SEMICLASSICAL RESOLVENT ESTIMATES FOR BOUNDED POTENTIALS

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Abstract. We study the cut-off resolvent of semiclassical Schrödinger operators on $\mathbb{R}^d$ with bounded compactly supported potentials V. We prove that for real energies λ^2 in a compact interval in $\mathbb{R}_+$ and for any smooth cut-off function χ supported in a ball near the support of the potential V, for some constant $C>0$, one has

$$\|\chi(-h^2\Delta + V - \lambda^2)^{-1}\chi\|_{L^2\to H^1} \leq C\,\mathrm{e}^{Ch^{-4/3}\log\frac{1}{h}}.$$

This bound shows in particular an upper bound on the imaginary parts of the resonances λ, defined as a pole of the meromorphic continuation of the resolvent $(-h^2\Delta+V-\lambda^2)^{-1}$ as an operator $L^2_{\mathrm{comp}} \to H^2_{\mathrm{loc}}$: any resonance λ with real part in a compact interval away from 0 has imaginary part at most

$$\operatorname{Im}\lambda \leq -C^{-1}\,\mathrm{e}^{Ch^{-4/3}\log\frac{1}{h}}.$$

This is related to a conjecture by Landis: The principal Carleman estimate in our proof provides as well a lower bound on the decay rate of L^2 solutions u to $-\Delta u = Vu$ with $0 \not\equiv V \in L^\infty(\mathbb{R}^d)$. We show that there exist a constant $M>0$ such that for any such u, for $R>0$ sufficiently large, one has

$$\int_{B(0,R+1)\setminus\overline{B(0,R)}} |u(x)|^2 dx \geq M^{-1}R^{-4/3}\mathrm{e}^{-M\|V\|_\infty^{2/3}R^{4/3}}\|u\|_2^2.$$

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1. Introduction

In quantum mechanics the study of scattering systems naturally leads to the study of *quantum resonances* or *scattering poles*, which can be defined as the complex-valued poles of the meromorphic continuation of the scattering matrix or of the resolvent of the Hamiltonian into the "nonphysical sheet" of the complex plane. They can also be seen as a generalization of eigenvalues of a bounded system in which energy can scatter to infinity. A typical associated resonance state has then a rate of oscillation and a rate of decay or "inverse life-time" which can be associated to the imaginary part of the resonance. In wave scattering for instance, one can describe the long-time dynamics of a wave, scattered on an obstacle or a potential, via the resonances and the associated resonant states. It is then the resonances closest to real axis, i.e. those with the longest "life-time", whose contribution in the scattered wave "survives" the longest. Therefore, the study of the resonances close to the real axis is in some sense the most pertinent one.

We consider the semiclassical Schrödinger operator on $\mathbb{R}^d$

$$P_V \overset{\text{def}}{=} -h^2\Delta + V \tag{1.1}$$

where $h \in (0,1]$ denotes the semiclassical parameter and the potential $V \in L^\infty_{\text{comp}}(\mathbb{R}^d;\mathbb{R})$ is real-valued with compact support. The potential $V(x) = V(x;h)$ may depend on $h > 0$. However, in this case we suppose that

$$\|V\|_\infty = C_V < +\infty \tag{1.2}$$

and that the support of V is contained in the ball $B(0,R_0) \Subset \mathbb{R}^d$ of radius $R_0 > 0$, with both constants $C_V > 0$ and $R_0 > 0$ independent of $h > 0$.

1.1. **Resolvent bounds.** We prove the following resolvent estimate:

Theorem 1. *Let I be a compact interval in $\mathbb{R}\backslash\{0\}$. Let $R > R_0$ and assume that the dimension $d \geq 2$. Then there exists constants $C > 0$ and $h_0 \in (0,1]$, so that for all $0 < h \leq h_0$, all $v \in L^2_{comp}(B(0,R))$ and all $\lambda \in I$*

$$\|(P_V - \lambda^2)^{-1}v\|_{H^1(B(0,R))} \leq C\,\mathrm{e}^{Ch^{-4/3}\log\frac{1}{h}}\|v\|_2. \tag{1.3}$$

In dimension $d = 1$ a stronger result is known: there we have that

$$\|(P_V - \lambda^2)^{-1}v\|_{H^1(B(0,R))} \leq C\,\mathrm{e}^{Ch^{-1}}\|v\|_2, \tag{1.4}$$

see for instance the proof in [DZ, Theorem 2.29]. From our proof of Theorem 1 in dimension $d \geq 2$ we get actually that the statement holds when we replace $H^1(B(0,R))$ on the left hand side of (1.3) by $H^1(B(0,Rh^{-1/3})$ for any $R > 0$.

Equivalently, we can formulate the statement of Theorem 1 as an estimate on the cut-off resolvent. More precisely, we have for any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ near $B(0,R_0)$ that there exist constants $C > 0$ and $h_0 \in (0,1]$ such that

$$\|\chi(P_V - \lambda^2)^{-1}\chi\|_{L^2\to H^1} \leq C\,\mathrm{e}^{Ch^{-4/3}\log\frac{1}{h}}. \tag{1.5}$$

Shapiro [Sha18] obtained independently from our work a quantitative limiting absorption principle for P_V, with $V \in L^\infty_{\text{comp}}(\mathbb{R}^d;\mathbb{R})$, in dimension $d \geq 1$. Shapiro proved that for fixed positive energy $E > 0$ and $s > 1/2$ one has for $h > 0$ small enough and any $\varepsilon > 0$ that

$$\|\langle|x|\rangle^{-s}(P_V - E - i\varepsilon)^{-1}\langle|x|\rangle^{-s}\|_{L^2\to H^2} \leq \mathrm{e}^{Ch^{-4/3}\log h^{-1}}, \tag{1.6}$$

for some constant $C > 0$ depending only on the L^∞ norm of V, the energy E, the dimension d and s.

For any other resolvent estimates so far, one assumed at least that not only V but also the radial derivative $\partial_r V$ are bounded: Datchev [Dat14] proved a quantitative limiting absorption principal in dimension $d \neq 2$ for L^∞ potentials V with radial derivative $\partial_r V \in L^\infty$ satisfying the decay conditions $V \leq \langle r\rangle^{-\delta_0}$ and $\partial_r V \leq \langle r\rangle^{-1-\delta_0}$, i.e.

$$\|\langle|x|\rangle^{-s}(P_V - E - i\varepsilon)^{-1}\langle|x|\rangle^{-s}\|_{L^2\to L^2} \leq \mathrm{e}^{C_1 h^{-1}}, \tag{1.7}$$

for $E > 0$, any $s > 1/2$, $h > 0$ small enough and any $\varepsilon > 0$. In dimension $d = 2$ Shapiro [Sha16] proved (1.7) replacing the above assumptions on $\partial_r V$ with $\nabla V \in L^\infty$ and $|\nabla V| \leq \langle r\rangle^{-1-\delta_0}$. Vodev [Vod14] proved a bound similar to (1.7) for potentials satisfying the decay conditions $\sup_{\mathbb{R}^d}\langle x\rangle^{1+\delta}|V(x,h)| \leq Ch^\nu$ and $\partial_r V \leq Ch^\nu\langle r\rangle^{-1-\delta}$ for some constants $C,\nu,\delta > 0$. Dyatlov and Zworski [DZ] simplified Datchev's proof for $V, \partial_r V \in L^\infty_{\text{comp}}$ in dimension $d \neq 2$ and showed

$$\|\chi(P_V - \lambda^2)^{-1}\chi\|_{L^2\to L^2} \leq C_0\,\mathrm{e}^{C_1 h^{-1}}. \tag{1.8}$$

Similar results were proven for various cases of short-range and long-range perturbations of the Laplacian $-h^2\Delta$ under stronger regularity assumptions. Burq [Bur98, Bur02] proved (1.8) for smooth V decaying sufficiently quickly near infinity and on domains of type $\mathbb{R}^d\backslash\overline{O}$

for some compact obstacle O with smooth boundary. Different proofs of Burq's theorem, providing some simplifications and extensions were given by Vodev [Vod00] and Sjöstrand [Sjö02]. Moreover, Cardoso and Vodev [CV02] provide a version of Burq's theorem on a class of infinite volume Riemannian manifolds with cusps.

1.2. **Resonance free regions.** As a consequence of Theorem 1 we get that there is a resonance free region below the real axis away from 0.

There are various ways of defining *resonances* of a quantum Hamiltonian, see for instance [DZ] for an overview. One way is to define them as the poles of the a meromorphic continuation of the resolvent through the essential spectrum. More precisely, we have the following well-known result [Sjö02, Proposition 2.1] and [DZ, Theorem 3.6]:

Theorem 2. *The meromorphic family of operators*

$$(P_V - \lambda^2)^{-1}: \ L^2_{\mathrm{comp}}(\mathbb{R}^d) \longrightarrow H^2_{\mathrm{loc}}(\mathbb{R}^d), \quad \operatorname{Im}\lambda > 0,$$

has a meromorphic extension from the upper half-plane $\operatorname{Im}\lambda > 0$ *to*

(1) $\lambda \in \mathbb{C}\backslash\{0\}$, *when* $d = 1$,
(2) $\lambda \in \mathbb{C}$, *when* $d \geq 3$ *is odd,*
(3) λ *in the logarithmic covering space of* $\mathbb{C}\backslash\{0\}$, *when* $d \geq 2$ *is even.*

The *resonances* of P_V are then defined as the poles of this extension with possibly the exception of the L^2 eigenvalues of P_V situated on the imaginary axis $i[0, +\infty)$. See Section 2 below for more details.

We prove the following

Theorem 3. *Let* I *be a compact interval in* $\mathbb{R}\backslash\{0\}$ *and suppose that* $d \geq 2$, *then there exists constants* $C > 0$ *and* $h_0 \in (0, 1]$ *such that for* $0 < h \leq h_0$ *there are no resonances of* P_V *in the set of* $\lambda \in \mathbb{C}$ *with*

$$\operatorname{Re}\lambda \in I, \quad \operatorname{Im}\lambda \geq -C^{-1}\,\mathrm{e}^{-Ch^{-4/3}\log\frac{1}{h}}.$$

In the case of dimension one $d = 1$ we have a stronger result: there exist constants $C_0, C_1 > 0$ and $h_0 \in (0, 1]$ such that for $0 < h \leq h_0$ there are no resonances of P_V in the set of $\lambda \in \mathbb{C}$ with

$$\operatorname{Re}\lambda \in I, \quad \operatorname{Im}\lambda \geq -C^{-1}\,\mathrm{e}^{-Ch^{-1}},$$

see for instance [DZ, Theorem 2.29]. This bound is optimal as can be seen for the study of resonances for cut off random potentials [Klo16].

1.3. **Remark on Landis' conjecture and decay of eigenfunctions.** We do not think that the bounds in (1.3) and in Theorem 3 are optimal. The $h^{-4/3}$ in the exponent comes from a Carleman estimate in a ball $B(0, R)$ with $R > R_0$ which cannot distinguish between real-valued and complex-valued potentials, see Lemma 9 below. Yet in the proof of Theorem 1 we crucially use that the potential V is assumed to be real-valued in flux norm estimate on outgoing solution in Lemma 13. We now present a slightly modified version of our main Carleman estimate:

Lemma 4. *(see Lemma 9) Let* P_V *be as in* (1.1) *with* $V \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d, \mathbb{C})$ *a bounded (possibly) complex valued potential with compact support satisfying* (1.2). *Let* $I \Subset \mathbb{R}$ *be a compact interval. Let* $R > R_0$. *Then, there exists a real-valued smooth function* $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ *and a constants* $C > 0$ *and* $h_0 \in (0, 1]$, *such that for all* $u \in \mathcal{C}^\infty_c(B(0, R))$, *all* $\lambda \in I$ *and all* $0 < h \leq h_0$

$$\int \mathrm{e}^{2\phi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \leq \frac{C}{h^{2/3}}\int \mathrm{e}^{2\phi/h^{4/3}}|(P_V - \lambda^2)u|^2 dx.$$

Here, the exponent $h^{-4/3}$ is optimal, since there we can allow for complex-valued potentials. This can be seen from a counter example to Landis conjecture [KL88] by Meshkov [Mes92]: Landis conjectured that if u is a bounded solution to $-\Delta u + Vu = 0$ in $\mathbb{R}^d$, with $\|V\|_\infty = \mathcal{O}(1)$ and $|u(x)| \leq C\exp(-c|x|^{1+})$, then $u \equiv 0$. The conjecture holds in dimension $d = 1$ which is consistent with the estimate (1.4). However, in general this conjecture was disproved by Meshkov [Mes92] who constructed a complex-valued bounded potential V and a complex-valued function u which solve $-\Delta u + Vu = 0$ in $\mathbb{R}^2$ such that $|u(x)| \leq C\exp(-c|x|^{4/3})$, $x \in \mathbb{R}^2$. Meshkov [Mes92] also proved a quantitative unique continuation principal: if u is a bounded solution to $-\Delta u + Vu = 0$ and decays faster than $\exp(-\tau|x|^{4/3})$ for any $\tau > 0$ as $|x| \to +\infty$, then necessarily $u \equiv 0$.

As a consequence of Lemma 9 we get the following lower bound on the decay of eigenfunctions of Schrödinger operator with L^∞ potentials.

Theorem 5. *Let $\|V\|_\infty \leq C_V$ with $V \not\equiv 0$. Then, there exist a constant $M > 0$ such that for any solution u to*

$$-\Delta u + Vu = 0 \quad \text{in } \mathbb{R}^d \tag{1.9}$$

satisfying $\|u\|_2 = 1$, for $R > 0$ sufficiently large

$$\int_{B(0,R,R+1)} |u(x)|^2 dx \geq M^{-1} R^{-4/3} \mathrm{e}^{-M\|V\|_\infty^{2/3} R^{4/3}}, \tag{1.10}$$

where $B(0,R,R+1) = B(0,R+1)\backslash\overline{B(0,R)} \Subset \mathbb{R}^d$ denotes the annulus of inner radius R and outer radius $R+1$ centered at 0.

If in Lemma 4 we had a weight $\exp(2\phi h^{-4/3+})$, then this would imply a corresponding lower bound $\exp(-M\|V\|_\infty^{2/3} R^{4/3-})$ in (1.10) which would be in contradiction with Meshkov's counter example to Landis' conjecture.

Let us remark that Bourgain and Kenig [JB05] proved the following more local estimate for u, a solution to (1.9),

$$\int_{B(j,1)} |u(x)|^2 dx \geq C\mathrm{e}^{-c|j|^{4/3}\log|j|}, \quad \text{for } |j| \to +\infty. \tag{1.11}$$

The lower bound (1.10) is a slight improvement over (1.11) since we loose the logarithm yet we pay the price of taking averages in a large annulus rather than in a small ball.

In a series of works by Nakić, Táufer, Tautenhahn and Veselić [NTTV15, NTTV18] a scale free unique continuation principal was proven. The authors consider an equidistributed sequence of balls $B(z_j,\delta)$ centered at $z_j \in \mathbb{R}^d$, with $j \in \mathbb{Z}^d$, and of radius $\delta \in (0, G/2)$, for some $G > 0$, so that $B(z_j,\delta) \Subset (-G/2, G/2)^d + j$. They showed that there exists a constant $N = N(d) > 0$ depending only on the dimension d, such that for all $G > 0$, all $\delta \in (0, G/2)$, all equidistributed sequences of balls as above, all $V \in L^\infty(\mathbb{R}^d;\mathbb{R})$, all $L \in G\mathbb{N}$, any energy $E_0 \geq 0$ and all $\phi \in \operatorname{ran}(\mathbf{1}_{(-\infty,E_0]}(H|_{\Lambda_L}))$

$$\|\phi\|^2_{S_\delta\cap\Lambda_L} \geq \left(\frac{\delta}{G}\right)^{N(1+G^{4/3}\|V\|_\infty^{2/3}+G\sqrt{E_0})} \|\phi\|^2_{\Lambda_L}, \tag{1.12}$$

where $S_\delta = \bigcup_{j\in(G\mathbb{Z})^d} B(z_j,\delta)$ and $\Lambda_L = (-L/2, L/2)^d$. This results extends previous results by Rojas-Molina and Veselić [RMV13], Combes, Hislop and Klopp [CHK07] and Klein [Kle23].

Tautenhahn and Veselić [TV15] extended the above result to $\psi \in \operatorname{ran}(\mathbf{1}_I(H))$, for any interval $I \subset (-\infty, E_0]$, i.e.

$$\|\psi\|^2_{S_\delta} \geq \frac{1}{2}\left(\frac{\delta}{G}\right)^{N(1+G^{4/3}(2\|V\|_\infty+E_0)^{2/3})} \|\psi\|^2_{\mathbb{R}^d}, \tag{1.13}$$

In a recent paper by Borsiv, Tautenhahn and Veselić [BTV17] a more general scale free unique continuation principal was proven for second order elliptic differential operators.

It is striking that in the above results the dependence of the exponent on the potential is only $\|V\|_\infty^{2/3}$. This agrees very well with our results (1.10). However, we do not know whether this dependence is optimal.

Meshkov's example uses fundamentally that the potential is complex-valued. Since Lemma 9 below cannot distinguish between real-valued and complex-valued potentials, we cannot improve the exponent $h^{-4/3}$ in Theorem 1 with our method in spite of the fact that, there, the potential is assumed to be real-valued which is crucial for a flux norm estimate on outgoing solution in Lemma 13 below.

Finally, let us remark that Landis' conjecture may still hold true for real-valued bounded potentials V and real-valued functions u. In fact some recent developments have been made by Davey, Kenig and Wang [DKW17] in dimension $d = 2$.

1.4. **Notation.** Let $\chi_1, \chi_2 \in \mathcal{C}_c^\infty(\mathbb{R}^d; [0,1])$. When we write $\chi_1 \prec \chi_2$, we mean that $\chi_2 \equiv 1$ in a small neighborhood of the support of χ_1. We extend this definition in the obvious way to include indicator functions of open sets.

Depending on the context we will denote by $|x|$ norm of x as a vector in some Banach space or the absolute value of x as a complex variable. Similarly, we will denote by $(x|y)$ the inner product of x, y as elements of some vector space.

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2. Meromorphic continuation of the resolvent

Let $h \in (0, 1]$ be the semiclassical parameter and consider the operator

$$P_V - \lambda^2 = -h^2\Delta + V(x; h) - \lambda^2 \quad \text{on } L^2(\mathbb{R}^d), \tag{2.1}$$

where $V = V(\cdot; h) \in L^\infty_{\mathrm{comp}}(\mathbb{R}^d, \mathbb{R})$ is a bounded real-valued compactly supported potential which may depend on the semiclassical parameter $h > 0$. We will often suppress the dependence on h and simply write V. We assume

$$\|V\|_\infty \leq C_V < +\infty \tag{2.2}$$

and that the support of V is contained in the ball $B(0, R_0) \Subset \mathbb{R}^d$ of radius $R_0 > 0$,

$$\operatorname{supp} V \subset B(0, R_0) \Subset \mathbb{R}^d, \tag{2.3}$$

where both constants $C_V > 0$ and $R_0 > 0$ are are independent of $h > 0$. Moreover, we assume that λ is in a compact interval I away from 0, i.e. we suppose that

$$\lambda \in I = [a, b] \Subset \mathbb{R}\backslash\{0\}. \tag{2.4}$$

Since the potential V is bounded and has compact support, it follows that the essential spectrum of P_V is given by $[0, +\infty)$ and that in $(-\infty, 0)$ are only isolated eigenvalues of finite multiplicity.

For let $\lambda \in \mathbb{C}$ with $\operatorname{Im} \lambda > 0$ the resolvent

$$R(\lambda) \stackrel{\text{def}}{=} (P_V - \lambda^2)^{-1} : \; L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d) \tag{2.5}$$

is a bounded linear operator. In this notation, we find the negative eigenvalues of P_V on $i\mathbb{R}_+$ given by $\lambda_j = i\mu_j$.

2.1. **Holomorphic continuation of the resolvent of the free Laplacian P_0.** Seen as an operator $L^2_{\rm comp}(\mathbb{R}^d) \to H^2_{\rm loc}(\mathbb{R}^d)$, it is possible to meromorphically continue the resolvent across the real axis. In the following we will recall some well-known results. We begin with the meromorphic continuation of the free resolvent

$$R_0(\lambda) \stackrel{\rm def}{=} (-h^2\Delta - \lambda^2)^{-1} : \; L^2(\mathbb{R}^d) \longrightarrow L^2(\mathbb{R}^d), \quad \operatorname{Im}\lambda > 0. \tag{2.6}$$

Theorem 6. *The family of operators*

$$R_0(\lambda) = (-h^2\Delta - \lambda^2)^{-1} : \; L^2_{\rm comp}(\mathbb{R}^d) \longrightarrow H^2_{\rm loc}(\mathbb{R}^d), \quad \operatorname{Im}\lambda > 0,$$

has a holomorphic extension from the upper half-plane $\operatorname{Im}\lambda > 0$ *to*

(1) $\lambda \in \mathbb{C}\backslash\{0\}$, when $d = 1$,
(2) $\lambda \in \mathbb{C}$, when $d \geq 3$ is odd,
(3) λ in the logarithmic (universal) covering space of $\mathbb{C}\backslash\{0\}$, when $d \geq 2$ is even.

Moreover, for any $\Omega \Subset \mathbb{C}\backslash\{0\}$ and any $\chi \in \mathcal{C}_c^\infty(\mathbb{R}^d)$ there exist constants $C_0, C_1 > 0$ such that for all $\lambda \in \Omega$ and $h > 0$ small enough

$$\|\chi R_0(\lambda)\chi\|_{L^2\to H^1} \leq C_0\, \mathrm{e}^{C_1/h}. \tag{2.7}$$

Proof. See for instance [Sjö02, Section 2.1], [DZ, Theorem 3.1]. □

In dimension $d = 1$ the free resolvent $R_0(\lambda)$ has a simple pole at $\lambda = 0$. It can be extended meromorphically to the entire plane $\mathbb{C}$. However, in this paper we will be interested in energies away from 0, therefore we will not need this particular result.

2.2. **Meromorphic continuation of the resolvent of P_V.** When adding a bounded potential V with compact support we can no longer extend the resolvent $R(\lambda)$ holomorphically since poles appear. More precisely, we have the following result.

Theorem 7. *The family of operators*

$$R_V(\lambda) \stackrel{\rm def}{=} (-h^2\Delta + V - \lambda^2)^{-1} : \; L^2_{\rm comp}(\mathbb{R}^d) \longrightarrow H^2_{\rm loc}(\mathbb{R}^d), \quad \operatorname{Im}\lambda > 0,$$

has a meromorphic extension from the upper half-plane $\operatorname{Im}\lambda > 0$ *to*

(1) $\lambda \in \mathbb{C}\backslash\{0\}$, when $d = 1$,
(2) $\lambda \in \mathbb{C}$, when $d \geq 3$ is odd,
(3) λ in the logarithmic (universal) covering space of $\mathbb{C}\backslash\{0\}$, when $d \geq 2$ is even.

Proof. See for instance [Sjö02, Proposition 2.1], [DZ, Theorem 3.6]. □

By definition, resonances or scattering poles of P_V are the poles of this extension with exception of the L^2 eigenvalues of P_V at $\lambda = i\mu_j$.

Let Ω be an open set in $\mathbb{C}$ or in a covering surface over some open set in $\mathbb{C}$. Then we say that a function $\Omega \ni z \mapsto P(z)$ with values in the space of linear operators $L^2_{\rm comp} \to H^2_{\rm loc}$ is *holomorphic* if $\chi_1 P(z)\chi_2$ is holomorphic as a function with values in the space of bounded linear operators $L^2 \to H^2$, for all $\chi_j \in \mathcal{C}_c^\infty$.

Correspondingly, we say that a function $\Omega \ni z \mapsto P(z)$ with values in the space of linear operators $L^2_{\rm comp} \to H^2_{\rm loc}$ is *meromorphic* if it is holomorphic on $\Omega\backslash S$, where S is a discrete subset of Ω, and such that if $z_0 \in S$, then near z_0 we have

$$P(z) = \sum_{j=1}^{N} \frac{A_j}{(z - z_0)^j} + B(z)$$

where N is finite, $B(z)$ is a holomorphic function with values in the space of linear operators $L^2_{\rm comp} \to H^2_{\rm loc}$ and $A_j : \; L^2_{\rm comp} \to H^2_{\rm loc}$ are of finite rank and continuous, in the sense that $\chi_1 A_j \chi_2$ is bounded for all $\chi_j \in \mathcal{C}_c^\infty$.

2.3. **Absence of resonances on the real axis.** We end this section by recalling the following result.

Proposition 8. *P_V has no resonances in $\mathbb{R}\backslash\{0\}$.*

Proof. The proof is based on the fact that λ_0 is a resonance of P_V if and only if there exists a solution u to $(P_V - \lambda_0^2)u = 0$ of the form $u = R_0(\lambda_0)w$ for some $w \in L^2_{\text{comp}}$. Such solutions are called outgoing solutions. This is then combined with the Paley-Wiener theorem and the Carleman estimate in Lemma 9 below to conclude the result. One can follow line by line (using Lemma 9) the standard proof which can be found for instance in [Sjö02, Theorem 2.4], [DZ, Theorem 3.30]. □

3. Resolvent estimate

In this section we will present the proof of Theorem 1. The global strategy of this proof was inspired by the approach to Carleman estimates in [Sjö02, Section 4].

3.1. **Local Carleman estimate in a ball.** From now on we suppose that $d \geq 2$ and we work under the assumption (2.2) and (2.3). The first step in the proof of Theorem 1 is to give a local Carleman estimate in a ball.

Lemma 9. *Let $I \Subset \mathbb{R}$ be a compact interval. Then, for any $R > 0$, there exists a real-valued smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and constants $C > 0$ and $h_0 \in (0, 1]$, such that for all $u \in \mathcal{C}_c^\infty(B(0, R))$, all $\lambda \in I$ and all $0 < h \leq h_0$*

$$\int e^{2\phi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \leq \frac{C}{h^{2/3}} \int e^{2\phi/h^{4/3}}|(P_V - \lambda^2)u|^2 dx.$$

Proof. The basic Carleman estimate [Sjö02, Lemma 4.2] for the semiclassical Laplacian $-\widetilde{h}^2\Delta$ is as follows: Let $R > 0$. Then, there exists a smooth real-valued function $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$, and constants $C_0 > 0$ and $\widetilde{h}_0 \in (0, 1]$ such that for all $v \in \mathcal{C}_c^\infty(B(0, R))$ and all $0 < \widetilde{h} \leq \widetilde{h}_0$

$$\widetilde{h} \int (|v|^2 + |\widetilde{h}\nabla v|^2)dx \leq C_0 \| e^{\phi/\widetilde{h}}(-\widetilde{h}^2\Delta)e^{-\phi/\widetilde{h}}u\|^2. \tag{3.1}$$

Next, let $h \in (0, 1]$ and let $C_1 > 0$ be so that $C_1 \geq 4C_0 \max\{\|V - \lambda^2\|_\infty^2, 1\}$ for all $\lambda \in I$. Define

$$P_V(\widetilde{h}) \stackrel{\text{def}}{=} -\widetilde{h}^2\Delta + \left(\frac{\widetilde{h}}{C_1}\right)^{1/2}(V - \lambda^2) \tag{3.2}$$

with $\widetilde{h} = h^{4/3}C_1^{-1/3}$. Notice that

$$P_V(\widetilde{h}) = \left(\frac{h}{C_1}\right)^{2/3}(P_V - \lambda^2). \tag{3.3}$$

Then, by (3.1), (3.2), we have that for all $0 < \widetilde{h} \leq \min\{\widetilde{h}_0, C_1^{-1/3}\}$, for all $v \in \mathcal{C}_c^\infty(B(0, R))$ and any $\lambda \in I$

$$\begin{aligned}
\|e^{\phi/\widetilde{h}}P_V(\widetilde{h})e^{-\phi/\widetilde{h}}v\|_2 &\geq \|e^{\phi/\widetilde{h}}(-\widetilde{h}^2\Delta)e^{-\phi/\widetilde{h}}v\|_2 - \frac{\widetilde{h}^{1/2}}{C_1^{1/2}}\|(V - \lambda^2)v\|_2 \\
&\geq \frac{\widetilde{h}^{1/2}}{C_0^{1/2}}\left(\int(|v|^2 + |\widetilde{h}\nabla v|^2)dx\right)^{1/2} - \frac{\widetilde{h}^{1/2}}{C_1^{1/2}}\|(V - \lambda^2)\|_\infty\|v\|_2 \\
&\geq \frac{\widetilde{h}^{1/2}}{2C_0^{1/2}}\left(\int(|v|^2 + |\widetilde{h}\nabla v|^2)dx\right)^{1/2}.
\end{aligned} \tag{3.4}$$

Setting $u = e^{\phi/\widetilde{h}}v$, we get by (3.4)

$$\int e^{2\phi/\widetilde{h}}(|u|^2 + |\widetilde{h}\nabla u|^2)dx \le \frac{C}{\widetilde{h}}\int e^{2\phi/\widetilde{h}}|P_V(\widetilde{h})u|^2 dx. \tag{3.5}$$

for some constant $C > 0$. Set $\widetilde{\phi} = C_1^{1/3}\phi$. Then, by (3.3), (3.5)

$$\int e^{2\widetilde{\phi}/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \le \frac{C}{h^{2/3}}\int e^{2\widetilde{\phi}/h^{4/3}}|(P_V - \lambda^2)u|^2 dx, \tag{3.6}$$

which concludes the proof of the Lemma. □

Next we will get rid of the assumption of compact support on u in Lemma 9. Suppose that $R_0 < R_1 < R_2$, let $u \in \mathcal{C}^\infty(B(0,R_2))$, let $1_{B(0,R_1)} \prec \chi \in \mathcal{C}_c^\infty(B(0,R_2),[0,1])$ and apply Lemma 9 to χu to get

$$\begin{aligned}\int_{B(0,R_1)} e^{2\phi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \le& \frac{C}{h^{2/3}}\int_{B(0,R_2)} e^{2\phi/h^{4/3}}|(P_V - \lambda)u|^2 dx\\ &+ \frac{C}{h^{2/3}}\int_{B(0,R_2)} e^{2\phi/h^{4/3}}|[-h^2\Delta,\chi]u|^2 dx.\end{aligned} \tag{3.7}$$

We denote by $B(0,R_1,R_2) \subset \mathbb{R}^d$ the open annulus $B(0,R_2)\backslash\overline{B(0,R_1)}$. Since

$$\begin{aligned}|[-h^2\Delta,\chi]u|^2 &= |(-h^2\Delta\chi)u - 2(h\nabla\chi|h\nabla u)|^2\\ &\le C_1(h^4|u|^2 + h^2|h\nabla u|^2),\end{aligned}$$

for some constant $C_1 > 0$, and since $\operatorname{supp}\nabla\chi \subset B(0,R_1,R_2)$, we obtain from (3.7)

$$\begin{aligned}\int_{B(0,R_1)} e^{2\phi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \le \frac{C}{h^{2/3}}&\int_{B(0,R_2)} e^{2\phi/h^{4/3}}|(P_V - \lambda)u|^2 dx\\ &+ CC_1h^{4/3}\int_{B(0,R_1,R_2)} e^{2\phi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx.\end{aligned} \tag{3.8}$$

3.2. **Carleman estimate in a shell away from the support of the potential V.** We will begin with the following

Lemma 10. *Let $w = w(r) = r^2$ for $r \ge 0$. Let I be as in (2.4) and let $\lambda \in I$. Let $A, B > 0$ be constants (to be determined later on) and set*

$$R_c = R_c(h) = \frac{\sqrt{2A}}{h^{1/3}|\lambda|}. \tag{3.9}$$

Let $h > 0$ be small enough so that $2R_0 \le R_c$. Then, there exists a smooth real-valued function $\phi_0 \in \mathcal{C}^\infty(]0,+\infty[)$ and a constant $C_{\phi_0} > 0$ (independent of $h > 0$) so that $0 \le \phi_0'|_{[R_0,+\infty[} \le C_{\phi_0}$ and

$$\phi_0'(r) = \begin{cases}\left(Ar^{-2} - \frac{h^{2/3}\lambda^2}{2}\right)^{1/2}, & \text{for } R_0 \le r \le R_c - 2,\\ B^{-1}h^{1/3}, & \text{for } r \ge R_c - 1.\end{cases} \tag{3.10}$$

Moreover, there exists a constant $h_0 \in (0,1]$ and $C_0 > 0$, depending only on A, B, I and R_0, so that for any $0 < h \le h_0$

$$(w(h^{2/3}\lambda^2 + (\phi_0')^2 - h^{4/3}\phi_0''))' \ge h^{2/3}\frac{\lambda^2 w'}{C_0}, \quad \text{for } r \ge R_0. \tag{3.11}$$

Proof. **Step 1**. Set $\psi = (\phi_0')^2$. To simplify the notation we will suppose that $\lambda > 0$ and we will work with $\widetilde{h} = h^{2/3}$ with $0 < h \le h_0$ for some $h_0 \in (0,1]$. Then, (3.11) is equivalent to

$$G(r) \stackrel{\text{def}}{=} \psi + \widetilde{h}\lambda^2 - \widetilde{h}^2\phi_0'' + \frac{r}{2}(\psi' - \widetilde{h}^2\phi_0''') \ge \widetilde{h}\frac{\lambda^2}{C_0}, \quad \text{for } r \ge R_0. \tag{3.12}$$

Let $A > 0$, let $h_0 \in (0,1]$ be small enough so that

$$2R_0 \le R_c = \frac{\sqrt{2A}}{\widetilde{h}^{1/2}\lambda}, \tag{3.13}$$

compare with (3.9). Set

$$\psi_0(r) \stackrel{\text{def}}{=} \frac{A}{r^2} - \frac{\widetilde{h}\lambda^2}{2}, \quad r > 0. \tag{3.14}$$

Notice that $\psi_0(r) = 0$ precisely at $r = R_c$. Hence, for $h_0 \in (0,1]$ small enough, $\psi_0 \ge 0$ for $0 < r \le R_c$.

Next, let $B > 0$ and set

$$\psi_1 \stackrel{\text{def}}{=} \frac{\widetilde{h}}{B^2}. \tag{3.15}$$

Let $\chi \in \mathcal{C}^\infty([0,+\infty[;[0,1])$ be so that $\chi \equiv 1$ on $[0, R_c - 2]$, $\chi \equiv 0$ on $[R_c - 1, +\infty[$ and so that all derivatives of χ are bounded uniformly in h (and, thus, $\widetilde{h}$). We can choose χ such that $\chi' \le 0$. Then, set

$$\psi(r) \stackrel{\text{def}}{=} \psi_0(r)\chi(r) + \psi_1(r)(1-\chi(r)), \quad r > 0 \tag{3.16}$$

Since $\psi_0(R_c) = 0$, we have that for $\tau \in [-2,-1]$

$$\begin{aligned} \psi_0(R_c+\tau) &= -2A\tau \int_0^1 (R_c + t\tau)^{-3} dt = -2A\tau R_c^{-3}(1 + \mathcal{O}(R_c^{-1})) \\ &= \frac{|\tau|\widetilde{h}^{3/2}\lambda^3}{(2A)^{1/2}}(1 + \mathcal{O}(\widetilde{h}^{1/2})) \end{aligned} \tag{3.17}$$

Since ψ_0 is a strictly decreasing function, by (3.16), (3.15), there exist constants $h_0 \in (0,1]$ and $C > 0$ (depending on A,B and I) such that for all $0 < h \le h_0$

$$\psi(r) \ge \frac{1}{C}\widetilde{h}^{3/2}, \quad r > 0 \tag{3.18}$$

Step 2. We estimate ϕ_0''. Assume first that $R_0 \le r \le R_c - 2$. Then,

$$\phi_0''(r) = \frac{\psi_0'(r)}{2\sqrt{\psi_0(r)}} = \frac{-\sqrt{2}A}{\sqrt{2Ar^4 - \widetilde{h}\lambda^2 r^6}} \stackrel{\text{def}}{=} \frac{-\sqrt{2}A}{m(r)^{1/2}} < 0. \tag{3.19}$$

Notice that $m'(r) = r^3(8A - \widetilde{h}\lambda^2 6r^2)$. Thus, $m(r)$ has its unique critical point at

$$r_1 = \frac{2\sqrt{A}}{\lambda\sqrt{3\widetilde{h}}} = \sqrt{\frac{2}{3}}R_c < R_c$$

where by (3.13) we have that $r_1 > \sqrt{2}R_0$. Hence, for $h_0 \in (0,1]$ small enough, $m'(r) > 0$ on $[R_0, r_1[$ and $m'(r) < 0$ on $]r_1, +\infty[$. This implies that $m(r)^{-1/2}$ is decreasing on $[R_0, r_1]$ and increasing on $[r_1, R_c - 2]$. Therefore, $|\phi_0''|$ is bounded by the maximum of $|\phi_0''(R_0)|$, $|\phi_0''(r_1)|$ and $|\phi_0''(R_c - 2)|$.

By (3.19), for $h_0 \in (0,1]$ small enough there exists a constant $C > 0$ (depending as well on I, A and R_0) such that

$$|\phi_0''(R_0)| \le C$$

A straight forward computation shows that

$$\phi_0''(r_1) = -\frac{3\sqrt{3}A}{2R_c^2} = -\frac{3\sqrt{3}\,\widetilde{h}\lambda^2}{4\sqrt{A}} = \mathcal{O}_{A,I}(\widetilde{h})$$

and Taylor expansion shows that

$$\phi_0''(R_c - 2) = \mathcal{O}_{A,I}(\widetilde{h}^{1/4})$$

for all $0 < h \le h_0$ with $h_0 \in (0,1]$ small enough.

Remark 11. If the constant in the big O notation depends on one of the parameters mentioned in the hypotheses of Lemma 10, then we add them as subscripts to keep track of the dependencies.

In conclusion, we have that for all $0 < h \leq h_0$, with $h_0 \in (0,1]$ small enough,

$$\phi_0''(r) = \mathcal{O}_{A,I,R_0}(1), \text{ for } R_0 \leq r \leq R_c - 2. \tag{3.20}$$

Next, suppose that $r \in [R_c - 2, R_c - 1]$. There,

$$\phi_0'' = \frac{\psi_0'\chi + (\psi_0 - \psi_1)\chi'}{2\sqrt{\psi}}. \tag{3.21}$$

By (3.14),

$$|\psi_0'(r)| \leq \frac{2A}{R_c^3}(1 + \mathcal{O}(R_c^{-1})) \leq \mathcal{O}_{A,I}(\widetilde{h}^{3/2}).$$

Since $\chi' \leq 0$, by (3.17) for all $h > 0$, sufficiently small, we have

$$0 \leq (\psi_0 - \psi_1)\chi' \leq \mathcal{O}_{A,B,I}(\widetilde{h}). \tag{3.22}$$

Combining the above two estimates with (3.21) and (3.18), we get that

$$\phi_0''(r) = \mathcal{O}(\widetilde{h}^{1/4}), \quad r \in [R_c - 2, R_c - 1]. \tag{3.23}$$

Notice that $\phi_0''(r) = 0$ for $r \geq R_c - 1$. Then, putting this together (3.20) and (3.23), we that for all $0 < h \leq h_0$, with $h_0 \in (0,1]$ small enough,

$$\phi_0''(r) = \begin{cases} \mathcal{O}_{A,I,R_0}(1), & R_0 \leq r \leq R_c - 2, \\ \mathcal{O}_{A,B,I}(\widetilde{h}^{1/4}), & R_c - 2 \leq r \leq R_c - 1, \\ 0, & r \geq R_c - 1. \end{cases} \tag{3.24}$$

Step 3. Recall that $\psi = (\phi_0')^2$. Hence, by (3.18)

$$-r\phi_0''' = -r\frac{\psi''}{2\sqrt{\psi}} + \frac{r(\psi')^2}{4\psi^{3/2}} \geq -r\frac{\psi''}{2\sqrt{\psi}} \stackrel{\text{def}}{=} -f. \tag{3.25}$$

We will show that f is bounded. Suppose first that $r \in [R_0, R_c - 2]$. There, by (3.16)

$$f(r) = \frac{r\psi_0''}{2\sqrt{\psi_0}} = \frac{3\sqrt{2}A}{m(r)^{1/2}} \geq 0,$$

with m as in (3.19). Considering the critical point of f, as in the discussion following (3.19), we get that $f(r)$ is bounded by the maximum of $f(R_0)$, $f(r_1)$ and $f(R_c - 2)$. Performing similar computations as for (3.20), we get that for all $0 < h \leq h_0$, with $h_0 \in (0,1]$ small enough,

$$0 \leq f(r) \leq \mathcal{O}_{A,I,R_0}(1) \quad \text{for } r \in [R_0, R_c - 2].$$

Next, suppose that $r \in [R_c - 2, R_c - 1]$. By (3.16),

$$r\psi'' = r(\psi_0''\chi + 2\chi'\psi_0' + (\psi_0 - \psi_1)\chi'').$$

We will estimate each term separately. First, using (3.14) and Taylor expansion, we see that

$$|r\psi_0''| = \frac{6A}{R_c^3}(1 + \mathcal{O}(R_c^{-1})) = \mathcal{O}_{A,I}(\widetilde{h}^{3/2})$$

and

$$|r\psi_0'| = \frac{6A}{R_c^2}(1 + \mathcal{O}(R_c^{-1})) = \mathcal{O}_{A,I}(\widetilde{h}).$$

By (3.17), we get that for all $0 < h \leq h_0$, with $h_0 \in (0,1]$ small enough,

$$r|(\psi_0 - \psi_1)\chi''| \leq r\,\mathcal{O}_{I,B}(\widetilde{h}) \leq \mathcal{O}_{A,B,I}(\widetilde{h}^{1/2}).$$

Combining the above three estimates with (3.18) and (3.25), we have that for all $0 < h \leq h_0$, with $h_0 \in (0, 1]$ small enough,

$$|f(r)| \leq \mathcal{O}_{A,B,I}(\widetilde{h}^{-1/4}), \quad \text{for } r \in [R_c - 2, R_c - 1].$$

Finally notice that $\phi_0'''(r) = 0$ for $r \geq R_c - 1$. Therefore,

$$-\widetilde{h} r \phi_0'''(r) \geq \begin{cases} \mathcal{O}_{A,I,R_0}(\widetilde{h}), & R_0 \leq r \leq R_c - 2, \\ \mathcal{O}_{A,I,B}(\widetilde{h}^{3/4}), & R_c - 2 \leq r \leq R_c - 1, \\ 0, & r \geq R_c - 1. \end{cases} \tag{3.26}$$

Step 4. We check that ψ, see (3.16), with $\psi = (\phi_0')^2$ satisfies (3.12). Suppose first that $r \in [R_0, R_c - 2]$. By (3.14),

$$\psi_0 + \frac{r}{2}\psi_0' = -\frac{\widetilde{h}\lambda^2}{2}.$$

Then, by (3.12) (3.26), (3.24) and (3.14) there exist constants $h_0 \in (0, 1]$ and $C_1 > 0$ (depending on A, I, R_0 and B) such that for all $0 < h \leq h_0$

$$\begin{aligned} G(r) &= \psi_0(r) + \widetilde{h}\lambda^2 - \widetilde{h}^2\phi_0''(r) + \frac{r}{2}(\psi_0'(r) - \widetilde{h}^2\phi_0'''(r)) \\ &\geq \frac{\widetilde{h}\lambda^2}{2} + \mathcal{O}_{A,I,R_0}(\widetilde{h}^2) \\ &\geq \frac{\lambda^2\widetilde{h}}{C_1}. \end{aligned} \tag{3.27}$$

Next, assume that $r \in [R_c - 2, R_c - 1]$. Then, by (3.12), (3.26), (3.24), (3.18), (3.22) and (3.16) there exist constants $h_0 \in (0, 1]$ and $C_2 > 0$ (depending on A, I, R_0 and B) such that for all $0 < h \leq h_0$

$$\begin{aligned} G &\geq \psi_0\chi + \psi_1(1 - \chi) + \widetilde{h}\lambda^2 + \frac{r}{2}\psi_0'\chi + \frac{r}{2}(\psi_0 - \psi_1)\chi' \\ &\quad + \mathcal{O}_{A,B,I}(\widetilde{h}^{2+1/4}) + \mathcal{O}_{A,B,I}(\widetilde{h}^{3/2+1/4}) \\ &\geq \frac{1}{C}\widetilde{h}^{3/2} + \frac{\widetilde{h}\lambda^2}{2} + \mathcal{O}_{A,B,I}(\widetilde{h}^{1+3/4}) \\ &\geq \frac{\lambda^2\widetilde{h}}{C_2}. \end{aligned} \tag{3.28}$$

Finally, suppose that $r \geq R_c - 1$. Then, by (3.26), (3.24) and (3.16),

$$G(r) = \frac{\widetilde{h}}{B^2} + \widetilde{h}\lambda^2 \tag{3.29}$$

In conclusion, ψ is a positive smooth function on $]0, +\infty[$ and satisfies (3.12). □

Lemma 12. *Let I be as in (2.4). Let $R_3 > R_2$ and let $P_0 = -h^2\Delta$. Let ϕ_0, $C_{\phi_0} > 0$, $A > 0$ and $B > 0$ be as in Lemma 10. Then, there exists a constant $C = C(I, R_0, A, B, C_{\phi_0}) > 0$ and an $h_0 \in (0, 1]$ such that for all $u \in \mathcal{C}_c^\infty(B(0, R_0, R_3))$ and all $0 < h \leq h_0$*

$$\int e^{2\phi_0/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \leq \frac{CR_3^3}{h^{2+2/3}} \int e^{2\phi_0/h^{4/3}}|(P_0 - \lambda^2)u|^2 dx, \tag{3.30}$$

where we write $\phi_0 = \phi_0(|x|)$.

Proof. The proof is an adaption of the proof of a global Carleman estimate by Datchev [Dat14]. We begin by passing to spherical coordinates, where

$$-h^2\Delta = -h^2\partial_r^2 - \frac{d-1}{2r}h^2\partial_r - r^{-2}h^2\Delta_{S^{d-1}},$$

where $-\Delta_{S^{d-1}} \geq 0$ denotes the Laplace-Beltrami operator on the $(d-1)$-dimensional sphere S^{d-1}. Set

$$P_{\phi_0} \stackrel{\text{def}}{=} \mathrm{e}^{\phi_0/h^{4/3}} r^{\frac{d-1}{2}} (P_0 - \lambda^2) r^{-\frac{d-1}{2}} \mathrm{e}^{-\phi_0/h^{4/3}}. \tag{3.31}$$

A straight forward computation shows that

$$P_{\phi_0} = -h^2\partial_r^2 + 2\phi_0' h^{2/3}\partial_r + V_{\phi_0} + \Lambda - \lambda^2, \tag{3.32}$$

where $\phi_0' = \partial_r \phi_0$ and

$$V_{\phi_0} \stackrel{\text{def}}{=} h^{2/3}\phi_0'' - h^{-2/3}(\phi_0')^2 \tag{3.33}$$

and

$$\Lambda = h^2 r^{-2}\left(-\Delta_{S^{d-1}} + \frac{d-1}{2}\frac{d-3}{2}\right) \tag{3.34}$$

which is a positive semidefinite operator for $d \geq 3$, and $\geq -\frac{h^2}{4r^2}$ for $d = 2$. Next, set $w = w(r) = r^2$ and let $f' = \partial_r f$ denote the radial derivative, and write for $v \in \mathcal{C}_c^\infty(B(0, R_0, R_3))$

$$F(r) \stackrel{\text{def}}{=} \|hv'(r\cdot)\|^2_{S^{d-1}} - ((\Lambda + V_{\phi_0} - \lambda^2)v(r\cdot)|v(r\cdot))_{S^{d-1}}, \quad r > 0, \tag{3.35}$$

where the norm and the scalar product are the norm and scalar product of $L^2(S^{d-1})$. Since the support of v is compact, we have that

$$\int_0^\infty (w(r)F(r))' dr = 0. \tag{3.36}$$

Since Λ is self-adjoint, we get by (3.32),

$$\begin{aligned} F' &= 2\mathrm{Re}\,(h^2\partial_r^2 v|v')_{S^{d-1}} - 2\mathrm{Re}\,((\Lambda + V_{\phi_0} - \lambda^2)v|v')_{S^{d-1}} \\ &\quad + 2r^{-1}(\Lambda v|v)_{S^{d-1}} - (V_{\phi_0}' v|v)_{S^{d-1}} \\ &= -2\mathrm{Re}\,(P_{\phi_0}v|v')_{S^{d-1}} + 4h^{-4/3}\phi_0'\|hv'\|^2_{S^{d-1}} \\ &\quad + 2r^{-1}(\Lambda v|v)_{S^{d-1}} - (V_{\phi_0}' v|v)_{S^{d-1}}. \end{aligned} \tag{3.37}$$

Recall that we are working in $0 < R_0 \leq r \leq R_3$ and that $w = r^2$. Therefore, $w\phi_0' \geq 0$ and $2r^{-1}w - w' = 0$. Then, using as well the elementary inequality $\|a\|^2 - 2\mathrm{Re}\,(a|b) + \|b\|^2 \geq 0$, we get that

$$\begin{aligned} (wF)' &= -2w\mathrm{Re}\,(P_{\phi_0}v|v')_{S^{d-1}} + (4h^{-4/3}w\phi_0' + w')\|hv'\|^2_{S^{d-1}} \\ &\quad + (2wr^{-1} - w')(\Lambda v|v)_{S^{d-1}} + ((w(\lambda^2 - V_{\phi_0}))'v|v)_{S^{d-1}} \\ &\geq -\frac{r^3}{2h^2}\|P_{\phi_0}v\|^2_{S^{d-1}} + ((w(\lambda^2 - V_{\phi_0}))'v|v)_{S^{d-1}}. \end{aligned} \tag{3.38}$$

Integrating (3.38) with respect to r, we get by (3.36), (3.11) and (3.33)

$$\int_0^\infty \int_{S^{d-1}} |v|^2 dr d\sigma \leq \frac{C_0 R_3^3}{4R_0\lambda^2 h^2}\int_0^\infty \int_{S^{d-1}} |P_{\phi_0}v|^2 dr d\sigma. \tag{3.39}$$

Here, we used as well that $\operatorname{supp} v \subset B(0, R_0, R_3)$. Moreover, recall from Lemma 10 that the constant C_0 depends only on the energy interval I and the constants A, B, R_0.

Setting $u = \mathrm{e}^{\phi_0/h^{4/3}} r^{(d-1)/2} v$, we get by (3.31) that

$$\int \mathrm{e}^{2\phi_0/h^{4/3}}|u|^2 dx \leq \frac{C_0 R_3^3}{4R_0\lambda^2 h^2}\int \mathrm{e}^{2\phi_0/h^{4/3}}|(P_0 - \lambda^2)u|^2 dx \tag{3.40}$$

Integration by parts yields that

$$\int \mathrm{e}^{2\phi_0/h^{4/3}}|h\nabla u|^2 dx = -\mathrm{Re}\int h\,\mathrm{div}(\mathrm{e}^{2\phi_0/h^{4/3}} h\nabla u)\overline{u} dx. \tag{3.41}$$

The right hand side is bounded from above by

$$
\begin{aligned}
-\int \mathrm{e}^{2\phi_0/h^{4/3}} 2\mathrm{Re}\left(\sqrt{2}\phi_0'(|x|)\overline{u}h^{-1/3}\frac{x}{|x|}\Big|\frac{h}{\sqrt{2}}\nabla u\right)dx \\
+\int \mathrm{e}^{\phi_0/h^{4/3}}|(P_0-\lambda^2)u|\,|u|dx + \lambda^2\int \mathrm{e}^{\phi_0/h^{4/3}}|u|^2dx.
\end{aligned}
\tag{3.42}
$$

Using the elementary inequality $2\mathrm{Re}\,(a|b) \leq |a|^2+|b|^2$, we get by (3.41), (3.42) that

$$
\begin{aligned}
\int \mathrm{e}^{2\phi_0/h^{4/3}}|h\nabla u|^2dx \leq (2\lambda^2+1+4\|\phi'\|_\infty^2 h^{-2/3})\int \mathrm{e}^{2\phi_0/h^{4/3}}|u|^2dx \\
+\int \mathrm{e}^{2\phi_0/h^{4/3}}|(P_0-\lambda^2)u|^2dx.
\end{aligned}
\tag{3.43}
$$

Let λ_∞ denote the minimum of the absolute value of the supremum and infimum of the interval I. Then, by (3.43), (3.40), we have that, for $h>0$ small enough,

$$
\int \mathrm{e}^{2\phi_0/h^{4/3}}(|u|^2+|h\nabla u|^2)dx \leq \frac{4C_0R_3^3\|\phi_0'\|_\infty^2}{R_0\lambda_\infty^2 h^{2+2/3}}\int \mathrm{e}^{2\phi_0/h^{4/3}}|(P_0-\lambda^2)u|^2dx. \tag{3.44}
$$

Recall from Lemma 10 that $\|\phi_0'|_{[R_0,+\infty[}\|_\infty \leq C_{\phi_0}$. This concludes the proof of Lemma 12. ☐

3.3. **Combining Carleman estimates.** Next, let $R_3 = R_3(h) \asymp h^{-1/3}$ and let $u \in \mathcal{C}^\infty(B(0,R_3))$ so that

$$
(P_V-\lambda^2)u = v \in \mathcal{C}_c^\infty(B(0,R)) \tag{3.45}
$$

and suppose that $R_0 < R < R_1 - 2$. Recall (3.8) and set $M=\phi(R_2)$. Then,

$$
\begin{aligned}
\int_{B(0,R_1)}(|u|^2+|h\nabla u|^2)dx \leq \frac{C\mathrm{e}^{2M/h^{4/3}}}{h^{2/3}}\int |v|^2dx \\
+ Ch^{4/3}\mathrm{e}^{2M/h^{4/3}}\int_{B(0,R_1,R_2)}(|u|^2+|h\nabla u|^2)dx.
\end{aligned}
\tag{3.46}
$$

Let $1_{B(0,R_1-1,R_3-1)} \prec \chi \in \mathcal{C}_c^\infty(B(0,R_1-2,R_3);[0,1])$ so that all derivatives of χ are bounded (uniformly in h). Applying (3.30) to χu, we obtain similar to (3.8) that

$$
\begin{aligned}
\int_{B(0,R_1,R_3-1)}\mathrm{e}^{2\phi_0/h^{4/3}}(|u|^2+|h\nabla u|^2)dx \leq \frac{CR_3^3}{h^{2/3}}\int_{B(0,R_1-2,R_1-1)}\mathrm{e}^{2\phi_0/h^{4/3}}(|u|^2+|h\nabla u|^2)dx \\
+\frac{CR_3^3}{h^{2/3}}\int_{B(0,R_3-1,R_3)}\mathrm{e}^{2\phi_0/h^{4/3}}(|u|^2+|h\nabla u|^2)dx.
\end{aligned}
\tag{3.47}
$$

Here we used as well that

$$
(P_0-\lambda^2)u = (P_V-\lambda^2)u = v = 0, \quad \text{on } B(0,R_1-2,R_3),
$$

which follows from (3.45) and the assumption that $\operatorname{supp} V \Subset B(0,R_0)$, see the discussion after (1.2).

Recall (3.10) and let $\eta>0$. Then, by shifting ϕ_0 by a constant and by choosing $A>0$ large enough, we can arrange that, for $h>0$ small enough,

$$
\begin{aligned}
&\phi_0(|x|) \leq -\eta, \quad \text{for } |x| \leq R_1-1, \\
&\phi_0(|x|) \geq M, \quad \text{for } |x| \geq R_1.
\end{aligned}
$$

Thus,

- the second term on the right hand side of (3.46) is bounded by the a constant times the left hand side of (3.47);
- the first term on the right hand side of (3.47) is bounded by a factor $\mathcal{O}(\mathrm{e}^{-1/Ch})$ times the left hand side of (3.46).

Therefore, adding (3.46) and (3.47) we get for $h > 0$ small enough

$$\int_{B(0,R_3-1)} \mathrm{e}^{2\psi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \leq e^{2M/h^{4/3}} \frac{C}{h^{2/3}} \int |v|^2 dx + \frac{CR_3^3}{h^{2/3}} \int_{B(0,R_3-1,R_3)} \mathrm{e}^{2\psi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx, \tag{3.48}$$

with

$$\psi(x) = \begin{cases} 0, & \text{for } |x| \leq R_1, \\ \phi_0(x), & \text{for } |x| \geq R_1. \end{cases} \tag{3.49}$$

3.4. **Outgoing solutions and flux norm.** Now assume that

$$u = R_V(\lambda)v \tag{3.50}$$

with $v \in L^2_{\mathrm{comp}}(B(0,R))$ is an outgoing solution with $R > 0$ as above. By Theorem 7 and analytic continuation we see that u satisfies $(P_V - \lambda^2)u = v$. Moreover, by a density argument, we see that u and v satisfy (3.48). In particular, since u is outgoing, there exists a $w \in L^2_{\mathrm{comp}}(B(0,R))$ so that

$$u(x) = R_0(\lambda)w(x), \quad \text{for } |x| \geq R_1.$$

Hence, u is a solution to the free Helmholtz equation $(-h^2\Delta - \lambda^2)u = 0$ outside the ball $B(0,R_1)$.

Let $R_c \stackrel{\text{def}}{=} \widetilde{R_c}(\lambda)h^{-1/3}$ be as in Lemma 10. Recall (2.4), let $C_r > 1$ be a constant and set

$$R_3 \stackrel{\text{def}}{=} \widetilde{R}_3 h^{-1/3} \stackrel{\text{def}}{=} C_r\widetilde{R_c}(a)h^{-1/3}. \tag{3.51}$$

Recall (3.10) and write for $r \geq R_c$

$$\begin{aligned}\phi_0(r) &= A^{1/2}\int_{R_0}^{R_c-2}(t^{-2} - R_c^{-2})^{1/2}dt + \int_{R_c-2}^{R_c-1}\phi_0'(t)dt + \int_{R_c-1}^{r} B^{-1}h^{1/3}dt \\ &\stackrel{\text{def}}{=} I_1 + I_2 + I_3.\end{aligned} \tag{3.52}$$

The first integral in (3.52) is bounded by

$$|I_1| \leq \frac{A^{1/2}}{3}\log\frac{1}{h} + A^{1/2}\log\frac{\sqrt{2A} - 2|\lambda|h^{1/3}}{|\lambda|\, R_0}.$$

The second integral $|I_2| \leq C_{\phi_0}$, see Lemma 10, and the third integral $I_3 = h^{1/3}B^{-1}(r - R_c + 1)$. Hence,

$$\phi_0(|x|) = C_0(h) + \frac{h^{1/3}}{B}|x|, \quad \text{for } |x| \geq R_c \tag{3.53}$$

where $C_0(h)$ depends on A, I, B, R_0, C_{ϕ_0} and $h > 0$ satisfying

$$|C_0(h)| \leq \frac{A^{1/2}}{3}\log\frac{1}{h} + \mathcal{O}_{A,I,B,R_0}(1) \tag{3.54}$$

for $h > 0$ small enough. Using Lemma 13 below and (3.51), we see that for $C_r > 1$ large enough, the second term on the right hand side of (3.48) is bounded from above by

$$\begin{aligned}\mathcal{O}(h^{-5/3})\mathrm{e}^{2\psi(R_3)/h^{4/3}} &\int_{B(0,R_3-1,R_3)} (|u|^2 + |h\nabla u|^2)dx \\ &\leq \mathcal{O}(h^{-3})\mathrm{e}^{2\psi(R_3)/h^{4/3}}\mathrm{Im}\,(v|u) \\ &\quad + \mathcal{O}(h^{-5/3})\mathrm{e}^{(2\psi(R_3)-\delta)/h^{4/3}}\int_{A(\widetilde{R}_3/4,1,h)} (|u|^2 + |h\nabla u|^2)\, dx.\end{aligned} \tag{3.55}$$

for some $\delta > 0$. Using (3.53) and (3.51) we get that

$$2\psi(R_3) - \delta = 2C_0 + \frac{1}{B}(2\widetilde{R}_3 - B\delta) \leq 2\psi(|x|) - \frac{\delta_1}{B}, \quad \text{for } |x| \geq (\widetilde{R}_3/4 - 1)h^{-1/3}.$$

where in the second to the last inequality we chose $B > 0$ to be large enough so that $2\widetilde{R}_3 - \delta B \leq 2(\widetilde{R}_3/4 - 1) - \delta_1$ for some $\delta_1 > 0$. Hence, for $h > 0$ small enough, we can absorb the second term on the right hand side of (3.48) into the term on the left hand side of (3.48). Hence,

$$\begin{aligned}\int_{B(0,R_3-1)} \mathrm{e}^{2\psi/h^{4/3}}(|u|^2 + |h\nabla u|^2)dx \leq e^{2M/h^{4/3}}\frac{C}{h^{2/3}}\int |v|^2 dx \\ + \mathcal{O}(h^{-3})\mathrm{e}^{2\psi(R_3)/h^{4/3}}\mathrm{Im}\,(v|u).\end{aligned} \tag{3.56}$$

By the Cauchy-Schwartz inequality and (3.50) we get that

$$\begin{aligned}\mathrm{Im}\,(v|u) &\leq \|v\|\,\|u\|_{L^2(B(0,R))} \\ &\leq \frac{h^3}{C}\mathrm{e}^{-2\psi(R_3)/h^{4/3}}\|u\|^2_{L^2(B(0,R))} + Ch^{-3}\mathrm{e}^{2\psi(R_3)/h^{4/3}}\|v\|^2.\end{aligned} \tag{3.57}$$

In view of (3.49),(3.53), (3.54) by (3.56) and (3.57) there exists constant $C, C' > 0$ such that

$$\int_{B(0,R_3-1)} (|u|^2 + |h\nabla u|^2)dx \leq C' e^{Ch^{-4/3}\log\frac{1}{h}} \int |v|^2 dx$$

which together with (3.50) concludes the proof of Theorem 1.

Lemma 13. *Assume* (3.50)*. Then, for any $\widetilde{R} > 0$ (independent of $h > 0$) and any $0 < \eta < 3\widetilde{R}/16$ there exist constants $C, C', \delta, h_0 > 0$ such that for any $\lambda \in I$ and any $0 < h < h_0$*

$$\begin{aligned}\int_{A(\widetilde{R},\eta,h)} (|u|^2 + |h\nabla u|^2)\,dx \leq Ch^{-4/3}\mathrm{Im}\,(v|u) \\ + C'\mathrm{e}^{-\delta/h^{4/3}}\int_{A(\widetilde{R}/4,\eta,h)} (|u|^2 + |h\nabla u|^2)\,dx,\end{aligned}$$

where $A(\widetilde{R},\eta,h) \stackrel{\text{def}}{=} B(0,(\widetilde{R}-\eta)h^{-1/3},(\widetilde{R}+\eta)h^{-1/3})$.

Proof. Let $U_h : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$ be the unitary map defined by

$$(U_h\phi)(x) = h^{-d/6}\phi(h^{-1/3}x). \tag{3.58}$$

Using (2.1) we rescale the operator $P_V - \lambda^2$ by $h^{-1/3}$, i.e.

$$\begin{aligned}U_h(P_V - \lambda^2)U_h^* &= -h^{2+2/3}\Delta + V(h^{-1/3}x;h) - \lambda^2 \\ &\stackrel{\text{def}}{=} -\widetilde{h}^2\Delta + \widetilde{V}(x;h) - \lambda^2 \\ &\stackrel{\text{def}}{=} (\widetilde{P}_{\widetilde{V}} - \lambda^2).\end{aligned} \tag{3.59}$$

Let u be as in (3.50). As discussed there, u is a solution to the free Helmholtz equation $(-h^2\Delta - \lambda^2)u = 0$ outside the ball $B(0,R_1)$. Set $\widetilde{u} \stackrel{\text{def}}{=} U_h u$. Then, we have that outside the ball $B(0,R_1h^{1/3})$

$$(\widetilde{P}_0 - \lambda^2)\widetilde{u} = 0. \tag{3.60}$$

Hence, by [Bur98, Proposition 2.2], it follows that for any $\widetilde{R}_2 > \widetilde{R}_1 > 0$ (constants independent of $h > 0$) there exist $C, C', \delta, h_0 > 0$ such that for any $\lambda \in I$ and any

$0 < h \leq h_0$

$$-\mathrm{Im}\int_{r=\widetilde{R}_2} \widetilde{h}\partial_r\widetilde{u}\cdot\overline{\widetilde{u}}\,d\sigma \geq C\lambda\int_{r=\widetilde{R}_2}(|\widetilde{u}|^2+\lambda^{-2}|\widetilde{h}\nabla\widetilde{u}|^2)\,d\sigma \\ - C'\mathrm{e}^{-\delta|\lambda|/\widetilde{h}}\int_{r=\widetilde{R}_1}(|\widetilde{u}|^2+\lambda^{-2}|\widetilde{h}\nabla\widetilde{u}|^2)\,d\sigma, \tag{3.61}$$

where $d\sigma$ is the surface measure on $\partial B(0,\widetilde{R}_2)$, respectively on $\partial B(0,\widetilde{R}_1)$, induced from the Lebesgue measure on $\mathbb{R}^d$. Let $\eta > 0$ be as in the hypothesis, then $\widetilde{R}/4+\eta < \widetilde{R}/2-\eta$. The mean value theorem implies that there exists a $\widetilde{R}_2 \in [\widetilde{R}-\eta,\widetilde{R}+\eta]$ such that

$$\int_{r=\widetilde{R}_2}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)\,d\sigma = \frac{1}{2}\int_{B(0,\widetilde{R}-\eta,\widetilde{R}+\eta)}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)\,dx. \tag{3.62}$$

Next, set $\widetilde{R}_1 = \widetilde{R}/4$ and let $1_{B(0,\widetilde{R}_1-\eta/2,\widetilde{R}_1+\eta/2)} \prec \chi \in \mathcal{C}_c^\infty(B(0,\widetilde{R}_1-\eta,\widetilde{R}_1+\eta);[0,1])$. Then, there exist constants $c,\widetilde{c}>0$ such that

$$\int_{B(0,\widetilde{R}_1-\eta,\widetilde{R}_1+\eta)}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)dx \geq c\int_{B(0,\widetilde{R}_1)}(|\chi\widetilde{u}|^2+|\widetilde{h}\nabla\chi\widetilde{u}|^2)dx \\ \geq \widetilde{c}\int_{r=\widetilde{R}_1}|\widetilde{u}|^2\,d\sigma,$$

where in the last inequality we use that the trace map $\tau : H^1(B(0,\widetilde{R}_1)) \to L^2(\partial B(0,\widetilde{R}_1))$ is continuous. Similarly, using (3.60), we get that

$$\int_{r=\widetilde{R}_1}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)\,d\sigma \leq \mathcal{O}(1)\int_{B(0,\widetilde{R}_1-\eta,\widetilde{R}_1+\eta)}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)dx. \tag{3.63}$$

Recall that $\lambda \in [a,b] \Subset \mathbb{R}\backslash\{0\}$, see (2.4), and assume for simplicity that $a>0$. Hence $a^{-2} \geq \lambda^{-2} \geq b^{-2} > 0$. Then, applying (3.61) with $\widetilde{R}_2$ and $\widetilde{R}_1$ as in (3.62) and (3.63) yields that there exist constants $C, C', \delta, h_0 > 0$ such that for any $\lambda \in I$ and any $0 < h \leq h_0$

$$-\mathrm{Im}\int_{r=\widetilde{R}_2}\widetilde{h}\partial_r\widetilde{u}\cdot\overline{\widetilde{u}}\,d\sigma \geq C\lambda\min\{1,b^{-2}\}\int_{r=\widetilde{R}_2}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)\,d\sigma \\ - \max\{1,a^{-2}\}C'\mathrm{e}^{-\delta|\lambda|/\widetilde{h}}\int_{r=\widetilde{R}_1}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)\,d\sigma \\ \geq \frac{Ca}{2}\min\{1,b^{-2}\}\int_{B(0,\widetilde{R}-\eta,\widetilde{R}+\eta)}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)\,dx \\ - \max\{1,a^{-2}\}C'\mathrm{e}^{-\delta a/\widetilde{h}}\int_{B(0,\widetilde{R}_1-\eta,\widetilde{R}_1+\eta)}(|\widetilde{u}|^2+|\widetilde{h}\nabla\widetilde{u}|^2)dx.$$

Then scaling back yields

$$-h^{-1/3}\mathrm{Im}\int_{r=\widetilde{R}_2h^{-1/3}}h\partial_r u\cdot\overline{u}\,d\sigma \geq C_1\int_{A(\widetilde{R},\eta,h)}(|u|^2+|h\nabla u|^2)\,dx \\ - C_2\mathrm{e}^{-\delta a/h^{4/3}}\int_{A(\widetilde{R}_1,\eta,h)}(|u|^2+|h\nabla u|^2)\,dx, \tag{3.64}$$

for some constants $C_1, C_2 > 0$. By (3.50) we get that

$$\int_{B(0,\widetilde{R}_2h^{-1/3})}v\overline{u}dx = \int_{B(0,\widetilde{R}_2h^{-1/3})}(P_V-\lambda^2)u\cdot\overline{u}dx \\ = \int_{B(0,\widetilde{R}_2h^{-1/3})}((V-\lambda^2)|u|^2+|h\nabla u|^2)dx - \int_{r=\widetilde{R}_2h^{-1/3}}h^2\partial_r u\cdot\overline{u}\,d\sigma.$$

Taking the imaginary part yields that

$$-\mathrm{Im} \int_{r=\widetilde{R}_2 h^{-1/3}} h\partial_r u \cdot \overline{u}\, d\sigma = h^{-1}\mathrm{Im}\,(v|u).$$

This together with (3.64) yields the statement of Lemma 13. □

4. Resonance free region

In this section we give a proof of Theorem 3 and show that away from 0 there are no resonances super-exponentially close to the real axis. The proof is standard and can be found for instance in [Sjö02]. We will present it here for the reader's sake. The principal idea is that assuming (1.3) we can extend the resolvent $(P_V - \mu^2)^{-1}$ holomorphically to μ in an exponentially small disc centered at λ as an operator $L^2_{comp} \to H^2_{loc}$.

Here, we are only interested in the poles of the resolvent close to the real axis. Therefore, let $\lambda \in I \Subset \mathbb{R}\backslash\{0\}$ so that (1.3) is valid and let $\Omega \subset \mathbb{C}\backslash i\mathbb{R}$ be a complex open neighborhood of I such that the resolvent

$$R_V(\mu) \stackrel{\text{def}}{=} (P_V - \mu^2)^{-1}, \quad \mu \in \Omega$$

is holomorphic for $\mathrm{Im}\,\mu > 0$ and continues meromorphically to Ω, see Theorem 7. Next, notice that for $\mu \in \Omega$

$$R_V(\mu) = R_0(\mu)(1 + Q(\mu))^{-1}, \tag{4.1}$$

where $Q(\mu) = VR_0(\mu)$. This expression makes sense since it holds for $\mathrm{Im}\,\mu > 0$ and by analytic Fredholm theory $(1+Q(\mu))^{-1}$ continues meromorphically from $\mathrm{Im}\,\mu > 0$ to $\mu \in \Omega$. To see this let first $\mathrm{Im}\,\mu > 0$. Since Ω does not contain any discrete spectrum of P_V - which is situated on $i\mathbb{R}_+$ in the μ variable - we have that $Q(\mu)$ is a holomorphic family compact operator $L^2 \to L^2$ for $\mathrm{Im}\,\mu > 0$. Recall (2.3), let $R > R_0$ and let $1_{B(0,R_0)} \prec \chi \prec 1_{B(0,R)}$. Recall from Theorem 7 that $\chi R_0(\mu)\chi : L^2(\mathbb{R}^d) \to H^2_0(B(0,R))$ is a holomorphic family of operators for $\mu \in \Omega$. Hence, by the Rellich-Kondrachov theorem $\chi R_0(\mu)\chi$ is a holomorphic family of compact operators $L^2 \to L^2$. Since $V = V\chi$ it follows that $1 + Q\chi$ is a holomorphic family of Fredholm operators $L^2 \to L^2$ for $\mu \in \Omega$. Since $(1 + Q\chi)^{-1}$ for $\mathrm{Im}\,\mu \gg 1$ exists by a Neumann series argument, it follows by analytic Fredholm theory that $(1 + Q\chi)^{-1} : L^2 \to L^2$ extends to a meromorphic family of Fredholm operators to $\mu \in \Omega$.

Next, notice that $(1 + Q) = (1 + Q(1 - \chi))(1 + Q\chi)$ and that $(1 + Q(1 - \chi))^{-1} = (1 - Q(1 - \chi))$ has a holomorphic extension from $\mathrm{Im}\,\mu > 0$ to $\mu \in \Omega$ as an operator $L^2_{\mathrm{comp}}(\mathbb{R}^d) \to L^2_{\mathrm{comp}}(\mathbb{R}^d)$. Hence, $(1+Q(\mu))^{-1}$ has a meromorphic extension from $\mathrm{Im}\,\mu > 0$ to $\mu \in \Omega$ as an operator $L^2_{\mathrm{comp}}(\mathbb{R}^d) \to L^2_{\mathrm{comp}}(\mathbb{R}^d)$ and, thus, (4.1) holds.

Let us now turn to the proof of Theorem 3. Suppose that $\Omega \Subset \mathbb{C}\backslash i\mathbb{R}$ is a relatively compact open complex neighborhood of the interval I. Let $\mu \in \Omega$, suppose that $\mathrm{Im}\,\mu \geq 0$ and assume that $R > R_0 + 1$ and let $1_{B(0,R-1)} \prec \chi_0 \prec \chi_1 \prec \chi_2 \prec \chi_3 \prec 1_{B(0,R)}$ with $\chi_j \in \mathcal{C}^\infty_c(\mathbb{R}^d)$. We approximate the interior part of the resolvent $R_V(\mu)\chi_1$ by

$$A(\mu) \stackrel{\text{def}}{=} \chi_2 R_V(\lambda)\chi_1 - R_0(\mu)[P_0, \chi_2]R_V(\lambda)\chi_1.$$

Then,

$$\begin{aligned}(P_V - &\mu^2)A(\mu) \\ &= \chi_1 + [P_0, \chi_2]R_V(\lambda)\chi_1 + \chi_2(\lambda^2 - \mu^2)R_V(\lambda)\chi_1 - (1 + VR_0(\mu))[P_0, \chi_2]R_V(\lambda)\chi_1 \\ &= \chi_1 + \chi_2(\lambda^2 - \mu^2)R_V(\lambda)\chi_1 - VR_0(\mu)[P_0, \chi_2]R_V(\lambda)\chi_1.\end{aligned} \tag{4.2}$$

Next, set $u = R_V(\lambda)\chi_1$. Then, since $1_{\operatorname{supp} V} \prec 1_{B(0,R-1)} \prec \chi_0 \prec \chi_1 \prec \chi_2 \prec \chi_3 \prec 1_{B(0,R)}$, we see that

$$\begin{aligned}(P_0 - \lambda^2)(1-\chi_2)u &= (1-\chi_2)\chi_1 - [P_0, \chi_2]u \\ &= -[P_0, \chi_2]R_V(\lambda)\chi_1,\end{aligned}$$

which implies that

$$(1-\chi_2)u = -R_0(\lambda)[P_0, \chi_2]R_V(\lambda)\chi_1. \tag{4.3}$$

A priori the above two expressions make sense for $\operatorname{Im}\lambda > 0$, however, by analytic continuation, they hold as well for $\lambda \in I$.

Next, notice that the support of the term on the right hand side of (4.3) is contained in $\operatorname{supp}(1-\chi_2)$ which has empty intersection with the support of the potential V. Hence,

$$VR_0(\lambda)[P_0, \chi_2]R_V(\lambda)\chi_1 = 0.$$

Thus, by (4.2), we deduce that

$$\begin{aligned}(P_V - \mu^2)A(\mu) &= \chi_1 + \chi_2(\lambda^2 - \mu^2)R_V(\lambda)\chi_1 + T \\ \text{with } T &\overset{\text{def}}{=} -V\chi_3(R_0(\mu) - R_0(\lambda))\chi_3[P_0, \chi_2]R_V(\lambda)\chi_1.\end{aligned} \tag{4.4}$$

By (2.7), we have that $\chi_3 R_0(\mu)\chi_3$, as an operator from $L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^d)$, is of norm $\mathcal{O}(\mathrm{e}^{C/h})$ uniformly for $\mu \in \Omega$. Then, the Cauchy inequalities imply that

$$\|\partial_\mu(\chi_3 R_0(\mu)\chi_3)\|_{L^2\to L^2} = \mathcal{O}(\mathrm{e}^{C/h})$$

uniformly for $\mu \in \widetilde{\Omega}$, where $\widetilde{\Omega} \Subset \Omega$ is a slightly smaller complex open neighborhood of I strictly contained in Ω. Thus, for any $\lambda \in I$ and any $\mu \in \widetilde{\Omega}$,

$$\|\chi_3 R_0(\mu)\chi_3 - \chi_3 R_0(\lambda)\chi_3\|_{L^2\to L^2} = \mathcal{O}(|\mu - \lambda|\mathrm{e}^{C/h}). \tag{4.5}$$

By (1.5), we see that

$$\|[P_0, \chi_2]R_V(\lambda)\chi_1\|_{L^2\to L^2} = \mathcal{O}\left(\mathrm{e}^{Ch^{-4/3}\log\frac{1}{h}}\right),$$

which in combination with (4.4), (4.5) and (1.2) gives that

$$\|T\|_{L^2\to L^2} = \mathcal{O}\left(|\mu - \lambda|\mathrm{e}^{Ch^{-4/3}\log\frac{1}{h}}\right). \tag{4.6}$$

Notice that $\operatorname{supp} T \subset \operatorname{supp} V \subset B(0, R_0)$, which yields that T maps $L^2(\mathbb{R}^d) \to L^2_{\mathrm{comp}}(B(0,R))$.

For the exterior part of the resolvent $R_V(\mu)(1-\chi_1)$ we use the approximation

$$B(\mu) \overset{\text{def}}{=} (1-\chi_0)R_0(\mu)(1-\chi_1) + A(\mu)[P_0, \chi_0]R_0(\mu)(1-\chi_1).$$

Then,

$$\begin{aligned}(P_V - \mu^2)B(\mu) &= 1 - \chi_1 - [P_0, \chi_0]R_0(\mu)(1-\chi_1) \\ &\quad + (\chi_1 + \chi_2(\lambda^2 - \mu^2)R_V(\lambda)\chi_1 + T)[P_0, \chi_0]R_0(\mu)(1-\chi_1) \\ &= (1-\chi_1) + (\chi_2(\lambda^2 - \mu^2)R_V(\lambda) + T)[P_0, \chi_0]R_0(\mu)(1-\chi_1).\end{aligned} \tag{4.7}$$

Here, we used as well that $\chi_0 \prec \chi_1$.

Put $\widetilde{R}(\mu) \overset{\text{def}}{=} A(\mu) + B(\mu) : L^2_{\mathrm{comp}}(\mathbb{R}^d) \to H^2_{\mathrm{loc}}(\mathbb{R}^d)$. Then, combining (4.4) and (4.7) gives

$$(P_V - \mu^2)\widetilde{R} = 1 + K \tag{4.8}$$

with

$$K = \chi_2(\lambda^2 - \mu^2)R_V(\lambda)(\chi_1 + [P_0, \chi_0]R_0(\mu)(1-\chi_1)) + T[P_0, \chi_0]R_0(\mu)(1-\chi_1). \tag{4.9}$$

Using (2.7) we have that $[P_0,\chi_0]R_0(\mu)$ as an operator from $L^2_{\rm comp}(B(0,R))\to L^2_{\rm comp}(B(0,R))$ is of norm $\mathcal{O}(\mathrm{e}^{C/h})$ uniformly for $\mu\in\Omega$. It then follows by (1.3) and (4.6) that

$$K:\ L^2_{\rm comp}(B(0,R))\longrightarrow L^2_{\rm comp}(B(0,R)) \tag{4.10}$$

has operator norm $\leq \mathcal{O}(|\lambda-\mu|\,\mathrm{e}^{Ch^{-4/3}\log h^{-1}})$ for some constant $C>0$. Therefore, if $|\lambda-\mu|\leq \mathrm{e}^{-2Ch^{-4/3}\log h^{-1}}$, for $h>0$ small enough, it follows that $(1+K)$ has a bounded inverse

$$(1+K)^{-1}:\ L^2_{\rm comp}(B(0,R))\longrightarrow L^2_{\rm comp}(B(0,R)) \tag{4.11}$$

and we get that

$$R_V(\mu)=\widetilde{R}(\mu)(1+K)^{-1}:\ L^2_{\rm comp}(B(0,R))\longrightarrow H^2_{\rm loc}(\mathbb{R}^d) \tag{4.12}$$

is holomorphic for $|\lambda-\mu|\leq \mathrm{e}^{-2Ch^{-4/3}\log h^{-1}}$ for $h>0$ small enough. For μ still in the same set, it follows by (4.1) that

$$\begin{aligned} R_V(\mu)&=R_0(\mu)-R_V(\mu)Q(\mu)\\ &=R_0(\mu)-R_V(\mu)\chi_3Q(\mu). \end{aligned}$$

Since both $Q(\mu):L^2_{\rm comp}(\mathbb{R}^d)\to L^2_{\rm comp}(\mathbb{R}^d)$ and $R_0(\mu):L^2_{\rm comp}(\mathbb{R}^d)\to H^2_{\rm loc}(\mathbb{R}^d)$ are holomorphic families of operators, it follows by (4.12) that

$$R_V(\mu):\ L^2_{\rm comp}(\mathbb{R}^d)\longrightarrow H^2_{\rm loc}(\mathbb{R}^d) \tag{4.13}$$

is holomorphic for $|\lambda-\mu|\leq \mathrm{e}^{-2Ch^{-4/3}\log h^{-1}}$, for $h>0$ small enough, which completes the proof of Theorem 3.

5. Decay of eigenfunctions of Schrödinger operators with bounded potentials

In this section we prove Theorem 5. Let $d\geq 2$, let $0\not\equiv W\in L^\infty(\mathbb{R}^d)$ with $\|W\|_\infty\leq C_W$ and let u be a bounded solution to

$$-\Delta u+Wu=0\quad \text{in } \mathbb{R}^d \tag{5.1}$$

and suppose that u admits the estimate $|u(x)|\leq C\exp(-c|x|)$ for $|x|>1$ and some constants $C,c>0$. Notice that in particular $u\in H^2_h(\mathbb{R}^d)$, the semiclassical Sobolev space. Let $h\in(0,1]$ and let $U_h:L^2(\mathbb{R}^d)\to L^2(\mathbb{R}^d)$ be the unitary map defined by

$$(U_h\phi)(x)=h^{d/2}\phi(hx). \tag{5.2}$$

Then,

$$U_h^*(-\Delta+W)U_h=-h^2\Delta+W(h^{-1}x). \tag{5.3}$$

Let $\psi\in\mathcal{C}_c^\infty(\mathbb{R};[0,1])$ be such that $\operatorname{supp}\psi\subset(1+1/4,1+1/2)$ and $\int\psi dx=1$. Then, set

$$\chi_h(x)\overset{\rm def}{=}\chi_h(|x|)\overset{\rm def}{=}1-\int_0^{|x|}\psi\left(1+\frac{t-1}{h}\right)dt.$$

Notice that $\chi_h\in\mathcal{C}_c^\infty(\mathbb{R}^d;[0,1])$ with support contained in the ball $B(0,2)$ independently of $h>0$. Moreover, $\chi_h\equiv 1$ on $\overline{B(0,1+h/4)}$ and $\chi_h=0$ outside $B(0,1+h/2)$. For any $\alpha\in\mathbb{N}^d\backslash\{0\}$ we have that the support of $\partial^\alpha\chi_h$ is contained in the annulus $B(0,1+h/4,1+h/2)$ with inner radius $1+h/4$ and outer radius $1+h/2$ and all derivatives satisfy the estimate $\|\partial^\alpha\chi_h\|_\infty=\mathcal{O}(h^{1-|\alpha|})$ for $\alpha\neq 0$. Similarly, we can construct a $\widetilde{\chi}_h\in\mathcal{C}_c^\infty(\mathbb{R}^d;[0,1])$ so that $\widetilde{\chi}_h\equiv 1$ on $\operatorname{supp}\nabla\chi_h$ and $\widetilde{\chi}_h=0$ outside the annulus $B(0,1,1+h)$. Moreover, we can arrange so that all derivatives satisfy the estimate $\|\partial^\alpha\widetilde{\chi}_h\|_\infty=\mathcal{O}(h^{1-|\alpha|})$, for $\alpha\neq 0$.

Set

$$\widetilde{u}(x)=(U_h^*u)(x).$$

Then, by (5.1), (5.3)

$$\begin{aligned}(-h^2\Delta + W(h^{-1}x)\mathbf{1}_{B(0,2)}(x))\chi_h(x)\widetilde{u}(x) \\ = \chi_h(x)(-h^2\Delta + W(h^{-1}x))\widetilde{u}(x) + [-h^2\Delta, \chi_h(x)]\widetilde{u}(x) \\ = [-h^2\Delta, \chi_h(x)]\widetilde{u}(x).\end{aligned} \tag{5.4}$$

Notice that

$$\begin{aligned}|[-h^2\Delta, \chi_h]\widetilde{u}|^2 &= |(-h^2\Delta\chi_h)\widetilde{u} - 2h\nabla\chi_h \cdot h\nabla\widetilde{u}|^2 \\ &\leq 2|(-h^2\Delta\chi_h)\widetilde{u}|^2 + 8|h\nabla\chi_h|^2|h\nabla\widetilde{u}|^2.\end{aligned}$$

Using that that $\widetilde{\chi}_h \equiv 1$ on $\operatorname{supp}\nabla\chi_h \subset B(0, 1+h/4, 1+h/2)$ and the estimate on its derivatives, we see by integration by parts shows that

$$\begin{aligned}\int |[-h^2\Delta, \chi_h]\widetilde{u}|^2 dx &\leq \mathcal{O}(h^2)\int_{B(0,1,1+h)} |\widetilde{u}|^2 dx + \mathcal{O}(h^2)\int_{\operatorname{supp}\nabla\chi_h} |h\nabla\widetilde{u}|^2\, dx \\ &\leq \mathcal{O}(h^2)\int_{B(0,1,1+h)} |\widetilde{u}|^2 dx + \mathcal{O}(h^2)\int_{B(0,1,1+h)} |h^2\Delta\widetilde{u}|^2\, dx \\ &\leq \mathcal{O}(h^2)\int_{B(0,1,1+h)} |\widetilde{u}|^2 dx,\end{aligned} \tag{5.5}$$

where in the last line we used as well that $-h^2\Delta\widetilde{u} = -W(h^{-1}x)\widetilde{u}$ by (5.1), (5.3). Hence, setting $V(x;h) = W(h^{-1}x)\mathbf{1}_{B(0,2)}(x) \in L^\infty(\mathbb{R}^d)$ with $\operatorname{supp} V \subset B(0,2)$, we get by (5.4), (5.5) and (5.2) that

$$\|(-h^2\Delta + V)\chi_h\widetilde{u}\|^2 = \mathcal{O}(h^2)\int_{B(0,h^{-1},h^{-1}+1)} |u|^2 dx \stackrel{\text{def}}{=} \varepsilon(h) \tag{5.6}$$

Next, we apply Lemma 9 with $R = 3$: there exists a real-valued smooth function $\phi \in \mathcal{C}^\infty(\mathbb{R}^d)$ and constants $C > 0$ and $h_0 \in (0,1]$ such that for any $v \in \mathcal{C}_c^\infty(B(0,3))$ and all $0 < h \leq h_0$

$$\int e^{2\phi/h^{4/3}}(|v|^2 + |h\nabla v|^2)dx \leq \frac{C}{h^{2/3}}\int e^{2\phi/h^{4/3}}|(-h^2\Delta + V)v|^2 dx. \tag{5.7}$$

Notice in particular from the proof of Lemma 9 that $\phi = \max\{\|V\|_\infty, 1\}^{2/3}\phi_0$ where ϕ_0 is a smooth real-valued function which does not depend on the potential V as it stems from the Carleman estimate for the free Laplacian. In fact ϕ_0 is a non-constant function since one requires $|d\phi_0| \neq 0$ for the Carleman estimate to work, see for instance [Sjö02]. Furthermore, since we assume that $W \not\equiv 0$ we obtain by an easy modification of the proof of Lemma 9 that we can take $\phi = \|V\|_\infty^{2/3}\phi_0$ for $h > 0$ small enough. Let $M \stackrel{\text{def}}{=} \max_{B(0,3)}\phi_0 - \min_{B(0,3)}\phi_0 > 0$, then, applying (5.7) to $\chi_h\widetilde{u}$, we get in combination with (5.6) that

$$\int_{B(0,h^{-1})} |u|^2 dx \leq Ch^{4/3}e^{2M\|V\|_\infty^{2/3}/h^{4/3}}\int_{B(0,h^{-1},h^{-1}+1)} |u|^2 dx. \tag{5.8}$$

Since we assumed that u that $\|u\|_2 = 1$, we get that for $h > 0$ small enough

$$\int_{B(0,h^{-1},h^{-1}+1)} |u|^2 dx \geq 2Ch^{-4/3}e^{-2M\|V\|_\infty^{2/3}/h^{4/3}}. \tag{5.9}$$

Setting $R = h^{-1}$, we conclude formula (1.10) and hence the proof of Theorem 5.

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