Berkovich-Gelfand duality in constructive mathematics

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Abstract

We define constructively the Berkovich space of a Banach ring, as a locale, and the Berkovich-Gelfand transform of its elements. We define constructively the sheaf of sets of continuous functions. This defines a sheaf of generalized rings, in a certain sense. We then proceed to define local analytic functions.

Contents

1 Introduction

There is a well known important duality between algebra and geometry, that is present in various domains of mathematics. This is the case, for example, of Gelfand's duality theory for commutative Banach algebras over $\mathbb{C}$, and of the duality between rings and their prime spectrum in algebraic geometry. Our motivation in this work is to try to answer the following question: how many of these constructive results can be adapted to the setting of (analytic) arithmetic geometry.

In algebraic arithmetic geometry, the spectrum $\mathrm{Spec}(A)$ of a finitely presented algebra $A = \mathbb{Z}[x_1, \ldots, x_n]/(f_1, \ldots, f_m)$ over $\mathbb{Z}$ mainly encodes non-archimedean algebraic information. The aim of Berkovich arithmetic analytic geometry (also called global analytic geometry) is to define another space $\mathcal{M}(A)$ that encodes both archimedean and non-archimedean analytic information. This seems important for applications, because many results of arithmetic geometry use archimedean analytic methods. Hodge theory give us an important example. Another example is given by the proof of Fermat's last theorem, that uses in an essential way analytic modular forms on the Poincaré Half plane $\mathbb{H} := \{z \in \mathbb{C},\ \mathrm{Im}(z) > 0\}$. More generally, the analytic theory of automorphic forms and their L-functions can't be formulated in a purely algebraic setting.

We first recall the main results of constructive algebraic geometry. Joyal proposed in [Joy76] (see also the work of Espanol [Esp83]) a constructive definition of the Zariski spectrum $\mathrm{Spec}(A)$ of a ring A, as the locale (formal topological space without points) generated by the basic opens $D(f)$ for $f \in A$. In [CLS09], Coquand, Lombardi and Schuster gave a constructive definition of spectral schemes as ringed locales. These works have led to the complete formalization of affine schemes in the proof assistant cubical agda (using the constructive cubical version of univalent homotopy type theory) by Mörtberg and Zeuner (see [ZM23] for a description of this work).

On the analytic side, Coquand and Spitters developped a constructive proof of the constructive Gelfand duality in [CS08], and Henry studied it further in the setting of locales in [Hen23].

The aim of this note is to try to adapt part of the above results on the duality between algebra and geometry to the context of arithmetic geometry, i.e., to the context of Berkovich's geometry over an arbitrary Banach ring $(A, |\cdot|_A)$. This will define a Berkovich (localic) space $\mathcal{M}(A, |\cdot|_A)$ together with a Gelfand transform that sends $a \in A$ to an element

$$\hat{a} : \mathcal{M}(A, |\cdot|) \to \mathbb{A}^1_A$$

of the *set* $\mathcal{C}^0$ of continuous functions on $\mathcal{M}(A, |\cdot|)$. Another important aim of this note is to define a workable notion of sheaf of analytic functions $\mathcal{O}$ on the Berkovich space.

Let us first explain classically what the Berkovich-Gelfand duality is about. The standard reference for this subject is Berkovich's book [Ber90] (see also the book [LP22] for more recent results), where those spaces were first defined and studied, in the setting of classical mathematics. The space that replaces the Gelfand spectrum

$$\mathcal{C}(A) = \mathrm{Hom}_{\mathrm{BanAlg}_{\mathbb{C}}}(A, \mathbb{C})$$

of a $\mathbb{C}$-Banach algebra A, when A is a general Banach algebra is given by the Berkovich spectrum

$$\mathcal{M}(A) \subset \prod_{a \in A} \mathbb{R}_+,$$

whose classical points are bounded multiplicative seminorms on $(A, |\cdot|_A)$. For every $a \in A$, there is a natural continuous projection $\mathrm{ev}_a : \mathcal{M}(A) \to \mathbb{R}_+$ on the a-component of the product.

One may also define, following Berkovich, the affine space $\mathbb{A}^n_A$ as the space of multiplicative seminorms on $A[X_1, \ldots, X_n]$ whose restriction to A are bounded, and the multiplicative group $\mathbb{G}_{m,A}$ as the space of multiplicative semi-norms on $A[Y, Y^{-1}]$ whose restriction to A are bounded, or equivalently (*classically*), those multiplicative seminorms y on $A[Y]$ (points of the affine line) such that $|Y(y)| > 0$.

To every element $a \in A$, the Berkovich-Gelfand transform associates the continuous section $\hat{a} : \mathcal{M}(A) \to \mathbb{A}^1_A$ associated to the algebra morphism $P(X) \mapsto P(a)$. More generally, to every pair $(a, b) \in A$, the Berkovich-Gelfand transform associates the continuous section $\widehat{(a,b)} : \mathcal{M}(A) \to \mathbb{A}^2_A$ associated to the A-algebra morphism $P(X, Y) \mapsto P(a, b)$.

A rational function without poles on an open subspace $U \subset \mathcal{M}(A)$ may be defined as a local section $(a/b) : U \to \mathbb{A}^1_A$ of the natural projection $\mathbb{A}^1_A \to \mathcal{M}(A)$ obtained from the datum of a section $\widehat{(a,b)}_{|U} : U \to (\mathbb{A}^1 \times \mathbb{G}_m)_A$ (where the target space is *classically* the subspace of $\mathbb{A}^2_A$ given by multiplicative seminorms x on $A[X, Y, Y^{-1}]$ that are bounded on A) by application of the functorial inverse multiplication map $(\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{A}^1_A$ given by $(a, b) \mapsto ab^{-1}$ to a section associated to a pair $(a, b) \in A^2$. For example, $1/b$ is *classically* a rational function without poles on $\{x, |b(x)| > \epsilon\} = \mathrm{ev}_b^{-1}(]\epsilon, +\infty[)$ for every $\epsilon > 0$ because we have *classically* the inclusion:

$$\{x, |X(x)| > \epsilon\} \subset \{x, |X(x)| > 0\} = \mathbb{G}_{m,A} \subset \mathbb{A}^1_A$$

for every $\epsilon > 0$. We denote $\mathcal{K}(U)$ the set of rational functions without poles on U and $\mathcal{K}_b(U)$ its subset of bounded rational functions.

An analytic function on an open subspace $U \subset \mathcal{M}(A)$ is then a continuous section $s : U \to \mathbb{A}^1_U$ of the natural projection $p : \mathbb{A}^1_A \to \mathcal{M}(A)$ over U that is locally the uniform limit of a bounded sequence of rational functions without poles. This gives a sheaf $\mathcal{O}$ on $\mathcal{M}(A)$ called the sheaf of Berkovich analytic functions.

We first give an abstract definition of the constructive Berkovich spectrum, together with two different constructive sheaves of analytic functions: the sheaf of *generalized* rings $\mathcal{O}^{cl}$ of classical analytic functions (defined essentially as above), together with a sheaf morphism $\mathcal{O}^{cl} \to \mathcal{C}^0$, and the sheaf of rings $\mathcal{O}^c$ of constructive analytic functions, defined using locally some equivalence classes of rational fractions without poles (that can't be constructively related to $\mathcal{C}^0$). *Classically*, there should be a natural morphism of sheaves

$$\mathcal{O}^c \to \mathcal{O}^{cl},$$

and this morphism should be an isomorphism.

We finish this introduction by remarking that there may be a third possible approach to constructive Berkovich geometry, given by the use of the semi-analytic Grothendieck topology (usually called the G-topology in the non-archimedean situation) instead of the usual Berkovich topology. It may correspond to a spectral locale (a kind of Zariski-Riemann space) that one can hope to totally describe by generators and relations (in a similar fashion to Joyal's constructive description of the Zariski spectrum and to Coquand and Spitters' description of the Gelfand spectrum from [CS08]). In such a more concrete approach, one may probably get a sheaf of (ind-)Banach rings of analytic functions under some additional hypothesis (Tate's acyclicity hypothesis) on A. We leave this research direction for future developments.

2 Some rudiments of point-free topology

We refer to the book [Joh86] (see also [Joh02]) for more details on point-free topology. In constructive (i.e., intuitionistic) mathematics (i.e., without the law of excluded middle and thus, the axiom of choice), there are may examples of "point-free topological spaces". For example, it is not always possible to show that the spectrum of a ring A has points (because without choice, we don't have Zorn's lemma), but one may still define (following Joyal) the corresponding "point-free topological space" $\mathrm{Spec}(A)$.

Definition 1. A frame $\mathcal{T}$ is a poset with all joins $\vee$ (suprema) and finite meets $\wedge$ (infima), which satisfy the infinite distributivity law

$$x \wedge (\vee_i y_i) = \vee_i (x \wedge y_i).$$

A frame morphism is a function $\varphi : \mathcal{T} \to \mathcal{T}'$ that preserves all joins and finite meets. The category opposite to the category of frames is called the category of locales (aka "point-free topological spaces"). If X is an object of this category, we denote $\mathcal{T}(X)$ the corresponding frame, and if $f : X \to Y$ is a morphism of locales, we denote f^{-1} the corresponding frame morphism.

Example 1. The datum of a topological space $(X, \mathcal{T}(X))$ contains its frame $\mathcal{T}(X)$ of open subspaces. For example, the one point space $\mathbb{1} = \{*\}$ defines the one point locale $\mathbb{1}$, whose frame is $\mathcal{T}(\mathbb{1}) = \{\emptyset, \{*\}\}$, and one defines a classical point of a locale X to be a locale morphism $x : \mathbb{1} \to X$. Classical points of a locale define a topological space, and a topological space is called sober if it is homeomorphic to the set of points of its associated locale. In particular, any Hausdorf space is sober.

We refer to the book loc. cit. for the proof of the following result, that we will use in our definition of the constructive Berkovich spectrum.

Theorem 1. *The category of locales has all small limits and colimits.*

Remark 1. The functor from topological spaces to locales doesn't always commute with limits. In particular, the product $X \times Y$ of two locales associated to two topological spaces X and Y is not always the locale associated to the product topological space $X \times_{top} Y$. This however doesn't lead to substantial problems.

Remark 2. One may define the notion of a compact locale, and the main advantage of point-free topology on classical topology is that an arbitrary product of compact spaces may be proved to be compact, without assuming the axiom of choice. This is a constructive version of Tychonov's theorem (that is known to be equivalent to the axiom of choice in classical mathematics).

3 The constructive Berkovich spectrum

Adapting Coquand and Spitters' work [CS08] to Berkovich's setting, we define a constructive Banach ring to be a set A together with a map $|\cdot|_A : A \to \mathbb{R}_+$ where $\mathbb{R}_+$ denotes the non-negative upper reals, such that $|a| = 0$ if and only if $a = 0$, $|a - b| \leq |a| + |b|$, $|ab| \leq |a| \cdot |b|$ for every $a, b \in A$. We assume that A is complete: any cauchy approximation has a unique limit. Depending on the context, we will consider the space $\mathbb{R}_+$ as an ordered set (of upper reals, with an infimum for any of its inhabited subsets) or as a locale. We start with the localic viewpoint.

The first important result is the description of the Berkovich space that avoids the use of its points.

Proposition 1. *Let A be a ring and $[A, \mathbb{R}_+] = \prod_{a \in A} \mathbb{R}_+$ be the corresponding (product) locale of maps $A \to \mathbb{R}_+$. There exists a sublocale $\mathcal{M}(A) \subset [A, \mathbb{R}_+]$ that corresponds classically to multiplicative seminorms on A. If $f : A \to B$ is a ring morphism it induces a continuous map $\mathcal{M}(f) : \mathcal{M}(B) \to \mathcal{M}(A)$, in a functorial way. The locale $\mathcal{M}(A)$ is locally compact.*

Proof. The locale $\mathbb{R}_+$ is an ordered semiringed locale, i.e., we have addition and multiplication maps $+, \cdot : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, and a closed relation sublocale $\mathbb{R}_{+,\leq} \subset \mathbb{R}_+ \times \mathbb{R}_+$. There is for each $a_0 \in A$ a natural projection

$$p_{a_0} : \prod_{a \in A} \mathbb{R}_+ \to \mathbb{R}_+.$$

This induces, for every $(a_0, b_0) \in A^2$, natural projections

$$p_{a_0 \times b_0} : \prod_{a \in A} \mathbb{R}_+ \to \mathbb{R}_+$$

and

$$p_{a_0 - b_0} : \prod_{a \in A} \mathbb{R}_+ \to \mathbb{R}_+.$$

From this, we define two natural maps

$$p_\times : \prod_{a \in A} \mathbb{R}_+ \to \prod_{a,b \in A} \mathbb{R}_+$$

and

$$p_- : \prod_{a,b \in A} \mathbb{R}_+ \to \prod_{a,b \in A} \mathbb{R}_+.$$

that send (x_a) to (x_{ab}) and (x_{a-b}) respectively. There are also natural maps

$$+, \times : \prod_{a \in A} \mathbb{R}_+ \to \prod_{a,b \in A} \mathbb{R}_+$$

sending (x_a) to $(x_a + x_b)$ and $(x_a \cdot x_b)$ respectively. We may ask, using a fiber product of locales morphisms, that $p_\times = \times$ on $\mathcal{M}(A)$, which corresponds to the multiplicativity $|ab| = |a| \cdot |b|$ of the map. More precisely, we may ask that there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{M}(A) & \longrightarrow & \prod_{a \in A} \mathbb{R}_+ \\ \downarrow & & \downarrow{\scriptstyle p_\times} \\ \prod_{a \in A} \mathbb{R}_+ & \xrightarrow[\times]{} & \prod_{a,b \in A} \mathbb{R}_+ \end{array}$$

We may also ask, using a similar fiber product of locales, that $p_- \leq +$, meaning that $(p_-, +) : \prod_{a \in A} \mathbb{R}_+ \to \prod_{a,b \in A} \mathbb{R}_+^2$ lands into $\prod_{a,b \in A} \mathbb{R}_{+,\leq}$, which corresponds to the triangular inequality $|a - b| \leq |a| + |b|$. More precisely, we may ask that there is a cartesian diagram

$$\begin{array}{ccc} \mathcal{M}(A) & \longrightarrow & \prod_{a,b \in A} \mathbb{R}_{+,\leq} \\ \downarrow & & \downarrow{\scriptstyle p_\times} \\ \prod_{a \in A} \mathbb{R}_+ & \xrightarrow[(p_-,+)]{} & \prod_{a,b \in A} \mathbb{R}_+^2 \end{array}$$

Using the natural projection $p_0, p_1 : \prod_{a \in A} \mathbb{R}_+ \to \mathbb{R}_+$ and the two points $0, 1 : * \to \mathbb{R}_+$, that give constant functions $\underline{0}, \underline{1} : \prod_{a \in A} \mathbb{R}_+ \to \mathbb{R}_+$, we may further ask that $|0| = 0$ and $|1| = 1$, i.e., $p_0 = \underline{0}$ and $p_1 = \underline{1}$. Setting universally all these conditions defines a limit diagram in locales whose limit is $\mathcal{M}(A)$. The fact that the locale $\mathcal{M}(A)$ is locally compact follows from the fact that it is a closed sublocale of the locally compact product $\prod_{a \in A} \mathbb{R}_+$ of the locally compact locales $\mathbb{R}_+$. $\square$

Corollary 1. *Let $(A, |\cdot|_A)$ be a Banach ring and B be an A-algebra. There exists a sublocale $\mathcal{M}_{(A,|\cdot|_A)}(B)$ of $\mathcal{M}(B)$ that classifies multiplicative seminorms on B that are bounded by $|\cdot|_A$ on A. If $f : B \to B'$ is a morphism of A-algebras, it defines a continuous map $\mathcal{M}_{(A,|\cdot|_A)}(B') \to \mathcal{M}_{(A,|\cdot|_A)}(B)$ in a functorial way.*

Proof. The structural map $\varphi : A \to B$ defines a natural map $\prod_{b\in B} \mathbb{R}_+ \to \prod_{a\in A} \mathbb{R}_+$ by $(x_b) \mapsto (x_{\varphi(a)})$ and the norm $|\cdot|_A : A \to \mathbb{R}_+$ defines a point $|\cdot|_A : \{*\} \to \prod_{a\in A} \mathbb{R}_+$. We may ask, using a fiber product, for an element in $\prod_{a\in A} \mathbb{R}_+$ to be smaller than $|\cdot|_A$, meaning that it is in the fiber of the natural maps $\prod_{a\in A} \mathbb{R}_{+,\leq} \to \prod_{a\in A} \mathbb{R}^2_+$ and

$$\prod_{a\in A} \mathbb{R}_+ \cong \prod_{a\in A} \mathbb{R}_+ \times \{*\} \stackrel{(\mathrm{id},|\cdot|_A)}{\longrightarrow} \prod_{a\in A} \mathbb{R}^2_+.$$

This defines a locale that maps to $\mathcal{M}(B)$ and that is clearly functorial in B. □

4 The generalized sheaf of rings of continuous functions

Continuous functions on the Berkovich space of an arbitrary ring should not be taken with values in $\mathbb{C}$. It is more natural to see them as continuous functions with values in the affine line $\mathbb{A}^1_A$ over A, i.e., as continuous sections of the natural projection $\mathbb{A}^1_A \to \mathcal{M}(A)$. However, such continuous sections may not be added or multiplied in general. We will now explain precisely what kind of structure do this space of continuous functions have.

Definition 2. For $n \geq 0$, we define

1. the affine space on the Banach ring $(A, |\cdot|_A)$ by $\mathbb{A}^n_A := \mathcal{M}_{(A,|\cdot|_A)}(A[X_1, \ldots, X_n])$,
2. the multiplicative group by $\mathbb{G}_{m,A} := \mathcal{M}_{(A,|\cdot|_A)}(A[X, X^{-1}])$,
3. a continuous function on an open subspace $U \subset \mathcal{M}(A)$ to be a continuous section $s : U \to \mathbb{A}^1_A$ of the natural projection $\mathbb{A}^1_A \to \mathcal{M}(A, |\cdot|_A)$.
4. an invertible continuous function on an open subspace $U \subset \mathcal{M}(A)$ to be a continuous section $s : U \to \mathbb{G}_{m,A}$ of the natural projection $\mathbb{G}_{m,A} \to \mathcal{M}(A, |\cdot|_A)$.
5. more generally, if B is an A-algebra, a continuous $\mathcal{M}(B)$-valued function is a continuous section $s : U \to \mathcal{M}_{(A,|\cdot|_A)}(B)$ of the natural projection $p_B : \mathcal{M}_{(A,|\cdot|_A)}(B) \to \mathcal{M}(A)$.

The continuous $\mathcal{M}(B)$-valued functions on $\mathcal{M}(A)$ form a sheaf of sets $\mathcal{C}^0_{\mathcal{M}(A)}(B)$ and we denote its local sections on $U \subset \mathcal{M}(A)$ by $\mathcal{C}^0_U(B)$.

Remark 3. It is not possible to add or to multiply arbitrary usual pairs of continuous functions $f_1, f_2 : U \to \mathbb{A}^1_A$. However, it is possible to add and multiply sections $f = (f_1 : f_2) : U \to \mathbb{A}^2_A$ by simply defining

$$+(f_1 : f_2) := + \circ f \text{ and } \times (f_1 : f_2) := \times \circ f$$

where $+, \times : \mathbb{A}^2_A \to \mathbb{A}^1_A$ are defined as $\mathcal{M}(+)$ and $\mathcal{M}(\times)$ with $+, \times : A[X] \to A[X, Y]$ the A-algebra morphisms sending X to $X + Y$ and $X \cdot Y$ respectively.

If we want to be more precise about the structure carried by continuous functions on $\mathcal{M}(A)$, we may say that the map $B \mapsto \mathcal{C}^0_{\mathcal{M}(A)}(B)$ defines a functor on the category whose objects are A-algebras, with values in sheaves of sets on $\mathcal{M}(A)$. This means that it is a kind of sheaf of A-algebras on $\mathcal{M}(A)$ in a (very) generalized sense (the given functor doesn't a priori commute with colimits of algebras).

Proposition 2. *The natural functor from the category* $\textsc{Alg}_A^{op}$ *to the category of locales given by* $B \mapsto \mathcal{M}_{(A,|\cdot|_A)}(B)$ *induces a natural functor*

$$\mathcal{C}^0_{\mathcal{M}(A)} : \textsc{Alg}_A^{op} \to \mathrm{Sh}(\mathcal{M}(A, |\cdot|_A)).$$

Proof. This follows from the definition of $\mathcal{C}^0$ as local sections of the natural projection $\mathcal{M}_{(A,|\cdot|_A)}(B) \to \mathcal{M}(A, |\cdot|_A)$ and from the functoriality of $\mathcal{M}_{(A,|\cdot|_A)}(B)$ in the A-algebra B. □

Remark 4. We may replace the category $\textsc{Alg}_A$ by the simpler subcategory $\textsc{Poly}_A$ of polynomial algebras over A, and this makes the sheaf of continuous functions a generalized (it doesn't commute with products) model of the finitary Lawvere theory of A-algebras.

Remark 5. We may also replace the category $\textsc{Alg}_A$ of algebras by the category $\textsc{BanAlg}_A$ of Banach algebras. This would then make the sheaf of continuous functions a sheaf of generalized Banach algebras.

5 The Berkovich-Gelfand transform

Definition 3 (Berkovich-Gelfand transform)**.** The natural map

$$G : A \to \mathcal{C}^0_{\mathcal{M}(A)}(A[x]) = \Gamma_{\mathcal{C}^0}(\mathcal{M}(A), \mathbb{A}^1_A)$$

that sends $a \in A$ to the section $G(a) \equiv \hat{a} : \mathcal{M}(A) \to \mathbb{A}^1_A$ dual to the A-algebra map $A[X] \to A$ sending X to a is called the Berkovich-Gelfand transform.

We now want to say that the Berkovich-Gelfand transform is a morphism of generalized rings in a certain sense. This will only be possible if we restrict our test category for generalized rings to the category $\textsc{Poly}_A \subset \textsc{Alg}_A$ of polynomial algebras over A. So define a functor

$$\underline{A} : \textsc{Poly}_A^{op} \to \textsc{Sets}$$

by $\underline{A}(B) = \mathrm{Hom}_{\textsc{Alg}_A}(B, A)$. For $B = A[x_1, \ldots, x_n]$, we get $\underline{A}(B) \cong A^n$. It is clear that $\underline{A}$ commutes with finite products (this is the model of the Lawvere theory of A-algebras corresponding to the algebra A). However, the corresponding functor of continuous functions

$$\mathcal{C}^0_{\mathcal{M}(A)} : \textsc{Poly}_A^{op} \to \textsc{Sets}$$

given by $\mathcal{C}^0_{\mathcal{M}(A)}(B) = \Gamma_{\mathcal{C}^0}(\mathcal{M}(A), \mathcal{M}(B))$ does not commute with product. We still have the following result, that means that the above Gelfand transform is compatible with addition and multiplication, and that is is a morphism of generalized finitary A-algebras.

Proposition 3. *There is a natural morphism of functors* $\textsc{Poly}_A^{op} \to \textsc{Sets}$ *denoted*

$$G : \underline{A} \to \mathcal{C}^0_{\mathcal{M}(A)}$$

given sending $(a_1, \ldots, a_n) \in A^n$ *to the corresponding continuous section* $G(a_1, \ldots, a_n)$ *of* $\mathbb{A}^n_A$.

Proof. This just follows from the fact that a morphism $f : A[x_1, \ldots, x_n] \to A[x_1, \ldots, x_m]$ induces a commutative diagram showing the result. □

6 The local Berkovich-Gelfand transform

Definition 4. A representative for a rational function without poles on $U \subset \mathcal{M}(A)$ is a triple (a, b, f) composed of a pair $(a, b) \in A^2$ and a section $f : U \to (\mathbb{A}^1 \times \mathbb{G}_m)_A$ such that if $i : (\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{A}^2_A$ is the natural map, we have

$$i \circ f = G(a, b)_{|U} : U \to \mathbb{A}^2_A.$$

We denote $\tilde{\mathcal{K}}_U(A[x])$ the set of representatives for rational functions. More generally, if $A[x_1, \ldots, x_n]$ is a polynomial algebra, we denote $\tilde{\mathcal{K}}_U(A[x_1, \ldots, x_n])$ the set of similar triples $((a_1, \ldots, a_n), (b_1, \ldots, b_n), f)$ with $(a_i) \in A^n$, $(b_i) \in A^n$ and $f : U \to (\mathbb{A}^1 \times \mathbb{G}_m)^n_A$ fulfilling a similar relation with the multiple Gelfand transform.

Proposition 4. *The representatives for rational functions without poles define a functor*

$$\tilde{\mathcal{K}}_U : \textsc{Poly}_A^{op} \to \textsc{Sets}.$$

Proof. We just describe how addition, multiplication, and their units are implemented. The units are given by $(1, 1, G(1)_{|U})$ and $(0, 1, G(0)_{|U})$. The addition and multiplication of representatives of fractions (a, b) and (c, d) are given by $(ad+bc, bd)$ and (ac, bd) as usual, and these may also be implemented on the third part f since they are purely algebraic. □

Proposition 5. *There is a natural morphism of functors (i.e., of generalized finitary A-algebras)*

$$G_U : \tilde{\mathcal{K}}_U \to \mathcal{C}^0_U$$

called the local Berkovich-Gelfand transform. It is obtained by precomposing the third component f of an element of $\tilde{\mathcal{K}}_U$ with the natural "take the fraction" map

$$(\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{A}^1_A$$

that sends (a, b) to ab^{-1}.

Remark 6. A problem with the above local construction is that for example if we take $U \subset \mathcal{M}(A)$ to be given by $\{x,\ |b(x)| > \epsilon\} := |b|^{-1}(]\epsilon, +\infty[)$, we are not able to show that the section $G(b) : \mathcal{M}(A) \to \mathbb{A}^1_A$ factors on U through $\mathbb{G}_{m,A}$, which is exactly what we need to invert it and define $1/b : U \to \mathbb{G}_{m,A}$. Another way to say it is that we are not able (without using points) to define some representative in $\tilde{\mathcal{K}}_U$ of such a very natural and simple rational function without poles. In the classical situation, using residue fields at points, we may show easily that $\mathbb{G}_{m,A} \subset \mathbb{A}^1_A$ identifies with $\{x, |X(x)| > 0\}$, which solves this problem. For this reason, we will later use a more formal approach to the definition of the representatives of rational function without poles on opens, so that this kind of problem doesn't occur.

Remark 7. If $A = \ell^1(\mathbb{N}, \mathbb{C})$ is the complex Banach algebra of power series converging on the unit disc, then we don't really need to use rational fraction without poles to define analytic functions on an open subspace U of its Berkovich spectrum, because it is classically covered by (its intersection with) complex open discs. We may then define analytic functions locally by converging power series. This of course doesn't work for a more general Banach ring such as $(\mathbb{Z}, |\cdot|_\infty)$.

7 The norm on continuous functions

In this section, we consider $\mathbb{R}_+$ as the set of upper reals, that has the property that any of its inhabited subsets has an infimum.

For each $U \subset \mathcal{M}(A)$, there is a natural "norm map"

$$|\cdot| : \mathcal{C}^0_U(U) \to \mathcal{C}^0(U, \mathbb{R}_+)$$

obtained by sending $s : U \to \mathbb{A}^1_A$ to $\mathrm{ev}_X \circ s$ where $\mathrm{ev}_X : \mathbb{A}^1_A \to \mathbb{R}_+$ is induced by the natural projection $\prod_{P \in A[X]} \mathbb{R}_+ \xrightarrow{\mathrm{ev}_X} \mathbb{R}_+$ on the X-factor of the product.

Definition 5. A continuous function $s \in \mathcal{C}^0_{\mathbb{A}^1}(U)$ on $U \subset \mathcal{M}(A)$ is called bounded on U if there exists $C > 0$ such that $|s|^{-1}([0, C[) = U$ where $|s| : U \to \mathbb{R}_+$ is the norm of s on U. If s is a bounded continuous function on U, we define $\|s\|_{\infty,U}$ to be the infimum of such $C > 0$. More generally, a continuous function $s \in \mathcal{C}^0_{\mathbb{A}^n}(U)$ is bounded on U if each its components are bounded on U. We denote $\mathcal{C}^0_{b,\mathbb{A}^n}(U)$ the set of such sections.

Remark that the natural substraction map $- : \mathcal{C}^0_{\mathbb{A}^2}(U) \to \mathcal{C}^0_{\mathbb{A}^1}(U)$ is compatible with the corresponding norm $| \cdot |$, meaning that $|s - t| \leq |s| + |t| : U \to \mathbb{R}_+$. This follows from the triangular inequality $|(X - Y)(x)| \leq |X(x)| + |Y(x)|$ true by definition for x a formal point of the space $\mathbb{A}^2_A$. This implies that the formal sum of two bounded continuous functions is bounded and that the supremum norm fulfils the triangular inequality

$$\|s - t\|_\infty := \| - \circ (s : t)\|_\infty \leq \|s\|_\infty + \|t\|_\infty.$$

Similar reasonings imply that

$$\|st\|_\infty := \| \times \circ (s : t)\|_\infty \leq \|s\|_\infty \cdot \|t\|_\infty,$$

and

$$\|1\|_\infty = 1 \text{ and } \|0\|_\infty = 0.$$

8 The generalized sheaf of rings of analytic functions

We are now able to define the notion of rational function without pole on an open subspace $U \subset \mathcal{M}(A, | \cdot |_A)$.

Remark 8 (Preliminary warning). The problem with the following definition is that even if it is constructive, it is not efficient if we are not using classical mathematics. Indeed, if $U = \{|b| > \epsilon\} = |b|^{-1}(]\epsilon, +\infty[)$, we are not even able to define $1/b$ constructively as a rational function without poles on U. This is related to the fact that we can only prove classically (see Berkovich's original book [Ber90], Chapter 1) that

$$\mathbb{G}_{m,A} = \{x \in \mathbb{A}^1_{\mathbb{A}},\ |X(x)| > 0\}.$$

In the smooth complex situation, analytic functions are locally given by converging power series on discs, so that we don't need to use rational fractions without poles. However, with arithmetic Banach rings such as $(\mathbb{Z}, | \cdot |_\infty)$, the situation is different, and the contents of this remark apply. We will thus take another approach to the definition of rational function without pole in a forthcoming section.

Definition 6. A representative for a rational function without pole on an open subspace $U \subset \mathcal{M}(A, |\cdot|_A)$ is the datum of a continuous section

$$s = (A : B) : U \to (\mathbb{A}^1 \times \mathbb{G}_m)_A := \mathcal{M}_{(A,|\cdot|_A)}(A[X, Y, Y^{-1}]),$$

i.e., an element of $\mathcal{C}^0_{A[X,Y,Y^{-1}]}(U)$ such that its precomposition $i \circ (A : B) : U \to \mathbb{A}^2_A$ with the natural map $i : (\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{A}^2_A$ is such that

$$i \circ (A : B) = \widehat{(a, b)}_{|U} : U \to \mathbb{A}^2_A$$

is the restriction of a section $\widehat{(a, b)} : \mathcal{M}(A) \to \mathbb{A}^2_A$ obtained by sending $P(X, Y)$ to $P(a, b)$ for some $(a, b) \in A^2$. The rational function without poles $A/B : U \to \mathbb{A}^1_A$ on U associated to $(A : B)$ is its precomposition with the map $(\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{A}^1_A$ given on polynomial variables by $X \mapsto XY^{-1}$.

Once the notion of rational function without pole is defined, we may follow Berkovich's approach to the definition analytic functions, as continuous functions that are local uniform limits of bounded rational functions without poles. The Berkovich space being locally compact, this definition using *bounded* rational functions is classically equivalent to the usual one given by Berkovich, that uses only usual rational functions.

Definition 7. A local representative for an analytic function on $U \subset \mathcal{M}(A)$ is a continuous section

$$s = (A_n : B_n : f) : U \to ((\mathbb{A}^1 \times \mathbb{G}_m)^{\mathbb{N}} \times \mathbb{A}^1)_A := \mathcal{M}(A[\{X_n, Y_n, Y_n^{-1}, n \in \mathbb{N}\}, Z])$$

such that each $(A_n : B_n) : U \to (\mathbb{A}^1 \times \mathbb{G}_m)_A$ is a representative for a bounded rational function without poles and $f : U \to \mathbb{A}^1_A$ is a bounded continuous section (so that all sections $(A_n/B_n - f) : U \to \mathbb{A}^1_A$ are bounded, by the triangular inequality for the supremum norm), and we further have

$$\lim_{n \to \infty} \|(A_n/B_n) - f\|_\infty = 0.$$

An analytic function on $U \subset \mathcal{M}(A)$ is a continuous section $f : U \to \mathbb{A}^1_A$ such that there exists a covering $U = \cup_i U_i$ such that $f_{|U_i} = f_i : U_i \to \mathbb{A}^1_A$ is part of a local representative $s_i = (A_{n,i}, B_{n,i}, f_i)$ for an analytic function. We denote $\mathcal{O}_{\mathbb{A}^1}(U)$ the set of analytic functions on U.

We want to say what it means to add two analytic functions, i.e., to see analytic functions as some kind of generalized ring. It is not possible to define $\mathcal{O}_B(U) \subset \mathcal{C}^0_B(U)$ for any A-algebra B. However, if $B = A[Z_1, \ldots, Z_m]$ is a polynomial ring, we may define $\mathcal{O}_{\mathbb{A}^m}(U) := \mathcal{O}_B(U) \subset \mathcal{C}^0_B(U)$. This allows us to define a functor

$$\mathcal{O}_*(U) : F\text{ALG}_A^{op} \to \text{SETS}$$

where $F\text{ALG}_A$ is the category of finitely generated polynomial A-algebras $A[Z_1, \ldots, Z_m]$ with A-algebras morphisms.

Definition 8. Let $B = A[Z_1, \dots, Z_m]$. A local representative for an analytic function on $U \subset \mathcal{M}(A)$ with values in B is a continuous section

$$s = (A_{n,k} : B_{n,k} : f_k) : U \to ((\mathbb{A}^m \times \mathbb{G}_m^m)^{\mathbb{N}} \times \mathbb{A}^m)_A$$

such that for each $k = 1, \dots, , m$, the corresponding component

$$(A_{n,k} : B_{n,k} : f_k) : U \to ((\mathbb{A}^1 \times \mathbb{G}_m)^{\mathbb{N}} \times \mathbb{A}^1)_A$$

is a local representative of an analytic function with values in $\mathbb{A}^1$. A continuous section $f : U \to \mathbb{A}^m_A$ is called an analytic function if it is locally of the form (f_k) for some local representative of an analytic function.

Proposition 6. *The map* $A[Z_1, \dots, Z_m] \mapsto \mathcal{O}_{\mathbb{A}^m}(U) = \mathcal{O}_{A[Z_1,\dots,Z_m]}(U)$ *gives a contravariant functor*

$$\mathcal{O}_*(U) : F\text{ALG}_A^{op} \to \text{SETS}$$

on the category of finitely generated polynomial A-algebras.

Proof. Let us first show that if $f : U \to \mathbb{A}^2_U$ is an analytic function, then $+ \circ f : U \to \mathbb{A}^1_U$ is also an analytic function. By restriction to a smaller U, we may suppose that $f : U \to \mathbb{A}^2_U$ is obtained from a local representative of analytic functions. Then we have

$$s = (A_{n,1} : A_{n,2} : B_{n,1} : B_{n,2} : f_1 : f_2) : U \to ((\mathbb{A}^2 \times \mathbb{G}_m^2)^{\mathbb{N}} \times \mathbb{A}^2)_A$$

such that $f = (f_1 : f_2)$ and $(\|A_{n,k}/B_{n,k} - f_k\|_\infty)_n$ tends to zero for $k = 1, 2$. Then the triangular inequality for the supremum norm

$$\|(A_{n,1}/B_{n,1} - A_{n,2}/B_{n,2}) - (f_1 - f_2)\|_\infty \leq \|A_{n,1}/B_{n,1} - f_1\|_\infty + \|A_{n,2}/B_{n,2}) - f_2\|_\infty$$

shows that $(A_{n,1} + A_{n,2} : B_{n,1} + B_{n,2} : f_1 + f_2)$ is a local representative for an analytic function, whose associated analytic function is $+ \circ f$ as required. Now we want to prove that if $f : U \to \mathbb{A}^2_U$ is an analytic function, then $\times \circ f : U \to \mathbb{A}^1_U$ is also an analytic function. The question is still local on U, so that we may restrict to local representatives of analytic functions. Let us denote $(f_{n,1} : f_{n,2} : f_1, f_2) : U \to ((\mathbb{A}^2)^{\mathbb{N}} \times \mathbb{A}^2)_A$ the sequences of rational functions without poles together with their limits. The point is to show that

$$\|f_{n,1} \cdot f_{n,2} - f_1 \cdot f_2\|_\infty$$

tends to zero as n tends to infinity. At least, it is well defined because $\| \cdot \|_\infty$ is submultiplicative and subadditive. Since converging sequences of bounded continuous functions are uniformly bounded, the usual trick of the trade does the rest. We may do similar proofs for mutliple sums and multiple products. It remains to show that the external multiplication map by $a \in A$, denoted $\cdot a : \mathbb{A}^n_A \to \mathbb{A}^n_A$, and defined by $X \mapsto aX$, also respects analytic functions, but this is true since $|a(x)| \leq |a|_A$ for every $x \in \mathcal{M}(A)$. Since arbitrary polynomials are constructed from additions, multiplications, and scalar multiplications, this shows by generator and relations that $A[Z_1, \ldots, Z_m] \mapsto \mathcal{O}_{\mathbb{A}^m}(U)$ is indeed a well defined functor. □

The intuitive meaning of the above functoriality statement is that the sheaf of analytic functions comes equipped with some kind of polynomial operations, and in particular, some kind of addition and multiplication, so that it is a ring in this generalized sense.

Remark 9. We carefully inform the reader that the above functor is not a usual ring structure (model of the Lawvere theory of polynomial rings over A) because it doesn't send coproducts of polynomial algebras to products.

9 The constructive sheaf of analytic functions

The Berkovich definition of analytic functions as local uniform limits of rational functions without poles is not constructive, because it uses the existence of points in $\mathcal{M}(A)$ and also the values of analytic (and in particular of rational) functions at those points in an essential way.

We will circumvent this problem by using (the setoid or higher inductive-inductive type of) equivalence classes of local uniform Cauchy sequences of bounded rational functions without poles. Classically, this doesn't change the resulting sheaf, but the new notion is purely constructive and avoids the use of the excluded middle and of the axiom of choice.

First recall that for $a \in A$, we denote $|a| : U \to \mathbb{R}_+$ the associated locale morphism, given by composition of the Berkovich-Gelfand transform $\hat{a} : \mathcal{M}(A) \to \mathbb{A}^1_A = \mathcal{M}_A(A[X])$ with the evaluation at X map $|X(\cdot)| : \mathbb{A}^1_A \to \mathbb{R}_+$. Let us first describe the constructive version of rational functions without poles.

Definition 9. Let A be a Banach ring and $U \subset \mathcal{M}(A)$ be an open sublocale of its Berkovich locale. A (representative for a) bounded rational function (without poles) on U is a tuple

$$f = (a, b, r, s, C) \in A^2 \times (\mathbb{Q}_+^*)^2 \times \mathbb{N}$$

such that

1. We have $|a| < r$ on U, i.e., $|\hat{a}|_{|U}^{-1}([0, r[) = U$.
2. We have $|b| > s$ on U, i.e., $|\hat{b}|_{|U}^{-1}(]s, +\infty[) = U$.
3. We have $r/s < C$.

We carefully inform the reader that we will often simply write

$$|f| < \frac{r}{s} < C$$

to encode the above three conditions. We denote $\tilde{\mathcal{K}}_b(U)$ the set of such local representatives of bounded rational functions f.

We may here point the fact that $\tilde{\mathcal{K}}_b(U)$ is a constructive version of the ring of bounded rational functions on U. We now remark that all we need to work with bounded fractions is already present on the set $\tilde{\mathcal{K}}_b(U)$.

Proposition 7. *There are natural associative addition and multiplication operations on $\tilde{\mathcal{K}}_b(U)$, given by the usual operations on fractions of elements of A, $\mathbb{R}_+^*$ and by addition and multiplication on $\mathbb{N}$. There is also a natural inversion operation given by $(a, b, r, s, C) = (-a, b, r, s, C)$. This makes $(\tilde{\mathcal{K}}_b(U), +, \cdot, -)$ a commutative semiring with inversion without any units. There are also*

1. *for every $r > 0$ and $s < 1$ and $r/s < C$, a corresponding additive partial unit*

$$0_{r,s,C} := (0, 1, r, s, C) \equiv (0/1, r/s, C),$$

2. *for every $r > 1$ and $s < 1$ such that $1 < r/s < C$, a corresponding multiplicative partial unit*

$$1_{r,s,C} := (1, 1, r, s, C) \equiv (1/1, r/s, C),$$

that give units on the fractional A-component.

Proof. The multiplication works fine because if $|a/b| < r/s$ and $|c/d| < t/u$ on U (in the above notation), then, if we look at the denominator of $(a/b) \cdot (c/d)$, we have

$$|a|_{|U}^{-1}([0, r[) = U \text{ and } |c|_{|U}^{-1}([0, t[) = U$$

and since $|ac| = |a| \cdot |c| : \mathcal{M}(A) \to \mathbb{R}_+$ by definition of $\mathcal{M}(A)$, we get

$$|ac|_{|U}^{-1}([0, rt[) = (|a|_{|U} \cdot |c|_{|U})^{-1}([0, rt[) \supset |a|_{|U}^{-1}([0, r[) \cap |c|_{|U}^{-1}([0, t[) = U$$

by compatibility of the multiplication on $\mathbb{R}_+$ with its order. This gives $|ac| < rt$. A similar reasoning shows that $|bd| > su$. For addition, if $|a/b| < r/s$ and $|c/d| < t/u$, then we have

$$|(a/b) + (c/d)| < (r/s) + (t/u)$$

since $| \cdot | : \mathcal{M}(A) \to \mathbb{R}_+$ fulfills the triangular inequality when applied to (Gelfand transforms of) elements of a, and since the addition on $\mathbb{R}_+$ is compatible with its order. □

Definition 10. The "fractional" equivalence relation on elements of $\tilde{\mathcal{K}}_b(U)$ given by

$$F = (a_1, b_1, r_1, s_1, C_2) \sim_{frac} (a_2, b_2, r_2, s_2, C_2) = G$$

if $C_1 = C_2$ in $\mathbb{N}$, $r_1/s_1 = r_2/s_2$ in $\mathbb{Q}_+^*$, and there exists $S, S^{-1} \in \mathcal{K}_b(U)$ such that $S \cdot S^{-1} = 1_{r,s,C}$ for some (r, s, C) and

$$S \cdot F = S \cdot G.$$

The quotient of $\tilde{\mathcal{K}}_b(U)$ by this equivalence relation $\sim_{frac}$ is denoted $\mathcal{K}_b(U)$. We will denote an element in $\mathcal{K}_b(U)$ by $F = (a/b, r/s, C)$.

Proposition 8. *The set $\mathcal{K}_b(U)$ is naturally equipped with a commutative semi-ring structure without units that makes $\tilde{\mathcal{K}}_b(U) \to \mathcal{K}_b(U)$ a morphism.*

There is a natural map $\mathcal{K}_b(U) \to \mathbb{Q}_+^* \times N$. It is completely formal to check that the natural forgetful map

$$|\cdot| : \mathcal{K}_b(U) \to \mathbb{R}_+$$

given by $|(a/b, r/s, C)| := r/s$ fulfills all the usual conditions for being a multiplicative seminorm for these operations, except that on partial units, we only get $|(0, r/s, C)| = r/s > 0$ and $|(1, r/s, C)| = r/s > 1$.

Remark 10. A representative for a bounded rational function without poles on A defines a continuous function

$$f : U \to \mathbb{A}_A^2 = \mathcal{M}(A[X, Y])$$

by the Berkovich-Gelfand transform associated to $(X, Y) \mapsto (a, b)$, and this continuous function factorizes through the open sublocale $|X|^{-1}([0, r_f[) \cap |Y|^{-1}(]s_f, +\infty[)$. We don't know a priori if the above function f factorizes through the natural map $(\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{A}_A^2$. If we make this additional (a priori not very constructive) hypothesis, we may define a continuous function $\widehat{(a/b)} : U \to \mathbb{G}_{m,A}$ by composition with the natural map $(\mathbb{A}^1 \times \mathbb{G}_m)_A \to \mathbb{G}_{m,A}$ given by $(X, Y) \mapsto XY^{-1}$. But we are able to prove this additional hypothesis constructively only in the case, say, of $(a, 1, r, 1/2, [2r] + 1)$, i.e., for the rational function $a/1$, and not even for $1/b$ when $|b| > \epsilon$. The main problem here is to show that the open sublocale $\{|X| > 0\}$ of $\mathbb{A}_A^1$ is naturally isomorphic to $\mathbb{G}_{m,A}$, and Berkovich's proof of this result is essentially non-constructive.

We are now able to define the local representatives of analytic functions, given by Cauchy sequences with moduli of bounded rational functions without poles.

Definition 11. Let $U \subset \mathcal{M}(A)$ be an open sublocale. A local representative of analytic function on U is the datum of a tuple

$$f = ((f_n), C, (C_{p,q}), k, N) \in \tilde{\mathcal{K}}_b(U)^{\mathbb{N}} \times \mathbb{N} \times (\mathbb{Q}_+^2)^{\mathbb{N}^2} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$$

composed of

1. a sequence (f_n) of rational functions in $\tilde{\mathcal{K}}_b(U)$ bounded by the same integer $C \in \mathbb{N}$, i.e., such that
$$|f_n| < C_{f_n} < C$$
for every n,

2. a family of additional real bounds $(C_{p,q}) = (r_{p,q}/s_{p,q})$,

3. an integer k,

4. a modulus of convergence (increasing function) $N : \mathbb{N} \to \mathbb{N}$,

such that:

1. for $p, q \geq 0$, we have that the differences $f_p - f_q$ of the rational functions with their new bounds
$$(f_p - f_q, C_{p,q})$$
are also elements of $\tilde{\mathcal{K}}_b(U)$, i.e., we have morally the inequality
$$|f_p - f_q| < C_{p,q}.$$

2. and all these elements for $p, q \geq 0$ define a Cauchy sequence with modulus, meaning that for every $n > 0$, if $p, q \geq N(n)$, we have
$$|f_p - f_q| < C_{p,q} < \frac{k}{n}.$$

We denote $\tilde{\mathcal{O}}^{loc}(U)$ the set of such local representatives f.

Proposition 9. *There is a natural inclusion $\tilde{\mathcal{K}}_b(U) \hookrightarrow \tilde{\mathcal{O}}^{loc}(U)$ and the semiring structure (without units) on $\tilde{\mathcal{K}}_b(U)$ naturally extends to $\tilde{\mathcal{O}}^{loc}(U)$.*

Proof. The natural map $i : \tilde{\mathcal{K}}_b(U) \to \tilde{\mathcal{O}}^{loc}(U)$ sends $(f, |f|, C)$ to the tuple
$$i(f, |f|, C) := ((f), C, (C_{p,q} = 1/(pq)), k = 1, N : n \mapsto n + 1).$$
We indeed know that for $n > 0$ and $p, q \geq n + 1$, we have
$$1/(pq) \leq 1/(n + 1)^2 < 1/n = k/n.$$
The sum of f and g is defined by
$$f + g = ((f_n + g_n), C^f + C^g, (C^f_{p,q} + C^g_{p,q}), k^f + k^g, \max(N^f, N^g)).$$
It is associative by construction because the norm fulfills the triangular inequality. To define the multiplication of f and g, we use, as usual, the triangular inequality
$$|f_p g_p - f_q g_q| \leq |f_p g_p - f_p g_q| + |f_p g_q - f_q g_q| \leq |f_p| \cdot |g_p - g_q| + |g_q| \cdot |f_p - f_q| < C^f \cdot C^g_{p,q} + C^g \cdot C^f_{p,q}.$$

The multiplication of f and g is thus defined by

$$f \cdot g = ((f_n \cdot g_n), C_{fg}, (C^{fg}_{p,q}), k_{fg}, N_{fg}),$$

with

$$\begin{array}{rcl} C_{fg} & = & C^f \cdot C^g \\ C^{fg}_{p,q} & = & C^f \cdot C^g_{p,q} + C^g \cdot C^f_{p,q} \\ k_{fg} & = & C^f \cdot k_g + C^g \cdot k_f \\ N_{fg} & = & \max(N^f, N^g) \end{array}$$

The associativities at the C, $C_{p,q}$, k and N levels work well. Let us check this at the k level: we have

$$k_{(fg)h} = C^{fg}k_h + C^h k_{fg} = C^f C^g k_h + C^h C^f k_g + C^h C^g k_f = C^f h_{gh} + C^{gh} k_f = k_{f(gh)}.$$

□

We now want to quotient $\tilde{\mathcal{O}}^{loc}(U)$ by the relation that identifies two Cauchy sequences f and g if their difference tends to zero. As in the above definition, we need to make additional choices to make this definition constructive, and this will not define an equivalence relation but a groupoid acting on $\tilde{\mathcal{O}}^{loc}(U)$, but the difference here is essentially technical.

The intuitive idea behind the following definition of the identification groupoid is that the triangular inequality gives

$$|f_n - h_n| = |f_n - g_n + g_n - h_n| \leq |f_n - g_n| + |g_n - h_n| < r_n/s_n + t_n/u_n < \frac{k_{f,g}}{m} + \frac{k_{g,h}}{m} = \frac{k_{f,g} + k_{g,h}}{m}$$

under the condition that $n \geq \max(N(m), M(m))$ when N and M are the moduli of convergence of $|f_n - g_n|$ and $|g_n - h_n|$ to 0.

Definition 12. Define the identification pre-groupoid (with only a partial unit) $\mathcal{R}(U)$ acting on $\tilde{\mathcal{O}}^{loc}(U)$ as the set of tuples

$$((f_n), (g_n), (r_n, s_n), k, N) \in \tilde{\mathcal{O}}^{loc}(U) \times \tilde{\mathcal{O}}^{loc}(U) \times (\mathbb{Q}^2_+)^{\mathbb{N}} \times \mathbb{N} \times \mathbb{N}^{\mathbb{N}}$$

such that, for every $n \geq 0$, we have morally (as above) the inequality

$$|f_n - g_n| < r_n/s_n$$

and r_n/s_n tends to zero with (up to the constant k) modulus of convergence $N : \mathbb{N} \to \mathbb{N}$, i.e., for every $m > 0$, if $n > N(m)$, then

$$r_n/s_n < \frac{k}{m}.$$

The source and the target maps $s, t : R(U) \to \tilde{\mathcal{O}}^{loc}(U)$ are the natural projections, and the composition is given by

$$\begin{array}{c}((f_n),(g_n),(r_n,s_n),k,N)\circ((g_n),(h_n),(t_n,u_n),l,M)\\ \| \\ ((f_n),(h_n),(r_nu_n+s_nt_n,s_nu_n),k+l,\max(N,M))\end{array}$$

The partial unit $e : \tilde{\mathcal{O}}^{loc} \to R(U)$ sends (f_n) to the diagonal Cauchy sequence

$$e((f_n)) = ((f_n),(f_n),(1,n),1,N : m \mapsto m)$$

Indeed, for $n > m$, we have $|f - f| < 1/n < 1/m$. This fulfills

$$e((f_n))\circ((f_n),(g_n),(r_n,s_n),k,N) = ((f_n),(g_n),(r_nn+s_n,s_nn),k+1,\max(N,\mathrm{id}_{\mathbb{N}})),$$

so that e is a left (and right) unit only on the $\tilde{\mathcal{O}}^{loc}(U)^2$ component. The obtained pre-groupoid, that encodes the constructive quotient, is denoted

$$\mathcal{O}^{loc}(U) := \left[\ \mathcal{R}(U) \underset{t}{\overset{s}{\rightrightarrows}} \tilde{\mathcal{O}}^{loc}(U)\ \right].$$

The presence of the modulus doesn't prevent the above composition map to be associative, because of the presence of the additional constant k. Indeed, if $((h_n),(k_n),(v_n,w_n),m,K)$ is in $\mathcal{R}(U)$ then the modulus of its composition with $((f_n),(h_n),(r_nu_n+s_nt_n,s_nu_n),k+l,L)$ given above is

$$\max(L,K) = \max(N,M,K),$$

and its additional constant is $k+l+m$. The modulus of the other composition in the associativity condition is also given by

$$\max(N,M,K)$$

and the additional constant is also $k+l+m$.

We are now able to show a completeness result for the formal/constructive quotient $\mathcal{O}^{loc}(U)$. As usual in constructive mathematics (see [BB85] and [TvD88]), to avoid the use of the axiom of choice, we use Cauchy sequences with moduli of Cauchy sequences with modulus to prove this result.

Definition 13. A Cauchy sequence with moduli of elements of $\tilde{\mathcal{O}}^{loc}(U)$ is a tuple

$$((f_n),(r_{1,p,q},s_{1,p,q}),(r_{2,s,p},r_{2,p,q}),(k_1,k_2),(M,K,L)) \in \tilde{\mathcal{O}}^{loc}(U)^{\mathbb{N}}\times(\mathbb{R}_+^4)^{\mathbb{N}^2}\times\mathbb{N}^2\times(\mathbb{N}^{\mathbb{N}})^2$$

such that for every m, if $p, q > M(m)$, we have

$$|f_p - f_q| < r_{1,p,q}/s_{1,p,q} < k_1/m,$$

i.e., for every $n > K(m)$ and $p, q > M(m)$, we have

$$|f_{p,n} - f_{q,n}| < r_{1,p,q}/s_{1,p,q} < k_1/m.$$

We further ask for a uniform modulus L for all the Cauchy sequences f_q, i.e., an $L : \mathbb{N} \to \mathbb{N}$ such that if $n > K(m)$ and $p, q > L(m)$, then

$$|f_{n,p} - f_{n,q}| < r_{2,p,q}/s_{2,p,q} < k_2/m.$$

We denote $\mathcal{C}auchy(\tilde{\mathcal{O}}^{loc}(U))$ the set of Cauchy sequences with moduli.

The usual triangular inequality

$$|f_{p,p} - f_{q,q}| \leq |f_{p,p} - f_{q,p}| + |f_{q,p} - f_{q,q}|$$

gives the following constructive completeness result for $\tilde{\mathcal{O}}^{loc}(U)$.

Proposition 10. *If $((f_n), (r_{1,p,q}, s_{1,p,q}), (r_{2,s,p}, r_{2,p,q}), (k_1, k_2), (M, K, L))$ is a Cauchy sequence with moduli of elements of $\tilde{\mathcal{O}}^{loc}(U)$, then the diagonal sequence*

$$\lim_n f_n := (f_{n,n}, r_{1,p,q}s_{2,p,q} + r_{2,q,p}s_{1,p,q}, s_{1,p,q}s_{2,p,q}, k_1 + k_2, \max(M, K, L))$$

is an element of $\tilde{\mathcal{O}}^{loc}(U)$. This defines a natural map

$$\lim : \mathcal{C}auchy(\tilde{\mathcal{O}}^{loc}(U)) \to \tilde{\mathcal{O}}^{loc}(U)$$

Remark 11. It is possible but cumbersome to define a constructive pre-groupoid $\mathcal{C}auchy\mathcal{R}(U)$ acting on $\mathcal{C}auchy(\tilde{\mathcal{O}}^{loc}(U))$ and a pre-groupoid morphism from it to $\mathcal{O}^{loc}(U)$.

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