STARK-WANNIER LADDERS AND CUBIC EXPONENTIAL SUMS

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On $L^2(\mathbb{R})$, we consider the Schrödinger operator

$$H_\epsilon = -\frac{\partial^2}{\partial x^2} + v(x) - \epsilon x, \tag{1.1}$$

where v is a real analytic 1-periodic function and ϵ is a positive constant. This operator is a model to study a Bloch electron in a constant electric field ([1]). The parameter ϵ is proportional to the electric field. The operator (1.1) was studied both by physicists (see, e.g., the review [6]) and by mathematicians (see, e.g., [9]). Its spectrum is absolutely continuous and fills the real axis. One of main features of H_ϵ is the existence of Stark-Wannier ladders. These are ϵ-periodic sequences of resonances, which are poles of the analytic continuation of the resolvent kernel in the lower half plane through the spectrum (see, e.g., [2]). Most of the mathematical work studied the case of small ϵ (see, e.g., [9, 3] and references therein). When ϵ is small, there are ladders exponentially close to the real axis. Actually, only the case of finite gap potentials v was relatively well understood. For these potentials, there is only a finite number of ladders exponentially close to the real axis. It was further noticed that the ladders non-trivially "interact" as ϵ changes, and conjectured that the behavior of the resonances strongly depends on number theoretical properties of ϵ (see, e.g., [1]).

In the present note, we only consider the periodic potential $v(x) = 2\cos(2\pi x)$ and study the reflection coefficient $r(E)$ of the Stark-Wannier operator (1.1) in the lower half of the complex plane of the spectral parameter E. The resonances are the poles of the reflection coefficient. We show that, as $\operatorname{Im} E \to -\infty$, the function $E \mapsto \dfrac{1}{r(E)}$ can be asymptotically described in terms of a regularized cubic exponential sum that is a close relative of the cubic exponential sums often encountered in analytic number theory. This explains the dependence of the reflection coefficient on the arithmetic

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nature of ϵ. For $\frac{\pi^2}{3\epsilon} \in \mathbb{Q}$, we describe the asymptotics of the Stark-Wannier ladders situated far from the real axis.

Let us recall the definition of the reflection coefficient for (1.1) following [2]. Consider the equation

$$-\psi''(x) + (v(x) - \epsilon x)\psi(x) = E\psi(x), \quad x \in \mathbb{C}, \tag{1.2}$$

For the sake of simplicity, assume that the potential v is entire. Assume also $\int_0^1 v(x)\,dx = 0$. For any $E \in \mathbb{C}$, there are unique solutions $\psi_\pm$ to (1.2) that admit the asymptotic representations

$$\begin{aligned} \psi_-(x,E) &= \tfrac{1}{\sqrt[4]{-\epsilon x - E}} e^{-\int_x^{-E/\epsilon} \sqrt{-\epsilon t - E}\,dt + o(1)}, \quad x \to -\infty, \\ \psi_+(x,E) &= \tfrac{1}{\sqrt[4]{\epsilon x + E}} e^{i\int_{-E/\epsilon}^{x} \sqrt{\epsilon t + E} dt + o(1)}, \quad x \to +\infty, \end{aligned} \tag{1.3}$$

where the determinations of $\sqrt{\cdot}$ and $\sqrt[4]{\cdot}$ are analytic in $\mathbb{C}\setminus\mathbb{R}_-$ and positive along $\mathbb{R}_+$. Consider also the solution $\psi_+^*(x,E) = \overline{\psi_+(\bar{x},\bar{E})}$. The solutions ψ_+ and ψ_+^* being linearly independent, one has

$$\psi_-(x,E) = w(E)\psi_+^*(x,E) + w^*(E)\psi_+(x,E), \quad x \in \mathbb{R}, \tag{1.4}$$

where the coefficient $w(E)$ is independent of x and the function $E \mapsto w(E)$ is entire. The ratio $r(E) = w^*(E)/w(E)$ is *the reflection coefficient.* It is an ϵ-periodic meromorphic function of E. The reflection coefficient is analytic in $\mathbb{C}_+$, and, for $E \in \mathbb{R}$, one has $|r(E)| = 1$. The poles of r are the resonances of H_ϵ.

Let us now state the first of our results. Represent $1/r$ by its Fourier series $1/r(E) = \sum_{m\in\mathbb{Z}} e^{2\pi niE/\epsilon} p(m)$ for $\operatorname{Im} E \le 0$. Let $a(\epsilon) = \sqrt{\frac{2}{\epsilon}}\,\pi e^{i\pi/4}$. One has

Theorem 1. *Let $v(x) = 2\cos(2\pi x)$. Then, as $m \to \infty$,*

$$p(m) = a(\epsilon)\sqrt{m}\, e^{-2\pi i\omega m^3 - 2m\log(2\pi m/e) + \delta(m)}, \quad \omega = \left\{\frac{\pi^2}{3\epsilon}\right\}, \tag{1.5}$$

where, for x real, $\{x\}$ denotes the fractional part of x, and $\delta(m) = O(\log^2 m/m)$. This estimate is locally uniform in $\epsilon > 0$.

Clearly, the asymptotic behavior of $1/r(E)$ as $\operatorname{Im} E \to -\infty$ is determined by the Fourier series terms with large positive m, and so, roughly,

$$\frac{1}{r(E)} \approx a(\epsilon)\,\mathcal{P}(E/\epsilon), \quad \mathcal{P}(s) = \sum_{m\ge 1} \sqrt{m}\, e^{-2\pi i\omega m^3 - 2m\log(2\pi m/e) + 2\pi ims}. \tag{1.6}$$

It is worth to compare the function $\mathcal{P}$ with the cubic exponential sums $\sum_{n=1}^{N} e^{-2\pi i\omega n^3}$. Such sums were extensively studied in analytic number theory, see, e.g., [4]. They were proved to depend strongly on the arithmetic nature of ω. This appears to be true in our case too. We have

Theorem 2. *Let $v(x) = 2\cos(2\pi x)$. Assume that $\omega \in \mathbb{Q}$ and represent it in the form $\omega = \frac{p}{q}$, where $0 \le p < q$ are co-prime integers. If $p = 0$, we take $q = 1$. For $\xi \in \mathbb{R}$, we set $I_q(\xi) := \{m \in \mathbb{Z} : |\xi - \frac{m}{q}| \le 1/2\}$. As $\operatorname{Im} E \to -\infty$, one has*

$$r^{-1}(E) = \tfrac{b(\epsilon)\,\rho}{q} \sum_{m\in I_q(\xi)} S_q(p,m) e^{\rho e^{i\pi(\xi-m/q)} + i\pi(\xi-m/q) + O(\log^2\rho/\rho)} + e^{O(\frac{\rho}{\ln\rho})}, \tag{1.7}$$

where $b(\epsilon) = \pi^{\frac{3}{2}} e^{i\pi/4}/\sqrt{2\epsilon}$, $\xi = \operatorname{Re} E/\epsilon$, $\rho = e^{-\pi \operatorname{Im} E/\epsilon}$, and

$$S_q(p,m) = \sum_{l=0}^{q-1} e^{-2\pi i \,\frac{pl^3 - ml}{q}}.$$

The error estimates are locally uniform in $\epsilon > 0$.

Let us discuss this result. First, assume that $\omega = 0$. By Theorem 2,

$$(b(\epsilon) r(E))^{-1} = \sqrt{z}\, e^{\sqrt{z} + O(\frac{\ln^2 z}{\sqrt{z}})} + e^{O(\frac{\sqrt{z}}{\ln z})}, \quad z = e^{2i\pi E/\epsilon}, \tag{1.8}$$

where the determination of $\sqrt{\cdot}$ is analytic in $\mathbb{C} \setminus \mathbb{R}_-$ and positive along $\mathbb{R}_+$. Recall that $1/r$ is ϵ-periodic. Let $B_\epsilon = \{E \in \mathbb{C} : \operatorname{Im} E \le 0,\ 0 \le \operatorname{Re} E \le \epsilon\}$. Representation (1.8) implies

Corollary 1. *Assume $\omega = 0$. The resonances located in B_ϵ have the following properties :*

- *for sufficiently large $y > 0$, the resonances with $\operatorname{Im} E < -\epsilon y$ are located in the domain $|\operatorname{Re} E - \epsilon/2| \le C\epsilon^2/|\operatorname{Im} E|$, where $C > 0$ is a constant;*
- *let $n(y)$ be the number of resonances in the rectangle $[0,\varepsilon] - i\,[0,\varepsilon y]$; then, one has*
$$n(y) = \frac{1}{\pi}\, e^{\pi y + o(1)} \quad \text{as} \quad y \to \infty.$$

The first statement immediately follows from Theorem 2; to prove the second one has to use Jensen formula and Levin lower bounds for the absolute values of entire functions, see, e.g., [8].
When $\omega = 0$, it is difficult to obtain the asymptotics of the resonances as, in a neighborhood of the line $\operatorname{Re} E/\epsilon = 1/2 \mod 1$, they are determined by the first Fourier coefficients of $1/r$, i.e., by $p(m)$ with $m = 1, 2, 3 \ldots$. Hence, the problem is not asymptotic in nature.

If $\omega \ne 0$, then the description of the resonances is determined by the values of $S_q(p,m)$ for $m = 1, 2, \ldots q-1$ (the map $m \to S_q(p,m)$ is q-periodic). The $S_q(p,m)$ are *cubic complete rational exponential sums*, see, e.g., [7]. One easily checks

Lemma 1. *For any $q \in \mathbb{N}$,* $\displaystyle\sum_{m=0}^{q-1} |S_q(p,m)|^2 = q^2.$

This implies that, for $q \geq 1$, there is at least one integer $0 \leq m_0 < q-1$ such that $S_q(p, m_0) \neq 0$.

If $S_q(p,m)$ is non zero for only one $0 \leq m_0 < q$ (this happens, for example, for $q = 2, 3, 6$), then one can characterize the resonances as when $\omega = 0$. Now, they live near the lines $\{\operatorname{Re} E/\epsilon = m_0/q + 1/2 + n\}$, $n \in \mathbb{Z}$.

For large q, there are actually many non-zero values $S_q(p,m)$:

Lemma 2. *There exists a constant $C > 0$ such that, for any co-prime $q > p > 0$, one has $\#\{0 \leq m < q :\ S_q(p,m) \neq 0\} \geq Cq^{\frac{2}{3}}$.*

This statement follows from Lemma 1 and the well-known upper bound for general complete rational exponential sums of Hua ([7]).

In general, the behavior of $m \mapsto S_q(p,m)$ is nontrivial; it is known to depend strongly on the prime factorization of q. Computer calculations lead to the following conjecture: *if q is prime, $0 < p < q$, and $0 < m < q$, then $S_q(p,m) \neq 0$.*

If $S_q(p,m)$ is non zero for at least two values of m such that $0 \leq m < q$, then, using (1.5), one can describe asymptotically all the resonances with sufficiently negative imaginary part. One has

Corollary 2. *Assume that, for some integers $m_1 < m_2$ such that $m_2 - m_1 < q$, one has $S_q(p,m_1) \neq 0$, $S_q(p,m_2) \neq 0$, and $S_q(p,m) = 0$ for all $m_1 < m < m_2$. Then, for sufficiently large $y > 0$, in the vertical half-strip*

$$\left\{E \in \mathbb{C} :\ -\operatorname{Im} E \geq \epsilon y,\ \ \frac{m_1}{q} \leq \frac{\operatorname{Re} E}{\epsilon} \leq \frac{m_2}{q}\right\},$$

there are resonances, and they are described by the asymptotic formulas:

$$\frac{E}{\epsilon} = -i\left(\frac{\ln(\pi k)}{\pi} - \ln\sin\frac{\pi(m_2 - m_1)}{q}\right) + \frac{m_2 + m_1}{q} + o(1), \quad k \in \mathbb{N}, \tag{1.9}$$

where $o(1) \underset{k \to +\infty}{\to} 0$.

This statement easily follows from Theorem 2.

Finally, let us describe very briefly the ideas leading to Theorems 1 and 2. Buslaev's solutions $\psi_\pm$ used to define the reflection coefficient (see (1.3)) are entire functions of x and E; they satisfy the relations $\psi_\pm(x+1, E) = \psi_\pm(x, E+\epsilon)$. It appears that the analytic properties of such solutions can naturally be described in terms of a system of two first order difference equations on the complex plane (see, for example, [5]). To get the asymptotics of the Fourier coefficients of the reflection coefficient, we study the solutions of this system far from the origin. The idea leading from Theorem 1 to Theorem 2 is analogous to one used to study the behavior of the exponential sums $\sum_{n=1}^{N} e^{-2\pi i\omega n^3}$ with $\omega \in \mathbb{Q}$ for large N, see [4]. However, to use it successfully, one has to carry out a non trivial analysis of properties of the error term in (1.5).

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