{aligned} \tag{3.22}$$

where $(a_l(z))_{l\in\mathbb{N}\cup\{0\}}$ and $(b_l(z)))_{l\in\mathbb{N}\cup\{0\}}$ are new coefficients independent of h. In particular, one has

$$\begin{aligned} a_0(z) &= a_0(z,0), & b_0(z) &= b_0(z,0), \\ a_1(z) &= a_1(z,0) + \frac{\partial a_0}{\partial h}(z,0), & b_1(z) &= b_1(z,0) + \frac{\partial b_0}{\partial h}(z,0). \end{aligned} \tag{3.23}$$

Now, to complete the proof of Proposition 3.1, it suffices to compute a_0, b_0, a_1, b_1.
8. Let us check that

$$a_0(z,0) = \frac{\partial a_0}{\partial h}(z,0) = 0. \tag{3.24}$$

Substituting (3.11) into (3.13), we get

$$\begin{aligned} a_0(z,h) = A(z+h)cosh &\left(\sqrt{\zeta(z)}\, \frac{\zeta(z+h) - \zeta(z)}{h}\right) \\ &+ A(z-h)cosh \left(\sqrt{\zeta(z)}\, \frac{\zeta(z-h) - \zeta(z)}{h}\right) + v(z)A(z). \end{aligned}$$

Thus,

$$a_0(z,0) = A(z)\left(2\, cosh\,(\sqrt{\zeta(z)}\zeta'(z)) + v(z)\right).$$

Recall that the complex momentum p is defined in (1.2). In view of (1.8) we get

$$2cosh\left(\sqrt{\zeta(z)}\zeta'(z)\right)+v(z)=0. \tag{3.25}$$

So, $a_0(z,0)=0$. As $a_0(z,h)$ is even in h, we also see that $\frac{\partial a_0}{\partial h}(z,0)=0$.
9. Let us check that

$$b_0(z,0)=0,\quad \tfrac{\partial b_0}{\partial h}(z,0)=2A'(z)\frac{sinh\left(\sqrt{\zeta(z)}\zeta'(z)\right)}{\sqrt{\zeta(z)}}+A(z)cosh\left(\sqrt{\zeta(z)}\zeta'(z)\right)\zeta''(z). \tag{3.26}$$

Substituting (3.11) into (3.14), we get

$$b_0(z,h)=A(z+h)\frac{sinh\left(\sqrt{\zeta(z)}\ \frac{\zeta(z+h)-\zeta(z)}{h}\right)}{\sqrt{\zeta(z)}}+A(z-h)\frac{sinh\left(\sqrt{\zeta(z)}\ \frac{\zeta(z-h)-\zeta(z)}{h}\right)}{\sqrt{\zeta(z)}}.$$

Clearly, $b_0(z,h)$ is odd in h, and so $b_0(z,0)=0$. Computing $\frac{\partial b_0}{\partial h}(z,0)$, we complete the proof of (3.26).
10. To compute $a_1(z,0)$ and $b_1(z,0)$, we first study f_0. Let $r>0$ be such that $|\zeta(z)|\leq r^2/2$ for all $z\in K$. Let $|t|=r$, $z\in K$ and $h\in V$. Formulas (3.11) and (3.25) imply that

$$F_0(t,z,h)=F_0(t,z)+O(h),\ \ F_0(t,z)=2A(z)\left(cosh\ (t\zeta'(z))-cosh\left(\sqrt{\zeta(z)}\zeta'(z)\right)\right).$$

This result, the formulas $a_0(z,0)=b_0(z,0)=0$ (see steps 9–10) and (3.12) imply that for $|t|=r$ one has

$$f_0(t,z,h)=\frac{F_0(t,z)}{t^2-\zeta(z)}+O(h).$$

By the maximum principle for analytic functions, this representation remains true for all $|t|\leq r$.
11. The result of the previous step and the Cauchy estimates for the derivatives of the analytic functions imply that, for $|t|\leq r/2$, $z\in K$ and $h\in V$, one has

$$F_1(t,z,h)=F_1(t,z)+O(h),\quad \text{where}\quad F_1(t,z)=\frac{\partial}{\partial t}\left(\frac{F_0(t,z)}{t^2-\zeta(z)}\right). \tag{3.27}$$

Therefore

$$a_1(z,0)=\frac{F_1\left(\sqrt{\zeta(z)},z\right)+F_1\left(-\sqrt{\zeta(z)},z\right)}{2},\quad b_1(z,0)=\frac{F_1\left(\sqrt{\zeta(z)},z\right)-F_1\left(-\sqrt{\zeta(z)},z\right)}{2\sqrt{\zeta(z)}}.$$

As $t\mapsto F_1(t,z)$ is odd, one has

$$a_1(z,0)=0,\qquad b_1(z,0)=\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{F_0(t,z)}{t^2-\zeta(z)}\right)\bigg|_{t=\sqrt{\zeta}}.$$

Elementary calculations yield

$$b_1(z,0)=\frac{A\zeta'}{2\zeta}\left(\zeta'(z)cosh\left(\sqrt{\zeta}\zeta'\right)-\frac{sinh\ (\sqrt{\zeta}\zeta')}{\sqrt{\zeta}}\right),\quad \zeta=\zeta(z).$$

The results of the steps 8, 9 and 11 imply that

$$a_0(z)=b_0(z)=a_1(z)=0,\qquad b_1(z)=A(z)g(z)\frac{d\log(A^2g)}{dz}(z),$$

where g is the defined in (1.14). Substituting these formulae into (3.22), we obtain (3.1) and (3.2). This completes the proof of Proposition 3.1. □

3.3. **Proof of Proposition 3.2.** The proof of Proposition 3.2 being parallel to that of Proposition 3.1, we concentrate only on the differences and omit details. We assume that B is analytic in U. Let w, γ and K be as in the proof of Proposition 3.1. We use the notations from (1.16). We assume that $z \in K$ and that h is sufficiently small. The derivation of the asymptotics of $H(Bh^{\frac{2}{3}}w')$ is split into several steps.

1. We get

$$H\left(Bh^{\frac{2}{3}}w_h'\right) = \int_\gamma e^{-\frac{1}{h}\left(\frac{t^3}{3}-t\zeta(z)\right)} G_0(t,z,h)\,dt, \tag{3.28}$$

$$G_0(t,z,h) = t\left(B(z+h)e^{\frac{\zeta(z+h)-\zeta(z)}{h}t} + B(z-h)e^{\frac{\zeta(z-h)-\zeta(z)}{h}t} + v(z)B(z)\right). \tag{3.29}$$

2. Fix an $N \in \mathbb{N}$. Reasoning as in steps 1-6 of the proof of Proposition 3.1, instead of (3.21), for $z \in K$ and for sufficiently small h, we prove that

$$H\left(B\,h^{\frac{2}{3}}w_h'\right) = h^{\frac{1}{3}}w_h \sum_{l=0}^{N} h^l c_l(z,h) + h^{\frac{2}{3}}w_h' \sum_{l=0}^{N} h^l d_l(z,h) + O\left(h^{N+1+\frac{1}{3}}w_h\right) + O\left(h^{N+1+\frac{2}{3}}w_h'\right),$$

where, for $l \in \mathbb{N}\cup\{0\}$, one computes

$$c_l(z,h) = \tfrac{1}{2}\left(G_l(\sqrt{\zeta(z)},z,h) + G_l(-\sqrt{\zeta(z)},z,h)\right), \tag{3.30}$$

$$d_l(z,h) = \frac{1}{2\sqrt{\zeta(z)}}\left(G_l(\sqrt{\zeta(z)},z,h) - G_l(-\sqrt{\zeta(z)},z,h)\right), \tag{3.31}$$

$$g_l(t,z,h) = \frac{G_l(t,z,h) - c_l - d_l t}{t^2 - \zeta(z)}, \quad G_{l+1} = \frac{\partial g_l}{\partial t}. \tag{3.32}$$

In (3.30) and (3.31), we use one and the same branch of $\sqrt{\zeta(z)}$.
Approximating the $(c_l)_{l\in\mathbb{N}\cup\{0\}}$ and $(d_l)_{l\in\mathbb{N}\cup\{0\}}$ as functions of h by Taylor polynomials, we get

$$H\left(B\,h^{\frac{2}{3}}w_h'\right) = h^{\frac{1}{3}}w_h \sum_{l=0}^{N} h^l c_l(z) + h^{\frac{2}{3}}w_h' \sum_{l=0}^{N} h^l d_l(z) + O\left(h^{N+1+\frac{1}{3}}w_h\right) + O\left(h^{N+1+\frac{2}{3}}w_h'\right). \tag{3.33}$$

One has

$$\begin{aligned} c_0(z) &= c(z,0), & d_0(z) &= d(z,0),\\ c_1(z) &= c_1(z,0) + \frac{\partial c_0}{\partial h}(z,0), & d_1(z) &= d_1(z,0) + \frac{\partial d_0}{\partial h}(z,0). \end{aligned} \tag{3.34}$$

3. Substituting (3.29) into formula (3.31) with $l = 0$, we obtain the formulas

$$d_0(z,0) = \frac{\partial d_0}{\partial h}(z,0) = 0 \tag{3.35}$$

in the same way as we obtained (3.24).

4. Let us check that

$$c_0(z,0) = 0, \quad \tfrac{\partial c_0}{\partial h}(z,0) = \sqrt{\zeta}\left(2B'(z)sinh\left(\sqrt{\zeta}\zeta'\right) + B(z)cosh\left(\sqrt{\zeta}\zeta'\right)\sqrt{\zeta}\zeta''\right), \tag{3.36}$$

where $\zeta = \zeta(z)$. Substituting (3.29) into formula (3.30) with $l = 0$, we get

$$c_0(z,h) = B(z+h)\sqrt{\zeta(z)}\, sinh\left(\sqrt{\zeta(z)}\,\frac{\zeta(z+h)-\zeta(z)}{h}\right) + B(z-h)\sqrt{\zeta(z)}\, sinh\left(\sqrt{\zeta(z)}\,\frac{\zeta(z-h)-\zeta(z)}{h}\right).$$

The coefficient $c_0(z,h)$ is odd in h, and so $c_0(z,0) = 0$. Computing $\frac{\partial c_0}{\partial h}(z,0)$, we complete the proof of (3.36).

5. As G_1 is computed in the same way as F_1 in steps 11–12 of the proof of Proposition 3.1, we omit the details and write down the result:

$$G_1(t,z,h) = G_1(t,z) + O(h), \quad G_1(t,z) = 2B\frac{\partial}{\partial t}\frac{t\left(cosh\,(t\zeta') - cosh\,(\sqrt{\zeta}\zeta')\right)}{t^2-\zeta}, \tag{3.37}$$

where $\zeta = \zeta(z)$. This representation is locally uniform in t and uniform in $z \in K$.

6. Having described G_1, we easily get the formulas

$$d_1(z,0) = 0, \quad c_1(z,0) = \frac{B\zeta'}{2}\left(\zeta' cosh\left(\sqrt{\zeta}\zeta'\right) + \frac{sinh\,(\sqrt{\zeta}\zeta')}{\sqrt{\zeta}}\right), \quad \zeta = \zeta(z).$$

We again omit elementary details and only note that the first formula follows from the evenness of the function $t \mapsto G_1(t,z)$.

7. Substituting the results of steps 3,4 and 6 into (3.34), we get

$$c_0(z) = d_0(z) = d_1(z) = 0, \qquad c_1(z) = \zeta(z)B(z)g(z)\frac{d\log(\zeta B^2 g)}{dz}(z),$$

where g is the function from (1.14). Substituting these formulas into (3.33), we prove the statement of Proposition 3.2. □

4. Properties of asymptotic solutions

We now study basic properties of the asymptotic solutions. More precisely, we fix an integer L and study the functions $(W_j)_{j\in\mathbb{Z}_3}$, i.e., the functions W from Theorem 1.3 corresponding to the chosen L and to the Airy functions $w = w_j$, $j \in \mathbb{Z}_3$.

4.1. Functional relations.

We recall that the function $(W_j)_{j\in\mathbb{Z}_3}$ are defined in a domain U satisfying the assumptions from sections 1.3 and 1.6.1.

Lemma 4.1. *One has*

$$W_0(z) + W_1(z) + W_2(z) = 0, \qquad \forall z \in U. \tag{4.1}$$

Proof. Formula (1.10) and the definitions of the integration paths $(\gamma_j)_{j\in\mathbb{Z}_3}$ (see Fig. 1) imply that

$$w_0(\zeta) + w_1(\zeta) + w_2(\zeta) = 0, \qquad \zeta \in \mathbb{C}. \tag{4.2}$$

As the function ζ and all the coefficients $(A_l)_{l\in\mathbb{N}\cup\{0\}}$ and $(B_l)_{l\in\mathbb{N}\cup\{0\}}$ in representations (1.16)– (1.17) are independent of the choice of w, the solution of the Airy equation in this representation, the relation (4.2) implies (4.1). □

Relation (4.1) implies that

$$(W_0(z), W_1(z)) = (W_1(z), W_2(z)) = (W_2(z), W_0(z)), \qquad \forall\{z, z+h\} \subset U, \tag{4.3}$$

where $(f(z),\, g(z)) = f(z+h)g(z) - g(z+h)f(z)$ is the difference Wronskian of f and g.

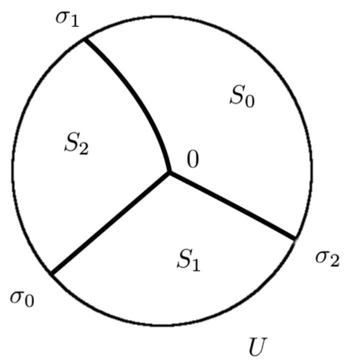

FIGURE 2. Stokes lines and sectors

4.2. **Estimates of W_j.** To prove the existence of analytic solutions that admit asymptotic expansions of the form (1.19), we need rough estimates of $(W_j)_{j\in\mathbb{Z}_3}$ in U. Therefore, we first introduce some tools.

4.2.1. *Geometry.* We recall that the function ζ defined in (1.7) is analytic in U and bijectively maps U onto $V = \zeta(U)$, $\zeta(z_0) = 0$.
We put

$$\sigma_j = \zeta^{-1}(V \cap a_j), \quad a_j = e^{-2\pi i j/3}\,\mathbb{R}_-, \quad j \in \mathbb{Z}_3, \tag{4.4}$$

where $\mathbb{R}_- = (-\infty, 0]$. The curves $(\sigma_j)_{j\in\mathbb{Z}_3}$ are analytic. They all begin at z_0. Any two of them do not intersect except at z_0. The angles between these curves at z_0 are equal to $2\pi/3$.
The curves $(\sigma_j)_{j\in\mathbb{Z}_3}$ cut the domain U (a neighborhood of z_0) into three simply connected subdomains that we call *sectors*. We denote them by S_0, S_1 and S_2 so that the sector S_0 is bounded by σ_1 and σ_2, S_1 is bounded by σ_2 and σ_0, and S_2 is bounded by σ_0 and σ_1, see Fig. 2. Let

$$U_j = U \setminus \sigma_j, \quad j \in \mathbb{Z}_3. \tag{4.5}$$

These domains do not contain branch points of the complex momentum p: the only branch point of p in U is $z = z_0$. We shall use

Lemma 4.2. *For $j \in \mathbb{Z}_3$, there exists a branch p_j of the complex momentum that is analytic in U_j and such that $p_j(z_0) = 0$ and*

(1) $\operatorname{Im} \int_{z_0}^{z} p_j(z)\,dz > 0$ *inside S_j;*
(2) $\operatorname{Im} \int_{z_0}^{z} p_j(z)\,dz < 0$ *inside the two other sectors;*
(3) $\operatorname{Im} \int_{z_0}^{z} p_j(z)\,dz = 0$ *along the curves σ_1, σ_2 and σ_0 (in the case of σ_j, we mean the boundary values);*

Moreover, one has

$$p_1 = -p_0 \text{ in } \sigma_0 \cup S_1 \cup \sigma_2 \cup S_0 \cup \sigma_1, \quad p_2 = -p_0 \text{ in } \sigma_0 \cup S_2 \cup \sigma_1 \cup S_0 \cup \sigma_2. \tag{4.6}$$

In the WKB method, the curves σ_j, $j \in \mathbb{Z}_3$, are called *Stokes lines.*

Proof. Let us check the first three points of Lemma 4.2 for $j = 0$. We recall that ζ is an analytic branch of the function (1.7). We can assume that in (1.7) p is a branch of the complex momentum analytic in U_0 and such that $p(z_0) = 0$.
Formulas (4.4) and the definition of ζ imply that $\operatorname{Im} \int_{z_0}^{z} p(z)\,dz = 0$ on any of the Stokes lines. We note that

$$\zeta(S_j) = \{v \in V \;:\; v \neq 0,\ \arg v \in -2\pi j/3 + (-\pi/3, \pi/3)\}, \quad j \in \mathbb{Z}_3. \tag{4.7}$$

This and the definition of ζ imply that $\text{Im} \int_{z_0}^{z} p(z)\, dz \neq 0$ in each of the sectors.
In view of the analysis made in section 1.3.1, in U_0, we can choose an analytic branch p_0 of the complex momentum so that $\text{Im} \int_{z_0}^{z} p_0(z)\, dz > 0$ in S_0. For p_0, the statements 1. and 3. of Lemma 4.2 are obviously valid.
To prove point 2., it suffices to check that $\text{Im} \int_{z_0}^{z} p_0(z)\, dz < 0$ in the sectors S_1 and S_2. Therefore, we note that as $z \neq z_0$, $z \sim z_0$, crosses σ_2 moving from S_0 to S_1 the argument of $\zeta(z)$ decreases (ζ vanishes only at $z = z_0$) as does the argument of $\int_{z_0}^{z} p_0(z)\, dz$. Therefore, point 2. of Lemma 4.2 follows from points 1. and 3.
To complete the proof of Lemma 4.2, we choose p_1 in the following way. First, we restrict p_0 to S_1; then, in S_1 we choose $p_1 = -p_0$ and continue p_1 analytically from S_1 to U_1. For the thus chosen p_1, we have

$$\text{Im} \int_{z_0}^{z} p_1(z)\, dz = -\text{Im} \int_{z_0}^{z} p_0(z)\, dz > 0, \quad z \in S_1.$$

This proves point 1 for p_1. Point 2 and point 3 for p_1 are proved as for p_0 and p.
To choose p_2, first, we restrict p_0 to S_2, then, in S_2 we choose $p_2 = -p_0$ and p_2 analytically from S_2 to U_2. To complete the proof of Lemma 4.2 for p_2, we reason as for p_1. We omit further details. □

4.2.2. *Estimates.* For $j \in \mathbb{Z}_3$ and $z \in U_j$, we set

$$\rho_j(z) = e^{\frac{i}{h} \int_{z_0}^{z} p_j(z')\, dz'}. \tag{4.8}$$

We note that ρ_j is continuous up to the cut along σ_j, and the boundary values of its absolute value $|\rho_j|$ on both the sides of the cut equal one. So, below, we consider $|\rho_j|$ as a continuous function in U.

Let us recall that H is defined by (1.13). We set

$$\delta_j(z) = [H(W_j)](z), \quad z \in U, \qquad j \in Z_3. \tag{4.9}$$

Proposition 4.1. *For each $j \in \mathbb{Z}_3$, one has*

$$|W_j(z)| \leq Ch^{1/3} |\rho_j(z)|, \quad z \in U, \tag{4.10}$$

$$|\delta_j(z)| \leq Ch^{L+2+1/3}\, |\rho_j(z)|, \quad \{z, z+h, z-h\} \subset U, \tag{4.11}$$

where L is the order entering the definition of W_j, see (1.17).

Proposition 4.1 immediately follows from formulas (1.16)–(1.18) with $w = w_j$ and

Lemma 4.3. *Let $j \in \mathbb{Z}_3$. Then one has*

$$|w_j(h^{-\frac{2}{3}}\zeta(z))| \leq C|\rho_j(z)|, \quad |w_j'(h^{-\frac{2}{3}}\zeta(z))| \leq Ch^{-\frac{1}{6}}|\rho_j(z)|, \qquad z \in U. \tag{4.12}$$

Proof. We prove (4.12) only for $j = 0$. The other cases are treated similarly. We recall that $w_0 = \text{Ai}$, that ζ bijectively maps U onto its image and that $\zeta(z_0) = 0$ (see (1.7)). Clearly,

$$w_0(h^{-\frac{2}{3}}\zeta(z)) = O(1) \text{ and } w_0'(h^{-\frac{2}{3}}\zeta(z)) = O(1) \text{ if } |\zeta(z)| \leq h^{\frac{2}{3}}. \tag{4.13}$$

Now we turn to the case where $|\zeta(z)| \geq h^{\frac{2}{3}}$. It suffices to prove (4.12) in U_0.
The asymptotic formulas (1.12) imply that, for $Z \in \{Z \in \mathbb{C} \setminus \mathbb{R}_- \,:\, |Z| \geq 1\}$, one has

$$|w_0(Z)| \leq C|Z|^{-\frac{1}{4}} \left| e^{-\frac{2}{3} Z^{\frac{3}{2}}} \right| \text{ and } |w_0'(Z)| \leq C|Z|^{\frac{1}{4}} \left| e^{-\frac{2}{3} Z^{\frac{3}{2}}} \right|, \tag{4.14}$$

where the $Z \mapsto Z^{\frac{3}{2}}$ is analytic in $\mathbb{C} \setminus \mathbb{R}_-$ and positive when $Z > 0$.
Estimate (4.14) and the definition of U_0, see (4.5), imply that, for $z \in U_0$ such that $|\zeta(z)| \geq h^{\frac{2}{3}}$, one has

$$|w_0(h^{-\frac{2}{3}}\zeta(z))| \leq C \left| e^{-\frac{2}{3h} \zeta(z)^{\frac{3}{2}}} \right| \text{ and } |w_0'(h^{-\frac{2}{3}}\zeta(z))| \leq Ch^{-\frac{1}{6}} \left| e^{-\frac{2}{3h} \zeta(z)^{\frac{3}{2}}} \right|, \tag{4.15}$$

where $z \to \zeta(z)^{3/2}$ is analytic in U_0 and positive along $\alpha_0 = \zeta^{-1}((0,\infty))$.
In view of the analysis made in section 1.3.1,

$$\zeta(z)^{\frac{3}{2}} = \pm \frac{3i}{2} \int_{z_0}^{z} p_0(z')\, dz', \quad z \in U_0.$$

As $\alpha_0 = \zeta^{-1}((0,\infty)) \subset S_0$, along α_0 one has $\mathrm{Im} \int_{z_0}^{z} p_0(z')\, dz' > 0$. Therefore, in U_0 $\zeta(z)^{\frac{3}{2}} = -\frac{3i}{2} \int_{z_0}^{z} p_0(z')\, dz'$, and $\left| e^{-\frac{2}{3h}\zeta(z)^{\frac{3}{2}}} \right| = |\rho_0(z)|$. This and (4.15) imply (4.12) for $|\zeta(z)| \geq h^{-2/3}$. This and (4.13) imply the statement of the lemma. □

4.3. **Wronskians.** Below $\mathcal{C} \subset U$ is a closed disk independent of h with the center at z_0. We now prove

Lemma 4.4. *For $\{z, z+h\} \subset \mathcal{C}$, as $h \to 0$ one has*

$$(W_0(z), W_1(z)) = h(w_0'(z) w_1(z) - w_0'(z) w_1(z)) + O(h^{\frac{5}{3}}).$$

Before proving Lemma 4.4, we check

Lemma 4.5. *Let $j \in \mathbb{Z}_3$, and let $w = w_j$. For $\{z, z+h\} \subset \mathcal{C}$, as $h \to 0$ one has*

$$h^{\frac{1}{3}} w_h \Big|_{z+h} = h^{\frac{1}{3}} cosh\,(\sqrt{\zeta(z)}\zeta'(z))\, w_h + g(z) h^{\frac{2}{3}}\, w_h' + O(h^{\frac{4}{3}} w_h) + O(h^{\frac{5}{3}} w_h')\,,$$

where we use the notations from (1.16) *and g is defined in* (1.14).

Proof of Lemma 4.5. We proceed as in the proof of Proposition 3.1. Thus, we omit some details and concentrate on the new computations.
Let $\gamma = \gamma_j$ be the integration path in (1.10). Also, below we assume that $z \in \mathcal{C}$ and that h is sufficiently small. The proof is broken into several steps.
1. We compute

$$h^{\frac{1}{3}} w_h \Big|_{z+h} = \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} E(t,z,h)\, dt, \quad E(t,z,h) = e^{\frac{t}{h}(\zeta(z+h)-\zeta(z))}. \tag{4.16}$$

2. We represent $E(t,z,h)$ in the form

$$E(t,z,h) = \alpha(z,h) + \beta(z,h) t + (t^2 - \zeta(z))\phi(t,z,h) \tag{4.17}$$

with

$$\alpha(z,h) = \frac{E(\sqrt{\zeta(z)}, z, h) + E(-\sqrt{\zeta(z)}, z, h)}{2} = cosh\, \left(\sqrt{\zeta} \frac{\zeta(z+h) - \zeta(z)}{h} \right),$$

$$\beta(z,h) = \frac{E(\sqrt{\zeta(z)}, z, h) - E(-\sqrt{\zeta(z)}, z, h)}{2\sqrt{\zeta(z)}} = \frac{1}{\sqrt{\zeta}}\, sinh\, \left(\sqrt{\zeta} \frac{\zeta(z+h) - \zeta(z)}{h} \right).$$

3. Clearly,

$$\alpha(z,h) = cosh\,(\sqrt{\zeta}\zeta') + O(h), \quad \beta(z,h) = g(z) + O(h). \tag{4.18}$$

4. Substituting (4.17) into (4.16) and integrating by parts, we get

$$\begin{aligned} h^{\frac{1}{3}} w_h \Big|_{z+h} &= \alpha h^{\frac{1}{3}} w_h + \beta h^{\frac{2}{3}} w_h' + h \int_{\gamma} e^{-\frac{1}{h}\left(\frac{t^3}{3} - t\zeta(z)\right)} E_1(t,z,h)\, dt, \\ E_1(t,z,h) &= \frac{\partial \phi}{\partial t}(t,z,h). \end{aligned} \tag{4.19}$$

Reasoning as when proving Proposition 3.1, we check that the last term in the right hand side of (4.19) is $O(h^{\frac{4}{3}} w) + O(h^{\frac{5}{3}} w')$. Lemma 4.5 follows from this estimate, asymptotics (4.18) and representation (4.19). □

Now, we turn to the proof of Lemma 4.4.

Proof of Lemma 4.4. Below we assume that $\{z, z+h\}\subset\mathcal{C}$ and that h is sufficiently small. Using (1.17) and Lemma 4.5, we compute

$$\begin{aligned}(W_0(z),&W_1(z)) = A_0^2(z)\\ &\times\Big((h^{\frac13} cosh\,(\sqrt{\zeta}\zeta')\,w_0+gh^{\frac23}\,w_0'+O(h^{\frac43}w_0)+O(h^{\frac53}w_0'))\\ &\qquad\cdot (h^{\frac13}w_1+O(h^{\frac43}w_1)+O(h^{\frac53}w_1'))\\ &\quad-(h^{\frac13} cosh\,(\sqrt{\zeta}\zeta')\,w_1+gh^{\frac23}\,w_1'+O(h^{\frac43}w_1)+O(h^{\frac53}w_1'))\\ &\qquad\cdot(h^{\frac13}w_0+O(h^{\frac43}w_0)+O(h^{\frac53}w_0'))\Big).\end{aligned}$$

Here, $w_j = w_j(h^{-\frac23}\zeta(z))$, $j\in\mathbb{Z}_3$, $\zeta=\zeta(z)$, and $g=g(z)$.
Now, we assume that $z\in\sigma_0\cup S_1\cup\sigma_2\cup S_0\cup\sigma_1$. Then, by (4.12) and (4.6)

$$|w_0w_1|\le C,\quad |w_0'w_1|\le Ch^{-\frac16},\quad |w_0w_1'|\le Ch^{-\frac16},\quad |w_0'w_1'|\le Ch^{-\frac13},\tag{4.20}$$

and we get

$$(W_0(z),W_1(z)) = hg(z)A_0^2(z)(w_0'w_1-w_0w_1')+O(h^{\frac53}).$$

In view of (1.15), Lemma 4.4 is proved for $z\in\sigma_0\cup S_1\cup\sigma_2\cup S_0\cup\sigma_1$.
When $z\in\sigma_0\cup S_2\cup\sigma_1\cup S_0\cup\sigma_2$, we similarly get

$$(W_0(z),W_2(z)) = hg(z)A_0^2(z)(w_0'w_2-w_2'w_0)+O(h^{\frac53}).\tag{4.21}$$

In view of (4.3), $(W_0,W_2) = -(W_0,W_1)$ and relation (4.2) imply that $w_0'w_2 - w_0w_2' = -(w_0'w_1-w_0w_1')$. Therefore, Lemma 4.4 for $z\in S_2$ follows from (4.21). This completes the proof of Lemma 4.4. □

5. Solutions to (1.1) on precanonical domains

Fix $L\in\mathbb{N}$. Here we construct solutions $(\psi_j)_{j\in\mathbb{Z}_3}$ to equation (1.1) that, up to $O(h^L)$, coincide with $(W_j)_{j\in\mathbb{Z}_3}$, the functions from Theorem 1.3. The result of this section is preliminary: we only construct the $(\psi_j)_{j\in\mathbb{Z}_3}$ on some subdomains of U.

5.1. **The result of this section.**

5.1.1. *Notations and some definitions.* First, to formulate the results of this section, we introduce some notations and recall some definitions related to the complex WKB method for difference equations, see, for example, [13].
For $z\in\mathbb{C}$, we let $x=\operatorname{Re} z$ and $y=\operatorname{Im} z$.
A curve $\gamma\subset\mathbb{C}$ is called *vertical*, if z is a piecewise continuously differentiable function of y along γ.
Let $\gamma\subset U$ be a regular vertical curve parameterized by $z=z(y)$. Let p be a branch of the complex momentum continuous on γ. The curve γ is *precanonical* with respect to p, if the function $y\mapsto \operatorname{Im}\int_{z_0}^{z(y)} p(z)\,dz$ is non decreasing and the function $y\mapsto\operatorname{Im}\int_{z_0}^{z(y)}(p(z)-\pi)\,dz$ is non increasing.
Let $d>0$. For $M\subset C$ we define the *horizontal d-neighborhood* of M to be the set $M^d := M+[-d,d]$ and $M^{-d} := (M^d-d)\cap M\cap(M^d+d)$.
We recall that, for $j\in\mathbb{Z}_3$, the sector S_j and the Stokes line σ_j are shown in Fig. 2.
For $j\in\mathbb{Z}_3$, we denote by $S_{j,j+1}$ the closure of the domain $S_j\cup S_{j+1}$ without the boundary of U. For example, one has

$$S_{1,2}=\sigma_1\cup S_2\cup\sigma_0\cup S_1\cup\sigma_2.$$

We also note that relations (4.6) imply that

$$|\rho_j(z)\rho_{j+1}(z)|=1,\qquad z\in S_{j,j+1},\quad j\in\mathbb{Z}_3.\tag{5.1}$$

Let $r_1<r_2$. We set $S(r_1,r_2)=\{z\in\mathbb{C}\,:\,r_1<\operatorname{Im} z<r_2\}$.

5.1.2. *The main result of the section.* One has

Theorem 5.1. *Let $j \in \mathbb{Z}_3$, $L \in \mathbb{N}$, $c \in (1,2)$ and $r > 0$. Let $K \subset S_{j,j+1}$ be a regular simply connected domain bounded by two curves having common endpoints z_1 and z_2 and both precanonical with respect to either the branch p_j or p_{j+1}.*
Then, for sufficiently small h, there exist two solutions ψ_j and ψ_{j+1} to (1.1) that are analytic in K^{ch} and that, in $K^{ch} \cap S(\operatorname{Im} z_1 + rh, \operatorname{Im} z_2 - rh)$ admit the asymptotic representations

$$\psi_l(z) = W_l(z) + O(|\rho_l| h^{L+1+\frac{1}{3}}), \quad l \in \{j, j+1\}, \tag{5.2}$$

where W_l is the function described in Theorem 1.3 and corresponding to w_l and the order L.

Let us discuss the solutions ψ_j and ψ_{j+1} described in Theorem 5.1.

Corollary 5.1. *In the case of Theorem 5.1, the solutions ψ_j and ψ_{j+1} can be analytically continued to $U \cap S(\operatorname{Im} z_1, \operatorname{Im} z_2)$. Let $r > 0$. As $h \to 0$, one has*

$$(\psi_j(z), \psi_{j+1}(z)) = (W_j(z), W_{j+1}(z)) + O(h^{L+1+\frac{2}{3}}) \tag{5.3}$$

in $K^{ch} \cap S(\operatorname{Im} z_1 + rh, \operatorname{Im} z_2 - rh)$.

Proof. The solutions being analytic in K^{ch} with $c > 1$, they can be analytically continued to $U \cap S(\operatorname{Im} z_1, \operatorname{Im} z_2)$ just be means of equation (1.1).
We fix $l \in \mathbb{Z}_3$ and note that, for all z in a compact set $\mathcal{C} \subset U$, for sufficiently small h, one has $|\rho_l(z+h)|/|\rho_l(z)| \leq C$. For $z \in K^{ch} \cap S(\operatorname{Im} z_1 + rh, \operatorname{Im} z_2 - rh)$, representation (5.3) follows from this observation and from (4.10), (5.2) and (5.1). □

The remainder of this section is devoted to the proof of Theorem 5.1.
For the sake of definiteness, when proving Theorem 5.1, we assume that $j = 0$ and that the two curves from Theorem 5.1 are precanonical with to respect to the branch p_0. The other cases are treated similarly.
Below, K is as in the theorem (for $j = 0$), it is bounded by the precanonical curves γ_1 and γ_2, and their common endpoints satisfy $\operatorname{Im} z_1 < \operatorname{Im} z_2$. Finally, h is supposed to be sufficiently small.

5.2. **Ideas of the proof.** In the present section, we describe the construction of the solution ψ_0. The solution ψ_1 is constructed similarly.
Let us assume that ψ_0 is a solution to (1.1) analytic in K^{ch} that we expect to be close to W_0. Let us recall that $\delta_0 = HW_0$. Clearly, $\Delta_0 := W_0 - \psi_0$ satisfies the equation

$$H(\Delta_0)(z) = \delta_0(z), \quad \{z - h,\, z,\, z + h\} \subset K^{ch}. \tag{5.4}$$

For $z \in K^{ch}$, let $\gamma(z)$ denote a vertical curve in K^{ch} that contains z and connects z_1 and z_2. We construct a solution to the equation for Δ_0 in the form

$$\Delta_0 = R_0\, g_0 \quad \text{where} \quad R_0 g_0\,(z) := \int_{\gamma(z)} r_0(z,\zeta) g_0(\zeta)\, d\zeta, \tag{5.5}$$

$$r_0(z,\zeta) = \frac{1}{2ih} \frac{W_0(z)W_1(\zeta) - W_0(\zeta)W_1(z)}{(W_0(\zeta),\, W_1(\zeta))}\, \theta_0\left(\frac{\zeta - z}{h}\right), \quad \theta_0(t) = \cot(\pi t) - i. \tag{5.6}$$

Here, $(W_0(\zeta), W_1(\zeta))$ is the difference Wronskian of W_0 and W_1. The choice of Ansatz (5.5) is explained by

Lemma 5.1. *Let $0 < \beta < 1$. Let f be a function defined and analytic in $U \cap S(\operatorname{Im} z_1, \operatorname{Im} z_2)$ and such that the expression*

$$f_\beta(z) = (z - z_1)^\beta (z - z_2)^\beta f(z) \tag{5.7}$$

is bounded. Then, if $\{z-h, z, z+h\} \subset U \cap S(\operatorname{Im} z_1, \operatorname{Im} z_2)$*, one has*

$$HR_0 f(z) = f(z) + D_0 f(z), \quad D_0 f(z) = \int_{\gamma(z)} d_0(z,\zeta) f(\zeta)\, d\zeta \tag{5.8}$$

where

$$d_0(z,\zeta) = \frac{1}{2ih} \frac{\delta_0(z) W_1(\zeta) - W_0(\zeta)\delta_1(z)}{(W_0(\zeta),\, W_1(\zeta))}\, \theta_0\left(\frac{\zeta - z - 0}{h}\right), \tag{5.9}$$

and $\delta_j := HW_j$ *are the "error" terms estimated in (4.11). The function* $D_0 f$ *is analytic in* $U \cap S(\operatorname{Im} z_1, \operatorname{Im} z_2)$*.*

Proof. The analyticity of f and the boundedness of f_β imply that $D_0 f$ is well defined and analytic in $U \cap S(\operatorname{Im} z_1, \operatorname{Im} z_2)$. The relation $HR_0 f = f + D_0 f$ follows from the residue theorem. We omit further details. □

We note that an operator similar to R_0 was introduced in [14], but, was not studied for small h.
In view of Lemma 5.1 and the formulas $\Delta_0 = R_0 g_0$ and $H(\Delta_0) = \delta_0$, we can expect that in K^{ch}

$$g_0 + D_0 g_0 = \delta_0. \tag{5.10}$$

Roughly, to prove Theorem 5.1, we consider (5.10) as an equation on a vertical curve γ. It appears that if γ is precanonical, the operator D_0 is small. This enables us to construct a solution ψ_0 to equation (5.10) on γ_1. Next, we check that it is analytic in K^{ch}, satisfies (1.1) and admits the asymptotic representation (5.2). The solution ψ_1 is constructed similarly.

5.3. **The integral operator D_0.** The aim of this section is to estimate the operator norm of D_0 in a suitable functional space.
Let γ be either γ_1 or γ_2. We fix $\alpha \in (0,1)$ and define the strip

$$\Pi_{\gamma,\alpha} = \gamma \setminus \{z_1, z_2\} + [-\alpha h, \alpha h].$$

We recall that $z \to |\rho_0(z)|$ defined in U_0 is a continuous function in U. We fix $0 < \beta < 1$ and let $H_{\gamma,\alpha,\beta}$ be the linear space of functions analytic in $\Pi_{\gamma,\alpha}$ and having finite norm

$$\|f\| = \sup_{z \in \Pi_{\gamma,\alpha}} \frac{|f_\beta(z)|}{|\rho_0(z)|} \tag{5.11}$$

f_β being defined in (5.7). Obviously, endowed with this norm, $H_{\gamma,\alpha,\beta}$ is a Banach space.
For $f \in H_{\gamma,\alpha,\beta}$, we define $D_0 f$ by the formula in (5.8), where $\gamma(z)$ is a vertical curve that connects the points z_1 and z_2 in $\Pi_{\gamma,\alpha}$ and passes through z. The function $D_0 f$ is then analytic in $\Pi_{\gamma,\alpha}$. One has

Proposition 5.1. *For sufficiently small h*

$$\|D_0\|_{H_{\gamma,\alpha,\beta} \to H_{\gamma,\alpha,\beta}} \le Ch^{L+\frac{2}{3}}.$$

The remainder of this subsection is devoted to the proof of Proposition 5.1. Therefore, for $f \in H_{\gamma,\alpha,\beta}$, we estimate $Rf(z)$. Up to the end of this subsection, we assume that $\{z,\zeta\} \subset \Pi_{\gamma,\alpha}$ and that h is sufficiently small.

5.3.1. *Auxiliary lemma.* When estimating $Rf(z)$, we use

Lemma 5.2. *For* $q > 0$*, there exists* $C > 0$ *such that*

$$\sup_{\substack{\{z,\zeta\} \subset \Pi_{\gamma,\alpha} \\ \min_{k\in\mathbb{Z}} |\zeta - z - kh| \ge qh}} \left| \frac{\rho_0(\zeta)}{\rho_0(z)}\, d_0(z,\zeta) \right| \le Ch^{L+\frac{2}{3}}. \tag{5.12}$$

Proof. We proceed in several steps.

1. Proposition 4.1 and Lemma 4.4 imply that

$$\left|\frac{\rho_0(\zeta)}{\rho_0(z)}\,d_0(z,\zeta)\right| \le Ch^{L+\frac{2}{3}}\left(|\rho_1(\zeta)\rho_0(\zeta)| + \frac{|\rho_1(z)\rho_0^2(\zeta)|}{|\rho_0(z)|}\right)\left|\theta_0\left(\frac{\zeta - z - 0}{h}\right)\right|. \quad (5.13)$$

2. Recall that in $S_{0,1}$ one has $\rho_0(z)\rho_1(z) = 1$ (see (5.1)). As γ_1, $\gamma_2 \subset S_{0,1}$, one has $|\rho_0(z)||\rho_1(z)| \le C$ for $z \in \Pi_{\gamma,\alpha}$. Therefore,

$$\left|\frac{\rho_0(\zeta)}{\rho_0(z)}\,d_0(z,\zeta)\right| \le Ch^{L+\frac{2}{3}}\,(1 + e(z,\zeta))\left|\theta_0\left(\frac{\zeta - z - 0}{h}\right)\right|, \quad e(z,\zeta) = \left|\frac{\rho_0(\zeta)}{\rho_0(z)}\right|^2. \quad (5.14)$$

3. For $z \in \Pi_{\gamma,\alpha}$, we define $z_\perp \in \gamma$ so that $\operatorname{Im} z_\perp = \operatorname{Im} z$. We have

$$|e(z,\zeta)| \le C\left|\exp\left(\frac{2i}{h}\int_{z_\perp}^{\zeta_\perp} p_0(z')\,dz'\right)\right|.$$

4. On the complex plane outside a fixed neighborhood of the points $\mathbb{Z}$, we have the estimate

$$|\theta_0(z)| = |\cot(\pi z) - i| \le C\begin{cases} 1, & \operatorname{Im} z \ge 0; \\ e^{2\pi \operatorname{Im} z}, & \operatorname{Im} z \le 0. \end{cases}$$

Therefore, for ζ outside the (qh)-neighborhood of $z + h\mathbb{Z}$, we get

$$\left|\theta_0\left(\frac{\zeta - z - 0}{h}\right)\right| \le C \quad \text{and}$$

$$\left|e(z,\zeta)\,\theta_0\left(\frac{\zeta - z - 0}{h}\right)\right| \le C\begin{cases} e^{-\frac{2}{h}\operatorname{Im}\int_{z_\perp}^{\zeta_\perp} p\,dz} & \text{if } \operatorname{Im}(\zeta - z) \ge 0; \\ e^{\frac{2}{h}\operatorname{Im}\int_{\zeta_\perp}^{z_\perp}(p-\pi)\,dz} & \text{if } \operatorname{Im}(\zeta - z) \le 0. \end{cases}$$

5. As γ is a precanonical curve, we finally get

$$\left|\theta_0\left(\frac{\zeta - z - 0}{h}\right)\right| \le C, \quad \text{and} \quad \left|e(z,\zeta)\,\theta_0\left(\frac{\zeta - z - 0}{h}\right)\right| \le C. \quad (5.15)$$

This and (5.14) imply (5.12). □

5.3.2. *Estimates in the strip* $S(\operatorname{Im} z_1 + h/2, \operatorname{Im} z_2 - h/2)$. When $z \in S(\operatorname{Im} z_1 + h/2, \operatorname{Im} z_2 - h/2)$, we prove

$$|\rho_0(z)^{-1} D_0 f(z)| \le Ch^{L+\frac{2}{3}}\|f\|. \quad (5.16)$$

First, we assume that z is between the curves $\gamma + \alpha h/2$ and $\gamma + \alpha h$. Then, one can deform the integration path $\gamma(z)$ in (5.8) to γ. The distance between the poles of d_0 and γ is larger than Ch. This, (5.11) and (5.12) imply (5.16).

Next, we assume that z is either between the curves γ and $\gamma + ah/2$ or on one of them. In this case, in (5.8) we can replace the integration path γ by $\tilde\gamma$ where

- $\tilde\gamma$ is a continuous curve that connects z_1 to z_2,
- $\tilde\gamma$ coincides with $\gamma - \alpha h/2$ in the strip $\{\operatorname{Im} z_1 + h/2 \le \operatorname{Im} z \le \operatorname{Im} z_2 - h/2\}$,
- outside this strip, $\tilde\gamma$ consists of two segments of straight lines.

Reasoning as above on this new integral, we again obtain (5.16).

Let us assume now that z is to the left of γ. We note that, by the Residue theorem, the integral in (5.8) decomposes as the sum of

$$-\frac{\delta_0(z)W_1(z) - W_0(z)\delta_1(z)}{(W_0(z),\, W_1(z))}\,f(z) = O(h^{L+1+\frac{2}{3}})\,f(z)$$

and the integral defined by (5.8)–(5.9) with $\theta((\zeta - (z+0))/h)$ replaced with $\theta((\zeta - (z-0))/h)$.

This new integral for z to the left of γ is analyzed as above. This completes the proof of (5.16).

5.3.3. *Estimates in* $S(\operatorname{Im} z_1, \operatorname{Im} z_1 + h/2)$ *and* $S(\operatorname{Im} z_2 - h/2, \operatorname{Im} z_2)$. Both domains are treated similarly. So, we detail only the analysis for the first one. We prove that

$$|\rho_0(z)^{-1}(D_0 f)_\beta(z)| \leq C\, h^{L+\frac{2}{3}}\, \|f\|. \tag{5.17}$$

For z between $\gamma + \alpha h/2$ and $\gamma + \alpha h$, reasoning as in section 5.3.2, one obtains (5.16) that implies (5.17).

For z between γ and $\gamma + \alpha h/2$, by contour deformation, the integration path in (5.8) is replaced with $\tilde{\gamma}$ defined in section 5.3.2. We, thus, write $D_0 f$ as the sum of an integral, say A, over the part of $\tilde{\gamma} \cap \{\operatorname{Im}\zeta \leq z_1 + h/2\}$ and an integral, say B, over the part of $\tilde{\gamma} \cap \{\operatorname{Im} z_1 + h/2 \leq \operatorname{Im}\zeta\}$.

Reasoning as in section 5.3.2, we estimate B and obtain

$$|\rho_0(z)^{-1} B| \leq C h^{L+\frac{2}{3}} \|f\|. \tag{5.18}$$

Let us turn to A. We again use (5.13). Now, both $|z - z_1|$ and $|\zeta - z_1|$ are bounded by Ch; thus, $|\rho_0(z)/\rho_0(\zeta)| \leq C$. Furthermore, for such z, only one pole of the integrand, the pole at the point z, can approach the integration path in A; the other poles stay at a distance greater than Ch from it. Therefore, we get

$$|\rho_0(z)^{-1} A| \leq C\, h^{L+1+\frac{2}{3}}\, \|f\| \int_{\operatorname{Im} z \leq \operatorname{Im} z_1 + h/2} \frac{|d\zeta|}{|z-\zeta|\,|\zeta - z_1|^\beta},$$

where we integrate along $\tilde{\gamma}$. Changing variable $t = (\zeta - z_1)/|z - z_1|$, one checks that the last integral is bounded by $C/|z - z_1|^\beta$. Thus,

$$|\rho_0(z)^{-1} A| \leq C\, h^{L+1+\frac{2}{3}}\, \|f\|/|z - z_1|^\beta.$$

This and (5.18) yields (5.17).

We omit further details and note only that, to prove (5.17) when z is to the left of γ, we first transform the integral from (5.8) as when doing the estimations in the strip $S(\operatorname{Im} z_1 + h/2, \operatorname{Im} z_2 - h/2)$ (see the end of the section 5.3.2).

5.3.4. *Completing the proof of Proposition 5.1.* Proposition 5.1 follows from estimates (5.16) and (5.17).

5.4. **Solutions to the integral equation (5.10).** Consider the integral equation (5.10) in $H_{\gamma,\alpha,\beta}$. Proposition 5.1 and the estimate for δ_0 from (4.11) imply

Lemma 5.3. *For sufficiently small* h*, the equation* (5.10) *has a unique solution* g_0 *in* $H_{\gamma,\alpha,\beta}$. *It satisfies*

$$\|g_0(z)\| = O(h^{L+2+\frac{1}{3}}). \tag{5.19}$$

Moreover, one has

Lemma 5.4. *The solution* g_0*, constructed in Lemma 5.3 for the curve* $\gamma = \gamma_1$*, can be analytically continued to the domain* $K^{\alpha h}$*. It then satisfies (5.10) and in* $K^{\alpha h}$

$$|(z_1 - z)(z_2 - z)|^\beta \frac{|g_0(z)|}{|\rho_0(z)|} \leq C h^{L+2+\frac{1}{3}}. \tag{5.20}$$

Proof. The proof is divided into four parts.

1. As g_0 is analytic in $\Pi_{\gamma_1,\alpha}$, it suffices to continue it to the right of γ_1. The function $\zeta \to \theta_0\left(\frac{\zeta - z - 0}{h}\right)$ has all its poles in $z + 0 + h\mathbb{Z}$. Hence, for z between γ_1 and $\gamma_1 + h$, we can define $D_0 g_0$ by means of (5.8) with $\gamma(z) = \gamma_1$, and $D_0 g_0$ appears to be analytic between γ_1 and $\gamma_1 + h$.

As g_0 is analytic between γ and $\gamma + \alpha h$, to define $D_0 g_0$ for z between $\gamma_1 + \alpha h$

and $\gamma_1 + (1+\alpha)h$, we can deform the path of the integral in (5.8) to a vertical curve connecting z_1 to z_2 staying between γ_1 and $\gamma_1 + \alpha h$. Thus, (5.8) implies that $D_0 g_0$ is analytic in z between γ_1 and $\gamma_1 + \alpha h + h$. In view of equation (5.10), this implies that g_0 itself is analytic to the left of $\gamma + (\alpha+1)h$. Reasoning in this way inductively, one shows that g_0 and $D_0 g_0$ are analytic between γ and $\gamma + (\alpha+2)h$, between γ and $\gamma + (\alpha+3)h$ and so on. As a result, one sees that g_0 and $D_0 g_0$ are analytic in $K^{\alpha h}$ to the right of γ and satisfy (5.10) for all $z \in K^{\alpha h}$.
2. Clearly, g_0 is analytic in $\Pi_{\gamma_2,\alpha}$, the expression $|(z_1 - z)(z_2 - z)|^\beta |g_0(z)|$ stays bounded in $\Pi_{\gamma_2,\alpha}$ (as γ_1 and γ_2 have common ends), and g_0 satisfies equation (5.10) along $\gamma = \gamma_2$. By Lemma 5.3, for sufficiently small h, this equation has a unique solution in $H_{\gamma_2,\alpha,\beta}$ which, thus, coincides with g_0. Hence, g_0 satisfies (5.19) with the norm of $H_{\gamma_2,\alpha,\beta}$.
3. In view of the previous step, g_0 satisfies (5.20) in $\Pi_{\gamma_1,\alpha} \cup \Pi_{\gamma_2,\alpha}$. This and the maximum principle for analytic functions imply that g_0 satisfies (5.20) also in the domain bounded by γ_1 and γ_2, i.e., in K.
The proof of Lemma 5.4 is complete. □

5.5. **The solution to the difference equation.** We define Δ_0 by (5.5) in terms of g_0 constructed in section 5.4. One has

Lemma 5.5. *The function Δ_0 can be analytically continued to $K^{(1+\alpha)h}$ where it satisfies equation* (5.4).
Let $0 < c < 1+\alpha$ and $r > 0$. In $K^{ch} \cap S(\operatorname{Im} z_1 + rh, \operatorname{Im} z_2 - rh)$, one has

$$|\Delta_0(z)| \le C|\rho_0(z) h^{L+1}|. \tag{5.21}$$

Proof. By (5.20) the function $z \mapsto |(z_1 - z)(z_2 - z)|^\beta |g_0(z)|$ is bounded in $K^{\alpha h}$. For a given z, the poles of the kernel in (5.5) are contained in $z + h(\mathbb{Z} \setminus \{0\})$. Thus, the function Δ_0 is analytic in $\left(K^{\alpha h}\right)^h = K^{(1+\alpha)h}$.
By means of the Residue theorem, one checks that $H\Delta_0 = HR_0 g_0$ is equal to $g_0 + D_0 g_0$ if $z, z \pm h \in K^{(1+\alpha)h}$. As g_0 satisfies (5.10) in $K^{\alpha h}$, we obtain (5.4) if $z, z \pm h \in K^{(1+\alpha)h}$.
To prove (5.21), we estimate $R_0 g_0$ in the same way as in section 5.3.2 we estimated $D_0 f$. So, we omit further details and only note that

(1) outside (Ch)-neighborhood of the set $z + h\mathbb{Z}$, instead of (5.12) we obtain
$$\left| \frac{\rho_0(\zeta)}{\rho_0(z)}\, r_0(z,\zeta) \right| \le C h^{-\frac{4}{3}};$$
(2) on the diagonal $\{\zeta = z\}$, r_0, the kernel of R_0, is analytic whereas d_0, the kernel of D_0, has a pole. This simplifies the estimates of $(R_0 g_0)(z)$ to the left of γ_1.

□

Having constructed Δ_0, we construct a solution ψ_0 to equation (1.1) setting $\psi_0 = W_0 - \Delta_0$, see (5.4). Let $c \in (0,2)$. In view of (5.21), one has

$$\psi_0(z) = W_0(z) + O(|\rho_0(z)| h^{L+1}), \qquad z \in K^{ch} \cap S(\operatorname{Im} z_1 + rh, \operatorname{Im} z_2 - rh). \tag{5.22}$$

In view of (4.12), estimate (5.22) implies (5.2) with L replaced with $L-1$. As we could choose a larger L, this actually completes the proof of the statement of Theorem 5.1 on the solution ψ_0.

5.6. **The second solution.** Mutatis mutandis, the construction of the solution ψ_1 repeats that of ψ_0. We omit further details and mention only that, in this case,

- we set $\psi_1 = W_1 - R_1 g_1$ where R_1 is an integral operator with the kernel

$$r_1(z,\zeta) = \frac{1}{2ih}\,\frac{W_0(z)W_1(\zeta) - W_0(\zeta)W_1(z)}{(W_0(\zeta),\, W_1(\zeta))}\,\theta_1\left(\frac{\zeta - z}{h}\right), \quad \theta_1(t) = \cot(\pi t) + i;$$

- instead of (5.11), we use the norm $\|f\| = \sup_{z\in\Pi_{\gamma,\alpha}} \dfrac{|f_\beta(z)|}{|\rho_1(z)|}$.

6. Proof of the main Theorem

In this section we finally prove Theorem 1.4. We recall that in U there are three Stokes lines beginning at z_0. They are analytic curves, and the angle between any two of them at z_0 is equal to $2\pi/3$. So, possibly reducing U somewhat, we can assume that at least two of them form a vertical curve. We prove the theorem in the case where these are σ_1 and σ_2, and σ_1 goes upwards from z_0, i.e., the vector tangent to σ_1 at z_0 is directed in the upper half-plane. Mutatis mutandis, the other cases are treated in the same way. Moreover, we assume that the tangent vector to σ_0 is either directed in the lower half-plane or is parallel to the real line and directed to the left. Then the curves σ_j, $j \in \mathbb{Z}_3$, correspond to Fig. 2. The complimentary case is studied similarly.
Below we assume that h is sufficiently small.

6.1. **Two geometric lemmas.** To prove Theorem 1.4, we shall use the following two lemmas.

Lemma 6.1. *There exist two curves in $S_{1,2}$ precanonical with respect to p_2 and having common endpoints, and $\overset{\circ}{U}_1 \subset U$, a neighborhood of z_0, such that*
- *the domain K_1 bounded by the two curves is simply connected,*
- $K_1 \cap \overset{\circ}{U}_1 = S_{1,2} \cap \overset{\circ}{U}_1$.

and

Lemma 6.2. *There are exist two curves in $\sigma_2 \cup S_0 \cup \sigma_1$ precanonical with respect to p_0 and having common endpoints, and $\overset{\circ}{U}_0 \subset U$, a neighborhood of z_0, such that*
- *the domain K_0 bounded by the two curves is simply connected,*
- $K_0 \cap \overset{\circ}{U}_0 = (\sigma_2 \cup S_0 \cup \sigma_1) \cap \overset{\circ}{U}_0$.

We prove these two lemmas in section 7.1.
We define $\overset{\circ}{U} = \overset{\circ}{U}_0 \cap \overset{\circ}{U}_1$.

6.1.1. *Solution ψ_1.* We denote by $\psi_{0,0}$ and $\psi_{1,0}$ the solutions ψ_0 and ψ_1 constructed by Theorem 5.1 for the domain K_0, and consider the solution ψ_1 constructed in Theorem 5.1 for the domain K_1. In view of Corollary 5.1, in $\overset{\circ}{U}$ (possibly reduced somewhat), all the three solutions are analytic, the Wronskian of $\psi_{0,0}$ and $\psi_{1,0}$ does not vanish (see also Lemma 4.4), and one has

$$\psi_1 = a\psi_{1,0} + b\psi_{0,0}, \tag{6.1}$$

where a and b are h-periodic coefficients (see section 2). We prove

Lemma 6.3. *One can reduce $\overset{\circ}{U}$ so that for $z \in \overset{\circ}{U}$*

$$a(z) = 1 + O(h^{L+\frac{2}{3}}), \quad b(z) = O(h^{L+\frac{2}{3}}), \qquad h \to 0. \tag{6.2}$$

Proof. In $\overset{\circ}{U}$ (possibly reduced somewhat), the coefficients a and b are described by (2.3) with $\psi = \psi_1$, $f = \psi_{1,0}$ and $g = \psi_{0,0}$.

Let $\gamma_{12} = (\sigma_1 \cup \sigma_2) \cap \overset{\circ}{U}$. By Lemmas 6.1 and 6.2 one has $\gamma_{12} \subset K_0$ and $\gamma_{12} \subset K_1$. First, we fix $c \in (1, 2)$ and assume that $\{z, z+h\} \subset (\gamma_{12})^{ch}$.
In view of Lemma 4.2, one has $|\rho_1| = |\rho_2| = 1$ on γ_{12}. This and the definitions of $|\rho_1|$ and $|\rho_2|$, see section 4.2.2, imply that there exists $C > 0$ such that $|\rho_1(z)|, |\rho_2(z)| \leq C$ in $(\gamma_{12})^{ch}$.
As $(\gamma_{12})^{ch}$ is a subset of both K_0^{ch} and K_1^{ch}, by means of (5.2) and (4.10), we get

$$a = \frac{(\psi_1,\, \psi_{0,0})}{(\psi_{1,0}, \psi_{0,0})} = \frac{(W_1 + O(h^{L+\frac{4}{3}}),\, W_0 + O(h^{L+\frac{4}{3}}))}{(W_1 + O(h^{L+\frac{4}{3}}),\, W_0 + O(h^{L+\frac{4}{3}}))} = \frac{(W_1, W_0) + O(h^{L+\frac{5}{3}})}{(W_1, W_0) + O(h^{L+\frac{5}{3}})}.$$

Lemma 4.4, then, yields the asymptotic representation for a from (6.2). Reasoning similarly, we get

$$b = \frac{(\psi_{1,0}, \psi_1)}{(\psi_{1,0}, \psi_{0,0})} = \frac{(W_1, W_1) + O(h^{L+\frac{5}{3}})}{(W_1, W_0) + O(h^{L+\frac{5}{3}})} = O(h^{L+\frac{2}{3}}).$$

This is the estimate for b from (6.2).

Let c_1 and c_2 correspond to the minimal strip $S(c_1, c_2)$ containing $(\gamma_{12})^{ch}$. We proved estimates (6.2) for $a(z)$ and $b(z)$ in the case where $z, z+h \in (\gamma_{12})^{ch}$. As $c > 1$ and as a and b are h-periodic, these estimates remain valid in $S(c_1, c_2)$. This implies Lemma 6.3. □

In view of Lemma 6.1, the solution ψ_1 admits representation (5.2) with $l = 1$ in $S_{1,2} \cap \overset{\circ}{U}$. Let us prove that it admits this representation in $S_0 \cap \overset{\circ}{U}$.
In view of Lemma 6.2, the solutions $\psi_{0,0}$ and $\psi_{1,0}$ admit representations (5.2) with $l = 0$ and $l = 1$ in $S_0 \cap \overset{\circ}{U}$. Substituting (6.2) and these representations into (6.1) and using (4.10), we get for $z \in S_0 \cap \overset{\circ}{U}$

$$\begin{aligned}\psi_1(z) &= (1 + O(h^{L+\frac{2}{3}}))(W_1(z) + O(h^{L+1+\frac{1}{3}}\rho_1(z))) \\ &\qquad\qquad + O(h^{L+\frac{2}{3}})(W_0(z) + O(h^{L+1+\frac{1}{3}}\rho_0(z))) \\ &= W_1(z) + O(h^{L+1}\rho_1(z)) + O(h^{L+1}\rho_0(z)).\end{aligned}$$

In view of Lemma 4.2, in S_0 one has $|\rho_0(z)| \leq |\rho_1(z)|$. For ψ_1 in $S_0 \cap \overset{\circ}{U}$, this implies representation (5.2) with L replaced with $L - 1$. As we can increase L, we actually proved (5.2) for ψ_1 in the whole domain $\overset{\circ}{U}$.
Now, we note that

$$h^{\frac{1}{2}}|\rho_0(z)| \leq C|h^{\frac{1}{3}} w_0(h^{-\frac{2}{3}}\zeta(z))| + C|h^{\frac{1}{3}} w_0'(h^{-\frac{2}{3}}\zeta(z))|, \quad z \in U. \tag{6.3}$$

For sufficiently large values of $h^{-\frac{2}{3}}|\zeta(z)|$, this estimate follows from the definition of ρ_0 and the asymptotic formulas (1.12). For bounded $h^{-\frac{2}{3}}|\zeta(z)|$, it follows from the fact that w and w' do not have common zeros.
Estimates (5.2) and (6.3) imply (1.22) with L replaced with $L - 1$. As we can increase L, this completes the proof of the statement of Theorem 1.4 on the solution ψ_1 in the case that we consider.

6.1.2. *Solution* ψ_0. Let $\psi_{1,1}$ and $\psi_{2,1}$ be the solutions ψ_1 and ψ_2 constructed by Theorem 5.1 for the domain K_1, and let ψ_0 be the solution constructed by Theorem 5.1 for the domain K_0. For $z \in \overset{\circ}{U}$ (possibly reduced somewhat) one has

$$\psi_0 = a\psi_{1,1} + b\psi_{2,1}, \tag{6.4}$$

where a and b are h-periodic. One proves

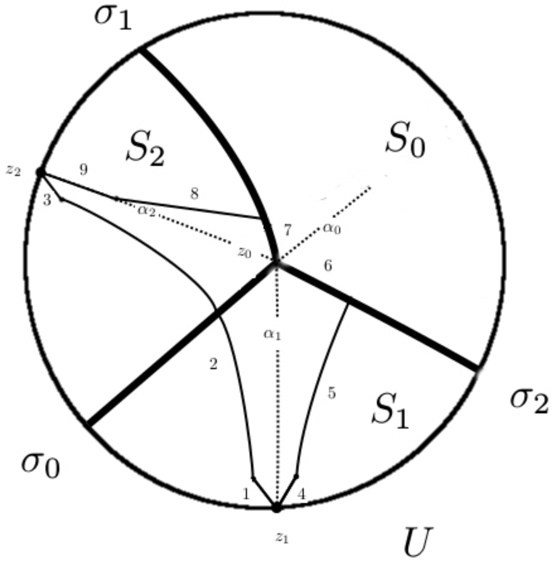

FIGURE 3. Domain K_1

Lemma 6.4. *One can reduce $\overset{\circ}{U}$ so that, for $z \in \overset{\circ}{U}$, one has*

$$a(z) = -1 + O(h^{L+\frac{2}{3}}), \quad b(z) = -1 + O(h^{L+\frac{2}{3}}), \quad h \to 0. \tag{6.5}$$

Proof. We omit details explained in the course of the proof of Lemma 6.3. We fix $c \in (1, 2)$ and assume that $z,\ z + h \in (\gamma_{12})^{ch}$. For the coefficient a from (6.4), we get

$$a = \frac{(\psi_0,\, \psi_{2,1})}{(\psi_{1,1}, \psi_{2,1})} = \frac{(W_0,\, W_2) + O(h^{L+\frac{5}{3}})}{(W_1,\, W_2) + O(h^{L+\frac{5}{3}})} = \frac{(-W_1 - W_2,\, W_2) + O(h^{L+\frac{5}{3}})}{(W_1,\, W_2) + O(h^{L+\frac{5}{3}})},$$

where, in the last step, we used relation (4.1). Continuing, we get $a = -1 + O(h^{L+\frac{2}{3}})$. Similarly one proves that $b = -1 + O(h^{L+\frac{2}{3}})$. So, (6.5) is proved for z we considered. Reasoning as in the completion of the proof of Lemma 6.3, we complete the proof of Lemma 6.4. □

By Theorem 5.1 and Lemma 6.2, the solution ψ_0 admits representation (5.2) with $l = 0$ in $(\sigma_1 \cup S_0 \cup \sigma_2) \cap \overset{\circ}{U}$. Estimates (6.5) and (4.10) imply that in $S_{1,2} \cap \overset{\circ}{U}$ one has

$$\psi_0 = -W_1 - W_2 + O((|\rho_1| + |\rho_2|)h^{L+1}) = W_0 + O((|\rho_1| + |\rho_2|)h^{L+1}).$$

In view of Lemma 4.2 and the definitions of $|\rho_j|$, in $S_{1,2}$ one has $|\rho_1| + |\rho_2| \le C|\rho_0|$ which yields (5.2) with $l = 0$ in $S_2 \cap \overset{\circ}{U}$. Reasoning as in the completion of section 6.1.1, we complete the proof of Theorem 1.4 for ψ_0 in the case that we consider.

6.1.3. *Solution ψ_2.* One proves the main theorem for ψ_2 using the same techniques as for ψ_0 and ψ_1. So, we omit the proof and note only that in S_0 one represents ψ_2 as a linear combination of $\psi_{1,0}$ and $\psi_{0,0}$, and computes the coefficients in this linear combination as in the case of ψ_0.

7. Proof of the geometric lemmas

7.1. **Proof of Lemma 6.1.** This is done in several steps.
Below, all the precanonical lines are precanonical with respect to the branch p_2 of the complex momentum. We recall that p_2 is defined and analytic in the domain U_2 and continuous up to its boundary.

7.1.1. *AntiStokes lines.* We recall that the Stokes lines σ_j are defined by (4.4). The *AntiStokes lines*, $(\alpha_j)_{j\in\mathbb{Z}_3}$, are defined as

$$\alpha_j := \zeta^{-1}(V \cap e^{-2\pi ij/3}\,[0,+\infty)). \tag{7.1}$$

For $j \in \mathbb{Z}_3$, $\sigma_j \cap \alpha_j = \{z_0\}$ and the curve $\sigma_j \cup \alpha_j$ is analytic. The angles between any two of the AntiStokes lines at z_0 equal $2\pi/3$.
In the case we study, the Stokes and AntiStokes lines are pictured in Fig. 3; the AntiStokes lines are represented by dotted lines. In particular, α_2 goes up from z_0, and α_1 goes down from z_0.
Reducing U if necessary, we assume that the AntiStokes lines α_1 and α_2 are vertical in U. As in Fig. 3, let z_1 be the lower end of α_1 and z_2 the upper end of α_2.
One has

Lemma 7.1. *Along the AntiStokes lines α_0, α_1 and α_2, one has $\mathrm{Re}\int_{z_0}^{z} p_2\,dz = 0$. The vector field $z \mapsto v(z) = \nabla \mathrm{Im}\int_{z_0}^{z} p_2\,dz$ vanishes only at $z = z_0$. The AntiStokes lines are tangent to this vector field at $z \neq z_0$. As z moves away from z_0, $\mathrm{Im}\int_{z_0}^{z} p_2\,dz$ monotonously increases along α_2 and monotonously decreases along α_1 and α_0.*

Proof. The statement on $\mathrm{Re}\int_{z_0}^{z} p_2\,dz$ follows directly from the definitions of the function ζ and of the AntiStokes lines. We note that $\|v(z)\| = |p_2(z)|$, and that $p_2(z)$ vanishes only at z_0 (the complex momentum vanishes modulo π only at turning points and z_0 is the only turning point in U). Therefore, the vector field v vanishes only at $z = z_0$. The statement on $\mathrm{Re}\int_{z_0}^{z} p_2\,dz$ and the Cauchy-Riemann equations imply that the AntiStokes lines are tangent to the vector field v where it does not vanish. This and the first two points of Lemma 4.2 imply the statements of Lemma 7.1 on $\mathrm{Im}\int_{z_0}^{z} p_2\,dz$. □

We also use

Lemma 7.2. *There exists $\tilde U \subset U$, a neighborhood of z_0, such that the lines $\alpha_1 \cap \tilde U$ and $\alpha_2 \cap \tilde U$ are precanonical.*
Let us parametrize $(\alpha_1 \cup \alpha_2) \cap \tilde U$ by $y = \mathrm{Im}\,z$, $z = z(y) = x(y) + iy$. Then, if $y \neq \mathrm{Im}\,z_0$, one has

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z(y)} p_2(z)\,dz > 0, \tag{7.2}$$

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z(y)} (p_2(z) - \pi)\,dz < 0. \tag{7.3}$$

Proof. As α_1 and α_2 are vertical, inequality (7.2) follows from Lemma 7.1. Furthermore, one has

$$\frac{d}{dy}\mathrm{Im}\int_{z_0}^{z(y)} (p_2 - \pi)\,dz = \mathrm{Im}\,(z'(y)p_2(z)) - \pi.$$

Therefore, as $p_2(z_0) = 0$, reducing U somewhat if necessary, we ensure (7.3).
Since $\alpha_1 \cup \alpha_2$ is vertical, (7.2) and (7.3) imply that the curve $\alpha_1 \cup \alpha_2$ is precanonical. □

Below, we assume that $\tilde U = U$ (if necessary we reduce U somewhat).

7.1.2. *Precanonical line* γ_1. We now construct a precanonical line $\gamma_1 \subset S_{1,2}$. It consists of three segments 1,2 and 3 shown in Fig. 3. Let us describe them.
The segments 1 and 3. To construct these segments, we use

Lemma 7.3. *Let γ be a compact vertical C^1-curve parameterized by $y = \operatorname{Im} z$, $z = z(y) = x(y) + iy$. We assume that (7.2)– (7.3) hold along γ. Then, any compact C^1-curve sufficiently close in C^1-topology to γ is precanonical.*

This statement immediately follows from the definition of the precanonical curves.
The segment 1. It is a segment of a compact precanonical C^1-curve $c_1 \subset S_{1,2}$ that begins at z_1 and above z_1 goes to the left of α_1. When choosing c_1, we take an internal point of α_1 as $\tilde{z}_1$, and, as c_1, we take a C^1-curve close enough in C^1-topology to α_1 between z_1 and $\tilde{z}_1$. Lemmas 7.2 and 7.3 guarantee that c_1 is a precanonical line.
The segment 3. Similarly, the segment 3 is a segment of a compact precanonical C^1-curve $c_3 \subset S_{1,2}$, having the upper end at z_2 and going to the left of α_2 below the point z_2.
The segment 2. We note that $\alpha_1 \cup \alpha_2$ is a level curve of the harmonic function $z \to \operatorname{Re} \int_{z_0}^{z} p_2(z)\, dz$ in $S_{1,2}$. The segment 2 is a segment of another level curve c_2 of this function in $S_{1,2}$. This curve is located to the left of $\alpha_1 \cup \alpha_2$. It does not contain the point z_0, the only point in $S_{1,2}$ where p_2 vanishes. So, c_2 is smooth. We choose c_2 sufficiently close to $\alpha_1 \cup \alpha_2$ to ensure that

- c_2 is vertical (as α_1 and α_2 are);
- one has (7.2) along c_2 (the vector field $\nabla \operatorname{Im} \int_{z_0}^{z} p_2(z)\, dz$ does not vanish along c_2 and is tangent to c_2);
- (7.3) holds along c_2 (as it holds along $\alpha_1 \cup \alpha_2$);
- c_2 intersects both c_1 and c_3.

Clearly, c_2 is precanonical.
The curve γ_1. The segment 1 is the segment of c_1 between z_1 and the point of intersection of c_1 and c_2, the segment 2 is the segment of c_2 between the segment 1 and the point of intersection of c_2 and c_3, and the segment 3 is the segment of c_3 connecting the segment 2 with z_2. Clearly, the curve γ_1 made of segments 1–3 is precanonical.

7.1.3. *The sign of* $\operatorname{Im} p_2$ *in* S_2. The only place where we use our assumption on the direction of the tangent vector to σ_0 at z_0 is the proof of

Lemma 7.4. *Both in S_2 between the curves α_2 and σ_1 and on these curves, near z_0 one has $\operatorname{Im} p_2(z) < 0$ if $z \neq z_0$.*

Proof. Below we assume that either z is in S_2 between the curves α_2 and σ_1 or on one of these curves. In view of (1.6), we can write

$$p_2(z) = k_1 \tau (1 + O(\tau)), \qquad \int_{z_0}^{z} p_2(z)\, dz = \frac{2}{3} k_1 \tau^3 (1 + O(\tau)), \qquad z \to z_0, \tag{7.4}$$

where $k_1 \neq 0$ and τ is the branch of $\sqrt{z - z_0}$ analytic in U_2 and positive if $z > z_0$. Let $0 < \theta_2 < \pi$ be the angle at z_0 between the line $\{z \geq z_0\}$ and the curve α_2 . Note that the angle between σ_0 and α_2 equals $\pi/3$. Therefore, as the tangent vector to σ_0 at z_0 is either directed downwards or parallel to the real line and directed to the left, one has $2\pi/3 \leq \theta_2 < \pi$.
In view of Lemma 7.1, along α_2, $\operatorname{Re} \int_{z_0}^{z} p_2 dz = 0$ and $\operatorname{Im} \int_{z_0}^{z} p_2 dz$ is monotonously increasing. This and the second formula in (7.4) imply that

$$\arg k_1 + \frac{3}{2} \theta_2 = \frac{\pi}{2} \operatorname{mod} 2\pi. \tag{7.5}$$

Let $z - z_0 = |z - z_0| e^{i\theta}$. Using (7.5) and the first formula in (7.4), we get near z_0

$$\frac{\operatorname{Im} p_2(z)}{|p_2(z)|} = \sin\left(\arg k_1 + \frac{\theta}{2} + o(1)\right) = \cos\left(\theta_2 - \frac{\theta-\theta_2}{2} + o(1)\right). \tag{7.6}$$

Now, we note that, for z we consider, near z_0 one has $\theta_2 - \pi/3 + o(1) \le \theta \le \theta_2 + o(1)$. Therefore, for z sufficiently close to z_0, one has

$$\frac{2\pi}{3} + o(1) \le \theta_2 + o(1) \le \theta_2 - \frac{\theta-\theta_2}{2} \le \theta_2 + \frac{\pi}{6} + o(1) < \frac{7\pi}{6}.$$

This and (7.6) implies the statement of Lemma 7.4. □

7.1.4. *Precanonical line γ_2.* The precanonical line γ_2 is located in $S_{1,2}$ and consists of six segments 4–9 shown in Fig. 3. Let us describe them.

The segments 4-5-6-7. The segment 4 is a segment of a compact precanonical C^1-curve $c_4 \subset S_{1,2}$. This curve begins at z_1 and above z_1 goes to the right of α_1. It is constructed as the curve c_1 containing the segment 1.

The segment 5 is a segment of a level curve c_5 of the function $z \to \operatorname{Re} \int_{z_0}^z p_2(z)\,dz$ in $S_{1,2}$. The construction of c_5 is similar to one of c_2. The curve c_5 is located to the right of α_1. We choose c_5 sufficiently close to α_1. Then, c_5 is a precanonical curve and intersects both c_4 and the Stokes line σ_2.

The segment 4 is the segment of c_4 between z_1 and the point of intersection of c_4 and c_5. The segment 5 connects this point with a point of σ_2.

We prove

Lemma 7.5. *Let γ be a vertical curve, let $a \in \gamma$ and let p be a branch of the complex momentum continuous on γ. If, on γ, either* $\operatorname{Im} \int_a^z p(z)\,dz = 0$ *or* $\operatorname{Im} \int_a^z (p(z) - \pi)\,dz = 0$, *then γ is precanonical with respect to p.*

Proof. Assume that $\operatorname{Im} \int_a^z p(z)\,dz = 0$ on γ. Then, $z \mapsto \operatorname{Im} \int_a^z (p(z) - \pi)\,dz = -\pi \operatorname{Im}(z-a)$ is decreasing along γ when $\operatorname{Im} z$ increases. Thus, γ is precanonical. If $\operatorname{Im} \int_a^z (p(z) - \pi)\,dz = 0$, then $z \mapsto \operatorname{Im} \int_a^z p(z)\,dz = \operatorname{Im} \int_a^z \pi\,dz = \operatorname{Im}(z-a)$ is increasing along γ when $\operatorname{Im} z$ increases. Thus, γ is precanonical. □

The segment 6 is the segment of $c_6 = \sigma_2$ between the upper end of the segment 5 and the point z_0. The segment 7 is the segment of $c_7 = \sigma_1$ between z_0 and an internal point a of σ_1. We describe this point later. Lemma 7.5 implies that the segments 6 and 7 are precanonical.

Segment 8. This segment is a segment of c_8, the level curve $\gamma(a)$ of the harmonic function $z \to \operatorname{Im} \int_{z_0}^z (p_2(z) - \pi)\,dz$ that contains $a \in \sigma_1$. To choose the segment 8, we check

Lemma 7.6. *If $a \in \sigma_1 \setminus \{z_0\}$ is sufficiently close to z_0, then $\gamma(a)$ intersects σ_1 transversally at a, enters at a in S_2 going upwards, intersects α_2 and, up to intersection and at the intersection point, remains vertical.*

Proof of Lemma 7.6. Below, we identify the vectors on $\mathbb{R}^2$ with the complex numbers in the standard way, and the bar denotes complex conjugation. The Stokes line σ_1 is tangent to the vector field $z \mapsto v_0(z) = \overline{p_2(z)}$ at $z \ne z_0$ ($p_2(z_0) = 0$). The curve $\gamma(a)$ is tangent to the vector field $z \mapsto v_\pi(z) = \overline{p_2(z)} - \pi$.

Let $a \in \sigma_1 \setminus z_0$ be sufficiently close to the point z_0. In view of Lemma 7.4, $\operatorname{Im} p_2(a) < 0$. Therefore, $\gamma(a)$ is vertical at a. Moreover, both the vectors $v_0(a)$ and $v_\pi(a)$ are directed upwards and $v_\pi(a)$ is directed to the left of $v_0(a)$. Therefore, at a, the curve $\gamma(a)$ intersects σ_1 transversally and enters S_2 going upwards.

Furthemore, in view of Lemma 7.4, as long as $\gamma(a)$ stays in S_2 near z_0 between the curves α_2 and σ_1 or on them, it remains vertical.

To complete the proof, it suffices to show that if a is sufficiently close to z_0, then

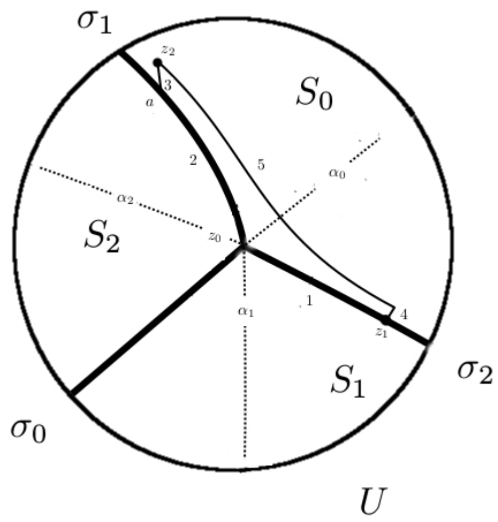

FIGURE 4. Domain K_0

$\gamma(a)$ intersects α_2 remaining vertical. Therefore, we note that $v_\pi(z_0) = -\pi$. So, at z_0 the vector tangent to $\gamma(z_0)$ is parallel to $\mathbb{R}$, and the curve $\gamma(z_0)$ intersects the analytic curve $\alpha_2 \cup \sigma_2$ transversally. Depending continuously on a, $\gamma(a)$ intersects this curve also for all a sufficiently close to z_0. But, if $\operatorname{Im} a > \operatorname{Im} z_0$ and a is sufficiently close to z_0, the curve $\gamma(a)$ goes upward from a. Therefore, for a sufficiently close to z_0, the curve $\gamma(a)$ intersects α_2 still remaining vertical. This completes the proof of Lemma 7.6. □

The segments 8 and 9. We choose the point a, the end of the segment 7 and the beginning of the segment 8, so that $c_8 = \gamma(a)$ intersects α_2 as described in Lemma 7.6. The end of the segment 8 is the point of intersection of c_8 and α_2. By Lemma 7.5, the segment 8 is precanonical. The segment 9 is the segment of α_2 connecting the upper end of the segment 8 to the point z_2. It precanonical by Lemma 7.2.

The domain K_1 bounded by γ_1 and γ_2 is the one described in Lemma 6.1, the proof of which is complete.

7.2. **Proof of Lemma 6.2.** The proof uses the same techniques as the proof of Lemma 6.1. Therefore, we omit standard details. The construction of the curves γ_1 and γ_2 bounding the domain K_0 from Lemma 6.2 is illustrated by Fig. 4. Below, all the precanonical lines are precanonical with respect to p_0.

7.2.1. *Curve* γ_1. This curve consists of segments 1–3. Let us describe them.
We take an internal point of σ_2 as z_1, and we fix a, an internal point of σ_1. The segment 1 is the segment of σ_2 between z_1 and z_0, and the segment 2 is the segment of σ_1 between z_0 and a.
To describe the segment 3, we consider $\gamma_0(a)$, the curve in $\sigma_2 \cup S_0 \cup \sigma_1$ described by the equation $\operatorname{Im} \int_a^z (p_0(z) - \pi)\, dz = 0$. We suppose that a is sufficiently close to z_0. Then, $\gamma_0(a)$ intersects σ_1 at a transversally, enters in S_0 going upwards and is vertical in a neighborhood a (To prove this, one uses the observation that near z_0 on σ_1 one has $\operatorname{Im} p_0(z) > 0$. The proof of this observation is similar to one of Lemma 7.4.) The segment 3 is a segment of $\gamma_0(a)$ connecting in this neighborhood a to a point $z_2 \in S_2$. We choose z_2 later.

Lemma 7.5 imply that the segments 1–3 are precanonical.
The points z_1 and z_2 are the ends of γ_1.

7.2.2. *Curve* γ_2. This curve consists of two segments, segments 4 and 5.
The segment 4 is a segment of c_4, a level curve of the function $z \to \mathrm{Re} \int_{z_1}^{z} p_0(z)\,dz$ in $S_0 \cup \sigma_2$ that contains the point z_1. The curve c_4 is orthogonal to σ_2 at z_1.
Let us note that, under our assumptions on σ_0 and σ_2 (see the very beginning of section 6), the angle at z_0 between σ_2 and the horizontal line $\{z \geq z_0\}$ belongs to $(0, \pi/3)$. Possibly reducing U somewhat, we assume that, at any point $\zeta \in \sigma_2$, the angle between σ_2 and the line $\{z \geq \zeta\}$ belongs to $(0, \pi/2)$. Then, c_4 is vertical at least in a neighborhood of the point z_1 and goes upward from z_1 into S_0.
The segment 5 is a segment of a level curve c_5 of the function $z \to \mathrm{Im} \int_{z_0}^{z} p_0(z)\,dz$ in S_0. It is located to the right of $\sigma_2 \cup \sigma_1$ (which is also a level curve of this function). We choose the curve c_5 sufficiently close to $\sigma_2 \cup \sigma_1$. Then it is vertical, intersects $\gamma_0(a)$ and c_4, and the segments of these curves between $\sigma_2 \cup \sigma_1$ and the intersection points are vertical.
The point z_2 is the point of intersection of $\gamma_0(a)$ and c_5. The segment 4 is the segment of c_4 between $\sigma_2 \cup \sigma_1$ and c_5, and the segment 5 is the segment of c_5 connecting c_4 to z_2.
The segment 5 is precanonical in view of Lemma 7.5. Arguing as when proving Lemma 7.2 and reducing somewhat U if necessary, we check that the segment 4 is precanonical.

The domain K_0 bounded by the curves γ_1 and γ_2, is the one described in Lemma 6.2. Its proof is complete.

References

[1] V. Babich, M. Lyalinov and V. Grikurov, *Diffraction theory: the Sommerfeld-Malyuzhinets technique.* Oxford, Alpha Science, 2008.

[2] V. Buslaev and A. Fedotov. The monodromization and Harper equation. *Séminaires Équations aux Dérivées Partielles, 1993–1994*, XXI:1-23, École Polytechnique, Palaiseau, 1994.

[3] V. Buslaev and A. Fedotov. Complex WKB method for Harper equation. *St. Petersburg Math. J.*, 6(3):495-517, 1995.

[4] N. Bleistein. Uniform asymptotic expansions of integrals with many nearby stationary points and algebraic singularities. *J. Math. Mech.*, 17:533-559, 1967.

[5] O. Costin and R. Costin. Rigorous WKB for finite-order linear recurrence relations with smooth coefficients. *SIAM J. math. an.*, 27:110-134, 1996.

[6] S. Yu. Dobrokhotov, A. V. Tsvetkova. On Lagrangian manifolds related to asymptotics of Hermite polynomials. *Math notes*, 110(6), to appear.

[7] A. Fedoryuk. *Asymptotic Analysis. Linear Ordinary Differential Equations.* Springer-Verlag, Berlin, Heidelberg, 2009.

[8] A. Fedotov. Monodromization method in the theory of almost-periodic equations. *St. Petersburg Mathematical Journal*, 2014, 25:303-325, 2014.

[9] A. Fedotov and F. Klopp. A complex WKB method for adiabatic problems. *Asymptotic analysis*, 27:219-264, 2001.

[10] A. Fedotov and F. Klopp. Anderson transitions for a family of almost periodic Schrödinger equations in the adiabatic case. *Communications in Mathematical Physics*, 227:1-92, 2002.

[11] A. Fedotov and F. Klopp. On the absolutely continuous spectrum of an one-dimensional quasi-periodic Schrödinger operator in adiabatic limit. *Transactions of AMS*, 357:4481-4516, 2005.

[12] A. Fedotov and F. Klopp Difference equations, uniform quasiclassical asymptotics and Airy functions *Proceedings of the conference "Days on Diffraction 2018", IEEE, St. Petersburg, 2018*, 98-101, to appear in 2018.

[13] A. Fedotov and E. Shchetka. The complex WKB method for difference equations in bounded domains. *J. of Math. Sciences*, 438:236-254, 2015.

[14] A. Fedotov and E. Shchetka. A Complex WKB Method for Difference Equations in Bounded Domains. *Journal of Mathematical Sciences(New York)*, 224:157-169, 2017.

[15] A. Fedotov and E. Shchetka. Complex WKB method for the difference Schrödinger equation with the potential being a trigonometric polynomial. *St. Petersburg Mathematical Journal*, 29:363-381, 2018.

[16] J. S. Geronimo, O. Bruno and W. Van Assche. WKB and turning point theory for second-order difference equations. *Operator theory: advances and app.*,69:269-301, 1992.

[17] J. P. Guillement, B. Helffer and P. Treton. Walk inside Hofstadter's butterfly. *J. Phys. France,* 50:2019-2058, 1989.

[18] F. W. J. Olver. *Asymptotics and Special Functions.* Academic Press, New York, 1974.

[19] G. A. Tsyganov. Asymptotic behavior of the solution of a linear difference system with small difference in the presence of a turning point. *Diff. equations (in Russian)*, 10:1312-1321, 1974.

[20] A. B. Vasilieva, E. Yuldashev. Linear difference systems with small lag. *Diff. equations (in Russian)*, 6:2267-2269, 1970.

[21] W. Wasow. Asymptotic expansions for ordinary differential equations. Dover Publications, New York, 1987.

[22] R. Wong. *Asymptotic approximations of integrals.* SIAM, Philadelphia, 2001.

[23] R.Wong, Z. Wang. Asymptotic expansions for second-order linear difference equations with a turning point. *Numerische Mathematik*, 94: 147-194, 2003.

(Alexander Fedotov) St. Petersburg State University, 7/9 Universitetskaya nab., St.Petersburg, 199034, Russia

Email address: `a.fedotov@spbu.ru`

(Frédéric Klopp) Sorbonne Université, Université Paris Diderot, CNRS, Institut de Mathématiques de Jussieu - Paris Rive Gauche , F-75005, Paris, France

Email address: `frederic.klopp@imj-prg.fr`