Global analytic geometry and the functional equation[1]

(master course and exercices)

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[1]This text was typed in the flow of a course during one month at IMPA and thus certainly contains many typos, that will be corrected in proper time.

Contents

Acknowledgements

These notes were prepared for a master course at IMPA (Rio de Janeiro) during the month of February 2010 (six times two hours). Their aim is to give a survey, with some proofs and many exercices, of the main conjectures of arithmetic geometry, with an orientation towards global analytic methods. Please kindly inform me of any mistake, so that i can correct it.

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Chapter 1

Motivations

There is a strong analogy between the ring of integers $\mathbb{Z}$ and the ring of polynomials $\mathbb{F}_p[X]$ (where $\mathbb{F}_p := \mathbb{Z}/p\mathbb{Z}$ is a finite field with a prime number p of elements). Both of them have a euclidean division, are thus noetherian (ascending chain condition), and are of Krull dimension 1 (prime ideals are maximal).

This analogy that continues to hold on the level of completions with respect to various seminorms, is at the heart of various deep problems in analytic number theory.

Indeed, one can associate to $\mathbb{F}_p[X]$ and $\mathbb{Z}$ their respective Dedekind zeta $\zeta_{\mathbb{F}_p[X]}(s)$ and Riemann zeta $\zeta_{\mathbb{Z}}(s)$ functions. Both these functions fulfil a functional equation, whose proof can be done essentially in a unified setting, first settled by Tate and Iwasawa, using harmonic analysis (Poisson summation formula) on the locally compact group of adeles.

However, this analytic proof is restricted to dimension 1, because one needs locally compact groups to do integration theory. The idea of automorphic representation theory is to use these locally compact harmonic analytic methods with non-commutative groups to transpose (among other things) the proof of the functional equation to higher dimensional varieties. This has proved valuable for various examples but the relation between varieties and the functional equation of their zeta functions is then a very involved program, called Langlands correspondence, that is not even fully settled at the time being. It has however given deep results as Fermat's last theorem, to cite the most famous one.

A good thing to do before developing further analytic number theoretical methods is to better understand the geometry of numbers, since it has proved essential in Grothendieck's proof of the functional equation for general Dedekind zeta functions, and has also allowed Deligne to prove the analog of Riemann hypothesis in this case. There are various hints in the litterature to the fact that this problem remains open in the case of $\mathbb{Z}$ because we don't understand enough the relation between the archimedean place and prime ideals. Global analytic geometry, whose basic ideas were grounded by Kurschak, Ostrowski, E. Artin, Zariski and many others, and that have been given a decisive new impetus by V. Berkovich, gives a pragmatic way to approach this problem, and is certainly a nice setting to better understand the relation between geometry and analytic number theory.

1.1 Functions and numbers

We here develop a bit on the analogy between the ring of polynomial functions $\mathbb{C}[X]$ and the ring of integers $\mathbb{Z}$.

1.1.1 What is a number?

This section can be read on a third degree basis. Its aim is to put the reader in front of the fact that usual mathematical numbers are not as natural as they look like. The above question is not well formulated. The author of these notes thinks of any mathematical entity as a machine that was constructed by human mind to solve a given problem of (almost) everyday life. One should thus ask

What problems are solved by the numbers we use?

The answer will lead us naturally to the basic notions of global analytic geometry.

- The set of non-zero natural numbers $\mathbb{N}_{>0}$ solves the problem of counting things, in a universal way: counting potatoes is the same as counting people, even if these are quite different notions. It's first use dates from prehistoric time, around 35000 years ago.
- Rational numbers solve the problem of cutting a cake in parts. As natural numbers, they are as old as prehistoric times.
- Irrational numbers were first aluded to around 800-500 B.C.

- The zero number solves the problem of talking about nothing. It's use was "invented" in mesopotamia around 300 B.C.
- Relative numbers solve the problem of taking some potatoes out of a given basket full of them. They were first used around 100 B.C.
- Real numbers solve the problem of measuring lengths (or higher dimensional analogs, e.g. surfaces, volumes, etc...). For example, the algebraic number $\sqrt{2}$ measures the length of the triangle's hypothenuse and the real number π measures the length of the unit circle. Their proper mathematic definition dates from 1878 (Weierstrass, Dedekind and Cantor, among others).

As you can see, real numbers were very hard to find and we arrived to their construction after solving many interesting problems: it took around 35000 years to get to this notion. Actually, the study and definition of nice sets of numbers that solve problems from every day life or natural science remains one of the main focus of present mathematical research. For example,

- periods are length of curves defined by rational polynomials (or more generally integrals of rational differential forms). For example π is the length of the curve
$$S^1 = \{(x, y) \in \mathbb{R}^2,\, x^2 + y^2 = 1\}.$$
- exponential periods are related to rational differential equations, for example e is the value at one of the solution e^x of the differential equation
$$y' = y.$$

For a very nice account of these sets of numbers and parts of the mysteries that surround them, see Kontsevich and Zagier's overview [ZK01].

We will see along this course that global analytic geometry gives us a new geometrical way to look at numbers, that will be useful to better understand them, and to prove some elegant theorems.

1.1.2 Seminorms and completions

Since we want to explain in detail the relation between integer rings and polynomial rings, we will give back precise definitions for all the objects in play.

Definition 1. *A* seminorm *on a group A is a map $|.| : A \to \mathbb{R}_+$ such that for all $a, b \in A$, we have*

1. $|0| = 0$,
2. $|-a| \leq |a|$,
3. $|a + b| \leq |a| + |b|$ *"triangle inequality".*

If moreover $|a| = 0$ implies $a = 0$, the seminorm is called a norm. *If $(A, +, \times, 0, 1)$ is a ring, a* ring seminorm *on A is a seminorm on $(A, +, 0)$ such that*

4. $|1| = 1$,
5. $|ab| \leq |a|.|b|$.

We will later replace the triangle inequality by another (almost) equivalent condition.

If $|.| : A \to \mathbb{R}_+$ is a norm (resp. a ring norm), one can define the completion of A with respect to $|.|$ as the quotient of the set of Cauchy sequences in A by null sequences.

1.1.3 Functional examples

Seminorms are the necessary building blocs for almost all interesting spaces of functions in analysis.

For example, the ring $\mathcal{C}^0(K, \mathbb{R})$ of continuous real valued functions on a compact subset $K \subset \mathbb{R}$ can be defined (this is the so-called Stone-Weierstrass theorem, see [Rud76], theorem 7.24) as the completion of the polynomial ring $\mathbb{R}[X]$ with respect to the sup norm on K, i.e., for

$$\|P\|_{\infty,K} := \sup_{x \in K} |P(x)|.$$

More generally, the ring $\mathcal{C}^0(\mathbb{R}, \mathbb{R})$ of continuous real valued functions on $\mathbb{R}$ can be defined as the completion of the polynomial ring $\mathbb{R}[X]$ with respect to the family of seminorms

$$\{\|.\|_{\infty,K},\ K \text{ compact subset in } \mathbb{R}\}.$$

Similarly, one can define the ring $\mathcal{O}(D(0,1))$ of analytic functions on the unit disc $D(0,1) = \{z \in \mathbb{C} |\, |z| \leq 1\}$ in $\mathbb{C}$ continuous on the boundary as the completion of the ring of polynomials with respect to the sup norm $\|.\|_{\infty,D(0,1)}$ (This is Mergelyan's theorem, see Rudin [Rud87], Theorem 20.5). The ring of analytic functions $\mathcal{O}(\mathbb{C})$ on $\mathbb{C}$ is also the completion of $\mathbb{C}[X]$ with respect to the family of seminorms

$$\{\|.\|_{\infty,D(z,r)},\ z \in \mathbb{C},\ r > 0\}.$$

If we drop the submultiplicativity condition $|ab| \leq |a|.|b|$, we can also construct $L^1([0,1])$ as the completion of $\mathbb{R}[X]$ with respect to the norm

$$\|P\|_{1,[0,1]} = \int_0^1 |P(x)|dx$$

and $\mathcal{C}^1([0,1])$ as the completion of $\mathbb{R}[X]$ with respect to the norm

$$N_1(P) = \|P\|_{\infty,[0,1]} + \|P'\|_{\infty,[0,1]}.$$

One can generalize these examples to get locally integrable, smooth functions on a compact, and various other types of functions, that are the basic tools of functional analysis. The construction of Schwartz functions is more involved since it probably can't be obtained directly from polynomials, but only using a two step completion (for examples, of smooth functions with compact support).

These very generic examples show that the notion of seminorm is an important tool that allows to construct many (if not all) interesting functional spaces. They are however also very useful to study spaces of numbers.

1.1.4 Number theoretic examples

The main example of number theoretic completions are given by the completion of $\mathbb{Z}$ and $\mathbb{Q}$ with respect to their ring norms:

1. The archimedean norm $|.|_\infty : \mathbb{Z} \to \mathbb{R}_+$ is given by

   $$|n|_\infty := \max(n, -n).$$

   The completion of $\mathbb{Q}$ with respect to (the extension of) this norm is a possible definition of the ring $\mathbb{R}$ of real numbers.

2. If p is a prime number, the (normalized) p-adic norm $|.|_p : \mathbb{Z} \to \mathbb{R}_+$ is given (for $n \neq 0$) by
$$|n|_p := \frac{1}{p^{v_p(n)}}$$
where $v_p(n)$ is the only positive integer (whose existence and unicity is proved using euclidean division) such that $n = p^{v_p(n)}.m$ with p not dividing m. The completion of $\mathbb{Z}$ with respect to the p-adic norm is called the ring of p-adic numbers and denoted $\mathbb{Z}_p$. The completion of $\mathbb{Q}$ with respect to (the extension of) this norm is called the field of p-adic numbers and denoted $\mathbb{Q}_p$.

3. The trivial seminorm $|.|_0 : \mathbb{Z} \to \mathbb{R}_+$ is defined as being equal to 0 on 0 and to 1 on all other integers. The completion of $\mathbb{Q}$ with respect to this seminorm is $\mathbb{Q}$ itself, equipped with the discrete topology.

4. If p is a prime number, the residually trivial seminorm $|.|_{0,p}$ is defined as the composition of the projection $\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z}$ with the trivial seminorm on $\mathbb{Z}/p\mathbb{Z}$. The separated completion of $\mathbb{Z}$ with respect to this residual seminorm is $\mathbb{Z}/p\mathbb{Z}$, equiped with the discrete topology.

These new examples show that seminorms are also the central objects of number theory, since they are even necessary to define what a good notion of number is. We will see later that they also play a central role in the definition of adeles, that are the basic technical tool of analytic number theory.

1.1.5 Various types of seminorms

We have thus seen that seminorms are the basic tools of functional analysis and number theory. We now describe some important families of seminorms.

- If we have $|ab| = |a|.|b|$ for all $a, b \in A$, the seminorm is called *multiplicative*. The norm $P \mapsto |P(z)|$ on $\mathbb{C}[X]$ for $z \in \mathbb{C}$ fixed fulfils this condition. These kind of seminorms are also often called *real valued valuations*. They are deeply related with algebraic geometry and number theory since the multiplicativity condition implies that their kernel is a prime ideal.

- If we have $|a^n| = |a|^n$ for all $a \in A$, the seminorm is called *power-multiplicative*. The uniform norm $\|.\|_{\infty,D(0,1)}$ on the unit disc in $\mathbb{C}[X]$

defined above fulfils this condition. These kind of seminorms are very natural in analytic geometry, and are also called *uniform seminorms*, since they are often related to uniform convergence properties.

- If $|2| \leq 1$, the seminorm is called *non-archimedean*. The p-adic norm $|.|_p : \mathbb{Z} \to \mathbb{R}_+$ is non-archimedean.
- If $|2| > 1$, the seminorm is called *archimedean*. The archimedean norm $|.|_\infty : \mathbb{Z} \to \mathbb{R}_+$ is archimedean since $|2|_\infty = 2$.

1.1.6 Classification of seminorms

We will later prove that multiplicative (and power-multiplicative) seminorms are quite tractable objets in practice since one can classify them on our two main examples $\mathbb{C}[X]$ and $\mathbb{Z}$. These two results are deep and important theorem for number theory and functional analysis respectly.

Theorem 1 (Ostrowski)**.** *Let $|.| : \mathbb{Z} \to \mathbb{R}_+$ be a multiplicative seminorm on $\mathbb{Z}$. Then $|.|$ is one of the following (see examples in subsection 1.1.4):*

1. $|.|_p^t$ *for some prime p and $t \in \mathbb{R}_{>0}$,*
2. $|.|_\infty^t$ *for $t \in]0,1]$,*
3. $|.|_0 = \lim_{t\to 0} |.|_\infty^t$,
4. $|.|_{0,p} = \lim_{t\to\infty} |.|_p^t$ *for some prime p.*

This theorem can be summed up by the following drawing of the space of all multiplicative seminorms on $\mathbb{Z}$ (also called the Berkovich spectrum of $\mathbb{Z}$): We have here included the powers $|.|_\infty^t$ of the archimedean norm with $t \in [0,\infty[$ and the ∞ point corresponds to the infinite power $|.|_\infty^\infty$ that is well defined on $[-1,1]$, but not on $\mathbb{Z}$ itself.

Theorem 2 (Gelfand-Mazur)**.** *Let $|.| : \mathbb{C}[X] \to \mathbb{R}_+$ be a multiplicative seminorm on $\mathbb{C}[X]$ whose restriction to $\mathbb{C}$ is the usual norm $|.|_\infty$. Then there exists $z \in \mathbb{C}$ such that*

$$|P| = |P(z)|_\infty.$$

This theorem can be summed up by saying that the Berkovich spectrum of $\mathbb{C}[X]$ as a $(\mathbb{C}, |.|_\infty)$-algebra is equal to $\mathbb{C}$.

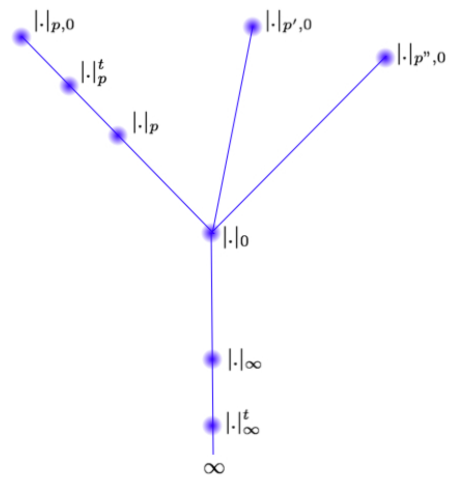

Figure 1.1: The analytic spectrum $\mathcal{M}(\mathbb{Z})$.

1.2 Analytic number theory and dynamical systems

The reader that is not familiar with analytic number theory can pass this motivation part and go directly to section 1. The given examples will be explained more precisely in the forthcoming sections.

1.2.1 Dynamical properties of p-adic rational maps

A theorem by Favre and Rivera-Letelier [FRL04] shows that the study of dynamics in the p-adic setting is very similar to the complex case, if one uses Berkovich's p-adic projective line.

Theorem 3. *Let R be a rational map of degree $D \geq 2$ defined over $\mathbb{C}_p$. There exists a probability measure ρ_R on $\mathbb{P}^{1,an}_{\mathbb{C}_p}$, which is invariant by R_*,* *mixing*[1]*, whose support equals the Julia set*[2] *of* R_* *in* $\mathbb{P}^1(\mathbb{C}_p)$*, and such that* $\lim_{n\to\infty} D^{-n} R_n^* \delta_z = \rho_R$*, for each point* $z \in \mathbb{P}^1(\mathbb{C}_p)$ *which is not totally invariant by* R *or* R^2*.*

1.2.2 The functional equation of Riemann's zeta function

Recall that Riemann's zeta function is given by the series

$$\zeta(s) := \sum_{n\geq 0} \frac{1}{n^s}.$$

This series converge for $\mathrm{Re}(s) > 1$ and has there an Euler product expression

$$\zeta(s) = \prod_p \frac{1}{1-p^{-s}}$$

indexed by prime ideals that converges also for $\mathrm{Re}(s) > 1$. This is a first explanation of the role that the zeta function plays in questions about prime numbers in $\mathbb{R}$.

If we denote

- $\zeta_p(s) := \frac{1}{1-p^{-s}}$ the prime $p < \infty$ local factor and
- $\zeta_\infty(s) := \pi^{-s/2}\Gamma(s/2)$,

[1]Ergodicity means

$$\forall f \in L^1(\mu),\ \lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f\left(R^k(x)\right) = \frac{1}{\mu(X)} \int_X f(y) d\mu(y)$$

and mixing means

$$\forall f, g \in L^1(\mu),\ \lim_{n\to\infty} \int f(R^n x) g(x) d\mu = \int f d\mu . \int g d\mu.$$

[2]The Julia set is the complementary of the Fatou set, that is the maximal subset on which the family of iterates of R form a normal family (every sequence contains a subsequence that converges uniformly on compact subsets).

the completed zeta function

$$\hat{\zeta}(s) := \prod_{p \leq \infty} \zeta_p(s)$$

fulfils the functional equation

$$\hat{\zeta}(s) = \hat{\zeta}(1-s).$$

This is hard to see on the given definition of $\hat{\zeta}$ but this function can also be seen as the Mellin transform of a periodic function on $\mathbb{R}$ (the θ function), and the functional equation then follows from Poisson's summation formula for θ on $\mathbb{R}$. This proof is already in Riemann's paper from 1859.

This will be treated in great details in this course.

The modern version of this proof [Tat67] is essentially the same, except that it replaces harmonic analysis on $\mathbb{Z} \subset \mathbb{R}$ by harmonic analysis on $\mathbb{Q} \subset \mathbb{A}$ where $\mathbb{A}$ denotes the topological group of adeles. We will come back to this point latter, since it is important to understand the functional equations of more general "non-commutative" spectral zeta functions (also called automorphic L-functions).

1.2.3 Dynamical description of local factors

Another motivation for looking for a geometry that treats all local factors of zeta functions on equality footing is the work of Deninger [Den98], that gives a uniform formula for archimedean and non-archimedean local factors of a (flat) affine variety over $\mathbb{Z}$. He also proposes a conjectural formalism (that however lacks for a proposed construction) to treat arithmetic zeta functions in way similar to the cohomological approach of Grothendieck through Lefchetz fixed point formula on foliated laminations.

This dynamical description can already be understood in the case of $\mathbb{Q}_p$ and $\mathbb{R}$.

Recall that $\zeta_p(s) = \frac{1}{1-p^{-s}}$. If one defines (ad hoc)

$$\mathcal{R}_p := \mathcal{C}^\infty(\mathbb{R}^*/p^{\mathbb{Z}}) \cong \mathcal{C}^\infty(\mathbb{R}/\log p\mathbb{Z})$$

and

$$\mathcal{R}_\infty := \mathbb{R}[e^{-2t}],$$

and equip these spaces for $p \leq \infty$ with the action of the endomorphism

$$\Theta := \frac{\partial}{\partial t} : \mathcal{R}_p \to \mathcal{R}_p,$$

the spectrum of Θ on $\mathcal{R}_p$ is $\frac{2i\pi}{\log p}\mathbb{Z}$ if $p < \infty$ and $-2\mathbb{N}$ otherwise. These are exactly the poles of the given local factor.

One can then use a zeta-regularized determinant to express both local factors in the form

$$\zeta_p(s) = \det_\infty \left(\frac{1}{2\pi}(s - \Theta)|\mathcal{R}_p \right)^{-1}, \ p \leq \infty.$$

Recall that if (V, Θ) is a vector space with endomorphism with discrete numerable spectrum, we let $\zeta_\Theta(s) = \sum_{0 \neq \alpha \in \mathrm{Sp}\,\Theta} \frac{1}{\alpha^s}$ be the corresponding spectral zeta function and

$$\det_\infty(\Theta|V) = \begin{cases} \exp(-\zeta'_\Theta(0)) & \text{si } 0 \notin \mathrm{Sp}(\Theta), \\ 0 & \text{si } 0 \in \mathrm{Sp}(\Theta). \end{cases}$$

1.2.4 The global spectral interpretation

These works due to Connes [Con99] and Meyer [Mey05] in the number field case and (in a less complete version) to Godement and Jacquet [GJ72] in the spectral theory of automorphic representations, tend to show that the zeroes of automorphic zeta function are related to orbits of a flow on some very special dynamical systems, like for example the action of the multiplicative group of rational numbers $\mathbb{Q}^*$ on the additive locally compact group of adeles $\mathbb{A}$, or more generally the action of $\mathrm{GL}_n(\mathbb{Q})$ on the additive group of adelic matrices $\mathrm{M}_n(\mathbb{A})$.

This approach to zeta functions can be roughly summed up by saying that for each type of zeta function $\zeta_0(s)$ (or more precisely L-function), there is a naturally defined family $\{f_i\}_{i\in I}$ of functions on $\mathbb{R}_+$ such that $\zeta_0(s)$ is the greatest common divisor of the family $\{M(f_i, s)\}_{i\in I}$ of Mellin transforms of this family, i.e.,

$$\zeta_0(s) := \mathrm{GCD}(M(f_i, s),\, i \in I).$$

The difficulty in this approach is to find a natural family $\{f_i\}$ that gives such a result, but the obtained results are quite easy to find once this is done.

For example, this way of proving the functional equation is very close to the geometric way, since it is based on a spectral identification of zeroes of the zeta functions in play. It even gives a new, fully spectral definition of the relevant zeta functions. Its relation with geometrical methods, at least in dimension 1, remains however to be better understood. Moreover, combining with Deninger's method, one gets a regularized determinant expression for the full zeta function. The problem is that this expression is not yet coming from a geometric Lefchetz trace formula.

Chapter 2

Global analytic spectra

2.1 The analytic spectrum

2.1.1 Definition

All groups and rings will be commutative and unital.

The ordering on $\mathbb{Z}$ induces an ordering on $\mathbb{Q}$. We denote $\mathbb{Q}_+$ the set of positive rational numbers. The positive real numbers $\mathbb{R}_+$ are given by Dedekind cuts in $\mathbb{Q}_+$. These are partitions

$$\mathbb{Q}_+ = X \coprod Y$$

such that X and Y are non-empty and every element of X is smaller than every element of Y.

Definition 2. *A* seminorm *on a group* $(A, +, -, 0)$ *is a map* $|.| : A \to \mathbb{R}_+$ *such that for all* $a, b \in A$*, we have*

1. $|0| = 0$,
2. $|-a| \leq |a|$,
3. $|a + b| \leq |2|. \max(|a|, |b|)$.

If moreover $|a| = 0$ *implies* $a = 0$*, the seminorm is called a* norm. *If* $(A, +, \times, 0, 1)$ *is a ring, a* ring seminorm *on* A *is a seminorm on* $(A, +, 0)$ *such that*

3. $|1| = 1$,

4. $|ab| \leq |a|.|b|$.

If $|.| : A \to \mathbb{R}_+$ is a seminorm (resp. a ring seminorm), its *kernel* $\mathfrak{a}_{|.|} := \{a \in A|\, |a| = 0\}$ is a subgroup (resp. an ideal of) A and there is a natural factorization

$$|.| : \bar{A} := A/\mathfrak{a}_{|.|} \to \mathbb{R}_+.$$

The *separated completion* of A with respect to $|.|$ is defined as the quotient topological group (resp. topological ring) $\widehat{(A, |.|)}$ of the space $\mathcal{C}_{(\bar{A},|.|)}$ of Cauchy sequences in $(A, |.|)$ by the subgroup (resp. ideal) of sequences converging to 0. There is a natural embedding

$$\bar{A} \to \hat{A}.$$

Definition 3. *A ring seminorm $|.| : A \to \mathbb{R}_+$ is called*

- multiplicative *if $|ab| = |a|.|b|$ for all $a, b \in A$,*
- power-multiplicative *if $|a^n| = |a|^n$ for all $a \in A$,*
- non-archimedean *if $|2| \leq 1$,*
- archimedean *if $|2| > 1$.*

Definition 4. *Let A be a ring. The analytic spectrum of A is the set $\mathcal{M}(A)$ of all multiplicative seminorms on A. If $(A, \|.\|)$ is a Banach ring (complete normed ring), the analytic spectrum of $(A, \|.\|)$ is the set $\mathcal{M}(A, \|.\|)$ of bounded multiplicative seminorms on A, i.e., the set of multiplicative seminorms $|.| : A \to \mathbb{R}_+$ such that there exists $C > 0$ with*

$$|a| \leq C\|a\| \text{ for all } a \in A.$$

For all $a \in A$, there is a natural evaluation map

$$\mathrm{ev}_a : \mathcal{M}(A) \to \mathbb{R}_+$$

given by $|.| \mapsto |a|$. The analytic spectrum is equiped with the coarsest topology that makes all evaluation maps continuous.

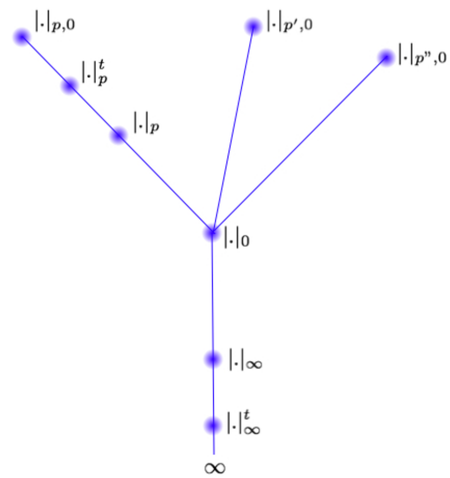

Figure 2.1: The analytic spectrum $\mathcal{M}(\mathbb{Z})$.

2.1.2 Ostrowski's theorem

We now explain Ostrowski's exhaustive classification of multiplicative seminorms on relative integers. This can be summarized by saying that the spectrum of $\mathbb{Z}$ is a tree with one branch per prime number and one archimedean branch, all of them being homeomorphic to $[0, 1]$ and pasted at one single point, as in figure 2.1.2.

Theorem 4 (Ostrowski). *Let $|.| : \mathbb{Z} \to \mathbb{R}_+$ be a multiplicative seminorms on $\mathbb{Z}$. Then $|.|$ is one of the following (see examples in subsection 1.1.4):*

1. *$|.|_p^t$ for some prime p and $t \in \mathbb{R}_{>0}$,*
2. *$|.|_\infty^t$ for $t \in \mathbb{R}_{>0}$,*
3. *$|.|_0 = \lim_{t \to 0} |.|_\infty^t$,*
4. *$|.|_{0,p} = \lim_{t \to \infty} |.|_p^t$ for some prime p.*

Proof. See Neukirch [Neu99], proposition 3.7, or Artin [Art67], section 1.5. Let $|.| : \mathbb{Z} \to \mathbb{R}_+$ be a multiplicative seminorm. Multiplicativity implies that the kernel $\mathfrak{p}_{|.|} := \{n \in \mathbb{Z}, |n| = 0\}$ of $|.|$ is a prime ideal, i.e., of the form (p) for some prime number p or (0). If $\mathfrak{p}_{|.|} = (p)$, then $|.|$ factorizes through

$|.| : \mathbb{Z}/p\mathbb{Z} \to \mathbb{R}_+$. Every element k in the multiplicative group $(\mathbb{Z}/p\mathbb{Z})^\times$ has finite order m, so that $|k|^m = |k^m| = |1| = 1$. This implies that $|k| = 1$ because the only root of 1 in $\mathbb{R}_+$ is 1. This shows that $|.|$ is equal to the trivial norm $|.|_{0,p}$.

Suppose now that $|.|$ has trivial kernel, i.e., is a norm. Let $m, n > 1$ be integers, and write m in the n-adic scale:

$$m = a_0 + a_1 n + \cdots + a_r n^r,$$

with $0 \leq a_i < n$, and $n^r \leq m$, i.e., $r \leq \frac{\log m}{\log n}$. We have $|a_i| < n.|1| = n$ and

$$|m| \leq \sum_{i=0}^{r} |a_i|.|n|^i \leq \sum |a_i| \max(1, |n|)^r \leq \sum n \max(1, |n|)^r \leq (1+r) n \max(1, |n|)^r$$

so that

$$|m| \leq \left(1 + \frac{\log m}{\log n}\right) n \max(1, |n|)^{\frac{\log m}{\log n}}.$$

Using this estimate for $|m|^s = |m^s|$, extracting the s-th root and letting $s \to \infty$ gives

$$|m| \leq \max(1, |n|)^{\frac{\log m}{\log n}}.$$

If $|n| = 1$ for all nonzero $n \in \mathbb{Z}$, we get the trivial norm $|.| = |.|_0$ on $\mathbb{Z}$. We suppose this is not the case.

We now consider two cases:

1. Suppose that $|n| > 1$ for all $n > 1$. Then the above inequality gives

$$|m|^{1/\log m} \leq |n|^{1/\log n}$$

and exchanging the role of n and m, we get

$$|m|^{1/\log m} = |n|^{1/\log n} = e^t$$

where t is a positive number. It follows that

$$|n| = e^{t \log n} = n^t$$

so that

$$|.| = |.|_\infty^t$$

where $|n|_\infty = \max(n, -n)$ for $n \in \mathbb{Z}$.

2. Suppose that there exists an integer $n > 1$ such that $|n| < 1$. Then the set $\mathfrak{p}$ of all integers n such that $|n| < 1$ form a prime ideal of $\mathbb{Z}$. Indeed, if $|xy| < 1$ then $|x|.|y| < 1$ so that either $|x| < 1$ or $|y| < 1$. We thus have $\mathfrak{p} = (p)$ for some prime number p. If $|p| = c$ and $m = p^\nu b$ with $(p, b) = 1$ then $|m| = c^\nu$. If we let $c = \frac{1}{p^t}$ for some positive real t, we thus have
$$|.| = |.|_p^t.$$

□

Corollary 1. *A power-multiplicative seminorm on a ring is non-archimedean, i.e., fulfils $|2| \leq 1$, if and only if it fulfils*

$$|a + b| \leq \max(|a|, |b|).$$

Proof. Suppose first that $|.|$ is non-archimedean, i.e., $|2| \leq 1$. Restricting $|.|$ to the ring of integers $\mathbb{Z}$ and using the inequality

$$|m| \leq \max(1, |n|)^{\frac{\log m}{\log n}}$$

shown in the proof of Ostrowski's theorem with $n = 2$ and any m, we show that $|m| \leq 1$ for all $m \in \mathbb{Z}$. Now consider the equality

$$|a + b|^n = |(a + b)^n| = |\sum_{i=0}^{n} \binom{i}{n} a^i b^{n-i}|,$$

we get
$$|a + b|^n \leq (n + 1) \max(|a|, |b|)^n.$$

Taking the n-th root and passing to the limit $n \to \infty$, we get

$$|a + b| \leq \max(|a|, |b|).$$

If this inequality is fulfilled, it is clear that $|2| = |1 + 1| \leq \max(|1|, |1|) = 1$. □

2.1.3 The spectrum $\mathcal{M}(\mathbb{Z})$

We use Poineau's nice introduction [Poi08a] to global analytic geometry.

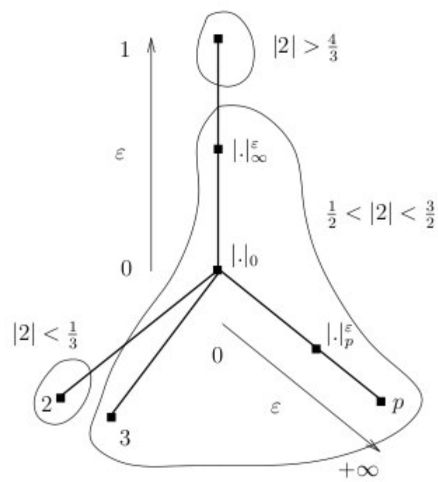

Figure 2.2: Open subsets of $\mathcal{M}(\mathbb{Z})$.

By definition, you get a base of the topology of $\mathcal{M}(\mathbb{Z})$ by taking the following sets :

$$\bigcap_{1\leq i\leq u}\left\{|.|_x\in\mathcal{M}(\mathbb{Z})\,\middle|\,|f_i|_x<r_i\right\}\cap\bigcap_{1\leq j\leq v}\left\{|.|_x\in\mathcal{M}(\mathbb{Z})\,\middle|\,|g_j|_x>s_j\right\},$$

with $u,v\in\mathbb{N}$, $f_1,\ldots,f_u,g_1,\ldots,g_v\in\mathbb{Z}$ and $r_1,\ldots,r_u,s_1,\ldots,s_v\in\mathbb{R}$.

Exercice 1. *Describe geometrically the following open subsets:*

1. $|3|>1$; $|3|\leq 1$; $|3|<1$,
2. $|2^n|<\frac{1}{2}$ *for* $n>>0$,
3. *an open subset that contains the trivial norm* $|.|_0$,
4. $\frac{1}{5}<|5|<\frac{3}{2}$,
5. $|28|>1$; $|28|\leq\frac{1}{2}$.

2.1.4 First properties of the Berkovich spectrum

Let $(A, \|.\|)$ be a Banach ring. The completion of $A[T]$ with respect to the norm

$$\| \sum_{i \geq 0} a_i T^i \|_r := \sum_{i \geq 0} \|a_i\| r^i$$

is the ring of convergent power series on A, denoted $A\langle r^{-1}T \rangle$.

Theorem 5. *The spectrum $\mathcal{M}(A)$ of a commutative Banach ring is a nonempty, compact Hausdorff space.*

Proof. The proof can be found in [Ber90], page 13-14. The compacity follows from the fact that the natural map

$$\mathcal{M}\left(\prod_{x \in \mathcal{M}(A)} \mathcal{K}(x) \right) \to \mathcal{M}(A)$$

is surjective and the left hand side is compact. We only show that $\mathcal{M}(A)$ is nonempty and Hausdorff. We can replace A by the quotient $A/\mathfrak{m}$ of A by a maximal ideal, an thus suppose that A is a Banach field with the residue norm $|\bar{f}| := \inf\{|g|, \, g \in \pi^{-1}(f)\}$ with $\pi : A \to A/\mathfrak{m}$ the quotient map. Let S be the set of all nonzero bounded seminorms on A. It is nonempty because the norm of A belongs to it. It is partially ordered and every chain admits an infimum so that, by Zorn's lemma, it admits a minimal element, denoted $|.|$. We replace A by its completion with respect to $|.|$. We will prove that $|.|$ is multiplicative.

Suppose there exists an element $f \in A$ with $|f^n| < |f|^n$ for some $n > 1$. The element $f - T$ is noninvertible in the Banach ring $A\langle r^{-1}T \rangle$ with $r = \sqrt[n]{|f|^n}$ because $\sum_{i \geq 0} |f^{-i}| r^i$ does not converge. If $i = pn + q$, where $0 \leq q \leq n - 1$, then $|f^i| \leq |f^n|^p |f^q|$ and

$$|f^{-i}| r^i \geq |f^i|^{-1} |f^n|^{n+p/n} \geq \frac{|f^n|^{q/n}}{|f^q|}.$$

If we denote $\epsilon := \min \left\{ \frac{|f^n|^{q/n}}{|f^q|}, 0 \leq q \leq n-1 \right\}$, we have $|f^{-i}| r^i \geq \epsilon > 0$ for all $i \geq 0$. Consider the morphism $\varphi : A \to A\langle r^{-1}T \rangle / (f - T)$. It is injective since A is a field, and $\|\varphi(f)\| = \|T\| \leq r < |f|$. But this is impossible since $|.|$ is a minimal bounded seminorm on A.

Now suppose that there exists an element $f \in A - \{0\}$ such that $|f|^{-1} < |f^{-1}|$. Then the element $f - T$ is noninvertible in $A\langle r^{-1}T\rangle$ with $r = |f^{-1}|^{-1}$. Indeed, since $|f^{-i}|r^i = |f^{-1}|^i|f^{-1}|^{-i} = 1$, the series $\sum_{i\geq 0}|f^{-i}|r^i$ does not converge. As above, for the homomorphism $\varphi : A \to A\langle r^{-1}T\rangle/(f - T)$, we have $\|\varphi(T)\| = |T| \leq r < |f|$. We get that $|f|^{-1} = |f^{-1}|$.

Now remark that proving $|f|.|g| \leq |fg|$ is equivalent to proving $|f| \leq |fg|.|g|^{-1}$ but this follows from $|fgg^{-1}| \leq |fg|.|g^{-1}| \leq |fg|.|g|^{-1}$. This shows that $|.|$ is multiplicative.

Let x_0 and x_1 be two points in $\mathcal{M}(A)$. Then there exists an element $f \in A$ such that $|f(x_0)| \neq |f(x_1)|$. Suppose that $|f(x_0)| < |f(x_1)|$ and let r be a real number such that $|f(x_0)| < r < |f(x_1)|$. Let $U_0 = \{x, |f(x)| < r\}$ and $U_1 = \{x, |f(x)| > r\}$. These two open subsets are disjoint and contain respectively x_0 and x_1, so that $\mathcal{M}(A)$ is separated. □

Proposition 1. *An element $f \in A$ is invertible if and only if $|f(x)| \neq 0$ for all $x \in \mathcal{M}(A)$.*

Definition 5. *Let $A = (A, \|.\|)$ be a commutative Banach ring and $f \in A$. The spectral radius of f is the number*

$$\rho(f) := \lim_{n\to\infty} \sqrt[n]{\|f^n\|}.$$

The uniform norm of f is

$$\|f\|_\infty := \max_{x\in\mathcal{M}(A)} |f(x)|.$$

Proposition 2. *There is an equality between the spectral radius and the uniform norm.*

Proof. The inequality $\|f\|_\infty \leq \rho(f)$ follows from the definition of $\mathcal{M}(A)$. To verify the reverse inequality, it suffices to show that if $|f(x)| < r$ for all $x \in \mathcal{M}(A)$, then $\rho(f) < r$. Consider the Banach ring $B := A\langle rT\rangle$. Since $\|T\| = r^{-1}$, we have $|T(x)| \leq r^{-1}$ for all $x \in \mathcal{M}(B)$. Therefore, $|(fT)(x)| < 1$. In particular, $(1 - fT)(x) \neq 0$ for all $x \in \mathcal{M}(B)$. This implies (non trivial) that $1 - fT$ is invertible in B. It follows that the series $\sum_{i\geq 0}\|f^i\|r^{-i}$ is convergent hence $\rho(f) < r$ since the convergence radius of this series is $\rho(f)$ (see Bourbaki [Bou67], I, §2, n. 4). □

Proposition 3. *Let $(K, |.|)$ be a (multiplicatively) normed field. The uniform norm on the ring $K\langle r^{-1}T\rangle$ of convergent power series is equal to*

1. $\rho(f) = \max\{|f(z)|, z \in \mathbb{C}\}$ *if* K *is archimedean and*

2. $\rho(f) = \max_{i\geq 0} |a_i|r^i$ *if* K *is non-archimedean.*

Moreover, in the second case, ρ *is a multiplicative seminorm.*

Proof. Indeed, let $|f|$ be the number defined on the right side. Since $|.|$ is a bounded power-multiplicative norm on A, we have $|f| \leq \rho(f)$. It suffices to verify the reverse inequality for polynomials. Let $f = \sum_{i=0}^{d} a_i T^i$. In both case, we have $|a_i| \leq r^{-i}|f|$ (obvious in case 2 and follows from Cauchy's formula in case 1). Hence, $\|f\| \leq (d+1)|f|$. Applying this to f^n, taking the n-th root and passing to the limit for $n \to \infty$ gives the desired inequality.

Now we prove that $|.| = \rho$ is a multiplicative seminorm in the second case. Let $P = \sum a_i T^i$ and $Q = \sum b_j T^j$ be two polynomials in $K[T]$. We have

$$|P.Q| = |\sum a_i b_j T^{i+j}| = \max(|a_i b_j| r^{i+j}) = \max(|a_i|r^i.|b_j|r^j)$$

because $|.|$ is multiplicative on K so that $|a_i b_j| = |a_i|.|b_j|$. This maximum can only be reached when $|a_i|r^i$ and $|b_j|r^j$ are maximum. This means that

$$|PQ| = \max(|a_i|r^i.|b_j|r^j) = \max(|a_i|r^i).\max(|b_j|r^j) = |P|.|Q|,$$

so that $|.|$ is multiplicative. □

Proposition 4. *There is a bijection between equivalence classes of bounded morphisms*

$$\chi : (A, \|.\|) \to (K, |.|)$$

to a normed Banach field and points of $\mathcal{M}(A)$*.*

Theorem 6. *The spectrum* $\mathcal{M}(A)$ *is locally arcwise connected.*

Proof. See Berkovich [Ber90], Theorem 3.2.1. □

2.1.5 The affine line

We refer to Poineau's thesis [Poi07], and his more recent paper [Poi08b] for a thorough study of affine analytic spaces. Let A be a ring. The affine space of dimension n over A is the analytic spectrum

$$\mathbb{A}^n_A := \mathcal{M}(A[T_1, \ldots, T_n])$$

of the polynomial ring over A.

If $A = (A, |.|)$ is a Banach ring, we also denote $\mathbb{A}^1_A$ the subspace of $\mathcal{M}(A[T])$ of seminorms whose restriction to A are bounded by the given norm on A.

If $K = (K, |.|)$ be a complete normed field. The affine line over $(K, |.|)$ is the subset $\mathbb{A}^1_K$ of the $\mathcal{M}(K[T])$ given by multiplicative seminorms $|.| : K[T] \to \mathbb{R}_+$ whose restriction to K is equal to the given norm on K.

If $\rho > 0$ is a real number and $\mathfrak{p}$ is a prime ideal in $K[T]$ (of the form $\mathfrak{p} = (P)$ for some irreducible polynomial, for example $\mathfrak{p} = (T - a)$ with $a \in K$), we define the disc of center $\mathfrak{p}$ and radius ρ as the subset

$$D(\mathfrak{p}, \rho) := \{x \in \mathbb{A}^1_K,\ |P(x)| \leq \rho\}.$$

To simplify this construction, we usually restrict to $(K, |.|)$ being complete and algebraically closed like $\mathbb{C}_p := \widehat{\overline{\mathbb{Q}_p}}$ with $\mathbb{Q}_p := \widehat{(\mathbb{Q}, |.|_p)}$. This implies that $\mathfrak{p} = (X - x_0)$ for some $x_0 \in K$.

Definition 6. *Let K be a field and $\mathfrak{p} \subset K[T]$ be a prime ideal. Let $D(\mathfrak{p}, \rho)$ be a disc with center $\mathfrak{p}$. The norm*

$$\|.\|_{\infty, D(\mathfrak{p},\rho)} : K[T] \to \mathbb{R}_+$$

defined by

$$\|f\|_{\infty, D(\mathfrak{p},\rho)} := \max_{x \in D(\mathfrak{p},\rho)} |f(x)|$$

is called the uniform norm *on the given disc. The completion $K\langle \mathfrak{p}, \rho^{-1}T\rangle$ of the polynomial ring for this norm is a Banach ring called the* affinoid algebra *of functions on the disc. If $\mathfrak{p} = (X - 0)$, we denote it $K\langle \rho^{-1}T\rangle$.*

If K is a non-archimedean field, one can show that this seminorm on $K[T]$ is equal to

$$\|F\|_{\infty, D(\mathfrak{p},\rho)} = \max_n |a_n|\rho^n$$

where the decomposition $F = \sum_{n \geq 0} a_n P^n$ is obtained by euclidean division with respect to the irreducible polynomial P.

If $\{D_i = D(\mathfrak{p}_i, \rho_i), i \in I\}$ is a family of embedded closed discs in $\mathbb{A}^1_K$, one defines

$$|F|_{\{D_i\}_I} := \inf_{i \in I} |F|_{D_i}.$$

2.1.6 Gelfand-Mazur theorem

Theorem 7. *The only complete archimedean normed fields are the real numbers and the complex numbers.*

Proof. See Neukirch [Neu99], theorem 4.2, or Artin [Art67], section 2.3. □

Theorem 8 (Gelfand-Mazur)**.** *Let* $|.| : \mathbb{C}[X] \to \mathbb{R}_+$ *be a multiplicative seminorm on* $\mathbb{C}[X]$ *whose restriction to* $\mathbb{C}$ *is the usual norm* $|.|_\infty$*. Then there exists* $z \in \mathbb{C}$ *such that*

$$|P| = |P(z)|_\infty.$$

This theorem can be summed up by saying that the affine line over $(\mathbb{C}, |.|_\infty)$ *is*

$$\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}.$$

2.1.7 The p-adic affine line

Consider the Banach ring $(\mathbb{Q}_p, |.|_p)$. We now describe the p-adic affine line $\mathbb{A}^1_{\mathbb{Q}_p}$.

We first describe the classical points of the affine line.

If L/K is a finite extension, we define the norm map

$$\mathrm{Nm} : L \to K$$

as the map that sends $l \in L$ to the determinant $\mathrm{Nm}(l) = \det(m_l)$ of the multiplication map $m_l \in \mathrm{End}_K(L)$.

Theorem 9. *Let* $(K, |.|_K)$ *be a complete normed field. Let* L/K *be a finite extension. The map*

$$|.|_L := |\mathrm{Nm}(.)|_K : L \to \mathbb{R}_+$$

is the unique extension of the seminorm L *to* K*.*

Proof. See Artin [Art67], chapter 2. □

Corollary 2. *Let* $|.| : \mathbb{Q}_p[T] \to \mathbb{R}_+$ *be a valuation whose restriction to* $\mathbb{Q}_p$ *is* $|.|_p$ *and whose kernel is a maximal ideal* $\mathfrak{p} = (P)$*. Then* $|.| = |.|_\mathfrak{p}$ *is the unique extension of* $|.|_p$ *to the (finite) field extension* $\mathbb{Q}_p[T]/(P)$ *of* $\mathbb{Q}_p$*.*

Theorem 10. *The points of* $\mathbb{A}^1_{\mathbb{Q}_p}$ *are of the following form*

1. $|.|_\mathfrak{p}$ *for some (maximal) prime ideal* $\mathfrak{p} \subset \mathbb{Q}_p[X]$*,*

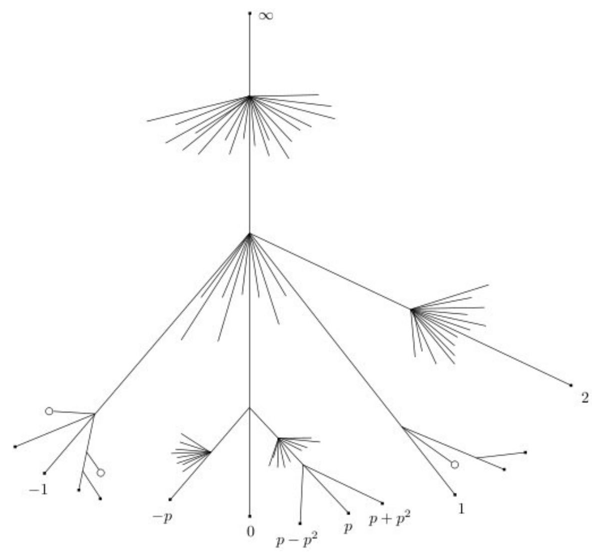

Figure 2.3: The p-adic projective line $\mathbb{P}^1_{\mathbb{C}_p}$.

2. $|.|_{\infty,D(\mathfrak{p},\rho)}$ *for some prime ideal* $\mathfrak{p}$ *and some* $\rho > 0$,

3. $|.|_{\infty,D_i} = \inf_i |.|_{\infty,D_i}$ *where* D_i *is a family of embedded closed discs.*

2.2 Analytic functions

The space $\mathbb{Q}_p$ being totally disconnected, it is not reasonable to define analytic functions as special continuous maps $f : \mathbb{Q}_p \to \mathbb{Q}_p$. As we will see, it is much more natural to define them as special continuous maps

$$f : \mathbb{A}^1_{\mathbb{Q}_p} \to \mathbb{A}^1_{\mathbb{Q}_p}$$

or equivalently as special continuous sections of the projection map

$$p : \mathbb{A}^2_{\mathbb{Q}_p} \to \mathbb{A}^1_{\mathbb{Q}_p}.$$

This definition nicely extends to the spectrum of a (Banach) ring.

Let A be a commutative ring.

2.2.1 Rational functions and convergent series

Let $U \subset \mathcal{M}(A)$ be an open subset.

The ring of fractions of A without poles on U, denoted

$$\mathcal{O}^{alg}(U) \subset \mathrm{Frac}(A),$$

is the ring of fractions $\frac{a}{b}$ such that $|b(x)| \neq 0$ for all $x \in U$.

If we suppose that U has compact closure in $\mathcal{M}(A)$, we get a (power-multiplicative) seminorm on $\mathcal{O}^{alg}(U)$ by

$$\begin{array}{rccc} \|.\|_{\infty,U}: & \mathcal{O}^{alg}(U) & \to & \mathbb{R}_+ \\ & f & \mapsto & \sup_{x\in U} |f(x)|. \end{array}$$

The ring $\mathcal{O}\langle U\rangle$ of convergent series on U is defined as the completion of $\mathcal{O}^{alg}(U)$ with respect to this supremum seminorm.

2.2.2 The standard definition

To every point $x \in \mathcal{M}(A)$, we associate the residue field

$$\mathcal{K}(x) := \mathrm{Frac}(\widehat{A/\mathrm{Ker}}(x))$$

given by the completion of the residue field. We consider the value of $f \in A$ in $\mathcal{K}(x)$ as its value $f(x)$ at the point x.

Definition 7. *Let U be an open subset of $\mathcal{M}(A)$. A function*

$$f : U \to \coprod_{x\in U} \mathcal{K}(x)$$

is called analytic if U can be covered by a family U_i of open subsets such that f_{U_i} is a uniform limit of rational functions without poles in U_i, meaning that for all $\epsilon > 0$, there exists $g \in \mathcal{O}^{alg}(U_i)$ such that

$$\sup_{x\in U_i} |(f-g)(x)| < \epsilon.$$

The ring of analytic functions on U is denoted $\mathcal{O}(U)$. The pair $(\mathcal{M}(A), \mathcal{O})$ is called the global analytic spectrum of A.

One could also define analytic functions on $U \subset \mathcal{M}(A)$ as continuous sections of the natural projection

$$p : \mathbb{A}^1_U \to U$$

that are locally a uniform limit of rational functions. This gives to analytic function an existing value space that replaces the complex plane of complex analytic function theory: the affine line.

2.2.3 Usual complex analytic functions

One can understand the relation between global analytic functions and usual complex analytic functions with help of the two following theorems.

Theorem 11 (Mergelyan)**.** *Let K be a compact subset of the complex plane $\mathbb{A}^1_{\mathbb{C}} = \mathbb{C}$ whose complement is connected. Let $\|.\|_{\infty,K} : \mathbb{C}[X] \to \mathbb{R}_+$ be the sup norm on K. The completion of $(\mathbb{C}[X], \|.\|_{\infty,K})$ is the space of continuous functions on K that are holomorphic in its interior.*

Proof. See Rudin [Rud87], theorem 20.5. □

Theorem 12 (Runge)**.** *Let Ω be an open subset of $\mathbb{C}$. Let $F(\Omega)$ be the ring of rational functions on $\mathbb{C}$ without poles in Ω. The completion of the space $F(\Omega)$ for the topology of uniform convergence on all compact subsets of Ω (which is induced by the family of seminorms $\|.\|_{\infty,K}$ for $K \subset \Omega$ compact) is the space of complex analytic functions on Ω.*

Proof. See Rudin [Rud87] for a finer result. The original result follows from Cauchy's formula: the corresponding (Riemann) integral is the limit of its finite sums that are rational functions. □

2.2.4 p-adic analytic functions

Usual analytic functions $f : \mathbb{Q}_p \to \mathbb{Q}_p$ are not very nice because $\mathbb{Q}_p$ is totally discontinuous (connected components are points). An analytic function will be defined as continuous maps

$$f : \mathbb{A}^1_{\mathbb{Q}_p} \to \mathbb{A}^1_{\mathbb{Q}_p}$$

that are locally uniform limit of polynomials. The p-adic affine line has much more points that $\mathbb{Q}_p$ and this makes it very similar to the set of usual complex numbers in complex analytic geometry.

2.2.5 All numbers are analytic functions

Ostrowski's theorem 4 gives a classification of all multiplicative seminorms on $\mathbb{Z}$. The corresponding residue fields are:

- $\mathcal{K}(|.|_0) = (\mathbb{Q}, |.|_0)$,
- $\mathcal{K}(|.|_{0,p}) = (\mathbb{F}_p, |.|_{0,p})$,
- $\mathcal{K}(|.|_p^t) = (\mathbb{Q}_p, |.|_p^t)$ for all $t > 0$,
- $\mathcal{K}(|.|_\infty^t) = (\mathbb{R}, |.|_\infty^t)$ for all $0 < t \leq 1$.

If an open subset contains the trivial norm, it contains almost all vertices of $\mathcal{M}(\mathbb{Z})$, so that the condition of being a uniform limit of rational functions imply that a holomorphic function is a rational function. In particular, global analytic functions on $\mathcal{M}(\mathbb{Z})$ are simply the integers. Moreover, the germs of analytic functions at a trivial seminorm $|.|_{0,p}$ are $\mathbb{Z}_p$ and $\mathbb{Q}$ is the space of germs of analytic functions at $|.|_0$.

We give in figure 2.2.5 a graphical representation of various spaces of analytic functions on $\mathcal{M}(\mathbb{Z})$ (found in [Poi08a]).

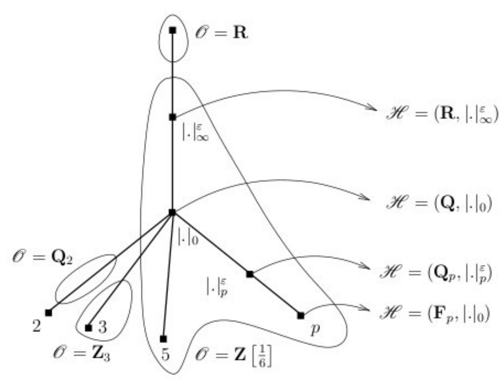

Figure 2.4: Analytic functions over $\mathcal{M}(\mathbb{Z})$.

2.2.6 Adelic spaces and analytic functions

Definition 8. *Let $j : U \hookrightarrow \mathcal{M}(\mathbb{Z})$ be the inclusion of the open complement of the trivial norm $|.|_0$. The ring of adèles is the topological ring of germs*

$$\mathbb{A} := (j_* \mathcal{O}_U)_{|.|_0}$$

of analytic functions on U at the point $|.|_0$. Similarly, the ring of idèles is the topological ring of germs

$$\mathbb{A}^\times := (j_*\mathcal{O}_U^\times)_{|.|_0}.$$

One can easily give an explicit description of the ring of adèles. An open subset V of $\mathcal{M}(\mathbb{Z})$ that contains the trivial norm contains almost all branches. A basis of open subsets containing the trivial norm is given by the open subsets $|n| > 0$ for $n \in \mathbb{Z}$. These are parametrized by finite subsets S (dividors of n) of the set of all prime numbers (that will be supposed also to contain the archimedean absolute value), and denoted V_S. By definition, the direct image $j_*\mathcal{O}_U$ has as sections on such an open subset V_S given by

$$j_*\mathcal{O}_U(V_S) := \mathcal{O}(U \cap V_S).$$

Remark now that $U \cap V_S$ is a disjoint union of intervals so that

$$j_*\mathcal{O}_U(V_S) = \prod_{p \notin S} \mathbb{Z}_p \times \prod_{p \in S} \mathbb{Q}_p.$$

The germs are given by the direct limit of this family of topological groups, which gives exactly the usual definition of adèles, like they appeared for example in Tate's thesis [Tat67].

Definition 9. *If A is a commutative ring, define*

$$\mathrm{GL}_n(A) := \{(M, N) \in \mathrm{M}_n(A),\ MN = NM = \mathrm{Id}\}.$$

An algebraic subgroup of $\mathrm{GL}_{n,\mathbb{Z}}$ is a set of polynomial equations $F_1 = \cdots = F_m = 0$ for $F_i \in \mathbb{Z}[N_{i,j}, M_{i,j}]$ such that for all commutative ring A, the subset

$$G(A) := \{(M, N),\ F_i(M, N) = 0\}$$

of $\mathrm{GL}_n(A)$ is a subgroup, and for all ring morphism $f : A \to B$, the corresponding map $f : G(A) \to G(B)$ is a group morphism (one says in modern language we have a closed sub-group scheme).

If $G \subset \mathrm{GL}_{n,\mathbb{Z}}$ is a subgroup, the topology on $G(\mathbb{A})$ is defined as being induced by the product topology and the natural embedding

$$G(\mathbb{A}) \to \mathrm{M}_n(\mathbb{A}) \times \mathrm{M}(\mathbb{A})$$

given by $g \mapsto (g, g^{-1})$ (which is quite natural with our description of the general linear group).

The following proposition is due to Berkovich.

Proposition 5. *Let $G \subset \mathrm{GL}_{n,\mathbb{Z}}$ be an algebraic subgroup. There is a natural topological isomorphism*

$$(j_* G(\mathcal{O}_U)/G(\mathcal{O}_U))(\mathcal{M}(\mathbb{Z})) \cong G(\mathbb{A})/G(\mathbb{Q}).$$

Proof. This relies on the fact that $j_* G(\mathcal{O}_U)$ and $G(\mathcal{O}_U)$ have the same germs at every points except at the trivial norm where $j_* G(\mathcal{O}_{U,|\cdot|_0}) = G(\mathbb{A})$ is the set of adelic points of G and $G(\mathcal{O}_{U,|\cdot|_0}) = G(\mathbb{Q})$ is the set of rational points. □

We can apply this to the additive group and the multiplicative group to get a global analytic description of $\mathbb{A}/\mathbb{Q}$ and $\mathbb{A}^\times/\mathbb{Q}^\times$. This construction of adèles and adelic homogeneous spaces of course applies to more general algebraic number fields.

Chapter 3

Global analytic varieties

3.1 Categories and functors

The reader is invited to do the exercices on category theory, that give many examples, to make the reading of this section easier.

Definition 10. *A category C is given by the following data:*

1. *a class $\mathcal{O}b(C)$ called the objects of C,*
2. *for each pair of objects X, Y, a set $\mathrm{Hom}(X, Y)$ called the set of morphisms,*
3. *for each object X a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ called the identity,*
4. *for each triple of objects X, Y, Z, a composition law for morphisms*

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z).$$

One supposes moreover that this composition law is associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$ and that the identity is a unit, i.e., $f \circ \mathrm{id} = f$ et $\mathrm{id} \circ f = f$.

Definition 11. *A universal property* [1] *for an object Y of C is an explicit description (compatible to morphisms) of $\mathrm{Hom}(X, Y)$ (or $\mathrm{Hom}(Y, X)$) for every object X of C.*

[1] Every object has exactly two universal properties, but we will usually only write the simplest one.

To be more precise about universal properties, we need the notion of "morphism of categories".

Definition 12. *A (covariant) functor $F : C \to C'$ between two categories is given by the following data:*

1. *For each object X in C, an object $F(X)$ in C',*

2. *For each morphism $f : X \to Y$ in C, a morphism $F(f) : F(X) \to F(Y)$ in C'.*

One supposes moreover that F is compatible with composition, i.e., $F(f \circ g) = F(f) \circ F(g)$, and with unit, i.e., $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Definition 13. *A natural transformation φ between two functors $F : C \to C'$ and $G : C \to C'$ is given by the following data:*

1. *For each object X in C, a morphism $\varphi_X : F(X) \to G(X)$,*

such that if $f : X \to Y$ is a morphism in C, $G(f) \circ \varphi_X = \varphi_Y \circ F(f)$.

We can now improve definition 42 by the following.

Definition 14. *A universal property for an object Y of C is an explicit description of the functor* $\mathrm{Hom}(X, .) : C \to$ SETS *(or* $\mathrm{Hom}(., X) : C \to$ SETS*).*

3.2 Sheaves and spaces

The basic idea of Grothendieck's functorial approach to geometry is that any geometrical object X should be defined by its universal property, i.e., by its functor of points $\mathrm{Hom}(., X)$ or functor of function $\mathrm{Hom}(X, .)$ that is defined on some category LEGOS of simple geometric building blocs. In algebraic geometry, the category LEGOS is simply the category of commutative unital rings. In global analytic geometry, we will use another category of building blocs, constructed from closed subsets of global analytic spectra of general rings. One can in fact define any kind of variety (and much more general spaces) working in the following general setting.

Definition 15. *Let* LEGOS *be a category with fiber products. A Grothendieck topology τ on* LEGOS *is the data, for every lego U, of covering families $\{f_i : U_i \to U\}_{i \in I}$, fulfilling:*

1. *(Base change) For every morphism* $f : V \to U$ *and every covering family* $\{f_i : U_i \to U\}$ *of* U, $f \times_U f_i : V \times_U V_i \to V$ *is a covering family.*

2. *(Local character) If* $\{f_i : U_i \to U\}$ *is a covering family and* $\{f_{i,j} : U_{i,j} \to U_i\}$ *are covering families, then* $\{f_i \circ f_{i,j} : U_{i,j} \to U\}$ *is a covering family.*

3. *(Isomorphisms) If* $f : U \to V$ *is an isomorphism, it is a covering family.*

A space modeled on Legos *for the Grothendieck topology* τ *is simply contravariant functor*

$$X : \text{Legos} \to \text{Sets}$$

that is a sheaf for τ*, i.e., such that for each covering family* $\{f_i : U_i \to U\}$ *the sequence*

$$X(U) \longrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)$$

is exact.

One always assume that for every lego U, the functor

$$\underline{U} := \text{Hom}(.,U) : \text{Legos} \to \text{Sets}$$

is a sheaf. The Grothendieck topology is then called sub-canonical. Of course, the functor

$$\text{Legos} \to \text{Spaces}$$

is fully faithful (this is Ioneda's lemma), so that spaces generalize legos.

One can easily define open subspaces in smooth spaces.

Definition 16. *A space morphism* $f : U \to X$ *is called an open immersion if*

1. *it is injective on points, i.e.,* $f_V : U(V) \to X(V)$ *is injective for all* $V \in$ Legos*, and*

2. *it is universally open, i.e., for every* $V \in$ Legos *and every map* $i : V \to X$*, the pull-back morphism* $i^*f : V \times_X U \to V$ *given by the fiber product*

$$\begin{array}{ccc} V \times_X U & \longrightarrow & U \\ \downarrow{\scriptstyle i^*f} & & \downarrow{\scriptstyle f} \\ V & \xrightarrow{i} & X \end{array}$$

is an inclusion of open subsets.

A space X is a variety modeled on (Legos, τ) *if there is an open covering* $\coprod U_i \to X$ *(surjective on points and each $U_i \subset X$ are open)*

As an example, the opposed category to the category of rings equiped with the Zariski topology generated by localizations $A \to A[f^{-1}]$ is the lego category of algebraic schemes.

3.3 Arithmetic varieties

Since the main interest of this course is in the relation between geometry and analytic number theory, we will concentrate on the geometric approach of zeta functions. We give here a short review of the basics on arithmetic varieties. More details are given in an exercice sheet.

An affine variety of finite type over $\mathbb{Z}$ can be defined by the data of a family of polynomials $P_1, \ldots, P_m \in \mathbb{Z}[X_1, \ldots, X_n]$. The set of points of the variety X with values in a ring R is the set of zeroes in R^n of the given polynomials, i.e., the set

$$X(R) := \{(x_1, \ldots, x_n) \in R^n \,|\, P_j(x_1, \ldots, x_n) = 0, \forall j = 1, \ldots, m\}.$$

Essentially by definition of the polynomial and quotient rings, there is a natural bijection

$$X(R) \xrightarrow{\sim} \mathrm{Hom}_{\text{Rings}}(A, R),$$

where $A := \mathbb{Z}[X_1, \ldots, X_n]/(P_1, \ldots, P_m)$ is the ring of polynomial functions on X. This shows that X only depends on the ideal generated by the given polynomials. The study of X is essentially equivalent to the study of the ring A. We thus denote X by $\mathrm{Spec}(A)$.

A closed point of X is a maximal ideal $\mathfrak{m}_x$ of A. It corresponds to a morphism $x : A_X \to k_x$ with values in the finite field $k_x := A/\mathfrak{m}_x$, i.e., to a point of X with values in k_x. The set of closed points of X is denoted by $|X|$.

More generally, a covariant functor $X : \text{Rings} \to \text{Sets}$ can be thought of as a (very general notion of) algebraic space if it fulfils the Zariski sheaf condition: for every family of localisation maps $\{f_i : A \to A[f_i^{-1}]\}_{i \in I}$ such that

$$\coprod_{i \in I} \mathrm{Spec}(A[f_i^{-1}]) \to \mathrm{Spec}(A)$$

is surjective on points, the natural restriction sequence

$$X(A) \longrightarrow \prod_i X(A[f_i^{-1}]) \rightrightarrows \prod_{i,j} X(A[f_i^{-1}, f_j^{-1}])$$

is exact.

This allows one to define projective varieties by pasting affine ones conveniently in the category of algebraic spaces.

Definition 17. *An algebraic space X is projective if it can be defined from a family of homogeneous polynomials $P_1, \dots, P_n \in \mathbb{Z}[X_0, \dots, X_n]$ in the following way: the set of points with values in a ring is given by the values of the Zariski sheaf associated to the presheaf that sends a ring R to the set*

$$\tilde{X}(R) := \left\{ (x_0 : \dots : x_n) \in R^{n+1} \,\middle|\, \begin{array}{l} R.(x_0 : \dots : x_n) \subset R^{n+1} \text{ is a free module and} \\ P_j(x_0, \dots, x_n) = 0, \forall j = 1, \dots, m \end{array} \right\} / R^\times.$$

If K is a field, we actually have

$$X(K) = \tilde{X}(K) = \left\{ (x_0 : \dots : x_n) \in K^{n+1} - \{0\}/K^\times \,|\, P_j(x_1, \dots, x_n) = 0, \forall j = 1, \dots, m \right\},$$

so that the sheaf issue can be forgotten for what we need (counting points).

Most of the geometric properties of algebraic varieties can be seen on their underlying algebraic space. We here give the example of smoothness.

Definition 18. *An algebraic space X is smooth if for every ring A and every ideal I in A, the map*

$$X(A/I^2) \to X(A/I)$$

is surjective.

There is a very concrete test for smoothness in the case of affine or projective variety. The equivalence with the above definition can be found in EGA IV [Gro67]. It is given by saying that the defining family of polynomials for the given variety

$$\{F_1(X_0, \dots, X_n), \dots, F_m(X_0, \dots, X_n)\}$$

has nonsingular differential, meaning that the jacobian matrix

$$\left[\frac{\partial F_i}{\partial X_j} \right]_{i=1,\dots,m; j=1,\dots,n}$$

is invertible, for all $(X_0 : \dots : X_n) \in X(A)$, for every ring A. This can usually be checked on fields.

3.4 Analytic varieties

As we will see in a forthcoming section, the use of algebraic geometry prevents us from a proper understanding of the archimedean local factor $\zeta_{\infty,X}(s)$ of a given arithmetic variety $X/\mathbb{Z}$. The global analytic approach gives much more information since it contains the archimedean analytic space $X(\mathbb{C})^{an}/\sigma$ (quotient of complex points by complex conjugation) as archimedean fiber.

Definition 19. *A global analytic arithmetic variety is a locally ringed space* $(X, \mathcal{O})$ *that has a covering by analytic spectra*

$$(U_i, \mathcal{O}_{U_i}) = (\mathcal{M}(A_i), \mathcal{O})$$

of some rings $A_i/\mathbb{Z}$ *smooth and of finite type (quotient of a polynomial ring) with algebraic pastings. A global analytic space is a locally ringed space* $(X, \mathcal{O})$ *that has a covering by open subsets* $(V_i, \mathcal{O}_{V_i})$ *that are closed analytic subsets (support of a closed ideal of analytic functions)*

$$(\mathrm{Supp}(\mathcal{I}_i), \mathcal{O}_{U_i}/\mathcal{I}_i) \subset (U_i, \mathcal{O}_{U_i})$$

of given open subsets of global analytic arithmetic varieties.

Chapter 4

Analytic number theory

4.1 The functional equation of Riemann's zeta function

This was proved originally in Riemann's paper from 1858. We propose in appendix an exercice sheet that gives a complete proof of the functional equation.

Recall that Riemann's zeta function is defined for $\mathrm{Re}(s) > 1$, by the absolutely convergent series

$$\zeta(s) = \sum_{n>0} \frac{1}{n^s}.$$

Its basic properties (that will be proved in the exercices in appendix) are summed up as follows:

1. this series converges for $\mathrm{Re}(s) > 1$, uniformly for $\mathrm{Re}(s) > 1 + \delta$ for all $\delta > 0$;

2. it meromorphically continues (i.e. can be written as a quotient of two holomorphic functions) in the half plane $\mathrm{Re}(s) > 0$;

3. one can write it as an infinite product

$$\zeta(s) = \prod_p \zeta_p(s)$$

indexed by prime numbers with $\zeta_p(s) := \frac{1}{1-p^{-s}}$.

4. the gamma function $\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$ continues analytically to $\mathbb{C} - \mathbb{N}_-$ and fulfils there $\Gamma(s+1) = s\Gamma(s)$;

5. the completed zeta function

$$\hat{\zeta}(s) = \zeta_\infty(s)\zeta(s)$$

where

$$\zeta_\infty(s) := 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$$

admits the integral representation

$$\hat{\zeta}(s) = \frac{1}{2\sqrt{2}} \int_0^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y},$$

where

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} = 1 + 2 \sum_{n=1}^{\infty} e^{\pi i n^2 z}$$

is Jacobi's theta series.

6. Jacobi's theta series $\theta(z)$ uniformly converges for $\mathrm{Im}(z) \geq \delta$ for all $\delta > 0$ and Poisson's summation formula implies the functional equation

$$\theta(-1/z) = \sqrt{z/i}\,\theta(z).$$

7. the completed zeta function holomorphically continues to $\mathbb{C} - \{0, 1\}$ and fulfils in this domain the functional equation

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s).$$

The meaning of this functional equation remain to be well understood, because it does not generalize to zeta functions of arithmetic varieties. This comes from the fact that the definition of the zeta function (as an integral or as a series) does not give enough information to study it in a geometrical setting. We now turn to the modern approach to analytic number theory that was grounded by Tate's marvelous thesis.

4.2 Weil's explicit formula for Riemann's zeta function

Let $\varphi \in \mathcal{C}_0^\infty([1, +\infty[)$, that we extend by zero to function in $\mathcal{C}_0^\infty([1, +\infty[)$ and denote

$$\Phi(s) = \int_1^\infty \varphi(y) y^{s-1} dy.$$

Theorem 13 (Weil's explicit formula)**.** *There is an equality*

$$\Phi(0) - \sum_{\hat{\zeta}(\rho)=0} \Phi(\rho) + \Phi(1) = W_\infty(\varphi) + \sum_p W_p(\varphi)$$

where

$$W_p(\varphi) = \log(p) \sum_{k \geq 1} \varphi(p^k) \qquad et \qquad W_\infty(\varphi) = \int_1^\infty \frac{\varphi(y)}{y - y^{-1}} dy.$$

Proof. This follows from a careful use of the residue theorem for a rectangle path with left and right boundaries $\{\mathrm{Re}(s) = -1\}$ and $\{\mathrm{Re}(s) = 2\}$ and upper and lower boundaries $\{\mathrm{Im}(s) = r\}$ and $\{\mathrm{Im}(s) = -r\}$. This is treated in a detail exercice sheet in appendix. □

4.3 Tate's thesis

One of the main input of Tate's thesis [Tat67] is that he studies zeta functions putting all non trivial absolute values on an equality footing. This opens the road for the development of noncommutative adelic harmonic analysis, whose main achievments are contained in the developments of the so-called Langlands program of automorphic representation theory.

Since the proof of the functional equation is quite similar to the usual one due to Riemann, we only give a short overview of Tate's thesis, leaving to the reader the great pleasure to read in the text this masterpiece of twentieth century mathematics, which is strickingly accessible to any good undergraduate student.

The first basic properties of adèles we need are the following.

Proposition 6. *The quotient space $\mathbb{A}/\mathbb{Q}$ is compact. If $|.| : \mathbb{A}^\times \to \mathbb{R}_+^*$ is the standard adelic norm, and J denotes its kernel, the quotient space $J/\mathbb{Q}^\times$ is compact.*

Proof. See Weil's book [Wei95a]. □

The second result implies and superseeds the finiteness of class numbers of number fields because idèles classes surject on the group of ideal classes.

We first need a definition of a nice class of functions on a locally compact commutative topological group G. We admit here the existence and unicity of a Haar measure (positive invariant functional)

$$\mu : \mathcal{C}_c^0(G) \to \mathbb{C}$$

on the group G. In the computations, we will actually only need the restriction of μ to the space $\mathcal{S}(G)$ of Schwartz functions, but the full measure is needed to define this space.

In the cases of interest, i.e., for G of the form A or $A^\times$ for $A = \mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}$, we have the following explicit descriptions of the Haar measure μ:

- $\mathbb{R}$: the usual Lebesgue measure dx normalized by $\mu([-1, 1]) = 1$,
- $\mathbb{R}^\times$: the measure $\frac{dx}{|x|}$,
- $\mathbb{Q}_p$: the measure that gives to $\mathbb{Z}_p$ measure 1, and thus to the disc $D(0, \frac{1}{p^k}) = p^k\mathbb{Z}_p$ measure $\frac{1}{p^k}$.
- $\mathbb{Q}_p^\times$: the measure that gives to $\mathbb{Z}_p^*$ measure 1, given by $d^*x = \frac{1}{1-1/p}\frac{dx}{|x|_p}$.
- $\mathbb{A}$: the tensor product measure of all the above local measures. The space of linear combinations of functions f whose restriction to $\mathbb{Q}_p$ are equal to the characteristic function of $\mathbb{Z}_p$ for almost all p and that are Schwartz functions (see the following) on the remaining places is dense in the space of test functions. For $f = \otimes f_v$ of this form, we define the adelic measure by

  $$\int_{\mathbb{A}} f(a)da := \prod_v f_v(a_v)da_v,$$

  the terms in the product being almost all equal to 1.
- $\mathbb{A}^\times$: a tensor product measure of the local multiplicative measures.

Definition 20. *The quasi-character space is defined as*

$$\Omega(G) := \mathrm{Hom}_{cont}(G, \mathbb{C}^*).$$

The character group $\hat{G}$ of G is defined as

$$\hat{G} := \mathrm{Hom}_{cont}(G, S^1).$$

If $f \in L^1(G)$, we define its fourier transform $\hat{f} : \hat{G} \to \mathbb{C}$, if it exists, as the integral

$$\hat{f}(\chi) := \int_G f(x)\chi(x)dx.$$

Consider for G one of the group $\mathbb{A}^\times$, $\mathbb{Q}_p^\times$ or $\mathbb{R}^\times$. They are all equiped with a natural "norm map"

$$|.| : G \to \mathbb{R}_+^\times$$

(the standard normalized absolute value). Let U be the kernel of this norm map (group of units). Let W denote the image of the norm map $|.| : G \to \mathbb{R}_+^*$. It is called the residual Weil group, and it equals $\mathbb{R}_+^*$ for $A = \mathbb{A}$ or $\mathbb{R}$, and $p^{\mathbb{Z}}$ for $A = \mathbb{Q}_p$. There is an exact sequence

$$1 \to U \to G \to W \to 1.$$

The choice of a section of the right map induces a topological isomorphism

$$G \cong U \times W.$$

The map

$$\begin{array}{ccc} \mathbb{C} & \to & \Omega(\mathbb{R}_+^*) \\ s & \mapsto & [x \mapsto x^s] \end{array}$$

is a topological isomorphism. The space of quasi-characters we will be interested in, corresponding to the above groups, have some kind of laminated space structure induced by the above exact sequence. The natural maps above induce a sequence

$$1 \to \mathbb{C} \cong \Omega(W) \to \Omega(G) \to \Omega(U) \to 1$$

that is "exact", in the sense that every quasi-character c of G can be written in the form

$$c(a) = |a|^s.c_0(a_u)$$

where a_u is the U component of $a \in G$ in the non-canonical decomposition $G \cong U \times \mathbb{R}_+^*$. The power s that appears above is canonical and is called the exponent of the given quasi-character c.

Before going further into the adelic study of zeta functions, we want to describe in some details the analogy between the real and adelic numbers by the following table:

	Real numbers	Adelic numbers
Ring A	$\mathbb{R}$	$\mathbb{A}$
Discrete subgroup	$\mathbb{Z}$	$\mathbb{Q}$
Discrete units	$\mathbb{Z}^\times = \{\pm 1\}$	$\mathbb{Q}^\times$
Compact quotient	$\mathbb{R}/\mathbb{Z} = S^1$	$\mathbb{A}/\mathbb{Q}$
$\mathrm{Ker}(\lvert.\rvert) \subset A^\times$	$\{\pm 1\}$	J
Compact quotient	$\mathrm{Ker}(\lvert.\rvert)/\mathbb{Z}^\times = 1$	$J/\mathbb{Q}^*$
Dual group	$\hat{\mathbb{R}} \cong \mathbb{R}$	$\hat{\mathbb{A}} \cong \mathbb{A}$
Quasicharacters	$\mathbb{C} = \Omega(\mathbb{R}_+^*)$	$\Omega(\mathbb{A}^*)$

Following this analogy, we will use Fourier transform on the additive group $\mathbb{A}$ (which is the analog of $\mathbb{R}$), and zeta functions will be functions on the space of quasi-characters (which is the analog of $\mathbb{C}$).

A natural class of functions to consider on a commutative locally-compact topological group G is that of Schwarz functions, because it is stable by Fourier transform. This definition can be found in [Osb75].

Definition 21. *A function $f : G \to \mathbb{C}$ is called a rapid decay function if*

1. *it is essentially bounded, i.e., $f \in L^\infty(G)$,*

2. *it has rapid decay off some powers of a given compact subset: there exists a compact subset $C(f) \subset G$ such that, for all $n > 0$, there is a constant M_n such that for each integer $k \geq 1$,*

$$\|f_{|G-C(f)^k}\|_\infty \leq M_n k^{-n}.$$

We say that f is a Schwartz function if both f and $\hat{f}$ are rapid decay functions. We denote $\mathcal{S}(G)$ the space of Schwartz functions.

All we have to know on these spaces of functions is that they are stable by Fourier transform. However, here are their explicit description for the cases in play:

- If $G = \mathbb{R}$, the space of Schwartz functions is often used in analysis because it is the smallest space that is stable by differentiation, Fourier transform, and that contains smooth functions with compact support. It can be described more explicitely as the space of smooth functions

  $$\mathcal{S}(\mathbb{R}) = \{f \in \mathcal{C}^\infty(\mathbb{R}) \mid \|f\|_{i,j} < \infty\}$$

  for $\|f\|_{i,j} := \sup_{x\in\mathbb{R}} |x^i \partial_x^j f(x)|$.

- If $G = \mathbb{Q}_p$, the space of Schwartz functions is simply the space of locally constant functions with compact support.

- If $G = \mathbb{A}$, the space of Schwartz functions contains is the space of finite linear combinations of infinite products $\prod_v f_v$ where the function f_v
  - depends only on the v-adic component,
  - is for almost all v's the characteristic function of the integers in K_v,
  - and is a Schwartz function at the remaining places.

Lemma 1. *Suppose that A is one of the rings $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}$, with its natural topology. The ring A is locally compact and the choice of an additive character $\chi : A \to \mathbb{C}$ induces an isomorphism*

$$\begin{array}{cccc} i_\chi : & G & \to & \hat{G} \\ & a & \mapsto & [b \mapsto \chi(ba)]. \end{array}$$

between the additive group of A and its character group $\hat{A}$.

In usual Fourier analysis, it is common to choose as a basic additive character of $\mathbb{R}$ the map

$$x \mapsto e^{2i\pi x}.$$

The above proposition says that the map

$$x \mapsto [t \mapsto e^{2i\pi tx}]$$

is an isomorphism between $\mathbb{R}$ and its character group that allows to identify the Fourier transform of a function on $\mathbb{R}$ with a function on $\mathbb{R}$.

The object of main interest for modern analytic number theory are some kinds of adelic Mellin transforms.

Definition 22. *Let A be one of the topological rings $\mathbb{R}$, $\mathbb{Q}_p$ or $\mathbb{A}$, with its natural topology. Let $f \in \mathcal{S}(A)$ be a Schwartz function on the additive group of A. The zeta function associated to f is the complex valued functions of characters of invertible elements of A defined by*

$$\begin{array}{cccc} \zeta(f,.): & \Omega(A^\times) & \to & \mathbb{C} \\ & c & \mapsto & \int_{A^\times} f(a)c(a)d^*a. \end{array}$$

If $c \in \Omega(A^\times)$, we denote $\hat{c}(a) := |a|c^{-1}(a)$.

A very nice input of this approach is the following uniform description of local factors.

Proposition 7. *For $\mathrm{Re}(s) > 0$, the local factor $\zeta_v(s)$ of Riemann's zeta function can be written as*

$$\zeta_v(s) = \int_{\mathbb{Q}_v^\times} f_v(x)|x|_v^s d^*x$$

for:

- *$f_v(x) = \mathbb{1}_{\mathbb{Z}_p}$ if $v = p$ if a finite prime and*
- *$f_v(x) = e^{-\pi x^2}$ if $v = \infty$.*

For $\mathrm{Re}(s) > 1$, the completed zeta function can be written as

$$\hat{\zeta}(s) = \int_{\mathbb{A}^*} f_0(x)|x|^s d^*x$$

where $f_0(x) = \prod_v f_v(x)$.

Proof. For the archimedean factor, if we use the multiplicative Haar measure $d^*x = \frac{dx}{|x|}$ on $\mathbb{R}^*$, we get

$$\begin{array}{rcl} \int_{\mathbb{R}^*} e^{-\pi x^2}|x|^s \frac{dx}{|x|} & = & 2\int_0^\infty e^{-\pi x^2} x^s \frac{dx}{x}, \\ & = & \int_0^\infty e^{-\pi t} t^{s/2} \frac{dt}{t}, \\ & = & \pi^{-s/2}\Gamma(s/2). \end{array}$$

For the non-archimedean factor, if we use the Haar measure d^*x on $\mathbb{Q}_p^*$ that gives $\mathbb{Z}_p^*$ measure one, i.e., $d^*x = \frac{1}{1-1/p}\frac{dx}{|x|_p}$ where dx is usual Haar measure on $\mathbb{Q}_p$ that gives measure one to $\mathbb{Z}_p$, we get

$$\begin{array}{rcl}\int_{\mathbb{Q}_p^*} \mathbb{1}_{\mathbb{Z}_p}(x)|x|_p^s d^*x & = & \int_{\mathbb{Z}_p - 0} |x|_p^s d^*x \\ & = & \sum_{n\geq 0} \int_{p^n\mathbb{Z}_p^*} \frac{d^*x}{p^{ns}} \\ & = & \sum_{n\geq 0} \frac{1}{p^{ns}} \int_{p^n\mathbb{Z}_p*} d^*x \\ & = & \sum_{n\geq 0} \frac{1}{p^{ns}} . \int_{\mathbb{Z}_p*} d^*x \text{ by multiplicative invariance of } d^*x \\ & = & \sum_{n\geq 0} \frac{1}{p^{ns}} .1 \\ & = & \frac{1}{1-1/p^s}. \end{array}$$

The statement for the full zeta function follows from the Euler product expansion

$$\hat{\zeta}(s) = \prod_v \zeta_v(s)$$

and the local statement since for $f_0 = \prod_v f_v$, we have

$$\int_{\mathbb{A}^*} f_0(x)|x|^s d^*x = \prod_v \int_{\mathbb{Q}_v^*} f_v(s)|x|_v^s d^*x.$$

□

The local version of the functional equation, which follows from Fubini's theorem, is given by the following theorem.

Theorem 14. *If $A = \mathbb{R}$ or $\mathbb{Q}_p$, this integral converges for quasi-characters of exponent bigger than 0. Moreover, it extends analytically to the domain of all quasi-characters by a functional equation of the form*

$$\zeta(f, c) = \Gamma(c)\zeta(\hat{f}, \hat{c}),$$

with Γ independent of f.

The global version of the functional equation, whose proof is very similar to Riemann's proof for the usual zeta function, follows from Poisson's summation formula for the embedding of topological rings

$$\mathbb{Q} \subset \mathbb{A}.$$

Theorem 15 (Poisson's summation formula)**.** *Let $f \in \mathcal{S}(\mathbb{A})$. Then we have the equality*

$$\sum_{q\in\mathbb{Q}} f(q) = \sum_{q\in\mathbb{Q}} \hat{f}(q),$$

and if $g \in \mathbb{A}^\times$, we have

$$\sum_{q \in \mathbb{Q}} f(qg) = \frac{1}{|g|} \sum_{q \in \mathbb{Q}} \hat{f}(qg^{-1}).$$

Remark that the above formula directly generalizes to the embedding $\mathrm{M}_n(\mathbb{Q}) \subset \mathrm{M}_n(\mathbb{A})$, where M_n is the algebra of n by n matrices, so that for a function $f \in \mathcal{S}(\mathrm{M}_n(\mathbb{A}))$ and an element $g \in \mathrm{GL}_n(\mathbb{A})$, we have

$$\sum_{m \in M_n(\mathbb{Q})} f(mg) = \frac{1}{|\det(g)|} \sum_{m \in M_n(\mathbb{Q})} \hat{f}(mg^{-1}).$$

Theorem 16. *If $A = \mathbb{A}$, the integral $\zeta(f, .)$ converges for quasi-characters of exponent bigger than 1. Moreover, it extends meromorphically to the domain of all quasi-characters by the formula*

$$\zeta(f, c) = \kappa \left\{ \frac{\hat{f}(0)}{s-1} - \frac{f(0)}{s} \right\} + \int_1^\infty \left[\zeta_t(f, c) + \zeta_t(\hat{f}, \hat{c}) \right]$$

where

- *κ is the volume of the compact quotient space $J/\mathbb{Q}^*$ with $J \subset \mathbb{A}^\times$ the idèles of norm 1.*

- *$\zeta_t(f, c) := \int_{\mathbb{A}_t^\times} f(a)c(a)da$ for $\mathbb{A}_t^\times$ the set of idèles of norm t.*

This formula clearly implies that $\zeta(f, c)$ fulfils the functional equation

$$\zeta(f, c) = \zeta(\hat{f}, \hat{c}).$$

The generalization of this proof of the functional equation to noncommutative groups (more precisely to cuspidal automorphic representations of GL_n) is very similar, appart from some technicalities, and can be found in the last chapter of Godement and Jacquet's book [GJ72]. The main difficulties of this work is the identification of the local factors, but if one uses the "spectral definition" of zeta functions, it is better not to care too much about these explicit descriptions to prove the functional equation.

4.4 Weil's adelic distribution theory

In his Bourbaki seminar [Wei95b], Weil interpreted Tate's thesis in terms of invariant distributions for the action of $\mathbb{A}^\times$ on $\mathbb{A}$. Here are the main aspect of his approach, that can also be found in 9.1 of [CM08].

Let A be one of the rings $\mathbb{A}$, $\mathbb{R}$, $\mathbb{Q}_p$. Let $c : A^\times \to \mathbb{C}^\times$ be a quasi-character. The problem denoted $P(A, c)$ considered by weil is to find all tempered distributions on A that have equivariance c with respect to the action of $A^\times$ on A, meaning that

$$g.\Delta = c(g).\Delta$$

where $g.\Delta(f) := \Delta(f(g^{-1}.))$.

Proposition 8. *In the local case, i.e., if $A = \mathbb{R}$ or $A = \mathbb{Q}_p$, there is up to a factor only one solution to the problem $P(A, c)$ for each c. It is given by the formula*

$$\Delta_c(f) := \int_{A^\times} f(x)c(x)d^*x$$

if the exponent of c is strictly positive. Otherwise, it is given by analytic continuation of Δ_c outside of its poles, and by the distributions with zero support, at the poles.

We now give a description of the distributions with support 0 on a local field, since they are useful to understand spectrally the local factors (spectral interpretation for their poles).

Let A be either $\mathbb{R}$ or $\mathbb{Q}_p$. Let $i : \mathcal{S}(A^\times) \to \mathcal{S}(A)$ be the extension by zero map. The space of distributions with zero support on A is the space $\mathcal{D}_0(A)$ given by the kernel of the dual map

$$i' : \mathcal{S}(A)' \to \mathcal{S}(A^\times)'.$$

Proposition 9. *The space of distributions with zero support have the following description*

1. *$\mathcal{D}_0(\mathbb{Q}_p) = \mathbb{C}.\delta_0$ and*

2. *$\mathcal{D}_0(\mathbb{R}) = \oplus_i \partial_x^i \delta_0$.*

As explained above, the distribution with zero support are in natural bijection (up to a factor) with the set of poles of the local zeta distribution.

Proposition 10. *In the global case, i.e., if $A = \mathbb{A}$, there is up to a factor only one solution to the problem $P(A, c)$ for each c. It is given by analytic continuation of the formula*

$$\Delta_c(f) := \int_{A^\times} f(x)c(x)d^*x.$$

This distributional approach to zeta functions is also used in the non-commutative setting at the end of Godement and Jacquet's book [GJ72].

4.5 The spectral definition of zeta functions

We only give here an overview of the definition of zeta functions. We refer to Godement and Jacquet's opus [GJ72], to Meyer's paper [Mey05], and to part 4 of Connes and Marcolli's book [CM08]. One can also use Soulé article [Sou01] for more details on the non-commutative case.

Let $C_\mathbb{Q}$ denote the idèle class group $\mathbb{A}^\times/\mathbb{Q}^\times$. For $I \subset \mathbb{R}$, denote $\mathcal{S}(C_\mathbb{Q})_I$ the subspace of $\mathcal{S}(C_\mathbb{Q})$ given by

$$\mathcal{S}(C_\mathbb{Q})_I := \{f : C_\mathbb{Q} \to \mathbb{C} \mid f|.|^\alpha \in \mathcal{S}(C_\mathbb{Q}) \text{ for all } \alpha \in I\}.$$

We denote

$$\mathcal{H}_- := \mathcal{S}(C_\mathbb{Q})_{]-\infty,+\infty[}$$

and

$$\mathcal{S}(\mathbb{A})_0 := \{f \in \mathcal{S}(\mathbb{A}) \mid f(0) = \hat{f}(0) = 0\}.$$

Proposition 11. *The summation map $\Sigma : f(x) \mapsto \sum_{q\in\mathbb{Q}^\times} f(qx)$ induces an $\mathbb{A}^\times$-equivariant map*

$$\Sigma : \mathcal{S}(\mathbb{A})_0 \to \mathcal{S}(C_\mathbb{Q}).$$

Proof. The fact that Σ is $\mathbb{A}^\times$-equivariant is clear. The proof that if f is a Schwartz function in $\mathcal{S}(\mathbb{A})_0$, $\Sigma(f)$ is also a Schwartz function can be found in [CM08], lemma 2.51, page 411. □

We denote $J_\mathbb{Q} \subset \mathbb{A}^\times$ the space of idèles of norm 1. Let $\mathcal{H}_+ \subset \mathcal{H}_-$ be the subspace of functions of the form $\Sigma(f)$ for some $f \in \mathcal{S}(\mathbb{A})_0$, and consider the space of $J_\mathbb{Q}$-invariants

$$\mathcal{H}^1(\mathbb{Z}) := (\mathcal{H}_-/\mathcal{H}_+)^{J_\mathbb{Q}}.$$

We now arrive to the theorem that gives the spectral definition of the zeta function.

Theorem 17. *The Mellin transform identifies $\mathcal{H}_-$ with the space of holomorphic functions on $\Omega(C_{\mathbb{Q}})$ that are Schwartz functions on the vectical lines $c_0|.|^{s+i\mathbb{R}}$ for c_0 a varying character. The subspace of $\mathcal{H}_+$ of $J_{\mathbb{Q}}$-invariant functions identifies with the ideal generated by the L-function of $\mathbb{Q}$*

$$L(s) := \hat{\zeta}(s) - \left[\frac{1}{s} + \frac{1}{1-s}\right].$$

Proof. The full proof can be found in Meyer [Mey05] and Connes and Marcolli [CM08], chapter 4, 4.116 and 4.125. □

We thus arrived to a new definition of the zeta function.

Definition 23. *The L-function of $\mathbb{Z}$ (completed zeta function with all quasi-characters, and with poles taken out) is the greatest common divisor of the family of Mellin transforms of elements in $\mathcal{H}_+$.*

Corollary 3. *The space $\mathcal{H}^1(\mathbb{Z})$ with its action of $\mathbb{R}_+^* \cong C_{\mathbb{Q}}/J_{\mathbb{Q}}$ gives a spectral interpretation of the zeroes of $\hat{\zeta}(s)$.*

Proof. We are considering functions of the form

$$M(F,s) = \int_{C_{\mathbb{Q}}} F(a)|a|^s d^*a$$

for $F \in \mathcal{H}_-$. Suppose that ρ is a zero of $\hat{\zeta}(s)$ of order 1. The evaluation map $F \mapsto M(F,\rho)$ induces a surjective map

$$\mathcal{H}^1(\mathbb{Z}) \to \mathbb{C}$$

that is moreover $C_{\mathbb{Q}}$-equivariant if we equip $\mathbb{C}$ with the action by $x \mapsto |.|^\rho x$. Indeed, we have

$$\begin{array}{rcl} M(F(y^{-1}.),\rho) & = & \int_{C_{\mathbb{Q}}} F(y^{-1}a)|a|^\rho d^*a \\ & = & \int_{C_{\mathbb{Q}}} F(a)|ya|^\rho d^*a \\ & = & |y|^\rho M(F,\rho) \end{array}$$

by a multiplicative change of variable with respect to the invariant measure da. The above surjective map induces an isomorphism between the $|.|^\rho$-eigenspace in $\mathcal{H}^1(\mathbb{Z})$ and $\mathbb{C}$. Multiple zeroes can be treated using jets of holomorphic functions at the corresponding points. One thus concludes that the spectrum of the $\mathbb{R}_+^*$-action on $\mathcal{H}^1(\mathbb{Z})$ is in natural bijection by the evaluation map $M(F,s) \mapsto M(F,\rho)$ with the set of zeroes of the completed zeta function. □

4.6 Zeta functions as regularized determinants

We now give a formula for the local and global zeta functions that only uses their spectral definition.

We first give a natural construction of the spectral interpretation for local poles of Riemann's zeta function.

Proposition 12. *Let $(K, |.|)$ be one of the normalized normed field $\mathbb{R}$ ($p = \infty$) or $\mathbb{Q}_p$ ($p < \infty$). Let $\mathcal{D}_0(K)$ be the space of distributions with support zero on K. We consider the suspension of the action of K^* on $\mathcal{D}_0(K)$ through the normalized absolute value, defined as*

$$\mathcal{R}_p := \Gamma_{\mathcal{C}^\infty}(\mathcal{D}_0(K) \times \mathbb{R}_+^* \to \mathbb{R}_+^*)^{K^*},$$

i.e., as the space of K^*-invariants in the space of smooth sections of the trivial bundle on $\mathbb{R}_+^*$ with fiber $\mathcal{D}_0(K)$. The spectrum of the action of $\mathbb{R}_+^*$ on $\mathcal{R}_p$ is identified with the set of poles of the local factor $\zeta_p(s)$ of Riemann's zeta function, $p \leq \infty$.

Proof. By direct computation, we see that $\mathcal{R}_p$ is equal to $\mathcal{C}^\infty(\mathbb{R}_+^*/p^{\mathbb{Z}})$ if p if finite and to $\mathcal{D}_0(\mathbb{R})^{\pm 1}$ if $p = \infty$. The spectrum of the action of $\mathbb{R}_+^*$ on the first space is given as $\frac{2i\pi}{\log p}\mathbb{Z}$ by the use of Fourier transformation. The spectrum of the action of $\mathbb{R}_+^*$ on the second space is given by $-2\mathbb{N}$. These are the poles of the local factors. □

Actually, Deninger defines (ad hoc)

$$\mathcal{R}_p := \mathcal{C}^\infty(\mathbb{R}^*/p^{\mathbb{Z}}) \cong \mathcal{C}^\infty(\mathbb{R}/\log p\mathbb{Z})$$

and

$$\mathcal{R}_\infty := \mathbb{R}[e^{-2t}],$$

and equip these spaces for $p \leq \infty$ with the action of the endomorphism

$$\Theta := \frac{\partial}{\partial t} : \mathcal{R}_p \to \mathcal{R}_p,$$

the spectrum of Θ on $\mathcal{R}_p$ is $\frac{2i\pi}{\log p}\mathbb{Z}$ if $p < \infty$ and $-2\mathbb{N}$ otherwise. These are exactly the poles of the given local factor. We gave above a more natural construction of the spaces $\mathcal{R}_p$.

Definition 24. *Let (V, Θ) be a vector space with endomorphism with discrete numerable spectrum, we define the spectral zeta function associated to Θ as*

$$\zeta_\Theta(s) = \sum_{0 \neq \alpha \in \mathrm{Sp}\,\Theta} \frac{1}{\alpha^s}$$

and the regularized determinant of Θ on V as

$$\det_\infty(\Theta|V) = \begin{cases} \exp(-\zeta'_\Theta(0)) & si\ 0 \notin \mathrm{Sp}(\Theta), \\ 0 & si\ 0 \in \mathrm{Sp}(\Theta). \end{cases}$$

Theorem 18. *One can use a zeta-regularized determinant to express both local factors in the form*

$$\zeta_p(s) = \det_\infty \left(\frac{1}{2\pi}(s - \Theta)|\mathcal{R}_p \right)^{-1},\ p \leq \infty.$$

Proof. It is an elementary computation of complex analysis that can be found in [Den94]. We leave it in appendix in a detailled exercice sheet. □

Chapter 5

Automorphic representations of GL_n

The aim of this section is to give a glimpse at the technicalities that appear if one wants to generalize Tate's thesis from GL_1 to GL_n. We use here mostly the second part of Godement-Jacquet's book [GJ72], and Soulé's work [Sou01].

5.1 Notations and definitions

We suppose that $n > 1$, since the case $n = 1$ was already treated in Tate's thesis.

Recall that if A is a ring, we denote

$$\mathrm{GL}_n(A) := \{(M, N) \in \mathrm{M}_n(A) \times \mathrm{M}_n(A),\ MN = NM = I\}.$$

The usual adelic topology on $\mathrm{M}_n(\mathbb{A}) \cong \mathbb{A}^{n^2}$ induces a topology on $\mathrm{GL}_n(\mathbb{A})$ through the embedding

$$\mathrm{GL}_n(\mathbb{A}) \subset \mathrm{M}_n(\mathbb{A}) \times \mathrm{M}_n(\mathbb{A}).$$

For the rest of this chapter, we denote:

1. $G := \mathrm{GL}_n$,

2. $G_0(\mathbb{A})$ the kernel of the norm map

$$|\det(.)| : G(\mathbb{A}) \to \mathbb{R}_+^*,$$

3. $K = G(\hat{\mathbb{Z}}).\mathrm{SO}_n(\mathbb{R})$ the standard maximal compact subgroup of $G(\mathbb{A})$, where $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$,

4. T the group of diagonal matrices in G,

5. and Z the center of G (which is the group of diagonal matrices with identical entries).

There is a natural adjoint action

$$\begin{array}{rccc} \mathrm{Ad}: & G & \to & \mathrm{GL}(\mathrm{M}_n) \\ & g & \mapsto & [m \mapsto gmg^{-1}]. \end{array}$$

The characters of T (algebraic morphisms $\chi : T \to \mathrm{GL}_1$) are all of the form

$$\chi_{i_1,\dots,i_n}(x_1,\dots,x_n) = x_1^{i_1}.\dots.x_n^{i_n},$$

i.e., their set is naturally identified with $\mathbb{Z}^n$.

The action of $T = \mathrm{GL}_1^n$ on M_n through the adjoint action Ad decomposes in eigenspaces and the eigencharacters of T for this action are called roots for G and their set is denoted $R \subset \mathbb{Z}^n$.

The simple roots of G are the characters $r_i : T \to \mathrm{GL}_1$ given by

$$r_i(x_1,\dots,x_n) = x_i x_{i+1}^{-1}, \text{ for } i = 1,\dots,n-1.$$

For $t \in T(\mathbb{A})$, we define

$$\nu(t) := \inf |r_i(t)|,\ k(t) = \sup |r_i(t)|.$$

Definition 25. *A function $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ will be said to be slowly increasing if for any compact of $G(\mathbb{A})$ and any $c > 0$, there is $p \geq 1$ and $c' > 0$ such that the relations*

$$g \in G(\mathbb{A}),\ t \in T(\mathbb{A}) \cap G_0(\mathbb{A}),;\nu(t) \geq c$$

imply

$$|\varphi(tg)| \geq c' k(t)^p.$$

Similarly, a function $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ will be said to be rapidly decreasing if for any compact C of $G(\mathbb{A})$, any $c > 0$ and any $p \geq 1$, there is c' so that the relations

$$g \in C,\ t \in T(\mathbb{A}) \cap G_0(\mathbb{A}),\ \nu(t) \geq c$$

imply

$$|\varphi(tg)| \leq c' k(t)^{-p}.$$

Let $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the space of all functions $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ such that

$$\int_{G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} |\varphi(g)|^2 dg < \infty.$$

The group $G(\mathbb{A})$ operates on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ by right translations.

Definition 26. *An automorphic representation of $G(\mathbb{A})$ is a unitary subrepresentation the representation of $G(\mathbb{A})$ on $L^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$.*

5.2 Cuspical automorphic representations

Definition 27. *A continuous function $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ is said to be cuspidal if the integral*

$$\int_{U(\mathbb{Q})\backslash U(\mathbb{A})} \varphi(ug) du$$

vanishes each time U is an algebraic subgroup of the group of upper triangular matrices in G. Let $L_0^2 = L_0^2(G(\mathbb{Q})\backslash G(\mathbb{A}))$ be the subset of cuspical elements. It is a closed invariant subspace. A cuspidal automorphic representation of $G(\mathbb{A})$ is a unitary subrepresentation the representation of $G(\mathbb{A})$ on L_0^2.

Theorem 19. *Let π be a cuspidal automorphic representation of $G(\mathbb{A})$. Every irreducible representation of K occurs in (the ω-covariant subspace of) π (for ω a character of $\mathbb{A}^\times/\mathbb{Q}^\times$) with finite multiplicity.*

Definition 28. *Let $\mathfrak{a}$ be the center of the enveloping algebra of the Lie group $G(\mathbb{R})$. The space $\mathcal{A}_0(G)$ of cuspidal automorphic functions on G is the subspace of L_0^2 of functions that are K-finite and $\mathfrak{a}$-finite. If ω is a character of $\mathbb{A}^\times/\mathbb{Q}^\times$ and χ a character of $\mathfrak{a}$, the space of (ω, χ)-covariant cuspidal automorphic functions is denoted $\mathcal{A}_0(G, \omega, \chi)$. Let $\mathcal{H}(G, K)$ be the Hecke convolution algebra of K-bi-invariant continuous functions on $G(\mathbb{A})$ with compact support.*

There is a natural action of $\mathcal{H}(G, K)$ on $\mathcal{A}_0(G)$ by convolution.

There is a natural $\mathcal{H}(G, K)$-equivariant pairing between $\mathcal{A}_0(G, \omega, \chi)$ and $\mathcal{A}_0(G, \omega^{-1}, \tilde{\chi})$ defined by

$$\langle \varphi, \tilde{\varphi} \rangle = \int_{G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(g)\tilde{\varphi}(g) dg.$$

The admissible coefficient corresponding to the pair $(\varphi, \tilde{\varphi})$ is given by

$$f(g) = \int_{G(\mathbb{Q})Z(\mathbb{A})\backslash G(\mathbb{A})} \varphi(hg)\tilde{\varphi}(g)dh.$$

More generally, any coefficient of the unitary representation π is of this form for φ and $\tilde{\varphi}$ in L_0^2.

5.3 The functional equation for coefficients

The methods of this section are very similar to those used in Tate's thesis.

Theorem 20 (Poisson's summation formula)**.** *For* $f \in \mathcal{S}(\mathrm{M}_n(\mathbb{A}))$ *and* $g \in \mathrm{GL}_n(\mathbb{A})$*, we have*

$$\sum_{m \in M_n(\mathbb{Q})} f(mg) = \frac{1}{|\det(g)|} \sum_{m \in M_n(\mathbb{Q})} \hat{f}(mg^{-1}).$$

Let $\varphi : G(\mathbb{Q})\backslash G(\mathbb{A}) \to \mathbb{C}$ be

1. a continuous function K-finite on the right and
2. ω-covariant
3. that is rapidely decreasing
4. and cuspidal.

The proof of the following theorem is almost identical to Tate's thesis, but much more technical.

Theorem 21. *For* $f \in \mathcal{S}(\mathrm{M}_n(\mathbb{A}))$*, consider the zeta-integral*

$$Z(f, \varphi, s) := \int_{G(\mathbb{A})} f(x)\varphi(x)|\det(x)|^s dx.$$

It converges for $\mathrm{Re}(s) > n$*, analytically continues to* $\mathbb{C}$ *and fulfils the functional equation*

$$Z(f, \varphi, s) = Z(f, \check{\varphi}, s)$$

where $\check{\varphi}(g) = \varphi(g^{-1})$*.*

5.4 Spectral definition of automorphic L-functions

The spectral definition of automorphic L-functions can be shown by Godement and Jacquet's main theorem to be equivalent to the explicit formulas (which don't treat all places on equal footing) usually used.

Definition 29. *The L-function $L(\pi, s)$ of a cuspidal automorphic representation π is defined as the greatest common divisor of the family of analytic functions*

$$\{Z(f, \varphi, s)\}$$

for φ an admissible coefficient of π and $f \in \mathcal{S}(\mathrm{M}_n(\mathbb{A}))$.

The following theorem follows for the study of the zeta functions of coefficients of a given automorphic L-function.

Theorem 22. *The L-function $L(\pi, s)$ of a cuspidal automorphic representation fulfils the functional equation*

$$L(\pi, s) = \epsilon(\pi, s)L(\pi, s)$$

where $\epsilon(\pi, s)$ is an entire function without zeroes.

There is a strong analogy between automorphic L-functions and (partition functions of) quantum fields because both of them can be viewed as operator valued distributions in the sense of the following proposition.

Proposition 13. *Let π be the representation of $G(\mathbb{A})$ on L^2_0. The zeta integrals define a holomorphic family of operator valued distributions*

$$Z(\pi, s) : \mathcal{S}(\mathrm{M}(\mathbb{A})) \to \mathrm{Aut}(\pi, \langle ., . \rangle)$$

on $\mathrm{M}_n(\mathbb{A})$ such that

$${}^t\hat{Z}(\tilde{\pi}, n - s) = Z(\pi, s)$$

where $\hat{Z}$ denotes the Fourier-transformed distribution.

Chapter 6

Arithmetic geometry and dynamical systems

Two of the main problems of arithmetic geometry, both of them being financially interesting (one million dollars), lie in the study of general geometric zeta functions. Both of them are related to the study (or construction) of some generalized dynamical systems related to these zeta functions.

6.1 General geometric zeta functions

The (non-completed) zeta function of an arithmetic variety X is defined as the product

$$\zeta_X(s) := \prod_{x \in |X|} \frac{1}{1 - |k_x|^{-s}}.$$

The completed zeta functions is defined by multipliying the above by a convenient archimedean factor $\zeta_{X,\infty}(s)$ (definition due to Serre), whose definition uses Hodge theoretical informations on the complex analytic variety of complex valued points $X(\mathbb{C})$. The bad understanding of this archimedean factor is at the heart of many difficulties with arithmetic zeta functions.

Example 1. *It is easy to describe explicitely the zeta functions of the most simple varieties:*

1. *If $X = \mathrm{Spec}(\mathbb{Z}) = \mathbb{P}^0_{\mathbb{Z}}$, i.e., $A = \mathbb{Z}$, we get the (non-completed) Riemann zeta function*

$$\zeta_X(s) = \zeta(s)$$

and the archimedean factor is $\zeta_\infty(s) := \pi^{-s/2}\Gamma(s/2)$.

2. *If* $X = \mathbb{A}^n_{\mathbb{Z}}$*, we get the twisted zeta function*

$$\zeta_X(s) = \zeta(s-n)$$

and the archimedean factor is $\zeta_{X,\infty}(s) = \zeta_\infty(s-n)$. *The completed zeta function fulfils the functional equation*

$$\hat{\zeta}_X(s) = \hat{\zeta}_X(n-s),$$

which can be a guiding formula for smooth varieties of dimension n.

3. *If* $X = \mathbb{P}^n_{\mathbb{Z}}$*, the decomposition* $\mathbb{P}^n_{\mathbb{Z}} = \coprod_{i=0}^n \mathbb{A}^n_{\mathbb{Z}}$ *implies that*

$$\zeta_X(s) = \prod_{i=0}^n \zeta(s-i).$$

The completed zeta function also fulfils the functional equation

$$\hat{\zeta}_X(s) = \hat{\zeta}_X(n-s).$$

One can think of Grothendieck 's etale topology on a given projective smooth variety as a tools to translate the proof of the functional equation of its zeta function in a local question, which is then trivially solved by the above computation of the functional equation of the affine space (because every smooth variety is etale locally isomorphic to the affine space).

To describe the archimedean local factor in full generality, we first need the main theorem of Hodge theory, that gives us the linear algebra invariant necessary to compute archimedean local factors.

Theorem 23. *Let* $X/\mathbb{C}$ *be a projective smooth variety. There is a natural isomorphism*

$$\oplus_{p+q=n} H^p(X,\Omega^q) \cong H^n_{dR}(X/\mathbb{C})$$

between the Hodge cohomology and the de Rham cohomology. This isomorphism induces a filtration $F^\bullet$ *defined by*

$$F^p = \oplus_{i\geq p} H^p(X,\Omega^q)$$

on the de Rham cohomology called the Hodge filtration.

Proof. This can be done either by using complex analysis and harmonic differential forms (original proof, see [GH94]) or by a reduction mod p argument (see [Ill96]). The complex analytic proof relies on identifying $H^p(X, \Omega^q)$ with the space of Harmonic (p, q) forms, that fulfil Laplace's equation

$$\Delta\omega = 0$$

for $\Delta = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$. This gives unique representing form for the corresponding cohomology classes. □

We cannot cite this theorem without giving you the following one millions dollar problem.

Conjecture 1 (Hodge's conjecture). *Let $X/\mathbb{C}$ be a projective smooth algebraic variety over $\mathbb{C}$. The for every p, the space*

$$H^{p,p} := H^p(X, \Omega^p) \cap H^*(X, \mathbb{Q}) \subset H^*(X, \mathbb{C})$$

is spanned by the cohomology classes $[Z]$ of algebraic cycles Z in X of codimension p.

Remark that this conjecture gives a link between an algebraic, i.e., non-archimedean or perhaps global analytic invariant (the space of algebraic cycles) and a purely archimedean invariant. There is a similar conjecture in the purely non-archimedean setting called Tate's conjecture (that uses Galois action on etale cohomology). One can define an arithmetic variety $X/\mathbb{Z}$ whose archimedean component has the given variety as direct factor. The proper definition of a combined Hodge-Tate conjecture is of course directly linked to the proper understanding of the functional equation of the geometric zeta function $\zeta_X(s)$.

We refer to Deninger's work [Den94] for the following definition of the archimedean factor.

Definition 30. *Let $X/\mathbb{R}$ be a projective smooth variety and $H := H^*(X_{\mathbb{C}}, \mathbb{R})$ be its real cohomology. There is a natural map $F_\infty : X_{\mathbb{C}} \to X_{\mathbb{C}}$ given by the action on coefficients that gives an endomorphism*

$$F_\infty : H \to H.$$

The archimedean filtration on H is defined by

$$\mathcal{V}^i H := (H \cap F^i H_{\mathbb{C}})^{(-1)^i} \oplus (H \cap F^{i+1} H_{\mathbb{C}})^{(-1)^{i+1}}$$

where the exponent ± 1 denotes the ± 1-eigenspace of F_∞.

Remark here that there is certainly a relation between the filtration by

$$F^i H_{\mathbb{C}}^{(-1)^i} \oplus F^{i+1} H_{\mathbb{C}}^{(-1)^{i+1}}$$

and Hodge theory on the quotient analytic space $X(\mathbb{C})/F_\infty$, which is the natural archimedean fiber of the global analytic space associated to X, if it is defined over $\mathbb{Z}$.

Definition 31. *The archimedean local factor of X is defined as*

$$\zeta_{X,\infty}(s) := \prod_{n\in\mathbb{Z}} \zeta_\infty(s-n)^{d_n}$$

where $d_n := \dim \mathrm{Gr}^n \mathcal{V}(H)$ is the dimension of the graded part of degree n for the filtration $\mathcal{V}$ and $\zeta_\infty(s-n) := \pi^{-s/2}\Gamma(s/2)$. The completed zeta function of X is given by the product

$$\hat{\zeta}_X(s) := \zeta_X(s).\zeta_{X,\infty}(s).$$

6.2 The main conjectures of arithmetic geometry

We give here the main motivation for the geometric and dynamical study of the zeta function of X. Except for Riemann hypothesis, the following conjecture is known is some very special case, using the methods of automorphic representation theory that generalize Tate's thesis (see above). One could say that (appart some very special cases), the only know cases for the functional equation are ring of integers of number fields, elliptic curves over $\mathbb{Q}$, and varieties over finite fields. The proof for varieties over finite fields is dynamical (see next section). For elliptic curve it is simply crazy! (this follows from Taylor-Wiles proof of Fermat's last theorem, more precisely from the fact that elliptic curve has the same L-function as some automorphic representations of $\mathrm{GL}_2(\mathbb{A})$).

Conjecture 2. *Let X be a smooth variety over $\mathbb{Z}$.*

1. *(Functional equation) The completed zeta function $\hat{\zeta}(s)$ has a meromorphic continuation to $\mathbb{C}$ and there is a functional equation*

$$\hat{\zeta}_X(s) = \epsilon(X,s)\hat{\zeta}^*_X(1-s)$$

where $\epsilon(X, s) = ae^{bs}$ *for some real* a *and* b, *and* $\hat{\zeta}_X^*$ *is obtained from* $\hat{\zeta}_X$ *by some twisting at each cohomological level on* X.

2. $\hat{\zeta}(s)$ *has only finitely many poles that are integers.*

3. *(Riemann Hypothesis) The zeroes of the factor of* $\hat{\zeta}_X(s)$ *that corresponds to* $H^k(X)$ *lie on the line* $\mathrm{Re}(s) = \frac{k+1}{2}$.

6.3 The dynamics of Frobenius in finite characteristic

We refer to appendix C of Hartshorne's book [Har77] for a more detailled account of the Weil's conjecture, that are shortly explained in this subsection.

Let X be a projective smooth variety over $\mathbb{Z}$. If we suppose that one of the defining polynomials for X is constant and equal to a prime number $p \in \mathbb{Z}$, we get a homogeneous coordinate ring A of the form

$$A = \mathbb{F}_p[X_0, \dots, X_n]/(P_1, \dots, P_m).$$

This means that the variety X is defined over the finite field $\mathbb{F}_p$. In this case, the completed Dedekind zeta function has no archimedean factor, since the archimedean component of X is empty.

Let $N_r(X) := |X(\mathbb{F}_{p^r})|$ be the number of points of X with values in a finite field $\mathbb{F}_{p^r}$ (which is unique up to non-unique isomorphism). It is also equal to the number of fixed points by the r-th power of the frobenius map

$$F : X(\bar{\mathbb{F}}_p) \to X(\bar{\mathbb{F}}_p)$$

which sends $(x_1, \dots, x_n)$ to $(x_1^p, \dots, x_n^p)$. This fact is at the basis for a Lefchetz fixed point formula that allows to prove geometrically all the desired results on the geometric zeta function $\zeta_X(s)$.

The completed Dedekind zeta function of X is then equal to

$$\zeta_X(s) = \exp\left(\sum_{m\geq 1} N_m(X)\frac{p^{-ms}}{m}\right).$$

It can be shown to converge for $\mathrm{Re}(s) > \dim(X)$.

The main conjecture of arithmetic is completely proved in this case, following Dwork, Grothendieck and Deligne. The proof is based on the following

Lefschetz trace formula in etale cohomology and on a devissage to the case of curve (for Riemann hypothesis), which was already known since Weil's work [Wei95a].

The main mathematical input of the proof is the construction of etale cohomology theory, which associates to a (say projective smooth) variety $X/\mathbb{F}_p$ a graded cohomology with coefficients in $\mathbb{Q}_\ell$

$$H^*(X) = H^*(X_{\bar{\mathbb{F}}_p})$$

equiped with a natural action of the Frobenius endomorphism F. This can be found in Grothendieck's famous SGA [AGV73], and is based on the notion of torsion sheaves on space with respect to the étale topology.

We just recall the definition of an etale morphism, which is an algebraic version of a galois covering of analytic variety. The definition of etale cohomology is very similar to the definition of usual sheaf cohomology with constant coefficients $\mathbb{Z}/\ell^n\mathbb{Z}$, except that one has to use a generalization of topological space called a Grothendieck site, which is outside of the scope of these notes.

Definition 32. *Let $f : X \to S$ be a morphism of algebraic spaces. The morphism f is called smooth (resp. étale, resp. unramified) is if for every ring A, every ideal I in A and every morphism $x : \mathrm{Spec}(A/I) \to S$ the natural map*

$$X_S(A/I^2) \to X_S(A/I)$$

is a surjective (resp. bijective, resp. injective), where X_S denotes the set of morphisms that respect the map to S.

The following theorem is due to Grothendieck and Verdier.

Theorem 24 (Lefchetz fixed point formula)**.** *There is an equality*

$$N_r(X) = \sum_{i=0}^{2n} (-1)^i \mathrm{Tr}(F \mid H^i(X))$$

between the number of fixed points for the Frobenius endomorphism and its (super)-trace on etale cohomology $H^*(X)$.

The following corollary is an easy result of linear algebra.

Corollary 4. *The zeta function can be defined from the spectrum of the frobenius endomorphism acting on etale cohomology by the formula*

$$\zeta_X(s) = \prod_{i=0}^{2\dim(X)} \det(1 - p^{-s} F | H^i(X))^{(-1)^{i+1}}.$$

Theorem 25 (Poincaré duality)**.** *There is a perfect frobenius-equivariant cup-product Pairing*

$$H^i(X) \times H^{2n-i}(X) \to H^{2n}(X)$$

and an equivariant trace isomorphism $\mathrm{Tr} : H^{2n}(X) \xrightarrow{\sim} \mathbb{Q}_\ell(n)$.

Corollary 5. *The zeta function of X fulfils the functional equation*

$$\zeta_X(n - s) = p^{E(\frac{n}{2} - s)} \zeta_X(s)$$

where E is a constant uniquely determined by X (self intersection number of the diagonal Δ in $X \times X$).

6.4 The dynamics of archimedean local factors

We here follow the approach proposed by Deninger in [Den01], using a nice result of Simpson. The translation to the global analytic setting is new and makes this approach even more natural.

Let X be a smooth analytic variety over a ring. Let $\Delta : X \hookrightarrow X \times X$ be the diagonal embedding. Consider the ideal $\mathcal{I} \subset \mathcal{O}_{X \times X}$ of this closed embedding. We filter $\mathcal{O}_{X \times X}$ by integral power of $\mathcal{I}$ with $\mathcal{I}^n = \mathcal{O}_{X \times X}$ for $n \leq 0$. Let M_2 be the defined as the $X \times X$-affine scheme

$$M_{2,X} := \mathrm{Spec}(\oplus_{i \in \mathbb{Z}} z^{-i} \mathcal{I}^i).$$

By construction, there is a natural map

$$M_{2,X} \to \mathbb{A}^1_{X \times X} := \mathrm{Spec}(\mathcal{O}_{X \times X}[z])$$

and a natural action of $\mathrm{GL}_{1,X \times X}$ on $M_{2,X} \to \mathbb{A}^1_{X \times X}$.

Definition 33. *The analytification M_X of the inverse image of $M_{2,X}$ through the diagonal embedding $\Delta : X \to X \times X$ is called the deformation to normal cone for X. It has a natural projection to the analytic line $\mathbb{A}^1_X$ and a natural action of the analytic group $\mathrm{GL}_{1,X}$. The Simpson cohomology sheaf is the GL_1-sheaf $\mathcal{F}^k_{Simpson}(X)$ on $\mathbb{A}^1$ given by*

$$\mathcal{F}^k_{Simpson}(X) := \mathbb{R}^k\pi_*\Omega^\bullet_{M/\mathbb{A}^1}.$$

Now let $X/\mathbb{Z}$ be an arithmetic variety and consider the deformation to normal cone M_X on $\mathbb{A}^1_X$. If $X = \mathrm{Spec}(\mathbb{Z})$ then $M_X = \mathbb{A}^1_{\mathbb{Z}}$. The archimedean fiber $M_{\mathbb{R}}$ of M_X has a natural projection

$$\pi_\infty : M_{\mathbb{R}} \to \mathbb{A}^1_{\mathbb{R}} \cong \mathbb{C}/\sigma,$$

where σ denotes complex conjugation and Poincaré's lemma gives us an isomorphism

$$\mathbb{R}^k\pi_*\mathbb{R}_M \otimes_{\mathbb{R}_{\mathbb{A}^1}} \mathcal{O}_{\mathbb{A}^1} \xrightarrow{\sim} \mathbb{R}^k\pi_*\Omega^\bullet_{M/\mathbb{A}^1} =: \mathcal{F}^k_{Simpson}(X).$$

Now let $i : \mathbb{R}/\{\pm 1\} \to \mathbb{A}^1_{\mathbb{R}}$ be defined by sending $x \in \mathbb{R}$ to $ix \in \mathbb{C}$. We also denote $i : M \to M_{\mathbb{R}}$ the pull-back of $M_{\mathbb{R}}$ along i. Let $\mathcal{A}_{\mathbb{R}/\{\pm 1\}}$ be the ring of real analytic functions and let $\mathcal{DR}_{X/\mathbb{R}}$ be the cokernel of the natural inclusion of complexes

$$\pi^{-1}\mathcal{A}_{\mathbb{R}/\{\pm 1\}} \to i^{-1}\Omega^\bullet_{M_{\mathbb{R}}/\mathbb{A}^1_{\mathbb{R}}}.$$

The following theorem, whose proof can be found (up to minor translations) in [Den01], gives a dynamical interpretation of the archimedean local factor.

Theorem 26. *Let $X/\mathbb{R}$ be a projective smooth variety together with a hyperplane section. The action of $\mathbb{R}^*_+$ on the space*

$$\mathbb{H}^*_{ar}(X_{\mathbb{R}}) := \Gamma\left(\mathbb{R}/\{\pm 1\}, \underline{\mathrm{Hom}}_{\mathcal{A}_{\mathbb{R}}}(\mathbb{R}^*\pi_*\mathcal{DR}_{X/\mathbb{R}}, \mathcal{A}_{\mathbb{R}/\{\pm 1\}})\right)$$

gives a spectral interpretation for the poles of archimedean local factor $\zeta_{X,\infty}(s)$.

To be continued...

6.5 A glimpse at Langlands' program

From the point of view of this course, Langlands' program is a mathematical machinery that gives you a method to try to prove the functional equation of the completed zeta function $\hat{\zeta}_X(s)$ of a given arithmetic variety X, by using the proof of the functional equation for cuspidal automorphic representations of GL_n. Remark that this proof also uses a non-commutative "dynamical system" given by the action of $P_n(\mathbb{A})$ on $P_n(\mathbb{A})/P_n(\mathbb{Q})$ (and quotients/subspaces of this) where

$$P_n := \mathrm{M}_n \rtimes \mathrm{GL}_n.$$

Here is how the program works.

Let $X/\mathbb{Z}$ be an arithmetic variety. One associates to it its Grothendieck motive (universal linear invariant that contains both archimedean and non-archimedean information) $h^*(X_{\mathbb{Q}})$. The non-archimedean information about $h^*(X_{\mathbb{Q}})$ is completely contained in the $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ representation given by the etale cohomology group $H^*(X_{\bar{\mathbb{Q}}})$. Choose an irreductible part M of $h^*(X_{\mathbb{Q}})$ (which corresponds to an irreductible Galois representation say of dimension n on the corresponding subspace of etale cohomology). This corresponds to a factor $L(M,s)$ of $\hat{\zeta}_X(s)$ that can be explicitely computed.

Conjecture 3 (Langlands correspondence)**.** *There exists a unique cuspidal automorphic representation π_M of GL_n such that*

$$L(M,s) = L(\pi_M,s).$$

The direct corollary of Godement and Jacquet's theorem (that generalizes Tate's thesis to dimension n) is that $L(M,s)$, and thus $\hat{\zeta}_X(s)$ fulfils a functional equation that can be computed explicitely.

One of the main difficulties of this program is that present approaches are based on the systematic use of non-archimedean tools (algebraic geometry of schemes, Galois representations, etc...) that break the natural symmetry between places present in arithmetic. Since the local p-adic version used Berkovich's analytic geometry, one can guess that global analytic geometry is much better adapted than algebraic geometry to the study of this problem.

The standard methods for approaching this program is the use of so-called Shimura varieties, and can be understood roughtly in the case $n = 2$ by saying the following. One finds a morphism of arithmetic varieties $f : E \to S$ whose complex points $f : E(\mathbb{C}) \to S(\mathbb{C})$ are given by the adelic quotient

$$P_2(\mathbb{A})/P_2(\mathbb{Q}) \to \mathrm{GL}_2(\mathbb{A})/\mathrm{GL}_2(\mathbb{Q}).$$

The construction gives two commuting actions

$$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \circlearrowleft X_{\bar{\mathbb{Q}}} \circlearrowright \mathrm{GL}_2(\mathbb{A}_f)$$

(where $\mathbb{A}_f$ is the non-archimedean component of $\mathbb{A}$) that allow to decompose certain cohomological invariants of the morphism $f : E \to S$ in tensor products

$$M \otimes \pi_M$$

and Langlands' correspondence is given by sending M to π_M. These methods thus apply only to very particular case, the widely known example being the following:

Theorem 27 (Breuil,Diamond,Taylor,Wiles)**.** *If E is an elliptic curve over $\mathbb{Q}$, there exists an automorphic representation π of GL_2 such that*

$$L(E, s) = L(\pi, s).$$

The conclusion of this section is that global analytic spaces could also be very useful to improve our understanding of Langlands' program (that uses, for the time being, the purely non-archimedean methods of Galois theory), by giving a formulation that treats all places on equal footing.

Chapter 7

Arithmetic compactifications

Another main theme of arithmetic geometry is the counting of solutions to diophantine equations, i.e., of cardinals of sets of integral $X(\mathbb{Z})$ or rational $X(\mathbb{Q})$ points of a given arithmetic variety over $\mathbb{Z}$.

There is a materially interesting problem in this area (one million dollars) that related analytic number theory and the study of rational points. Birch and Swinnerton-Dwyer's conjecture relates the order of the zero at $s = 1$ of the zeta function of an elliptic curve E over $\mathbb{Q}$ with the rank of the group $E(\mathbb{Q})$. We will not talk about this problem here because its formulation does not treat all places on an equality footing (see however Lichtenbaum's approach [Lic09] that is more symmetric).

Another stricking result on rational points was obtained by Faltings, answering a long standing problem:

Theorem 28 (Mordell's conjecture)**.** *Let $C/\mathbb{Q}$ be a curve of genus $g > 1$. Then the set $X(\mathbb{Q})$ of rational points is finite.*

The proof of this theorem used very strongly the analogy between $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}[X]$, and particularly the so called arithmetic compactifications (introduced by Arakelov) that play the role for $\mathbb{Z}$ analogous to the role played by the projective line $\mathbb{P}^1_{\mathbb{Z}/p\mathbb{Z}}$ for $\mathbb{Z}/p\mathbb{Z}[X]$.

In Tate's thesis, there was a perfect dichotomy between the characteristic function $f_p(x) := \mathbb{1}_{\mathbb{Z}_p}(x)$ and $f_\infty(x) = e^{-\pi x^2}$ (they share the property of being Fourier-stable) since

$$\zeta_p(s) = \int_{\mathbb{Q}_p} f_p(x)|x|_p^s d^*x, \text{ for all } p \leq \infty.$$

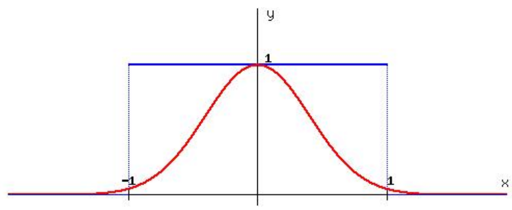

Figure 7.1: Analytic and arithmetic characteristic function of $\mathbb{Z}_\infty$.

We could thus think of the valuation ring of $\mathbb{R}$ as being a probabilistic subset of $\mathbb{R}$ whose "analytic characteristic function" is the gaussian $e^{-\pi x^2}$. However, in the theory of arithmetic compactifications, we use as a definition of the (generalized) valuation ring of $\mathbb{R}$ the interval $\mathbb{Z}_\infty := \{x \in \mathbb{R},\ |x|_\infty \leq 1\} = [-1, 1]$. The arithmetic characteristic function is thus defined to be $\mathbb{1}_{[-1,1]}$. The difference between the two is shown in figure 7. We thus insist on the fact that the use of the arithmetic compactifications in analytic number theory is, for the moment, only very hypothetical (see however proposition 16).

7.1 Monads and $\mathbb{F}_1$

We give here a very funny application of abstract category theory. We refer to the short survey of J. Frésan [Fre09] and to Durov's thesis [Dur07] for motivations. One can sum up our motivations by the following: the space $\mathbb{Z}_p$ of p-adic numbers with norm smaller than 1 is a ring. On the contrary, in $\mathbb{R}$, the interval $\mathbb{Z}_\infty = [-1, 1]$ of real numbers of norms smaller than 1 is a multiplicative monoid, but it does not have a globally defined addition. However, one can define an operation by

$$1/2\{1\} + 1/2\{2\} : \begin{array}{ccc} [-1,1] \times [-1,1] & \to & [-1,1] \\ (x,y) & \mapsto & \frac{x}{2} + \frac{y}{2} \end{array}$$

The systematic study of formal operations like this one is called universal algebra, and can be done using the theory of monads. This formalism can be based on the notion of free object.

Let C be a category whose objects are sets with additionnal data. For example, C can be the category of groups, topological spaces or rings.

Definition 34. *The free object in C on a given set X is defined, if it exists, as an object $\langle X\rangle$ of C such that for every object A of C, there is a natural bijection*

$$\mathrm{Hom}_{\textsc{Sets}}(X, A) \to \mathrm{Hom}_C(\langle X\rangle, A).$$

If all the free objects exist, the monad associated to the category C is the functor

$$\Sigma_C : \textsc{Sets} \to \textsc{Sets}$$

that sends a set X to the underlying set of the free object $\langle X\rangle$ in C.

Remark that the monad associated to C has a natural unity (natural transformation)

$$\mathbb{1} : \mathrm{Id}_{\textsc{Sets}} \to \Sigma_C$$

and multiplication

$$\mu : \Sigma_C \circ \Sigma_C \to \Sigma_C$$

that is associative. Concretely, $\mathbb{1}$ is given by the natural maps

$$\mathbb{1}_X : X \to \langle X\rangle$$

of sets and μ is given by the maps

$$\mu_X : \langle\langle X\rangle\rangle \to \langle X\rangle$$

associated to the identity map of the underlying set of $\langle X\rangle$.

Exercice 2. *Write down the unit and associativity axiom for such a monad.*

Example 2. *1. The monad associated to the category* Grab *of abelian groups is given by the free abelian group (formal linear combinations with finite support on X)*

$$\mathbb{Z}(X) := \Sigma_{\textsc{Grab}}(X) := \mathbb{Z}^{(X)}.$$

2. More generally, the category associated to modules over a given ring A is given by the free module

$$A(X) := \Sigma_{\mathrm{Mod}(A)}(X) := A^{(X)}.$$

Actually, if A is any other algebraic structure that has a natural notion of module over it, the monad associated to A is also given by the free module construction.

3. *The generalized ring of integer of* $\mathbb{R}$ *(with respect to the standard absolute value* $|.|_\infty$*) is defined by*

$$\mathbb{Z}_\infty(X) := \{\sum_{x \in X} a_x\{x\},\ \sum_{x \in X} |a_x| \leq 1\}.$$

Remark now that if $\mathbf{n} = \{1, \ldots, n\}$ is a finite set, every element $m \in A(\mathbf{n})$ defines for every set X an operation

$$m : A(X)^n \to A(X).$$

Indeed, the indexes of a family $(x_1, \ldots, x_n) \in A(X)^n$ define a map of sets $\mathbf{n} \to A(X)$ and using the composition of the corresponding map

$$m : A(\mathbf{n}) \to A(A(X))$$

with multiplication $\mu : A(A(X)) \to A(X)$, we get an operation

$$m : A(X)^n \to A(X).$$

Proposition 14. *The functor*

$$\text{RINGS} \to \text{MONADS}$$

that sends a ring A *to the monad associated to the category of* A*-modules is fully faithful.*

Proof. Let A be a ring. Remark that we find back the set A as $A(1)$. Every element in $A(1)$ gives an operation $A(1) \to A(1)$ which is just multiplication by this element. The unit is given by

$$\mathbb{1}_{\mathbf{1}}(1) \in A(1).$$

The zero is given by the image of $A(0) = \{0^{[0]}\}$ in $A(1) = A$. Remark that the distinguished element $+^{[2]} := \{1\} + \{1\}$ in $A(2)$ induces usual addition on A and the element $-^{[1]} := -\{1\}$ in $A(1)$ induces the inversion operation. This shows that the ring operations on A are uniquely determined by the monad structure. If $f : A \to B$ is a ring morphism, f is uniquely determined, as a map, by $f_1 : A(1) \to B(1)$. Now, let $f : A \to B$ be a monad morphism. This means that $f \circ \mu = \mu \circ f * f$ where $f * f : A \circ A \to B \circ B$ is induced by f. This implies that all operations are conserved, so that $f_1 : A(1) \to B(1)$ is a ring morphism. □

Definition 35. *A monad $A : \textsc{Sets} \to \textsc{Sets}$ is called algebraic if it commutes with inductive limits.*

Example 3. *The following are algebraic monads.*

1. $\mathbb{F}_\emptyset := \mathrm{Id}_{\textsc{Sets}}$,

2. *For every ring A, the monad A is algebraic,*

3. *The monad $\mathbb{N}$ associated to the semi-ring of positive integers.*

4. $\mathbb{Z}_\infty$,

5. $\mathbb{F}_1(X) := X \coprod 0$ *with the operation induced by the equality of monads* $\mathbb{F}_1 = \mathbb{N} \cap \mathbb{Z}_\infty$.

6. $\mathbb{F}_{\pm 1}(X) := X \coprod \{0\} \coprod -X$ *with the operation induced by the equality of monads* $\mathbb{F}_{\pm 1} = \mathbb{Z} \cap \mathbb{Z}_\infty$.

7. *The monad of words sending a set X to the set $W(X)$ of all finite words on the alphabet X.*

8. *The monad $W_U(X) := W(U \coprod X)$ of words with constants from U.*

Definition 36. *For A an algebraic monad, we denote $\|A\| = \coprod_{n \geq 0} A(i)$. The free monad*

$$\mathbb{F}_\emptyset\langle \underline{U} \rangle$$

on a graded set $\underline{U} = \coprod_{n \geq 0} U_n$ is the submonad of W_U, with $W_U(X)$ given by the set of valid expressions constructed using operations in U, variables from X, written in prefix notation.

For example for $u^{[1]} \in U_1$ and $v^{[2]}$ in U_2, the expression

$$u^{[1]}\{x\}u^{[2]}\{x\}\{y\}$$

is valid for $x, y \in X$ and describes the object obtained by applying the two given operations (of respective arity 1 and 2) to the given elements, from the left.

For a more precise explanation of this definition, we refer to Durov [Dur07], 4.5.2. One can also use relations compatible to the given operations to define a free monad on a graded set with relations.

Definition 37. *An algebraic monad A is called commutative if any two of its operations $t \in A(n)$ and $s \in A(m)$ commute, meaning that for every $\{x_{ij}\}$ indexed by $i = 1, \ldots, n$ and $j = 1, \ldots, m$, we have*

$$t(s(x_{11}, \ldots, x_{1m}), \ldots, s(x_{n1}, \ldots, x_{nm})) = s(t(x_{11}, \ldots, x_{n1}), \ldots, t(x_{1m}, \ldots, x_{nm})).$$

For example, the monad associated to a monoid is commutative if the monoid is commutative. If a ring gives a commutative monad, it's addition and multiplication automatically fulfil the distributivity axiom.

The idea of Durov is that one can study algebraic geometry in the context of commutative algebraic monads.

7.2 The analytic compactification of $\mathcal{M}(\mathbb{Z})$

We give here a construction inspired by Durov [Dur07] and Poineau [Poi05]. The following formulation is however original.

Definition 38. *Let $\mathbb{R}_+$ be the algebraic monad associated to the semiring of positive real numbers. For each set X, $\mathbb{R}_+(X)$ is partially ordered (by the ordering on components). Let A be a monad. A seminorm on A is a unital sub-morphism*

$$|.| : A \to \mathbb{R}_+,$$

i.e., a natural transformation such that for every X,

$$|.|(X) \circ \mathbb{1}_A(X) = 1_{\mathbb{R}}(X)$$

and

$$|.|(X) \circ \mu_A \leq \mu_{\mathbb{R}_+} \circ |.| * |.|.$$

We also ask that the comparison maps

$$\begin{array}{ccc} A(n) & \to & A(1)^n \\ t & \mapsto & \left(t(0, \ldots, \underbrace{1}_{i}, \ldots, 0)\right)_{i=1,\ldots,n} \end{array}$$

and $\mathbb{R}_+(n) \to \mathbb{R}_+^n$ are stricly respected by $|.|$. A seminorm on A is called multiplicative if the induced map $|.| : A(1) \to \mathbb{R}_+(1)$ is multiplicative.

Definition 39. *The analytic spectrum of a commutative algebraic monad A is the set $\mathcal{M}(A)$ of multiplicative seminorms on A equiped with the coarsest topology that makes, for every $a \in A(1)$, the map*

$$|.| \mapsto |a|$$

continuous.

Proposition 15. *The analytic spectrum $\mathcal{M}(\mathbb{Z}_\infty)$ is identified with the closed interval $[0, +\infty]$ through the map $r \mapsto |.|_\infty^r$.*

Proof. If $|.| : \mathbb{Z}_\infty \to \mathbb{R}_+$ has a non-trivial kernel $\mathfrak{m}$ then this kernel is equal to the unique non-trivial prime ideal $\mathfrak{m}_\infty :=]-1, 1[$ in $\mathbb{Z}_\infty$ (see Durov for a proof of this uniqueness result). The norm $|.|$ is thus equal to $|.|_{0,\infty} := |.|_\infty^{+\infty}$, at least on $\mathbb{Z}_\infty(1)$. Since $\mathbb{Z}_\infty(n) \subset \mathbb{Z}_\infty(1)^n$ and $\mathbb{R}_+(n) \subset \mathbb{R}_+(1)^n$, and since the comparison maps are respected by $|.|$ the value of $|.|$ is uniquely determined by its value on $A(1)$, which concludes the proof. □

There is a natural inclusion $\mathcal{M}(\mathbb{R}) \subset \mathcal{M}(\mathbb{Z}_\infty)$ whose image is $[0, +\infty[$.

We define now ad hoc (that's a pitty) the sheaf of analytic functions on $\mathcal{M}(\mathbb{Z}_\infty)$ to be given by usual sheaf on $\mathcal{M}(\mathbb{R})$ and to have germs at $|.|_{0,\infty} := |.|_\infty^{+\infty}$ given by $\mathbb{Z}_\infty$.

Definition 40. *The analytic compactification $\overline{\mathcal{M}(\mathbb{Z})}$ is the ringed space obtained by pasting $\mathcal{M}(\mathbb{Z})$ and $\mathcal{M}(\mathbb{Z}_\infty)$ along $\mathcal{M}(\mathbb{R})$.*

The following proposition shows that arithmetic compactifications could also be useful to study geometrically analytic number theory.

Proposition 16. *The Picard group (i.e., the group of locally free sheaves of rank 1) of the analytic compactification of $\mathcal{M}(\mathbb{Z})$ is*

$$\mathrm{Pic}(\overline{\mathcal{M}(\mathbb{Z})}) \cong \mathbb{R}_+^*.$$

Proof. Let

$$j : \overline{\mathcal{M}(\mathbb{Z})} - \{|.|_0\} \to \overline{\mathcal{M}(\mathbb{Z})}$$

be the inclusion of the open complement of the trivial norm. Let $\mathcal{L}$ be a locally free $\mathcal{O}$-module of rank 1 on $\overline{\mathcal{M}(\mathbb{Z})}$. Its sections on the standard open subsets $U_v := \{|.|_v^t,\ t \in]0, +\infty]\}$, where v is a place of $\mathbb{Q}$ are isomorphic to the sections of $\mathcal{O}$, i.e., to $\mathbb{Z}_v$. Its sections on $\{|.|_v^t,\ t \in]0,\infty[\}$ are given by $\mathbb{Q}_v$. This implies that we can choose an isomorphism

$$f : (j_*\mathcal{L})_{(|.|_0)} \xrightarrow{\sim} \mathbb{A}$$

between germs at the trivial norm of $j_*\mathcal{L}$ and adèles. Now remark that we can also choose an isomorphism

$$g : (\mathcal{L})_{(|.|_0)} \xrightarrow{\sim} \mathbb{Q}$$

between germs at the trivial norm of $\mathcal{L}$ and rational numbers, which is moreover compatible with f through the canonical maps $\mathbb{Q} \to \mathbb{A}$ and $\mathcal{L} \to j_*\mathcal{L}$. Now denote $\mathcal{U}_\mathcal{L}$ the subsheaf of sections of $\mathcal{L}$ that are everywhere of norm smaller than 1 (with respect to some, or equivalently any local trivialization of $\mathcal{L}$). There is another isomorphism

$$h : (j_*\mathcal{U}_\mathcal{L})_{(|.|_0)} \xrightarrow{\sim} \prod_v \mathbb{Z}_v = (j_*\mathcal{U}_\mathcal{O})_{(|.|_0)}$$

compatible with f through the canonical embeddings $\mathcal{U}_\mathcal{L} \subset \mathcal{L}$ and $\prod_v \mathbb{Z}_v \subset \mathbb{A}$. The triple (f, g, h), called the marking of $\mathcal{L}$, determines uniquely the bundle $\mathcal{L}$ up to isomorphism because it determines an isomorphism of the sections on the standard open subsets with $\mathbb{Q}_v$ and $\mathbb{Z}_v$ respectively, germs at the trivial norm and f determines pasting isomorphisms between all these local sections. Now any element of $\mathbb{A}^*$ acts transitively on the moduli space of marked bundles and forgetting the marking corresponds to quotienting by its automorphisms that are

$$\mathrm{Aut}(\mathcal{O}_{(|.|_0)}) = \mathbb{Q}^* \text{ and } \mathrm{Aut}((j_*\mathcal{U}_\mathcal{O})_{(|.|_0)}) = \{\pm 1\} \times \widehat{\mathbb{Z}}^* = K.$$

This shows that the moduli space of rank 1 bundles on $\overline{\mathcal{M}(\mathbb{Z})}$ is isomorphic to

$$\mathbb{Q}^*\backslash\mathbb{A}^*/K \overset{|.|}{\xrightarrow{\sim}} \mathbb{R}_+^*.$$

□

7.3 Arakelov models for arithmetic varieties

If $F_1, \ldots, F_m \in \mathbb{Z}[X_1, \ldots, X_n]$ are the defining polynomials for a given affine variety X^{an}, on can find an integer $n \in \mathbb{N}$ such that all coefficients of the polynomial $G_i := \frac{F_i}{n} \in \mathbb{Q}[X_0, \ldots, X_n]$ are in $\mathbb{Z}_\infty(1) := [-1, 1]$.

Let X_∞ be the analytic space over $\mathcal{M}(\mathbb{Z}_\infty)$ obtained by

$$X_\infty = \mathcal{M}(\mathbb{Z}_\infty[X_1, \dots, X_n]/(G_i)).$$

One can paste X_∞ with X^{an} along an open subset of $X^{an}_{\mathbb{R}}$ given by

$$U = \frac{1}{3n} < \left|\frac{1}{n}\right| < \frac{1}{2n}$$

for example.

Pasting functions is another "pair de manches" that will be treated latter.

7.4 Arithmetic analytic rings and Arakelov geometry

We now generalize the Lawvere-Dubuc approach to analytic geometry, through functor of functions (see [Pau12] for an introduction to these methods), to define an analytic version of the ring $\mathbb{Z}_\infty$, with free module of rank n exactly given by the n-ball

$$\mathbb{Z}_\infty(n) = B^n(0,1) := \mathrm{Hom}(\mathrm{Spa}(\mathbb{R}), B^n(0,1)_{\mathbb{R}}),$$

where $B^n(0,1)_{\mathbb{R}} \subset \mathbb{A}^n_{\mathbb{R}}$ is defined as the spectrum of the spectral norm on the ring of formal power series with the pseudo-norm given by the sum of the series of absolute values of coefficients.

Remark that this definition is a bit complicated. There is a natural way to define a norm on matrices, which may be a better way to approach the generalization of rings, like in Haran's work [Har07].

There are various types of analytic rings: the bounded and the unbounded ones. They correspond to open subsets of the balls B^n in $\mathbb{A}^n$, or open subsets of $\mathbb{A}^n$ for various n.

If $f : U \to V$ is a morphism between two subsets of $\mathbb{A}^n$, and $f_{|B}$ is its restriction to the ball, there exists $\lambda \in \mathbb{R}_+$ such that $|f_{|B}(u)| \leq \lambda |u|$. We may work with functions $f : U \to V$ such that $f_{|B}$ has an image contained in $V \cap B'$ (i.e., the λ above can be chosen to be equal to 1). We call such a function a contracting analytic function.

There is a natural functor of restriction from contracting analytic opens to opens of analytic discs, that is obtained by forgetting what happens in the exterior of the disc.

Remark that a contracting analytic function may not be bounded. For example, the identity function of $\mathbb{A}^1$ is disc-bounded, but not bounded.

Remark also that a contracting analytic function is not determined by its value on the disc, because it may be equal to anything on the other disc of the open disjoint union

$$U = D^\circ(0, 1+\epsilon) \coprod D^\circ(3,1) \subset \mathbb{A}^1,$$

for example. So this notion of contracting analytic function is not so trivial, and it gives an intermediary between analytic functions between opens of balls, and analytic functions between opens of the affine space.

We will call $\textsc{Open}_{an}$ the category of all open subsets of $\mathbb{A}^n$, and $\textsc{Open}_{an}^c$ the category of all open subsets of $\mathbb{A}^n$, but with contracting maps between them.

There is a natural faithful but not full functor

$$\textsc{Open}_{an}^c \to \textsc{Open}_{an}.$$

There is also a forgetful functor

$$\textsc{Open}_{an}^c \to \textsc{Open}_{an}^d$$

from contracting opens to opens of discs of all dimensions.

We may now define three associated categories of analytic rings, denoted $\mathcal{A}lg_{an}^c$, $\mathcal{A}lg_{an}^d$, and $\mathcal{A}lg_{an}$. There are natural functors

$$\mathcal{A}lg_{an} \to \mathcal{A}lg_{an}^c$$

and

$$\mathcal{A}lg_{an}^d \to \mathcal{A}lg_{an}^c.$$

If A is an analytic ring (in the sense of Dubuc), the datum of the associated contracting analytic ring may be essentially equivalent to the datum of the ring itself. So we may have an embedding

$$\mathcal{A}lg_{an} \hookrightarrow \mathcal{A}lg_{an}^c.$$

The analytic ring $\mathbb{R}$ is defined by

$$\mathbb{R}(U) := \mathrm{Hom}(\mathrm{Spa}(\mathbb{R}), U).$$

Its contracting version is the same, but with functoriality operations only given by contracting maps. This new $\mathbb{R}$, that we denote by the same letter, has a contracting subring

$$\mathbb{Z}_\infty(U) := \mathrm{Hom}(\mathrm{Spa}(\mathbb{R}), U \cap B(0,1)).$$

One may then think of $\mathbb{Z}_\infty(B^n(0,1))$ (defined using overconvergent functions, i.e., a limit of $\mathbb{Z}_\infty(B(0,1+\epsilon))$ for $\epsilon > 0$), as the free rank n module over $\mathbb{Z}_\infty$.

Exercices sheets for the course

Appendix A

The functional equation of zeta

IMPA - Rio de Janeiro — Master course
Year 2009-2010

Riemann's zeta function

1.*Basic properties and the functional equation.*

The aim of this exercice sheet is to give a precise statement of the famous two hundred years old problem called the Riemann hypothesis.

Riemann's zeta function is defined for $s \in \mathbb{R}$, $\mathrm{Re}(s) > 1$, by

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}.$$

We propose to prove that

1. this series converges for $\mathrm{Re}(s) > 1$, uniformly for $\mathrm{Re}(s) > 1 + \delta$ for all $\delta > 0$;

2. that it meromorphically continues (i.e. can be written as a quotient of two holomorphic functions) in the half plane $\mathrm{Re}(s) > 0$;

3. that one can write it as an infinite product

$$\zeta(s) = \prod_p \zeta_p(s)$$

indexed by prime numbers with $\zeta_p(s) := \frac{1}{1-p^{-s}}$.

4. that the gamma function $\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$ continues analytically to $\mathbb{C} - \mathbb{N}_-$ and fulfils there $\Gamma(s+1) = s\Gamma(s)$;

5. that the completed zeta function

$$\hat{\zeta}(s) = \zeta_\infty(s)\zeta(s)$$

where

$$\zeta_\infty(s) := 2^{-1/2}\pi^{-s/2}\Gamma(s/2)$$

admits the integral representation

$$\hat{\zeta}(s) = \frac{1}{2\sqrt{2}} \int_0^\infty (\theta(iy) - 1) y^{s/2} \frac{dy}{y},$$

where

$$\theta(z) := \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} = 1 + 2 \sum_{n=1}^{\infty} e^{\pi i n^2 z}$$

is Jacobi's theta series.

6. that Jacobi's theta series $\theta(z)$ uniformly converges for $\mathrm{Im}(z) \geq \delta$ for all $\delta > 0$ and fulfils

$$\theta(-1/z) = \sqrt{z/i}\, \theta(z).$$

7. that the completed zeta function holomorphically continues to $\mathbb{C} - \{0, 1\}$ and fulfils in this domain the functional equation

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s).$$

We will then be able to state precisely the following:

Conjecture 4. ("Riemann hypothesis")

All the zeroes of $\hat{\zeta}(s)$ *are on the line* $\mathrm{Re}(s) = 1/2$.

Exercice 3. (convergence for $\mathrm{Re}(s) > 1$**)**

1. *For* $i \geq 0$*, we denote* $N_i = \sum_{j=0}^{i} 2^j$*. For* $s \in \mathbb{R}$*,* $s > 1$ *and* $i \geq 0$*, show the estimation*
$$\sum_{n=1}^{N_i} \frac{1}{n^s} \leq \sum_{j=0}^{i} \left(\frac{1}{2^{s-1}}\right)^j.$$

2. *Deduce that the series* $\zeta(s)$ *converges uniformly on the domain* $\mathrm{Re}(s) > 1 + \delta$ *for all* $\delta > 0$*.*

3. *Show that the series* $\zeta(s)$ *diverges for* $\mathrm{Re}(s) \leq 1$*.*

Exercice 4. (continuation to $\mathrm{Re}(s) > 0$**)**

1. *We denote* $a_n(s) = \frac{1}{n^s} - \int_n^{n+1} t^{-s} dt$*. Show the estimation*
$$|a_n(s)| \leq \int_n^{n+1} |s|(t-n)n^{-\mathrm{Re}(s)-1} dt.$$

2. *Deduce that for* $\mathrm{Re}(s) > 0$*, we have the inequality*
$$\sum_{n>0} |a_n(s)| \leq \frac{|s|}{2}\zeta(\mathrm{Re}(s)+1).$$

3. *Denote* $f(s) = \sum_{n>0} a_n(s)$*. Show that* f *is holomorphic for* $\mathrm{Re}(s) > 0$ *and that for* $\mathrm{Re}(s) > 1$*, we have*
$$\zeta(s) = f(s) + \frac{1}{s-1}.$$

4. *Conclude that* $\zeta(s)$ *meromorphically continues to the half plane* $\mathrm{Re}(s) > 0$ *with a pole of order* 1 *at* $s = 1$*.*

Exercice 5. (Euler's identity) *We denote* $E(s)$ *the product* $\prod_p \zeta_p(s)$ *indexed by the prime numbers with* $\zeta_p(s) := \frac{1}{1-p^{-s}}$*.*

1. *Prove the formal identity*
$$\mathrm{Log}(E(s)) = \sum_p \sum_{n=1}^{\infty} \frac{1}{np^{ns}}.$$

2. *Let* $\delta > 0$*. For* $\mathrm{Re}(s) > 1+\delta$*, show the estimation*

$$\sum_p \sum_{n=1}^{\infty} \left|\frac{1}{np^{ns}}\right| \leq 2\sum_p \frac{1}{p^{1+\delta}} \leq 2\,\zeta(1+\delta).$$

3. *Deduce the uniform convergence of* $\mathrm{Log}(E(s))$ *for* $\mathrm{Re}(s) \geq 1+\delta$*, for all* $\delta > 0$*.*

4. *Use the expansion* $\frac{1}{1-x} = 1 + x + x^2 \cdots$ *to prove*

$$\prod_{p\leq N} \frac{1}{1-p^{-s}} = \sum_n{}' \frac{1}{n^s}$$

where $\sum'$ *is the sum on integers only divisible by primes* $p \leq N$*.*

5. *Deduce the estimation*

$$\left|\prod_{p\leq N} \frac{1}{1-p^{-s}} - \zeta(s)\right| \leq \sum_{n>N} \frac{1}{n^{1+\delta}}.$$

6. *Conclude that* $E(s)$ *converges to* $\zeta(s)$ *for* $\mathrm{Re}(s) > 1$*, i.e. one has* **Euler's identity**

$$\zeta(s) = \prod_p \zeta_p(s).$$

Exercice 6. (functional equation of the theta series) *Show that Jacobi's theta series*

$$\theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z}$$

converges absolutely and uniformly for $\mathrm{Im}(z) \geq \delta$ *for all* $\delta > 0$ *and fulfils*

$$\theta(-1/z) = \sqrt{\frac{z}{i}}\,\theta(z).$$

Exercice 7. (The Gamma function)

1. *Show that* $\Gamma(s) = \int_0^\infty e^{-y} y^s \frac{dy}{y}$ *is well defined for* $\mathrm{Re}(s) > 0$.

2. *Show that* $s \mapsto \int_1^\infty e^{-y} y^s \frac{dy}{y}$ *is holomorphic on* $\mathbb{C}$.

3. *by developping* $t \mapsto e^{-t}$ *in power series, show that for* $\mathrm{Re}(s) > 0$, *one has*

$$\int_0^1 e^{-y} y^s \frac{dy}{y} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{s+n}.$$

4. *Deduce that the function*

$$\tilde{\Gamma}(s) := \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \frac{1}{s+n} + \int_1^\infty e^{-y} y^s \frac{dy}{y}$$

holomorphically continues $\Gamma(s)$ *to* $\mathbb{C} - \mathbb{N}_-$. *For* $s \in \mathbb{C} - \mathbb{N}_-$, *we now denote* $\Gamma(s) = \tilde{\Gamma}(s)$.

5. *Show that for* $s \in \mathbb{C} - \mathbb{N}_-$, *one has the functional equation*

$$\Gamma(s+1) = s\Gamma(s).$$

Exercice 8. (Integral representation)

1. *Use the substitution* $y \mapsto \pi n^2 y$ *to get the equation*

$$\pi^{-s}\Gamma(s)\frac{1}{n^{2s}} = \int_0^\infty e^{-\pi n^2 y} y^s \frac{dy}{y}.$$

2. *Deduce the equality*

$$\pi^{-s}\Gamma(s)\zeta(2s) = \int_0^\infty \sum_{n=1}^{\infty} e^{-\pi n^2 y} y^s \frac{dy}{y}.$$

3. *Conclude that the completed zeta function*

$$\hat{\zeta}(s) = \zeta_\infty(s)\zeta(s)$$

where

$$\zeta_\infty(s) := \pi^{-s/2}\Gamma(s/2)$$

admits the **integral representation**

$$\hat{\zeta}(s) = \frac{1}{2}\int_0^\infty (\theta(iy) - 1)y^{s/2}\frac{dy}{y},$$

where

$$\theta(z) := \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z}$$

is Jacobi's theta function.

Exercice 9. (The functional equation for Riemann's zeta function) *Consider the following decomposition of the integral*

$$\int_0^\infty (\theta(iy) - 1)y^s\frac{dy}{y} = \int_0^1 (\theta(iy) - 1)y^s\frac{dy}{y} + \int_1^\infty (\theta(iy) - 1)y^s\frac{dy}{y}.$$

1. *Show that for $y \geq 1$, there exists a constant $B > 0$ such that one has the estimation*

$$|(\theta(iy) - 1)y^{s-1}| \leq Be^{-\pi y}y^{\mathrm{Re}(s)+1}y^{-2}.$$

2. *Show that for $y \geq 1$ and s in a compact, the real number $e^{-\pi y}y^{\mathrm{Re}(s)+1}$ is bounded by a constant independent of s.*

3. *Deduce the absolute and uniform convergence on all compacts of*

$$\int_1^\infty (\theta(iy) - 1)y^s\frac{dy}{y}.$$

4. *Use the change of variable $y \mapsto 1/y$ and the functional equation for theta to show that*

$$\frac{1}{2}\int_0^1 (\theta(iy) - 1)y^s\frac{dy}{y} = -\frac{1}{2s} + \frac{1}{2s-1} + \frac{1}{2}\int_1^\infty (\theta(iy) - 1)y^{-s+1/2}\frac{dy}{y}.$$

5. *Show that*

$$\frac{1}{2}\int_0^\infty (\theta(iy) - 1)y^s\frac{dy}{y} = -\frac{1}{2s} + \frac{1}{2s-1} + F(s)$$

where

$$F(s) = \frac{1}{2}\int_1^\infty [(\theta(iy) - 1)y^s + (\theta(iy) - 1)y^{1/2-s}]\frac{dy}{y}.$$

6. *Show that the function F is holomorphic on $\mathbb{C}$.*

7. *Show that $F(s) = F(1/2 - s)$.*

8. *Deduce that the zeta function meromorphically continues to $\mathbb{C}$ with simple poles in 0 and 1 and fulfils on $\mathbb{C} - \{0, 1\}$ the* **functional equation**

$$\hat{\zeta}(s) = \hat{\zeta}(1 - s).$$

Appendix B

The Riemann-Weil explicit formula

Riemann's zeta function

2. The zeroes and the explicit formula.

We will now study in greater details the zeroes of $\hat{\zeta}(s)$ showing that

1. its zeroes lie in the critical strip $0 \le \mathrm{Re}(s) \le 1$,
2. its zeroes and poles fulfil the "explicit formula", which shows that their distribution in complex plane is deeply related to properties of prime numbers.

Exercice 10. (The "critical strip")

1. *Use the Euler-product description of ζ to show that it has no zeroes for* $\mathrm{Re}(s) > 1$.
2. *Deduce that the zeroes of $\hat{\zeta}(s)$ are in the "critical strip"* $0 \le \mathrm{Re}(s) \le 1$.

Exercice 11. (Mellin's inversion theorem) *Let $f : \mathbb{R}_+^* \to \mathbb{C}$ be a $\mathcal{C}^\infty$ function such that for all $\sigma \in]\alpha, \beta[$, $x \mapsto f(x)x^{\sigma-1}$ is integrable.*

1. *Show that the function*

$$M(f, s) := \int_{\mathbb{R}_+^*} f(x)x^{s-1}dx$$

exists and is analytic in the strip $\mathrm{Re}(s) \in]\alpha, \beta[$.

2. *Using the change of variable $x = e^{2\pi u}$ and setting $s = \sigma + it$, show that*

$$M(f, \sigma + it) = 2\pi \int_{\mathbb{R}} g(u)e^{2i\pi tu}du,$$

i.e., that for σ fixed, $t \mapsto M(f, \sigma + it)$ if the Fourier transform of $g(u) = f(e^{2\pi u})e^{2\pi\sigma u}$.

3. *Use Fourier's inversion formula to show that*

$$f(x) = \lim_{t\to\infty} \frac{1}{2i\pi} \int_{\sigma-it}^{\sigma+it} M(f,s)x^{-s}ds.$$

Solution: See Patterson, Appendix 2.

Exercice 12. (The digamma function) *Denote* $\Psi(s) := \frac{\Gamma'(s)}{\Gamma(s)}$.

1. *Denote* $I_n(s) = \int_0^n (1-\frac{t}{n})^n t^{x-1}dt$. *Show that* $I_n(s)$ *tends to* $\Gamma(s)$ *uniformly on all compact subset for* $\mathrm{Re}(s) > 0$.

2. *Show that for all* s *with* $\mathrm{Re}(s) > 1$ *and all integer* $n \geq 1$, *one has*

$$I_n(s) = n^s \frac{n!}{\prod_{0\leq j\leq n}(s+j)} = n^s \frac{n!}{s(s+1)\dots(s+n)}.$$

3. *Compute* $\mathrm{dlog} I_n(s) := \frac{I_n'(s)}{I_n(s)}$.

4. *Deduce that*

$$\Psi(s) = \lim_{n\to\infty} \log(n) + \sum_{j=0}^{n} \int_1^\infty x^{s+j-1}dx.$$

5. *Deduce an estimation of*

$$\Psi_{\mathbb{R}}(s) := \mathrm{dlog}\zeta_\infty(s) = \frac{\zeta_\infty'(s)}{\zeta_\infty(s)}.$$

Exercice 13. (The "explicit formula") *Let* $\varphi \in \mathcal{C}_0^\infty([1,+\infty[)$, *that we extend by zero to function in* $\mathcal{C}_0^\infty([1,+\infty[)$ *and denote*

$$\Phi(s) = \int_1^\infty \varphi(y)y^{s-1}dy.$$

*The aim of this exercice is to prove the "***explicit formula***"*

$$\Phi(0) - \sum_{\hat{\zeta}(\rho)=0} \Phi(\rho) + \Phi(1) = W_\infty(\varphi) + \sum_p W_p(\varphi)$$

where

$$W_p(\varphi) = \log(p)\sum_{k\geq 1}\varphi(p^k) \qquad et \qquad W_\infty(\varphi) = \int_1^\infty \frac{\varphi(y)}{y-y^{-1}}dy.$$

1. *For $t > 0$, denote Γ_t the rectangle with vertical boundaries the lines $\mathrm{Re}(s) = -1$ and $\mathrm{Re}(s) = 2$ and horizontal boundaries the lines $\mathrm{Im}(s) = t$ and $\mathrm{Im}(s) = -t$. Orient Γ_t in the direct sense. Compute the complex integral*
$$I(t,\varphi) = \oint_{\Gamma_t} \Phi(s)\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)}ds$$
using the residues formula.

2. *Deduce that*
$$\lim_{t\to\infty} I(t,\varphi) = \Phi(0) - \sum_{\hat{\zeta}(\rho)=0} \Phi(\rho) + \Phi(1).$$

3. *Denote $D_t := \{\mathrm{Re}(s) = 2, -t \leq \mathrm{Im}(s) \leq t\}$ the right boundary of the rectangle Γ_t, and H_t the union of the two horizontal boundaries oriented as before. Use the function equation of $\hat{\zeta}(s)$ to show that $I(t,\varphi) = W(t,\varphi) + R(t,\varphi)$ où*
$$W(t,\varphi) := \oint_{D_t} [\Phi(s)+\Phi(1-s)]\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)}ds \qquad \text{and} \qquad R(t,\varphi) := \oint_{H_t} \Phi(s)\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)}ds.$$

4. *Show that $R(t,\varphi)$ tends to 0 when t tends to infinity.*

5. *Use the Euler product expansion of $\hat{\zeta}$ to show that*
$$\frac{\hat{\zeta}'(s)}{\hat{\zeta}(s)} = \frac{\zeta_\infty'}{\zeta_\infty}(s) - \sum_p \log p \sum_{k\geq 1} p^{-ks}.$$

6. *Use Mellin's inversion theorem (exercice 11) to show that*
$$\oint_{\mathrm{Re}(s)=2} \Phi(s)p^{-ks} = \varphi(p^k)$$
and deduce that
$$\oint_{\mathrm{Re}(s)=2} \Phi(s)\frac{\zeta'(s)}{\zeta(s)}ds = \sum_p \log(p) \sum_{k\geq 1} \varphi(p^k).$$

7. *Use the estimation of the digamma function given in exercice 12 to show that*
$$\oint_{\mathrm{Re}(s)=2} \Phi(s)\frac{\zeta'(s)}{\zeta(s)}ds = \int_1^\infty \frac{\varphi(y)}{y-y^{-1}}dy.$$

8. *Combine the two preceeding results to show that* $W(t,\varphi)$ *tends to* $W_\infty(\varphi)+\sum_p W_p(\varphi)$ *when* t *tends to infinity and conclude.*

Appendix C

Regularized products

IMPA - Rio de Janeiro — Master Course
Year 2009-2010

Regularized products

Let $(a_k)_{k\in I}$ be a countable family of complex numbers. We think of a_k as the logarithm of the non-zero complex number $\lambda_k = e^{a_k}$ whose argument is given by $\arg(\lambda_k) = \mathrm{Im}(a_k)$. The regularized product of these complex numbers λ_k (equiped with the given choice of argument) is said to exist if the Dirichlet series

$$\zeta_{(a_k)}(s) := \sum e^{-sa_k}$$

converges for $\mathrm{Re}(s) >> 0$ and analytically continues around zero. The regularized product of the λ_k's is then given by

$$\prod_k \lambda_k := e^{-\zeta'_{(a_k)}(0)}.$$

If one of the λ_k is zero, the regularized product is defined as being 0. We refer to Cartier's course [Car92] for basics, to Deninger [Den94] for Lerch's formula and to Jorgenson and Lang's book [JL93] for a general theory.

Euler-Maclaurin's formula is a basic tools to evaluate the analytic continuations of zeta functions.

Exercice 14. (Euler-Maclaurin formula) *Define the Bernouilli polynomials by the generating series*

$$\sum_{n=0}^{\infty} B_n(x)\frac{z^n}{n!} = \frac{ze^{xz}}{e^z - 1}.$$

1. *Compute $B_1(0)$ and $B_n(0)$ for any odd number $n \geq 3$.*
2. *Show that $B_1(x) = x - \frac{1}{2}$.*
3. *Show that $\partial_x B_n(x) = nB_{n-1}(x)$.*
4. *Let $f \in \mathcal{C}^n([0,1])$. By integrating by parts, show that*

$$f(0) = \int_0^1 f(x)dx + B_1(f(1) - f(0)) + \int_0^1 B_1(x)df(x).$$

5. *Using that the derivative of* $\frac{B_{n+1}}{(n+1)!}$ *equals* $\frac{B_n}{n!}$*, prove by induction on* n *that*

$$f(0) = \int_0^1 f(x)dx + \sum_{k=1}^{n} \frac{B_k}{k!} \left(f^{(k-1)}(1) - f^{(k-1)}(0)\right) + \frac{(-1)^{n-1}}{n!} \int_0^1 B_n(x) f^{(n)}(x) dx.$$

6. *If* $\bar{B}_n(x) := B_n(x - [x])$*, deduce from the above formula that for* $f \in \mathcal{C}^n([a,b])$ *with* $a, b \in \mathbb{Z}$*, we have* Euler-Maclaurin formula

$$\begin{aligned}\sum_{r=a}^{b-1} f(r) = \int_a^b f(x)dx + & \sum_{k=1}^{n} \frac{B_k}{k!} \left(f^{(k-1)}(b) - f^{(k-1)}(a)\right) \\ & + \tfrac{(-1)^n}{n!} \int_a^b \bar{B}_n(x) f^{(n)}(x) dx.\end{aligned}$$

7. *Applying Euler-Maclaurin formula to* $f(x) = x^{-s}$ *on* $[a,b] = [1,N]$ *and sending* N *to infinity, show that*

$$\begin{aligned}\zeta(s) = \tfrac{1}{s-1} + \tfrac{1}{2} \quad & + \sum_{k=2}^{n} \frac{B_k(0)}{k!} s(s+1)\dots(s+k-2) \\ & - \tfrac{1}{n!} s(s+1)\dots(s+n-1) \int_1^\infty \bar{B}_n(x) x^{-s-n} dx.\end{aligned}$$

8. *Show that the above formula gives an analytic continuation of* $\zeta(s)$ *for* $\mathrm{Re}(s) > 1-n$*.*

Exercice 15. (basic regularized products)

1. *Show that if the family* (a_k) *is finite and* $\lambda_k = e^{a_k}$*, there is an equality*

$$\prod_k \lambda_k = \prod_k \lambda_k.$$

2. *Derive from Euler-Maclaurin's formula that for* $\mathrm{Re}(s) > 0$*, we have*

$$\zeta(s) = \frac{1}{1-s} + \frac{1}{2} - s \int_1^\infty (x - [x] - \frac{1}{2}) x^{-s-1} dx.$$

3. *Deduce that*
$$\zeta'(0) = -1 - \int_1^\infty (x - [x] - \frac{1}{2})x^{-1}dx.$$

4. *Conclude that*
$$\prod_{n=1}^{\infty} n = \sqrt{2\pi}.$$

Exercice 16. (Hurwitz zeta functions) *For* $\mathrm{Re}(s) > 1$ *and* $z \notin -\mathbb{N}$, *define the Hurwitz zeta function by*

$$\zeta(s,z) := \sum_{n=0}^{\infty}(n+z)^{-s},\ \pi < \arg(n+z) \leq \pi.$$

1. *Show that* $\zeta(s,z)$ *converges normally on* $\mathrm{Re}(s) > 1+\delta$ *for* $\delta > 0$, *so that* $\zeta(s,z)$ *is holomorphic in* $\mathrm{Re}(s) > 1$.

2. *Show that* $\zeta(s,1) = \zeta(s)$ *and* $\zeta(s,v+1) = \zeta(s,v) - v^{-s}$.

3. *By using Euler-Maclaurin's formula for* $f(x) = (x+n)^{-s}$, $a = 0$, $b = N$ *and sending* N *to infinity, show that*

$$\zeta(s,z) = \tfrac{z^{1-s}}{s-1} + \tfrac{z^{-s}}{2} \quad + \sum_{k=2}^{n} \frac{B_k(0)}{k!} s(s+1)\dots(s+k-2)z^{-s-k+1}$$
$$-\tfrac{1}{n!}s(s+1)\dots(s+n-1)\int_1^\infty \bar{B}_n(x)(x+z)^{-s-n}dx.$$

4. *Conclude that the above formula gives a meromorphic continuation of* $\zeta(s,x)$ *with only pole* 1 *on* $\mathrm{Re}(s) > 1-n$ *for every real number* $x > 0$.

5. *Using the above formula for* $n = 1$, *show that*
$$\zeta(s,z) = \frac{z^{1-s}}{s-1} + \frac{z^{-s}}{2} - s\int_0^\infty (x - [x] - \frac{1}{2})(x+z)^{-s-1}dx.$$

6. *Deduce that* $\zeta(0,z) = \frac{1}{2} - z$.

7. *Show that* $\partial_s\zeta(0,z) = \log\Gamma(z) - \frac{1}{2}\log(2\pi)$.

8. *Conclude that*

$$\prod_{n=0}^{\infty}(z+n) = \frac{\sqrt{2\pi}}{\Gamma(z)}.$$

Exercice 17. (Lerch's formula) *Let $c \in \mathbb{C}^*$ and $z \in \mathbb{C}$ be given. The aim of this exercice is to prove* Lerch's formula

$$\prod_{n\in\mathbb{Z}} c(z+n) = \begin{cases} 1-e^{-2i\pi z} & \textit{if} \quad \mathrm{Im}(c)>0 \textit{ or } c>0,\, \mathrm{Im}(z)<0, \\ & \textit{or} \qquad\qquad c<0,\, \mathrm{Im}(z)\le 0; \\ 1-e^{2i\pi z} & \textit{if} \quad \mathrm{Im}(c)<0 \textit{ or } c>0,\, \mathrm{Im}(z)\ge 0, \\ & \textit{or} \qquad\qquad c<0,\, \mathrm{Im}(z)>0. \end{cases}$$

For $c \neq 0$, consider the "modified Hurwitz zeta function"

$$\zeta_c(s,z) := \sum_{n=0}^{\infty} \frac{1}{((c(z+n))^s}, \; -\pi < \arg(c(z+n)) \le \pi.$$

1. *Show that if $c \neq 0$ is not a negative real number,*

$$\mathrm{Arg}(c(z+n)) = \mathrm{Arg}(c) + \mathrm{Arg}(z+n) \textit{ for almost all } n \ge 0.$$

2. *Deduce that $\zeta_c(s,z) = c^{-s}\tilde{\zeta}(s,z)$ where $\tilde{\zeta}$ differs from the usual Hurwitz zeta function only by taking finitely many non-principale arguments in the definition of $(z+n)^{-s}$.*

3. *Conclude that*

$$\zeta_c(s,z) = \frac{1}{2} - z \textit{ and } \exp(-\partial_s\zeta(0,z) = c^{1/2-z}\left(\frac{\Gamma(z)}{\sqrt{2\pi}}\right)^{-1}.$$

4. *Similarly, show that if $c<0$, we have $\zeta_c(0,z) = \frac{1}{2} - z$ and*

$$\exp(-\partial_s\zeta_c(0,z)) = \begin{cases} |c|^{1/2-s}e^{i\pi(1/2-s)}\left(\frac{\Gamma(z)}{\sqrt{2\pi}}\right)^{-1} & \textit{if } \mathrm{Im}(z)\le 0, \\ |c|^{1/2-s}e^{-i\pi(1/2-s)}\left(\frac{\Gamma(z)}{\sqrt{2\pi}}\right)^{-1} & \textit{if } \mathrm{Im}(z)>0. \end{cases}$$

5. *Show the formula*

$$\frac{1}{z}\left(\frac{\Gamma(z)}{\sqrt{2\pi}}\right)^{-1}\left(\frac{\Gamma(-z)}{\sqrt{2\pi}}\right)^{-1} = i(e^{i\pi z} - e^{-i\pi z}).$$

6. *Conclude by a proof of Lerch's formula.*

Exercice 18. (Regularized determinants) *Let (V, Θ) be a vector space with endomorphism with discrete numerable spectrum. The regularized determinant of Θ is defined as the regularized product*

$$\det_{\infty}(\Theta|H) := \coprod \alpha.$$

Exercice 19. (non-archimedean local factors) *For p a finite prime, denote $\mathcal{R}_p = \mathcal{C}^{\infty}(\mathbb{R}_+^*/p^{\mathbb{Z}})$ and $\Theta = td/dt$.*

1. *Compute the spectrum of Θ on $\mathcal{R}_p$.*

2. *Using Lerch's formula, show that*

$$\zeta_p(s) = \det_{\infty}\left(\frac{1}{2\pi}(s - \Theta)|\mathcal{R}_p\right)^{-1}.$$

Exercice 20. (archimedean local factors) *Denote $\mathcal{R}_{\infty} = \mathbb{R}[-2t]$ and $\Theta = td/dt$.*

1. *Compute the spectrum of Θ on $\mathcal{R}_{\infty}$.*

2. *Using Lerch's formula, show that*

$$\zeta_{\infty}(s) = \det_{\infty}\left(\frac{1}{2\pi}(s - \Theta)|\mathcal{R}_{\infty}\right)^{-1}.$$

Appendix D

Cuspidal automorphic representations

Cuspidal automorphic representations

We will now treat the material necessary to prove the functional equation of a cuspidal automorphic representation of $\mathrm{GL}_n(\mathbb{A})$ as it is done at the end of the book of Godement and Jacquet [GJ72].

Exercice 21. *(Cuspical automorphic representations)*

Exercice 22. *(Convergence properties of Theta functions)*

Exercice 23. *(The functional equation of Theta functions)*

Exercice 24. *(The functional equation for coefficients)*

Exercice 25. *(The functional equation)*

Appendix E

Categories and universal properties

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Categories and universal properties

Definition 41. *A category C is given by the following data:*

1. *a class $\mathcal{O}b(C)$ called the objects of C,*
2. *for each pair of objects X, Y, a set $\mathrm{Hom}(X,Y)$ called the set of morphisms,*
3. *for each object X a morphism $\mathrm{id}_X \in \mathrm{Hom}(X,X)$ called the identity,*
4. *for each triple of objects X, Y, Z, a composition law for morphisms*

$$\circ : \mathrm{Hom}(X,Y) \times \mathrm{Hom}(Y,Z) \to \mathrm{Hom}(X,Z).$$

One supposes moreover that this composition law is associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$ and that the identity is a unit, i.e., $f \circ \mathrm{id} = f$ et $\mathrm{id} \circ f = f$.

Definition 42. *A universal property* [1] *for an object Y of C is an explicit description (compatible to morphisms) of $\mathrm{Hom}(X,Y)$ (or $\mathrm{Hom}(Y,X)$) for every object X of C.*

Example 4. *Here are some well known examples.*

1. Sets *whose objects are sets and morphisms are maps.*
2. Grp *whose objects are groups and morphisms are group morphisms.*
3. Grab *whose objects are abelian groups and morphisms are group morphisms.*
4. Rings *whose objets are commutative unitary rings and whose morphisms are ring morphisms.*

[1] Every object has exactly two universal properties, but we will usually only write the simplest one.

5. TOP *whose objets are topological spaces and morphisms are continuous maps.*

Principle 1. *(Grothendieck)*

Ce main interest in mathematics are not the mathematical objects,
but their relations
(i.e., morphisms).

Exercice 26. (Universal properties) *What is the universal property of*

1. *the empty set?*
2. *the one point set?*
3. $\mathbb{Z}$ *as a group?*
4. $\mathbb{Z}$ *as a commutative unitary ring?*
5. $\mathbb{Q}$ *as a commutative unitary ring?*
6. *the zero ring?*

To be more precise about universal properties, we need the notion of "morphism of categories".

Definition 43. *A (covariant) functor* $F : C \to C'$ *between two categories is given by the following data:*

1. *For each object* X *in* C*, an object* $F(X)$ *in* C'*,*
2. *For each morphism* $f : X \to Y$ *in* C*, a morphism* $F(f) : F(X) \to F(Y)$ *in* C'*.*

One supposes moreover that F *is compatible with composition, i.e.,* $F(f \circ g) = F(f) \circ F(g)$*, and with unit, i.e.,* $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$*.*

Definition 44. *A natural transformation* φ *between two functors* $F : C \to C'$ *and* $G : C \to C'$ *is given by the following data:*

1. *For each object* X *in* C*, a morphism* $\varphi_X : F(X) \to G(X)$*,*

such that if $f : X \to Y$ *is a morphism in* C*,* $G(f) \circ \varphi_X = \varphi_Y \circ F(f)$*.*

We can now improve definition 42 by the following.

Definition 45. *A universal property for an object Y of C is an explicit description of the functor* $\mathrm{Hom}(X,.) : C \to \textsc{Sets}$ *(or* $\mathrm{Hom}(.,X) : C \to \textsc{Sets}$*).*

The following triviality is at the heart of the understanding of what a universal property means.

Exercice 27. **(Ioneda's lemma)** *Let $C^\vee$ be the "category" whose objects are functors $F : C \to \textsc{Sets}$ and whose morphisms are natural transformations.*

1. *Show that there is a natural bijection*

$$\mathrm{Hom}_C(X,Y) \to \mathrm{Hom}_{C^\vee}(\mathrm{Hom}(X,.), \mathrm{Hom}(Y,.)).$$

2. *Deduce from this the an object X is determined by* $\mathrm{Hom}(X,.)$ *uniquely up to a unique isomorphism.*

Exercice 28. **(Free objects)** *Let C be a category whose objects are described by finite sets equiped with additional structures (for example,* Sets*,* Grp*,* Grab*,* Rings *or* Top*). Let X be a set. A free object of C on X is an object $L(X)$ of C such that for all object Z, one has a natural bijection*

$$\mathrm{Hom}(L(X), Z) \cong \mathrm{Hom}_{\mathrm{Ens}}(X, Z).$$

Let X be a given set. Describe explicitely

1. *the free group on X,*

2. *the free abelian group on X,*

3. *the free $\mathbb{R}$-module on X,*

4. *the free unitary commutative ring on X,*

5. *the free commutative unitary $\mathbb{C}$-algebra on X,*

6. *the free associative unitary $\mathbb{C}$-algebra on X.*

Exercice 29. (Products and sums) *The product (resp. the sum) of two objects X and Y is an object $X \times Y$ (resp. $X \coprod Y$, sometimes denoted $X \oplus Y$) such that for all object Z, there is a bijection natural in Z*

$$\begin{array}{c}\mathrm{Hom}(Z, X \times Y) \cong \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) \\ (\textit{resp.}\ \mathrm{Hom}(X \coprod Y, Z) \cong \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)).\end{array}$$

Explicitely describe

1. *the sum and product of two sets,*

2. *the sum and product of two abelian groups, and then of two groups,*

3. *the sum and product of two unitary associative rings.*

4. *the sum and product of two unitary commutative rings.*

Exercice 30. (Fibered products and amalgamed sums) *The fibered product (resp. amalgamed sum) of two morphims $f : X \to S$ and $g : Y \to S$ (resp. $f : S \to X$ and $f : S \to Y$) is an object $X \times_S Y$ (resp. $X \coprod_S Y$, sometimes denoted $X \oplus_S Y$) such that for all object Z, there is a natural bijection*

$$\begin{array}{rcl}\mathrm{Hom}(Z, X \times_S Y) & \cong & \{(h, k) \in \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y) | f \circ h = g \circ k\} \\ (\textit{resp.}\ \mathrm{Hom}(X \coprod_S Y, Z) & \cong & \{(h, k) \in \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z) | h \circ f = k \circ g\}).\end{array}$$

1. *Answer shortly the questions of the previous exercice with fibered products and amalgamed sums.*

2. *Let $a < b < c < d$ be three real numbers. Describe explicitely the sets*

$$]a, c[\times_{]a,d[}]b, d[\qquad \textit{and} \qquad]a, c[\coprod_{]b,c[}]b, d[.$$

3. *Describe explicitely the abelian group*

$$\mathbb{Z} \times_{\mathbb{Z}} \mathbb{Z}$$

where $f, g : \mathbb{Z} \to \mathbb{Z}$ are given by $f : n \mapsto 2n$ and $g : n \mapsto 3n$.

Exercice 31. (Projective limits) *Let $(I, \leq)$ be a partially ordered set. A projective system indexed by I is a family*

$$A_\bullet = ((A_i)_{i \in I}, (f_{i,j})_{i \leq j})$$

of objects and for each $i \leq j$, morphims $f_{i,j} : A_j \to A_i$ such that $f_{i,i} = \mathrm{id}_{A_i}$ and $f_{i,k} = f_{i,j} \circ f_{j,k}$ (such a data is a functor $A_\bullet : I \to C$ to the given category C). A projective limit for $A_\bullet$ is an object $\varprojlim_I A_\bullet$ such that for all object Z, one has a natural bijection

$$\mathrm{Hom}(Z, \varprojlim_I A_\bullet) \cong \varprojlim_I \mathrm{Hom}(A_i, Z)$$

where $\varprojlim_I \mathrm{Hom}(A_i, Z) \subset \prod_i \mathrm{Hom}(A_i, Z)$ denotes the families of morphisms h_i such that $f_{i,j} \circ h_j = h_i$. One defines inductive limits $\varinjlim A_\bullet$ in a similar way by interverting source and target of the morphims.

1. *Show that product and fibered products are particular cases of this construction.*

2. *Describe the ring $\varprojlim_n \mathbb{C}[X]/(X^n)$.*

3. *Describe the ring $\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.*

Exercice 32. (Localization) *Let A be a unitary commutative ring, $S \subset A$ a multiplicative subset (stable by multiplication and containing 1_A). The localization $A[S^{-1}]$ of A with respect to S is defined by the universal property*

$$\mathrm{Hom}_{\mathrm{RINGS}}(A[S^{-1}], B) = \{f \in \mathrm{Hom}_{\mathrm{RINGS}}(A, B) | \forall s \in S,\ f(s) \in B^\times\},$$

where $B^\times$ is the set of invertible elements in the ring B.

1. *Describe $\mathbb{Z}[1/2] := \mathbb{Z}[\{2^{\mathbb{Z}}\}^{-1}]$.*

2. *Is the morphism $\mathbb{Z} \to \mathbb{Z}[1/2]$ finite (i.e. is $\mathbb{Z}[1/2]$ a finitely generated $\mathbb{Z}$-module)? Of finite type (i.e. can $\mathbb{Z}[1/2]$ be described as a quotient of a polynomial ring over $\mathbb{Z}$ with a finite number of variables)?*

3. *Construct a morphism $\mathbb{Z}[1/2] \to \mathbb{Z}_3$ where $\mathbb{Z}_3$ are the 3-adiques integers defined in the previous exercice.*

4. *Does there exist a morphism* $\mathbb{Z}[1/3] \to \mathbb{Z}_3$*?*

Exercice 33. (Epimorphisms and monomorphisms) *An epimorphism (resp. monomorphism)* $p : X \to Y$ *(resp.* $i : X \to Y$*) is a morphims which is right (resp. left) cancelable, i.e. such that*

$$f \circ p = g \circ p \Rightarrow f = g$$
$$(\textit{resp. } i \circ f = i \circ g \Rightarrow f = g).$$

1. *Show that a group morphism is an epimorphism (resp. a monomorphism) if and only if it is surjective (resp. injective).*

2. *Is the morphism of unitary rings* $\mathbb{Z} \to \mathbb{Q}$ *a monomorphism? an epimorphism?*

3. *Give an example of morphism that is not an isomorphism but that is an epimorphism and a monomorphism.*

Appendix F

Sheaves and spaces

Sheaves and spaces

We will now treat the material necessary to the functorial study of geometric spaces (schemes, analytic spaces). We refer to the exercices of Harshorne's book [Har77], Section II.1 for a complete introduction to classical sheaves, and to the book of Kashiwara and Shapira [KS90] for sheaves on Grothendieck topologies.

Definition 46. *Let* Legos *be a category and* $U \in$ Legos *be an object. A sieve* $\mathcal{U}$ *on* X *is a subfunctor of* $\underline{U} := \mathrm{Hom}(., U)$*. A Grothendieck topology* τ *on a category* Legos *is the data, for each object* U *of families of sieves* $\mathcal{U} \subset \underline{U}$ *called covering sieves, verifying the following axioms:*

- *(Base change) If* $\mathcal{U}$ *is a covering sieve on* U *and* $f : V \to U$ *is a morphism then the inverse image* $f^*\mathcal{U} := \mathcal{U} \times_{\underline{U}} \underline{V}$ *is also a covering sieve on* V*.*
- *(Local character) Let* $\mathcal{U}$ *be a covering sieve on* U *and* $\mathcal{U}'$ *be any sieve on* U*. Suppose that for each object* W *of* Legos *and each arrow* $f : W \to U$ *in* $\mathcal{U}(W)$*, the pullback sieve* $f^*\mathcal{U}'$ *is a covering sieve on* W*. Then* $\mathcal{U}'$ *is a covering sieve on* U*.*
- *(Identity)* $\mathrm{Hom}(., X)$ *is a covering sieve on* X *for any object* X *in* Legos*.*

Exercice 34. (Relation with usual topology) *Let* (X, τ) *be a topological space. Let* $\mathcal{O}(X)$ *be the category whose objects are open subsets of* X *and whose morphisms are inclusions. A sieve* $\mathcal{U} \subset \underline{U}$ *is defined as a subfunctor the fulfils: if* W *is the union of opens* V *such that* $\mathcal{U}(V)$ *is non-empty, then* $W = U$*.*

1. *Show that the base change axiom is equivalent to the fact that if* $\{U_i\}$ *is a covering of* U *and* $V \subset U$ *then* $\{U_i \cap V\}$ *is a covering of* V*.*
2. *Show that the local character axiom is equivalent to the fact that if* U_i *covers* U *and* V_{ij}*,* $j \in J_i$ *covers* U_i *for each* i*, then the collection* V_{ij} *for all* i *and* j *should cover* U*.*

3. Explain the identity axiom in topological terms.

Definition 47. *A sheaf on a category* LEGOS *with Grothendieck topology* τ*, also called a space (if the Grothendieck topology is understood), is a contravariant functor (also called a presheaf)* $X : \text{LEGOS} \to \text{SETS}$ *such that for all objects* U *and all covering sieves* $\mathcal{U} \subset \underline{U}$*, the natural map*

$$X(\mathcal{U}) \to X(U)$$

is a bijection, i.e. the value of the sheaf is uniquely determined by its value on a covering.

Exercice 35. (Sheaf on a topological space) *Let* $\text{OUV}(X)$ *be the Grothendieck topology of a topological space. Show that a functor contravariant* $\mathcal{F} : \text{OUV}(X) \to \text{SETS}$ *is a sheaf if and only if for all covering* $\{U_i\}$ *of an open* U *and all family* $f_i \in \mathcal{F}(U_i)$ *of sections such that*

$$f_{i|U_i \cap U_j} = f_{j|U_i \cap U_j} \text{ for all } i, j,$$

there exists a unique $f \in \mathcal{F}(U)$ *such that* $f_{|U_i} = f_i$*.*

Exercice 36. (The punctual definition of smooth varieties)

Exercice 37. (The punctual definition of schemes)

Exercice 38. (The punctual definition of global analytic spaces)

Exercice 39. (The exponential exact sequence)

Exercice 40. (An exact sequence of étale sheaves)

Appendix G

Cohomology of sheaves

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Cohomology of sheaves

Exercice 41. (Free resolutions of modules)

Exercice 42. (Injective resolutions of abelian sheaves)

Exercice 43. (Derived functors)

Exercice 44. (De Rham cohomology)

Exercice 45. (Etale cohomology)

Exercice 46. ()

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