INVERSE TUNNELING ESTIMATES AND APPLICATIONS TO THE STUDY OF SPECTRAL STATISTICS OF RANDOM OPERATORS ON THE REAL LINE

FRÉDÉRIC KLOPP

Abstract. We present a proof of a Minami type estimate for one dimensional random Schrödinger operators valid at all energies in the localization regime provided a Wegner estimate is known to hold. The Minami type estimate is then applied to two models to obtain results on their spectral statistics.
The heuristics underlying our proof of Minami type estimates is that close by eigenvalues of a one-dimensional Schrödinger operator correspond either to eigenfunctions that live far away from each other in space or they come from some tunneling phenomena. In the second case, one can undo the tunneling and thus construct quasi-modes that live far away from each other in space.

Résumé. Nous démontrons une inégalité de type Minami pour des opérateurs de Schrödinger aléatoires uni-dimensionnel dans toute la région localisée si une estimée de Wegner est connue. Cette estimée de type de Minami est alors appliquée pour obtenir les statistiques spectrales pour deux modèles.
L'heuristique qui guide ce travail est que des valeurs propres proches pour un opérateur de Schrödinger sur un intervalle sont soient localisées loin l'une de l'autre soit sont la conséquence d'un phénomène d'"effet tunnel". Dans le second cas, on peut, en "défaisant" cet effet tunnel construire des quasi-modes qui sont localisés loin l'un de l'autre.

0. Introduction

Consider the following two random operators on the real line

- the Anderson model

$$H^A_\omega = -\frac{d^2}{dx^2} + W(\cdot) + \sum_{n\in\mathbb{Z}} \omega_n V(\cdot - n) \tag{0.1}$$

  where
  - $W: \mathbb{R} \to \mathbb{R}$ is a bounded, continuous, $\mathbb{Z}$-periodic function;
  - $V: \mathbb{R} \to \mathbb{R}$ is a bounded, continuous, compactly supported, non negative, not identically vanishing function;
  - $(\omega_n)_{n\in\mathbb{Z}}$ are bounded i.i.d random variables, the common distribution of which admits a continuous density.
- the random displacement model

$$H^D_\omega = -\frac{d^2}{dx^2} + \sum_{n\in\mathbb{Z}} V(\cdot - n - \omega_n) \tag{0.2}$$

  where

The author is partially supported by the grant ANR-08-BLAN-0261-01.

- $V : \mathbb{R} \to \mathbb{R}$ is a smooth, even function that has a fixed sign and is compactly supported in $(-r_0, r_0)$ for some $0 < r_0 < 1/2$;
- $(\omega_n)_{n\in\mathbb{Z}}$ are bounded i.i.d random variables, the common distribution of which admits a density supported in $[-r, r] \subset [-1/2 + r_0, 1/2 - r_0]$, that is continuously differentiable in $[-r, r]$ and which support contains $\{-r, r\}$.

Let $\bullet \in \{A, D\}$. For $L > 0$, consider $H^\bullet_{\omega,L}$ the operator $H^\bullet_\omega$ restricted to the interval $[-L/2, L/2]$ with Dirichlet boundary conditions. The spectrum of this operator is discrete and accumulates at $+\infty$; we denote it by

$$E^\bullet_1(\omega, L) < E^\bullet_2(\omega, L) \le \cdots \le E^\bullet_n(\omega, L) \le \cdots$$

It is well known (see e.g. [36]) that, ω almost surely, the limit

$$N^\bullet(E) = \lim_{L\to+\infty} \frac{\#\{n;\ E^\bullet_n(\omega, L) \le E\}}{L} \tag{0.3}$$

exists and is independent of ω. $N^\bullet$ is the *integrated density of states* of the operator $H^\bullet_\omega$. This non decreasing function is the distribution function of a non negative measure, say, $dN^\bullet$ supported on $\Sigma^\bullet$, the almost sure spectrum of $H^\bullet_\omega$.
Moreover, it is known that, under our assumptions, N^A is Lipschitz continuous on $\mathbb{R}$ (see [12]) and there exists $\tilde E^D \in (\inf \Sigma^D, +\infty)$ such that, for any $\alpha \in (0, 1)$, N^D is Lipschitz continuous on $(-\infty, \tilde E^D)$ (see Theorem 5.2 in section 5.2).
For a fixed energy E_0, one defines the *locally unfolded levels* to be the points

$$\xi^\bullet_n(E_0, \omega, L) = L\,[N^\bullet(E^\bullet_n(\omega, L)) - N^\bullet(E_0)].$$

Out of these points form the point process

$$\Xi^\bullet(\xi; E_0, \omega, L) = \sum_{n\ge1} \delta_{\xi^\bullet_n(E_0,\omega,L)}(\xi),$$

The local level statistics are described by

Theorem 0.1. *There exists an energy* $\inf \Sigma^\bullet < E^\bullet \le \tilde E^\bullet$ *and such that, if* $E_0 \in (-\infty, E^\bullet) \cap \Sigma^\bullet$ *satisfies, for some* $\rho \in [1, 4/3)$*, one has*

$$\forall a > b,\ \exists C > 0,\ \exists \varepsilon_0 > 0, \forall \varepsilon \in (0, \varepsilon_0),\ |N^\bullet(E_0 + a\varepsilon) - N^\bullet(E_0 + b\varepsilon)| \ge C\varepsilon^\rho \tag{0.4}$$

then, when $L \to +\infty$*, the point process* $\Xi(E_0, \omega, L)$ *converges weakly to a Poisson process on* $\mathbb{R}$ *with intensity the Lebesgue measure.*

One easily checks that, if E_0 is such that $E \mapsto N(E)$ is differentiable at E_0 and its derivative is positive, then (0.4) is satisfied. For both models, this is the case for Lebesgue almost all points in $[\inf \Sigma^\bullet, E^\bullet) \cap \Sigma^\bullet$. To the best of our knowledge, Theorem 0.1 gives the first instance of a model that is not of alloy type for which local Poisson statistics have been proved.

As is to be expected from e.g. [35, 32, 18] and as we shall see in section 1, the local Poisson statistics property holds over the localized region of the spectrum i.e. the energy $E^\bullet$ is the energy such that $H^\bullet_\omega$ is localized in $(-\infty, E^\bullet)$. In particular, the conclusions of Theorem 0.1 also holds in any region of localization of $H^\bullet_\omega$ where a Wegner type estimate is known to hold.
When $\bullet = A$,, in section 5.1, extending the analysis done in [14], we show that the localized region (in the sense of (Loc)) extends over the whole real axis. Thus, we can take $E^A = +\infty$.

When $\bullet = D$, it was proved in [5] that the localization region also extends over the whole real axis except for possibly a discrete set; here localization did not mean (Loc) but a weaker statement, namely, that the spectrum is pure point associated to exponentially decaying eigenvalues. The analysis in [5] works under assumptions less restrictive than those made above. In section 5.2, extending the analysis done in [30], we show that H^D_ω satisfies (Loc) in some neighborhood of $\inf \Sigma^D$.
Note that, to obtain the local Poisson statistics near an energy E_0, we do not require the density of states, i.e. the derivative of $N^\bullet$, not to vanish at E_0; we only require that $N^\bullet$ not be too flat near E_0.
Following the ideas of [18, 28], using the Minami type estimates that we present in section 1, we can obtain a host of other results on the asymptotics of the statistics of the eigenvalues of the random operator $H^\bullet_{\omega,L}$ in the localized regime. We now give a few of those.
Fix $\bullet \in \{A, D\}$. For $J = [a, b]$, a compact interval s.t. $|N^\bullet(J)| := N^\bullet(b) - N^\bullet(a) > 0$ and a fixed configuration ω, consider the point process

$$\Xi^\bullet_J(\omega, t, L) = \sum_{E^\bullet_n(\omega,L)\in J} \delta_{|N^\bullet(J)|L[N^\bullet_J(E^\bullet_n(\omega,L))-t]}$$

under the uniform distribution in $[0, 1]$ in t; here we have set

$$N^\bullet_J(\cdot) := \frac{N^\bullet(\cdot) - N^\bullet(a)}{N^\bullet(b) - N^\bullet(a)}.$$

This process was introduced in [33, 34]; we refer to these papers for more references, in particular, for references to the physics literature. The values $(N^\bullet_J(E^\bullet_n(\omega, L)))_{n\geq1}$ are called the *J-unfolded eigenvalues* of the operator $H^\bullet_{\omega,L}$.
Following [28], one proves

Theorem 0.2. *Fix $J = [a, b] \subset (-\infty, E^\bullet)\cap\Sigma^\bullet$ a compact interval such that $|N^\bullet(J)| > 0$. Then, ω-almost surely, as $L \to +\infty$, the probability law of the point process $\Xi^\bullet_J(\omega, \cdot, L)$ under the uniform distribution $\mathbf{1}_{[0,1]}(t)dt$ converges to the law of the Poisson point process on the real line with intensity 1.*

As is shown in [34], Theorem 0.2 implies the convergence of the unfolded level spacings distributions for the levels in J. More precisely, define the n-th unfolded eigenvalue spacings

$$\delta N^\bullet_n(\omega, L) = L|N^\bullet(J)|(N^\bullet_J(E_{n+1}(\omega, L)) - N^\bullet(E_n(\omega, L))) \geq 0. \tag{0.5}$$

Define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS^\bullet(x; J, \omega, L) = \frac{\#\{j;\ E^\bullet_n(\omega, L) \in J,\ \delta N^\bullet_n(\omega, L) \geq x\}}{N^\bullet(J, \omega, L)} \tag{0.6}$$

where $N^\bullet(J, \omega, L) := \#\{E^\bullet_n(\omega, L) \in J\} = |N^\bullet(J)|L(1 + o(1))$ as $L \to +\infty$ (see e.g. [19]).

Theorem 0.3. *Under the assumptions of Theorem 0.2, ω-almost surely, as $L \to +\infty$, $DLS^\bullet(x; J, \omega, L)$ converges uniformly to the distribution $x \mapsto e^{-x}$.*

One can also obtain results for the eigenvalues themselves i.e. when they are not unfolded; we refer to [18, 28] for more details.
Finally we turn to results on level spacings that are local in energy (in the sense of Theorem 0.1). Fix $E_0 \in (-\infty, E^\bullet) \cap \Sigma^\bullet$. Pick $I_L = [a_L, b_L]$, a small interval

centered near 0. With the same notations as above (see (0.5)), define the empirical distribution of these spacings to be the random numbers, for $x \geq 0$

$$DLS^\bullet(x; I_L, \omega, L) = \frac{\#\{j;\ E_j^\bullet(\omega, L) - E_0 \in I_L,\ \delta N_j^\bullet(\omega, L) \geq x\}}{N^\bullet(E_0 + I_L, L, \omega)}. \tag{0.7}$$

We prove

Theorem 0.4. *Assume that $E_0 \in [\inf \Sigma^\bullet, E^\bullet)$. Fix $(I_L)_L$ a decreasing sequence of intervals such that $\sup_{I_L} |x| \underset{L\to+\infty}{\to} 0$. Assume that, for some $\delta > 0$ and $\tilde\rho \in [1, 4/3)$, one has*

$$N(E_0 + I_L) \cdot |I_L|^{-\tilde\rho} \geq 1, \tag{0.8}$$

and

$$L^{1-\delta} \cdot N(E_0 + I_L) \underset{L\to+\infty}{\to} +\infty, \quad \frac{N(E_0 + I_{L_{L+o(L)}})}{N(E_0 + I_{L_L})} \underset{L\to+\infty}{\to} 1. \tag{0.9}$$

Then, with probability 1, as $L \to +\infty$, $DLS(x; I_L, \omega, L)$ converges uniformly to the distribution $x \mapsto e^{-x}$, that is, with probability 1,

$$\sup_{x\geq 0} \left| DLS(x; I_L, \omega, L) - e^{-x} \right| \underset{L\to+\infty}{\to} 0. \tag{0.10}$$

As condition (0.4), condition (0.8) is satisfied for $\tilde\rho = 1$ for almost every $E_0 \in [\inf \Sigma^\bullet, E^\bullet)$. Condition (0.9) on the intervals $(I_L)_L$ ensures that they contain sufficiently many eigenvalues for the empirical distribution to make sense and that this number does not vary too wildly when one slightly changes the size of I_L.
The main technical result of the present paper that we turn to below entail a number of other consequences about the spectral statistics in the localized region. We refer to [18, 28] for more such examples and more references.

1. The setting and the results

Let us now turn to the main result of this paper. It concerns random operators on the real line and consist in Minami type estimates valid for all energies in the localization region of general one dimensional random operators satisfying a Wegner estimate. It can be summarized as follows:

- for one dimensional random Schrödinger operators, in the localization region, a Wegner estimate implies a Minami estimate.

The statement does not depend on the specific form of the random potential.
Let us start with a description of our setting. From now on, on $L^2(\mathbb{R})$, we consider random Schrödinger operators of the form

$$H_\omega u = -\frac{d^2}{dx^2} u + q_\omega\, u \tag{1.1}$$

where q_ω is an almost surely bounded $\mathbb{Z}^d$-ergodic random potential.

Remark 1.1. The boundedness assumption may be relaxed so as to allow local singularities and growth at infinity. We make it to keep our proofs as simple as possible.

It is well known (see e.g. [36]) that H_ω then admits an integrated density of states, say, N, and, an almost sure spectrum, say, Σ. We now fix I an open interval in Σ and the subsequent assumptions and statements will be made on energies in I.
Let $H_\omega(\Lambda)$ be the random Hamiltonian H_ω restricted to the interval $\Lambda := [0, L]$ with periodic boundary conditions.
We now state our main assumptions and comment on the validity of these assumptions for the models $H_\omega^{A,D}$ defined respectively in (0.1) and (0.2).
Our first assumption will be a independence assumption for local Hamiltonians that are far away from each other, that is,

(IAD): There exists $R_0 > 0$ such that for $\mathrm{dist}(\Lambda, \Lambda') > R_0$, the random Hamiltonians $H_\omega(\Lambda)$ and $H_\omega(\Lambda')$ are independent.

Remark 1.2. This assumption may be relaxed to asking some control on the correlations between the random Hamiltonians restricted to different cubes. To keep the proofs as simple as possible, we assume (IAD).

Next, we assume that

(W): a Wegner type estimate holds i.e. there exists $C > 0$, $s \in (0, 1]$ and $\rho \geq 1$ such that, for $J \subset I$, and Λ, an interval in $\mathbb{R}$, one has

$$\mathbb{E}\left[\mathrm{tr}(\mathbf{1}_J(H_\omega(\Lambda)))\right] \leq C|J|^s|\Lambda|^\rho. \tag{1.2}$$

Here, $|\cdot|$ denotes the length of the interval $\cdot$.

Remark 1.3. In many cases e.g. for the operators $H_\omega^{A,D}$, assumption (W) is known to hold for $s = 1$ and $\rho = 1$. In the case of H_ω^A, we can take $I = \Sigma^A$ (see e.g. [12]). For Anderson type Hamiltonians with single site potentials that are not of fixed sign, Wegner estimates with arbitrary $s \in (0, 1)$ and $\rho = 1$ have been proved near the bottom of the spectrum and at spectral edges (see [27, 25]).
In the case of H_ω^D, it holds for any $s \in (0, 1)$ and $\rho = 1$ near the infimum of Σ^D (see section 5.2 and [30]).

The second assumption crucial to our study is the existence of a localization region to which I belongs i.e. we assume

(Loc): for any $\xi \in (0, 1)$, one has

$$\sup_{\substack{L>0\\ \mathrm{supp} f\subset I\\ |f|\leq 1}} \mathbb{E}\left(\sum_{n\in\mathbb{Z}} e^{|n|^\xi} \|\mathbf{1}_{[-1/2,1/2]} f(H_\omega(\Lambda_L))\mathbf{1}_{[n-1/2,n+1/2]}\|_2\right) < +\infty. \tag{1.3}$$

Here, the supremum is taken over all Borel functions $f : \mathbb{R} \to \mathbb{C}$ which satisfy $|g| \leq 1$ pointwise.

Remark 1.4. For the models $H_\omega^{A,D}$, the spectral theory has been studied under various assumptions on V and $(\omega_\gamma)_\gamma$ (see e.g. [6, 13, 38, 22]). The existence of a region of localized states is well known and, in many cases, this region extends over the whole spectrum. In the case of H_ω^A, in [14], this is proved under a more restrictive support condition on V, namely, that the support of V is contained in $(-1/2, 1/2)$; that this condition can be removed is proved in section 5.1. Actually, for this model we get a stronger form of (Loc), namely, for any $I \subset \mathbb{R}$ compact, there exists $\xi = \xi_I > 0$ such that

$$\sup_{\substack{L>0\\ \text{supp} f\subset I\\ |f|\leq 1}} \mathbb{E}\left(\sum_{n\in\mathbb{Z}} e^{\xi|n|}\,\|\mathbf{1}_{[-1/2,1/2]}f(H_\omega(\Lambda_L))\mathbf{1}_{[n-1/2,n+1/2]}\|_2\right) < +\infty. \tag{1.4}$$

In the case of model H^D_ω, localization (in a sense weaker than assumption (Loc) above) has been proved at all energies except possibly at a discrete set (see [5]). In dimension $d\geq 2$, localization at the bottom of the spectrum for H^D_ω has been proved in [30]. This proof does not work directly in dimension $d=1$. In section 5.2, we show prove that, under our assumptions, H^D_ω satisfies (Loc) at the bottom of the spectrum.

There are other random models for which localization (in a possibly weaker sense than (Loc) above) has been proved e.g. the Russian model ([20]), the Bernoulli Anderson model ([13, 4, 15]), the Poisson model ([40, 17]), more general displacement models ([5]), matrix valued models ([3]), etc. For many of these models, the validity of (W) is still an open question.

1.1. **A Minami type estimate in the localization region.** Our main technical result is the following Minami type estimate

Theorem 1.1. *Assume (W) and (Loc). Fix J compact in I the region of localization. Then, for $\eta>1$, $\beta>\max(1+4s,\rho)$ and $\rho'>\rho$ (recall that ρ and s are defined in (W)), there exists $L_{\eta,\beta,\rho'}>0$ and $C=C_{\eta,\beta,\rho'}>0$ s.t., for $E\in J$, $L\geq L_{\eta,\beta,\rho'}$ and $\varepsilon\in(0,1)$, one has*

$$\begin{aligned}\sum_{k\geq 2}\mathbb{P}\left(\text{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right)\\ \leq C\left[\left(\varepsilon^s\,L\,\ell^\beta+e^{-\ell/8}\right)^2 e^{C\,\varepsilon^s\,L\,\ell^{\rho'}}+e^{-s\ell/9}\right].\end{aligned} \tag{1.5}$$

where $\ell:=(\log L)^\eta$.

The estimate (1.5) only becomes useful when $\varepsilon^s L$ is small; as $\rho\geq 1$, this is also the case for the Wegner type estimate (W). Note that, as $s\leq 1$, $\varepsilon^s L(\log L)^\beta$ is small only when εL is small. Finally, note that, as $\rho>1$, the factor $(\varepsilon^s L(\log L)^\beta)^2$ is better i.e. smaller than $(\varepsilon^s L^\rho)^2$, the square of the upper bound obtained by the Wegner type estimate (W). This improvement is a consequence of localization.

The estimate (1.5) is weaker than the Minami type estimate found in [32, 2, 21, 9] which gives a bound on $\displaystyle\sum_{k\geq 2} k\,\mathbb{P}\left(\text{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right)$. The estimate (1.5) is nevertheless sufficient to repeat the analysis done in [18, 28]. In particular, it is sufficient to obtain the description of the eigenvalues of $H_\omega(\Lambda_L)$ in terms of the "approximated eigenvalues" i.e. the eigenvalues of H_ω restricted to smaller cubes and to compute the law of those approximated eigenvalues (see [18, Lemma 2.1, Theorem 1.15 and 1.16], [28]).

Let us now say a word how (1.5) can be used to apply the analysis done in [18, 28] to the models H^A_ω and H^D_ω studied in the introduction.

One checks that Theorem 1.1 implies that, for any $s'\in(0,s)$ and $\eta>1$, there exists

$L_{\eta,s'} > 0$ s.t., for $E \in J$, $L \geq L_{\eta,s'}$ and $\varepsilon \in \left(0, L^{-1/s'}\right)$, one has

$$\sum_{k\geq 2} \mathbb{P}\left(\operatorname{tr}\left[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))\right] \geq k\right) \leq C\left(\varepsilon^{2s'}L^2 + e^{-(\log L)^\eta/8}\right). \tag{1.6}$$

The estimate (1.6) differs from the Minami estimate used in [18, 28] in three ways. First, in [18, 28], it was assumed that $s = 1$ and $\rho = 1$ in (W) and (1.6). For H^A_ω and H^D_ω, in (1.6), we have $\rho = 1$ but only have $s' < 1$ even though arbitrary. Second, we only have (1.6) under a smallness condition on ε (i.e. $\varepsilon \leq L^{-1/s'}$). Third, in (1.6), there is an additional error term $e^{-(\log L)^\eta/8}$. As already mentioned above, in the analysis performed in [18, 28], the Minami type estimate is used in two ways: to control the occurrence of two eigenvalues in a small interval for the operator restricted to a given box and to control the law of approximated eigenvalues. For the first use, the crucial thing is that if L is the size of the box and ε the size of the interval, the bound in the Minami estimate should be very small (see [18, Theorem 1.15 and 1.16] and [28, section 3]). For the second use, the crucial thing is that the term given by the Minami estimate should be smaller than than the main term giving the law of these approximate eigenvalues which is of size εL (see [18, Lemma 2.1] and [28, Lemma 2.2]). In this application, the box size L and ε in (1.6) are related by a power law i.e. $L = \varepsilon^{-\kappa}$ for some $\kappa < 1$. So taking s' sufficiently close to 1 (which, for H^A_ω and H^D_ω is possible as $s = 1$ in (W)) guarantees that the condition $\varepsilon \leq L^{-1/s'}$ is satisfied and, for L large,

$$\varepsilon^{2s'}L^2 + e^{-(\log L)^\eta/8} = o(\varepsilon L)$$

Before explaining the heuristics guiding the proofs of Theorems 1.1, let us very briefly describe some consequences for random operators. Essentially, all the conclusions described for the models H^A_ω and H^D_ω in the introduction hold for any general one dimensional random model satisfying the assumptions (IAD), (W) and (Loc). As said in the introduction, following [18, 28], more results on the spectral statistics can be obtained. As assumptions (IAD) and (Loc) have been proved for many models e.g. the Poisson model (see [17, 16]), the Bernoulli Anderson model (see [4]) or general Anderson models with non trivial distributions (see [15]), it remains to understand Wegner type estimates (W) or replacements of such estimates for those models.

1.2. **Inverse tunneling and the Minami type estimates.** To the best of our knowledge, up to the present work, the availability of decorrelation estimates of the type (1.5) relied on the fact that the single site potential was rank one ([32, 2, 21, 9]) or had the effective weight of a rank one potential as was shown in [10] in the Lifshitz tails regime. In the present paper, we exhibit a heuristic why such estimates should hold quite generally and use it to develop a different approach. This approach makes crucial use of localization to reduce the complexity of the problem i.e. to study the random Hamiltonian restricted to some much smaller cube. Such ideas were already used in [29] to study spectral correlations at distinct energies. We now turn to the heuristic we referred to earlier. The basic mechanism at work in our heuristics is what we call "inverse tunneling". Let us explain this and therefore, first recall some facts on tunneling.

Fix $\ell \in \mathbb{R}$ and $q : [0, \ell] \to \mathbb{R}$ a real valued function bounded by $Q > 0$. On $[0, \ell]$, consider the Dirichlet eigenvalue problem defined by the differential expression

$-u''+qu$ i.e. the eigenvalue problem

$$-\frac{d^2}{dx^2}u(x)+q(x)u(x)=Eu(x),\quad u(0)=u(\ell)=0. \tag{1.7}$$

Tunneling estimates can be described as follows. Assume that the interval $[0,\ell]$ can be split into two intervals, say, $[0,\ell']$ and $[\ell',\ell]$, such that the Dirichlet eigenvalue problem for each of those intervals share a common eigenvalue and such that the associated eigenfunctions are "very" (typically exponentially) small near ℓ' then the eigenvalue problem (1.7) has two eigenvalues that are "very" close together. The closeness of the eigenvalues and the smallness of the eigenfunctions are related; they are in general measured in terms of some parameter e.g. a coupling constant in front of the potential q, a semi-classical parameter in front of the kinetic energy $-d^2/dx^2$ or the length of the interval ℓ (see e.g. [31, 24, 7, 8]). The tunneling effect is well illustrated by the double well problem (see e.g. [23]).
In the present paper, we discuss a converse to the above description i.e. we assume that the eigenvalue problem (1.7) has two (or more) close together eigenvalues, say, 0 and E small associated respectively to u and v. Let $r_u:=\sqrt{|u|^2+|u'|^2}$ and $r_v:=\sqrt{|v|^2+|v'|^2}$ be the Prüfer radii for u and v (see e.g. [41]). Then, either of two things happen:

(1) no tunneling occurs i.e. $r_u\cdot r_v$ is small on the whole interval $[0,\ell]$. In this case, the eigenfunctions u and v live in separate space regions and, thus, don't see each other.
(2) tunneling occurs i.e. $r_u\cdot r_v$ becomes large in some region of space. In the connected components of such regions, u and v are roughly proportional. Thus, we show that it is possible to construct linear combinations of u and v that live in distinct space regions, that is, we undo the tunneling; these linear combinations are not true eigenfunctions anymore but they almost satisfy the eigenvalue equation as E is close to 0.

In both cases, we construct quasi-modes that live in distinct space regions (see section 2.2). Thus, we derive (1.5) using the Wegner type estimate (W) in each of these regions (see section 3). This yields Theorem 1.1.

1.3. **Universal estimates.** We now turn to deterministic estimates that are related to our analysis of Minami estimates in one dimension. These estimates control the minimal spacing between any two eigenvalues of a Schrödinger operator on $[0,\ell]$ (with Robin boundary conditions). By extension, they also give an upper bound on the maximal number of eigenvalues a Schrödinger operator of $[0,\ell]$ can put inside an interval of size ε. Though we do not know any reference for such estimates, we are convinced that they are well known to the specialists.
For the sake of simplicity, assume $q:[0,\ell]\to\mathbb{R}$ is bounded. On $[0,\ell]$, consider the operator $Hu=-u''+qu$ with self-adjoint Robin boundary conditions at 0 and ℓ (i.e. $u(0)\cos\alpha+u'(0)\sin\alpha=0$). Then, one has

Theorem 1.2. *Fix J compact. There exists a constant $C>0$ (depending only on $\|q\|$ and J) such that, for $\ell\geq 1$, if $\varepsilon\in(0,1)$ is such that $|\log\varepsilon|\geq C\ell$, then, for any $E\in J$, the interval $[E-\varepsilon,E+\varepsilon]$ contains at most a single eigenvalue of H.*

One can generalize Theorem 1.2 to

Theorem 1.3. *Fix $\nu > 2$ and J compact. There exists $\ell_0 > 1$ and $C > 0$ (depending only on $\|q\|_\infty$ and J) such that, for $\ell \geq \ell_0$, if $\varepsilon \in (0, \ell^{-\nu})$ then, for $E \in J$, the number of eigenvalues of H in the interval $[E-\varepsilon, E+\varepsilon]$ is bounded by* $\max(1, C\ell/|\log\varepsilon|)$.

These a-priori bounds prove that there is some repulsion between the level for arbitrary Schrödinger operators in dimension one. For random systems in the localized phase, this repulsion takes place at a length scale of size $e^{-C\ell}$; it is much smaller than the typical level spacings that is of size $1/\ell$.
Similar results hold for discrete operators (see e.g. [37]).

2. Inverse tunneling estimates

Fix $\ell \in \mathbb{R}$ and $q : [0,\ell] \to \mathbb{R}$ a real valued function bounded by $Q > 0$. On $[0,\ell]$, consider the Sturm-Liouville eigenvalue problem defined by

$$(Hu)(x) := -\frac{d^2}{dx^2}u(x) + q(x)u(x) = Eu(x), \quad u(0) = 0 = u(\ell). \tag{2.1}$$

Remark 2.1. Here, we use Dirichlet boundary conditions; the same analysis goes through with general Robin boundary conditions.

For u, a solution to (2.1), define the Prüfer variables (see e.g. [41]) by

$$r_u(x)\begin{pmatrix}\sin(\varphi_u(x))\\ \cos(\varphi_u(x))\end{pmatrix} := \begin{pmatrix}u(x)\\ u'(x)\end{pmatrix}, \quad r_u(x) > 0, \quad \varphi_u(x) \in \mathbb{R}$$

Bu the Cauchy-Lipschitz Theorem, if u does not vanish identically, r_u does not vanish. φ_u is chosen so as to be continuous. If u is a solution to (2.1), then we set $\varphi_u(0) = 0$ and $\varphi_u(\ell) = k\pi$ (for some $k \in \mathbb{N}^*$). Rewritten in terms of the Prüfer variables, the eigenvalue equation in (2.1) becomes

$$\varphi_u'(x) = 1 - (1 + (q(x) - E))\sin^2(\varphi_u(x)) \tag{2.2}$$

$$\frac{r_u'(x)}{r_u(x)} = (1 + (q(x) - E))\sin(\varphi_u(x))\cos(\varphi_u(x)). \tag{2.3}$$

Let us now compare eigenfunctions associated to close by eigenvalues.

2.1. General estimates for eigenfunctions associated to close by eigenvalues.

Consider now u and v two normalized eigenfunctions of the Sturm-Liouville problem (2.1) associated to two consecutive eigenvalues, say, 0 and E. We assume $0 < E \ll 1$. Sturm's oscillation theorem then guarantees that $\varphi_u(x) < \varphi_v(x)$ for $x \in (0,\ell)$ and $\varphi_v(\ell) = \varphi_u(\ell) + \pi$ (see e.g. [41]). Define

$$\delta\varphi(x) = \varphi_v(x) - \varphi_u(x). \tag{2.4}$$

Thus, $\delta\varphi(0) = 0$, $\delta\varphi(\ell) = \pi$ and $\delta\varphi(x) \in (0,\pi)$ for $x \in (0,\ell)$.
The function $\delta\varphi$ satisfies the following differential equation

$$\begin{aligned}(\delta\varphi)'(x) &= (1 + q(x))[\sin^2(\varphi_v(x)) - \sin^2(\varphi_u(x))] - E\sin^2(\varphi_v(x))\\ &= (1 + q(x))\sin(\delta\varphi(x))\sin(2\varphi_v(x) - \delta\varphi(x)) - E\sin^2(\varphi_v(x)).\end{aligned} \tag{2.5}$$

The first property we prove is that, on intervals where $\sin(\delta\varphi(x))$ is small, r_u and r_v are essentially proportional to each other, that is,

Lemma 2.1. *Fix* $\varepsilon > 0$. *Assume that, for* $x \in [x_-, x_+]$, *one has* $\sin(\delta\varphi(x)) \leq \varepsilon$. *Then, there exists* $\lambda > 0$ *such that, for* $x \in [x_-, x_+]$, *one has*

$$e^{-[(Q+1)\varepsilon+E]\ell} \leq \frac{r_v(x)}{r_u(x)}\frac{1}{\lambda} \leq e^{[(Q+1)\varepsilon+E]\ell}. \tag{2.6}$$

Proof. Comparing (2.3) for u and v yields

$$\left[\log\left(\frac{r_v(x)}{r_u(x)}\right)\right]' = (1+q(x))\sin(\delta\varphi(x))\cos(2\varphi_v(x)-\delta\varphi(x)) - E\sin(2\varphi_v(x)) \tag{2.7}$$

As, for $x \in [x_-, x_+]$ one has $0 \leq \sin(\delta\varphi(x)) \leq \varepsilon$, (2.7) yields, for $x \in [x_-, x_+]$,

$$\left|\left[\log\left(\frac{r_v(x)}{r_u(x)}\right)\right]'\right| \leq (1+Q)\varepsilon + E.$$

Integrating this equation, for $(x,y) \in [x_-, x_+]^2$, one obtains

$$e^{-[(Q+1)\varepsilon+E](y-x)}\frac{r_v(y)}{r_u(y)} \leq \frac{r_v(x)}{r_u(x)} \leq e^{[(Q+1)\varepsilon+E](y-x)}\frac{r_v(y)}{r_u(y)}. \tag{2.8}$$

This in particular immediately yields (2.6) and completes the proof of Lemma 2.1. □

Next we prove that the Wronskian of u and v does not vary much on intervals over which $\sin(\delta\varphi(x))$ is "large", that is,

Lemma 2.2. *Fix* $\varepsilon > 0$ *such that* $E < \varepsilon < 1$. *Assume that*

- *for* $x \in [x_-, x_+]$, *one has* $\sin(\delta\varphi(x)) \geq \varepsilon$;
- $\sin(\delta\varphi(x_\pm)) = \varepsilon$.

Then, for $x \in [x_-, x_+]$, *one has* $w(v,u)(x) > 0$ *and*

$$\max_{x\in[x_-,x_+]}\left[1 - \frac{w(v,u)(x)}{\max\limits_{x\in[x_-,x_+]} w(v,u)(x)}\right] \leq \frac{E}{\varepsilon}(x_+ - x_-) \leq \frac{E\ell}{\varepsilon}.$$

Proof. Consider the Wronskian of v and u, that is, $w(v,u)(x) = u'(x)v(x) - v'(x)u(x)$. As u and v are eigenfunctions for the same Sturm-Liouville problem for the eigenvalues 0 and E, $w(v,u)$ satisfies the equation $[w(v,u)]' = Euv$. Thus, for $(x,y) \in [0,\ell]^2$, one has

$$w(v,u)(x) = r_u(x)r_v(x)\sin(\delta\varphi(x)), \tag{2.9}$$

and

$$\begin{aligned} r_u(x)r_v(x)\sin(\delta\varphi(x)) &- r_u(y)r_v(y)\sin(\delta\varphi(y)) \\ &= E\int_y^x r_u(t)r_v(t)\sin(\varphi_u(t))\sin(\varphi_v(t))dt. \end{aligned} \tag{2.10}$$

The positivity of $w(v,u)$ is a direct consequence of (2.9) and the assumption on $[x_-, x_+]$.
As for $x \in [x_-, x_+]$, one has $\sin(\delta\varphi(x)) \geq \varepsilon$, using (2.9), for $(x,y) \in [x_-, x_+]^2$, one can rewrite (2.10) as

$$w(v,u)(y) - w(v,u)(x) = \frac{E}{\varepsilon}\int_x^y w(v,u)(t)g(t)dt \quad \text{where} \quad \sup_{t\in[x,y]}|g(t)| \leq 1.$$

Thus, for y such that $w(v,u)(y) = \max_{x\in[x_-,x_+]} w(v,u)(x)$, we obtain

$$0 \le 1 - \frac{w(v,u)(x)}{\max_{x\in[x_-,x_+]} w(v,u)(x)} = \frac{E}{\varepsilon}\int_x^y \frac{w(v,u)(t)}{\max_{x\in[x_-,x_+]} w(v,u)(x)} g(t)dt \le \frac{E}{\varepsilon}|y-x|$$

This completes the proof of Lemma 2.2. □

Next, we give another result showing that the Wronskian of u and v does not vary much on intervals over which $\sin(\delta\varphi(x))$ is "large", namely,

Lemma 2.3. *Fix $\varepsilon > 0$ such that $E < \varepsilon < 1$. Assume that*

- *for $x \in [x_-, x_+]$, one has $\sin(\delta\varphi(x)) \ge \varepsilon$;*
- $\sin(\delta\varphi(x_\pm)) = \varepsilon$.

Then, for any $a > 1$,

- *either $r_u(x_+)r_v(x_+) + r_u(x_-)r_v(x_-) \le 2a\,\ell(E/\varepsilon)^2$,*
- *or*

$$1 - \frac{1}{1+a} \le \frac{r_u(x_-)r_v(x_-)}{r_u(x_+)r_v(x_+)} \le 1 + \frac{1}{a}.$$

Proof. Recall that the system (u, v) is orthonormal in $L^2([0,\ell])$; thus, one has

$$\int_0^l r_u^2(x)\sin^2(\varphi_u(x))dx = 1 = \int_0^l r_v^2(x)\sin^2(\varphi_v(x))dx, \tag{2.11}$$

$$\int_0^l r_u(x)r_v(x)\sin(\varphi_u(x))\sin(\varphi_v(x))dx = 0. \tag{2.12}$$

As $w(v,u)(0) = w(v,u)(\ell) = 0$, this and (2.10) implies that

$$0 < \max_{x\in[0,\ell]} r_u(x)r_v(x)\sin(\delta\varphi(x)) \le E. \tag{2.13}$$

As for $x \in [x_-, x_+]$, one has $\sin(\delta\varphi(x)) \ge \varepsilon/2$, one obtains

$$0 < \max_{x\in[x_-,x_+]} r_u(x)r_v(x) \le 2E/\varepsilon. \tag{2.14}$$

Inserting this estimate into (2.10) for $(x,y) = (x_-, x_+)$ and using the fact that $\sin(\delta\varphi(x_-)) = \varepsilon = \sin(\delta\varphi(x_+))$, one obtains

$$|r_u(x_-)r_v(x_-) - r_u(x_+)r_v(x_+)| \le 2\ell(E/\varepsilon)^2.$$

This implies the alternative asserted by Lemma 2.3. □

2.2. **An inverse "splitting" result.** We prove

Theorem 2.1. *Fix $S > 0$ arbitrary and $J \subset \mathbb{R}$ a compact interval. There exists $\varepsilon_0 > 0$ and $\ell_0 > 0$ (depending only on $\|q\|_\infty$, J and S) such that, for $\ell \ge \ell_0$ and $0 < \varepsilon\ell^4 \le \varepsilon_0$, for $E \in J$, if the operator H defined in (2.1) has two eigenvalues in $[E-\varepsilon, E+\varepsilon]$, then there exists two points x_+ and x_- in the lattice segment $\varepsilon_0\mathbb{Z}\cap[0,\ell]$ satisfying $S < x_+ - x_- < 2S$ such that, if H_-, resp. H_+, denotes the second order differential operator H defined by (1.7) and Dirichlet boundary conditions on $[0, x_-]$, resp. on $[x_+, \ell]$, then H_- and H_+ each have an eigenvalue in the interval $[E - \varepsilon\ell^4/\varepsilon_0, E + \varepsilon\ell^4/\varepsilon_0]$.*

Theorem 2.1 is a consequence of Propositions 2.1 and 2.3 that are respectively proved in sections 2.2.1 and 2.2.2. Let us now explain the ideas guiding the proof of Theorem 2.1.
Up to a shift in energy and potential q, we may assume that $E = 0$ and that, changing the notations, the eigenvalues considered in Theorem 2.1 are 0 and $E > 0$. All the estimates we will prove only depend on $\|q\|_\infty$ in this new setting, thus, only depend on $\|q\|_\infty$ and J in the old setting. Note that, in the new notations we have $E \leq \varepsilon$.
Let u and v be the eigenfunctions associated respectively to 0 and E. The goal is then to prove that we can find two independent linear combinations of u and v such that

- they vanish at two points, say, x_- and x_+ satisfying the statement of Theorem 2.1,
- in each of these intervals $[0, x_-]$ and $[x_+, \ell]$, the masses of the combinations are of size of order $\ell^{-\alpha}$ (for some $\alpha > 0$).

Therefore, we consider two cases:

(1) if $r_u \cdot r_v$ becomes "large" over $[0, \ell]$ which we dub the "tunneling case".
(2) if $r_u \cdot r_v$ stays "small" over $[0, \ell]$ which we dub the "non tunneling case".

In the first case, u and v put mass at the same locations in $[0, \ell]$. This is typically what happens in a tunneling situation (see e.g. [31, 24, 7, 8]). In this case, there is a strong "interaction" between u and v and the estimates obtained in section 2 enable us to show that u and v are quite similar up to a phase change. Although they are linearly independent (as they are eigenfunctions associated to distinct eigenvalues of the same self-adjoint operator), they are similar in the sense that r_u and r_v are similar (see Lemma 2.1). Their orthogonality comes mainly from the phase difference. We analyze this phase difference to prove that the claims of Theorem 2.1 hold in this case.
In the second case, u and v live "independent lives"; they are of course orthogonal but $|u|$ and $|v|$ (actually, r_u and r_v too) are also almost orthogonal. So, u and v roughly live on disjoint sets; this makes it quite simple to construct the functions whose existence is claimed in Theorem 2.1: one just needs to restrict u and v to their "essential supports".

2.2.1. *When there is tunneling.* The case when there is tunneling can be described by the fact that the function u and v are "large" at the same location or equivalently by the fact that $r_u \cdot r_v$ becomes "large" at some point of the interval $[0, \ell]$. Clearly, as u and v are normalized, r_u and r_v need each to be at most only of size $1/\sqrt{\ell}$. So one can say that $r_u \cdot r_v$ becomes "large" if and only if $r_u \cdot r_v \gtrsim \ell^{-1}$ somewhere in $[0, \ell]$.
We prove

Proposition 2.1. *Fix $S > 0$ arbitrary. There exists $\eta_0 > 0$ (depending only on S and $\|q\|_\infty$) such that, for $\eta \in (0, \eta_0)$ and ℓ sufficiently large (depending only on η, S and $\|q\|_\infty$), if u and v are as in section 2, that is, eigenfunctions of H associated respectively to the eigenvalues 0 and E, and, assume that $E\ell^4 \leq \eta^4$ and that one has*

$$\exists\, x_0 \in [0, \ell], \quad r_u(x_0) \cdot r_v(x_0) \geq \frac{\eta}{\ell}, \tag{2.15}$$

then, there exists two points x_+ and x_- in the lattice segment $\eta\mathbb{Z} \cap [0, \ell]$ satisfying

$$|\log(E\ell^2)|/C < x_- < x_+ < \ell - |\log(E\ell^2)|/C \quad \text{and} \quad S < x_+ - x_- < 2S \tag{2.16}$$

such that, if H_- (resp. H_+) denotes the second order differential operator H defined by (2.1) and Dirichlet boundary conditions on $[0, x_-]$ (resp. on $[x_+, \ell]$), then H_- and H_+ have an eigenvalue in the interval $[-E\ell^4\eta^{-4}, E\ell^4\eta^{-4}]$.

Proof. We keep the notations of section 2.1. By (6.2) for u and v, (2.15) implies that there exists $C > 0$ (depending only on $\|q\|_\infty$) such that

$$\forall x \in [x_0 - 1, x_0 + 1] \cap [0, \ell] \quad r_u(x) \cdot r_v(x) \geq \frac{\eta}{C\ell}, \tag{2.17}$$

Note that, by (2.15) and (2.13), one has

$$\forall x \in [x_0 - 1, x_0 + 1] \cap [0, \ell], \quad \sin(\delta\varphi(x)) \lesssim E\ell/\eta. \tag{2.18}$$

For the sake of definiteness, we assume that

$$\forall x \in [x_0 - 1, x_0 + 1] \cap [0, \ell], \quad 0 \leq \delta\varphi(x) \lesssim E\ell/\eta, \tag{2.19}$$

the case $0 \leq \pi - \delta\varphi(x) \lesssim E\ell/\eta$ being dealt with in the same way.
As u and v are normalized and orthogonal to each other, one proves

Lemma 2.4. *There exists $C > 0$ (depending only on $\|q\|_\infty$) and $x_2 \in [0, \ell]$ such that, for $x \in [x_2 - 1, x_2 + 1] \cap [0, \ell]$, one has*

$$r_u(x) \cdot r_v(x) \geq \frac{\eta}{C\ell^2} \quad \text{and} \quad 0 \leq \pi - \delta\varphi(x) \lesssim E\ell^2/\eta. \tag{2.20}$$

Remark 2.2. When $0 \leq \pi - \delta\varphi(x) \lesssim E\ell/\eta$ on $[x_0 - 1, x_0 + 1] \cap [0, \ell]$, in (2.20), the statement $0 \leq \pi - \delta\varphi(x) \lesssim E\ell^2/\eta$ is replaced with $0 \leq \delta\varphi(x) \lesssim E\ell^2/\eta$.

Proof. Indeed, by (2.17) and (2.19), one has

$$\begin{aligned} \Bigg| \int_{[x_0-1,x_0+1]\cap[0,\ell]} & r_u(x) r_v(x) \sin^2(\varphi_u(x)) dx \\ & + \int_{[0,\ell]\setminus[x_0-1,x_0+1]} r_u(x) r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \Bigg| \lesssim \frac{E\ell}{\eta}. \end{aligned} \tag{2.21}$$

Hence, by (6.3) in Lemma 6.1 and (2.17), as $E\ell^4 \leq \eta_0$, we get that, for some $C > 0$ (depending only on $\|q\|_\infty$), one has

$$\begin{aligned} \int_{[0,\ell]\setminus[x_0-1,x_0+1]} r_u(x) r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx &\leq -\frac{\eta}{C\ell}\left(1 - \frac{C\eta^2}{\ell^2}\right) \\ &\lesssim -\frac{\eta}{\ell} \end{aligned} \tag{2.22}$$

for ℓ sufficiently large.
Write

$$\begin{aligned} &\int_{[0,\ell]\setminus[x_0-1,x_0+1]} r_u(x) r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \\ &\quad = \int_{\substack{x\in[0,\ell]\setminus[x_0-1,x_0+1] \\ r_u(x)r_v(x)\leq\eta/\ell^2}} r_u(x) r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \\ &\qquad + \int_{\substack{x\in[0,\ell]\setminus[x_0-1,x_0+1] \\ r_u(x)r_v(x)>\eta/\ell^2}} r_u(x) r_v(x) \sin(\varphi_u(x)) \sin(\varphi_v(x)) dx \end{aligned} \tag{2.23}$$

By (2.13), on the set $\{x \in [0,\ell];\ r_u(x)r_v(x) > \eta/\ell^2\}$, one has $\sin(\delta\varphi(x)) \leq E\ell^2/\eta$. Thus, as $E\ell^4 \leq \eta$, (2.23) yields

$$\begin{aligned}\int_{\substack{x\in[0,\ell]\setminus[x_0-1,x_0+1]\\ r_u(x)r_v(x)>\eta/\ell^2\\ \sin(\delta\varphi(x))\leq E\ell^2/\eta}} r_u(x)r_v(x)\sin^2(\varphi_u(x))\cos(\delta\varphi(x))dx &\leq -\frac{\eta}{2C\ell}\left(1-\frac{C\eta^2}{\ell^2}\right)\\ &\lesssim -\frac{\eta}{\ell}.\end{aligned} \tag{2.24}$$

This and (6.2) then proves Lemma 2.4. □

Clearly, by (2.19) and (2.20), one has $[x_0-1,x_0+1]\cap[x_2-1,x_2+1]=\emptyset$. For the sake of definiteness, assume that $x_0+1 < x_2-1$. By (2.19) and (2.22), as $x \mapsto \delta\varphi(x)$ is continuous, there exists $x_0+1 < x_1 < x_2-1$ such that $\sin(\delta\varphi(x_1)) = 1$, that is, $\delta\varphi(x_1) = \pi/2$.
Fix now $\varepsilon = \eta/(C\ell^2)$. By (2.18) and (2.20), there exists two intervals $[x_0^-, x_0^+]$ and $[x_2^-, x_2^+]$ such that,

- $[x_0-1,x_0+1]\cap[0,\ell] \subset [x_0^-,x_0^+] \subset [0,\ell]$;
- $[x_2-1,x_2+1]\cap[0,\ell] \subset [x_2^-,x_2^+] \subset [0,\ell]$;
- for $x \in [x_0^-,x_0^+]\cup[x_2^-,x_2^+]$, one has $\sin(\delta\varphi(x)) \leq \varepsilon$;
- $\sin(\delta\varphi(x_0^\pm)) = \sin(\delta\varphi(x_2^\pm)) = \varepsilon$.

As $x_0+1 < x_2-1$, one has $0 < x_0^- < x_0^+ < x_2^- < x_2^+ < \ell$. This also implies that $[x_0, x_0+1] \subset [0,\ell]$ and $[x_2-1,x_2] \subset [0,\ell]$. Moreover, there exists a segment $[x_1^-,x_1^+]$ such that

- $x_1 \in [x_1^-,x_1^+] \subset [x_0^+,x_2^-] \subset [0,\ell]$,
- for $x \in [x_1^-,x_1^+]$, one has $\sin(\delta\varphi(x)) \geq \varepsilon$ and $\sin(\delta\varphi(x_1^\pm)) = \varepsilon$.

As $\sin(\delta\varphi(x_1)) = 1$, by Lemma 6.2, for some $C > 0$ (depending only on S and $\|q\|_\infty$), one has

$$\min_{x\in[x_1-2S,x_1+2S]} \sin(\delta\varphi(x)) \geq \frac{1}{C}. \tag{2.25}$$

Thus, for ℓ sufficiently large, as $\varepsilon < 1/C$, one has $[x_1-2S, x_1+2S] \in [x_1^-,x_1^+]$.
By Lemma 6.2, we know that

$$C^{-1}|\log(E\ell^2)| \leq x_0^+ - x_0^- \quad \text{and} \quad C^{-1}|\log(E\ell^3)| \leq x_2^+ - x_2^- \tag{2.26}$$

We apply Lemma 2.1 to $[x_0^-,x_0^+]$ and $[x_2^-,x_2^+]$. Hence, for ℓ sufficiently large, (2.6) implies that there exists $\lambda_0 > 0$ and $\lambda_2 > 0$ such that, for $i \in \{0,2\}$, one has

$$\frac{\lambda_i}{1+C\eta_0/\ell} \leq \min_{x\in[x_i^-,x_i^+]}\left(\frac{r_u(x)}{r_v(x)}\right) \leq \max_{x\in[x_i^-,x_i^+]}\left(\frac{r_u(x)}{r_v(x)}\right) \leq [1+C\eta_0/\ell]\,\lambda_i. \tag{2.27}$$

Moreover, as r_u and r_v are bounded by a constant depending only on $\|q\|_\infty$, by (2.26), (2.17) and (2.20), one has $\dfrac{\eta}{C\ell^2} \leq \lambda_0, \lambda_2 \leq \dfrac{C\ell^2}{\eta}$.
By Lemma 2.2, on $[x_1^-,x_1^+]$, one has

$$w(u,v)(x) = M\left(1+O\left(\frac{E\ell}{\varepsilon}\right)\right) \quad \text{where} \quad M := \max_{x\in[x_1^-,x_1^+]} w(u,v)(x). \tag{2.28}$$

We prove

Lemma 2.5. *There exists $\eta_0 > 0$ (depending only on $\|q\|_\infty$) such that, for $\eta \in (0,\eta_0)$, there exists $(k_-,k_+) \in \mathbb{N}^2$ such that*

(1) $\frac{2S}{3} < x_1 - k_-\eta < \frac{3S}{4}$ *and* $\frac{2S}{3} < k_+\eta - x_1 < \frac{3S}{4}$;
(2) *there exists* $\lambda_\pm \in \mathbb{R}$ *s.t. for* $\bullet \in \{+,-\}$, *one has*
  - *either* $u(k_\bullet\eta) = \lambda_\bullet v(k_\bullet\eta)$ *and*
    - $|\lambda_- - \lambda_0| \geq \eta_0\eta$,
    - $|\lambda_+ + \lambda_2| \geq \eta_0\eta$.
  - *or* $v(k_\bullet\eta) = \lambda_\bullet u(k_\bullet\eta)$ *and*
    - $|\lambda_- - 1/\lambda_0| \geq \eta_0\eta$,
    - $|\lambda_+ + 1/\lambda_2| \geq \eta_0\eta$.

Proof. The proofs of the existence of k_- and k_+ being the same up to obvious modifications, we only give the details for k_-.
By Lemma 6.3, there exists $\eta_0 > 0$ (depending only on $\|q\|_\infty$) such that, for $\eta \in (0,\eta_0)$, $|\sin(\varphi_u(x))|$ and $|\sin(\varphi_v(x))|$ can stay smaller than η only on intervals of length less than η/η_0. Thus, there exits $\eta_0 > 0$ such that, for $\eta \in (0,\eta_0)$, one can find an integer k such that

$$\frac{2S}{3} < x_1 - k\eta < \frac{3S}{4} \quad \text{and} \quad \begin{matrix} |\sin(\varphi_u(x))| \geq \eta \\ |\sin(\varphi_v(x))| \geq \eta \end{matrix} \quad \text{for } x \in [(k-1)\eta,(k+1)\eta]. \tag{2.29}$$

This, in particular, implies that $u(x) \neq 0 \neq v(x)$ for $x \in [(k-1)\eta,(k+1)\eta]$. Note also that, by (2.26) and (2.29), for ℓ large, one has $k\eta \in [x_1 - 2S, x_1 + 2S] \subset [x_1^-, x_1^+]$. To fix ideas, assume moreover that $r_u(k\eta) \geq r_v(k\eta)$; the reverse case is treated similarly interchanging u and v, and, λ_0 and $1/\lambda_0$. This in particular implies that, for some constant $C > 0$ (depending only on $\|q\|_\infty$, see equation (2.7)), one has

$$r_u(x) \geq r_v(x)e^{-C\eta} \quad \text{for} \quad x \in [(k-1)\eta,(k+1)\eta]. \tag{2.30}$$

Assume that the first point of (2) in Lemma 2.5 does not hold i.e. assume now that

$$\exists\, \lambda \in [\lambda_0 - \eta_0\eta, \lambda_0 + \eta_0\eta] \quad \text{such that} \quad u(k\eta) = \lambda v(k\eta). \tag{2.31}$$

As $v^2 \cdot (u/v)' = w(u,v)$, we compute

$$\begin{aligned} \frac{u(k\eta+\eta)}{v(k\eta+\eta)} &= \frac{u(k\eta)}{v(k\eta)} + \eta \int_0^1 \frac{w(u,v)(k\eta+\eta t)}{v^2(k\eta+\eta t)} dt \\ &= \lambda + \eta \int_0^1 \frac{w(u,v)(k\eta+\eta t)}{v^2(k\eta+\eta t)} dt. \end{aligned}$$

Using successively

- the uniform estimate on the growth rate of r_v given by equation (2.3),
- the estimate (2.28) on the Wronskian $w(u,v)$,
- the assumption $r_u(k\eta) \leq r_v(k\eta)$,
- the bound (2.25),
- and, presumably, a reduction of the value η_0,

we compute

$$\begin{aligned} \int_0^1 \frac{w(u,v)(k\eta+\eta t)}{v^2(k\eta+\eta t)} dt &\geq \frac{1}{Cr_v^2(k\eta)} \int_0^1 w(u,v)(k\eta+\eta t)dt \geq \frac{M}{Cr_v^2(k\eta)} \\ &\geq \frac{M}{Cr_v(k\eta)r_u(k\eta)} \geq \frac{1}{C}\frac{M}{w(u,v)(k\eta)} \geq \frac{1}{C} \geq 2\eta_0. \end{aligned}$$

Thus, one has

$$u(k\eta+\eta) = (\lambda+\delta\lambda)v(k\eta+\eta) \quad \text{with} \quad \lambda+\delta\lambda-\lambda_0 \geq \delta\lambda - |\lambda-\lambda_0| \geq \eta\eta_0$$

and we set $k_- = k+1$.
If (2.31) does not hold, it suffices to set $k_- = k$.
This completes the proof of Lemma 2.5. □

To complete the proof of Proposition 2.3, we check the assertion about H_-; the one about H_+ is checked likewise except for the fact that ℓ has to be replaced by ℓ^2, compare (2.20) in Lemma 2.4 with (2.17).
The proof of Proposition 2.3 now depends on which of the alternatives of Lemma 2.5 is realized. First, assume that, in Lemma 2.5, it is the function $u - \lambda_- v$ that vanishes at $x_- = k_-\eta$. So, the function $u - \lambda_- v$ satisfies Dirichlet boundary conditions on the interval $[0, x_-]$. One computes

$$\|(H_- - E)(u - \lambda_- v)\|_{L^2([0,x_-])} = E\|u\|_{L^2([0,x_-])} \leq E.$$

Moreover, by the defining property of $[x_0^-, x_0^+]$ and Lemma 2.1, as $|\lambda_- - \lambda_0| \geq \eta_0\eta$, using (2.27), for $x \in [x_0^-, x_0^+]$, one has

$$\begin{aligned} u(x) - \lambda_- v(x) &= r_u(x)\sin(\varphi_u(x)) - \lambda_- r_v(x)\sin(\varphi_v(x)) \\ &= [r_u(x) - \lambda_- r_v(x)]\sin(\varphi_v(x)) + O(E|u'(x)|) \\ &= [\lambda_0 - \lambda_-]v(x) + O(E|u'(x)|) + O(E^2|u(x)|) + O(\eta_0/\ell|v(x)|) \end{aligned}$$

Possibly reducing η_0, one then computes

$$\|u - \lambda_- v\|_{L^2([0,x_-])} \geq \eta_0(\eta - 1/\ell)\|v\|_{L^2([x_0-1,x_0+1])} - CE\ell \geq \eta_0\eta^3\ell^{-2} - CE\ell.$$

Thus, we know that H_- has an eigenvalue at distance at most $E\ell^2/(2\eta_0\eta^3)$ from E if $\eta_0\eta^3\ell^{-2} \gtrsim E > 0$.
When, in Lemma 2.5, it is the function $v - \lambda_- u$ that vanishes at $x_- = k_-\eta$, one computes $\|H_-(v - \lambda_- u)\|_{L^2([0,x_-])}$ and the remaining part of the proof is unchanged.
This completes the proof of Proposition 2.1. □

When we use Theorem 2.1 to derive Theorem 1.1, it will be of importance to have two points x_- and x_+ that are well separated from each other. But, minor changes in the proof of Proposition 2.1 also yield the following result

Proposition 2.2. *Fix $S > 0$ arbitrary. There exists $\eta_0 > 0$ such that, for $\eta \in (0, \eta_0)$ and ℓ sufficiently large (depending only on η, S and $\|q\|_\infty$), if u and v are as section 2 and such that (2.15) is satisfied, then, there exists a points $\overline{x}$ in the lattice $\eta\mathbb{Z}$ satisfying*

$$|\log\eta|/C < \overline{x} < \ell - |\log\eta|/C$$

such that, if H_- (resp. H_+) denotes the second order differential operator H defined by (2.1) and Dirichlet boundary conditions on $[0, \overline{x}]$ (resp. on $[\overline{x}, \ell]$), then H_- and H_+ have an eigenvalue in the interval $[-E\ell^4\eta^{-4}, E\ell^4\eta^{-4}]$.

2.2.2. *When there is no tunneling.* The case when there is no tunneling can be described by the fact that both function u and v are "large" only at distinct location or equivalently by the fact that $r_u \cdot r_v$ stays small all over the interval $[0, \ell]$. Clearly, as u and v are normalized, r_u and r_v need each to be at most only of size $1/\sqrt{\ell}$. So one can say that $r_u \cdot r_v$ stays small if and only if $r_u \cdot r_v \ll \ell^{-1}$ all over $[0, \ell]$. We prove

Proposition 2.3. *Fix $S > 0$ arbitrary. There exists $\eta_0 > 0$ (depending only on $\|q\|_\infty$) such that, for $\eta \in (0, \eta_0)$ and ℓ sufficiently large (depending only on η, S and*

$\|q\|_\infty$), if u and v are as section 2 that is, eigenfunctions of H associated respectively to the eigenvalues 0 and E, and if $E\ell \le \eta^{1/4}$ and one has that

$$\forall x \in [0,\ell], \quad r_u(x)\cdot r_v(x) \le \frac{\eta}{\ell}, \tag{2.32}$$

then, there exists two points x_+ and x_- in the lattice $\eta\mathbb{Z}$ satisfying

$$|\log\eta|/C < x_- < x_+ < \ell - |\log\eta|/C \quad \textit{and} \quad S < x_+ - x_- < 2S \tag{2.33}$$

such that, if H_- (resp. H_+) denotes the second order differential operator H defined by (2.1) and Dirichlet boundary conditions on $[0,x_-]$ (resp. on $[x_+,\ell]$), then H_- and H_+ have an eigenvalue in the interval $[-E\ell\eta^{-1/4}, E\ell\eta^{-1/4}]$.

Proof. As u and v are normalized, one can pick x_u (resp. x_v) s.t. $r_u(x_u) \ge \ell^{-1/2}$ (resp. $r_v(x_v) \ge \ell^{-1/2}$). Thus, by (2.32), one has $r_u(x_v) \le \eta\ell^{-1/2}$ and $r_v(x_u) \le \eta\ell^{-1/2}$. To fix ideas, assume $x_u < x_v$. Note that, as r_u satisfies equation (2.3), one has $|\log\eta|/C \le x_v - x_u$ (for some C depending only on $\|q\|_\infty$). Hence, as $x \mapsto (r_u/r_v)(x)$ is continuous, there exists $x_u < x_0 < x_v$ such that $r_u(x_0) = r_v(x_0)$. Define $x_\pm$ to be respectively the points in the lattice $\eta\mathbb{Z}$ closest to $x_0 \pm S/2$. Then, there exists $C > 0$ (depending only on S and $\|q\|_\infty$) such that

$$\frac{1}{C} \le (r_u/r_v)(x_\pm) \le C \quad \text{and} \quad |\log\eta|/C \le \inf(x_v - x_+, x_- - x_u). \tag{2.34}$$

We will start with H_- on the interval $[0,x_-]$; the case of H_+ on the interval $[x_+,\ell]$ is dealt with in the same way.
Assume that $|\sin(\varphi_v(x_-))| \ge \sqrt[4]{\eta}$. Then, we pick $\lambda = \dfrac{u(x_-)}{v(x_-)}$ and set $w_- = u - \lambda v$. Thus, w vanishes at the points 0 and x_- and, one computes

$$\|H_- w_-\|_{L^2([0,x_-])} \le E\lambda \le \frac{E}{\sqrt[4]{\eta}}$$

and, using $r_u(x_u) \ge \ell^{-1/2}$ and (6.1) in Lemma 6.1, for η sufficiently small

$$\begin{aligned}\|w\|^2_{L^2([0,x_-])} &= \int_0^{x_-} (u(x) - \lambda v(x))^2 dx \\ &\ge \int_0^{x_-} u^2(x)dx - 2\lambda \int_0^{x_-} r_u(x) r_v(x) dx \ge \ell^{-1}/C - \eta^{3/4}\ell^{-1} \ge \frac{1}{2C\ell}\end{aligned}$$

for η sufficiently small. Hence, as H_- is self-adjoint, we have proved the statement of Proposition 2.3 if $|\sin(\varphi_v(x_-))| \ge \sqrt[4]{\eta}$.
Assume now that $|\sin(\varphi_v(x_-))| \le \sqrt[4]{\eta}$. Then, for η sufficiently small, point (2) of Lemma 6.3 for φ_v guarantees that, for some $x_0 \in \eta\mathbb{Z}$ such that $x_- - 2\sqrt[8]{\eta} \le x_0 \le x_- - \sqrt[8]{\eta}$, one has $|\sin(\varphi_v(x_0))| \ge \sqrt[4]{\eta}$. So, we can do the computations done above replacing x_- with x_0.
To obtain the counterpart of this analysis for H_+ on $[x_+,\ell]$, we proceed as above except for the fact that we set $w_+ = v - \lambda^{-1}u$ where λ is chosen as before and estimate $\|(H_+ - E)w_+\|_{L^2([x_+,\ell])}$.
This completes the proof of Proposition 2.3. □

When we use Theorem 2.1 to derive Theorem 1.1, it will be of importance to have two points x_- and x_+ that are well separated from each other. Sight changes in the proof of Proposition 2.3 yield the following result

Proposition 2.4. *There exists $\eta_0 > 0$ (depending only on $\|q\|_\infty$) such that, for $\eta \in (0, \eta_0)$ and ℓ sufficiently large, if u and v are as section 2 and if (2.32) is satisfied, then, there exists a points $\overline{x}$ in the lattice $\eta\mathbb{Z}$ satisfying*

$$|\log \eta|/C < \overline{x} < \ell - |\log \eta|/C$$

such that, if H_- (resp. H_+) denotes the second order differential operator H defined by (2.1) and Dirichlet boundary conditions on $[0, \overline{x}]$ (resp. on $[\overline{x}, \ell]$), then, H_- and H_+ have an eigenvalue in the interval $[-E\ell\eta^{-1/4}, E\ell\eta^{-1/4}]$.

2.2.3. *Completing the proof of Theorem 2.1.* It suffices to pick η so small that both Propositions 2.1 and 2.3 hold. Recall that there is a change of notations between Theorem 2.1 and Propositions 2.1 - 2.3. In Theorem 2.1, $E - \varepsilon$ (resp. $E + \varepsilon$) plays the role that 0 (resp. E) plays in Propositions 2.1 and 2.3, 2ε that of E and ε_0 that of a power of η that is now fixed. □

3. The proof of Theorems 1.1

The basic idea of the proof follows the basic idea of [29] i.e. use localization to reduce the complexity of the problem by reducing it to studying eigenvalues of H_ω restricted to cubes of size roughly $(\log L)^{1/\xi}$ for $\xi \in (0, 1)$.

3.1. Reduction to localization cubes.

Pick J a compact interval where (Loc) is satisfied. Thus, we know

Lemma 3.1 ([18])**.** *Under assumption (W) and (Loc), for any $\xi' \in (0, 1)$ and $\xi'' \in (0, \xi')$, for $L \geq 1$ sufficiently large, with probability larger than $1 - e^{-L^{\xi''}}$, if*

(1) *$\varphi_{n,\omega}$ is a normalized eigenvector of $H_\omega(\Lambda_L)$ associated to $E_{n,\omega} \in J$,*
(2) *$x_n(\omega) \in \Lambda_L$ is a maximum of $x \mapsto \|\varphi_{n,\omega}\|_x^2 = \int_{[x-1,x+1]\cap\Lambda_L} |\varphi_{n,\omega}(y)|^2 dy$ in Λ_L,*

then, for $x \in \Lambda_L$, one has

$$\|\varphi_{n,\omega}\|_x \leq e^{2L^{\xi''}} e^{-|x-x_n(\omega)|^{\xi'}}. \tag{3.1}$$

So, with good probability, all the eigenfunctions essentially live in cubes of size of order $(\log L)^{1/\xi'}$ for any $\xi' \in (0, 1)$. Thus, they only see the configuration ω in such cubes. To fix ideas, we define the center of localization of an eigenfunction φ as the left most maximum of $x \mapsto \|\varphi\|_x$.

Remark 3.1. When (Loc) takes the form (1.4), the estimate (3.1) can be replaced with $\|\varphi_{n,\omega}\|_x \leq e^{2L^{\xi'}} e^{-\xi|x-x_n(\omega)|}$.

We prove

Lemma 3.2. *Assume (W) and (Loc). Fix J compact in $\overset{\circ}{I}$. Then, for any $\xi \in (0, 1)$ and $\xi' \in (\xi, 1)$, there exists $C = C_{\xi,\xi'} > 0$ and $L_{\xi,\xi'} > 0$ s.t., for $E \in J$, $L \geq L_{\xi,\xi'}$*

and $\varepsilon \in (0,1)$*, one has*

$$\sum_{k\geq 2}\mathbb{P}\left(\mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right) \leq e^{-s\ell^{\xi'}/9}+\frac{L^2}{\ell}\mathbb{P}_{2,9\ell,\ell}(\varepsilon) + \left(\frac{L}{\ell}\right)^2\left(\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)+e^{-\ell^{\xi'}/8}\right)^2 e^{L\,\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)/\ell} \tag{3.2}$$

where $\ell=(\log L)^{1/\xi}$ *and, for* $j\geq 1$*, one has set*

$$\mathbb{P}_{j,\ell,\ell'}(\varepsilon) := \sup_{\gamma\in\ell'\mathbb{Z}\cap[0,L]}\mathbb{P}\left(\mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_\ell(\gamma)))]\geq j\right). \tag{3.3}$$

Remark 3.2. When (Loc) takes the form (1.4), in Lemma 3.2, one can pick $\ell = K\log L$ with K sufficiently large.

Proof of Lemma 3.2. Pick $E\in J$. First, by standard bounds on the eigenvalue counting function of $-\Delta$, there exists $C>0$ depending only on J such that, for $\varepsilon\in(0,1)$, one has

$$\mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\leq CL. \tag{3.4}$$

Pick $\xi'\in(\xi,1)$ and $\xi''\in(0,\xi)$. Let $\mathcal{Z}_{\xi',\xi''}$ be the set of configurations ω defined by Lemma 3.1 for the exponents ξ' and ξ''. It has probability at least $1-e^{-L^{\xi''}}$. Thus, by (3.4), we estimate, for L sufficiently large,

$$\sum_{k\geq 2}\mathbb{P}\left(\left\{\omega\notin\mathcal{Z}_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)\leq CLe^{-L^{\xi''}}\leq e^{-\ell^{\xi'}} \tag{3.5}$$

as $\ell=(\log L)^{1/\xi}$.
Let us now estimate $\mathbb{P}(\{\omega\in\mathcal{Z}_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\})$.
For $\omega\in\mathcal{Z}_{\xi',\xi''}$, by Lemma 3.1, for each φ eigenfunction of $H_\omega(\Lambda_L)$ associated to an eigenvalue $E\in J$, we define the center of localization associated to φ as in the remarks following Lemma 3.1. We consider the events $\Omega^b_{\xi',\xi''}:=\mathcal{Z}_{\xi',\xi''}\setminus\Omega^g_{\xi',\xi''}$ and

$$\Omega^g_{\xi',\xi''}:=\left\{\omega\in\mathcal{Z}_{\xi',\xi''};\ \begin{array}{r}\text{no two centers of localization of eigenfunctions}\\ \text{associated to eigenvalues in }[E-\varepsilon,E+\varepsilon]\\ \text{are at a distance less than }4\ell\text{ from each other}\end{array}\right\}.$$

Note that, for $\omega\in\Omega^g_{\xi',\xi''}$, $H_\omega(\Lambda_L)$ has at most $[L/(4\ell)]+1$ eigenvalues in $[E-\varepsilon,E+\varepsilon]$; here, $[\cdot]$ denotes the integer part of $\cdot$.
We prove

Lemma 3.3. *Fix* $0<\xi''<\xi<\xi'<1$*. Then, there exists* $L_{\xi,\xi',\xi''}>0$ *such that, for* $\ell=(\log L)^{1/\xi}$*, for* $L\geq L_{\xi,\xi',\xi''}$ *and* $k\geq 2$*, one has*

$$\mathbb{P}\left(\left\{\omega\in\Omega^b_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)\leq\frac{L}{\ell}\mathbb{P}_{2,9\ell,\ell}(\varepsilon)+e^{-s\ell^{\xi'}/9} \tag{3.6}$$

and, for $k\leq [L/(4\ell)]+1$,

$$\mathbb{P}\left(\left\{\omega\in\Omega^g_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)\leq\binom{[L/\ell]}{k}\left(\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)+e^{-\ell^{\xi'}/8}\right)^k \tag{3.7}$$

where $\mathbb{P}_{j,\ell,\ell'}(\varepsilon)$ *is defined in* (3.3).

We postpone the proof of Lemma 3.3 to complete that of Lemma 3.2. We pick $q\geq 1$ and sum (3.6) and (3.7) for $k\geq 2$ to get, for some $C>0$

$$\begin{aligned}
&\frac{1}{C}\sum_{k\geq 2}\mathbb{P}\left(\mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right)\\
&\leq\left(\frac{L}{\ell}\right)^2\left(\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)+e^{-(|\log L|)^{\xi'/\xi}/8}\right)^2\left(1+\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)+e^{-(|\log L|)^{\xi'/\xi}/8}\right)^{L/\ell}\\
&\qquad\qquad+e^{-s\ell^{\xi'}/9}+\frac{L^2}{\ell}\mathbb{P}_{2,9\ell,\ell}(\varepsilon)\\
&\leq C\left(\left(\frac{L}{\ell}\right)^2\left(\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)+e^{-(|\log L|)^{\xi'/\xi}/8}\right)^2 e^{L\,\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)/\ell}\right.\\
&\qquad\qquad\left.+e^{-s\ell^{\xi'}/9}+\frac{L^2}{\ell}\mathbb{P}_{2,9\ell,\ell}(\varepsilon)\right).
\end{aligned}$$

Here, we have used the following bound, for $(x,y)\in(\mathbb{R}^+)^2$ and $m\leq n$ integers,

$$\sum_{k=m}^{n}\binom{n}{k}x^k y^{n-k}\leq\binom{n}{m}x^m(x+y)^{n-m}. \tag{3.8}$$

This completes the proof of Lemma 3.2. □

Proof of Lemma 3.3. We will use

Lemma 3.4. *For* $0<\xi''<\xi<\xi'<1$, *there exists* $L_{\xi,\xi',\xi''}>0$ *such that for* $\ell=(\log L)^{1/\xi}$ *and* $L\geq L_{\xi,\xi',\xi''}$ *and* $\omega\in\mathcal{Z}_{\xi',\xi''}$, *for any* $\gamma\in\Lambda_L$, *if* $H_\omega(\Lambda_L)$ *has* k *eigenvalues in* $[E-\varepsilon,E+\varepsilon]$ *with localization center in* $\Lambda_{4\ell/3}(\gamma)$, *then* $H_\omega(\Lambda_{3\ell/2}(\gamma))$ *has* k *eigenvalues in* $\left[E-\varepsilon-e^{-\ell^{\xi'}/8},E+\varepsilon-e^{-\ell^{\xi'}/8}\right]$.

We postpone the proof of Lemma 3.4 to complete that of Lemma 3.3. Pick $k\geq 2$. We first estimate $\mathbb{P}\left(\left\{\omega\in\Omega^b_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)$. Clearly, one has

$$\begin{aligned}
&\mathbb{P}\left(\left\{\omega\in\Omega^b_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)\\
&\qquad\leq\mathbb{P}\left(\left\{\omega\in\Omega^b_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq 2\right\}\right).
\end{aligned}$$

Thus, we take $k=2$.
By the definition of $\Omega^b_{\xi',\xi''}$ and Lemma 3.4, one clearly has

$$\begin{aligned}
&\mathbb{P}\left(\left\{\omega\in\Omega^b_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq 2\right\}\right)\\
&\leq \mathbb{P}\left(\left\{\exists\gamma\in\ell\mathbb{Z}\cap[0,L];\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon-e^{-\ell^{\xi'}/8},E+\varepsilon+e^{-\ell^{\xi'}/8}]}(H_\omega(\Lambda_{9\ell}(\gamma)))]\geq 2\right\}\right)\\
&\leq \sum_{\gamma\in\ell\mathbb{Z}\cap[0,L]}\mathbb{P}\left(\left\{\mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon-e^{-\ell^{\xi'}/8},E+\varepsilon+e^{-\ell^{\xi'}/8}]}(H_\omega(\Lambda_{9\ell}(\gamma)))]\geq 2\right\}\right)\\
&\leq \frac{L}{\ell}\mathbb{P}_{2,9\ell,\ell}\left(\varepsilon+e^{-\ell^{\xi'}/8}\right)\leq\frac{L}{\ell}\mathbb{P}_{2,9\ell,\ell}\left(\varepsilon\right)+e^{-s\ell^{\xi'}/9}
\end{aligned}$$

for L sufficiently large as $\ell=(\log L)^{1/\xi}$ and $\xi'>\xi$; in the last step, we have used the Wegner estimate (W). This completes the proof of (3.6).
Let us now estimate $\mathbb{P}\left(\left\{\omega\in\Omega^g_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)$. We cover the cube Λ_L by cubes $(\Lambda_{4\ell/3}(\gamma))_{\gamma\in\Gamma}$ i.e. $\Lambda_L=\cup_{\gamma\in\Gamma}\Lambda_{4\ell/3}(\gamma)$ in such a way that $[3L/(4\ell)]\leq\#\Gamma\leq[L/\ell]$.
Assume now that $\omega\in\Omega^g_{\xi',\xi''}$ is such that $\mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k$. Thus, the localization centers for any two eigenfunctions being at least 4ℓ away from each other, by Lemma 3.4, we can find k points in Γ, say $(\gamma_j)_{1\leq j\leq k}$ such that

- for $1\leq j\leq k$, $H_\omega(\Lambda_{3\ell/2}(\gamma_j))$ has exactly one eigenvalue in the interval $\left[E-\varepsilon-e^{-\ell^{\xi'}/8},E+\varepsilon+e^{-\ell^{\xi'}/8}\right]$;
- for $1\leq j<j'\leq k$, one has $\mathrm{dist}(\Lambda_{3\ell/2}(\gamma_j),\Lambda_{3\ell/2}(\gamma_{j'}))>\ell/2$.

Hence, by (IAD), for ℓ sufficiently large, the operators $(H_\omega(\Lambda_{3\ell/2}(\gamma_j)))_{1\leq j\leq k}$ are stochastically independent. Hence, we have the bound

$$\mathbb{P}\left(\left\{\omega\in\Omega^g_{\xi',\xi''};\ \mathrm{tr}\,[\mathbf{1}_{[E-\varepsilon,E+\varepsilon]}(H_\omega(\Lambda_L))]\geq k\right\}\right)\leq\binom{\#\Gamma}{k}\left(\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon)+e^{-\ell^{\xi'}/8}\right)^k.$$

As $[3L/(4\ell)]\leq\#\Gamma\leq[L/\ell]$ and $k\leq[L/(4\ell)]$, this completes the proof of (3.7) and, thus, of Lemma 3.3. □

Proof of Lemma 3.4. Analogous results can be found in [29, 18].
If φ is an eigenfunction of $H_\omega(\Lambda_L)$ associated to e an eigenvalue in $[E-\varepsilon,E+\varepsilon]$ that has localization center in $\Lambda_{4\ell/3}(\gamma)$, then, by (3.1) in Lemma 3.1, we have that, for χ a smooth cut-off that is 1 on $\Lambda_{10\ell/9}(\gamma)$ and vanishing outside $\Lambda_{3\ell/2}(\gamma)$, one has, for L sufficiently large,

$$\left\|H_\omega(\Lambda_{3\ell/2}(\gamma))-e)(\chi\varphi)\right\|\leq e^{2\ell^{\xi''}}e^{-(\ell/6)^{\xi'}}\leq e^{-\ell^{\xi'}/8}.$$

Recall that $\xi'/\xi>1$. On the other hand, if one has k such eigenvalues, say, $(\varphi_j)_{1\leq j\leq k}$, then $k\leq CL$ and one computes the Gram matrix in the same way

$$\begin{aligned}
(((\langle\chi\varphi_j,\chi\varphi_{j'}\rangle))_{1\leq j,j'\leq k}&=(((\langle\varphi_j,\varphi_{j'}\rangle))_{1\leq j,j'\leq k}+O\left(k^2e^{-\ell^{\xi'}/8}\right)\\
&=\mathrm{Id}_k+O\left(L^2e^{-\ell^{\xi'}/8}\right).
\end{aligned}$$

as k is bounded by CL. This completes the proof of Lemma 3.4. □

3.2. **The proof of Theorem 1.1.** We use Lemma 3.2. Recall that $\ell = (\log L)^{1/\xi}$. In (3.2), to estimate $\mathbb{P}_{1,3\ell/2,4\ell/3}(2\varepsilon)$, we use the Wegner type estimate (W) and obtain

$$\mathbb{P}_{1,3\ell/2,4\ell/3}(\varepsilon) \leq C\varepsilon^s (\log L)^{\rho/\xi}. \tag{3.9}$$

To estimate $\mathbb{P}_{2,9\ell,\ell}(2\varepsilon)$, we use Theorem 2.1 and the Wegner type estimate (W). The point $(x_\pm)$ are not known but we know that they belong to the lattice segment $\varepsilon_0\mathbb{Z}\cap[0,\ell]$ (independent of the potential q_ω) so there are at most $(\ell/\varepsilon_0)^2$ possible pairs of points. We choose the constant $S > R_0$ defined by (IAD); hence, as the points $x_+ - x_- \geq S$, the operators $H_- := H^D_{\omega|[0,x_-]}$ and $H_+ := H^D_{\omega|[x_+,\ell]}$ are stochastically independent. Thus, applying the Wegner type estimate (W) for the operators $H_\pm$ and summing over the pairs of points in $\varepsilon_0\mathbb{Z}\cap[0,\ell]$ yields

$$\mathbb{P}_{2,9\ell,\ell}(2\varepsilon) \leq C(\log L)^{2/\xi}\left(\varepsilon(\log L)^{4/\xi}\right)^{2s}. \tag{3.10}$$

Plugging this and (3.9) into (3.2) yields (1.5) with

$$\eta := \xi'/\xi, \quad \beta := \max(1+4s,\rho)/\xi' = \frac{\max(1+4s,\rho)}{\eta\xi} \quad \text{and} \quad \rho' := \rho/\xi' = \frac{\rho}{\eta\xi}.$$

As $\xi < \xi' < 1$ can be chosen arbitrary, this completes the proof of Theorem 1.1. □

4. Proofs of the universal estimates

We now prove Theorems 1.2 and 1.3. By a shift in energy, it suffices to prove the results for $E = 0$ and see that the constants only depend on $\|q\|_\infty$. From now on, we assume the energy interval under consideration is centered at $E = 0$.

Proof of Theorem 1.2. Pick $\varepsilon \in (0,1)$. Assume H has at least two eigenvalues, say, E and $\tilde{E}$ in $[-\varepsilon,\varepsilon]$. By shifting the potential by a constant less than 1, without loss of generality, we may assume that $\tilde{E} = 0$ and $E > 0$. Let v and w be the fundamental solutions to the equation $-u'' + qu = 0$ (i.e. $v(0) = 1 = w'(0)$ and $v'(0) = 0 = w(0)$) and let $S_0(y,x)$ be the resolvent matrix associated to (v,w) i.e.

$$S_0(y,x) = \begin{pmatrix} v(y) & w(y) \\ v'(y) & w'(y) \end{pmatrix} \begin{pmatrix} w'(x) & -w(x) \\ -v'(x) & v(x) \end{pmatrix}.$$

Clearly S_0 solves

$$\frac{d}{dy}S_0(y,x) = \begin{pmatrix} 0 & 1 \\ q(y) & 0 \end{pmatrix} S_0(y,x), \quad S_0(x,x) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Obviously, as q is bounded, for some C depending only on $\|q\|_\infty$, one has

$$\|S_0(y,x)\| \leq e^{C|y-x|}. \tag{4.1}$$

Let u be a $L^2([0,\ell])$-normalized solution to $Hu = Eu$. Hence, we have

$$\begin{pmatrix} u(x) \\ u'(x) \end{pmatrix} = S_0(x,0)\begin{pmatrix} u(0) \\ u'(0) \end{pmatrix} + \int_0^x S_0(y,0)B(y)dy \tag{4.2}$$

where

$$B(y) = E\begin{pmatrix} 0 \\ u(x) \end{pmatrix}.$$

The eigenfunction, say, u_0, associated to H and 0 can be written as

$$\begin{pmatrix} u_0(x) \\ u_0'(x) \end{pmatrix} = S_0(x,0)\begin{pmatrix} u_0(0) \\ u_0'(0) \end{pmatrix}.$$

As u and u_0 satisfy the same boundary conditions, using (4.2), (4.1) and the normalization of u, we get that, for some $\lambda > 0$, one has

$$\left\| \begin{pmatrix} u \\ u' \end{pmatrix} - \lambda \begin{pmatrix} u_0 \\ u_0' \end{pmatrix} \right\|_\infty \leq C\varepsilon e^{C\ell}. \tag{4.3}$$

If $\varepsilon \in (0, 1)$ such that $|\log \varepsilon| \geq K\ell$ where K is taken such that, for $\ell \geq 1$, one has $Ce^{(C-K)\ell}\ell < 1$. By (4.3), as, on $[0, \ell]$, u and u_0 are normalize and orthogonal to each other, we get $\lambda^2 + 1 < 1$ which is absurd. This completes the proof of Lemma 1.2. □

Proof of Theorem 1.3. Assume H has $N + 1$ eigenvalues in $[-\varepsilon, \varepsilon]$. As q is bounded, standard comparison with the Laplace operator $H_0 = -d^2/dx^2$ implies that $N \leq C\ell$ for some $C > 0$ depending only $\|q\|_\infty$.
As in the proof of Theorem 1.2, we may assume that the smallest one of them be 0, thus, that the other be positive. Let $(u_j)_{0\leq j\leq N}$ be the associated normalized eigenfunctions, u_0 being the one associated to the eigenvalue 0.
Fix $1 \leq \tilde\ell < \ell$ to be chosen later. Partition the interval $[0, \ell]$ into A intervals of length approximately $\tilde\ell$ i.e. $[0, \ell] = \cup_{1\leq\alpha\leq A} I_\alpha$ where $I_\alpha = [x_\alpha, x_{\alpha+1}]$ and $x_{\alpha+1} - x_\alpha \asymp \tilde\ell$; hence, $A \asymp \ell/\tilde\ell$.
As in Lemma 1.2, let (v, w) be the fundamental solutions to $-u'' + qu = 0$. Formula (4.2) and (4.1) show that there exists constants $((\lambda_j^\alpha))_{\substack{1\leq j\leq N\\1\leq\alpha\leq A}}$ and $((\beta_j^\alpha))_{\substack{1\leq j\leq N\\1\leq\alpha\leq A}}$ such that, for $0 \leq j \leq N$ and $1 \leq \alpha \leq A$, we have

$$\sup_{x\in I_\alpha} \left| \begin{pmatrix} u_j(x) \\ u_j'(x) \end{pmatrix} - \lambda_j^\alpha \begin{pmatrix} v(x) \\ v'(x) \end{pmatrix} + \beta_j^\alpha \begin{pmatrix} w(x) \\ w'(x) \end{pmatrix} \right| \leq C\varepsilon e^{C\tilde\ell}. \tag{4.4}$$

Let $\langle\cdot,\cdot\rangle$ denote the standard scalar product on $L^2([0, \ell])$ and $\langle\cdot,\cdot\rangle_\alpha$ that on $L^2(I_\alpha)$. One has

$$\mathrm{Id}_{N+1} = ((\langle u_i, u_j\rangle))_{\substack{0\leq i\leq N\\0\leq j\leq N}} = \sum_{\alpha=1}^{A} ((\langle u_i, u_j\rangle_\alpha))_{\substack{0\leq i\leq N\\0\leq j\leq N}}. \tag{4.5}$$

Using (4.4), we compute

$$M_\alpha := ((\langle u_i, u_j\rangle_\alpha))_{\substack{0\leq i\leq N\\0\leq j\leq N}} = \sum_{n=1}^{4} M_{\alpha,n} + S_\alpha \tag{4.6}$$

where

$$M_{\alpha,1} = \langle v, v\rangle_\alpha \left((\lambda_i^\alpha\lambda_j^\alpha)\right)_{\substack{0\leq i\leq N\\0\leq j\leq N}}, \quad M_{\alpha,2} = \langle v, w\rangle_\alpha \left((\lambda_i^\alpha\beta_j^\alpha)\right)_{\substack{0\leq i\leq N\\0\leq j\leq N}}, \tag{4.7}$$

$$M_{\alpha,3} = \langle w, v\rangle_\alpha \left((\beta_i^\alpha\lambda_j^\alpha)\right)_{\substack{0\leq i\leq N\\0\leq j\leq N}}, \quad M_{\alpha,4} = \langle w, w\rangle_\alpha \left((\beta_i^\alpha\beta_j^\alpha)\right)_{\substack{0\leq i\leq N\\0\leq j\leq N}}, \tag{4.8}$$

$$\|S_\alpha\| \leq C\varepsilon N e^{C\tilde\ell}\tilde\ell \leq C\varepsilon\,\ell\,\tilde\ell\,e^{C\tilde\ell}. \tag{4.9}$$

Pick $\tilde\ell = |\log \varepsilon|/K$ for some K sufficiently large; as $0 < \varepsilon \leq \ell^{-\nu}$ with $\nu > 2$, for ℓ sufficiently large, by (4.9), one has

$$\sum_{\alpha=1}^{A} \|S_\alpha\| \leq C\varepsilon\ell^2 e^{C\tilde\ell} \leq C\ell^{2-\nu(1-C/K)} \leq 1/2.$$

By (4.7) and (4.8), the matrices $(M_{\alpha,n})_{\alpha,n}$ are all of rank at most 1. Hence, equation (4.5) implies that $4A \geq N + 1$ which yields $N + 1 \leq C\ell/|\log \varepsilon|$ for some $C > 0$. This completes the proof of Theorem 1.3. □

One can wonder whether the bounds given in Theorems 1.2 and 1.3 are optimal. Examples build using semi-classical ideas show that the orders of magnitudes are. The precise values of the constants depend on the details of the potential q.

5. Localization for the models H^A_ω and H^D_ω

In the present section, we establish that the models H^A_ω and H^D_ω satisfy (Loc) as claimed in the introduction.

5.1. Localization for the model H^A_ω.

In the present section, we show how to extend the results of [14] to our assumptions.

Let

$$\tilde{H}_\omega = -\frac{d^2}{dx^2} + \tilde{W}(\cdot) + \sum_{n\in\mathbb{Z}} \tilde{\omega}_n \tilde{V}(\cdot - n)$$

where

- $(\tilde{\omega}_n)_{n\in\mathbb{Z}}$ and $\tilde{V}$ satisfy the assumptions that $(\omega_n)_{n\in\mathbb{Z}}$ and V satisfy for H^A_ω in the introduction, section 0,
- $\tilde{V}$ has its support in $(-1/2, 1/2)$,
- $\tilde{W}$ is uniformly continuous on $\mathbb{R}$.

Then, the main result of [14] can be rephrased in the following way: H_ω satisfies (Loc) (see (1.4)) for any compact interval I (see Lemma 2.1 and Proposition 2.2 in [14]).

Consider now H^A_ω as defined in section 0. Let $n_0 \in \mathbb{N}$ be such that supp$V \subset (-n_0/2, n_0/2)$. Doing the change of variable $x = n_0 y$, we can rewrite

$$H^A_\omega = n_0^{-2}\left(-\frac{d^2}{dy^2} + \tilde{W}(\cdot) + \sum_{n\in\mathbb{Z}} \tilde{\omega}_n \tilde{V}(\cdot - n)\right) \tag{5.1}$$

where

- $\tilde{V}(\cdot) = n_0^2 V(n_0\,\cdot)$, thus, $\tilde{V}$ has its support in $(-1/2, 1/2)$,
- $\tilde{\omega}_n = \omega_{n\cdot n_0}$ for $n \in \mathbb{Z}$,
- $\tilde{W}(\cdot) = W(\cdot) + n_0^2 \sum_{n\in\mathbb{Z}\setminus n_0\mathbb{Z}} \omega_n V(n_0\,\cdot - n)$, thus, $\tilde{W}$ is uniformly continuous on $\mathbb{R}$ for any realization $(\omega_n)_{n\in\mathbb{Z}\setminus n_0\mathbb{Z}}$ (as the random variables are bounded).

So, for any realization $(\omega_n)_{n\in\mathbb{Z}\setminus n_0\mathbb{Z}}$, we know that H^A_ω satisfies assumption (Loc) on any compact interval I when the expectation is taken with respect to the random variables $(\omega_n)_{n\in n_0\mathbb{Z}}$. A priori, the constant in the right hand side of (1.4) may depend on the realization $(\omega_n)_{n\in\mathbb{Z}\setminus n_0\mathbb{Z}}$. The proof of Theorem 1 in [14] shows that this is not the case. More precisely, as $\tilde{W}$ stays uniformly bounded independently of the realization $(\omega_n)_{n\in\mathbb{Z}\setminus n_0\mathbb{Z}}$, the estimates of the operator T_1 and its continuity with respect to the potential $\tilde{W}$ (W_0 in [14]) yield that the right hand side of (1.4) is bounded uniformly in $(\omega_n)_{n\in\mathbb{Z}\setminus n_0\mathbb{Z}}$. Thus, H^A_ω satisfies (Loc) on any compact interval I.

5.2. Localization for the model H^D_ω.

The purpose of this section is to prove that, in the setting of the introduction, there exists $\tilde{E}^D > \inf \Sigma^D$ such that assumption (1.3) is satisfied in $(\inf \Sigma^D, \tilde{E}^D]$ for H^D_ω. Actually we will prove this under assumptions weaker than those made in the introduction.

Consider the random displacement model (0.2) where

- $V : \mathbb{R} \to \mathbb{R}$ is a smooth, even function that is compactly supported in $(-r_0, r_0)$ for some $0 < r_0 < 1/2$;
- $(\omega_n)_{n\in\mathbb{Z}}$ are bounded i.i.d random variables, the common distribution of which admits a density supported in $[-r, r] \subset [-1/2 + r_0, 1/2 - r_0]$, that is continuously differentiable in $[-r, r]$ and which support contains $\{-r, r\}$.

For $a \in [-r, r]$, consider $H_1(a) = -\Delta + q(x - a)$ on $L^2(-1/2, 1/2)$ with Neumann boundary condition and let $E_0(a) = \inf \sigma(H_1(a))$ be the lowest eigenvalue of $H_1(a)$. Note that, by symmetry of q, $E_0(a)$ is even.
We prove

Theorem 5.1. *Assume that $E_0(a)$ does not vanish identically for $a \in [-r, r]$. Then, there exists $\delta > 0$ such that H_ω almost surely has pure point spectrum in $I = [E_0, E_0+\delta]$ with exponentially decaying eigenfunctions. Moreover, H_ω is dynamically localized in I, in the sense that for every $\zeta < 1$, (1.3) holds.*

In [1], it is proved that if V has a fixed sign, then $E_0(a)$ does not vanish identically for $a \in [-r, r]$. Thus, under our assumptions in the introduction, we obtain that assumption (Loc) holds in some neighborhood of the bottom of the spectrum of H^D_ω. In [30], Theorem 5.1 was proved when $d \geq 2$. Here, we are going to extend the ideas used to prove it to the one-dimensional case.
The proof of Theorem 5.1 follows a well known strategy: to prove localization in some energy region I, one only needs to prove that, in I, the operator satisfies a Wegner estimate and the resolvent of its restriction to a finite cube satisfies a smallness estimate with a good probability (see e.g. [26, Theorem 5.4]). This strategy is the one followed in [30] that we also follow below.
For any $s \in (0, 1)$ and $\rho = 1$, the Wegner estimate (W) for our model was proved in [30, Theorem 4.1] under no restriction on the dimension. In dimension 1, the same analysis can be improved to give

Theorem 5.2. *There exists $\delta > 0$ and $C > 0$ such that, for any $L > 1$ and any interval $I \subset [\inf \Sigma^D, \inf \Sigma^D + \delta]$, one has*

$$\mathbb{E}(\operatorname{tr} \chi_I(H^D_{\omega,L})) \leq C|I|L. \tag{5.2}$$

Thus, in $[\inf \Sigma^D, \inf \Sigma^D + \delta]$, the integrated density of states $E \mapsto N^D(E)$ is Lipschitz continuous.

To obtain (5.2), it suffices to follow the proof of [30, Theorem 4.1] and in [30, (53)] to use the boundedness of the spectral shift function $E \mapsto \xi(E; -\Delta + V, -\Delta + V + V_0)$ in dimension 1 when V is bounded, and V_0 is bounded and of compact support (see [11, Remark 3.1]).
Recall that $N(E) = N^D(E)$ denotes the integrated density of states of H^D_ω (see (0.3)). The proof of the "smallness" of the resolvent usually relies on a so-called "Lifshitz tail" estimate for $N(E)$. Such an estimate says roughly that, at the bottom of the spectrum (resp. at a so called fluctuational edge of the almost sure spectrum (see e.g. [36])), the function $E \mapsto N(E)$ vanishes very quickly (resp. is very flat).
In dimension 1, in [1, Theorem 4.1], it was proved that such a quick vanishing of N fails for displacement model H^D_ω when the random variables $(\omega_n)_{n\in\mathbb{Z}}$ have a Bernoulli distribution supported in $\{-r, r\}$. It was also conjectured that, when this is not the case, the integrated density of states should be infinitely flat at $\inf \Sigma^D$. This is not the case. Indeed, we prove that, if we assume V to be as above and that

- $(\omega_n)_{n\in\mathbb{Z}}$ are i.i.d random variables supported in $[-r,r]\subset[-1/2+r_0,1/2-r_0]$ which support contains $\{-r,r\}$.

then one has

Theorem 5.3. *In the above setting assume that $\mathbb{P}(\omega_0=r)\mathbb{P}(\omega_0=-r)>0$. Then, there exists $n\geq 0$ such that one has*

$$\lim_{\substack{E\to\inf\Sigma^D\\ E>\inf\Sigma^D}}\frac{N(E)}{(E-\inf\Sigma^D)^n}\to+\infty. \tag{5.3}$$

Under the same conditions on V and $(\omega_n)_{n\in\mathbb{Z}}$, we also prove

Theorem 5.4. *In the above setting assume that $\mathbb{P}(\omega_0=r)+\mathbb{P}(\omega_0=-r)=0$. Then, for any $n\geq 0$, one has*

$$\lim_{\substack{E\to\inf\Sigma^D\\ E>\inf\Sigma^D}}\frac{N(E)}{(E-\inf\Sigma^D)^n}=0. \tag{5.4}$$

Theorem 5.4 is not optimal: for it to be optimal, (5.4) should hold under the weaker assumption $\mathbb{P}(\omega_0=r)\mathbb{P}(\omega_0=-r)=0$.
Let us now complete the proof of Theorem 5.1 using Theorem 5.4. Clearly, under the assumptions in the introduction i.e. when the random variables admit a density, one has $\mathbb{P}(\omega_0\in\{-r,r\})=0$.
We will use the following classical two-sided bound on the integrated density of states obtained using Dirichlet-Neumann bracketing (see e.g. [36, 39]): there exists $C>0$ such that, for $L\geq 1$, one has

$$\frac{1}{L}\mathbb{P}\{E_{D,L}(\omega)\leq E\}\leq N(E)\leq C\,\mathbb{P}\{E_{N,L}(\omega)\leq E\} \tag{5.5}$$

where

- $E_{D,L}(\omega)$ is the ground state of $H^D_{\omega,L}$ with Dirichlet boundary conditions,
- $E_{N,L}(\omega)$ is the ground state of $H^D_{\omega,L}$ with Neumann boundary conditions,
- C is a constant depending only on $\|V\|_\infty$.

We now use it to obtain the initial length scale estimate needed in addition to the Wegner estimate to apply [26, Theorem 5.4]. Indeed, by (5.4) and (5.5), for any $a>0$ and $b\in(0,1)$ there exists $\tilde L=\tilde L(a,b)$ such that, for all $L\geq\tilde L$.

$$\mathbb{P}(H^D_{\omega,L}\text{ (with Dirichlet b.c.) has an eigenvalue less than }\inf\Sigma^D+L^{-b})\leq L^{-a}.$$

Using standard Combes-Thomas estimates (see e.g. [39]), this implies that there exists $C>0$ such that, with probability, as least $1-L^{-a}$, one has

$$\sup_{E\leq\inf\Sigma^D+L^{-b}/2}\|\chi_x(H^D_{\omega,L}-E)^{-1}\chi_y\|\leq e^{-L^{-b}|x-y|/C}$$

where $\chi_x=\mathbf{1}_{[x-1/2,x+1/2]}$.
This estimate immediately yields that assumption (5.7) of [26, Theorem 5.4]) is satisfied in some neighborhood of $\inf\Sigma^D$ for the model H^D_ω considered in the introduction. Thus, we obtain Theorem 5.1.

Let us now return to Theorems 5.3 and 5.4. Before proving these results, let us give a more precise result under a simple additional assumption on the random variables $(\omega_n)_{n\in\mathbb{Z}}$. We prove

Theorem 5.5. *Assume that the common distribution of the displacements* $(\omega_n)_{n\in\mathbb{Z}}$ *satisfies* $\mathbb{P}(\omega_0=-r)+\mathbb{P}(\omega_0=r)=0$ *and*

$$\lim_{\varepsilon\to 0^+}\frac{\log|\log\mathbb{P}(\omega_0\in[-r,-r+\varepsilon])+\log\mathbb{P}(\omega_0\in[r-\varepsilon,r])|}{\log|\log\varepsilon|}=1. \tag{5.6}$$

Then, one has

$$\lim_{\substack{E\to\inf\Sigma^D\\ E>\inf\Sigma^D}}\frac{\log|\log N(E)|}{\log|\log(E-\inf\Sigma^D)|}=2. \tag{5.7}$$

Up to terms of smaller order, the limit (5.7) should be interpreted as

$$N(E)\sim e^{-C|\log(E-\inf\Sigma^D)|^2},$$

and assumption (5.6) as, for some $n_+>n_->0$ and ε positive sufficiently small, one has

$$\varepsilon^{n_-}\leq\mathbb{P}(\omega_0\in[-r,-r+\varepsilon])\leq\varepsilon^{n_+}\quad\text{and}\quad\varepsilon^{n_-}\leq\mathbb{P}(\omega_0\in[r-\varepsilon,r])\leq\varepsilon^{n_+}. \tag{5.8}$$

When the common distribution of the $(\omega_n)_{n\in\mathbb{Z}}$ is even, a lower bound for $N(E)$ was obtained in [1, section 4] (even though it was not stated explicitly); it was of size $e^{-C|\log(E-\inf\Sigma^D)|^3}$.

Remark 5.1. In Theorems 5.4 and 5.5, the smoothness assumption on V can be relaxed quite a bit (see e.g. [1]).

Let us now turn to the proof of Theorems 5.3, 5.4 and 5.5. For $L>0$, consider $H^D_{\omega,L}$ the operator H^D_ω restricted to the interval $[-L+1/2,L+1/2]$; the boundary conditions will be made precise below.
Our main tools to prove Theorems 5.3, 5.4 and 5.5 are the two following lemmas

Lemma 5.1. *There exists* $C>1>c>0$, $\tau\in(0,1)$ *and* $\varepsilon_0>0$ *such that, for* $\varepsilon\in(0,\varepsilon_0)$ *and* $L\geq\varepsilon_0^{-1}$, *one has*

$$\begin{aligned}\mathbb{P}\{E_{D,L}(\omega)\leq\inf\Sigma^D+C(\varepsilon+\tau^{2L})\}&\\ \geq\left[\mathbb{P}\left(\omega_0\in[-r,-r+c\varepsilon]\right)\mathbb{P}\left(\omega_0\in[r-c\varepsilon,r]\right)\right]^L&.\end{aligned} \tag{5.9}$$

and

Lemma 5.2. *Set* $p:=\mathbb{P}(\omega_0\in[-r,0]\in(0,1)$. *Then, there exists* $C>1$ *such, for* $\varepsilon\in(0,1)$ *and* $L\geq 1$, *one has*

$$\mathbb{P}\{E_{N,L}(\omega)\leq\inf\Sigma^D+\varepsilon\}\leq\sum_{k=0}^{L}\sum_{\substack{K\subset\{-L+1,\cdots,L\}\\ \#K=k}}P_{\mathcal{K},L}(\varepsilon) \tag{5.10}$$

where

$$\begin{aligned}P_{\mathcal{K},L}(\varepsilon):=\sum_{m=0}^{L}\prod_{n\in\mathcal{K}}\mathbb{P}\left(\omega_0\in\left[-r,-r+CL\,e^{C|n-m|}\varepsilon\right]\right)&\\ \prod_{n\notin\mathcal{K}}\mathbb{P}(\omega_0\in[r-CL\,e^{C|n-m|}\varepsilon,r]).&\end{aligned} \tag{5.11}$$

Let us now show how these lemmas are used to prove Theorems 5.3, 5.4 and 5.5. We start with Theorem 5.3 and the lower bound in Theorem 5.5. Pick ε positive small and L such that

$$L-1\leq\alpha|\log\varepsilon|\leq L. \tag{5.12}$$

where $\alpha>0$. If we pick $\alpha\geq(2\log\tau)^{-1}$ then $\tau^{2L}\leq\varepsilon$. Under the assumptions of Theorem 5.3, the bound (5.9) and the lower bound in (5.12) yield, for some $\nu\in(0,1)$,

$$N(\inf\Sigma^D+2C\varepsilon)\geq\nu^{|\log\varepsilon|}=\varepsilon^{|\log\nu|}.$$

One completes the proof of Theorem 5.3 by taking $n>|\log\nu|$.
The lower bound in (5.7) in Theorem 5.5 is obtained in the same way. For α in (5.12) sufficiently large, we obtain that

$$\begin{aligned}\log|\log N(\inf\Sigma^D+2C\varepsilon)|&\\&\geq\log L+\log\left|\log\mathbb{P}\left(\omega_0\in[-r,-r+c\varepsilon]\right)+\log\mathbb{P}\left(\omega_0\in[r-c\varepsilon,r]\right)\right|.\end{aligned}$$

Thus, assumption (5.6) and the bound (5.12) immediately yield the lower bound in (5.7).
Let us now turn to the proof of Theorem 5.4 and the upper bound in Theorem 5.5. We again pick ε positive small and L such that (5.12) be satisfied for some $\alpha>0$. Now α is chosen so small that $C\alpha<1/4$ where C is given by Lemma 5.2. Thus, for ε small and $(n,m)\in\{-L+1,\cdots,L\}^2$, one has $CL\,e^{C|n-m|}\varepsilon\leq\sqrt{\varepsilon}$ and (5.10) becomes

$$\begin{aligned}&\mathbb{P}\{E_{N,L}(\omega)\leq\inf\Sigma^D+\varepsilon\}\\&\qquad\leq L^2\,2^L\,\left(\mathbb{P}\left(\omega_0\in\left[-r,-r+\sqrt{\varepsilon}\right]\right)+\mathbb{P}\left(\omega_0\in\left[r-\sqrt{\varepsilon},r\right]\right)\right)^{2L}\end{aligned} \tag{5.13}$$

Under the assumptions of Theorem 5.4 or Theorem 5.5, for ε small, we get

$$\begin{aligned}&\log\mathbb{P}\{E_{N,L}(\omega)\leq\inf\Sigma^D+\varepsilon\}\\&\qquad\leq-\alpha|\log\varepsilon|\log\left(\mathbb{P}\left(\omega_0\in\left[-r,-r+\sqrt{\varepsilon}\right]\right)+\mathbb{P}\left(\omega_0\in\left[r-\sqrt{\varepsilon},r\right]\right)\right).\end{aligned}$$

This immediately gives (5.4) under the assumptions of Theorem 5.4 and the upper bound in (5.7) under those of Theorem 5.5. Hence, the proofs of Theorem 5.4 and Theorem 5.5 are complete.

The proof of Lemma 5.1. As V is smooth and compactly supported, we know that there exists $c\in(0,1)$, such that for any admissible $(\omega_n)_{-L+1\leq n\leq L}$ and $(\omega'_n)_{-L+1\leq n\leq L}$, one has

$$\begin{aligned}\|H^D_{\omega,L}-H^D_{\omega',L}\|&=\sup_{[-L+1/2,L+1/2]}\left|\sum_{n\in\mathbb{Z}}V(\cdot-n-\omega_n)-\sum_{n\in\mathbb{Z}}V(\cdot-n-\omega'_n)\right|\\&\leq c^{-1}\sup_{-L+1\leq n\leq L}|\omega_n-\omega'_n|.\end{aligned} \tag{5.14}$$

Here, $\|\cdot\|$ denotes the operator norm and the estimate does not depend on the boundary conditions used to define $H^D_{\omega',L}$ (provided we use the same boundary conditions for $H^D_{\omega',L}$ and $H^D_{\omega,L}$).
Recall that, for $a\in[-r,r]$, we have defined $H_1(a)=-\Delta+q(x-a)$ on $L^2(-1/2,1/2)$ with Neumann boundary condition and $E_0(a)=\inf\sigma(H_1(a))$ to be the lowest eigenvalue of $H_1(a)$. Let $\psi_0(a;x)$ be the associated positive ground state. Note that, by symmetry, one has $\psi_0(-a;x)=\psi_0(a;-x)$. By [1, Lemma 3.2], we know that

$\psi_0(a;-1/2) \neq \psi_0(a;1/2)$ as $a \mapsto E_0(a)$ is supposed not to be constant. For $a = r$, assume that

$$0 < \frac{\psi_0(r;-1/2)}{\psi_0(r;1/2)} := \tau < 1$$

If this is not the case, in the construction that follows, we invert the parts of r and $-r$.

By the results of [1], we know that $E(-r) = E(r) = \inf \Sigma^D$.

Consider the event

$$\Omega_{L,\varepsilon} = \begin{Bmatrix} \forall n \in \{-L+1, 0\}, & |\omega_n + r| \leq \varepsilon \\ \forall n \in \{1, L\}, & |\omega_n - r| \leq \varepsilon \end{Bmatrix};$$

The $(\omega_n)_{n\in\mathbb{Z}}$ being independent, the probability of this event is bounded from below

$$\mathbb{P}(\Omega_{L,\varepsilon}) \geq [\mathbb{P}(\omega_0 \in [-r, -r+\varepsilon])\mathbb{P}(\omega_0 \in [r-\varepsilon, r])]^L. \tag{5.15}$$

For the realization $(\omega_n^r)_{-L+1\leq n\leq L}$ defined by $\omega_n^r = -r$ if $n \in \{-L+1, 0\}$ and $\omega_n^r = r$ if $n \in \{1, L\}$, we know (see [1]) that $\psi_{\omega^r,L}$, the normalized positive ground state of $H^D_{\omega,L}$ with Neumann boundary conditions, is given by

$$\psi_{\omega^r,L}(x) = \frac{1}{C_0}\begin{cases} \tau^{-n}\,\psi_0(r; n-x) \text{ if } n \in \{-L+1, 0\} \\ \tau^{n-1}\,\psi_0(r; x-n) \text{ if } n \in \{1, L\} \end{cases} \quad \text{for } -\frac{1}{2} \leq x - n \leq \frac{1}{2}$$

where

$$C_0^2 = \sum_{n=0}^{L-1} \tau^{2n} \int_{-1/2}^{1/2} |\psi_0(r;x)|^2 dx = \frac{1-\tau^{2L}}{1-\tau^2} > 1.$$

Here, we have used the symmetries of $(a, x) \mapsto \psi_0(a; x)$ and the fact that it is normalized.

Pick $\chi : (-L+1/2, L+1/2) \to \mathbb{R}^+$ smooth such that $0 \leq \chi \leq 1$, $\chi \equiv 1$ on $(-L+1, L-1)$ and it vanishes identically near $L+1/2$ and $-L+1/2$. Consider the function $\phi = \chi\psi_{\omega^r,L}$. It satisfies Dirichlet boundary conditions at $L+1/2$ and $-L+1/2$. Moreover, using (5.14), for $\omega \in \Omega_{r,c\varepsilon}$ (recall that c is defined in (5.14)), one computes that $1 - \tau^{2L}C_0^{-2} \leq \|\phi\|^2 \leq 1$ and there exists $C > 0$ such that

$$\|(H^D_{\omega,L} - E_0(r))\phi\|^2 \leq C\left(\tau^{2L} + \varepsilon\right)^2 \leq C^2\left(\tau^{2L} + \varepsilon\right)^2 \|\phi\|^2. \tag{5.16}$$

This and estimate (5.15) immediately yields (5.9) and completes the proof of Lemma 5.1. □

The proof of Lemma 5.2. We are going to rely on the analysis done for the Lifshitz tails regime in [30, section 3]. Define the random variable ω_0^e and $\tilde{\omega}_0$ as $\omega_0^e = r\left(\mathbf{1}_{\omega_0>0} - \mathbf{1}_{\omega_0\leq 0}\right)$ and $\tilde{\omega}_0 = |\omega_0 - \omega_0^e|$ conditioned on ω_0^e. Note that, under the assumptions of Theorems 5.4 and 5.5, $\tilde{\omega}_0$ is not identically vanishing. In the same way, for any $n \in \mathbb{Z}$, define ω_n^e and $\tilde{\omega}_n$. Then, though not stated directly in this way, the following result is proved in [30, section 3]

Lemma 5.3 ([30])**.** *There exists $C > 0$ such that*

$$H^D_{\omega,L} - \inf \Sigma^D \geq \frac{1}{C}\left(H^D_{\omega^e,L} - \inf \Sigma^D + V_{\tilde{\omega},L}\right)$$

where

$$V_{\tilde{\omega},L}(x) = \sum_{n=-L+1}^{L} \tilde{\omega}_n \mathbf{1}_{[-1/2,1/2]}(x-n).$$

Using this decomposition and recalling that $p = \mathbb{P}(\omega_0 \in [-r, 0] \in (0,1)$, we can write

$$\mathbb{P}\{E_{N,L}(\omega) \leq \inf \Sigma^D + \varepsilon\} \leq \sum_{k=0}^{L} p^k(1-p)^{L-k} \sum_{\substack{K\subset\{-L+1,\cdots,L\}\\ \#K=k}} \tilde{P}_{\mathcal{K},L}(\varepsilon) \tag{5.17}$$

where

$$\tilde{P}_{\mathcal{K},L}(\varepsilon) = \mathbb{P}\left\{ \begin{array}{c} \exists\varphi \in C^1,\ \|\varphi\| = 1 \text{and } \exists E \in [0, C\varepsilon] \text{ s.t.} \\ \left(H^D_{\omega^e,L} - \inf \Sigma^D + V_{\tilde{\omega},L} - E\right)\varphi = 0 \end{array} \middle| \begin{array}{l} \omega^e_n = -r \text{ for } n \in \mathcal{K} \\ \omega^e_n = r \text{ for } n \notin \mathcal{K} \end{array} \right\}.$$

Lemma 6.1 guarantees that there exists $C > 0$ (independent of L and the realization ω) such that, if φ is a solution to $(H^D_{\omega^e,L} - \inf \Sigma^D + V_{\tilde{\omega},L} - E)\varphi = 0$,

$$\forall (m,n) \in \{-L+1, \cdots, L\}, \quad \int_{-1/2}^{1/2} |\varphi(x-n)|^2 dx \leq e^{C|m-n|} \int_{-1/2}^{1/2} |\varphi(x-m)|^2 dx.$$

If φ is normalized, we know that one has $\displaystyle\int_{-1/2}^{1/2} |\varphi(x-n)|^2 dx \geq (2L)^{-1}$ for some $n \in \{-L+1, \cdots, L\}$.

As $H^D_{\omega^e,L} - \inf \Sigma^D \geq 0$, these two properties imply that

$$\begin{aligned} \tilde{P}_{\mathcal{K},L}(\varepsilon) &\leq \sum_{m=-L+1}^{L} \mathbb{P}\left\{ \sum_{n=-L+1}^{L} \tilde{\omega}_n e^{-C|m-n|} \leq 2CLE \middle| \begin{array}{l} \omega^e_n = -r \text{ for } n \in \mathcal{K} \\ \omega^e_n = r \text{ for } n \notin \mathcal{K} \end{array} \right\} \\ &\leq \sum_{m=-L+1}^{L} \prod_{n\in\mathcal{K}} \mathbb{P}\left(\omega_0 \in \left[-r, -r + 2CL\, e^{C|n-m|}\varepsilon\right] |\omega^e_n = -r\right) \\ &\qquad\qquad \prod_{n\notin\mathcal{K}} \mathbb{P}(\omega_0 \in [r - 2CL\, e^{C|n-m|}\varepsilon, r]|\omega^e_n = r). \end{aligned}$$

Using the definition of $(\omega^e_n)_{n\in\mathbb{Z}}$, we immediately obtain the bound (5.10) and thus complete the proof of Lemma 5.2. □

6. Appendix

In this appendix, we collect various technical results that were used in our study.

6.1. Some results on differential equations.

We recall some standard estimates on ordinary differential equations that are immediate consequences of equations (2.2) and (2.3), and, presumably well known (see e.g. [42, 14]). We use the notation of section 2.

Lemma 6.1. *There exists a constant $C > 0$ (depending only on $\|q\|_\infty$) such that, for u a solution to $Hu = 0$ (see (2.1)), if $I(x) := [x - 1/2, x + 1/2] \cap [0, \ell]$, one has*

$$\forall x \in [0,\ell], \quad \frac{1}{C}\int_{I(x)} u^2(y)dy \leq r_u^2(x) \leq C \int_{I(x)} u^2(y)dy, \tag{6.1}$$

$$\forall x \in [0,\ell], \quad \min_{I(x)} r_u \leq \max_{I(x)} r_u \leq C \min_{I(x)} r_u, \tag{6.2}$$

$$\forall x \in [0,\ell], \quad \|\sin(\varphi_u(\cdot))\|_{L^2(I(x))} \geq \frac{1}{C}. \tag{6.3}$$

Lemma 6.2. *Let $\delta\varphi$ be a solution to the equation (2.5). There exists $C > 0$ (depending only on $\|q\|_\infty$) such that, for $x_0 \in [0, \ell]$, one has*

$$\forall x \in [0, \ell], \quad |\sin(\delta\varphi(x))| \leq [|\sin(\delta\varphi(x_0))| + E\ell]e^{C|x-x_0|}.$$

Proof. Write $s(x) = |\sin(\delta\varphi(t))|$ and note that, integrating equation (2.5) implies that

$$s(x) \leq s(x_0) + E\ell + C\int_{x_0}^{x} s(t)dt.$$

The statement of Lemma 6.2 then follows from Gronwall's Lemma (see e.g. [41]). □

Lemma 6.3. *There exists $\eta_0 > 0$ depending only on $\|q\|_\infty$ such that, for $\eta \in (0, \eta_0)$ and φ_u, a solution to equation (2.2), one has*

(1) *if $y < y'$ are such that $\max_{x\in[y,y']} |\sin\varphi_u(x)| \leq \eta$, then $|y - y'| \leq \eta/\eta_0$;*

(2) *if $|\sin(\varphi_u(y))| \leq \eta$ then, for $4\eta \leq |x - y| \leq \sqrt{\eta}$, one has*

$$|\sin(\varphi_u(y))| \geq |x - y|/2.$$

(3) *if $y < y'$ are such that*

$$|\sin\varphi_u(y)| = |\sin\varphi_u(y')| = \eta \text{ and } \min_{x\in[y,y']} |\sin\varphi_u(x)| \geq \eta$$

then $|y - y'| \geq (\eta_0 - \eta)\eta_0$.

Proof. First, by equation (2.2), for some $C > 0$ depending only on $\|q\|_\infty$, one has $|\varphi_u'(x)| \leq C$ and, if $|\sin(\varphi_u(x))| \leq \eta$ then $1 - C\eta^2 \leq |\cos\varphi_u(x)|\varphi_u'(x)$. Pick $\eta_0 \in (0, 1)$ such that $1 - C\eta_0^2 \geq 1/2$.
To prove point (1), consider $y < y'$ such that $\max_{x\in[y,y']} |\sin\varphi_u(x)| \leq \eta$. As $\eta < \eta_0 < 1$, $\cos\varphi_u(x)$ does not change sign on $[y, y']$. Thus, one computes

$$2\eta \geq |\sin\varphi_u(y') - \sin\varphi_u(y)| = \int_y^{y'} |\cos\varphi_u(x)|\varphi_u'(x)dx \geq |y - y'|/2.$$

This proves (1) possibly diminishing the value of η_0.
To prove point (2), as by equation (2.2), for some $C > 0$ depending only on $\|q\|_\infty$, one has $|\varphi_u'(x)| \leq C$, there exists $\eta_0 > 0$ such that, for $\eta \in (0, \eta_0]$, if $|\sin(\varphi_u(y))| \leq \eta$, one has $|\sin(\varphi_u(x))| \leq \eta_0$ for $|x - y| \leq \eta_0$. Thus, at the possible cost of reducing η_0, $x \mapsto |\cos(\varphi_u(x))|$ stays larger than 9/10 on $[y - \eta_0, y + \eta_0]$, and, by equation (2.2), one has $d/dx[\sin(\varphi_u(x))] \geq 3/4$ on $[y - \eta_0, y + \eta_0]$. This, the assumption $|\sin(\varphi_u(y))| \leq \eta$ and the Taylor formula immediately entail point (2).
To prove point (3), note that, as $|\varphi_u'(x)| \leq C$, for $y < z < y + (\eta_0 - \eta)/C$, one has $|\sin(\varphi_u(z))| \leq \eta_0$. Thus, $x \mapsto \cos\varphi_u(x)$ keeps a constant sign on the interval $[y, \min(y', y + (\eta_0 - \eta)/C)]$. Moreover, as $\min_{x\in[y,y']} |\sin\varphi_u(x)| \geq \eta$, so does $x \mapsto \sin\varphi_u(x)$ and both signs are the same. Thus, for $y < z < y + (\eta_0 - \eta)/C$, we know that

$$\begin{aligned}|\sin\varphi_u(z)| &= |\sin\varphi_u(y)| + \int_y^z |\cos\varphi_u(x)|\varphi_u'(x)dx \\ &\geq \eta + \sqrt{1 - (\eta')^2}(z - y)/2 > \eta.\end{aligned}$$

Hence, one has $y' > y + (\eta_0 - \eta)/C$. This proves (2) at the expense of possibly changing η_0 again. This completes the proof of Lemma 6.3. □

References

[1] Jeff Baker, Michael Loss, and Günter Stolz. Low energy properties of the random displacement model. *J. Funct. Anal.*, 256(8):2725–2740, 2009.
[2] Jean V. Bellissard, Peter D. Hislop, and Günter Stolz. Correlation estimates in the Anderson model. *J. Stat. Phys.*, 129(4):649–662, 2007.
[3] Hakim Boumaza. Localization for a matrix-valued Anderson model. *Math. Phys. Anal. Geom.*, 12(3):225–286, 2009.
[4] Jean Bourgain and Carlos E. Kenig. On localization in the continuous Anderson-Bernoulli model in higher dimension. *Invent. Math.*, 161(2):389–426, 2005.
[5] Dirk Buschmann and Günter Stolz. Two-parameter spectral averaging and localization for non-monotonic random Schrödinger operators. *Trans. Amer. Math. Soc.*, 353(2):635–653 (electronic), 2001.
[6] René Carmona and Jean Lacroix. *Spectral theory of random Schrödinger operators*. Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1990.
[7] Jean-Michel Combes, Pierre Duclos, and Rudi Seiler. Convergent expansions for tunneling. *Commun. Math. Phys.*, 92:229, 1983.
[8] Jean-Michel Combes, Pierre Duclos, and Rudi Seiler. Kreĭn's formula and one-dimensional multiple-well. *J. Funct. Anal.*, 52(2):257–301, 1983.
[9] Jean-Michel Combes, François Germinet, and Abel Klein. Generalized eigenvalue-counting estimates for the Anderson model. *J. Stat. Phys.*, 135(2):201–216, 2009.
[10] Jean-Michel Combes, François Germinet, and Abel Klein. Poisson statistics for eigenvalues of continuum random Schrödinger operators. *Anal. PDE*, 3(1):49–80, 2010.
[11] Jean-Michel Combes, Peter D. Hislop, and Frédéric Klopp. Hölder continuity of the integrated density of states for some random operators at all energies. *Int. Math. Res. Not.*, (4):179–209, 2003.
[12] Jean-Michel Combes, Peter D. Hislop, and Frédéric Klopp. An optimal Wegner estimate and its application to the global continuity of the integrated density of states for random Schrödinger operators. *Duke Math. J.*, 140(3):469–498, 2007.
[13] David Damanik, Robert Sims, and Günter Stolz. Localization for one-dimensional, continuum, Bernoulli-Anderson models. *Duke Math. J.*, 114(1):59–100, 2002.
[14] David Damanik and Günter Stolz. A continuum version of the Kunz-Souillard approach to localization in one dimension, 2009. ArXiv: http://fr.arxiv.org/abs/arXiv:0912.3568.
[15] F. Germinet and A. Klein. A comprehensive proof of localization for continuous Anderson models with singular random potentials, 2011. ArXiv: http://fr.arxiv.org/abs/arXiv:1105.0213.
[16] François Germinet, Peter D. Hislop, and Abel Klein. Localization at low energies for attractive Poisson random Schrödinger operators. In *Probability and mathematical physics*, volume 42 of *CRM Proc. Lecture Notes*, pages 153–165. Amer. Math. Soc., Providence, RI, 2007.
[17] François Germinet, Peter D. Hislop, and Abel Klein. Localization for Schrödinger operators with Poisson random potential. *J. Eur. Math. Soc. (JEMS)*, 9(3):577–607, 2007.
[18] François Germinet and Frédéric Klopp. Spectral statistics for random Schrödinger operators in the localized regime. ArXiv http://arxiv.org/abs/1011.1832, 2010.
[19] François Germinet and Frédéric Klopp. Enhanced Wegner and Minami estimates and eigenvalue statistics of random Anderson models at spectral edges. ArXiv http://arxiv.org/abs/1111.1505, 2011.
[20] I. Ja. Gol′dšeĭd, S. A. Molčanov, and L. A. Pastur. A random homogeneous Schrödinger operator has a pure point spectrum. *Funkcional. Anal. i Priložen.*, 11(1):1–10, 96, 1977.
[21] Gian Michele Graf and Alessio Vaghi. A remark on the estimate of a determinant by Minami. *Lett. Math. Phys.*, 79(1):17–22, 2007.
[22] Eman Hamza, Robert Sims, and Günter Stolz. A note on fractional moments for the one-dimensional continuum Anderson model. *J. Math. Anal. Appl.*, 365(2):435–446, 2010.
[23] Evans M. Harrell. Double wells. *Comm. Math. Phys.*, 75(3):239–261, 1980.
[24] Bernard Helffer and Johannes Sjöstrand. Multiple wells in the semiclassical limit. I. *Comm. Partial Differential Equations*, 9(4):337–408, 1984.
[25] Peter D. Hislop and Frédéric Klopp. The integrated density of states for some random operators with nonsign definite potentials. *J. Funct. Anal.*, 195(1):12–47, 2002.

[26] Abel Klein. Multiscale analysis and localization of random operators. In *Random Schrödinger operators*, volume 25 of *Panor. Synthèses*, pages 121–159. Soc. Math. France, Paris, 2008.
[27] Frédéric Klopp. Localization for some continuous random Schrödinger operators. *Comm. Math. Phys.*, 167(3):553–569, 1995.
[28] Frédéric Klopp. Asymptotic ergodicity of the eigenvalues of random operators in the localized phase. ArXiv: http://fr.arxiv.org/abs/1012.0831, 2010.
[29] Frédéric Klopp. Decorrelation estimates for the eigenlevels of the discrete Anderson model in the localized regime. *Comm. Math. Phys.*, 303(1):233–260, 2011.
[30] Frédéric Klopp, Michael Loss, Shu Nakamura, and Günter Stolz. Localization for the random displacement model, 2010. ArXiv http://fr.arxiv.org/abs/arXiv:1007.2483, to appear in Duke Math. Jour.
[31] Lev Landau and Evgueniy Lifschitz. *Physique théorique ("Landau-Lifchitz"). Tome III: Mécanique quantique. Théorie non relativiste*. Éditions Mir, Moscow, 1967. Deuxième édition, Traduit du russe par Édouard Gloukhian.
[32] Nariyuki Minami. Local fluctuation of the spectrum of a multidimensional Anderson tight binding model. *Comm. Math. Phys.*, 177(3):709–725, 1996.
[33] Nariyuki Minami. Theory of point processes and some basic notions in energy level statistics. In *Probability and mathematical physics*, volume 42 of *CRM Proc. Lecture Notes*, pages 353–398. Amer. Math. Soc., Providence, RI, 2007.
[34] Nariyuki Minami. Energy level statistics: a formulation and some examples. In N. Minami, editor, *Spectra of random operators and related topics*, 2011. To appear.
[35] Stanislav A. Molchanov. The local structure of the spectrum of a random one-dimensional Schrödinger operator. *Trudy Sem. Petrovsk.*, 8:195–210, 1982.
[36] Leonid Pastur and Alexander Figotin. *Spectra of random and almost-periodic operators*, volume 297 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1992.
[37] A. Rivkind, Y. Krivolapov, S. Fishman, and A. Soffer. Eigenvalue repulsion estimates and some applications for the one-dimensional Anderson model. *Journal of Physics A Mathematical General*, 44:D5206, July 2011.
[38] Robert Sims and Günter Stolz. Localization in one dimensional random media: a scattering theoretic approach. *Comm. Math. Phys.*, 213(3):575–597, 2000.
[39] Peter Stollmann. *Caught by disorder*, volume 20 of *Progress in Mathematical Physics*. Birkhäuser Boston Inc., Boston, MA, 2001. Bound states in random media.
[40] Günter Stolz. Localization for random Schrödinger operators with Poisson potential. *Ann. Inst. H. Poincaré Phys. Théor.*, 63(3):297–314, 1995.
[41] Joachim Weidmann. *Spectral theory of ordinary differential operators*, volume 1258 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.
[42] Joachim Weidmann. *Spectral theory of ordinary differential operators*. Springer-Verlag, Berlin-New York, 1987.

(Frédéric Klopp) IMJ, UMR CNRS 7586 Université Pierre et Marie Curie Case 186, 4 place Jussieu F-75252 Paris cedex 05 France
E-mail address: klopp@math.jussieu.fr