Towards the mathematics of quantum field theory

(master course and exercices[1])

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[1]In preparation. This document was typed in a continuous flux along the IMPA course and certainly contains many typos. These will be corrected during the academic year 2010-2011.

Contents

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The purpose of this course is to gradually introduce graduate students in mathematics to the mathematical models underlying modern particle physics experiments. The subject being quite large, we had to make some drastic choices. The reader is adviced to refer to the existing litterature on this subject for alternative approaches. These notes are presently in constant evolution, so please kindly inform me of any mistake, so that i can correct it.

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[1]Workshop's archives: http://www.math.jussieu.fr/∼fpaugam/GdT-physique/GdT-physique.html.

Mello, P. Dingoyan, T. Drummond, M. Dubois-Violette, M. Egeileh, G. Ginot, F. Helein, B. Keller, Y. Kosmann-Schwarzbach, J. Krasilshchik, L. Lazzarini, G. Maltsiniotis, V. Minerbe, J. Oesterlé, P. Schapira, G. Skandalis, B. Toen, W. van Suijlekom, A. Verbovetsky, A. Vinogradov, L. Vitagliano, J. Zinn-Justin. I thank E. de Faria and W. De Mello for giving me a preliminary version of their book on the mathematics of quantum field theory.

Course summary

The purpose of this course is to gradually introduce master students in mathematics to the mathematical models underlying modern particle physics experiments. We will start with some basic differential geometry, and give an account of (non-local) functional calculus with bosonic and fermionic variables, followed by many physical examples of lagrangian variational problems: classical mechanics, general relativity, electromagnetism and Yang-Mills theory. We will then describe the general methods of local variational calculus, so called “jet bundle formalism”. We will explain how to find fundamental solutions of partial differential operators with constant coefficients, and a description of various classical examples (wave, Klein-Gordon). The Dirac equation solution will involve some basic structure and representation theory of reductive groups, which are the building blocs for symmetry in physics. In the above setting, we will explain the geometric definition of classical matter and interaction particles from a mathematical viewpoint. We will give a short account of quantum mechanics following von Neumann and pass rapidely to the functional integral quantization, in a Dyson-Schwinger equation flavor. We will then describe perturbative expansions and give a brief overview of the mathematical methods underlying the renormalization procedure, including the Connes-Kreimer Hopf algebra approach to BPHZ renormalisation and Costello's description of Wilson's effective action method. If time permits, we will describe roughly the BRST/BV method for quantizing gauge theories.

Remark

Few prerequisites are necessary to follow this course (a bit of differential geometry and commutative algebra). This course will provide the interested students necessary bases if they wish to pursue on this topic and continue their studies for a thesis, for instance in Paris VI.

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9. F Paugam. Les mathématiques de la physique moderne. *(in preparation) http://people.math.jussieu.fr/∼fpaugam/*, 2009.

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Chapter 1

Introduction

The modern formulation of classical and quantum mechanics is based on very simple principles. One works with a pair (M, C) composed of a parameter space M for trajectories and a configuration space C for trajectories. The space of trajectories T is a space of maps

$$x : M \to C.$$

Example 1.1. Here are the main examples of this general setting:

- In point mechanics, M is an interval $[t_0, t_1]$ of real numbers that represent proper time, and C is a subspace of $\mathbb{R}^3$ (classical) or $\mathbb{R}^4$ (relativistic). For example, for a three dimensional pendulum, C could be a sphere $S^2 \subset \mathbb{R}^3$.

- In linear fields mechanics (electromagnetism, for example), M is usually a space of the form $[t_0, t_1] \times M_0$ for M_0 a subspace of $\mathbb{R}^3$ and $[t_0, t_1] \subset \mathbb{R}$, and C is usually a fiber bundle over $\mathbb{R}^4$ (for example the cotangent bundle $T^*\mathbb{R}^4 \cong \mathbb{R}^4 \times \mathbb{R}^4$ for electromagnetism).

- In nonlinear field mechanics, M can be a general space and C also (for example, in string theory, M is a Riemann surface and C is a variety of higher dimension).

1.1 Classical variational principle

The natural setting to study general variational problems is the following (the definition of the necessary notion of space, on which one can define a differential calculus, will be precised during the course; one can think as if it was a smooth variety).

Definition 1.1. A variational problem is made of the following data:

1. A space M called the parameter space for trajectories,
2. A space C called the configuration space for trajectories,
3. A morphism $\pi : C \to M$ (often supposed to be surjective),
4. A subspace $H \subset \Gamma(M, C)$ of the space of sections of π

   $$\Gamma(M, C) := \{x : M \to C,\ \pi \circ x = \mathrm{id}\},$$

   called the space of histories,
5. A functional (partial function with a domain of definition) $S : H \to A$ (where A is a space in rings that is often the real line $\mathbb{R}$ or $\mathbb{R}[[\hbar]]$) called the action functional.

The description of classical physical systems can often be based on the following.

Principle 1 (Least action principle)**.** The space T of physical trajectories of a given classical variational problem is the subspace T of the space of histories H given by critical points of the action S, i.e.,

$$T = \{x \in H,\ d_x S = 0\}.$$

One can apply this general approach to the case of the so-called local action functionals, that are given by expressions of the form

$$S(x) = \int_M L(x(m), \partial_\alpha x(m))dm$$

for $L : J^\infty(C) \to \mathbb{R}$ a function called the lagrangian density, that depends on variables x_0 (functional variables) in the fiber of $\pi : C \to M$ and of formal additional variables x_α that play the role of formal derivatives of x_0. This will give a very nice description of the space T of physical trajectories for the classical system, as a solution space for a (usually nonlinear) partial differential equation called the Euler-Lagrange system. We will spend a good part of this course in studying the general mathematical setting for the study and derivation of this equation.

Example 1.2. Consider the variational problem of classical Newtonian mechanics for a particle moving in a potential: one has $M = [0, 1]$, $C = [0, 1] \times \mathbb{R}^3$, $\Gamma(M, C) \cong \mathcal{C}^\infty([0, 1], \mathbb{R}^3)$ and for x_0 and x_1 fixed in $\mathbb{R}^3$, $H := \{x \in \Gamma(M, C),\ x(0) = x_0,\ x(1) = x_1\}$. One equips $\mathbb{R}^3$ with its standard scalar product $\langle ., . \rangle$. The action functional is given by

$$S(x) = \int_M \frac{1}{2} m \langle \partial_t x(t), \partial_t x(t) \rangle - V(x(t))dt$$

where $V : \mathbb{R}^3 \to \mathbb{R}$ is a smooth function, called the potential. If $\vec{h} : \mathbb{R} \to \mathbb{R}^3$ is a function tangent to H at some point x, one shows that $h(0) = h(1) = 0$. Defining

$$d_x S(\vec{h}) := \lim_{\epsilon \to 0} \frac{S(x + \epsilon h) - S(x)}{\epsilon},$$

one gets

$$d_x S(\vec{h}) = \int_0^1 \langle m\partial_t x, \partial_t \vec{h}\rangle - \langle d_x V(x), \vec{h}\rangle dt$$

and by integrating by parts using that $h(0) = h(1) = 0$, finally,

$$d_x S(\vec{h}) = \int_0^1 \langle -m\partial_t^2 x - d_x V(x), \vec{h}\rangle dt$$

The space of physical trajectories is thus the space of maps $x : \mathbb{R} \to \mathbb{R}^3$ such that

$$m.\partial_t^2 x = -V'(x).$$

This is the standard law of newtonian mechanics. For example, if $V = 0$, the physical trajectories are those with constant speed, which corresponds to galilean inertial bodies.

As you see, classical systems have a very neat formalization. The formalization of quantum systems is a bit less satisfactory for a mathematician.

1.2 Quantum systems

Definition 1.2. An observable is a functional $F : \Gamma(M, C) \to A$ for A an $\mathbb{R}$-algebra.

Now suppose that the projection $\pi : C \to M$ from configuration space to the space of parameters is a linear bundle and that $H = \Gamma(M, C)$. One can then define the twisted dual bundle $\pi^* : C^* := \mathrm{Hom}(C, \wedge^m T^* M) \to M$ (where m is the dimension of M). We now denote $X = \Gamma(M, C)$ and $X^* = \Gamma(M, C^*)$. A coordinate on X^* is called a source by physicists and often denoted J. We also think of it as a function $J : x \mapsto \int_M J_m(x(m))$ on X (that is well defined if J has compact support for example).

The description of quantum physical systems by physicists can be based on the following marvelous idea of Dirac/Feynman.

Principle 2 (Quantum histories principle)**.** In a quantum process, all histories in H are physical trajectories of the given system. The probability of observing a given trajectory x is proportional to $e^{iS(x)}$. More precisely, if F is an observable quantity, the partition

function (i.e., the generating function for the families of mean values) for F is given by the functional

$$Z_F : X^* \to A$$

given by the functional Fourier-Laplace transform of $F(x)e^{iS(x)}$ on the space of histories, i.e., by the formal expression

$$Z_F(J) := \int_H F(x)e^{iS(x)+J(x)}[dx]$$

where $[dx]$ represents a formal analog of a normalized measure on the space H of histories.

The functional integral has a precise mathematical definition only in very special cases (perturbative expansions) and giving fully this definition for physically relevant theories requires the renormalization process, which can be the subject of a one year advanced master course in particle physics. It is thus a hard work that will be partially treated in these notes.

Example 1.3. Consider as before the case of a free particle $x : [0,1] \to \mathbb{R}^3$ of mass 1 moving in space with the action functional

$$S(x) = \int_0^1 \frac{1}{2}\langle \partial x, \partial x\rangle - k^2\|x\|^2.$$

If we suppose that $x(0) = x(1) = 0$, we can also write this as

$$S(x) = -\int_0^1 \frac{1}{2}\langle (\partial^2 + k^2)x, x\rangle.$$

The operator $\partial^2 + k^2 : \mathcal{S}(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$, that also gives the differential equation for classical trajectories, has a Schwarz kernel that is a distribution of two variables $P \in \mathcal{S}'(\mathbb{R} \times \mathbb{R})$ called the propagator. The generating functional for the corresponding quantum system is then defined by

$$Z(J) := e^{\frac{i}{2}\langle J,P,J\rangle}$$

for $J : [0,1] \to (\mathbb{R}^3)^* \cong \mathbb{R}^3$ a section of the dual bundle. This definition has been found by making an analogy between the given functional integral and the Fourier-Laplace transform of a finite dimensional gaussian, which was intuitively explained by Feynman in his thesis by the process of discretization and restriction to a compact time interval. This discretization process if a very powerful tool to understand the physicists' formal computations with functional integrals.

The quantum behavior of a "free" (here, the quadratic potential is considered as a free contribution to the particle movement) particle is thus completely determined by its classical behavior, and it is already interesting because it is used to prepare machines before (and to study the result of) an experiment in particle physics (in and out free particles). However, more interesting is the interacting situation, which corresponds in our example to a more general potential $V : \mathbb{R}^3 \to \mathbb{R}$, giving rise to the action

$$S(x) = \int_0^1 \frac{1}{2} m \|\partial x\|^2 - k^2 \|x\|^2 - V(x)$$

1.3 Quantum variational principle

Another non-perturbative approach to an understanding of the functional integral is given by Schwinger's quantum variational principle. It is based on the notion of functional derivative that will be treated in detail in this course. Its main interest is that it gives a way to understand bosonic and fermionic variables on equal footing.

Principle 3 (Quantum variational principle)**.** The partition function of a quantum system

$$Z := Z_1 : X^* \to A$$

is a function that fulfils the Dyson-Schwinger quantum equations of motion

$$\left(d_{\left[-i\frac{\delta}{\delta J}\right]} S.\vec{\delta}_m + J(m) \right) Z(J) = 0,$$

which looks as

$$\left(\frac{\delta S}{\delta x(m)} \left[-i \frac{\delta}{\delta J} \right] + J(m) \right) Z(J) = 0$$

in physics books.

This expression is not well defined for the two following reasons:

1. δ_m denotes Dirac's delta function at $m \in M$, which can't be used to define a vector field $\vec{\delta}_m$ on X. This problem is easily solved by replacing $\vec{\delta}_m$ by any vector field on X.
2. $\frac{\delta}{\delta J}$ denotes a kind of universal differential operator on X^*, and the process of replacing a trajectory x in $d_x S$ by this operator has to be explained more precisely, because it usually involves expressions of the form

$$\frac{\delta^2 Z}{\delta J(m)^2} := d_{.}(d_{.} Z.\vec{\delta}_m).\vec{\delta}_m,$$

i.e., second functional derivatives, which are even worsely ill defined. The replacement of every δ_m by any vector field on X will overcome this problem of definition, by the use of some kind of distributional differential operators.

Part I

Mathematical preliminaries

Chapter 2

Differential geometry and global functional calculus

This chapter's aim is to complete the reader's knowledge on differential geometry, so that he can understand the forthcoming, more physical chapters. The study of spaces of trajectories and space of fields, that are the central object of modern physics, and of functions on them, often called functionals, imposes a functorial approach to the notion of space. We will call global functional calculus what physicists call non-local functional calculus.

We will use basic notions of category theory. These can be learned using wikipedia and the exercice sheet in appendix.

2.1 What is a space?

2.1.1 Atlases

The above problem of definition in mathematics was first historically solved (Euclid, 300 B.C.) by saying that a space is a set X of points, equiped with additional structures.

In the geometrical models for spaces that appear in physics, one can often formalize these structures by using the notion of atlas with respect to a category of simple building blocs, that we will call here LEGOS. This method however does not work for fermionic variables.

Definition 2.1. Let LEGOS be a category whose objects are (an explicit family of) sets with natural additional structures. Let X be a set. An atlas for X on the category LEGOS

is the data of a family

$$f_i : U_i \to X,\ i \in I,\ U_i \in \textsc{Legos}$$

of injective maps of sets such that

$$\coprod_{i\in I} f_i : \coprod_{i\in I} U_i \to X$$

is surjective and the bijections

$$f_i \circ f_j^{-1} : U_i \cap U_j \to U_i \cap U_j$$

are induced by isomorphisms in the category Legos. A pair $(X, \{f_i\}_{i\in I})$ composed of a set and an atlas is called a variety modeled on the category Legos.

Normally, one defines smooth varieties as being given by countable atlases (because it allows the use of partitions of unity). Remark that, using this definition, it is hard to define a morphism of varieties, and to compare varieties, for example, saying if they are isomorphic.

Example 2.1.

1. Differential varieties can be defined as varieties modeled on the category of open subsets U of $\mathbb{R}^n$ for varying n, equiped with smooth maps.

2. Smooth analytic varieties can be defined as varieties modeled on the category of open subsets U of $\mathbb{C}^n$ for varying n, equiped with analytic maps.

3. Algebraic varieties can be defined as varieties modeled on the category of (Zariski open subsets in) prime ideal spectra $\mathrm{Spec}(A)$ of rings (set of prime ideals, equiped with the Zariski topology, generated by subsets $U_f = \{\mathfrak{p} \subset A,\ f \notin \mathfrak{p}\}$), with morphisms given by maps that are induced by ring morphisms.

2.1.2 Ringed spaces

Remark that in all these cases, there is a natural way to associate to an open subset U of a variety X, a ring $\mathcal{O}(U)$ of functions on U. The pair $(X, \mathcal{O})$ is called a ringed topological space, and gives another viewpoint of spaces, that is more intrinsic (it is then easy to define what is a morphism of variety, contrary to what one has to do with the atlas definition). We will sometimes use this viewpoint of spaces, particularly for functional constructions.

2.1.3 Parametrized families of points

The problem with the above definitions of spaces is that they are not well adapted to the study of spaces of functions (that are the spaces of main interest for physics) like for example the space $\mathcal{C}^\infty_c(\mathbb{R},\mathbb{R})$ of smooth functions from $\mathbb{R}$ to itself.

Even if one can model spaces of these kind on Banach or Frechet spaces (normed spaces that appear in functional analysis), this way of formalizing these, even if it is very powerfull for some aims (mainly in the setting of linear or perturbatively linear problems), has the drawback of introducing some technical problems related to convergence properties, that make them not well adapted to general non-linear problems that appear in physics.

Physicists usually study functionals, i.e., functions on functional spaces, for example:

$$F : \mathcal{C}^\infty(\mathbb{R},\mathbb{R}) \to \mathbb{R},$$

and use usual notions of differential calculus on these. For example, they compute the derivative of such a functional by using a parametrized family f_t of functions for $t \in [0,1]$, i.e., a smooth function

$$f : [0,1] \times \mathbb{R} \to \mathbb{R},$$

with $f_0 = g$ and define

$$\frac{\partial F}{\partial \vec{f_t}}(g) := \left.\frac{\partial F(f_t)}{\partial t}\right|_{t=0}$$

as the differential of the given functional along the path in functional space.

One can directly relate this example, ubiquitous in physics (as for example in the computation of critical points of a lagrangian action functional), to another definition of space, based on the notion of parametrized family of points, that is due to Grothendieck [AGV73], and was also used (in a particular case) by Souriau and his school (see [Sou97] and [IZ99]) in physics. To illustrate this notion, one says that *a family of points of* $\mathbb{R}$ *parametrized by an open subset* $U \subset \mathbb{R}^n$ *for some* n is simply a smooth map

$$x : U \to \mathbb{R}.$$

Other important motivations for studying spaces defined using the notion of parametrized families of points, is that they are also very useful for the study of (algebraic or Lie) symmetry groups in physics, and that one can not overcome them for the quantum study of fermionic systems (i.e., mater particles, for example, electrons). Usual functional analysis methods simply don't work in this setting. To be clear,

this is the only way one can understand mathematically properly the computations of physicists with functionals of fermionic variables.

2.1.4 Punctual smooth spaces

The content of previous section justifies the jump into abstraction that we will demand to the reader right now. To illustrate simply this viewpoint, we will describe this approach for smooth spaces. Let LEGOS be the category of open subsets $U \subset \mathbb{R}^n$ for varying n with smooth maps between them. There is a so-called Grothendieck topology τ on this category that is simply given by the usual topology on open subsets of $\mathbb{R}^n$.

Remark that if $X \subset \mathbb{R}^n$ is a fixed open subset, the (contravariant) functor of (parametrized families of) points

$$\underline{X} := \mathrm{Hom}(., X) : \textsc{Legos} \to \textsc{Sets}$$

that sends an open $U \subset \mathbb{R}^m$ to the set $\mathrm{Hom}(U, X)$ of smooth morphisms from U to X is fully faithful (this is a particular case of Yoneda's lemma), which means that you can find back uniquely X up to a unique isomorphism if you know $\underline{X}$ (in some sense, $\underline{X}$ is the universal property of X in LEGOS; see the exercice sheet on categories and universal properties for a better understanding of this fact).

We don't define for the moment the general notion of Grothendieck topology, since it is not useful for our particular example.

Definition 2.2. A smooth punctual space, also called a smooth space, is a functor

$$X : \textsc{Legos} \to \textsc{Sets}$$

that is a sheaf for the given topology on open subsets, meaning that for every covering $U = \cup_{i \in I} U_i$ of an open subset of $\mathbb{R}^n$ for some n by open subsets, the sequence

$$X(U) \longrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \cap U_j)$$

is exact. We denote SPACES the category of smooth spaces.

Remark that an open subset U of $\mathbb{R}^n$ of course defines a smooth space $\underline{U}$ in this sense. The sheaf assumption is a unique gluing assumption that means that a family of elements in $X(U_i)$ whose restrictions on double intersections $U_i \cap U_j$ are all equal can be glued to a unique element of $X(U)$. It is imposed to be sure that the pasting of spaces that correspond to usual open subsets are identified with the space associated to the usual pasting of the given open subsets. More precisely, if $U_1, U_2 \subset U$ are two open subsets, one has a natural isomorphism

$$\underline{U_1 \cup U_2} = \underline{U_1 \coprod_{U_1 \cap U_2} U_2} \xrightarrow{\sim} \underline{U_1} \coprod_{\underline{U_1} \times_{\underline{U}} \underline{U_2}} \underline{U_2}$$

between the space associated to the union of U_1 and U_2 in U and the pasting of the spaces $\underline{U_1}$ and $\underline{U_2}$ along the space $\underline{U}$. This would not be true if we took the pasting (coproduct) in the whole category of functors (presheaves).

Of course, the functor

$$\text{LEGOS} \to \text{SPACES}$$

is fully faithful (this is Yoneda's lemma), so that spaces generalize legos. One can easily define open subspaces in smooth spaces.

Definition 2.3. A space morphism $f : U \to X$ is called an open immersion if

1. it is injective on points, i.e., $f_V : U(V) \to X(V)$ is injective for all $V \in$ LEGOS, and
2. it is universally open, i.e., for every $V \in$ LEGOS and every map $i : V \to X$, the pull-back morphism $i^*f : V \times_X U \to V$ given by the fiber product

$$\begin{tikzcd} V \times_X U \arrow[r] \arrow[d, "i^*f"] & U \arrow[d, "f"] \\ V \arrow[r, "i"] & X \end{tikzcd}$$

is an inclusion of open subsets.

Varieties are then defined as particular types of spaces that are locally isomorphic to objects of LEGOS.

Definition 2.4. A smooth space X : LEGOS $\to$ SETS is called a variety if it admits a covering $U_i \subset X$ by open subspaces that are isomorphic to legos (i.e., open subsets of $\mathbb{R}^n$ for varying n).

It is easy to show that the embedding of the category LEGOS in the category of smooth spaces extends to an embedding of the category VAR of smooth varieties (in the usual sense) in smooth spaces (whose image is the category of smooth varieties in our sense) by the functor

$$\begin{array}{ccc} \text{VAR} & \to & \text{SPACES} \\ X & \mapsto & [\underline{X} := \mathrm{Hom}_{\text{VAR}}(.,X) : \text{LEGOS} \to \text{SETS}] \end{array}$$

of its parametrized points by any open subset of $\mathbb{R}^n$ for varying n. Indeed, one can cover the variety X by standard open subsets U_i, and this gives open subspaces $\underline{U_i}$, whose pasting is exactly the space $\underline{X}$. The verification for morphisms is also formal but we don't give it since we did not define morphisms of varieties defined by an atlas.

However, smooth spaces are a much bigger category that contain for example singular quotient spaces like the "noncommutative torus" $\mathbb{R}/\theta\mathbb{Z}$ for θ irrational. These kind of

spaces naturally arise in physical theories with symmetries (reduction of gauge theories). For us, the main advantage of the category of spaces is that mapping spaces are also naturally spaces.

Definition 2.5. If X and Y are two spaces, the sheaf associated to the presheaf

$$U \mapsto \mathrm{Hom}_U(X \times U, Y \times U) := \mathrm{Hom}(X \times U, Y)$$

of morphisms from X to Y parametrized by U is called the mapping space and denoted $\underline{\mathrm{Hom}}(X, Y)$.

In particular, if $\pi : C \to M$ is a map of spaces, as the one used to define a lagrangian variational problem, on can define the space $\underline{\Gamma}(M, C)$ of its sections by writing

$$\underline{\Gamma}(M, C) := \{s \in \underline{\mathrm{Hom}}(M, C),\ \pi \circ s = \mathrm{id}\}.$$

2.1.5 General punctual spaces

The definition we gave above for smooth spaces and varieties generalizes straigtforwardly to any category Legos with Grothendieck topology τ (that is subcanonical meaning that $\mathrm{Hom}(., X)$ is a sheaf for τ for every lego X). We will shortly explain this here, since:

- it is the only way one can understand mathematically superspaces, that are at the basis of fermionic functional calculus (the category Legos is then the category of super-algebras),
- it is also very useful to understand local functional calculus in a very concise way (the category Legos is then the category of $\mathcal{D}_M$-algebras, i.e., algebras in the monoidal category of $\mathcal{D}$-modules on the space M of parameters for trajectories).

Definition 2.6. Let Legos be a category with fiber products. A Grothendieck topology τ on Legos is the data, for every lego U, of covering families $\{f_i : U_i \to U\}_{i \in I}$, fulfilling:

1. (Base change) For every morphism $f : V \to U$ and every covering family $\{f_i : U_i \to U\}$ of U, $f \times_U f_i : V \times_U V_i \to V$ is a covering family.
2. (Local character) If $\{f_i : U_i \to U\}$ is a covering family and $\{f_{i,j} : U_{i,j} \to U_i\}$ are covering families, then $\{f_i \circ f_{i,j} : U_{i,j} \to U\}$ is a covering family.
3. (Isomorphisms) If $f : U \to V$ is an isomorphism, it is a covering family.

A space modeled on Legos for the Grothendieck topology τ is simply contravariant functor

$$X : \text{Legos} \to \text{Sets}$$

that is a sheaf for τ, i.e., such that for each covering family $\{f_i : U_i \to U\}$ the sequence

$$X(U) \longrightarrow \prod_i X(U_i) \rightrightarrows \prod_{i,j} X(U_i \times_U U_j)$$

is exact.

If one generalizes to homotopical functors on model categories with Grothendieck topology as in section 4 (see also [TV08b] and [TV08a]), any space used by physicist to formalize a field theory becomes a space in our sense. Remark that these methods are necessary, even for everyday life of a theoretist of particle physics, because they are at the heart of the BRST-BV theory that is necessary to prove renormalizability of non-commutative Yang-Mills theory, i.e., of the standard model of elementary particules, and particularly weak and strong interactions.

2.2 Differential forms: a contravariant construction

Contravariant geometrical constructions on Legos can often be extended to Spaces. To illustrate this fact, we will define differential forms on spaces. The main motivation for the introduction of differential forms is that it gives a way to study a natural extension of integration theory to the setting of varieties, or more generally of spaces. This comes from the fact that a differential form is a local object. See the first chapter of Taylor [Tay96] for a very nice introduction.

2.2.1 Local differential forms

As before, we let Legos be the category of open subsets of $\mathbb{R}^n$ for varying n equiped with its usual topology. If $U \subset \mathbb{R}^n$ is a lego, we denote $TU := U \times \mathbb{R}^n$, $T^*U := U \times (\mathbb{R}^n)^*$ and $\mathbb{R}_U := U \times \mathbb{R}$. There is a natural fiberwise duality map

$$\begin{array}{ccccc} TU & \underset{U}{\times} & T^*U & \to & \mathbb{R}_U \\ (x, \vec{v}) & , & (x, \omega) & \mapsto & (x, \omega(\vec{v})) \end{array}$$

Any map $f : U \to V$ for $V \subset \mathbb{R}^m$ induces a map called the differential

$$Df : TU \to TV$$

defined by $Df(x,\vec{v}) := (f(x), D_x f.\vec{v})$ where

$$D_x f.\vec{v} := \left[\sum_{i=1}^n \frac{\partial f_j}{\partial x_i}.v_i\right]_{j=1,\dots,m} \in \mathbb{R}^m.$$

Differential 1-forms on Legos are given by the association

$$U \mapsto \Omega^1(U) := \Gamma(U, T^*U),$$

which can be extended to a functor

$$\Omega^1 : \textsc{Legos} \to \textsc{Vect}_{\mathbb{R}}$$

by saying that if $f : U \to V$ is a map,

$$f^* : \Omega^1(V) \to \Omega^1(U)$$

is the pull-back of differential forms defined for $\omega \in \Omega^1(V)$ by

$$f^*\omega(x) := (\omega \circ f)(x) \circ D_x f \in (\mathbb{R}^n)^*.$$

In coordinates, if dx_j is a basis for $(\mathbb{R}^m)^*$, one writes

$$\omega = \sum a_j(x) dx_j$$

for a_j smooth functions on V and

$$f^*\omega(x) = \sum_{j,k} a_j(f(x)) \frac{\partial f_j}{\partial y_k}(x) dy_k.$$

Moreover, one can paste uniquely compatible differential forms on a given covering, meaning that the functor

$$\Omega^1 : \textsc{Legos} \to \textsc{Vect}_{\mathbb{R}}$$

is actually a sheaf.

More generally, if U is a lego, one defines k-differential forms on U as the exterior product

$$\Omega^k(U) := \wedge^k_{\mathcal{C}^\infty(U)} \Omega^1(U).$$

The exterior differential is defined as the map

$$\begin{array}{cccc} d: & \mathcal{C}^\infty(U) & \to & \Omega^1(U) \\ & f & \mapsto & df := \sum_i \frac{\partial f}{\partial x_i} dx_i \end{array}$$

and more generally

$$\begin{array}{cccc} d: & \Omega^k(U) & \to & \Omega^{k+1}(U) \\ & \omega = a(x)dx_{i_1} \wedge \ldots, \wedge dx_{i_k} & \mapsto & d\omega := \sum_i \frac{\partial a}{\partial x_i} dx_i \wedge dx_{i_1} \wedge \cdots \wedge dx_{i_k} \end{array}$$

and extending by linearity. One can check that $d^2 = 0$. We thus get a sheaf

$$\Omega^k : \textsc{Legos} \to \textsc{Vect}_{\mathbb{R}}$$

and the so-called de Rham complex of a lego

$$0 \to \mathcal{C}^\infty(U) \to \Omega^1(U) \xrightarrow{d} \ldots \xrightarrow{d} \Omega^k(U) \ldots$$

2.2.2 Differential forms on spaces

Definition 2.7. Let X be a space. A differential form ω on X is given by the following data:

1. for each morphism $x : U \to X$ from a lego U to X (also called a U-point of X), a differential form on U denoted $[x^*\omega]$ (because one thinks of it as the pull-back of the given differential form on X along the point x),

2. for each morphism of points, i.e., commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "x"'] & & V \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

   for U and V two legos, of an equality $f^*[y^*\omega] = [x^*\omega]$ of usual differential forms on legos.

If X is a variety, one gets the usual notion of differential form, since the notion is local and one can restrict to legos on which the result is clear using the differential form $\mathrm{id}^*\omega$ for $\mathrm{id} : U \to U$ the identity map of the given lego.

Remark that this construction can be generalized to any sheaf $\Omega : \textsc{Legos} \to C$, for C a category, meaning that if X is a space, one can define the notion of Ω-object on X by using exactly the same construction as above. This means that

to define a contravariant notion on spaces,
it is enough to define it on legos,

and actually, any geometric contravariant construction on Legos can be extended to Spaces. To illustrate the powerful generality of the above principle, we define the notion of bundle on a space, which is associated to the functor

$$\Omega = \textsc{Bun} : \textsc{Legos} \to \textsc{Cat}$$

that sends a lego to the category of bundles on it (there are some problems with saying that this is a functor, but we prefer to skip these technicalities here).

Definition 2.8. A (locally trivial) bundle with fiber a given space F on a lego U is a space $\pi : E \to U$ over U such that there exists a covering $\{U_i\}$ of U such that $\pi_{|U_i} : E_{|U_i} \to U_i$ is isomorphic to the canonical projection $\pi : U_i \times F \to U_i$.

Remark that if the space F giving the fiber is a group space (i.e., a space whose points $F : \textsc{Legos} \to \textsc{Sets}$ naturally factorize through the category Grp of groups), one can ask the comparison isomorphisms between p_{U_i} and p_{U_j} to respect this additional structure. This gives the notion of group bundle. For $F = \mathbb{R}^n$, we could ask also the comparison isomorphisms to be linear. This gives the notion of vector bundle. In any of these settings, one can define the notion of bundle on a general space by the following.

Definition 2.9. Let X and F be two differential spaces. A bundle over X with fiber F is given by the data, for each morphism $x : U \to X$ between a lego U and X, of a bundle $[x^*E]$ over U (which is thought as the pull-back bundle of the given bundle E along the map x), in a way that is consistent with pull-back of bundles along morphisms of legos, meaning that for each morphism of points, i.e., commutative diagram

$$\begin{tikzcd} U \arrow[rr, "f"] \arrow[dr, "x"'] & & V \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

for U and V two legos, there is an isomorphism

$$f^*[y^*E] \cong [x^*E].$$

2.3 Algebras and differential calculus

The notion of observable quantity is central in modern physics since it is the most intuitive bridge between the mathematical models of classical and quantum physics. It is also

deeply related with the notion of coordinate. One thus earns a lot in translating the notions of differential geometry in the algebraic language of observable algebras. Moreover, this cannot be avoided when one works with fermionic variables. We will now introduce a slight modification in our definition of space, that will make these algebraic constructions compatible with ours.

2.3.1 Spaces and algebras

Consider the smooth functions functor

$$\begin{array}{rccc} \mathcal{C}^\infty : & \textsc{Legos} & \to & \textsc{Alg}_{\mathbb{R}} \\ & U & \mapsto & \mathcal{C}^\infty(U) \end{array}$$

sending an open subset of $\mathbb{R}^n$ for some n to the real algebra of smooth real valued functions on U. It is a fully faithful embedding of the category $\textsc{Legos}$ in the category $\textsc{Alg}_{\mathbb{R}}$ of real algebras. Consider also the real spectrum functor

$$\begin{array}{rccc} \mathrm{Spec}_{\mathbb{R}} : & \textsc{Alg}_{\mathbb{R}} & \to & \textsc{Top} \\ & A & \mapsto & \mathrm{Spec}_{\mathbb{R}}(A) := \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(A, \mathbb{R}) \end{array}$$

that sends a real algebra to the set of morphisms from A to $\mathbb{R}$ equiped with the Zariski topology, that is generated by subsets of the form $\mathrm{Spec}_{\mathbb{R}}(A[f^{-1}]) := \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(A[f^{-1}], \mathbb{R})$, for $f \in A$.

Theorem 1. *The smooth function functor $\mathcal{C}^\infty$ is a fully faithful embedding of the category $\textsc{Legos}$ in the category $\textsc{Alg}_{\mathbb{R}}$. The real spectrum functor restricted to the image $\textsc{Alg}_{legos,\mathbb{R}}$ of $\mathcal{C}^\infty$ is a quasi-inverse*

$$\mathrm{Spec}_{\mathbb{R}} : \textsc{Alg}_{legos,\mathbb{R}} \xrightarrow{\sim} \textsc{Legos}$$

of the smooth function functor.

Proof. We only give the main steps of the proof, referring to Nestruev [Nes03] for more details. The main difficulty is to show that if $U \subset \mathbb{R}^n$, the natural map

$$U \to \mathrm{Spec}_{\mathbb{R}}(\mathcal{C}^\infty(U))$$

is a homeomorphism. It is injective since two different points can be separated by a smooth (say simply linear) function. Surjectivity follows from the fact, proved in forthcoming lemma 1, that there exists a smooth function f on U such that for any $\lambda \in \mathbb{R}$, the set $f^{-1}(\lambda)$ is compact. Given such a function f and a point $p : \mathcal{C}^\infty(U) \to \mathbb{R}$ of the real spectrum, the set $L = f^{-1}(\lambda)$ where $\lambda = p(f)$, is compact. Assume that p does not

correspond to a point $x \in U$. Then for any point $a \in U$, there exists a function f_a for which $f_a(a) \neq p(f_a)$. The sets

$$U_a = \{x \in U,\ f_a(x) \neq p(f_a)\},\ a \in L,$$

constitute an open covering of L. Since L is compact, we can choose a finite subcovering $U_{a_1}, \ldots, U_{a_m}$. Consider the function

$$g = (f - p(f))^2 + \sum_{i=1}^{m} (f_{a_i} - p(f_{a_i}))^2.$$

This is a smooth nonvanishing function on U, so that $1/g \in \mathcal{C}^\infty(U)$. Moreover, since p is an $\mathbb{R}$-algebra morphism, one has $p(g) = 0$. Since p is a unital $\mathbb{R}$-algebra morphism, we have

$$1 = p(1) = p(g.1/g) = p(g).p(1/g) = 0$$

which gives a contradiction. To get a bijection between open subsets of U of $\mathrm{Spec}_{\mathbb{R}}(\mathcal{C}^\infty(U))$ is then easy: by forthcoming lemma 2, there exists a function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f(x) = 0$ for $x \notin U$ and $f(x) > 0$ for $x \in U$. Then the Zariski open subset $\mathrm{Spec}_{\mathbb{R}}(\mathcal{C}^\infty(U)[f^{-1}])$ is identified with U by the above bijection. A map $g : U \to V$ is uniquely determined by the morphism of algebras $\mathcal{C}^\infty(g) : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(U)$, which is well defined since g is smooth if and only if its composition with any function $h : V \to \mathbb{R}$ is smooth. The same argument shows that if $f : \mathcal{C}^\infty(V) \to \mathcal{C}^\infty(U)$ is a morphism of algebra, the corresponding map $U \to V$ on the corresponding real spectra is smooth. □

Lemma 1. *Let $U \subset \mathbb{R}^n$ be an open subset. There exists a smooth real valued function f on U such that for any $\lambda \in \mathbb{R}$, the set $f^{-1}(\lambda)$ is compact.*

Proof. Denote by A_k the set of points $x \in U$ such that $\|x\| \leq k$ and the distance from x to the boundary of U is not less than $1/k$. All points of A_k are interior points of A_{k+1}. Hence A_k and $\overset{\circ}{A_k}{}^c$ are two closed nonintersecting sets. By forthcoming lemma 2, that there then exists a smooth function $f_k : \mathbb{R}^n \to [0, 1]$ whose values are 0 on A_k and 1 for $x \notin A_k$. Since any point $x \in U$ belongs to the interior of the set A_k for all sufficiently large k, the sum

$$f = \sum_{k=1}^{\infty} f_k$$

is well defined and smooth (locally a finite sum). Consider a point $x \notin U \backslash A_k$. Since all functions f_i are nonnegative and for $i < k$, we have $f_i(x) = 1$, it follows that $f(x) \geq k - 1$.

Hence, for any $\lambda \in \mathbb{R}$, the set $f^{-1}(\lambda)$ is a closed subset of the compact set A_k, where k is an integer such that $\lambda < k-1$. We conclude that f is the function we were looking for. $\square$

Lemma 2. *For any open subset $U \subset \mathbb{R}^n$, there exists a function $f \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f(x) = 0$ for $x \notin U$ and $f(x) > 0$ for $x \in U$.*

Proof. If $U = \{x \in \mathbb{R}, x > 0\}$, the function $f(x) = 0$ for $x \leq 0$ and $f(x) = e^{-1/x}$ does the job. If $U = B(a,r) \subset \mathbb{R}^n$ is the open ball with center $a \in \mathbb{R}^n$ and radius $r > 0$, the function $g(x) = f(r^2 - \|x-a\|^2)$ does the job. If $U = \mathbb{R}^n$, take $f \equiv 1$. If $U = \emptyset$, take $f \equiv 0$. Now suppose that U is a non-trivial subset of $\mathbb{R}^n$ and let $\{U_k\}$ be a covering of U by a countable collection of open balls. There exist smooth functions $f_k \in \mathcal{C}^\infty(\mathbb{R}^n)$ such that $f_k(x) > 0$ for $x \in U_k$ and $f_k(x) = 0$ otherwise. Put

$$M_k = \sup_{|\alpha| \leq k,\, x \in \mathbb{R}^n} \left| \frac{\partial_\alpha f_k}{\partial x^\alpha} \right| (x).$$

Note that $M_k < \infty$, since outside the compact set $\bar{U}_k$, the function f_k and all its derivatives vanish. The series

$$f = \sum_{k=1}^{\infty} \frac{f^k}{2^k M_k}$$

converges to a smooth function since for every α, the series

$$\sum_{k=1}^{\infty} \frac{f^k}{2^k M_k} \frac{\partial_\alpha f_k}{\partial x^\alpha}$$

converges uniformly (terms bounded by powers 2^{-k}). The function f does the job for U. $\square$

We can thus consider, by using the equivalence $\textsc{Legos} \cong \textsc{Alg}_{legos,\mathbb{R}}$, that a smooth space is a covariant functor

$$X : \textsc{Alg}_{legos,\mathbb{R}} \to \textsc{Sets}$$

that is moreover a sheaf for the Zariski topology on $\textsc{Alg}_{legos,\mathbb{R}}$. This approach is exactly the one that leads to the notion of algebraic spaces and schemes, and it is needed if one wants to work with fermionic variables.

2.3.2 Categories of smooth algebras

We will need to extend the category $\mathrm{ALG}_{legos,\mathbb{R}}$ of algebras of smooth functions on LEGOS to a category of quotient algebras of the form A/I for $A \in \mathrm{ALG}_{legos,\mathbb{R}}$ and I an ideal, to introduce nilpotent algebras similar to $\mathbb{R}[\epsilon]/(\epsilon^2)$ that are necessary to study differential calculus. The study of equations defined by smooth functions also urges us to define a category of smooth algebras that has an easily computable affine space. We will thus now define various types of algebras that will be useful to define various types of smooth spaces. Depending on the situation, one will use one or the other category to study a given space.

Recall that the affine space on a given algebra is defined by

$$\mathbb{A}^n(A) = A^n.$$

The algebraic affine space $\mathbb{A}^n : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SETS}$ is representable by the polynomial algebra $\mathbb{A}[x_1, \dots, x_n]$ (this is the universal property of the polynomial algebra), meaning that

$$\mathbb{A}^n(A) = A^n \cong \mathrm{Hom}_{\mathrm{ALG}_{\mathbb{R}}}(\mathbb{R}[x_1, \dots, x_n], A).$$

This makes the polynomial algebra the main tool for the study of algebraic equations, since it is the free commutative unitary algebra on a given set of variables. The analogous of the polynomial algebra in the smooth setting is given by the algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ of smooth functions on the affine space. We now define a category of algebras in which the algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ is a representative object of the affine space.

Definition 2.10. An algebra A is called

- smoothly affine if for every n, the natural map

$$\begin{array}{ccc} \mathrm{Hom}_{\mathrm{ALG}_{\mathbb{R}}}(\mathcal{C}^\infty(\mathbb{R}^n), A) & \to & \mathbb{A}^n(A) \\ f^* & \mapsto & f^* \circ i \end{array}$$

  (where $i : \mathbb{R}[x_1, \dots, x_n] \hookrightarrow \mathcal{C}^\infty(\mathbb{R}^n)$ is the natural embedding) is bijective.

- geometric if the evaluation map

$$\begin{array}{cccc} \mathrm{ev} : & A & \to & \mathrm{Hom}(\mathrm{Spec}_{\mathbb{R}}(A), \mathbb{R}) \\ & a & \mapsto & [x \mapsto x(a)] \end{array}$$

  is injective.

- smoothly closed if of every $a_1, \dots, a_n \in A^n$ and $f \in \mathcal{C}^\infty(\mathbb{R}^n)$, there exists an $a \in A$ such that the function

$$f \circ (\mathrm{ev}_{a_1} \times \cdots \times \mathrm{ev}_{a_n}) : \mathrm{Spec}_{\mathbb{R}}(A) \to \mathbb{R}$$

  equals ev_a.

We denote respectively $\textsc{Alg}_{sa,\mathbb{R}}$, $\textsc{Alg}_{sc,\mathbb{R}}$ and $\textsc{Alg}_{g,\mathbb{R}}$ the categories of smoothly affine, smoothly closed and smoothly geometric algebras. We denote $\textsc{Alg}_{scg,\mathbb{R}}$ the category of smoothly closed geometric algebras.

It follows from theorem 1 that $\mathcal{C}^\infty(\mathbb{R}^n)$ is smoothly affine and it is now clear from the above definition that the smooth affine space

$$\begin{array}{rccc} \mathbb{A}^n : & \textsc{Alg}_{sa,\mathbb{R}} & \to & \textsc{Sets} \\ & A & \mapsto & A^n \end{array}$$

is representable by $\mathcal{C}^\infty(\mathbb{R}^n)$.

Example 2.2. If U is a lego, the algebra $\mathcal{C}^\infty(U)$ is smoothly affine, smoothly closed and geometric. The tensor product of the algebras $\mathcal{C}^\infty(U)$ and $\mathcal{C}^\infty(V)$ of functions on two legos in the categories $\textsc{Alg}_{scg}$ is isomorphic to $\mathcal{C}^\infty(U \times V)$. The algebra $\mathbb{R}[\epsilon]/(\epsilon^2)$ is smoothly affine, smoothly closed but not geometric.

Proposition 1. *The forgetful functors from $\textsc{Alg}_{scg,\mathbb{R}}$ and $\textsc{Alg}_{g,\mathbb{R}}$ to $\textsc{Alg}_{\mathbb{R}}$ have natural adjoints, called respectively the geometrization and the smooth closure.*

Proof. The adjoint of $\textsc{Alg}_{g,\mathbb{R}} \to \textsc{Alg}_{\mathbb{R}}$ is given by sending an algebra A to its image A_g in the set of continuous functions from $\mathrm{Spec}_{\mathbb{R}}(A)$ to $\mathbb{R}$ (given by the evaluation maps). More precisely for $A \in \textsc{Alg}_{\mathbb{R}}$ and for $a \in A$, we denote

$$\mathrm{ev}_a : \mathrm{Spec}_{\mathbb{R}}(A) \to \mathbb{R}$$

the evaluation function given by $[x : A \to \mathbb{R}] \mapsto x(a)$. One then defines

$$A_g := \{ev_a \in \mathrm{Hom}(\mathrm{Spec}_{\mathbb{R}}(A), \mathbb{R}), a \in A\}.$$

The adjoint to $\textsc{Alg}_{scg,\mathbb{R}} \to \textsc{Alg}_{g,\mathbb{R}}$ is defined by sending a geometric algebra A to

$$A_{sc} := \{f = g \circ (\mathrm{ev}_{a_1} \times \cdots \times \mathrm{ev}_{a_k}) : \mathrm{Spec}_{\mathbb{R}}(A) \to \mathbb{R},\ \forall a_1, \dots, a_k \in A^k \text{ and } g \in \mathcal{C}^\infty(\mathbb{R}^n)\}.$$

The adjoint of $\textsc{Alg}_{\mathbb{R}} \to \textsc{Alg}_{scg}$ is the composition $A \mapsto (A_g)_{sc}$ of the two above adjoint functors. $\square$

Theorem 2. *The smooth real valued functions functor*

$$\mathcal{C}^\infty : \textsc{Var} \to \textsc{Alg}_\mathbb{R}$$

gives a fully faithful embedding of the category of smooth varieties to the category of real algebras, whose essential image is the category of smoothly closed geometric algebras and whose quasi-inverse is given by the real spectrum functor $\mathrm{Spec}_\mathbb{R}$.

Proof. This can be found in the very complete book of Nestruev [Nes03]. □

Corollary 1. *There is a natural embedding*

$$\textsc{Alg}_{scg,\mathbb{R}} \hookrightarrow \textsc{Alg}_{sa,\mathbb{R}}.$$

Proof. Let A be a smoothly closed geometric algebra. Let $a_1, \ldots, a_n \in A$ be elements. They identify with the corresponding functions

$$\mathrm{ev}_{a_i} : \mathrm{Spec}_\mathbb{R}(A) \to \mathbb{R}.$$

Now if $g \in \mathcal{C}^\infty(\mathbb{R}^n)$ is a function, the function $g \circ (\mathrm{ev}_{a_1} \times \cdots \times \mathrm{ev}_{a_n}) : \mathrm{Spec}_\mathbb{R}(A) \to \mathbb{R}$ comes from a unique element $a \in A$ by hypothesis. This defines a morphism of algebras $f_{a_1,\ldots,a_n} : \mathcal{C}^\infty(\mathbb{R}^n) \to A$ so that the map

$$\mathrm{Hom}_{\textsc{Alg}_\mathbb{R}}(\mathcal{C}^\infty(\mathbb{R}^n), A) \to \mathbb{A}^n(A)$$

is surjective. Its injectivity follows from theorem 2. □

Using the above theorem, one shows that the space

$$X : \textsc{Alg}_{legos,\mathbb{R}} \to \textsc{Sets}$$

associated to a variety can be uniquely extended to an algebraic space

$$X : \textsc{Alg}_\mathbb{R} \to \textsc{Sets}$$

by setting $X(A) := \mathrm{Hom}_{\textsc{Alg}_\mathbb{R}}(\mathcal{C}^\infty(X), A)$. If X and Y are two varieties, one can also extend naturally $\underline{\mathrm{Hom}}(X, Y)$ to the category $\textsc{Alg}_\mathbb{R}$ by

$$\underline{\mathrm{Hom}}(X, Y)(A) := \mathrm{Hom}_{A-\textsc{Alg}}(\mathcal{C}^\infty(Y) \otimes_\mathbb{R} A, \mathcal{C}^\infty(X) \otimes_\mathbb{R} A).$$

Remark however that this extension is not identical to our previous definition of the mapping space (using parametrization by Legos) because of the fact that the tensor product $\mathcal{C}^\infty(U) \otimes \mathcal{C}^\infty(V)$ has to be smoothly closed to give back $\mathcal{C}^\infty(U \times V)$. One can correct this problem by working with the restriction

$$X, Y, \underline{\mathrm{Hom}}(X, Y) : \textsc{Alg}_{scg,\mathbb{R}} \to \textsc{Sets}$$

of the above functors to the category of smoothly closed geometric algebras.

If one wants to study functor of points of equations defined by smooth functions, it is very natural to work at least with the category $\mathrm{ALG}_{sa,\mathbb{R}}$ of smoothly affine algebras. To do infinitesimal constructions, one needs nilpotent algebras of the form $\mathbb{R}[\epsilon]/(\epsilon^2)$ that are not geometric but are smoothly affine and smoothly closed.

Remark that if $A = \mathcal{C}^\infty(U)$ and $B = \mathcal{C}^\infty(V)$, the tensor product of A and B in the category of smoothly closed geometric algebras is $\mathcal{C}^\infty(U \times V)$. If one restricts spaces to smoothly closed algebras, one gets mapping spaces $\underline{\mathrm{Hom}}(X, Y)$ that are compatible with the inclusion

$$\mathrm{LEGOS} \to \mathrm{ALG}_{sc,\mathbb{R}}.$$

To conclude this section, one can say that the choice of a convenient category of smooth algebras depends on the needs of a given situation.

2.3.3 Differential operators

We refer to Nestruev [Nes03] and SGA 3 [DG62], exp. VII, for this presentation of differential operators.

Definition 2.11. Let A be an $\mathbb{R}$-algebra and P be an A-module. A derivation from A to P is an $\mathbb{R}$-linear map $D : A \to P$ such that for all $a, b \in A$,

$$D(ab) = D(a)b + aD(b).$$

We denote $\mathrm{Der}(A, P)$ the space of derivations.

Proposition 2. *Let $U \subset \mathbb{R}^n$ and $A = \mathcal{C}^\infty(U)$. Let $x_i : U \to \mathbb{R}$ be the given coordinate system. Then* $\mathrm{Der}(A, A)$ *is the free module of rank n over A generated by the partial derivatives*

$$\partial_{x_i} := \frac{\partial}{\partial x_i} : A \to A.$$

Proof. See Nestruev [Nes03], theorem 9.6 and section 9.43. □

Definition 2.12. Let A be an $\mathbb{R}$-algebra, P, Q be two A-modules and $f \in A$ and define

$$\begin{array}{rccc} \delta_f : & \mathrm{Hom}_{\mathbb{R}}(P, Q) & \to & \mathrm{Hom}_{\mathbb{R}}(P, Q) \\ & \Delta & \mapsto & [\Delta, m_f] := \Delta \circ m_f - m_f \circ \Delta \end{array}$$

where m_f denotes multiplication by f. An $\mathbb{R}$-linear map $\Delta : P \to Q$ is called a differential operator of order $\leq n$ if for any $f_0, \ldots, f_n \in A$, we have

$$(\delta_{f_0} \circ \cdots \circ \delta_{f_n})(\Delta) = 0.$$

Differential operators of order $\leq k$ from P to Q are denoted $\mathrm{Diff}^k(P, Q)$. The algebra of all differential operators from A to A is denoted $\mathcal{D}_A$.

Theorem 3. *Let $U \subset \mathbb{R}^n$ and $A = \mathcal{C}^\infty(U)$. Then $\mathcal{D}_A$ is the sub-algebra of $\mathrm{End}_{\mathbb{R}}(A)$ generated by left multiplications m_f by elements $f \in A$ and derivations on A (i.e. vector fields on U). More precisely, if $x_i : \mathbb{R}^n \to \mathbb{R}$ is the coordinate system, every differential operator of order n on U can be written*

$$D = \sum_{|\alpha| \leq n} a_\alpha(x) \partial_x^\alpha$$

where $\alpha \in \mathbb{N}^n$ is a multi-index, $|\alpha| = \sum_{i=1}^n \alpha_i$ and $\partial_x^\alpha := \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$.

Proof. See Nestruev [Nes03], theorem 9.62. □

2.3.4 Differential forms and jets

We will now show how to define differential forms in a more algebraic setting. If one works with spaces modeled on open subsets of $\mathbb{R}^n$, there is a way (due to Nestruev [Nes03]) to relate quite directly the construction here with the one given in the section 2.2. We also refer to Hartshorne [Har77], section II.8 and Matsumura [Mat80] for the algebraic approach.

Definition 2.13. Let A be an algebra and $\mathrm{MOD}_{a,A} \subset \mathrm{MOD}_A$ be a full subcategory of the category of A-modules called admissible modules. The module of (admissible) differential forms on A is the unique admissible module Ω^1_A equiped with a derivation $d : A \to \Omega^1_A$ such that the map

$$\mathrm{Hom}_A(\Omega^1_A, P) \to \mathrm{Der}_A(A, P)$$

sending $f : \Omega^1_A \to P$ to $f \circ d$ is a bijection for every admissible A-module P.

If $\mathrm{MOD}_{a,A} = \mathrm{MOD}_A$, i.e., every module is admissible, the concrete construction is the following:

Definition 2.14. Consider the kernel I of the diagonal map $\Delta : A \otimes A \to A$. The space of Kähler differential forms Ω^1_A is defined as I/I^2 with the A-module structure given by multiplication on the left side of $A \otimes A$. One lets $d : A \to \Omega^1_A$ to be given by $d(a) = a \otimes 1 - 1 \otimes a$.

Proposition 3. *The space of Kähler differential forms is the module of admissible differential forms for the category MOD_A of all A-modules.*

Proof. See Matsumura [Mat80], p182. □

Example 2.3. If $A = \mathbb{R}[X]$ is the ring of polynomial functions on $\mathbb{R}$, one gets the ideal

$$I = \{P(X_0, X) \in \mathbb{R}[X_0][X],\ P(X_0, X_0) = 0\} = (X - X_0)$$

and $I^2 = (X - X_0)^2$ by the unicity of the rest of euclidean division in the polynomial ring $\mathbb{R}[X_0][X] = \mathbb{R}[X, X_0]$. The same argument shows that $I/I^2 \cong A[X_0]$ as an $A[X_0]$-module. The total differential $d : A[X_0] \to I/I^2$ is given by sending $P(X_0)$ to

$$P(X) - P(X_0) = P'(X_0).(X - X_0) \in (X - X_0)/(X - X_0)^2.$$

Definition 2.15. Let A be an $\mathbb{R}$-algebra. An A-module P is called geometric if

$$\cap_{x \in \mathrm{Spec}_{\mathbb{R}}(A)} \mathfrak{m}_x M = 0,$$

where $\mathfrak{m}_x$ denotes the ideal of functions that annihilate at x, i.e., the kernel of the map $x : A \to \mathbb{R}$. We denote $\textsc{Mod}_{g,A}$ the category of geometric modules.

Theorem 4. *Let X be a smooth variety. Admissible differential forms for the category $\textsc{Mod}_{g,A}$ identify with usual differential forms on X. In particular, if $U \subset \mathbb{R}^n$ is a lego, there is an identification*

$$\Omega^1(U) \cong \Gamma(U, T^*U)$$

where $T^*U := U \times (\mathbb{R}^n)^*$ is the cotangent space on U.

Proof. See Nestruev [Nes03], theorem 1.43. □

The difference between Kähler and usual differential forms can be catched by remarking that

$$d\exp \neq \exp$$

if $d\exp$ is the Kähler differential of the exponential function

$$\exp : \mathbb{R} \to \mathbb{R}.$$

Definition 2.16. Let A be an algebra and $\textsc{Mod}_{a,A} \subset \textsc{Mod}_A$ be a full subcategory of the category of A-modules. Let P be an object of $\textsc{Mod}_{a,A}$. The module of (admissible) k-jets of elements of P is the unique module J^kP in $\textsc{Mod}_{a,A}$ equiped with an order k differential operator $j_k : P \to J^kP$ such that the map

$$\mathrm{Hom}_A(J^kP, Q) \to \mathrm{Diff}^k(P, Q)$$

sending $f : J^kP \to Q$ to $f \circ j_k$ is a bijection for every A-module Q in $\textsc{Mod}_{a,A}$. The natural map $j_k : P \to J^kP$ is called the jet (or the Taylor series) map.

In the algebraic setting, if A is a ring and $P = A$ is the free A-module of rank 1, one defines the jet module by

$$J^k P := (A \otimes A)/I^{k+1},$$

where, as before, $I \subset A \otimes A$ is the kernel of the diagonal map $A \otimes A \to A$ and $J^k P$ is equiped with the A-module structure on the left term in $A \otimes A$.

Example 2.4. If $A = \mathbb{R}[X]$ is the ring of polynomial functions on $\mathbb{R}$ and $P = A$ is the module of polynomial sections of the trivial bundle $\mathbb{R}^2 \to \mathbb{R}$, one gets

$$J^k P := \mathbb{R}[X_0][X]/(X - X_0)^{k+1}$$

and the jet map is given by sending $P(X)$ to its taylor polynomial

$$P(X_0) + P'(X_0)(X - X_0) + \cdots + \frac{P^{(k)}(X_0)}{k!}(X - X_0)^k.$$

If X is a smooth variety, one usually works with usual jet bundles of modules of sections of vector bundles, that are given by applying the above construction to the category $\mathrm{Mod}_{g,A}$ of geometric modules.

Example 2.5. Let $X = \mathbb{R}$, $A = \mathcal{C}^\infty(X)$ and P be the section of the trivial vector bundle $E = X \times \mathbb{R}$ on X. One can deduce from

- the computation of the algebraic case given in example 2.4 and
- the fact that the smooth closure of $\mathbb{R}[X_1, \ldots, X_n]$ is $\mathcal{C}^\infty(\mathbb{R}^n)$,

that the the k-th jet bundle for $E \to X$ is (non-canonically) identified with the space

$$J^k E = X \times \mathbb{R}[T]/(T^{k+1}) \cong X \times \mathbb{R}^{k+1}$$

with coordinates $(x, u_i)_{i=0,\ldots,k}$ that play the role of formal derivatives (i.e., coefficients of the taylor series) of the "functional variable" u_0. It is defined by the module of its sections, given by the module $J^k P$ in Mod_g. The canonical map $A \to J^k P$ corresponds geometrically to the map

$$\begin{array}{cccc} j_k : & \Gamma(X, E) & \to & \Gamma(X, J^k E) \\ & [x \mapsto f(x)] & \mapsto & \left[x \mapsto \sum_{i=0}^k \frac{\partial^i f}{\partial x^i}(x).\frac{(X-x)^i}{i!}\right], \end{array}$$

that sends a real valued function to its Taylor series. More generally, if $X = \mathbb{R}^n$ and $E = X \times \mathbb{R}^m$, one gets the algebra bundle

$$J^k E = X \times \oplus_{i \geq 0} \mathrm{Sym}^i((\mathbb{R}^m)^*)/((\mathbb{R}^m)^*)^{k+1}) \cong X \times \oplus_{i=0}^k \mathrm{Sym}^i((\mathbb{R}^m)^*),$$

with coordinates (x, u_α) for $x \in X$ and $u_\alpha \in \mathbb{R}$, α being a multi-index in $\mathbb{N}^m$ with $|\alpha| \leq k$. The jet map correspond to higher dimensional Taylor series

$$\begin{array}{cccc} j_k: & \Gamma(X, E) & \to & \Gamma(X, J^k E) \\ & [x \mapsto f(x)] & \mapsto & \left[x \mapsto \sum_{|\alpha| \leq k} (\partial_x^\alpha f)(x) . \frac{(X-x)^\alpha}{\alpha!}\right], \end{array}$$

where multi-index notations are understood.

2.3.5 Tangent space and vector fields

Definition 2.17. Let $X : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SETS}$ be an algebraic space modeled on the category $\mathrm{ALG}_{\mathbb{R}}$. The tangent space to X is defined as the functor

$$\begin{array}{cccc} TX: & \mathrm{ALG}_{\mathbb{R}} & \to & \mathrm{SETS} \\ & A & \mapsto & X(A[\epsilon]/(\epsilon^2)). \end{array}$$

A vector field on X is a section $\vec{v} : X \to TX$ of the natural projection $\pi : TX \to X$.

This definition also works for spaces modeled on the category $\mathrm{ALG}_{sc,\mathbb{R}}$ of smoothly closed algebras because if A is smoothly closed, then $A[\epsilon]/(\epsilon^2)$ also, since

$$\mathrm{Hom}(A[\epsilon]/(\epsilon^2), \mathbb{R}) \cong \mathrm{Hom}(A, \mathbb{R}).$$

The following proposition gives the relation of the above definition with the usual notion of vector field on a variety.

Proposition 4. *Let X be the algebraic space associated to a smooth variety with algebra of smooth functions A. Then the datum of a vector field on X is equivalent to the datum of an $\mathbb{R}$-derivation*

$$D : A \to A.$$

Proof. Let $\vec{v} : X \to TX$ be a vector field. In particular, it gives a map

$$\vec{v} : X(A) \to TX(A)$$

by which we can take the image of the identity map $\mathrm{id} \in X(A) := \mathrm{Hom}(A, A)$. We thus get a morphism $A \to A[\epsilon]/(\epsilon^2)$ which can be written in the form $\mathrm{id} + \epsilon D$ where $D : A \to A$ is an $\mathbb{R}$-linear map that fulfils the derivation condition. Conversely, if such a derivation D is given, one can extend it to a functorial map of sets

$$\vec{v}_D(B) : X(B) := \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B[\epsilon]/(\epsilon^2)) =: TX(B)$$

for every algebra B, by setting $\vec{v}_D(B)(f) = f + \epsilon . f \circ D$. $\square$

Corollary 2. *If X is a variety, there is a natural bracket operation*

$$[.,.]:\Gamma(X,TX)\times\Gamma(X,TX)\to\Gamma(X,TX)$$

on vector fields on X.

Proof. On the level of derivations, the bracket is just the commutator $[D_1,D_2]=D_1D_2-D_2D_1$. □

If X is a smooth space, a vector field $\vec{v}:X\to TX$ allows one to compute the derivative $D_xf.\vec{v}$ of a functional $f:X\to\mathbb{R}$ at a given point $x:U\to X$ where $U\subset\mathbb{R}^n$ is a lego, by

$$D_xf.\vec{v}:=Df\circ\vec{v}\circ x:U\to T\mathbb{R}$$

and by identifying $\mathrm{Hom}_{\mathrm{SPACES}}(U,T\mathbb{R})$ with

$$T\mathbb{R}(U):=\mathrm{Hom}\left(\mathcal{C}^\infty(\mathbb{R}),\mathcal{C}^\infty(U)[\epsilon]/(\epsilon^2)\right),$$

we get as ϵ-component a map

$$D_xf.\vec{v}:\mathcal{C}^\infty(\mathbb{R})\to\mathcal{C}^\infty(U)$$

which is a derivation from $\mathcal{C}^\infty(\mathbb{R})$ to the module $\mathcal{C}^\infty(U)$ (the module structure being given by $f\circ x:U\to\mathbb{R}$). If U is reduced to a point $\{.\}$, we get a tangent vector to $\mathbb{R}$ at the point $f\circ x(\{.\})$, which is the same as a real number, that gives the value of the derivative for the function f at the given point along the vector field $\vec{v}$.

2.4 Connections and curvature

The mathematical formulation of particle physics involve various notions of connections on bundles. The easiest way to relate them is to use the very general notion of Grothendieck connection. This is why we pass through this approach. In each concrete cases, the notion of connection is easier to handle, and we give for it a simpler description.

2.4.1 Grothendieck connections

We first give a general definition of connection, based on the notion of infinitesimal neighborhood.

Recall that if A is an algebra and $\mathrm{Mod}_{a,A}$ is a category of admissible modules, and if moreover A is an admissible A-module, we define the n-th jet space J^nA of A as the

A-module with differential operator $D : A \to J^n A$ such that for every $P \in \textsc{Mod}_{a,A}$, the natural map
$$\mathrm{Hom}(J^n A, P) \to \mathrm{Diff}^n(A, P)$$
induced by D is a bijection. The algebra operations on A induce algebra operations on $J^n A$. We then define the n-th infinitesimal neighborhood $X^{(n)}$ of the diagonal map $X \xrightarrow{\Delta} X \times X$ of $X = \mathrm{Spec}(A)$ as being the space associated to the algebra $\mathrm{Jet}^n A$.

If all modules are admissible and $X = \mathrm{Spec}(A) := \mathrm{Hom}(A, .)$ is associated to a real algebra $A \in \textsc{Alg}_{\mathbb{R}}$, the n-th infinitesimal neighborhood of the diagonal map is given by

$$X^{(n)} := \mathrm{Spec}(A \otimes A/I^{n+1})$$

for $I \subset A \otimes A$ the kernel of the diagonal map

$$A \otimes A \to A,\ a \otimes b \mapsto a.b.$$

This is the algebraic infinitesimal neighborhood.

If we work with a smoothly closed algebra A in $\textsc{Alg}_{sc,\mathbb{R}}$ and only geometric modules are admissible, we will find another jet module $\mathrm{Jet}^n A$, that corresponds to functions on the jet space of the trivial bundle $X \times \mathbb{R} \to \mathbb{R}$ if $A = \mathcal{C}^\infty(X)$ (X a smooth variety). It is also equiped with an algebra structure so that one can define the space $X^{(n)} := \mathrm{Spec}(\mathrm{Jet}^n A)$. This is the smooth infinitesimal neighborhood, and it is as different of the algebraic infinitesimal neighborhood as usual differential forms are different of Kähler differential forms.

For a more general space X modeled on $\textsc{Alg}_{\mathbb{R}}$ or $\textsc{Alg}_{sc,\mathbb{R}}$, one defines the subspace $X^{(n)} \subset X \times X$ by saying that a point $x \in X(A) \times X(A)$ is in $X^{(n)}(A)$ if and only if for every function $f : X \to \mathrm{Spec}(C)$, the corresponding morphism

$$\mathrm{Spec}(A) \xrightarrow{x} X \times X \xrightarrow{(f \times f)} \mathrm{Spec}(C \otimes C)$$

factorizes through $\mathrm{Spec}(C \otimes C/I^{n+1})$, giving a commutative diagram

$$\begin{tikzcd}
\mathrm{Spec}(A) \arrow[r, "x"] \arrow[drr] & X \times X \arrow[r, "f \times f"] & \mathrm{Spec}(C \otimes C) \\
& & \mathrm{Spec}(C \otimes C/I^{n+1}) \arrow[u]
\end{tikzcd}$$

If X comes from a ringed space, one can check in Grothendieck's [Gro67], Part 4, that one gets the usual notion of infinitesimal neighborhood.

Definition 2.18. Let $f : B \to X$ be a morphism of spaces modeled on $\textsc{Alg}_{\mathbb{R}}$ or $\textsc{Alg}_{sc,\mathbb{R}}$. A (Grothendieck) connection on $f : B \to X$ is the data of an isomorphism

$$\epsilon : p_1^* B \xrightarrow{\sim} p_2^* B$$

on $X^{(1)}$ that reduces to identity on X.

This notion of connection actually generalize to any kind of geometrical object on X that has a natural notion of pull-back, for example a linear connection on a vector bundle, a G-equivariant connection on a G-bundle, etc...

Example 2.6. Let $X = \mathbb{R}$ and $B = \mathbb{R}^2$ equiped with their polynomial algebras of functions $\mathcal{O}(X) = \mathbb{R}[X]$ and $\mathcal{O}(B) = \mathbb{R}[X, u]$. One then has

$$\mathcal{O}(X^{(1)}) := \mathbb{R}[X, X_0]/(X - X_0)^2,$$

$$\mathcal{O}(p_1^* B) = \mathbb{R}[X, X_0, u]/(X - X_0)^2,\ \mathcal{O}(p_2^* B) = \mathbb{R}[X_0, X, u]/(X - X_0)^2$$

A connection on B must be identity on X, i.e., modulo $(X - X_0)$. The trivial Grothendieck connection on $B \to X$ is given by the isomorphism

$$\mathrm{id} : \mathcal{O}(p_1^* B) \xrightarrow{\sim} \mathcal{O}(p_2^* B).$$

If $A \in (X - X_0)$ is an element (that represents a differential form on X), then $\epsilon : u \mapsto u + A$ is also a Grothendieck connection on B, whose inverse isomorphism is $u \mapsto u - A$.

Let $B \to X$ be a bundle of varieties. We will denote $X_3^{(1)}$ the infinitesimal neighborhood of the diagonal in $X \times X \times X$. Let $p_{1,2}, p_{2,3}, p_{1,3} : X_3^{(1)} \to X \times X \times X \to X \times X$ be the natural projections. Given a Grothendieck connection on $f : B \to X$, i.e., an isomorphism

$$\epsilon : p_1^* B \xrightarrow{\sim} p_2^* B$$

over $X^{(1)}$, one can pull-back it to $X_3^{(1)}$ through the projections $p_{i,j}$, getting isomorphisms

$$\epsilon_{i,j} : p_i^* B \xrightarrow{\sim} p_j^* B.$$

Definition 2.19. The curvature of the connection $\epsilon : p_1^* B \xrightarrow{\sim} p_2^* B$ is the isomorphism

$$C(\epsilon) := \epsilon_{1,3}^{-1} \circ \epsilon_{2,3} \circ \epsilon_{1,2} : p_1^* B \xrightarrow{\sim} p_1^* B.$$

Proposition 5. *If a connection τ on $B \to X$ has no torsion, the isomorphism $\tau = \tau_{(1)}$ can be extended to isomorphisms*

$$\tau_{(n)} : p_1^* B \xrightarrow{\sim} p_2^* B$$

on every $X^{(n)}$.

Proof. Can be found in [BO78]. □

2.4.2 Koszul connections

Definition 2.20. Let $F \to X$ be a vector bundle on a smooth variety. Let $A = \mathcal{C}^\infty(X)$ and M be the A-module $\Gamma(X, F)$ of sections of F. A Koszul connection on F is an $\mathbb{R}$-linear map

$$\nabla : M \to M \otimes_A \Omega^1_A$$

that fulfils Leibniz rule

$$\nabla(f.s) = f.\nabla(s) + df \otimes s.$$

Definition 2.21. Let X be a variety with function algebra A. Let $F \to X$ be a vector bundle and let $\nabla : \mathcal{F} \to \mathcal{F} \otimes_A \Omega^1_A$ be a Koszul connection. The curvature of ∇ is the composition

$$C(\nabla) := \nabla_2 \circ \nabla \in \mathrm{End}_A(\mathcal{F}) \otimes_A \Omega^2_A,$$

where ∇_i is defined by

$$\nabla_i := \mathrm{id}_{\mathcal{F}} \otimes d + (-1)^i \nabla \wedge \mathrm{id}_{\Omega^1_A} : \mathcal{F} \otimes_A \Omega^i_A \to \mathcal{F} \otimes_A \Omega^{i+1}_A.$$

Remark that the A-linearity of the curvature is not clear a priori.

Proposition 6. *Let $F \to X$ be a vector bundle on a smooth variety. The data of a linear Grothendieck connection and of a Koszul connection on F are equivalent.*

Proof. We refer to [BO78] for more details on the link between connections and infinitesimal neighborhoods. Just recall that the exterior differential is a map $d : A \to \Omega^1_A \subset J^1A$. Denote $\mathcal{F} = \Gamma(X, F)$ the space of sections of F. If $\tau : \mathcal{F} \otimes_A J^1A \to \mathcal{F} \otimes_A J^1A$ is a Grothendieck connection, and $d_{1,\mathcal{F}} := d \otimes \mathrm{id}_{\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes J^1A$, then the morphism $\theta = \tau \circ d_{1,\mathcal{F}} : \mathcal{F} \to \mathcal{F} \otimes J^1A$ allows us to construct a Koszul connection by $\nabla(f) = \theta(f) - f \otimes 1$. Indeed, one has $\nabla(f) = \mathcal{F} \otimes \Omega^1$ because τ is identity on X. The association $\tau \mapsto \nabla$ is grounded on various universal properties that makes it unique. $\square$

2.4.3 Ehresmann connections

We refer to the book [KMS93], 17.1, for a differential geometric approach to Ehresmann connections. We will mostly be interested by the more algebraic approach presented in the book of Krasilshchik and Verbovetsky [KV98], section 5.2.

Definition 2.22. Let $p : B \to M$ be a surjective submersion of varieties. An Ehresmann connection on B is the datum of a section a of the natural projection

$$J^1B \to B.$$

Proposition 7. *Let $p : B \to X$ be a surjective submersion of varieties. The the following data on B are equivalent:*

1. *An Ehresmann connection a,*
2. *A connection one form $v \in \Omega^1(B, TB) = \mathrm{End}(TB)$ such that $v^2 = v$ and the image of v is VB,*
3. *a section $a \in \Gamma_B(TB, VB)$ of the canonical exact sequence*

$$0 \to VB \to TB \to \pi^* TX \to 0$$

of bundles on B.

Proof. We first show that the two last data are equivalent. If $a : TB \to VB$ is a section of the natural map $VB \to TB$, the projection $v : TB \to TB$ on VB along the kernel of a gives a connection one form. Since its image is VB, the data of a and v are equivalent. Now we use the Krasilshchik-Verbovetsky approach to connections in [KV98], 5.2. Remark that the section $a : TB \to VB$ induces a natural morphism of functors on the category of admissible $\mathcal{O}_B$-modules (chosen to be the category of all modules in the case of algebraic varieties and the category of geometric modules for smooth varieties)

$$\nabla_a : \mathrm{Der}(\mathcal{O}_X, .) \to \mathrm{Der}(\mathcal{O}_B, .).$$

This corresponds on the representing objects to a morphism

$$\nabla_a : \mathcal{O}_B \to \mathcal{O}_B \otimes_{\mathcal{O}_X} \Omega^1_{B/X},$$

which can be extended to

$$s_a^* := \mathrm{id} + \nabla_a : \mathcal{O}_B \to \mathcal{O}_B \oplus \mathcal{O}_B \otimes_{\mathcal{O}_X} \Omega^1_{B/X} =: \mathcal{O}_{J^1B},$$

and then to a Grothendieck connection

$$\epsilon : \mathcal{O}_{J^1B} \xrightarrow{\sim} \mathcal{O}_{J^1B}.$$

Composing with the canonical projection $\mathcal{O}_{J^1B} \to \mathcal{O}_B$ gives a section $s : B \to J^1B$. $\square$

Definition 2.23. The curvature of an Ehresmann connection is given by

$$R = \frac{1}{2}[v, v]$$

where $[.,.]$ denotes the Frölicher-Nijenhuis bracket of $v \in \Omega^1(E, TE)$ with itself. Thus $R \in \Omega^2(E, TE)$ is defined by

$$R(X, Y) = v([(\mathrm{id} - v)X, (\mathrm{id} - v)Y)]).$$

Proposition 8. *Let $p : B \to X$ be a bundle given by a morphism of varieties. The following data are equivalent:*

1. *An Ehresmann connection on B,*
2. *A Grothendieck connection on B.*

Proof. Denote $\mathcal{A} = \mathcal{C}^\infty(X)$ and $\mathcal{B} = \mathcal{C}^\infty(B)$. Suppose given a Grothendieck connection

$$p_1^* B \xrightarrow{\sim} p_2^* B.$$

In this case, it is equivalent to a map

$$\mathcal{B} \otimes_{\mathcal{A}} J^1\mathcal{A} \xrightarrow{\sim} \mathcal{B} \otimes_{\mathcal{A}} J^1\mathcal{A}$$

that induces identity on $\mathcal{A}$, and that we can compose with the projection $\mathcal{B} \otimes_{\mathcal{A}} J^1\mathcal{A} \to \mathcal{B}$ to get a map

$$\mathcal{B} \otimes_{\mathcal{A}} J^1\mathcal{A} \to \mathcal{B}$$

that is also a map

$$J^1\mathcal{B} \to \mathcal{B}$$

and gives a section of the natural projection $J^1 B \to B$. These two data are actually equivalent because one can get back that Grothendieck connection by tensoring with $\mathcal{B}$ over $\mathcal{A}$. □

Corollary 3. *If $p : F \to X$ is a vector bundle on a smooth variety, the datum of a linear Ehresmann connection on F and of a Koszul connection on F are equivalent.*

2.4.4 Principal connections

We use here the definition of Giraud [Gir71], chapter III, 1.4, since it is adapted to general spaces.

Definition 2.24. Let G be a lie group and $P \to M$ be a space morphism equiped with an action $m : G \times_M P \to P$ of G. One says that (P, m) is a principal homogeneous space over M under G (also called a G-torsor over M) if

1. $P \to M$ is an epimorphism, i.e., there exists a covering family $\{U_i \to M\}$ such that $P_M(U_i)$ are non-empty (i.e., the bundle has locally a section on a covering of M),

2. the natural morphism

$$\begin{array}{ccc} G \times P & \to & P \times P \\ (g, p) & \mapsto & (p, gp) \end{array}$$

is an isomorphism (i.e., the action of G on P is simply transitive).

Definition 2.25. A principal G-connection on P is a G-equivariant connection on $P \to M$.

Proposition 9. *The following are equivalent:*

1. *A principal G-connection on $p : P \to M$.*

2. *An equivariant $\mathfrak{g}$-valued differential form A on P, i.e., $A \in \Omega^1(P, \mathfrak{g})^G$.*

Proof. The equivalence between an equivariant Ehresmann connection $A \in \Omega^1(TP, TP)^G$ and a differential form in $\Omega^1(P, \mathfrak{g})$ follows from the fact the derivative of the action map $m : G \times_M P \to P$ with respect to the G-variable at identity $e \in G$ defines a bundle map

$$i = D_e m : \mathfrak{g}_P := \mathfrak{g} \times P \to VP,$$

called a parallelization, between the vertical tangent bundle to P, defined by

$$0 \to VP \to TP \xrightarrow{Dp} p^*TM \to 0$$

and the trivial linear bundle with fiber the Lie algebra $\mathfrak{g}$ of G, that is an isomorphism. The fact that $i : \mathfrak{g}_P \to TP$ is valued in VP follows from the fact that G acts vertically on P. $\square$

Proposition 10. *The curvature of a principal G-connection A is identified with the form*

$$F = dA + [A \wedge A]$$

in $\Omega^2(P, \mathfrak{g})$ where the bracket exterior product is given by

$$[\omega \otimes h \wedge \nu \otimes k] = \omega \wedge \nu \otimes [h, k].$$

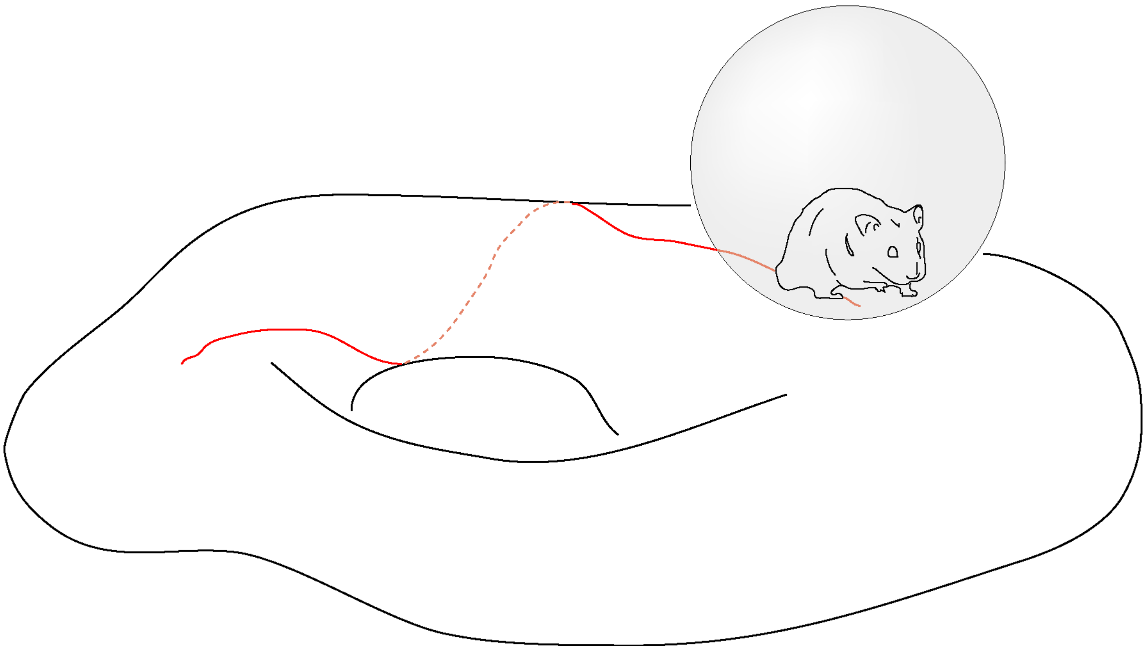


Figure 2.1: Cartan geometry and the hamster ball.

2.4.5 Cartan connections and moving frames

We refer to the excellent survey of Wise [Wis06] for a more complete description. Sharpe [Sha97] and Wise explain neatly the idea of Cartan geometry in a commutative diagram:

$$\begin{tikzcd}
\begin{array}{c}\text{Euclidean}\\ \text{Geometry}\end{array} \arrow[r, "\substack{\text{allow}\\ \text{curvature}}"] \arrow[d, swap, "\substack{\text{generalize}\\ \text{symmetry group}}"] & \begin{array}{c}\text{Riemannian}\\ \text{Geometry}\end{array} \arrow[d, "\substack{\text{generalize tangent}\\ \text{space geometry}}"] \\
\begin{array}{c}\text{Klein}\\ \text{Geometry}\end{array} \arrow[r, swap, "\substack{\text{allow}\\ \text{curvature}}"] & \begin{array}{c}\text{Cartan}\\ \text{Geometry}\end{array}
\end{tikzcd}$$

One can also say that a Cartan geometry is given by a space whose geometry is given by pasting infinitesimally some classical geometric spaces of the form G/H for $H \subset G$ two Lie groups. This can be nicely explained by the example of a sphere pasted infinitesimally on a space through tangent spaces, or, as in Wise's article [Wis06], by a Hamster ball moving on a given space:

Definition 2.26. Let M be a variety, $H \subset G$ be two groups. A Cartan connection on

M is the data of

1. a principal G-bundle Q on M,
2. a principal G-connection A on Q,
3. a section $s : M \to E$ of the associated bundle $E = Q \times_G G/H$ with fibers G/H,

such that the pullback $e = s^*A \circ ds : TM \to VE$, called the moving frame (vielbein), for $A : TE \to VE$ the associated connection, is a linear isomorphism of bundles.

The role of the section s here is to "break the G/H symmetry". It is equivalent to the choice of a principal H-subbundle $P \subset Q$.

In the cases of interest for this section, E is a linear bundle and the section s is simply the zero section that breaks the translation symmetry (action of $G/H = V$ on the sections of the vector bundle)

The first examples of Cartan connections are given by Klein geometries, i.e., by homogeneous spaces $E = G/H$ over the point space $M = \{.\}$. The corresponding Cartan connection is given by

1. the trivial principal G-bundle $Q = G$ on M,
2. the trivial G-connection A on Q,
3. a section $s : M \to E$ of the associated bundle $E = Q \times_G G/H = G/H \to M$, i.e., a point $x = s(.) \in G/H$.

The pull-back $e = s^*A : TM = M \times \{0\} \to VE = M \times \{0\}$ is an isomorphism.

Three examples of Klein geometries that are useful in physics are given by $H = \mathrm{SO}(n-1,1)$ and

$$G = \left\{ \begin{array}{cc} \mathrm{SO}(n,1) & \text{(de Sitter)} \\ \mathbb{R}^{n-1,1} \rtimes \mathrm{SO}(n-1,1) & \text{(Minkowski)} \\ \mathrm{SO}(n-1,2) & \text{(anti de Sitter)} \end{array} \right.$$

The de Sitter and anti de Sitter geometries are useful to study cosmological models with non-zero cosmological constant. Cartan geometries modeled on these Klein geometries are useful in general relativity. For example, the Minkowski Cartan geometry without torsion corresponds exactly to pseudo-Riemannian manifolds, that are the basic objects of general relativity.

Remark that the G-connection A on the principal G-bundle Q is equivalent to an equivariant $\mathfrak{g}$-valued differential form

$$A : TQ \to \mathfrak{g},$$

and its restriction to $P \subset Q$ gives an H-equivariant differential form

$$A : TP \to \mathfrak{g}.$$

This is the original notion of Cartan connection form.

Definition 2.27. The curvature of a Cartan connection is the restriction of the curvature of the corresponding Ehresman connection on the principal G-bundle Q to the principal H-bundle P. It is given by the formula

$$F_A := dA + \frac{1}{2}[A \wedge A] \in \Omega^2(P, \mathfrak{g}).$$

The torsion of the Cartan connection is given by the composition of F_A with the projection $\mathfrak{g} \to \mathfrak{g}/\mathfrak{h}$.

Suppose that one can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ in an H-equivariant way (the Cartan geometry is called reductive). The Cartan connection form thus can be decomposed in

$$A = \omega + e$$

for $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$. In this particular case, e can also be seen as

$$e \in \Omega^1(M, \underline{\mathfrak{g}/\mathfrak{h}})$$

for $\underline{\mathfrak{g}/\mathfrak{h}}$ the H-bundle associated to $\mathfrak{g}/\mathfrak{h}$. This form is called the vielbein by cartan. By definition of the Cartan connection, it gives an isomorphism

$$e : TM \to \underline{\mathfrak{g}/\mathfrak{h}}.$$

2.5 Superalgebras and fermionic variables

To formalize matter particles, so called fermions, one needs to generalize the notion of lego from usual algebras to super-algebras (with anticommuting coordinates). We will do this in the setting of general monoidal categories since it is not harder, avoids the "sign check nightmare" and allows to compute everything as if we were working with usual algebras. Moreover, we will need later (for the reduction of gauge theories) to work with more general types of algebras (graded, differential graded) that are also perfectly treated in this simple abstract setting.

2.5.1 Differential calculus in symmetric monoidal categories

We essentially follow Lychagin [LEP93] here. Let K be a base field of characteristic 0.

Definition 2.28. A symmetric tensor category over K is a tuple $(\mathcal{C}, \otimes, \mathbb{1}, \mathrm{un}, \mathrm{as}, \mathrm{com})$ composed of

1. an abelian K-linear category $\mathcal{C}$,
2. a K-linear bifunctor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$,
3. an object $\mathbb{1}$ of $\mathcal{C}$ called the unit object,
4. for each object A of $\mathcal{C}$, two unity isomorphisms $\mathrm{un}^r_A : A \otimes \mathbb{1} \to A$ and $\mathrm{un}^l_A : \mathbb{1} \otimes A \to A$.
5. for each triple (A, B, C) of objects of $\mathcal{C}$, an associativity isomorphism
$$\mathrm{as}_{A,B,C} : A \otimes (B \otimes C) \to (A \otimes B) \otimes C,$$
6. for each pair (A, B) of objects of $\mathcal{C}$, a commutativity isomorphism
$$\mathrm{com}_{A,B} : A \otimes B \to B \otimes A,$$

that are supposed to fulfil (for more details, we refer the reader to the article on monoidal categories of wikipedia)

1. a pentagonal axiom for associativity isomorphisms,
2. a compatibility of unity and associativity isomorphisms,
3. an hexagonal axiom for compatibility between the commutativity and the associativity isomorphisms.
4. the idempotency of the commutativity isomorphism: $\mathrm{com}_{A,B} \circ \mathrm{com}_{B,A} = \mathrm{id}_A$.

The tensor category is called closed if it has internal homomorphisms, i.e., if for every pair (B, C) of objects of $\mathcal{C}$, the functor
$$A \mapsto \mathrm{Hom}(A \otimes B, C)$$
is representable by an object $\underline{\mathrm{Hom}}(B, C)$ of $\mathcal{C}$.

The main example of a closed commutative tensor category is the category $\textsc{Vect}_K$ of K-vector spaces. The idea for defining differential calculus on algebras in an abstract symmetric tensor category is to formalize it for usual algebras using only the tensor structure and morphisms in $\textsc{Vect}_K$.

Consider now the category whose objects are graded vector spaces

$$V = \oplus_{k\in\mathbb{Z}} V^k$$

and whose morphisms are linear maps respecting the grading. We denote it $\textsc{Vect}_g$. A graded vector space restricted to degree 0 and 1 is called a super-vector space, and we denote $\textsc{Vect}_s$ the category of super-vector spaces. These are abelian and even K-linear categories. If $a \in V^k$ is a homogeneous element of a graded vector space V, we denote $\deg(a) := k$ its degree. The tensor product of two graded vector spaces V and W is the usual tensor product of the underlying vector spaces equiped with the grading

$$(V \otimes W)_k = \oplus_{i+j=k} V_i \otimes W_j.$$

There is a natural homomorphism object in $\textsc{Vect}_g$, defined by

$$\underline{\mathrm{Hom}}(V, W) = \oplus_{n\in\mathbb{Z}} \mathrm{Hom}^n(V, W)$$

where the degree n component $\mathrm{Hom}_n(V, W)$ is the set of all linear maps $f : V \to W$ such that $f(V^k) \subset W^{k+n}$. It is an internal homomorphism object meaning that for every X, there is a natural bijection

$$\mathrm{Hom}(X, \underline{\mathrm{Hom}}(V, W)) \cong \mathrm{Hom}(X \otimes V, W).$$

The tensor product of two internal homomorphisms $f : V \to W$ and $f' : V \to W'$ is defined using the Koszul sign rule on homogeneous components: We have

$$(f \otimes g)(v \otimes w) = (-1)^{\deg(g)\deg(v)} f(v) \otimes g(w).$$

The tensor product is associative with unit $\mathbf{1} = K$ in degree 0, the usual associativity isomorphisms of K-vector spaces. The main difference with the tensor category $(\textsc{Vect}, \otimes)$ of usual vector space is given by the non-trivial commutativity isomorphisms

$$c_{V,W} : V \otimes W \to W \otimes V$$

defined by extending by linearity the rule

$$v \otimes w \mapsto (-1)^{\deg(v)\deg(w)} w \otimes v.$$

One thus obtains a symmetric tensor category $(\textsc{Vect}_g, \otimes)$ which is moreover closed, i.e., has internal homomorphisms.

Definition 2.29. Let $(\mathcal{C}, \otimes)$ be a symmetric tensor category over K. An algebra in $\mathcal{C}$ is a triple (A, μ, ν) composed of

1. an object A of $\mathcal{C}$,
2. a multiplication morphism $\mu : A \otimes A \to A$,
3. a unit morphism $\nu : \mathbb{1} \to A$,

such that for each object V of $\mathcal{C}$, the above maps fulfil the usual associativity, commutativity and unit axiom with respect to the given associativity, commutativity and unity isomorphisms in $\mathcal{C}$. We denote $\textsc{Alg}_{\mathcal{C}}$ the category of algebras in $\mathcal{C}$.

In particular, a graded (resp. super)-algebra is an algebra in the tensor category $(\textsc{Vect}_g, \otimes)$ (resp. $(\textsc{Vect}_s, \otimes)$). Recall that the commutativity of a super algebra uses the commutativity isomorphism of the tensor category $\textsc{Vect}_s$, so that it actually means a graded commutativity: (A, μ) is commutative if $\mu \circ \mathrm{com}_{A,A} = \mu$.

From now on, let $\mathcal{C}$ be a symmetric tensor category over K and (A, μ) be an algebra in $\mathcal{C}$. We will now define differential invariants of (A, μ).

A left A-module is an object M of $\mathcal{C}$ equiped with an external multiplication map $\mu^l_M : A \otimes M \to M$. If A is a commutative algebra in $\mathcal{C}$, one can put on M a right A-module structure $\mu^r : M \otimes A \to M$ defined by $\mu^r_M := \mu \circ \mathrm{com}_{M,A}$. We will implicitly use this right A-module structure in the forthcoming

The intruition behing the definition of internal derivations is the following: an internal homomorphism $D : A \to M$ in $\mathcal{C}$ (be careful, this is not a well defined notion) will be called an internal derivation if

$$D(fg) = D(f)g + fD(g).$$

To be mathematically correct, we need to define internal derivations by only using the tensor structure.

Definition 2.30. Let M be an A-module. The internal derivation object $\underline{\mathrm{Der}}(A, M)$ is defined as the kernel of the Leibniz morphism

$$\begin{array}{rccc} \mathrm{Leibniz}: & \underline{\mathrm{Hom}}(A, M) & \to & \underline{\mathrm{Hom}}(A \otimes A, M) \\ & D & \mapsto & D \circ \mu - \mu^l_M \circ (\mathrm{id}_A \otimes D) + \mu^r_M \circ (D \otimes \mathrm{id}_A). \end{array}$$

Remark that for $\mathcal{C} = \textsc{Vect}_g$ this expression can be expressed as a graded Leibniz rule by definition of the right A-module structure on M above. More precisely, an internal morphism $D : A \to M$ is a graded derivation if it fulfils the graded Leibniz rule

$$D(ab) = D(f)g + (-1)^{\deg(D)\deg(f)} fD(g).$$

The restriction of this construction to the category $\textsc{Vect}_s$ of super vector spaces gives the notion of super-derivation.

The representing object for the functor $\underline{\mathrm{Der}} : A - \textsc{Mod}(\mathcal{C}) \to \mathcal{C}$ is the A-module Ω^1_A of 1-forms. One can restrict the derivation functor on A-modules to a subcategory $\mathcal{C}'$ of $\mathcal{C}$ and this allows to define various other types of differential forms on A.

We now define the differential operators on A following Lychagin in [LEP93]. Let M and N be two left A-modules. The internal homomorphisms $\underline{\mathrm{Hom}}(M,N)$ are naturally equiped with two A-modules structures

$$\mu^l : A \otimes \underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M,N) \text{ and } \mu^r : \underline{\mathrm{Hom}}(M,N) \otimes A \to \underline{\mathrm{Hom}}(M,N).$$

Define the morphism $\delta^l : A \otimes \underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M,N)$ by

$$\delta^l = \mu^l - \mu^r \circ \mathrm{com}_{\underline{\mathrm{Hom}}(M,N),A}$$

and define $\delta^r : \underline{\mathrm{Hom}}(M,N) \otimes A \to \underline{\mathrm{Hom}}(M,N)$ by $\delta^r = -\delta^l \circ \mathrm{com}_{A,\underline{\mathrm{Hom}}(M,N)}$. Let $\delta^l_n : A^{\otimes n} \otimes \underline{\mathrm{Hom}}(M,N) \to \underline{\mathrm{Hom}}(M,N)$ be the n-tuple composition of δ^l.

Definition 2.31. The module of (internal) differential operators of order n from M to N is the sub-object $\underline{\mathrm{Diff}}_n(M,N)$ of $\underline{\mathrm{Hom}}(M,N)$ given by intersecting the kernel of δ^l_n with $\underline{\mathrm{Hom}}(M,N)$ in $A^{\otimes n} \otimes \underline{\mathrm{Hom}}(M,N)$. If M is a fixed object of $A - \textsc{Mod}(\mathcal{C})$, the representing object for the functor on $\underline{\mathrm{Diff}}_n(M,.) : A - \textsc{Mod}(\mathcal{C}) \to \mathcal{C}$ is called the Jet module of M and denoted $J^n(M)$.

In the tensor category of usual vector spaces, this gives back the usual definition of differential operators and jet modules that is used for example in [Nes03]. We now finish by the definition of the notion of connection.

Definition 2.32. Let M be an A-module. A connection on M is a morphism

$$\nabla : M \to \Omega^1_A \otimes_A M$$

such that

$$\nabla(f.m) = f.\nabla(m) + df \otimes m,$$

i.e., such that

$$\nabla \circ \mu^l_M = \mu^l_{\Omega^1_A \otimes_A M} \circ (\mathrm{id}_A \otimes \nabla) + d \otimes \mathrm{id}_M.$$

We carefully advice the reader that the left multiplication $\mu^l_{\Omega^1_A \otimes_A M}$ by A is twisted by the commutativity isomorphism because the tensor product uses the right A-module structure on Ω^1_A. This implies the consistency of our definition with the one given by Verbovetsky in [Ver96] for the super case.

One can find more on super-algebras in [DM99].

2.5.2 Superspaces and fermionic calculus

One can base the fermionic calculus on classical differential calculus on super-algebras.

Definition 2.33. Let $\mathrm{ALG}_{s,\mathbb{R}}^{sc}$ be the category of super-algebras whose even component is smoothly closed, equiped with the Zariski topology generated by $A \to A[f^{-1}]$ for f even. A super-space is a sheaf

$$X : \mathrm{ALG}_{s,\mathbb{R}}^{sc} \to \mathrm{SETS}.$$

The main example of super-space is given by affine super-spaces: if $A = A^0 \oplus A^1$ is a super-algebra, we denote $\mathrm{Spec}(A)$ the functor on super-algebras defined by

$$\mathrm{Spec}(A) : B \mapsto \mathrm{Hom}(A, B).$$

In particular, if $\pi : S \to M$ is a usual $\mathbb{Z}/2$-graded vector bundle on a usual variety M, we denote

$$\mathcal{C}_s^\infty(S) := \mathcal{C}^\infty(S^0) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, \wedge^*(S^1)^*),$$

where $(S^1)^*$ is the dual vector bundle to S^1. The affine super-space associated to $\pi : S \to M$ is given by the functor

$$\begin{array}{rccc} \underline{S} := \mathrm{Spec}(\mathcal{C}_s^\infty(S)) : & \mathrm{ALG}_{s,\mathbb{R}}^{sc} & \to & \mathrm{SETS} \\ & A & \mapsto & \mathrm{Hom}_{\mathrm{ALG}_{\mathbb{R}}}(\mathcal{C}_s^\infty(S), A). \end{array}$$

There is a natural space morphism

$$\pi : \underline{S} \to M.$$

We carefully inform the reader that for example, the global sections of this map, which are also retractions of the map

$$\pi^* : \mathcal{C}^\infty(M) \to \mathcal{C}_s^\infty(S),$$

are trivial on the variables in S^1, so that we arrive to the following strange statement:

there are no non-trivial fermionic trajectories with coefficients in $\mathbb{R}$.

Physicists, and among them, DeWitt in [DeW03], solve this problem by using points of $\underline{\Gamma}(M, \underline{S})$ with values in the free odd algebra on a countable number of generators

$$A = \Lambda_\infty := \widehat{\wedge^* \mathbb{R}^{(\mathbb{N})}}$$

(of course, his formulation is a bit different of ours, but the idea is the same). There is no reason to choose this coefficient algebra and not another one, so that the natural space of section $\underline{\Gamma}(M, S)$ must really be thought of as the functor on super-algebras

$$A \mapsto \underline{\Gamma}(M, S)(A)$$

which sends a super-algebra to the set of sections of $\pi : \underline{S} \to M$ parametrized by $\mathrm{Spec}(A)$, which at the end, identify with morphisms of super-spaces

$$x : M \times \mathrm{Spec}(A) \to \underline{S}.$$

One often works with super-varieties locally defined as subspaces of the affine super space, that is defined by $\mathbb{A}^{n,m}(A) := (A^0)^n \oplus (A^1)^m$. The algebra of functions (i.e., representing algebra) for this super-space is $\mathcal{C}^\infty(\mathbb{R}^n, \wedge^*\mathbb{R}^m)$. It is identified with the algebra of functions

$$f : \mathbb{A}^{n,m} \to \mathbb{A}^{1,1}$$

of super-spaces. Such a function can always be uniquely written

$$f = \sum_I f_I \theta^I$$

where I is an increasing multi-index and f_I are smooth functions on $\mathbb{R}^n$. By definition, morphisms $f : \mathbb{A}^{n,m} \to \mathbb{A}^{p,q}$ are identified with elements

$$f \in \mathcal{C}^\infty(\mathbb{R}^n, \wedge^{2*}\mathbb{R}^m)^p \oplus \mathcal{C}^\infty(\mathbb{R}^n, \wedge^{2*+1}\mathbb{R}^m)^q.$$

2.5.3 Super-integrals and super action functionals

We use here the Princeton's seminar book [DF99] and the article of Penkov [Pen83] (see also [Ver96]) for the abstract approaches and the first chapter of Zinn-Justin's book [ZJ93] for a very effective presentation.

The notion of integration in the super-setting cannot be based on usual differential forms because there are no top differential forms (the de Rham complex is infinite because an odd exterior power is just a usual symmetric power in the classical setting): one has to use the Berezinian. This is necessary to understand supersymmetric lagrangian, but does not affect the case of fermionic bundles $\pi : S \to M$ for S a super-space and M a classical space (except if one wants to quantize using the fermionic functional integral). We will come back to this topic with more details in the chapter on local functional calculus.

Definition 2.34. Let A be an algebra in a symmetric monoidal category. Let V be an A-module and $\mathrm{Sym}^*_A(V) \to A$ be the augmentation morphism. The Berezinian of M is defined as

$$\mathrm{Ber}(V) := \underline{\mathrm{Ext}}_{\mathrm{Sym}^*_A(V)}(A, \mathrm{Sym}^*_A(V^*)).$$

If V is a free module of rank (p,q) on a commutative super algebra A, the only non-zero $\underline{\mathrm{Ext}}$ is $\underline{\mathrm{Ext}}^p$ and it is a free A-module of rank $(1,0)$ for q even and $(0,1)$ for q odd. If X is a smooth super-variety, one defines

$$\mathrm{Ber}(X) := \mathrm{Ber}(\Omega^1_X).$$

Remark that there is another approach to the Berezinian and to more general integral forms. It can be found in the work of Verbovetsky [Ver96] (see also [KV98]), and also in a $\mathcal{D}$-module language in the article of Penkov [Pen83]. We just cite the definition due to Penkov of the complex of integral forms in the setting of $\mathcal{D}$-modules, refering to the forthcoming chapter on local functional calculus for more details.

The following definition can't be understood with the material treated before but we will explain its terms in details in the chapter on local functional calculus.

Definition 2.35. Let X be a smooth super-variety of super dimension $n|m$. The Berezinian of X is defined as

$$\mathrm{Ber}(X) = \mathrm{Ext}^n_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{D}_X).$$

The complex of integral forms on X is defined as

$$\Sigma_\bullet X := \mathbb{R}\mathcal{H}om_{\mathcal{D}_X}(\mathrm{Ber}(\Omega^1_X), \mathrm{Ber}(\Omega^1_X)) = \mathcal{H}om(\mathrm{DR}^\bullet(\mathcal{D})[n], \mathrm{DR}^\bullet(\mathcal{D})[n]),$$

where $\mathrm{DR}^\bullet(\mathcal{D}) := \wedge^\bullet \Omega^1_X \otimes_{\mathcal{O}_X} \mathcal{D}_X$ is the universal de Rham complex.

Theorem 5. *Let X be a smooth variety (super-variety of dimension $n|0$). There are canonical isomorphisms*

$$\mathrm{Ber}(X) \cong \Omega^n_X \text{ and } \Sigma_\bullet X \cong \Omega^\bullet_X.$$

An explicit computation of the Berezinian of a matrix

$$M = \begin{pmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{pmatrix}$$

is given by

$$\mathrm{Ber}(M) = \frac{\det(M_{11} - M_{12}M_{22}^{-1}M_{21})}{\det(M_{22})}.$$

It satisfies

$$\mathrm{Ber}(MN) = \mathrm{Ber}(M)\mathrm{Ber}(N).$$

Recall that $\mathbb{R}^{n,m}$ is the super-space defined by $\mathbb{R}^{n,m}(A) = (A^0)^n \oplus (A^1)^m$. It is represented by the algebra $\mathcal{C}^\infty(\mathbb{R}^n, \wedge^*\mathbb{R}^m)$. A function $f : \mathbb{R}^{n,m} \to \mathbb{R}^{1,1}$ is thus by definition an element in $\mathcal{C}^\infty(\mathbb{R}^n, \wedge^*\mathbb{R}^m)$, that can be written uniquely as a formal sum

$$f = \sum_I f_I \theta^I$$

with $f_I : \mathbb{R}^n \to \mathbb{R}$ some smooth functions and $\theta = (\theta_1, \dots, \theta_m)$ the universal odd variable, that allows to define the Berezinian on a given super-variety (if its underlying variety is oriented).

The integral of such a function is defined as

$$\int_{\mathbb{R}^{m,n}} f\, dx d\theta := \int_{\mathbb{R}^m} f_{\theta_1 \dots \theta_m}(x) dx.$$

The main usefulness of the Berezinian is due to the following change of variable formula.

Theorem 6. *Let $U, V \subset \mathbb{R}^{n,m}$ be open subspaces and $\varphi : U \to V$ be a smooth isomorphism. If $g : V \to \mathbb{R}^{1,1}$ is a smooth function on V, one has the equality*

$$\int g\, dx d\theta = \int \mathrm{Ber}(d\varphi).g \circ \varphi\, dx d\theta.$$

Let M be a super variety and $\mathrm{Ber}_M := \mathrm{Ber}(\Omega^1_M)$ as a right $\mathcal{D}_M$-module. Let $E \to M$ be a super-bundle. If $J^\infty E \to M$ is the jet bundle and $\mathcal{A}$ is the corresponding left $\mathcal{D}_M$-algebra of functions on $J^\infty E$, we define $h(\mathcal{A}) := \mathrm{Ber}_M \otimes_{\mathcal{D}} \mathcal{A}$. If $\nu \in h(\mathcal{A})$, one defines the local super-action associated to ν as the map of super-spaces

$$S_\nu : \Gamma(M, E) \to \mathbb{R}$$

given by

$$S_\nu(x) = \int_M (j_\infty x)^* \nu.$$

If M is a classical variety, we find back usual action functionals. One also often asks the action functional (or the space of solutions) to be invariant by the action of a super-Poincaré group, in which case one calls the super-lagrangian super-symmetric.

Chapter 3

Symmetries

Before giving many examples of physical theories, we need to explain what physicist call symmetries of physical theories. The main reason for this is that one can not even define the Dirac operator (which is the basis of the formalization of matter particles, called fermions) or the Yang-Mills lagrangian (which is the basis of the formalization of interaction particles, called bosons) without knowing what a symmetry is. Physicists actually often construct physical theories not only basing on experiments, but also by symmetry considerations.

3.1 Structured spaces and group spaces

In our general setting for spaces, we can define a general notion of symmetry, that includes gauge symmetries of physics. We let $(\textsc{Legos}, \tau)$ be a category with Grothendieck topology. Let $C \to \textsc{Sets}$ be a category equiped with a natural (called forgetful) functor to sets. Most of the categories you know are of this type.

Definition 3.1. A C-space is a space $X : \textsc{Legos} \to \textsc{Sets}$ that is equiped with a C structure, meaning that it factors through the forgetful functor $C \to \textsc{Sets}$ from C to sets as a functor

$$X : \textsc{Legos} \to C.$$

In particular, one can talk of

- a group space for $C = \textsc{Grp}$,
- a lie algebra space for $C = \textsc{Lie}$, etc...

If one works in the algebraic category $(\text{Legos}, \tau) = (\text{Rings}, \tau_{Zar})$ given by (commutative unital) rings and their Grothendieck topology, one gets algebraic group spaces. A particular example of this one is given by the general linear group $\mathrm{GL}_n \subset M_n \times M_n$ whose points with value in a ring A are given by

$$\mathrm{GL}_n(A) := \{(M, N) \in M_n(A) \times M_n(A) \cong A^{n^2} \times A^{n^2} \,|\, MN = NM = I\}.$$

This group space is represented by the algebra

$$A_{\mathrm{GL}_n} := \mathbb{Z}\left[\{M_{i,j}\}_{i,j=1,\ldots,n}, \{N_{i,j}\}_{i,j=1,\ldots,n}\right]/(MN = NM = I)$$

meaning that there is natural bijection

$$\mathrm{GL}_n(A) \cong \mathrm{Hom}(A_{\mathrm{GL}_n}, A).$$

These kind of group spaces play an important role as building blocs for local gauge symmetries in Yang-Mills gauge theory, and also in the definition of the Dirac operator for matter particles.

Another good motivation to use the general notion of group spaces is that many examples of infinite dimensional groups are present in physics. If M is space of some kind, the space

$$\underline{\mathrm{Aut}}(M) : \text{Legos} \to \text{Grp}$$

of all its parametrized families of automorphisms is also a group space of the same kind. This kind of group naturally appears as symmetry group for example:

- in general relativity, where the gauge symmetry group is given by diffeomorphisms (i.e. reparametrizations) of the base manifold,
- in Yang-Mills theory where the gauge symmetry group is given by the space $\mathrm{Hom}(M, G)$ for M the space of parameters for trajectories and G a finite dimensional group space like for example the unitary group $U(n)$ to be studied in the next subsection.

There is no reason to avoid these kinds of groups by only studying finite dimensional group varieties, since the category of group spaces contain all useful symmetry groups of the physical theories we are interested in. This is another motivation for using the general notion of space instead of the classical notion of variety.

Remark that the notion of Lie algebra space is also very useful for gauge theory since physicists often replace computations in groups by computations in Lie algebras. For example, the Lie algebra of symmetries of a Yang-Mills theory is given by the space $\mathrm{Hom}(M, \mathfrak{g})$ for M the space of parameters and $\mathfrak{g}$ a finite dimensional Lie algebra.

We finish by giving a very general definition of gauge symmetry space, based on the notion of morphism of lagrangian variational problem, and that contains naturally the above examples.

Definition 3.2. The gauge symmetry group of a given lagrangian system is its space of automorphisms.

The notion of gauge symmetry used in physics is often what we would call here an infinitesimal *local* [1] gauge symmetry, which is something like a (local) vector field (section of the tangent bundle to the space of histories)

$$\vec{v} : H \to TH$$

that annihilates the action functional, i.e., such that

$$\langle dS, \vec{v} \rangle : H \to \mathbb{R}$$

is the zero functional.

3.2 Algebraic and Lie groups

The reader who wants to have a partial and simple summary can use Waterhouse's book [Wat79]. For many more examples and a full classification of algebraic groups on non-algebraically closed fields (like $\mathbb{R}$ for example, that is useful for physics), we refer to the very complete book of involutions [KMRT98].

Definition 3.3. A linear algebraic group over $\mathbb{R}$ is an algebraic group space that is representable by a quotient of a polynomial ring, i.e., it is a space

$$G : \textsc{Alg}_{\mathbb{R}} \to \textsc{Grp}$$

and a function algebra A_G such that for any algebra A, there is a natural bijection

$$G(A) \xrightarrow{\sim} \mathrm{Hom}_{\textsc{Alg}_{\mathbb{R}}}(A_G, A).$$

Similarly, a linear Lie group over $\mathbb{R}$ is a smooth group space that is representable by a smooth variety, i.e., it is a smooth group space

$$G : \textsc{Alg}_{sc,\mathbb{R}} \to \textsc{Grp}$$

that is representable by an algebra $A \in \textsc{Alg}_{sc,\mathbb{R}}$ of functions on a smooth variety G.

[1] See the chapter on local functional calculus for the meaning of the word local here.

One can show that any linear algebraic group G is a Zariski closed subgroup of the group $\mathrm{GL}_{n,\mathbb{R}}$ for some n that we defined in the previous subsection by

$$\mathrm{GL}_n(A) := \{(M, N) \in M_n(A) \times M_n(A) \cong A^{n^2} \times A^{n^2} |\, MN = NM = I\}.$$

For example, the additive group $\mathbb{G}_a$ whose points are given by

$$\mathbb{G}_a(A) = A$$

with its additive group structure is algebraic with function algebra $\mathbb{R}[X]$ and it can be embedded in GL_2 by the morphism

$$\begin{array}{rccc} r: & \mathbb{G}_a & \to & \to \mathrm{GL}_2 \\ & a & \mapsto & \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right). \end{array}$$

Remark that if V is a real vector space (without a particular choice of a basis, that would break the symmetry, as physicists use to say), one can also define $\mathrm{GL}(V)$ by

$$\mathrm{GL}(V)(A) := \{(M, N) \in \mathrm{End}_A(V \otimes A)^2 |\, MN = NM = \mathrm{id}_V\}.$$

A representation of an algebraic group, also called a G-module, is a group morphism

$$G \to \mathrm{GL}(V)$$

where V is a finite dimensional vector space.

Definition 3.4. The Lie algebra of the group G is the space

$$\begin{array}{rccc} \mathrm{Lie}(G): & \textsc{Rings} & \to & \textsc{Grp} \\ & B & \mapsto & \{X \in \mathrm{Hom}_{\textsc{Rings}}(A_G \otimes B, B[\epsilon]/(\epsilon^2) | X \mod \epsilon = I\}. \end{array}$$

One can think of elements of the Lie algebra as points $I + \epsilon X \in G(B[\epsilon]/(\epsilon^2))$ where X plays the role of a tangent vector to G at the identity point. It is equiped with a natural Lie bracket operation

$$[.,.] : \mathrm{Lie}(G) \times \mathrm{Lie}(G) \to \mathrm{Lie}(G)$$

given for $X \in G(R[\epsilon_1]/(\epsilon_1^2))$ and $Y \in G(R[\epsilon_2]/(\epsilon_2^2))$ by the commutator

$$XYX^{-1}Y^{-1} = I + \epsilon_1\epsilon_2[X, Y] \in G(R[\epsilon_1\epsilon_2]/(\epsilon_1\epsilon_2)^2) \cong G(R[\epsilon]/(\epsilon^2)).$$

One has a natural action of G on its Lie algebra called the adjoint action

$$\rho_{ad} : G \to \mathrm{GL}(\mathrm{Lie}(G))$$

defined by

$$g \mapsto \left[X \in G(R[\epsilon]/(\epsilon^2)) \mapsto g^{-1}Xg \in G(R[\epsilon]/(\epsilon^2))\right].$$

It is also equiped with a module structure over the ring space $\mathbb{A}$ given by the affine space $\mathbb{A}(B) = B$ for which the bracket is linear.

3.3 Algebras with involutions and algebraic groups

The study of algebraic groups on non-algebraically closed fields (for example unitary groups over $\mathbb{R}$) is easier to do in the setting of algebras with involutions because one can do there at once a computation that works for all classical groups. For example, the computation of Lie algebras is done only once for all classical groups (orthogonal, symplectic, linear and unitary) in this section. We refer to the book of involutions [KMRT98] for a very complete account of this theory.

On non-algebraically closed fields, algebraic groups are defined using a generalization of matrix algebras called central simple algebras, and involutions on them.

Definition 3.5. Let F be a field. A central simple algebra A over F is a finite dimensional nonzero algebra A with center F which has no two-sided ideals except $\{0\}$ and A. An involution on (a product of) central simple algebra(s) A is a map $\sigma : A \to A$ such that for all $x, y \in A$,

1. $\sigma(x + y) = \sigma(x) + \sigma(y)$,
2. $\sigma(xy) = \sigma(y)\sigma(x)$,
3. $\sigma^2 = \mathrm{id}$.

The center F of A is stable by σ and $\sigma_{|F}$ is either an automorphism of order 2 or the identity. If the first case, F is of degree 2 over the fixed subfield E.

Theorem 7 (Wedderburn)**.** *Let A be a finite dimensional algebra over F. The following conditions are equivalent:*

1. *A is a central simple algebra over F.*
2. *There exists a finite extension F'/F on which A becomes a matrix algebra, i.e., such that*
$$A_{F'} \cong M_n(F).$$

Corollary 4. *Let A be a central simple algebra of dimension n^2 over F. Let*

$$i : A \to \mathrm{End}_F(A)$$

by the action of A on itself by left multiplication. The characteristic polynomial of $i(a)$ is the n-th power of a polynomial

$$P_{A,a} = X^n - \mathrm{Tr}_A(a)X^{n-1} + \cdots + (-1)^n \mathrm{Nm}_A(a)$$

of degree n called the reduced characteristic polynomial. The coefficients Tr_A *and* Nm_A *are respectively called the reduced norm and reduced trace on A.*

Before giving some examples of algebras with involutions, we define the corresponding algebraic groups.

Definition 3.6. Let (A, σ) be an algebra (that is a product of central simple algebras) with involution over (a product of fields) F.

1. The group $\mathrm{Isom}(A, \sigma)$ of isometries of (A, σ) is defined by
$$\mathrm{Isom}(A, \sigma)(R) = \{a \in A_R^\times |\, a.\sigma_R(a) = 1\}.$$
2. The group $\mathrm{Aut}(A, \sigma)$ of automorphisms of (A, σ) is defined by
$$\mathrm{Aut}(A, \sigma)(R) = \{\alpha \in \mathrm{Aut}_R(A_R) | \alpha \circ \sigma_R = \sigma_R \circ \alpha\}.$$
3. The group $\mathrm{Sim}(A, \sigma)$ of similitudes of (A, σ) is defined by
$$\mathrm{Sim}(A, \sigma)(R) = \{a \in A_R^\times | a.\sigma(a) \in K_R^\times\}.$$

The first example of algebra with involution is the one naturally associated to a non-degenerate bilinear form $b : V \times V \to F$ on an F-vector space. One defines an algebra with involution (A, σ_b) by setting $A = \mathrm{End}_F(V)$ and the involution σ_b defined by
$$b(x, f(y)) = b(\sigma_b(f)(x), y)$$
for $f \in A$ and $x, y \in V$. If the bilinear form b is symmetric, the corresponding groups are the classical groups
$$\mathrm{Iso}(A, \sigma) = \mathrm{O}(V, b),\ \mathrm{Sim}(A, \sigma) = \mathrm{GO}(V, b),\ \mathrm{Aut}(A, \sigma) = \mathrm{PGO}(V, b).$$
If the bilinear form b is antisymmetric, the corresponding groups are
$$\mathrm{Iso}(A, \sigma) = \mathrm{Sp}(V, b),\ \mathrm{Sim}(A, \sigma) = \mathrm{GSp}(V, b),\ \mathrm{Aut}(A, \sigma) = \mathrm{PGSp}(V, b).$$

Another instructive example is given by the algebra
$$A = \mathrm{End}_F(V) \times \mathrm{End}_F(V^\vee)$$
where V is a finite dimensional vector space over F, equiped with the involutions sending (a, b) to $(b^\vee, a^\vee)$. It is not a central simple algebra but a product of such over $F \times F$ and $\sigma_{|F}$ is the exchange involution. One then gets
$$\begin{array}{ccc} \mathrm{GL}(V) & \xrightarrow{\sim} & \mathrm{Iso}(A, \sigma) \\ m & \mapsto & (m, (m^{-1})^\vee) \end{array}$$

and

$$\mathrm{PGL}(V) \cong \mathrm{Aut}(A,\sigma).$$

If A is a central simple algebra over a field F with σ-invariant subfield $E \subset F$ of degree 2, one gets the unitary groups

$$\mathrm{Iso}(A,\sigma) = \mathrm{U}(V,b),\ \mathrm{Sim}(A,\sigma) = \mathrm{GU}(V,b),\ \mathrm{Aut}(A,\sigma) = \mathrm{PGU}(V,b).$$

It is then clear that by scalar extension to F, one gets an isomorphism

$$U(V,b)_F \cong \mathrm{GL}(V_F)$$

by the above construction of the general linear group. One also defines $\mathrm{SU}(V,q)$ as the kernel of the reduced norm map

$$\mathrm{Nm} : \mathrm{U}(V,b) \to F^\times.$$

We now compute at once all the classical Lie algebras.

Proposition 11. *Let (A,σ) be a central simple algebra with involution. We have*

$$\mathrm{Lie}(\mathrm{Sim}(A,\sigma)) = \{m \in A,\ m + \sigma(m) = 0\}.$$

Proof. If $1 + \epsilon m \in \mathrm{Sim}(A,\sigma)(F[\epsilon]/(\epsilon^2))$ is a generic element equal to 1 modulo ϵ, the equation

$$(1 + m\epsilon)(1 + \sigma(m)\epsilon) = 0$$

implies $a + \sigma(a) = 0$. □

The above proposition gives us the simultaneous computation of the Lie algebras of $\mathrm{U}(V,b)$, $\mathrm{O}(V,b)$ and $\mathrm{Sp}(V,b)$.

3.4 Clifford algebras and spinors

This section is treated in details in the book of involutions [KMRT98], in Deligne's notes on spinors [Del99] and in Chevalley's book [Che97]. The physical motivation for the study of the Clifford algebra is the necessity to define a square root of the laplacian (or of the D'Alembertian) operator, as is nicely explained in Penrose's book [Pen05], section 24.6.

3.4.1 Clifford algebras

Let (V, q) be a symmetric bilinear space over a field K. The Clifford algebra $\mathrm{Cliff}(V, q)$ on the bilinear space (V, q) is the universal associative and unitary algebra that contains V and in which q defines the multiplication of vectors in V. More precisely, it fulfils the universal property

$$\mathrm{Hom}_{\mathrm{ALG}_K}(\mathrm{Cliff}(V, q), B) \cong \{j \in \mathrm{Hom}_{\mathrm{VECT}_K}(V, B) | j(v).j(w) + j(w).j(v) = q(v, w).1_B\}$$

for every associative unitary K-algebra B.

One defines explicitely $\mathrm{Cliff}(V, q)$ as the quotient of the tensor algebra $T(V)$ by the bilateral ideal generated by expression of the form

$$m \otimes n + n \otimes m - q(m, n).1 \text{ with } m, n \in V.$$

Remark that the degree filtration on the tensor algebra $T(V)$ induces a natural filtration F on the Clifford algebra whose graded algebra

$$\mathrm{gr}^F(\mathrm{Cliff}(V, q)) = \wedge^* V$$

is canonically isomorphic to the exterior algebra. One can extend this isomorphism to a linear isomorphism

$$\wedge^* V \xrightarrow{\sim} \mathrm{Cliff}(V, q)$$

by showing that the linear maps $f_k : \wedge^k V \to \mathrm{Cliff}(V, q)$ defined by

$$f_k(v_1, \ldots, v_k) = \frac{1}{k!} \sum_{\sigma \in S_k} \mathrm{sgn}\ (\sigma) v_{\sigma(1)} \ldots v_{\sigma(k)}$$

induces the above isomorphism of graded algebras.

Consequently, one has $\dim_K(\mathrm{Cliff}(V, q)) = 2^{\dim_K V}$. The automorphism $\alpha : V \to V$ given by $m \mapsto -m$ induces an automorphism $\alpha : \mathrm{Cliff}(V) \to \mathrm{Cliff}(V)$ such that $\alpha^2 = \mathrm{id}$. One thus gets a decomposition

$$\mathrm{Cliff}(V, q) = \mathrm{Cliff}^0(V, q) \oplus \mathrm{Cliff}^1(V, q),$$

giving $\mathrm{Cliff}(V, q)$ a super-algebra structure.

Proposition 12. *Let (V, q) be an even dimensional quadratic space. The clifford algebra $\mathrm{Cliff}(V, q)$ is a central simple algebra and $\mathrm{Cliff}^0(V, q)$ is either central simple over a quadratic extension of K or a product of two central simple algebras over K.*

Proof. The general structure theorem for even Clifford algebras can be found in the book of involution [KMRT98], page 88. To prove that $\mathrm{Cliff}(V,q)$ is central simple, one just has to do it on the algebraic closure of K. On an algebraically closed field, the structure of the Clifford algebra can by understood by the following: on such a field, the quadratic space (V,q) with V of dimension $2n$ is isomorphic to a hyperbolic quadratic space of the form $(H(U),q_h)$ with U a subspace of dimension n, U^* its linear dual and $H(U)=U\oplus U^*$ its hyperbolic quadratic space equiped with the bilinear form $q_h(v\oplus\omega)=\omega(v)$. Denote

$$S:=\wedge^*U.$$

For $u+\varphi\in H(U)$, let ℓ_u be the left exterior multiplication by u on S and d_φ be the unique derivation on S extending φ, given by

$$d_\varphi(x_1\wedge\cdots\wedge x_r)=\sum_{i=1}^r(-1)^{i+1}x_1\wedge\dots\hat{x}_i\wedge x_r\varphi(x_i).$$

One shows that the map

$$\begin{array}{rcl} H(U) & \to & \mathrm{End}(\wedge^*U)\\ \varphi+u & \mapsto & \ell_u+d_\varphi \end{array}$$

extends to an isomorphism

$$\mathrm{Cliff}(V,q)\xrightarrow{\sim}\mathrm{End}(S).$$

This isomorphism is actually an isomorphism

$$\mathrm{Cliff}(V,q)\xrightarrow{\sim}\underline{\mathrm{End}}(S)$$

of super-algebras. Moreover, if we denote

$$S^+:=\wedge^{2*}U \text{ and } S^-:=\wedge^{2*+1}U$$

the even and odd parts of S, the restriction of this isomorphism to $\mathrm{Cliff}^0(V,q)$ induces an isomorphism

$$\mathrm{Cliff}^0(V,q)\xrightarrow{\sim}\mathrm{End}_F(S^+)\times\mathrm{End}_F(S^-)$$

which can be seen as the isomorphism

$$\mathrm{Cliff}^0(V,q)\xrightarrow{\sim}\underline{\mathrm{End}}^0(S)$$

induced by the above isomorphism on even parts of the algebras in play. $\square$

Definition 3.7. Let $(V,q) = (\mathbb{H}(U), q_h)$ be a hyperbolic quadratic space. The representations S of $\mathrm{Cliff}(V,q)$ and S^+ and S^- of $\mathrm{Cliff}^0(V,q)$ are respectively called the spinorial and semi-spinorial representations.

Remark 3.1. The fact that $\mathrm{gr}^F\mathrm{Cliff}(V,q) \cong \wedge^* V$ can be interpreted by saying that the Clifford algebra is the canonical quantization of the exterior algebra. Indeed, the Weyl algebra (of polynomial differential operators) also has a filtration whose graded algebra is the polynomial algebra, and this is interpreted by physicists as the fact that the Weyl algebra is the canonical quantization of the polynomial algebra. More precisely, if V is a vector space, and ω is a symplectic 2-form on V (for example, the fiber of a cotangent bundle with its canonical 2-form), one can define the algebra of algebraic differential operators on V with the algebra that fulfils the universal property

$$\mathrm{Hom}_{\mathrm{ALG}_{\mathbb{R}}}(\mathcal{D}_V, B) \cong \{j \in \mathrm{Hom}_{\mathrm{VECT}_{\mathbb{R}}}(V,B) | j(v).j(w) - j(w).j(v) = \omega(v,w).1_B\}.$$

In this case, the graded algebra

$$\mathrm{gr}^F\mathcal{D}_V \cong \mathrm{Sym}^*(V)$$

is identified with the polynomial algebra $\mathrm{Sym}^*(V)$. The above isomorphism can be extended to a linear isomorphism

$$\mathcal{D}_V \to \mathrm{Sym}^*(V)$$

by using the symmetrization formula

$$a_1 \dots a_n \mapsto \frac{1}{n!} \sum_{\sigma \in S_n} a_{\sigma(1)} \dots a_{\sigma(n)}.$$

The corresponding product on $\mathrm{Sym}^*(V)$ is called the Moyal product. A combination of these two results can be given in the setting of Clifford super-algebras with respect to complex valued super-quadratic forms. The introduction of Clifford algebras in the definition of the Dirac operator and in fermionic field theory is thus, in this sense, equivalent to the use of classical anticommuting coordinates on fermionic bundles.

The above analogy between differential operators and elements of the Clifford algebra can be used in geometry, as is explained in the paper of Getzler [Get83] on the super proof of the Atiyah-Singer index theorem. Let M be a differential variety. One can think of differential forms $\Omega^*(M)$ as functions on the super-bundle $T[1]M$, and the Clifford algebra of $TM \oplus T^*M$, being isomorphic to $\mathrm{End}(\wedge^\bullet T^*M)$, gives operators on the state space $\Omega^*(M)$. This situation is analogous to the Weyl quantization of the symplectic variety (T^*M, ω) by the action of differential operators $\mathcal{D}(M)$ on $L^2(M)$.

3.4.2 Spin group and spinorial representations

The Clifford group of (V, q) is the group

$$\Gamma(V, q) := \{c \in \mathrm{Cliff}(V, q)^\times | cv\alpha(c)^{-1} \in v \text{ for all } v \in V\}.$$

One has by definition a natural action (by conjugation) of this group on V that induces a morphism

$$\Gamma(V, q) \to \mathrm{O}(V, q).$$

The special Clifford group of (V, q) is the subgroup $\Gamma^+(V, q)$ of even elements in $\Gamma(V, q)$ given by

$$\Gamma^+(V, q) = \Gamma(V, q) \cap \mathrm{Cliff}^0(V, q).$$

The tensor algebra $T(V)$ is naturally equiped with an anti-automorphism given by $m_1 \otimes \cdots \otimes m_k \mapsto m_k \otimes \cdots \otimes m_1$ on the homogeneous elements of degree k. Since the defining ideal for the Clifford algebra is stable by this anti-automorphism (because q is symmetric), one gets an antiautomorphism of $\mathrm{Cliff}(V, q)$ called the transposition and denoted $c \mapsto^t c$. The spinorial group is the subgroup of $\Gamma^+(V, q)$ defined by

$$\mathrm{Spin}(V, q) = \{c \in \Gamma^+(V, q) | \, {}^t cc = 1\}.$$

One has a natural action (by multiplication) of $\mathrm{Spin}(V, q)$ on the real vector space $\mathrm{Cliff}(V, q)$ that commutes with the idempotent $\alpha : \mathrm{Cliff}(V, q) \to \mathrm{Cliff}(V, q)$ and thus decomposes in two representations $\mathrm{Cliff}^0(V, q)$ and $\mathrm{Cliff}^1(V, q)$.

Definition 3.8. Let (V, q) be an even dimensional quadratic space. A rational representation S of $\mathrm{Spin}(V, q)$ will be called

1. a rational spinorial representation if it is a $\mathrm{Cliff}(V, q)$-irreducible submodule of the action of $\mathrm{Cliff}(V, q)$ on itself.

2. a rational semi-spinorial representation if it is a $\mathrm{Cliff}^0(V, q)$-irreducible submodule of the action of $\mathrm{Cliff}^0(V, q)$ on itself.

If K is algebraically closed and $(V, q) = (\mathbb{H}(U), q_h)$ is the hyperbolic quadratic space, one has $\mathrm{Cliff}(V, q) \cong \mathrm{End}(\wedge^* U)$, so that the spinorial representation is simply $\wedge^* U$ and the semi-spinorial representations are $S^+ = \wedge^{2*} U$ and $S^- = \wedge^{2*+1} U$.

In the particular case of the Lorentzian form $q(t, x) = -c^2 t^2 + x_1^2 + x_2^2 + x_3^2$ on Minkowski space $V = \mathbb{R}^{3,1}$, the algebra $\mathrm{Cliff}^0(V, q)$ is isomorphic to the algebra $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}}$ of complex matrices viewed as a real algebra. The group $\mathrm{Spin}(3, 1) := \mathrm{Spin}(V, q)$ then identifies to the group $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{SL}_{2,\mathbb{C}}$ of complex matrices of determinant 1 seen as a real algebraic

group. There is a natural representation of $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}M_{2,\mathbb{C}}$ on the first column $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$ of $\mathrm{Cliff}^0(V,q) \cong \mathrm{Res}_{\mathbb{C}/\mathbb{R}}M_{2,\mathbb{C}}$ and this is the representation of $\mathrm{Spin}(3,1)$ that we will call the real spinor representation. Its extension to $\mathbb{C}$ decomposes in two representations isomorphic to $\mathbb{C}^2$ that are exchanged by complex conjugation. These two representations are called the semi-spinorial representations and correspond in physics to the electron and the anti-electron.

For a more concrete and motivated approach to these matters, the reader is adviced to read section 24.6 of Penrose's book [Pen05].

3.4.3 Pairings of spinorial representations

Let (V,q) be an even dimensional quadratic space. Remark that the natural action of the vector space $V \subset \mathrm{Cliff}(V,q)$ on a rational spinorial representation S of $\mathrm{Cliff}(V,q)$ induces a morphism of $\mathrm{Spin}(V,q)$-representations

$$V \otimes S \to S.$$

This pairing will be useful to describe the Dirac operator on the spinor bundle.

If $(V,q) = (\mathbb{H}(U), q_h)$ is hyperbolic, one has two semispinorial representations S^+ and S^- of $\mathrm{Spin}(V,q)$ and the above morphism induces two morphisms of $\mathrm{Spin}(V,q)$ representations

$$V \otimes S^{\pm} \to S^{\mp}$$

which are also useful to define Dirac operators between semispinorial bundles. More general semispinorial representations have to be studied on a case by case basis (see [Del99], part I, for a more complete description).

We now need to define the two natural pairings that are necessary to define the Dirac lagrangian.

Proposition 13. *Let $(V,q) = (\mathbb{H}(U), q_h)$ be a hyperbolic even dimensional quadratic space. The spinor representation $S = \wedge^* U$ is naturally equiped with three pairings*

$$\epsilon : S \otimes S \to K,$$

and

$$\Gamma : S \otimes S \to V,\ \tilde{\Gamma} : S \otimes S \to V.$$

Proof. The form $\epsilon : S \otimes S \to K$ is the nondegenerate bilinear form on S for which

$$\epsilon(vs,t) = \epsilon(s,vt)$$

for all $v \in V$. It is the bilinear form whose involution on

$$\underline{\mathrm{End}}(S) \cong \mathrm{Cliff}(V, q)$$

is the standard anti-involution of $\mathrm{Cliff}(V, q)$. The Clifford multiplication map

$$c : V \otimes S \to S$$

induces a morphism $c : S^\vee \otimes S \to V^\vee$, that gives a morphism

$$\Gamma := q^{-1} \circ c \circ (\epsilon^{-1} \otimes \mathrm{id}_S) : S \otimes S \to V.$$

One the defines $\tilde{\Gamma}$ by

$$\tilde{\Gamma} := \Gamma \circ (\epsilon \otimes \epsilon) : S^\vee \otimes S^\vee \to V.$$

□

We now give the real Minkowski version of the above proposition, which is useful to define the super Poincaré group, and whose proof can be found in [Del99], theorem 6.1.

Proposition 14. *Let (V, q) be a quadratic space over $\mathbb{R}$ of signature $(1, n-1)$. Let S be an irreducible real spinorial representation of $\mathrm{Spin}(V, q)$. The commutant Z of $S_\mathbb{R}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.*

1. *Up to a real factor, there exists a unique symmetric morphism $\Gamma : S \otimes S \to V$. It is invariant unver the group Z^1 of elements of norm 1 in Z.*

2. *For $v \in V$, id $Q(v) > 0$, the form $(s, t) \mapsto q(v, \Gamma(s, t))$ on S is positive or negative definite.*

3.5 General structure of linear algebraic groups

We refer to the Grothendieck-Demazure seminar [DG62] for the general theory of root systems in linear algebraic groups and for the proof of the classification theorem for reductive groups.

Definition 3.9. An algebraic group is called:

1. reductive if the category of its representations is semi-simple, i.e., every exact sequence

$$0 \to U \to V \to W \to 0$$

   of G-module has an equivariant splitting, i.e., induces a direct sum decomposition

$$V \cong W \oplus U.$$

2. unipotent if it is a successive extension of additive groups.

3. a torus if after extending it to a (here the) finite extension $\mathbb{C}$ of $\mathbb{R}$, by $A_{G_\mathbb{C}} := A_G \otimes_\mathbb{R} \mathbb{C}$, it becomes isomorphic to GL_1^n (one says the torus splits on $\mathbb{C}$).

Let T be a torus. Its character group is the functor

$$\begin{array}{cccc} X^*(T): & \textsc{Alg}_\mathbb{R} & \to & \textsc{Grab} \\ & B & \mapsto & \mathrm{Hom}(T_B, \mathrm{GL}_{1,B}) \end{array}$$

with values in the category of $\textsc{Grab}$ of abelian groups and its cocaracter group is the functor

$$\begin{array}{cccc} X_*(T): & \textsc{Rings} & \to & \textsc{Grab} \\ & B & \mapsto & \mathrm{Hom}(\mathrm{GL}_{1,B}, T_B). \end{array}$$

One has a perfect pairing

$$\langle .,. \rangle : X_*(T) \times X^*(T) \to \underline{\mathbb{Z}}$$

given by $(r, s) \mapsto r \circ s$ with $\underline{\mathbb{Z}}$ the functor that associates to an algebra $A = \prod A_i$ product of simple algebras A_i the group $\mathbb{Z}^i$.

Lemma 3. *If $T = \mathrm{GL}_1^n$, a representation $T \to \mathrm{GL}(V)$ decomposes in a sum of characters: $V \cong \oplus \chi_i$ with $\chi_i \in X^*(T)$.*

Proposition 15. *Let G be a reductive group. The family of subtori $T \subset G$ has maximal elements that one calls maximal tori of G.*

The first structure theorem of linear algebraic groups gives a "dévissage" of general group in unipotent and reductive groups.

Theorem 8. *Let P be a linear algebraic group. There exists a biggest normal unipotent subgroup R_uP of P called the unipotent radical of P and the quotient P/R_uP is reductive.*

To give a more detailed study of the structure of reductive groups, one introduce new invariants called the roots.

Definition 3.10. Let G be a reductive group and T be a maximal torus of G. The roots of the pair (G, T) are the (non-trivial) weights of T in the adjoint representation of G on its Lie algebra. More precisely, they form a subspace of the space $X^*(T)$ of characters whose points are

$$R^* = \{\chi \in X^*(T) | \, \rho_{ad}(t)(X) = \chi(t).X\}.$$

The coroots $R_* \subset X_*(T)$ are the cocharacters of T that are dual to the roots with respect to the given perfect pairing between characters and cocharacters. The quadruple $\Phi(G, T) = (X_*(T), R_*, X^*(T), R^*, \langle .,. \rangle)$ is called the root system of the pair (G, T).

The main theorem of the classification theory of reductive group is that the root system determines uniquely the group and that every root system comes from a reductive group (whose maximal torus splits). Actually, the root system gives a system of generators and relations for the points of the give algebraic group. More precisely, every root correspond to a morphism

$$x_r : \mathbb{G}_a \to G$$

that is given by exponentiating the corresponding element of the Lie algebra, so that it fulfils

$$tx_r(a)t^{-1} = x_r(r(t)a)$$

and the group is generated by the images of these morphisms and by its maximal torus, the relations between them being given by the definition of a root. Actually, for any root $r \in R$, there is a homomorphism $\varphi_r : \mathrm{SL}_2 \to G$ such that

$$\varphi_r \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right) = x_r(a) \text{ and } \varphi_r \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix} \right) = x_{-r}(a),$$

the image of the diagonal matrices being given by the image of the dual coroot to the given root.

We don't want to define the abstract notion of root system, that would be a necessary step to explain that every abstract root system is the root system of a given reductive group. We just state the unicity result.

Theorem 9. *The functor $(G, T) \mapsto \Phi(G, T)$ from reductive groups to root systems is conservative: if two groups have the same root system, they are isomorphic.*

Remark that this theorem is often formalized in the setting of split reductive groups but the methods of SGA 3 [DG62] allow us to state it in general, if we work with spaces of characters as sheaves for the etale topology on the base field, i.e., as galois modules.

3.6 Representation theory of reductive groups

Because representation theory is very important in physics, and to be complete, we also recall the classification of representations of reductive groups (in characteristic 0). This can be found in the book of Jantzen [Jan87], part II.

A Borel subgroup B of G is a maximal closed and connected solvable subgroup of G. For example, the standard Borel subgroup of GL_n is the group of upper triangular matrices. If B is a Borel subgroup that contains the given maximal torus $T \subset G$, the

set of roots whose morphism $x_r : \mathbb{G}_a \to G$ have image in G are called positive roots and denoted $R^+ \subset R$. One defines an order on $X^*(T)$ by saying that

$$\lambda \leq \mu \Leftrightarrow \mu - \lambda \in \sum_{\alpha \in R^+} \mathbb{N}\alpha.$$

Any G-module V decomposes in weight spaces (representations of T, i.e., sums of characters) as

$$V = \oplus_{\lambda \in X^*(T)} V_\lambda.$$

For the given order on $X^*(T)$, every irreducible representation of G as a heighest non-trivial weight called the heighest weight of V. We define the set of dominant weights by

$$X^*(T)_{dom} := \{\lambda \in X^*(T),\ \langle \lambda, \alpha^\vee \rangle \geq 0 \text{ for all } \alpha \in R^+\}.$$

If λ is a dominant weight, we define the G-module $L(\lambda)$ by

$$L(\lambda) = \mathrm{ind}_B^G(\lambda) := (\lambda \otimes A_G)^B$$

where A_G denotes the algebra of functions on G.

The main theorem of classification of irreductible representations for reductive groups is the following.

Theorem 10. *Suppose that G is reductive and splitted, i.e., $T \cong \mathbb{G}_m^n$. The map*

$$\begin{array}{cccc} L: & X^*(T)_{dom} & \to & \textsc{RepIrr}/\sim \\ & \lambda & \mapsto & L(\lambda) \end{array}$$

is a bijection between the set of dominant weight and the set of isomorphism classes of irreducible representations of G.

If G is not splitted, i.e., if T is a twisted torus, one has to work a bit more to get the analogous theorem, that is due to Tits, and which can be found in the book of involutions [KMRT98], in the section on Tits algebras. Remark that many groups in physics are not splitted, and this is why their representation theory is sometimes tricky to handle.

3.7 Structure and representations of SL_2

As explained above, SL_2 is the main building bloc for any other algebraic groups. It is defined by

$$\mathrm{SL}_2 = \{M \in \mathrm{GL}_2, \det(M) = 1\}.$$

Its maximal torus is

$$T = \left\{ \left(\begin{smallmatrix} t & 0 \\ 0 & t^{-1} \end{smallmatrix} \right) \right\} \cong \mathrm{GL}_1.$$

The space of characters of T is identified with $\mathbb{Z}$ by $t \mapsto t^n$. The Lie algebra of SL_2 is given by

$$\mathrm{Lie}(\mathrm{SL}_2)(A) = \{I + \epsilon M \in \mathrm{SL}_2(A[\epsilon]/(\epsilon^2)),\ \det(I + \epsilon M) = 1\}.$$

If $M = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right)$, the determinant condition means $(1 + \epsilon a).(1 + \epsilon d) = 1 + \epsilon(a + d) = 1$, so that $\mathrm{Tr}(M) = 0$. We thus get

$$\mathrm{Lie}(\mathrm{SL}_2) = \{M \in M_2,\ \mathrm{Tr}(M) = 0\} = \left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix} \right) \right\}.$$

The adjoint action of $t \in T$ on $\mathrm{Lie}(\mathrm{SL}_2)$ is given by

$$\left\{ \left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix} \right) \right\} \mapsto \left\{ \left(\begin{smallmatrix} a & t^2 b \\ t^{-2} c & -a \end{smallmatrix} \right) \right\}$$

so that the roots are $r_2 : t \mapsto t^2$ and $r_{-2} : t \mapsto t^{-2}$. They correspond to the root morphisms

$$r_2, r_{-2} : \mathbb{G}_a \to \mathrm{SL}_2$$

given by

$$a \mapsto \left(\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix} \right) \text{ and } a \mapsto \left(\begin{smallmatrix} 1 & 0 \\ a & 1 \end{smallmatrix} \right).$$

Remark that SL_2 is generated by T and the images of the root morphisms.

One shows that the dominant weights of SL_2 are given by $\mathbb{Z}_{>0} \subset \mathbb{Z} = X^*(T)$ and the corresponding irreducible representations are given by

$$V_n := \mathrm{Sym}^n(V)$$

where V is the standard representation of SL_2.

For GL_2, the classification of representations is similar, and gives a family of irreducible representations

$$V_{n,m} := \mathrm{Sym}^n(V) \otimes (\det)^{\otimes m}$$

where det is the determinant representation and (n, m) is a pair of integers with $n > 0$ and $m \in \mathbb{Z}$.

3.8 Structure and representations of SU_2

Recall that SU_2 is defined by

$$\mathrm{SU}_2 = \{M \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}}, {}^t\bar{M}.M = M^t\bar{M} = 1\}.$$

Its Lie algebra can be computed by using

$$\mathfrak{su}_2 = \mathrm{Lie}(\mathrm{SU}_2) = \{M \in \mathrm{SU}_2(\mathbb{R}[\epsilon]/(\epsilon^2)),\ M = \mathrm{id} \mod \epsilon\}.$$

Indeed, if $I + \epsilon M$ is in $\mathfrak{su}_2$, it fulfils

$${}^t\bar{I}I + \epsilon[{}^t\bar{M}I +{}^t \bar{I}M] = 1$$

so that $\mathfrak{su}_2$ is given by

$$\mathfrak{su}_2 = \{M \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}},\ {}^t\bar{M} + M = 0\}.$$

Actually, if one defines the quaternion algebra by

$$\mathbb{H} := \{M \in \mathrm{Res}_{\mathbb{C}/\mathbb{R}} M_{2,\mathbb{C}}, \exists \lambda \in \mathbb{R},\ {}^t\bar{M} + M = \lambda.\mathrm{id}\},$$

one can show that $\mathbb{H}$ is a non-commutative field with group of invertibles

$$U_2 := \mathbb{H}^\times.$$

The Lie algebra of U_2 is $\mathbb{H}$ itself. There is a natural norm map $\mathrm{Nm} : \mathbb{H} \to \mathbb{R}$ sending M to $\mathrm{Nm}(M) :={}^t \bar{M}.M$ and one can identify SU_2 with the multiplicative kernel of

$$\mathrm{Nm} : U_2 = \mathbb{H}^\times \to \mathbb{R}^\times.$$

One has $\mathbb{H}_\mathbb{C} := \mathbb{H} \otimes_\mathbb{R} \mathbb{C} \cong \mathrm{M}_{2,\mathbb{C}}$, so that the special unitary group is a twisted version of SL_2, meaning that

$$\mathrm{SU}_2 \otimes_\mathbb{R} \mathbb{C} \cong \mathrm{SL}_{2,\mathbb{C}}.$$

This implies that their structure and representation theory are essentially equivalent. The maximal torus of SU_2 is isomorphic to SU_1, that is to the kernel of the norm map

$$\mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathrm{GL}_1 \to \mathrm{GL}_{1,\mathbb{R}}$$

that sends z to $z\bar{z}$. It is also a twisted version of the maximal torus GL_1 of SL_2.

The representations of SU_2 are given by taking irreductible representations of $\mathrm{SL}_{2,\mathbb{C}}$, i.e., the representations $V_n = \mathrm{Sym}^n(V)$ for V the standard representation, making their scalar restriction to $\mathbb{R}$, for example $\mathrm{Res}_{\mathbb{C}/\mathbb{R}} V_n$ and then taking irreducible subrepresentations of these for the natural action of $\mathrm{SU}(2)$.

For example, the representation $V_2 = \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{C}\langle X^2, XY, Y^2\rangle$ of real dimension 6 has a natural real subrepresentation of dimension 3 that corresponds to the morphism

$$\mathrm{SU}_2 \to \mathrm{SO}_3.$$

3.9 Structure and representations of $\mathrm{SO}(V, q)$

Let (V, q) be a quadratic space over $\mathbb{R}$. The special orthogonal group $\mathrm{SO}(V, q)$ is the subgroup of $\mathrm{GL}(V)$ defined by

$$\mathrm{SO}(V, q) = \{f \in \mathrm{GL}(V),\ q(f(v), f(w)) = q(v, w)\}.$$

Its lie algebra is given by

$$\mathfrak{so}(V, q) = \{m \in \mathrm{End}(V),\ q(v, m(w)) + q(m(v), w) = 0\}.$$

Indeed, if $\mathrm{id} + \epsilon m \in \mathrm{SO}(V, q)(\mathbb{R}[\epsilon]/(\epsilon^2))$ is a generic element that reduces to id modulo ϵ, the equation

$$q((\mathrm{id} + \epsilon m)(v), (\mathrm{id} + \epsilon m)(w)) = q(v, w)$$

gives

$$\epsilon[q(v, m(w)) + q(m(v), w)] = 0.$$

There is a canonical identification

$$\begin{array}{ccc} \wedge^2 V & \xrightarrow{\sim} & \mathfrak{so}(V, q) \\ v \otimes w & \mapsto & [x \mapsto q(w, x).v - q(v, x).w]. \end{array}$$

This identification can be better understood by using the natural isomorphism $\mathfrak{spin}(V, q) \xrightarrow{\sim} \mathfrak{so}(V, q)$, the natural embedding

$$\mathfrak{spin}(V, q) \to \mathrm{Cliff}(V, q)$$

and the linear isomorphism

$$\wedge^* V \xrightarrow{\sim} \mathrm{Cliff}(V, q)$$

given in degree 2 by

$$\begin{array}{ccc} \wedge^2 V & \to & \mathrm{Cliff}(V, q) \\ x \wedge y & \mapsto & \frac{1}{2}(xy - yx). \end{array}$$

Chapter 4

A glimpse at homotopical geometry

This section can be jumped at first reading and is here mostly for reference.

An example of obstruction that appear in physics is the natural obstruction, sometimes called anomaly, to the choice of a global section of a space bundle (called the gauge fixing), for example of a principal S^1-bundle in electromagnetism. Obstruction theory is necessary because it allows to simplify formulas in computations with bad (non-smooth) equations that are everywhere in physics. In geometry, obstructions to deformations are responsible of the non-smoothness of the spaces considered. They are thus necessary to give methods that apply in general and are not related to a particular example.

We first explain why one can not avoid to use homotopical spaces to understand geometrically the basic models of quantum field theory, as for example the BRST-BV method, that is necessary to renormalize the standard model of elementary particles and to make it a well defined quantum field theory. We then explain very shortly the basic ideas of homotopical geometry and give two examples of homotopical geometrical situations.

4.1 Homotopical algebra: a general obstruction theory

The first motivation for homotopical methods is that they give a systematic way to study (non-linear) obstruction problems. The analogous theory for linear problems is called homological algebra, and we will skip it since homotopical methods are more powerful (i.e., entail homological algebra) and are better adapted to the spaces used by physicists.

4.1.1 Model categories

The setting of model categories was first defined by Quillen to make homotopy theory functorial and symmetric, and to treat with unified methods:

- classical homotopy theory, which will involve a model category structure on the category of topological spaces, and

- homological algebra (derived categories), which will involve a model category structure on the category of complexes of modules over a ring.

For short presentations, we refer to Toen's various introductory articles on his webpage, and in particular the course [Toe] and to Keller's notes [Kel06], 4.1. For more details, we use Hovey [Hov99] and Dwyer-Spalinski [DS95].

The aim of the theory of model category is to give a workable notion of localization of a category M with respect to a given multiplicative class W of morphisms, whose elements will be denoted by arrows with tilde $\xrightarrow{\sim}$. This localization $M[W^{-1}]$ is the universal category in which all morphisms of W become isomorphisms, meaning that if $F : M \to M'$ is a functor from M to a category M' that sends W to isomorphisms, then there exists a unique functor $F' : M[W^{-1}] \to M'$ that makes the following diagram commute

$$\begin{tikzcd} M \arrow[r] \arrow[dr, "F"'] & M[W^{-1}] \arrow[d, dashed, "F'"] \\ & M' \end{tikzcd}$$

up to a unique equivalence.

The objects of this category are given by sequences

$$X_1 \xleftarrow{\sim} X_2 \longrightarrow \cdots \xleftarrow{\sim} X_{n-1} \longrightarrow X_n$$

where left arrows are in the class W. Two such strings are equivalent if, after forcing them to the same length, they can be connected by a commutative diagram with vertical

arrows in W of the form

$$\begin{tikzcd}
X_1 \arrow[d, equal] & X_2 \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & \cdots & X_{n-1} \arrow[l, "\sim"'] \arrow[r] \arrow[d, "\sim"] & X_n \arrow[d, equal] \\
X_1 \arrow[d, dotted, no head] & X_{2,1} \arrow[l, "\sim"'] \arrow[rr, dotted, no head] \arrow[d, dotted, no head] & & X_{2,n-1} \arrow[r] \arrow[d, dotted, no head] & X_n \arrow[d, dotted, no head] \\
X_1 \arrow[d, equal] & X_{m-1,1} \arrow[l, "\sim"'] \arrow[rr, dotted, no head] & & X_{m-1,n-1} \arrow[r] & X_n \arrow[d, equal] \\
X_1 & Y_2 \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"] & \cdots & Y_{n-1} \arrow[l, "\sim"'] \arrow[r] \arrow[u, "\sim"] & X_n
\end{tikzcd}$$

The main problem is that it is not even clear that this class of objects form a category, because their morphisms spaces are *a priori* not even sets. It is thus very hard to work with these general objects. The model category setting gives a method so replace them with simpler objects called fibrant or cofibrant resolutions. It is usually *very* hard to show that a given category fulfil the model category axioms, but in the examples of interest for physics, this work has already been done by topologists and algebraists (see for example [GM03]), so that we can use them as black boxes to make explicit computations with physical theories. The general definition of model categories can be found in [Hov99], 1.1.3.

Definition 4.1. A model category is a complete and cocomplete category M (i.e., a category that has all small limits and colimits) together with the following data

- three distinguished classes $(W, \mathrm{Fib}, \mathrm{Cof})$ of maps in M, (whose elements are respectively called weak equivalences, fibrations and cofibrations, and respectively denoted $\xrightarrow{\sim}$, $\twoheadrightarrow$ and $\rightarrowtail$);

- There are two functorial factorizations $f \mapsto (p, i')$, $f \mapsto (p', i)$ (i.e., $f = p \circ i'$ and $f = p' \circ i$, functorial in f),

subject to the following axioms

1. (2 out of 3) If (f, g) are composable arrows (i.e. $f \circ g$ exists), then all f, g and $f \circ g$ are in W if any two of them are;

2. (Left and right lifting properties) Calling maps in $W \cap \mathrm{Fib}$ (resp. $W \cap \mathrm{Cof}$) trivial fibrations (resp. trivial cofibrations), in each commutative square solid diagram of

the following two types

$$\begin{tikzcd} \cdot \arrow[r] \arrow[d, tail, "i"', "\sim"] & \cdot \arrow[d, "p"] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd} \qquad \begin{tikzcd} \cdot \arrow[r] \arrow[d, tail, "i"'] & \cdot \arrow[d, "p", "\sim"'] \\ \cdot \arrow[r] \arrow[ur, dotted] & \cdot \end{tikzcd}$$

the dotted arrow exists so as to make the two triangles commutative.

3. (Retracts) If a morphism f is a retract of a morphism g, meaning that there is a commutative diagram

$$\begin{tikzcd} \cdot \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] \arrow[d, "f"'] & \cdot \arrow[r] \arrow[d, "g"'] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r] \arrow[rr, bend right, "\mathrm{id}"'] & \cdot \arrow[r] & \cdot \end{tikzcd}$$

and f is a weak equivalence (resp. a fibration, resp. a cofibration), then g is a weak equivalence (resp. a fibration, resp. a cofibration);

4. (Factorizations) For the functorial factorizations $f \mapsto (p, i')$, $f \mapsto (p', i)$, p is a fibration, i' a trivial cofibration, p' is a trivial fibration and i a cofibration.

Remark that these axioms are self-dual, meaning that the opposite category is also a model category (with exchanged fibrations and cofibrations), and that the two classes Fib and Cof are determined by the datum of one of them and the class W of weak equivalences (by the lifting axiom). We now define the notion of homotopy of maps.

Definition 4.2. Let $X \in M$, a cylinder for X is a factorization

$$\begin{tikzcd} X \coprod X \arrow[r] \arrow[dr, tail, "i_0 \coprod i_1"'] & X \\ & \mathrm{Cyl}(X) \arrow[u, "u", "\sim"'] \end{tikzcd}$$

of the canonical map $X \coprod X \to X$ into a cofibration followed by a weak equivalence. A cylinder for X in the opposite category is called a path object or a cocylinder. If $f, g : X \to Y$ are maps, a left homotopy is a map $h : \mathrm{Cyl}(X) \to Y$ such that the following diagram commutes

$$\begin{tikzcd} X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\ \mathrm{Cyl}(X) \arrow[r, "h"] & Y \\ X \arrow[u, "i_1"] \arrow[ur, "g"'] & \end{tikzcd}$$

A right homotopy is a left homotopy in the opposite category. Left and right homotopies are denoted $h : f \sim_l g$ and $h : f \sim_r g$.

The factorization axiom of model categories ensures the existence of functorial cylinders and cocylinders.

Example 4.1. In the category TOP of (Hausdorff and compactly generated) topological spaces, one defines the classical cylinder of X by $\mathrm{Cyl}_c(X) = X \times [0,1]$. It is then clear that the injections $i_0, i_1 : X \to \mathrm{Cyl}_c(X)$ are closed and the projection $u : \mathrm{Cyl}_c(X) \to X$ is a homotopy equivalence because $[0,1]$ can be contracted to a point. One can use this classical cylinder to define in an ad-hoc way the notion of (left) homotopy of maps between topological spaces. Two maps $f, g : X \to Y$ are then called homotopic if there is a map $h : X \times [0,1] \to Y$ such that $h(0) = f$ and $h(1) = g$. This is the usual notion of homotopy. One then defines the homotopy groups of a pointed pointed topological space $(X, *)$ as the homotopy classes of pointed continuous maps

$$\pi_i(X) := \mathrm{Hom}_{\mathrm{TOP}}((S^i, *), (X, *))/ \sim_l$$

from the pointed spheres to $(X, *)$. These can be equiped with a group structure by identifying S^n with the quotient of the hypercube $[0,1]^n$ by the relation that identifies its boundary with a point and setting the composition to be the map induced by

$$(f * g)(x_1, \ldots, x_n) = \begin{cases} f(2x_1, x_2, ...x_n) \text{ for } x_1 \in [0, 1/2] \text{ and} \\ (f * g)(x_1, x_2, \ldots, x_n) = g(2x_1 - 1, x_2, \ldots, x_n) \text{ for } x_1 \in [1/2, 1] \end{cases}$$

on the quotient space. The main example of model category is given by the category TOP of topological spaces, where

- weak equivalences are given by weak homotopy equivalences (i.e., maps inducing isomorphisms on all the π_i's, for any choice of basepoint),
- fibrations (also called serre fibrations) $p : X \to Y$ have the right lifting property with respect to the maps $D^n \times \{0\} \to D^n \times I$ for all discs D^n:

$$\begin{tikzcd} D^n \arrow[r] \arrow[d, tail, "i"', "\sim"] & X \arrow[d, "p"] \\ D^n \times I \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd}$$

- cofibrations $i : X \to Y$ are closed immersions.

In this model category, the classical cylinder $\mathrm{Cyl}_c(X)$ is a cylinder for X. The mapping space $\mathrm{Cocyl}(X) := \underline{\mathrm{Hom}}([0,1], X)$, equiped with the topology generated by the subsets $V(K,U)$ indexed by opens U in X and compacts K in $[0,1]$, and defined by

$$V(K,U) = \{f : [0,1] \to X,\ f(K) \subset U\},$$

is a path (or cocylinder) for X.

For M a model category, the associated homotopy category is defined as before $\mathrm{Ho}(M) = M[W^{-1}]$. Since M is complete and cocomplete, it contains an initial object $\emptyset$ and a final object $*$. An object x is called fibrant (resp. cofibrant) if the natural map $x \to *$ (resp. $\emptyset \to x$) is a fibration (resp. a cofibration). The existence of functorial factorizations applied to these map imply the existence of two functors $R, Q : M \to M$ together with natural transformations $\mathrm{id} \to R, Q$ that are objectwise natural equivalences. The functor R is called the fibrant replacement functor and Q the cofibrant replacement. They are valued respectively in the subcategory M_f and M_c of fibrant and cofibrant objects. Denote M_{cf} the subcategory of objects that are fibrant and cofibrant.

The main structure theorem of model category theory is the following (see [Hov99], Theorem 1.2.10).

Theorem 11. *Let M be a model category.*

1. *(Simple description of objects) The natural inclusion $M_{cf} \to M$ induces an equivalence of categories*
$$\mathrm{Ho}(M_{cf}) \to \mathrm{Ho}(M)$$
whose quasi-inverse is induced by RQ.

2. *(Simple description of morphisms) The homotopy category $\mathrm{Ho}(M_{cf})$ is the quotient category of M_{cf} by the (left or right) homotopy relation $\sim$, i.e.,*
$$\mathrm{Hom}_{\mathrm{Ho}(M_{cf})}(x,y) \cong \mathrm{Hom}_{M_{cf}}(x,y)/\sim.$$

3. *There is a natural isomorphism*
$$\mathrm{Hom}_{\mathrm{Ho}(M)}(x,y) \cong \mathrm{Hom}_M(QX, RY).$$

This gives a nice description of maps in the homotopy category $\mathrm{Ho}(M)$ as homotopy classes of maps in M:

$$[x,y] := \mathrm{Hom}_{\mathrm{Ho}(M)}(x,y) \cong [RQx, RQy] \cong \mathrm{Hom}_M(RQx, RQy)/\sim \cong \mathrm{Hom}_M(Qx, Ry)/\sim.$$

4.1.2 Quillen functors and derived functors

One then wants to study functors between homotopy categories induced by morphisms of model categories. These are called Quillen functors.

Definition 4.3. Let C and D be two model categories, and

$$F : C \leftrightarrows D : G$$

be a pair of adjoint functors.

1. If F preserves cofibrations and G preserves fibrations, one calls the pair (F, G) a Quillen adjunction, F a left Quillen functor and G a right Quillen functor. If (F, G) is a Quillen adjunction, the total derived functors $\mathbb{L}F$ and $\mathbb{R}G$, defined to be the composites
$$\begin{tikzcd} \mathrm{Ho}(C) \arrow[r, "\mathrm{Ho}(Q)"] \arrow[rr, bend right, "\mathbb{L}F"'] & \mathrm{Ho}(C_c) \arrow[r, "\mathrm{Ho}(F)"] & \mathrm{Ho}(D) \end{tikzcd}$$
and
$$\begin{tikzcd} \mathrm{Ho}(D) \arrow[r, "\mathrm{Ho}(R)"] \arrow[rr, bend right, "\mathbb{R}G"'] & \mathrm{Ho}(D_f) \arrow[r, "\mathrm{Ho}(G)"] & \mathrm{Ho}(C) \end{tikzcd},$$
define an adjoint pair
$$\mathbb{L}F : \mathrm{Ho}(C) \leftrightarrows \mathrm{Ho}(D) : \mathbb{R}G.$$

2. Suppose in addition that for each cofibrant object A of C and fibrant object X of D, a map $f : A \to G(X)$ is a weak equivalence in C if and only if its adjoint $f^\sharp : F(A) \to X$ is a weak equivalence in D, then the pair (F, G) is called a Quillen equivalence and $\mathbb{L}F$ and $\mathbb{R}G$ are inverse equivalences of categories.

The main property of derived functors is that they are compatible with composition and identity.

Theorem 12. *For every model categories C, there is a natural isomorphism $\epsilon : \mathbb{L}(\mathrm{id}_C) \to \mathrm{id}_{\mathrm{Ho}(C)}$. For every pair of left Quillen functors $F : C \to D$ and $G : D \to E$ between model categories, there is a natural isomorphism $m_{GF} : \mathbb{L}G \circ \mathbb{L}F \to \mathbb{L}(G \circ F)$.*

The above compatibility isomorphism moreover fulfil additional associaticity and coherence conditions that can be found in [Hov99], theorem 1.3.7. One gets the right derived version of this theorem by passing to the opposite category.

4.1.3 Pointed model categories and homotopy exact sequences

We refer to the original work of Quillen [Qui67] for this section.

Definition 4.4. A pointed model category is a model category in which the initial and the final object coincide. This object is denoted 0.

Example 4.2. An example of a pointed model category is given by the category Top_* of pointed topological spaces. Remark that the product and coproduct in this setting are usually called smash product and wedge sum and denoted $\wedge$ and $\vee$.

If A and B are objects of a pointed model category, one denotes $0 : A \to 0 \to B$ the zero map and $\pi_1(A, B)$ the group of equivalence classes up to homotopy of homotopies $h : \mathrm{Cyl}(A) \to B$ between the zero morphism and itself.

From the basic axioms of model category, one gets the following theorem.

Theorem 13. *The functor $\pi_1 : C \times C \to \mathrm{Sets}$ can be derived and defines a functor $[.,.]_1 : \mathrm{Ho}(C) \times \mathrm{Ho}(C) \to \mathrm{Sets}$. There are two functors from $\mathrm{Ho}(C)$ to itself called the suspension functor Σ and the loop functor Ω such that one has functorial isomorphisms*

$$\mathrm{Hom}_{\mathrm{Ho}(C)}(\Sigma A, B) \cong [A, B]_1 \cong \mathrm{Hom}_{\mathrm{Ho}(C)}(A, \Omega B).$$

In the cases of interest, we will describe explicitly the suspension and loop functors.

The homotopical notion of exact sequence is given by fibration and cofibration sequences. These can be defined by using homotopy limits and colimits of fibrations and cofibrations, that are homotopical analogs of kernels and cokernels. Here is an explicit construction. Let $E \overset{p}{\twoheadrightarrow} B$ be a fibration with B fibrant and let $F \overset{i}{\rightarrowtail} E$ be the inclusion of the fiber of p in E. We describe this situation by saying that there is a short exact sequence

$$F \longrightarrow E \twoheadrightarrow B.$$

Construct a cocylinder for B, given by a factorization of the diagonal map into a cofibration and a weak equivalence:

$$\begin{tikzcd} X \arrow[r, "\Delta"] & X \times X \arrow[d, "u"', "\sim"] \\ & \mathrm{Cocyl}(X) \arrow[ul, "p_0 \times p_1"] \end{tikzcd}$$

Consider a factorization

$$E \overset{\sim}{\longrightarrow} E^I \twoheadrightarrow E \times_{p,B,p_0} \mathrm{Cocyl}(X) \times_{p_1,B,p} E$$

of the diagonal map into a fibration and a weak equivalence. There is a cartesian diagram

$$\begin{tikzcd} F \times_E E^I \times_E F \arrow[r] \arrow[d, "\pi"] & E^I \arrow[d] \\ F \times \Omega B \arrow[r] & E \times_{p,B,p_1} B^I \end{tikzcd}$$

and the axioms imply that π is a trivial fibration, so that there exists in $\mathrm{Ho}(C)$ a morphism

$$m : F \times \Omega B \to F$$

given by composing the third projection with the homotopical inverse of π.

Definition 4.5. A fibration sequence in $\mathrm{Ho}(C)$ is a sequence of the form

$$X \times \Omega Z \to X \to Y \to Z$$

that is isomorphic to a fibration sequence

$$F \times \Omega B \to F \to E \to B$$

for some fibration $p : E \to B$ in C_f. A cofibration sequence is a fibration sequence in the dual model category.

Theorem 14. *Let*

$$X \times \Omega Z \to X \to Y \to Z$$

be a fibration sequence and let A be any object in $\mathrm{Ho}(C)$ *then there is a long exact homotopy sequence*

$$\cdots \to \mathrm{Hom}(A, \Omega^{n+1} Z) \to \mathrm{Hom}(A, \Omega^n X) \to \mathrm{Hom}(A, \Omega^n Y) \to \mathrm{Hom}(A, \Omega^n Z) \to \cdots$$

Example 4.3. The category $\textsc{Top}_*$ of pointed (Hausdorff and compactly generated) topological spaces is an example of a pointed model category. In this setting, the suspension is given by the derived functor of the functor

$$X \mapsto \Sigma(X) := X \wedge S^1$$

of smash product with S^1 (product in the category $\textsc{Top}_*$ of pointed spaces) and the loop is given by the pointed mapping space

$$X \mapsto \Omega(X) := \underline{\mathrm{Hom}}_{\textsc{Top}_*}(S^1, X).$$

Example 4.4. The category $\textsc{Mod}_{dg}(A)$ of dg-modules on a given ring (see section 4.1.5) is another example of a pointed model category. In this setting, the suspension is given by the shift functor

$$X \mapsto X[1]$$

and the loop space is given by the functor

$$X \mapsto X[-1].$$

The mapping cone of a map $f : M \to N$ of complexes is given by the graded module

$$C(f) = M[1] \oplus N$$

equiped with the differential

$$d_{C(f)} = \begin{pmatrix} d_{A[1]} & 0 \\ f[1] & d_B \end{pmatrix}$$

acting on matrix vectors and the inclusion $N \to C(f)$ and projection $C(f) \to M[1]$ are just the natural ones. If $f : M_0 \to N_0$ is concentraded in degree 0, then $C(f)$ is just given by f itself with M_0 in degree -1, so that $H^0(C(f))$ is the cokernel of f and $H^{-1}(C(f))$ is its kernel. If $f : M_0 \to N_0$ is injective, i.e., a cofibration for the injective model structure, one gets an exact sequence

$$0 \to M_0 \xrightarrow{f} N_0 \to H^0(C_f) \to 0.$$

4.1.4 Simplicial sets

We refer to the book [GJ99] for this section. The notion of simplicial set allows to give a purely combinatorial description of the homotopy category of the category $\textsc{Top}$ of topological spaces. It is also a fundamental tool in homotopy theory, because it is the basis for the construction of model category structures on many categories of homotopical spaces (stacks and derived spaces to cite the most interesting).

Let Δ be the category whose objects are finite ordered sets $[n] = [0, \ldots, n-1]$ and whose morphisms are nondecreasing maps.

Definition 4.6. A simplicial set is a contravariant functor

$$X : \Delta^{op} \to \textsc{Sets}.$$

We denote $\textsc{SSets}$ the category of simplicial sets.

Define the n simplex to be $\Delta^n := \mathrm{Hom}(., [n])$. Yoneda's lemma imply that for every simplicial set X, one has

$$\mathrm{Hom}(\Delta^n, X) \cong X_n.$$

The boundary $\partial\Delta^n$ is defined by

$$(\partial\Delta^n)_m := \{f : [m] \to [n],\ \mathrm{im}(f) \neq [n]\}$$

and the k-th horn $\Lambda^n_k \subset \partial\Delta^n$ by

$$(\Lambda^n_k)_m := \{f : [m] \to [n],\ k \notin \mathrm{im}(f)\}.$$

The category SSETS has all limits and colimits (defined componentwise) and also internal homomorphisms defined by

$$\underline{\mathrm{Hom}}(X, Y) : [n] \mapsto \mathrm{Hom}_{\Delta_n}(X \times \Delta_n, Y \times \Delta_n) = \mathrm{Hom}(X \times \Delta^n, Y).$$

The geometric realization $|\Delta^n|$ is defined to be

$$|\Delta^n| := \{(x_0, \dots, x_n) \in [0,1]^n, \sum x_i = 1\}.$$

The geometric realization of a general simplicial set X is the colimit

$$|X| = \mathrm{colim}_{\Delta^n \to X} |\Delta^n|$$

indexed by the category of maps $\Delta^n \to X$ for varying n. The singular simplex of a given topological space Y is the simplicial set

$$S(Y) : [n] \mapsto \mathrm{Hom}_{\mathrm{TOP}}(|\Delta^n|, Y).$$

The geometric realization and singular simplex functor are adjoint meaning that

$$\mathrm{Hom}_{\mathrm{TOP}}(|X|, Y) \cong \mathrm{Hom}_{\mathrm{SSETS}}(X, S(Y)).$$

Definition 4.7. The simplicial cylinder of a given simplicial set is defined as $\mathrm{Cyl}(X) := X \times \Delta^1$. Let $f, g : X \to Y$ be morphisms of simplicial sets. A homotopy between f and g is a factorization

$$\begin{tikzcd}
X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\
\mathrm{Cyl}(X) \arrow[r, "h"] & X \\
X \arrow[u, "i_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$

Let X be a simplicial space and $x : \Delta^0 \to X$ be a base point of X. For $n \geq 1$, define the homotopy group $\pi_n(X, x)$ as the set of homotopy classes of maps $f : \Delta^n \to X$ that fit into a diagram

$$\begin{tikzcd} \Delta^n \arrow[r, "f"] & X \\ \partial\Delta^n \arrow[u] \arrow[r] & \Delta^0 \arrow[u] \end{tikzcd} .$$

The geometric realization of such maps correspond to maps

$$f : S^n \to |X|.$$

For $n = 0$, one denotes $\pi_0(X)$ the set of homotopy classes of maps $\Delta^0 \to X$.

The model structure on simplicial sets is defined by saying that

- Weak equivalences are given by maps of simplicial sets that induce isomorphisms on homotopy groups for any choice of base point $x : \Delta^0 \to X$.

- Fibrations are defined as maps that have the right lifting property with respect to all the standard inclusions $\Lambda^n_k \subset \Delta^n$, $n > 0$. This means that a map $p : X \to Y$ is a fibration if in every commutative square

$$\begin{tikzcd} \Lambda^n_k \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ \Delta^n \arrow[r] \arrow[ur, dotted] & Y \end{tikzcd} ,$$

  the dotted arrow exists to as to make the two triangle commutative.

- Cofibrations are defined as map that have the left lifting property with respect to fibrations.

As usual, it is very hard to prove that this indeed gives a model category.

Theorem 15. *The geometric realization and the singular simplex functor give a Quillen adjunction*

$$|.| : \mathrm{SS{\small ETS}} \leftrightarrows \mathrm{T{\small OP}} : S$$

whose derived version is an equivalence of categories

$$\mathbb{L}|.| : \mathrm{Ho}(\mathrm{SS{\small ETS}}) \overset{\sim}{\leftrightarrows} \mathrm{Ho}(\mathrm{T{\small OP}}) : \mathbb{R}S.$$

4.1.5 Derived categories and derived functors

Let A be an associative unital ring. Let $(\mathrm{M}\textsc{od}_{dg}(A), \otimes)$ be the monoidal category of graded (left) A-modules, equiped with a linear map $d : C \to C$ of degree -1 such that $d^2 = 0$, with graded morphisms that commute with d and tensor product $V \otimes W$ of graded vector spaces, endowed with the differential $d : d_V \otimes \mathrm{id}_W + \mathrm{id}_V \otimes d_W$ (tensor product of graded maps, i.e., with a graded Leibniz rule), and also the same anticommutative commutativity constraint.

Remark that this category is pointed, meaning that its initial and finite object are identified (the zero dg-module). It is moreover additive, meaning that morphisms are abelian groups, and even an abelian category. We would like to localize this category with respect to the class W of quasi-isomorphisms, (i.e., morphisms that induce isomorphisms of cohomology space $H^* := \mathrm{Ker}(d)/\mathrm{Im}(d)$). This is usually done in textbooks by the use of the formalism of derived and triangulated categories. We provide here directly the homotopical description of this construction.

The projective model structure on the category $\mathrm{M}\textsc{od}_{dg}(A)$ is defined by saying that:

- weak equivalences are quasi-isomorphisms,
- fibrations are degreewise surjections, and
- cofibrations are maps with the left lifting property with respect to trivial fibrations.

The injective model structure on the category $\mathrm{M}\textsc{od}_{dg}(A)$ is defined by saying that:

- weak equivalences are quasi-isomorphisms,
- cofibrations are degreewise injections, and
- fibrations are maps with the left lifting property with respect trivial cofibrations.

Remark that it is hard to prove that these really define model category structures on $\mathrm{M}\textsc{od}_{dg}(A)$ (see [Hov99]). Recall that a module P (resp. I) over A is called projective (resp. injective) if for all surjective (resp. injective) module morphism $M \to N$, the morphism

$$\mathrm{Hom}_{\mathrm{M}\textsc{od}(A)}(P, M) \to \mathrm{Hom}_{\mathrm{M}\textsc{od}(A)}(P, N)$$
$$(\text{resp. } \mathrm{Hom}_{\mathrm{M}\textsc{od}(A)}(M, I) \to \mathrm{Hom}_{\mathrm{M}\textsc{od}(A)}(N, I))$$

is surjective.

One can show that any cofibrant object in the projective model structure has projective components, and that having this property is enough to be cofibrant for bounded bellow dg-modules. Using this one can show that two morphisms $f, g : P \to Q$ between cofibrant

dg-modules are homotopic if and only if there is an A-linear morphism $r : X \to Y$ homogeneous of degree -1 such that

$$f - g = d \circ r + r \circ d.$$

This can be seen by remarking that if given a complex X, a path object for X is given by the Chain complex

$$\mathrm{Cocyl}(X)_n = X_n \oplus X_n \oplus X_{n+1}$$

with the differential

$$\partial(x, y, z) = (\partial x, \partial y, -\partial z + x - y).$$

One also defines the cylinder of X as the complex defines by

$$\mathrm{Cyl}(X)_n := X_n \oplus X_{n-1} \oplus X_n$$

equiped with the differential

$$\partial(x, y, z) := (\partial x - y, -\partial x, x + \partial y).$$

It can be used to show that the left homotopy in injective model structure is given by a chain homotopy.

If $\mathrm{Mod}_{dg,+}$ denotes the subcategory of bounded bellow dg-modules, on can define a (different from the standard) cofibrant replacement functor $Q : \mathrm{Mod}_{dg,+}(A) \to \mathrm{Mod}_{dg}(A)$ by

$$Q(M) = \mathrm{Tot}(L(M)),$$

where $L(M)$ is the free resolution of M given by setting $L(M)_0 = A^{(M)}$, $f_0 : L(M)_0 \to M$ the canonical morphism and

$$f_{i+1} : L(M)_{i+1} := A^{(\mathrm{Ker}(f_i))} \to L(M)_i,$$

and $\mathrm{Tot}(L(M))_k = \sum_{i+j=k} L(M)_{i,j}$ is the total complex associated to the bicomplex $L(M)$ (with one of the differential given by M and the other by the free resolution degree).

The homotopical category $\mathrm{Ho}(\mathrm{Mod}_{dg}(A))$ is called the derived category of A and denoted $D(A)$. Remark that in this case, a dg-module is cofibrant and fibrant if and only if it is fibrant. Applying theorem 11, one can compute $D(A)$ by taking the quotient category of the category of cofibrant dg-modules by the homotopy equivalence relation, i.e., the natural functor

$$\mathrm{Mod}_{dg,c}(A)/ \sim \longrightarrow \mathrm{Ho}(\mathrm{Mod}_{dg}(A))$$

is an equivalence. This gives back the usual notion of derived category.

If $F : \mathrm{Mod}(A) \to \mathrm{Mod}(B)$ is an additive functor that has a right adjoint, one can extend it to $F : \mathrm{Mod}_{dg}(A) \to \mathrm{Mod}_{dg}(B)$, one then defines its left derived functor by

$$\begin{array}{rccc} \mathbb{L}F : & \mathrm{Mod}_{dg}(A) & \to & \mathrm{Mod}_{dg}(B) \\ & M & \mapsto & F(Q(M)), \end{array}$$

where Q is the cofibrant replacement functor. This localizes to a functor $\mathbb{L}F : D(A) \to D(B)$ which is the usual derived functor of homological algebra.

For example, let A and B be two rings and M be an (A, B) bimodule. The pair

$$\_ \otimes_A M : \mathrm{Mod}_{dg,proj}(A) \leftrightarrows \mathrm{Mod}_{dg,inj}(B) : \mathrm{Hom}_B(M, \_)$$

is a Quillen adjunction that corresponds to a pair of adjoint derived functors

$$\_ \overset{\mathbb{L}}{\otimes}_A M : D(A) \leftrightarrows D(B) : \mathbb{R}\mathrm{Hom}_B(M, \_).$$

Remark that the category $\mathrm{Mod}_{dg,+}(A)$ of positively graded dg-modules can be equiped with a model structure (see [DS95], Theorem 7.2) by defining

- weak equivalences to be quasi-isomorphisms,
- cofibrations to be monomorphisms with componentwise projective cokernel,
- fibrations to be epimorphisms componentwise in strictly positive degree.

Simplicial A-modules are defined as contravariant functors $M : \Delta^{op} \to \mathrm{Mod}(A)$. There is an equivalence of categories

$$N : \mathrm{Mod}_s(A) \leftrightarrows \mathrm{Mod}_{dg,+}(A) : \Gamma$$

between the category $\mathrm{Mod}_s(A)$ of simplicial modules (which is equiped with the model structure induced by the one given on the underlying simplicial sets) and the category of positively graded complexes, which preserves the model structures. The dg-module $N(M)$ associated to a simplicial module M has graded parts

$$N(M)_n = \cap_{i=1}^n \mathrm{Ker}(\partial_n^i)$$

and is equiped with the differential

$$d_n(a) = \partial_n^0$$

where ∂_n^i is induced by the non-decreasing map $[0, \dots, \hat{i}, \dots, n+1] \to [0, \dots, n]$. The simplicial set associated to a given dg-module C is the family of sets

$$\Gamma(C)_n := \oplus_{n \twoheadrightarrow k} C_k$$

with naturally defined simplicial structure, for which we refer the reader to [GJ99], Chapter III. This gives a way to introduce simplicial methods in homological algebra.

The natural embedding

$$\textsc{Mod}_{dg,+}(A) \to \textsc{Mod}_{dg}(A)$$

into the projective model category of dg-modules induces a functor

$$D^+(A) := \mathrm{Ho}(\textsc{Mod}_{dg,+}(A)) \to D(A)$$

that can be used to compute total derived functors on bounded dg-modules. For example, the derived functor $\_ \overset{\mathbb{L}}{\otimes}_A M$ can be computed on bounded above objects by using the above defined projective cofibrant resolution. Dually, one can define the derived functor $\mathbb{R}\mathrm{Hom}_B(M, \_)$ on bounded bellow objects by using injective fibrant resolutions. This relates the above construction with usual derived functors.

4.1.6 Derived operations on sheaves

We refer to Hartshorne's book [Har77], Chapter II.1 and III.1, for a short review of basic notions on abelian sheaves and to Kashiwara and Schapira's book [KS90] for a more complete account. The extension of the above methods to derived categories of Grothendieck abelian categories (for example, categories of abelian sheaves on spaces) can be found in [CD09].

An abelian sheaf on a category $X = (\textsc{Legos}, \tau)$ with Grothendieck topology is a contravariant functor

$$\mathcal{F} : \textsc{Legos}^{op} \to \textsc{Grab}$$

that fulfils the sheaf condition given in definition 2.6. More precisely, the sheaf condition is that for every covering family $\{U_i \to U\}$, the sequence

$$\mathcal{F}(U) \to \prod_i \mathcal{F}(U_i) \to \prod_{i,j} \mathcal{F}(U_i \times_U U_j)$$

(where the last arrow sends $\{f_i\}_i$ to $\{f_i - f_j\}_{i,j}$) is exact.

We denote $\textsc{Shab}(X)$ the category of abelian sheaves on X.

Theorem 16. *(Beke) The category of complexes of abelian sheaves has a so called injective model category structure defined by setting*

- *weak equivalences to be quasi-isomorphisms, and*
- *cofibrations to be monomorphisms (injective in each degree).*

The projective model structure is harder to define because it involves a cohomological descent condition. We first define it in a simple but non-explicit way (explained to the author by D.-C. Cisinski) and then explain shortly the description of fibrations. We refer to [CD09] for more details on this result.

Theorem 17. *The category of complexes of abelian sheaves has a so called projective model category structure defined by setting*

- *weak equivalences to be quasi-isomorphisms, and*
- *cofibrations to have the left lifting property with respect to trivial fibrations, which are defined ad-hoc as morphisms $f : K \to L$ that are objectwise surjective in each degree (i.e., $f_{n,X} : K(X)_n \to L(X)_n$ is surjective for every n and X) and such that $\mathrm{Ker}(f_X)$ is acyclic).*
- *fibrations to have the right lifting property with respect to trivial cofibrations.*

We now give, just for the culture, the description of fibrations in the projective model structure on complexes of abelian sheaves.

Definition 4.8. Let $X \in \textsc{Legos}$. A hypercover of U is a simplicial lego $U : \Delta^{op} \to \textsc{Legos}$ equiped with a morphism $p : U \to X$ of simplicial legos such that each U_n is a coproduct of representables, and $U \to X$ is a local acyclic fibration (locally for the given topology on $\textsc{Legos}$, it is an objectwise acyclic fibration of simplicial sets).

Theorem 18. *The fibrations in the projective model category structure on the category of complexes of abelian sheaves are given by epimorphisms of complexes (surjective in each degree), whose kernel K has the cohomological descent property, meaning that for every object X in $\textsc{Legos}$ and every hypercovering $U \to X$, the natural morphism*

$$K(X) \to \mathrm{Tot}(N(K(U)))$$

is a quasi-isomorphism.

Another way to define the left bounded derived category of sheaves for the projective model structure is given by making a Bousfield homotopical localization (see 4.1.11) of the category of left bounded complexes of abelian presheaves (or equivalently, simplicial presheaves), with its standard model structure, by the morphisms of complexes of presheaves that induce isomorphisms on the cohomology sheaves.

Definition 4.9. Let $f : X \to Y$ be a morphism of topological spaces. The direct image of an abelian sheaf $\mathcal{F}$ on X is defined by

$$f_*\mathcal{F}(U) := \mathcal{F}(f^{-1}(U))$$

and the inverse image is defined as the sheaf associated to the presheaf

$$f^{-1}\mathcal{F}(U) := \lim_{f(U)\subset V} \mathcal{F}(V).$$

Theorem 19. *The direct and inverse image functor form a Quillen adjunction between categories of complexes of abelian sheaves*

$$f^* : C\textsc{Shab}(X)_{proj} \leftrightarrows C\textsc{Shab}(Y)_{inj} : f_*$$

that induce an adjoint pair of derived functors

$$\mathbb{L}f^* : D(\textsc{Shab}(X)) \leftrightarrows D(\textsc{Shab}(Y)) : \mathbb{R}f_*.$$

The above constructions can be extended to sheaves of modules over ringed spaces. A ringed space is a pair $(X, \mathcal{O}_X)$ of a space and of a sheaf of commutative rings, and a morphism of ringed spaces

$$f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$$

is a pair of a continuous map $f : X \to Y$ and a morphism of sheaves of rings $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

Definition 4.10. The inverse image functor $f^* : \textsc{Mod}(\mathcal{O}_Y) \to \textsc{Mod}(\mathcal{O}_X)$ is defined by

$$f^*\mathcal{F} := f^{-1}\mathcal{F} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X.$$

Theorem 20. *The model category structures on the category of complexes of abelian sheaves induce model category structures on the category of complexes of $\mathcal{O}_X$-modules on a given ringed space $(X, \mathcal{O}_X)$. The direct and inverse image functors form a Quillen adjunction*

$$f^* : C\textsc{Mod}(\mathcal{O}_X)_{proj} \leftrightarrow C\textsc{Mod}(\mathcal{O}_Y)_{inj} : f^*.$$

4.1.7 Symmetric monoidal model categories

A symmetric monoidal model category $(C, \otimes)$ is a model category equiped with a monoidal structure that is compatible with its model structure in a sense that is explained in Hovey's book [Hov99], 4.2.6. The conditions are that

- the tensor product is a Quillen bifunctor for the model structure, and
- the map obtained by tensoring a cofibrant resolution of the unit object by a cofibrant object is a weak equivalence.

One can then work with algebras in $(C, \otimes)$ up to homotopy (meaning that the category of commutative algebras is also equiped with a natural model category structure), that will be the category LEGOS of basic building blocs for homotopical spaces. Instead of giving the general theory, we now give the examples two monoidal model categories that appear naturally in physics.

The monoidal category $(\text{MOD}_{dg}(A), \otimes)$ of dg-modules over a given dg-algebra is a symmetric monoidal model category. The categories $(\text{MOD}_{dg,+}(A), \otimes)$ and $(\text{MOD}_s(A), \otimes)$ of positively graded or simplicial A-modules are also symmetric monoidal model categories. Monoid in these categories give the categories $\text{ALG}_{dg}(A)$ and$\text{ALG}_{dg,+}(A)$ of differential graded or $\text{ALG}_s(A)$ of simplicial A-algebras. These will be used in the BV procedure of homotopical poisson reduction of local field theories.

4.1.8 Differential graded algebras

A differential graded algebra is an algebra in the monoidal category $\text{MOD}_{dg,+}(\mathbb{R})$, i.e., a pair (A, d) composed of a graded algebra A and a derivation $d : A \to A$, i.e., a morphism of $\mathbb{R}$-module $A \to A$ of degree -1 and such that

$$d(ab) = d(a).b + (-1)^{\deg(a)} a.d(b)$$

and $d \circ d = 0$.

The model category structure on $\text{MOD}_{dg,+}(\mathbb{R})$ induces a model category structure on the category $\text{ALG}_{dg,\mathbb{R}}$ of differential graded algebra.

For computational purpose, we only need to know that the cofibrant replacement functor $Q : \text{ALG}_{dg,\mathbb{R}} \to \text{ALG}_{dg,\mathbb{R}}$ is given by the Koszul-Tate resolution

$$Q(A) = \underline{\text{Sym}}(Q_{mod}(A))$$

where $Q_{mod}(A)$ is the cofibrant resolution of the underlying module (defined in the preceeding section) and the differential on the graded symmetric algebra is given by a graded Leibniz rule.

4.1.9 Homotopy limits and homotopy colimits

We refer to Toen's notes [Toe]. Remark that one can not use the usual notion of limits and colimits in the a model category M to study limits and colimits in the homotopy category $\mathrm{Ho}(M)$, because the above are not compatible with weak equivalences. A slight generalization of the notion of model category, due to Cisinski and called derivable category, gives a functorial definition of the notion of homotopy limits and colimits.

We first recall some things about classical limits. Let I be a small category. If C is a category, we denote C^I the category of functors $F : I \to C$ with morphisms given by natural transformations. There is a natural constant functor

$$c : C \to C^I$$

and a limit functor on I is an adjoint $\lim_I : C^I \to C$ to c. More precisely, if $X_\bullet : I \to C$ is a functor (called an I-diagram in C), the limit of $X_\bullet$ is C, if it exists, is an object $\lim_I X_\bullet$ in C equiped with a natural transformation $c(\lim_I X_\bullet) \to X_\bullet$ that induces for every Y in C a natural isomorphism

$$\mathrm{Hom}_{C^I}(X_\bullet, c(Y)) \cong \mathrm{Hom}_C(\lim_I X_\bullet, c(Y)).$$

A colimit is a limit in the opposite category.

To extend this definition of limits and colimits to the homotopical setting of a model category M with weak equivalences W, one needs to consider the constant functor as a functor between pairs

$$c : (M, W) \to (M^I, W_I)$$

where W_I consists of natural transformations that are objectwise in W. One then gets a localized constant functor

$$c : M[W^{-1}] \to M^I[W_I^{-1}]$$

and a homotopy limit functor on I is an adjoint to this constant functor.

Theorem 21. *If C is a model category,* colim_I *and* colim_I *exist for every small category I.*

4.1.10 Mapping spaces

One wants to enrich the homotopy category of a given model category by putting a simplicial structure on morphisms whose π_0 gives usual morphisms, but that contains higher homotopical information. This is the so called simplicial localization method. We follow the approach explained in [Toe].

Let C be a model category. For a simplicial set $K : \Delta^{op} \to \textsc{Sets}$ and an object y of C, consider

$$y^K := \mathrm{colim}_\Delta(\Delta \to C : [n] \mapsto \prod_{K_n} y)$$

as an object in $W^{-1}C$; suppose that this gives a functor

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{S}\textsc{Sets}) & \to & W^{-1}C \\ K & \mapsto & y^K. \end{array}$$

For x, y in C, one says that the mapping space relative to the model category C between x and y exists if the functor

$$\begin{array}{cccc} \underline{\mathrm{Map}}_C(x,y) : & \mathrm{Ho}(\mathrm{S}\textsc{Sets}) & \to & \textsc{Sets} \\ & K & \mapsto & \mathrm{Hom}_{W^{-1}C}(x, y^K) \end{array}$$

is representable. The corresponding representative object in $\mathrm{Ho}(\mathrm{S}\textsc{Sets})$ will be denoted $\underline{\mathrm{Hom}}_C(x,y)$ and called the mapping space of x and y.

Theorem 22. *Mapping spaces always exist in model categories.*

4.1.11 Bousfield's homotopical localizations

An important operation in homotopical sheaf theory is the notion of Bousfield/homotopical localization. Given a model category M and a subset S of maps in M, find another model category M_S in which the maps in S are weak equivalences. The localization functor must be a left Quillen functor

$$f : M \to M_S$$

that is universal with respect to left Quillen functors $f : M \to N$ that send images of morphisms in S in $\mathrm{Ho}(M)$ to isomorphisms in $\mathrm{Ho}(N)$.

An object x of M is called S-local if it sees maps in S as weak equivalences, i.e., if it is fibrant and for any $y \to y'$ in S, the induced map

$$\underline{\mathrm{Hom}}(y', x) \to \underline{\mathrm{Hom}}(y, x)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{S}\textsc{Sets})$. A map $f : x \to x'$ is called an S-local equivalence if it is seens as an equivalence by any S-local map, i.e., if for any S-local object y in M, the induced map

$$\underline{\mathrm{Hom}}(x', y) \to \underline{\mathrm{Hom}}(x, y)$$

is an isomorphism in $\mathrm{Ho}(\mathrm{S}\textsc{Sets})$.

Theorem 23. *Let M be a standard model category. Then the following classes of maps in M*

- $W_S :=$ *S-local equivalence;*
- $\mathrm{Cof}_S :=$ *cofibrations in M;*

are part of a model structure on M, denoted by $L_S M$ and the identity functor induces a left Quillen functor

$$M \to L_S M$$

that is universal among left Quillen functors whose derived functor send morphisms in S to isomorphisms.

4.2 Homotopical differential geometry

This section can be jumped at first reading.

An example of obstruction that appear in physics is the natural obstruction, sometimes called anomaly, to the choice of a global section of a space bundle (called the gauge fixing), for example of a principal S^1-bundle in electromagnetism. Obstruction theory is necessary because it allows to simplify formulas in computations with bad (non-smooth) equations that are everywhere in physics. In geometry, obstructions to deformations are responsible of the non-smoothness of the spaces considered. They are thus necessary to give methods that apply in general and are not related to a particular example.

We first explain why one can not avoid to use homotopical spaces to understand geometrically the basic models of quantum field theory, as for example the BRST-BV method, that is necessary to renormalize the standard model of elementary particles and to make it a well defined quantum field theory. We then explain very shortly the basic ideas of homotopical geometry and give two examples of homotopical geometrical situations.

4.2.1 Motivations

In physics, for example, in gauge theory, one often has to study spaces defined by (say Euler-Lagrange) equations that are not smooth because of the so-called Noether relations (to be treated in the forthcoming chapter on local functional calculus). The idea of homotopical differential geometry is the following: take a smooth variety (for example the affine plane $\mathbb{A}^2 := \mathrm{Spec}(\mathbb{R}[X,Y])$) and equation in it whose solution space M is not smooth (for example $XY = 0$). Then one cannot do a reasonable differential calculus directly on M. However, one can replace the quotient ring $A = \mathbb{R}[X,Y]/(XY)$ by a dg-algebra

A' that is a cofibrant (meaning projective) resolution of A, and do differential calculus on this new dg-algebra. This will give a much better behaved non-smooth calculus. For example, the space of differential forms on A' gives the full cotangent complex of A.

To formalize this, one has to define a differential calculus "up to homotopy". This can be done in the general setting of algebras in monoidal *model* categories (see Toen-Vezzosi's seminal work [TV08b] and Hovey's book [Hov99]), which also encompases a good notion of higher stack, necessary to formalize correctly quotients and moduli spaces in covariant gauge theory. We will only give here a sketch of this theory, whose application to physics will certainly be very important, particularly gauge theory, where physicists independently discovered similar mathematical structures in the BRST-BV formalism.

Another good reason for using methods of homotopical geometry is the fact the quantization problem itself is in some sense a deformation problem: one wants to pass from a commutative algebra, say over $\mathbb{R}$, to an associative (or more generally a factorization) algebra, say over $\mathbb{R}[[\hbar]]$, by an infinitesimal deformation process (see Costello and Gwilliam's work [CG10]). The modern methods of deformation theory are deeply rooted in homotopical geometry, as one can see in the expository work of Lurie [Lur09], so that the conceptual understanding of the mathematical problem solved by the quantization and renormalization process uses homotopical methods.

4.2.2 Homotopical spaces

The main idea of homotopical geometry is to generalize the functor of point approach to spaces by studying homotopy classes of functors

$$X : (\textsc{Legos}, \tau, W)^{op} \to (C, W_C)$$

where $(\textsc{Legos}, W)$ is a model category, equipped with the homotopical analog of a Grothendieck topology τ, and (C, W_C) is a model category. The definition of the homotopy equivalence relation on these functors involves not only the weak equivalences W and W_C in Legos and C but also the topology τ. One gets back the usual theory of space by using a situation where the categories in play are usual Legos and $C = \textsc{Sets}$, equiped with trivial model structures (with W and W_C given by isomorphisms): the homotopy classes of functors then give the category of sheaves of sets of Legos, obtained as a localization of the category of presheaves by local equivalences.

The homotopical analog of a Grothendieck topology τ is the following.

Definition 4.11. Let $(\textsc{Legos}, W)$ be a model category with homotopy fiber products. A model topology τ on Legos is the data, for every lego U, of covering families $\{f_i : U_i \to U\}_{i\in I}$ of morphisms in $\mathrm{Ho}(\textsc{Legos})$, fulfilling:

1. (Homotopy base change) For every morphism $f : V \to U$ in Ho(LEGOS) and every covering family $\{f_i : U_i \to U\}$ of U, $f \times^h_U f_i : V \times^h_U V_i \to V$ is a covering family.

2. (Local character) If $\{f_i : U_i \to U\}$ is a covering family and $\{f_{i,j} : U_{i,j} \to U_i\}$ are covering families, then $\{f_i \circ f_{i,j} : U_{i,j} \to U\}$ is a covering family.

3. (Isomorphisms) If $f : U \to V$ is an isomorphism in Ho(LEGOS), it is a covering family.

We can now give a sketch of the definition of the category of general homotopical spaces. We refer to Toen-Vezzosi [TV08b] for more precise definitions.

Definition 4.12. Let (LEGOS, τ, W) be a model category equiped with a model topology. Let $(C, W_C) = (\text{SSETS}, W_{eq})$ be the model category of simplicial sets.

- The model category of C-valued prestacks on LEGOS is the category $C-\text{Pr}(\text{LEGOS})$ of functors
$$X : \text{LEGOS}^{op} \to C,$$
equiped with the model structure obtained by left Bousfield localization of the objectwise model structure (induced by the model structure W_C on C) by the weak equivalences W (seen as maps in C by the natural functor $h : \text{LEGOS} \to \text{SETS} - \text{Pr}(\text{LEGOS}) \to C - \mathcal{P}(\text{LEGOS})$).

- The model category of C-valued stacks on LEGOS is the left Bousfield localization $C-\text{SH}(\text{LEGOS})$ of $C-\text{Pr}(\text{LEGOS})$ by the class of homotopy τ-hypercovers (for which we refer to [TV], 4.4, 4.5. A τ hypercover is essentially similar to the simplicial nerve of a covering in the Grothendieck topology of the homotopy category, except that one allows arrows of the simplicial space themselves to be coverings for the topology).

4.2.3 Derived spaces and higher stacks

Chapter 5

Tools of analysis for partial differential equations

The spirit of this course is to use algebraic and functorial methods in physics whenever they are available. However, these abstract methods can only be used to prove what a physicist would call a qualitative result, or to go from the abstract mathematical data of a given physical model to the actual computations that one would like to do to check the model with some experiment.

If one would like to get quantitative results, i.e., to compute effectively real quantities, one can not avoid the use of the methods of analysis. To convince yourself of this fact, just recall that

a real number is a Cauchy sequence

and the use of complete metric spaces gives powerful computational tools to find explicitely (on a computer for example) the solutions to a given equation (think of newton's method or the fixed point theorem in analysis to cite the most famous and strickingly general results).

In this chapter, we will only give a short and sometimes sketchy survey of some of the main problems and methods of analysis of (linear) partial differential equations, refering to Evans' textbook [Eva98] for a more complete survey of the non-linear case.

5.1 Basic problems in the analysis of PDEs

The definition of a well posed problem was introduced by Hadamard. Roughly speeking, a partial differential equation E with initial/boundary value f in a topological space of functions $\mathcal{F}$ is called well posed if

1. there exists a solution to E around the given initial value,
2. this solution is unique,
3. the solution depends continuously on the initial value f.

In the case of evolution equations, one also often asks for a persistence condition (the solution remains in the same functional space as the initial condition).

The basic problem of the analysis of PDEs is to construct a large space $\mathcal{F}$ (for example a space of distribution) in which the existence of a solution for a given type of equation is assured, and to find a subspace (for example a sobolev space) in which the uniqueness is guaranteed. The existence and unicity can also be proved at once by invoquing a version of Banach's fixed point theorem in a complete normed vector space.

In the case of non-linear equations, one can also use the local inversion theorem and perturbative methods to see the equation as a linear one perturbed by a non-linear term with small parameter.

For more information on non-linear equations, we refer to Evan's textbook [Eva98].

5.2 Linear partial differential equations

In this section, we will study linear equations of motions, that can be associated to quadratic action functionals on flat space $\mathbb{R}^n$. We will give a general effective method to find solutions of such operators. Remark that true physical theories are never linear so that this very general result can not be used directly on them. However, perturbative theory is exactly a way to use linear equations to study non-linear equations.

5.2.1 Distributions

Recall that $\mathbb{R}^n$ is equiped with a natural Lebesgue measure μ. The idea of distributions is already at the heart of Dirac's work on quantum mechanics [Dir82]. The basic idea is to think of a give function $f : \mathbb{R}^n \to \mathbb{R}$ as defining a continuous linear form

$$\varphi \mapsto \int f(x)\varphi(x)dx$$

on a convenient space of functions on $\mathbb{R}^n$, and to then work with more general continuous linear forms, by thinking of them as if they were given by the above formula. The basic theorems of integration theory become definitions in the setting of distribution theory.

The space of test functions on an open subset U of $\mathbb{R}^n$ with values in $\mathbb{R}^m$ is the space $\mathcal{C}_c^\infty(U, \mathbb{R}^m)$ of smooth functions with compact support equiped with the inductive limit topology of the topologies on the subspaces $\mathcal{C}^\infty(K, \mathbb{R}^m)$ of smooth functions with support on a given compact $K \subset U$ induced by the norms

$$N_{K,k}(\varphi) = \sup_{x \in K, |\alpha| \leq k} |\partial_\alpha \varphi(x)|.$$

Remark that this family of seminorms indexed by compact subsets of U also define a natural topology on the space $\mathcal{C}^\infty(U, \mathbb{R}^m)$ of all smooth functions on U. A distribution on U (resp. with compact support on U) with values in $\mathbb{R}^m$ is a continuous linear form on $\mathcal{C}_c^\infty(U, \mathbb{R}^m)$ (resp. on $\mathcal{C}^\infty(U, \mathbb{R}^m)$). The space of distributions (resp. of distributions with compact support) is denoted $\mathcal{C}^{-\infty}(U, \mathbb{R}^m)$ (resp. $\mathcal{C}_c^{-\infty}(U, \mathbb{R}^m)$), or more simply $\mathcal{C}^{-\infty}(U)$ (resp. $\mathcal{C}_c^{-\infty}(U)$) if $m = 1$.

One has natural injections

$$i : \mathcal{C}^\infty(U, \mathbb{R}^m) \to \mathcal{C}^{-\infty}(U, (\mathbb{R}^m)^*) \text{ and } i : \mathcal{C}_c^\infty(U, \mathbb{R}^m) \to \mathcal{C}_c^{-\infty}((\mathbb{R}^m)^*)$$

given by $\langle i(f), g \rangle = \int_U \langle g, f \rangle d\mu$. This injection actually extends to a map

$$i : L^1_{loc}(U, \mathbb{R}^m) \to \mathcal{C}^{-\infty}(U, (\mathbb{R}^m)^*),$$

so that any locally integrable (for example L^p) function can be seen as a distribution.

If $T \in \mathcal{C}^{-\infty}(U, \mathbb{R}^m)$ is a distribution, one defines its derivative with respect to x_i, $i = 1, \ldots, m$ by

$$\langle \frac{\partial T}{\partial x_i}, \varphi \rangle = -\langle T, \frac{\partial \varphi}{\partial x_i} \rangle$$

for all $\varphi \in \mathcal{C}_c^\infty(U, \mathbb{R}^m)$ and more generally, if $\alpha \in \mathbb{N}^n$ is a multi-index,

$$\langle \partial_\alpha T, \varphi \rangle = (-1)^{|\alpha|} \langle T, \partial_\alpha \varphi \rangle.$$

We suppose given a pairing $\langle , \rangle : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ that could be for example the standard scalar product, or a Minkowski metric. If $f : \mathbb{R}^n \to \mathbb{R}$ is an L^1 function, we define its Fourier transform as the function

$$\hat{f}(y) := \int_{\mathbb{R}^n} e^{-2i\pi \langle x, y \rangle} f(x) d\mu(x).$$

The space $\mathcal{C}_c^\infty(U, \mathbb{R})$ is not stable by Fourier transform, meaning that the Fourier transform of a function with compact support need not be with compact support. (in quantum mechanical terms, it means that you can not measure at the same time the position and the impulsion of a given particle). To define a space that is stable by Fourier transform, it is easier to use purely measure theoretical terms, as is done in Osborne's paper [Osb75].

Definition 5.1. Let $\mathcal{A}(\mathbb{R}^n)$ be the space of rapid decay functions on $\mathbb{R}^n$, defined as functions $\varphi \in L^\infty(\mathbb{R}^n)$ such that there exists $R_\varphi > 0$ such that the following holds: for each positive integer n, there is a constant M_n such that for each integer $k \geq 1$,

$$\|f_{|\{x>R_\varphi^k\}}\|_\infty \leq M_n k^{-n}.$$

The Schwartz space of $\mathbb{R}^n$ is given by

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in \mathcal{A}(\mathbb{R}^n), \hat{\varphi} \in \mathcal{A}(\mathbb{R}^n)\}.$$

A more explicit description of the Schwartz space is given by

$$\mathcal{S}(\mathbb{R}^n) := \{\varphi \in \mathcal{S}(\mathbb{R}^n),\ \|x^\alpha \partial_\beta \varphi\|_\infty < \infty,\ \forall \alpha, \beta\}.$$

The space of Schwartz distributions is the continuous dual $\mathcal{S}'(\mathbb{R}^n)$ of $\mathcal{S}(\mathbb{R}^n)$. It is stable by the Fourier transform

$$\langle \hat{f}, \varphi \rangle := \langle T, \hat{\varphi} \rangle.$$

There are natural inclusions

$$\mathcal{C}_c^\infty(\mathbb{R}^n) \subset \mathcal{S}(\mathbb{R}^n) \subset \mathcal{C}^\infty(\mathbb{R}^n) \text{ and } \mathcal{C}_c^{-\infty}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{C}^{-\infty}(\mathbb{R}^n).$$

The main property of Fourier transform is that it replaces differential operators by multiplication operators. If $P(\xi) = \sum_{|\alpha| \leq k} a_\alpha x^\alpha$ is a polynomial on $\mathbb{R}^n$ with complex coefficients, we define a differential operator

$$P(D) = \sum_{|\alpha| \leq k} a_\alpha D^\alpha$$

with $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ and $D_i = \frac{1}{2i\pi} \partial_{x_i}$. We then have

$$\langle \widehat{P(D) f}, \varphi \rangle = \langle P(\xi) \hat{f}, \varphi \rangle.$$

5.2.2 Hyperfunctions

5.2.3 Schwartz's kernel theorem

It is often convenient to study operators between functional spaces

$$P : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}_c^\infty(\mathbb{R}^n)$$

defined by a kernel function $K \in \mathcal{C}^\infty(\mathbb{R}^n \times \mathbb{R}^n)$, i.e., by a formula of the form

$$P(f) = \int_{\mathbb{R}^n} K(x,y)f(x)dx.$$

Indeed, the properties of the operator can be read on the properties of its kernel. Schwartz's kernel theorem shows that essentially any reasonable operator is of this kind, if one uses distributions as kernels.

Theorem 24. *Let $P : \mathcal{C}_c^\infty(\mathbb{R}^n) \to \mathcal{C}^{-\infty}(\mathbb{R}^n)$ be a continuous operator. Then there exists a distribution $K \in \mathcal{C}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n)$ such that*

$$\langle Pf, \varphi\rangle = \langle K, f \otimes \varphi\rangle.$$

More generally, let $P : \mathcal{C}_c^\infty(\mathbb{R}^n, \mathbb{R}^l) \to \mathcal{C}^{-\infty}(\mathbb{R}^n, \mathbb{R}^k)$ be a continuous operator. Then there exists a distribution $K \in \mathcal{C}^{-\infty}(\mathbb{R}^n \times \mathbb{R}^n, (\mathbb{R}^l)^* \boxtimes \mathbb{R}^k)$ such that

$$\langle Pf, \varphi\rangle = \langle K, f \otimes \varphi\rangle.$$

5.2.4 The Sato-Bernstein polynomial

The Sato-Bernstein polynomial gives an elegant solution to the following problem of Gelfand: if P is a real polynomial in n variable with positive coefficients, show that the distribution

$$\langle P^s, f\rangle := \int_{\mathbb{R}^n} P(x)^s f(x)dx,$$

defined and holomorphic in s for $\mathrm{Re}(s) > 0$, can be meromorphically continued to the whole complex plane.

One can try to answer this for $P(x) = x^2$. An integration by parts gives

$$(2s+1)(2s+2)\langle x^{2s}, f\rangle = \int_{\mathbb{R}} x^{2(s+1)} f''(x)dx =: \langle \partial_x^2 x^{2(s+1)}, f\rangle.$$

This functional equation gives the desired meromorphic continuation to the complex plane with poles in the set $\{-1/2, -1, -3/2, -2, \dots\}$.

The generalization of this method to any real polynomial with positive coefficients can be done using a similar functional equation, that is given by the following theorem due to Sato and Bernstein.

Theorem 25. *If $f(x)$ is a no-zero polynomial in several variables with positive real coefficients, there is a non-zero polynomial $b(s)$ and a differential operator $P_x(s)$ with polynomial coefficients such that*

$$P_x(s).f(x)^{s+1} = b(s)f(x)^s.$$

Proof. A full and elementary proof of this theorem can be found in the memoire of Chadozeau and Mistretta [CM00]. □

As an example, if $f(x) = x_1^2 + \cdots + x_n^2$, one has

$$\sum_{i=1}^{n} \partial_i^2 f(x)^{s+1} = 4(s+1)\left(s + \frac{n}{2}\right) f(x)^s$$

so the Bernstein-Sato polynomial is

$$b(s) = (s+1)\left(s + \frac{n}{2}\right).$$

This gives a method to compute the fundamental solution of the laplacian.

More generally, if $f(x) = \sum_{i=1}^n a_i x_i^2$ is a quadratic form on $\mathbb{R}^n$, and $P_f = \sum_{i=1}^n a_i \partial_i^2$ is the second order differential operator with the same coefficients, one has

$$P_f.f(x)^{s+1} = \ldots f(x)^s.$$

This gives a method to compute the fundamental solution of the wave equation.

5.2.5 Fundamental solutions of partial differential operators

A fundamental solution for a partial differential operator $P(D)$ on $\mathbb{R}^n$ is a distribution f such that

$$P(D)f = \delta_0.$$

One calls this a fundamental solution of the partial differential equation because of the following: if g is a function that can be convoluted with the fundamental solution f (for example, if g has compact support), one can find a solution to the inhomogeneous equation

$$P(D)h = g$$

by simply putting $h = f * g$ to be the convolution of the fundamental solution and g.

Theorem 26 (Malgrange-Ehrenpreis)**.** *Let $P(D)$ be a differential operator in $\mathbb{R}^n$ with constant coefficients. There exists a distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ such that*

$$P(D)f = \delta_0.$$

Proof. By using the Fourier transform

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n),$$

we reduce the problem to finding a distributional inverse to the polynomial $P \in \mathbb{C}[x_1, \dots, x_n]$ that defines the differential operator $P(D)$. Indeed, the Fourier transformed equation of $P(D)f = \delta_0$ is

$$P(x).\mathcal{F}(f) = 1.$$

If the given polynomial has no real zeroes, its inverse is continuous on $\mathbb{R}^n$ and thus locally integrable. The corresponding distribution solves the problem. Otherwise, we need more efforts. Replacing P by $P.\bar{P}$ reduces the question to a polynomial P with positive coefficients. One can then apply the Malgrange-Ehrenpreis theorem 26 to P to get a non-zero polynomial $b(s)$ and a differential operator $P_x(s)$ with polynomial coefficients such that

$$P_x(s).f(x)^{s+1} = b(s)f(x)^s.$$

This gives meromorphic continuation of the distribution $P(x)^s$. If we take the constant term of the Laurent series of this meromorphic function at -1, we get a distribution D that fulfils

$$P(x).D = 1,$$

so that the problem is solved. □

The malgrange Ehrenpreis theorem is very elegant but one can often find fundamental solutions more easily by other methods.

Example 5.1. In euclidean field theory, linear trajectories are related to the solutions and eigenvalues of the laplacian $\Delta = \sum_{i=1}^n \partial_{x_i}^2$. This operator corresponds to the polynomial $\|x\|^2 = \sum_{i=1}^n x_i^2$. Using the formula

$$\int_{\mathbb{R}^n} f(x)dx = \int_{\mathbb{R}_+} \int_{S^{n-1}} f(x.\omega) r^{n-1} dr d\omega,$$

that is always true for a positive function, one gets that the function $\frac{1}{\|x\|^2}$ is locally integrable if $n > 2$ so that it defines a distribution D. One can check that it is tempered and that its inverse fourier transform f is a fundamental solution of the laplacian.

Example 5.2. The first model of electromagnetic waves is given by the wave operator

$$\frac{1}{c^2}\partial_t^2 - \sum_{i=2}^{n} \partial_{x_i}^2.$$

Its solutions are quite different of those of the laplacian. One first remarks that the corresponding polynomial $\frac{1}{c^2}t^2 - \sum_{i=2}^n x_i^2$ has real zeroes so that it cannot be naively inverted as a locally integrable function. The easiest way to solve the wave equation is to make the space Fourier-transform (on variable x) that transform it in ordinary differential operator

$$\frac{1}{c^2}\partial_t^2 - \|x\|^2.$$

One can also use the complex family of distributions parametrized by a real nonzero constant ϵ given by the locally integrable function

$$\frac{\mathbb{1}_X}{-\frac{1}{c^2}t^2 + \sum_{i=2}^n x_i^2 + i\epsilon}$$

where X is the light cone (where the denominator is positive). The fourier transform of the limit of this distribution for $\epsilon \to 0$ gives a fundamental solution of the wave equation called the Feynman propagator.

5.3 Spectral theory

Quantum mechanics being mainly based on spectral theory of operators on Hilbert spaces, we here recall the main lines of it, refering to von Neumann [vN96] for more details. This theory is a kind of generalization to infinite dimension of the theory of diagonalization of hermitian matrices.

Definition 5.2. A Hilbert space is a pair $\mathcal{H} = (\mathcal{H}, \langle .,. \rangle)$ composed of a topological vector space $\mathcal{H}$ and a sesquilinear pairing $\langle .,. \rangle : \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ such that $\mathcal{H}$ is complete for the norm $\|f\| = \sqrt{\langle f, f \rangle}$.

The main objective of spectral theory is the study of the spectrum of an operator on a Hilbert space.

Definition 5.3. If $A : \mathcal{H} \to \mathcal{H}$ is an operator (i.e., a linear map), its spectrum $\mathrm{Sp}(A)$ is the subset of $\mathbb{C}$ defined by

$$\mathrm{Sp}(A) = \{\lambda \in \mathbb{C} | \, A - \lambda.\mathrm{Id} \text{ non invertible}\}.$$

Definition 5.4. If $A : \mathcal{H} \to \mathcal{H}$ is an operator, its adjoint A^* is defined by

$$\langle Af, g\rangle = \langle f, A^*g\rangle.$$

We now define various classes of operations on the Hilbert space $\mathcal{H}$.

Definition 5.5. Let $A : \mathcal{H} \to \mathcal{H}$ be a linear operator.

1. One says that A is bounded if there exists a constant $C \geq 0$ such that

   $$\|Ax\| \leq C\|x\|,$$

   and a minimal such constant C will be denoted $\|A\|$.

2. One says that A is positive (bounded) if $\mathrm{Sp}(A) \subset [0, +\infty[\subset \mathbb{C}$.

3. One says that A is compact if the image of the unit ball $B(0,1) = \{f \in \mathcal{H} | \, \|f\| \leq 1\}$ de $\mathcal{H}$ by A has compact closure, i.e.,

   $$\overline{A(B(0,1))} \text{ compact.}$$

4. One says that A is autoadjoint if $A^* = A$.

5. One says that A is a projection if A is bounded and

   $$A = A^* = A^2.$$

We denote $\mathcal{B}(\mathcal{H})$ (resp. $\mathcal{B}(\mathcal{H})^+$, resp. $\mathcal{K}(\mathcal{H})$, resp. $\mathrm{Proj}(\mathcal{B}(\mathcal{H}))$, resp. $\mathcal{B}^{sa}(\mathcal{H})$) the set of bounded (resp. positive, resp. compacts, resp. projection, resp. bounded antoadjoint) operators on $\mathcal{H}$. If $A, B \in \mathcal{B}(\mathcal{H})$ are two bounded operators, one says that A is smaller than B if $B - A$ is positive, i.e.,

$$A \leq B \Leftrightarrow B - A \in \mathcal{B}(\mathcal{H})^+.$$

Theorem 27. *Let $A : \mathcal{H} \to \mathcal{H}$ be a bounded operator. One has an inclusion $\mathrm{Sp}(A) \subset \mathbb{R}$ is and only if A is autoadjoint. If $A \in \mathcal{K}(\mathcal{H})$ is compact then $\mathrm{Sp}(A) \subset \mathbb{C}$ is discrete.*

Definition 5.6. We denote Borel(Sp(A)) the borel σ-algebra on Sp(A). A spectral measure for a bounded operator $A \in \mathcal{B}(\mathcal{H})$ is a multiplicative measure on the spectrum of A with values projectors. More precisely, this is a map

$$\begin{array}{cccc} E: & \mathrm{Borel}(\mathrm{Sp}(A)) & \to & \mathrm{Proj}(\mathcal{B}(\mathcal{H})) \\ & B & \mapsto & E(B) \end{array}$$

such that

1. $E(\emptyset) = 0$,
2. $E(\mathrm{Sp}(A)) = 1$,
3. E is sigma-additive, i.e.,

$$E\left(\coprod_{i=1}^{\infty} B_i\right) = \sum_{i=1}^{\infty} E(B_i),$$

4. E is multiplicative, i.e., $E(B \cap C) = E(B).E(C)$.

The spectral theorem, that is a generalization to infinite dimension of the diagonalization theorem of symmetric or hermitian endomorphisms, can be formulated as follows.

Theorem 28. *Let $A \in \mathcal{B}(\mathcal{H})$ be a bounded autoadjoint operator. There exists a unique spectral measure*

$$E_A : \mathrm{Borel}(\mathrm{Sp}(A)) \to \mathrm{Proj}(\mathcal{B}(\mathcal{H}))$$

such that

$$A = \int_{\mathrm{Sp}(A)} \lambda dE_A(\lambda).$$

This theorem is also true for an unbounded autoadjoint operator (densely defined) and its proof can be found in Reed-Simon [RS80], chapters VII.2 et VIII.3.

The main interest of this theorem is that it gives a measurable functional calculus, that allows for example to evaluate a measurable positive function $f : \mathbb{R} \to \mathbb{R}_+$ on a given bounded autoadjoint operator A by

$$f(A) = \int_{\mathrm{Sp}(A)} f(\lambda) dE_A(\lambda).$$

This type of computation is very useful in quantum mechanics for the characteristic functions $\mathbb{1}_{[a,b]} : \mathbb{R} \to \mathbb{R}$, that allow to restrict to parts of the spectrum contained in an interval.

One also has the following theorem:

Theorem 29. *If two operators A and B of $\mathcal{B}(\mathcal{H})$ commute, there exists an operator $R \in \mathcal{B}(\mathcal{H})$ and two measurable functions $f, g : \mathbb{C} \to \mathbb{C}$ such that*

$$A = f(R) \textit{ et } B = g(R).$$

The Stone theorem allows one to translate the study of unbounded autoadjoint operators to the study of one parameter groups of unbounded operators. A proof can be found in [RS80], chapter VIII.4.

Theorem 30. *The datum of a strongly continuous one parameter family of unitary in $\mathcal{H}$, i.e., of a group morphism $U : \mathbb{R} \to \mathcal{U}(\mathcal{H})$ fulfilling*

$$\text{if } \varphi \in \mathcal{H} \text{ and } t \to t_0 \text{ in } \mathbb{R}, \text{ then } U(t)\varphi \to U(t_0)\varphi,$$

is equivalent with the datum of an autoadjoint operator A, by the map $A \mapsto e^{itA}$.

Part II

Classical trajectories and fields

Chapter 6

General lagrangian variational problems

Now that we have defined what is a general notion of space (superspace, homotopical superspace, etc...), we can just copy the definition from the introduction of a general lagrangian variational problem.

Definition 6.1. A variational problem is made of the following data:

1. A space M called the parameter space for trajectories,
2. A space C called the configuration space for trajectories,
3. A morphism $\pi : C \to M$ (often supposed to be surjective),
4. A subspace $H \subset \Gamma(M, C)$ of the space of sections of π
$$\Gamma(M, C) := \{x : M \to C,\ \pi \circ x = \mathrm{id}\},$$
called the space of histories,
5. A functional $S : H \to A$ (where A is a space in rings that is often the real line $\mathbb{R}$ or $\mathbb{R}[[\hbar]]$) called the action functional.

The space of classical trajectories for the variational problem is the subspace T of H defined by
$$T = \{x \in H |\ d_x S = 0\}.$$
If B is another space, a classical B-valued observable is a functional $F : T \to B$ and a quantum B-valued observable is a functional $F : H \to B$.

The claim is that any variational problem used by physicists to formalize any quantum field theory is in one of the classes introduced in chapter 2, and one needs all of them to understand geometrically the renormalization of non-commutative Yang-Mills theory, i.e., the quantum standard model of elementary particles.

Most of the lagrangian used in physics are local. This means that there exists a so called "lagrangian density" $L : J^\infty C \to \wedge^{max} T^*M$ where $J^\infty C \to M$ is the jet bundle of π, such that

$$S(x) = \int_M L \circ j_\infty x$$

for $j_\infty x : M \to J^\infty C$ the jet (i.e., local taylor series) of the given trajectory $x : M \to C$. Local functionals will be studied in details later, but this does not prevent us to write down our physical examples by using lagrangian densities.

Definition 6.2. A morphism of lagrangian variational problems

$$f : (C_1, M_1, \pi_1, H_1, S_1) \to (C_2, M_2, \pi_2, H_2, S_2)$$

is a map $f : \Gamma(M_1, C_1) \to \Gamma(M_2, C_2)$ such that

1. $f(H_1) \subset H_2$,
2. $S_2 \circ f = S_1 : H_1 \to \mathbb{R}$.

The composition rule for differentials implies that a morphism of variational problem sends solutions to the equations of motion to solutions to the equations of motion.

Chapter 7

Hamiltonian methods

We mainly use lagrangian methods in this course because their covariance is automatic. We thus only give a short account of the hamiltonian formalism. The reader can skip this part since it is not necessary for functional integral quantization. This presentation is however very useful to understand the basics of operator, so called "canonical" quantization.

7.1 Symplectic and Poisson manifolds

Definition 7.1. The algebra of multi vector fields on a given variety P is the antisymmetric algebra $\wedge^*\Theta_P$ on the space of vector fields.

Proposition 16. *There exists a unique extension*

$$[.,.]_{NS} : \wedge^*\Theta_P \otimes \wedge^*\Theta_P \to \wedge^*\Theta_P$$

of the Lie bracket of vector fields, that is moreover a bigraded derivation. More precisely, this bracket, called the Schouten-Nijenhuis bracket, fulfils:

1. *The graded Jacobi identity:*

$$[X, [Y, Z]] = [[X, Y], Z] + (-1)^{(|X|-1)(|Y|-1)}[Y, [X, Z]];$$

2. *The graded anticommutativity:*

$$[X, Y] = (-1)^{(|X|-1)(|Y|-1)}[Y, X];$$

3. *The bigraded derivation condition:*

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(|X|-1)(|Y|-1)}.$$

Definition 7.2. A symplectic variety is a pair (P, ω) of a variety P and a non-degenerate closed 2-form $\omega \in \Omega^2(P)$:

$$d\omega = 0, \quad \omega^\flat : TP \xrightarrow{\sim} T^*P^1.$$

A Poisson variety is a pair (P, π) of a variety P and a bivector $\pi \in \wedge^2\Theta_P$ whose Schouten-Nijenhuis bracket with itself

$$[\pi, \pi]_{SN} = 0$$

is zero.

Proposition 17. *The datum of a symplectic variety is equivalent to the datum of a Poisson structure that is non-degenerate, i.e., such that*

$$\pi^\natural : T^*P \to TP$$

is an isomorphism.

Proof. The inverse $\pi^\natural$ of $\omega^\flat : TP \to T^*P$ gives the Poisson bivector. One checks that the nullity of the bracket is equivalent to the fact that ω is closed. $\square$

Definition 7.3. If (P, π) is a Poisson manifold, we define its Poisson bracket by

$$\begin{array}{rccc} \{.,.\}_\pi : & \mathcal{O}_M \times \mathcal{O}_M & \to & \mathcal{O}_M \\ & (f, g) & \mapsto & \langle \pi, df \wedge dg \rangle. \end{array}$$

The main example of a symplectic manifold is given by the cotangent bundle $P = T^*X$ of a given manifold. If we define the Legendre 1-form by

$$\theta(v) = \mathrm{ev}(Dp \circ v)$$

where

- $Dp : T(T^*X) \to p^*TX := TX \times_X T^*X$ is the differential of $p : T^*X \to X$,
- $\mathrm{ev} : TX \times_X T^*X \to \mathbb{R}_M$ is the natural evaluation duality and,

[1]The notation $\flat$ lowers the index in physicist's notation and the notation $\natural$ puts the index up.

- and $v \in \Theta_{T^*X} = \Gamma(T^*X, T(T^*X))$ is a vector field.

The symplectic 2-form on P is defined by

$$\omega = -d\theta.$$

On $P = T^*\mathbb{R}^n$, with coordinates (q, p), the legendre form is given by

$$\theta = pdq = \sum_i p_i dq_i$$

and the symplectic form by

$$\omega = dq \wedge dp = \sum_i dq_i \wedge dp_i.$$

The Poisson bracket is then given by

$$\{f, g\} = \sum_i \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}.$$

7.2 Dynamics of a hamiltonian system

Definition 7.4. A hamiltonian system is a tuple $(P, \pi, I, \mathrm{Hist}, H)$ composed of

1. a Poisson manifold (P, π) called the phase space, whose functions are called observables,
2. an interval $I \subset \mathbb{R}$ called the time parameter space for trajectories,
3. a subspace $\mathrm{Hist} \subset \mathrm{Hom}(I, P)$ called the space of histories,
4. a function $H : P \to \mathbb{R}$ called the Hamiltonian.

If $x : I \to P$ is a history in the phase space and $f \in \mathcal{O}_P$ is an observable, Hamilton's equations for x along f are given by

$$\frac{\partial(f \circ x)}{\partial t} = \{H, f\} \circ x.$$

The Hamiltonian function H defines a Hamiltonian vector field

$$X_H := \{H, .\} : \mathcal{O}_P \to \mathcal{O}_P.$$

On $T^*\mathbb{R}^n$, the Hamiltonian vector field is given by

$$X_H = \sum_i \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \sum_i \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

If the Poisson structure π is non-degenerate, i.e., if (P,π) comes from a symplectic manifold (P,ω), the Hamilton equation on a history $x : I \to P$ are given by

$$x^*(i_{X_H}\omega) = x^*(dH).$$

On $T^*\mathbb{R}^n$, this corresponds to

$$\begin{cases} \frac{\partial q_x}{\partial t} &= \frac{\partial H}{\partial p}, \\ \frac{\partial p_x}{\partial t} &= -\frac{\partial H}{\partial q}. \end{cases}$$

Theorem 31. *(Symplectic Noether theorem) Given a hamiltonian system (P,π,H), a function f on P is constant along the trajectories of the Hamiltonian vector field if and only if the Hamiltonian is constant under the Hamiltonian vector field induced by f.*

Proof. Follows from the equality

$$X_f(H) = \{H,f\} = -\{f,H\} = X_H(f).$$

□

7.3 Relation with lagrangian variational problems

The main interest of the Hamilton formalism is that it translates the problem of solving the Euler-Lagrange equation of a lagrangian variational problem, i.e., a partial differential equation, in the problem of solving an ordinary differential equation: the Hamilton equation. It is however hard to formalize without coordinates for a general lagrangian variational problem. We refer to Vitagliano's paper [Vit09] for a jet space formulation of the relation between lagrangian mechanics and multisymplectic hamiltonian mechanics, giving in this section only a coordinate depend description of the lagrange transform.

Let $(\pi : C \to M, S)$ be a lagrangian variational problem in which

$$\pi : C = X \times I \to I = M$$

with $I \subset \mathbb{R}$ an interval and X a manifold. We suppose that the action functional $S : \Gamma(M,C) \to \mathbb{R}$ is local and given by

$$S(x) = \int_I L(t,x,\partial_t x)dt,$$

with $L : J^1C = I \times TX \to \mathbb{R}$ a lagrangian density.

We suppose now that we have a trivialization of TX given by coordinates (q, q_1). The legendre transformation is given by the map

$$\begin{array}{cccc} \mathbb{F}L : & TX & \to & T^*X \\ & (x, x_1) & \mapsto & (q,p) := (x, \frac{\partial L}{\partial x_1}). \end{array}$$

If this map is an isomorphism, one defines the Hamiltonian by

$$H = \langle p, x_1 \circ (\mathbb{F}L)^{-1}(q,p)\rangle - L(t, (\mathbb{F}L)^{-1}(q,p)).$$

In simplified notation, this gives

$$H(p,q,t) := pq_1 - L(t, q, q_1) \text{ with } p = \frac{\partial L}{\partial q_1}.$$

The Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_1^n}\right) - \frac{\partial L}{\partial x^n} = 0,$$

can be written more explicitely

$$(\partial_t^2 x^{n'})\frac{\partial^2 L}{\partial x_1^{n'}\partial x_1^n}(t, x, \partial_t x) = -(\partial_t x^{n'})\frac{\partial^2 L}{\partial x^{n'}\partial x_1^n}(t, x, \partial_t x) + \frac{\partial L}{\partial x^n}(t, x, \partial_t x).$$

If we suppose the non-degeneracy condition that the matrix

$$\left(\frac{\partial^2 L}{\partial x_1^{n'}\partial x_1^n}(t, x, \partial_t x)\right)_{n,n'}$$

is always invertible, we can compute $\partial_t^2 x^{n'}$ from all the other variables. This then gives a translation of the Euler-Lagrange equation through the Lagrange transform to the Hamilton equations

$$\left\{\begin{array}{ccc} \frac{\partial q_x}{\partial t} & = & \frac{\partial H}{\partial p}, \\ \frac{\partial p_x}{\partial t} & = & -\frac{\partial H}{\partial q}. \end{array}\right.$$

For example, if we start with the lagrangian

$$L(t, x, x_1) = \frac{1}{2}m(x_1)^2 - V(x)$$

of newtonian mechanics, we get the momentum variable $p = \frac{\partial L}{\partial x_1} = mx_1$, so that $x_1 = \frac{p}{m}$ and

$$H(t,q,p) = \sum p_i x_{i,1} - L(t,x,x_1) = \frac{p^2}{2m} + V(q).$$

The Hamilton equations are given by

$$\begin{cases} \frac{\partial q_x}{\partial t} & = & \frac{\partial H}{\partial p} = \frac{p}{m}, \\ \frac{\partial p_x}{\partial t} & = & -\frac{\partial H}{\partial q} = -V'(q) \end{cases}$$

and one recognize in them the usual equations of newtonian mechanics.

7.4 Hamilton-Jacobi equations

We only give here a coordinate description and refer to Vitagliano's paper [Vit10] for a coordinate free description of the Hamilton-Jacobi formalism. This construction can be found in Arnold's book [Arn99], p253-255.

Definition 7.5. Let $\pi : C = X \times I \to I = M$ be a bundle of classical mechanics as in last section and let $S : \Gamma(M,C) \to \mathbb{R}$ be a local action functional of the form $S(x) = \int_I L(t,x,\partial_t x)dt$. Let $(t_0,x_0) \in C$ be a fixed point. The Hamilton-Jacobi action function

$$S_{hj} : C \to \mathbb{R}$$

is defined as the integral

$$S_{hj,(t_0,x_0)}(t,x) = \int_I L(t,x_{ext},\partial_t x_{ext})dt$$

where $x_{ext} : M \to C$ is the extremal trajectory starting at (t_0,x_0) and ending at (x,t).

For this definition to make sense, one has to suppose that the mapping $(x_{0,1},t) \mapsto (x,t)$, given by solving the equations of motion with initial condition $x(0) = x_0$ and $\partial_t x(0) = x_{0,1}$, is nondegenerate. This can be shown to be possible in a small neighborhood of the initial condition. We thus stick to this case.

Theorem 32. *The differential of the Hamilton-Jacobi action function (for a fixed initial point) is equal to*

$$dS_{hj} = pdq - Hdt$$

where $p = \frac{\partial L}{\partial q_1}$ *and* $H = pq_1 - L$ *are defined with help of the terminal velocity* $q_1 := \partial_t x_{ext}(t,x)$ *of the extremal trajectory.*

Corollary 5. *The action function statisfies the Hamilton-Jacobi equation*

$$\frac{\partial S}{\partial t} + H\left(\frac{\partial S}{\partial x}, x, t\right) = 0.$$

7.5 Poisson reduction

We refer to the book [Ber01] for an introduction and to [GS90] for a more complete account of the theory in the symplectic setting. For the Poisson setting, we refer to [But07], Part A, 1, 6.1. We use also the article of Kostant-Sternberg [KS87] and Marsden-Weinstein [MW74].

Definition 7.6. Let G be a Lie group. A Hamiltonian G-space is a tuple $(M, \pi, \varphi, \delta)$ composed of

1. A Poisson variety (M, π),
2. A G-action $\varphi : G \times M \to M$ that is canonical, i.e., respects the Poisson bracket,
3. a linear map $\delta : \mathfrak{g} \to \mathcal{O}_M$ from the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$ to the Poisson algebra of functions on M such that
$$\{\delta(\xi), .\} = \xi_M(.)$$
where $\mathfrak{g} \to \Theta_M, \xi \mapsto \xi_M$ is the infinitesimal action.

The moment map of a Hamiltonian G-space is defined as the G-equivariant map

$$\begin{array}{rccc} J: & M & \to & \mathfrak{g}^* \\ & m & \mapsto & [\xi \mapsto \delta(\xi)(m). \end{array}$$

Following Marsden and Weinstein, one says that $\mu \in \mathfrak{g}^*$ is a weakly regular value of J if

1. $J^{-1}(\mu)$ is a submanifold of M and
2. for every $m \in J^{-1}(\mu)$, the inclusion $T_m(J^{-1}(\mu)) \subset \mathrm{Ker}(D_m J)$ is an equality.

The main theorem of Mardsen-Weinstein's symplectic reduction is the following.

Theorem 33. *Let $(M, \pi, \varphi, \delta)$ be a Hamiltonian G-space (that is supposed to be symplectic). Suppose that $\mu \in \mathfrak{g}^*$ is a regular value for the moment map J and that the isotropy group G_μ of μ acts freely and properly on the constraint surface $C_\mu := J^{-1}(\mu)$. Then the variety*

$$C_\mu / G_\mu$$

is equiped with a natural symplectic form.

An algebraic Poisson version of this construction can be given by the following.

Proposition 18. *Let $(M, \pi, \varphi, \delta)$ be a Hamiltonian G-space. Let $\mu \in \mathfrak{g}^*$ be an element and suppose that the ideal $\mathcal{I}_C$ of the constraint subspace $J^{-1}(\mu) \subset M$ is closed by the Poisson bracket (one talks of first class constraints). Then the algebra*

$$(\mathcal{O}_M/\mathcal{I}_C)^{\mathcal{I}_C}$$

of $\mathcal{I}_C$-invariant functions (such that $\{\mathcal{I}_C, f\} = 0$) in the quotient algebra is a Poisson algebra.

Remark that the above Poisson reduction identifies with the Mardsen-Weinstein reduction of the above theorem with its hypothesis.

7.6 The finite dimensional BRST formalism

We use here essentially the article of Kostant-Sternberg [KS87]. — Let $(M, \pi, \varphi, \delta)$ be a Hamiltonian G-space and suppose that $0 \in \mathfrak{g}^*$ is a regular value of the moment map $J : M \to \mathfrak{g}^*$. This implies that the ideal $\mathcal{I}$ of the subspace $C := J^{-1}(0)$ is generated by the image of δ.

We consider δ as a dg-module

$$\mathcal{C} := [\underset{-1}{\mathfrak{g} \otimes \mathcal{O}_M} \xrightarrow{\delta} \underset{0}{\mathcal{O}_M}]$$

and define the Koszul resolution as the symmetric algebra

$$\mathcal{K} := \mathrm{Sym}_{\mathcal{O}_M - dg}(\mathcal{C}).$$

As a graded algebra, we have $\mathcal{K} \cong \oplus \wedge^* \mathfrak{g} \otimes \mathcal{O}_M$, and the differential is given by defining $\delta(1 \otimes f) = 0$ and $\delta(\xi \otimes 1) = 1 \otimes \delta(\xi)$. One then has that

$$H^0(\mathcal{K}, \delta) = \mathcal{O}_M/\mathcal{I}_C = \mathcal{O}_C.$$

We now consider $\mathcal{K}$ as a module over $\mathfrak{g}$ and define

$$d : \mathcal{K} \to \mathfrak{g}^* \otimes \mathcal{K} = \mathrm{Hom}(\mathfrak{g}, \mathcal{K})$$

by

$$(dk)(\xi) = \xi k,\ \xi \in \mathfrak{g},\ k \in \mathcal{K}.$$

It is a morphism of modules over the dg-algebra $\mathcal{K}$. One extends this differential to the algebra

$$\mathcal{A}^{\bullet,\bullet} := \mathrm{Sym}_{\mathcal{K}-dg}\left([\underset{0}{K} \xrightarrow{d} \mathfrak{g}^* \underset{1}{\otimes} K] \right).$$

This bidifferential bigraded algebra is called the BRST algebra. One gets as zero cohomology

$$H^0(\mathcal{A}^{\bullet,\bullet}, d) = \mathcal{K}^{\mathfrak{g}}$$

for the differential d the space of $\mathfrak{g}$-invariants in $\mathcal{K}$ and the cohomology

$$H^0(H^0(\mathcal{A}^{\bullet,\bullet}, \delta), d) = \mathcal{O}_C^{\mathfrak{g}}$$

is the space of functions on the Poisson reduction on the given Hamiltonian G-space.

The total complex $(\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet}), D)$ is called the BRST complex. One can make the hypothesis that its zero cohomology is equal to the above computed space $\mathcal{O}_C^{\mathfrak{g}}$ of functions on the Poisson reduction.

Now remark that one has a canonical isomorphism

$$\wedge^* \mathfrak{g} \otimes \wedge^* \mathfrak{g}^* \cong \wedge^*(\mathfrak{g} \oplus \mathfrak{g}^*)$$

that induces a split scalar product on $\wedge^*(\mathfrak{g} \oplus \mathfrak{g}^*)$ given by the evaluation of linear forms. Let $C(\mathfrak{g} \oplus \mathfrak{g}^*)$ be the corresponding Clifford super-algebra.

Lemma 4. *The super-commutator in $C(\mathfrak{g} \oplus \mathfrak{g}^*)$ induces a super-Poisson structure on $\wedge^*(\mathfrak{g} \oplus \mathfrak{g}^*)$.*

Proof. Let c_i and c_j be representative elements in the Clifford algebra for some elements in the exterior algebra of respective degrees i and j. The class of the super-commutator $[c_i, c_j]$ in the degree $i + j$ exterior power depend only of the classes of c_i and c_j. This operation fulfils the axioms of a super Poisson algebra. □

The main theorem of homological perturbation theory is the following.

Theorem 34. *Under all the hypothesis given above, there exists an odd element $\Theta \in \mathcal{A}^{\bullet,\bullet}$, called the BRST generator, such that the Poisson bracket by Θ is precisely the BRST differential D, i.e.,*

$$\{\Theta, .\} = D$$

on the total complex $\mathrm{Tot}(\mathcal{A}^{\bullet,\bullet})$. Since $D^2 = 0$, this element fulfils the so called classical master equation

$$\{\Theta, \Theta\} = 0.$$

Remark that in the case of a group action on a point $X = \{\bullet\}$, the BRST generator $\Theta \in \wedge^3(\mathfrak{g} \oplus \mathfrak{g}^*)$ is simply given by the Lie bracket on $\mathfrak{g}$

$$\Theta = [.,.] \in \mathfrak{g}^* \wedge \mathfrak{g}^* \wedge \mathfrak{g}$$

and the BRST cohomology identifies with the Lie algebra cohomology.

Chapter 8

Local functional calculus

8.1 Differential modules and linear PDEs

Since we want to describe differential equations in a coordinate free presentation, we will use the standard language for this: $\mathcal{D}$-modules theory.

8.1.1 $\mathcal{D}$-modules

We refer to Schneiders [Sch94] for an introduction to $\mathcal{D}$-modules. Let M be a smooth variety of dimension n and $\mathcal{D}$ be the algebra of differential operators on M. Recall that locally on M, one can write an operator $P \in \mathcal{D}$ as a finite sum

$$P = \sum_{\alpha} a_\alpha \partial^\alpha$$

with $a_\alpha \in \mathcal{O}_M$,

$$\partial = (\partial_1, \dots, \partial_n) : \mathcal{O}_M \to \mathcal{O}_M^n$$

the universal derivation and α some multi-indexes.

To write down the equation $Pf = 0$ with f in an $\mathcal{O}_M$-module $\mathcal{S}$, one needs to define the universal derivation $\partial : \mathcal{S} \to \mathcal{S}^n$. This is equivalent to give $\mathcal{S}$ the structure of a $\mathcal{D}$-module. The solution space of the equation with values in $\mathcal{S}$ is then given by

$$\mathcal{S}ol_P(\mathcal{S}) := \{f \in \mathcal{S},\ Pf = 0\}.$$

Remark that

$$\mathcal{S}ol_P : \textsc{Mod}(\mathcal{D}) \to \textsc{Vect}_{\mathbb{R}_M}$$

is a functor that one can think of as representing the space of solutions of P. Denote $\mathcal{M}_P$ the cokernel of the $\mathcal{D}$-linear map

$$\mathcal{D} \xrightarrow{.P} \mathcal{D}$$

given by right multiplication by P. Applying the functor $\mathcal{H}om_{\mathcal{M}(\mathcal{D})}(.,\mathcal{S})$ to the exact sequence

$$\mathcal{D} \xrightarrow{.P} \mathcal{D} \longrightarrow \mathcal{M}_P \to 0,$$

we get the exact sequence

$$0 \to \mathcal{H}om_{\text{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S}) \to \mathcal{S} \xrightarrow{P.} \mathcal{S},$$

which gives a natural isomorphism

$$\mathcal{S}ol_P(\mathcal{S}) = \mathcal{H}om_{\text{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S}).$$

This means that the $\mathcal{D}$-module $\mathcal{M}_P$ represents the solution space of P, so that $\mathcal{D}$-module are a convenient setting for the functor of point approach to linear partial differential equations.

Remark that it is even better to consider the derived solution space

$$\mathbb{R}\mathcal{S}ol_P(\mathcal{S}) := \mathbb{R}\mathcal{H}om_{\text{Mod}(\mathcal{D})}(\mathcal{M}_P, \mathcal{S})$$

because it encodes also information on the inhomogeneous equation

$$Pf = g.$$

Recall that the sub-algebra $\mathcal{D}$ of $\text{End}_{\mathbb{R}}(\mathcal{O})$, is generated by the left multiplication by functions in $\mathcal{O}_M$ and by the derivation induced by vector fields in Θ_M. There is a natural right action of $\mathcal{D}$ on the $\mathcal{O}$-module Ω^n_M by

$$\omega.\partial = -L_\partial \omega$$

with L_∂ the Lie derivative.

There is a tensor product in the category $\text{Mod}(\mathcal{D})$ given by

$$\mathcal{M} \otimes \mathcal{N} := \mathcal{M} \otimes_{\mathcal{O}} \mathcal{N}$$

where the $\mathcal{D}$-module structure on the tensor product is given on vector fields $\partial \in \Theta_M$ by Leibniz's rule

$$\partial(m \otimes n) = (\partial m) \otimes n + m \otimes (\partial n).$$

There is also an internal homomorphism $\mathcal{H}om(\mathcal{M},\mathcal{N})$ given by the $\mathcal{O}$-module $\mathcal{H}om_{\mathcal{O}}(\mathcal{M},\mathcal{N})$ equiped with the action of derivations $\partial \in \Theta_M$ by

$$\partial(f)(m) = \partial(f(m)) - f(\partial m).$$

The functor

$$\mathcal{M} \mapsto \Omega^n_M \otimes_{\mathcal{O}} \mathcal{M}$$

induces an equivalence of categories between the categories $\textsc{Mod}(\mathcal{D})$ and $\textsc{Mod}(\mathcal{D}^{op})$ of left and right $\mathcal{D}$-modules whose quasi-inverse is

$$\mathcal{N} \mapsto \mathcal{H}om_{\mathcal{O}_M}(\Omega^n_M, \mathcal{N}).$$

Definition 8.1. Let $\mathcal{S}$ be a right $\mathcal{D}$-module. The de Rham functor with values in $\mathcal{S}$ is the functor

$$\mathrm{DR}_{\mathcal{S}} : \textsc{Mod}(\mathcal{D}) \to \textsc{Vect}_{\mathbb{R}_M}$$

that sends a left $\mathcal{D}$-module to

$$\mathrm{DR}_{\mathcal{S}}(\mathcal{M}) := \mathcal{S} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{M}.$$

The de Rham function with values in $\mathcal{S} = \Omega^n_M$ is denoted DR and simply called the de Rham functor. One also denotes $\mathrm{DR}^r_{\mathcal{S}}(\mathcal{M}) = \mathcal{M} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{S}$ if $\mathcal{S}$ is a fixed left $\mathcal{D}$-module and $\mathcal{M}$ is a varying right $\mathcal{D}$-module, and $\mathrm{DR}^r := \mathrm{DR}^r_{\mathcal{O}}$.

Proposition 19. *The natural map*

$$\begin{array}{ccc} \Omega^n_M \otimes_{\mathcal{O}} \mathcal{D} & \to & \Omega^n_M \\ \omega \otimes Q & \mapsto & \omega Q \end{array}$$

extends to a $\mathcal{D}^{op}$-linear quasi-isomorphism

$$\Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n] \xrightarrow{\sim} \Omega^n_M.$$

We will see that in the super setting, this proposition can be taken as a definition of the right $\mathcal{D}$-modules of volume forms, so called Berezinian.

Proposition 20. *Let $\mathcal{S}$ be a right coherent $\mathcal{D}$-module and $\mathcal{M}$ be a coherent left $\mathcal{D}$-module. There is a natural quasi-isomorphism*

$$\mathbb{R}\mathrm{Sol}_{\mathbb{D}(\mathcal{M})}(\mathcal{S}) := \mathbb{R}\mathrm{Hom}(\mathbb{D}(\mathcal{M}), \mathcal{S}) \cong \mathrm{DR}_{\mathcal{S}}(\mathcal{M}),$$

where $\mathbb{D}(\mathcal{M}) := \mathbb{R}\mathrm{Hom}(\mathcal{M}, \mathcal{D})$ is the $\mathcal{D}$-module dual of $\mathcal{M}$.

8.1.2 $\mathcal{D}$-modules on supervarieties and the Berezinian

We refer to Penkov's article [Pen83] for a complete study of the Berezinian in the $\mathcal{D}$-module setting.

Let M be a supervariety of dimension $n|m$. As explained in subsection 2.5.1 one defines Ω^1_M as the representing object for the internal derivation functor $\underline{\mathrm{Der}}(\mathcal{O}_M, .)$ on geometric $\mathcal{O}_M$-modules. One also defines Ω^*_M as the super-exterior power

$$\Omega^*_M := \wedge^* \Omega^1_M.$$

The super version of Proposition 19 can be taken as a definition of the Berezinian, as a complex of $\mathcal{D}$-modules, up to quasi-isomorphism.

Definition 8.2. The Berezinian of M is defined in the derived category of $\mathcal{D}_M$-modules by the formula

$$\mathrm{Ber}_M := \Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n].$$

The complex of integral forms $I_{*,M}$ is defined by

$$I_{*,M} := \mathbb{R}\mathrm{Hom}_{\mathcal{D}}(\mathrm{Ber}_M, \mathrm{Ber}_M).$$

The following proposition (see [Pen83], 1.6.3) gives a description of the Berezinian as a $\mathcal{D}$-module.

Proposition 21. *The Berezinian complex is concentraded in degree* 0 *and equal there to*

$$\mathrm{Ber}_M := \mathcal{E}xt^n_{\mathcal{D}}(\mathcal{O}, \mathcal{D}).$$

In the super-setting, the equivalence of left and right $\mathcal{D}$-modules is given by the functor

$$\mathcal{M} \mapsto \mathcal{M} \otimes_{\mathcal{O}} \mathrm{Ber}_M$$

of twist by the Berezinian right $\mathcal{D}$-module, which can be computed by using the definition

$$\mathrm{Ber}_M := \Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n]$$

and passing to degree 0 cohomology.

A more explicit description of the complex of integral forms (up to quasi-isomorphism) is given by

$$I_{*,M} := \mathbb{R}\mathcal{H}om_{\mathcal{D}}(\mathrm{Ber}_M, \mathrm{Ber}_M) \cong \mathcal{H}om_{\mathcal{D}}(\Omega^*_M \otimes_{\mathcal{O}} \mathcal{D}[n], \mathrm{Ber}_M)$$

so that we get

$$I_{*,M} \cong \mathcal{H}om_{\mathcal{O}}(\Omega^*_M[n], \mathrm{Ber}_M) \cong \mathcal{H}om_{\mathcal{O}}(\Omega^*_M[n], \mathcal{O}) \otimes_{\mathcal{O}} \mathrm{Ber}_M$$

and in particular $I_{n,M} \cong \mathrm{Ber}_M$.

Remark that proposition 19 shows that if M is a usual variety, then Ber_M is quasi-isomorphic with Ω^n_M, and this implies that

$$I_{*,M} \cong \mathcal{H}om_{\mathcal{O}}(\Omega^*_M[n], \mathcal{O}) \otimes_{\mathcal{O}} \mathrm{Ber}_M \cong \wedge^* \Theta_M \otimes_{\mathcal{O}} \Omega^n_M \xrightarrow{i} \Omega^*_M,$$

where i is the insertion morphism. This imply the isomorphism

$$I_{*,M} \cong \Omega^*_M,$$

so that in the purely even case, integral forms essentially identify with usual differential forms.

8.1.3 $\mathcal{D}$-modules on homotopical spaces and the Berezinian

Remark that the definition of the Berezinian in subsection 8.1.2 can be extended to spaces more general than super-varieties, that will occur in physical applications.

Let $(C, \mathrm{W}, \mathrm{Fib}, \mathrm{Cof}, \otimes)$ be a symmetric monoidal model category (for example the category of dg-modules on a given commutative ring), and if we take the category $\textsc{Legos}$ of commutative algebras in C (which are in this example given by commutative dg-algebras) equiped with its natural model category structure and its Zariski model topology, a derived space can be represented by (the homotopy class of) a functor

$$M : C \to \mathrm{S}\textsc{Sets}$$

that fulfils a sheaf condition up to homotopy. If this space is representable by a given dg-algebra A, it can be explicitely computed as the derived simplicial homomorphisms (i.e. all algebra morphisms, equiped with the simplicial structure induced by the dg-algebras structures)

$$M = \mathbb{R}\underline{\mathrm{Hom}}(A, .) := \underline{\mathrm{Hom}}(P_A, .)$$

where P_A is a cofibrant replacement of A. One then defines $\mathcal{D}_A$ as the algebra $\mathcal{D}_{P_A}$ up to homotopy and Ω^*_A also as $\Omega^*_{P_A}$. In particular, Ω^1_A is the cotangent complex of A. One can now define the Berezinian as is the super-setting as the homotopy class (with respect to the differential on P_A) of the complex of $\mathcal{D}_A$-modules

$$\mathrm{Ber}_M := \Omega^*_{P_A} \otimes_{P_A} \mathcal{D}_{P_A}[n].$$

What we will describe in the following sections on local functional calculus can thus be extended, with some care, to the setting of representable homotopical spaces, which have an important role to play in the BV quantization of gauge theories.

8.1.4 Inverse and direct image functors

We follow here the presentation of Penkov in [Pen83]. One can also use Schneiders' survey [Sch94] for the non-super case.

Let $g : X \to Y$ be a morphism of supermanifolds. The $(\mathcal{D}_X, g^{-1}\mathcal{D}_Y)$ module of relative inverse differential operators is defined as

$$\mathcal{D}_{X\to Y} := \mathcal{O}_X \otimes_{g^{-1}\mathcal{O}_Y} g^{-1}\mathcal{D}_Y.$$

The $(g^{-1}\mathcal{D}_Y, \mathcal{D}_X)$ module of relative direct differential operators is defined as

$$\mathcal{D}_{X\leftarrow Y} := \mathrm{Ber}_X \otimes_{\mathcal{O}_X} \mathcal{D}_{X\to Y} \otimes_{g^{-1}\mathcal{O}_Y} (\mathrm{Ber}_Y^*).$$

The inverse image functor of $\mathcal{D}$-modules is defined by

$$g^*_{\mathcal{D}}(.) := \mathcal{D}_{X\to Y} \overset{\mathbb{L}}{\otimes}_{g^{-1}\mathcal{O}_Y} g^{-1}(.) : D(\mathcal{D}_Y) \to D(\mathcal{D}_X).$$

If $g : X \hookrightarrow Y$ is a locally closed embedding, the direct image functor is defined by

$$g^{\mathcal{D}}_*(.) := \mathcal{D}_{Y\leftarrow X} \overset{\mathbb{L}}{\otimes}_{\mathcal{D}_X} (.) : D(\mathcal{D}_X) \to D(\mathcal{D}_Y).$$

More generally, for any morphism $g : X \to Y$, one defines

$$g^{\mathcal{D}}_*(.) := \mathbb{R}f_*(. \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{D}_{X\to Y}).$$

8.1.5 Local operations in $\mathcal{D}$-module theory

We refer to Beilinson-Drinfeld's book [BD04] for a complete and axiomatic study of general pseudo-tensor categories. We will only present here the tools of $\mathcal{D}$-module theory needed to understand local functional calculus. Local operations are new kinds of operations on $\mathcal{D}$-module that induce usual operations on their de Rham cohomology.

We will work in this section on a classical variety M of dimension n. The analogous results on super-varieties can be easily obtained replacing everywhere Ω^n_M by the Berezinian Ber_M. Recall that there is a natural equivalence of categories

$$\begin{array}{ccc} \textsc{Mod}(\mathcal{D}) & \to & \textsc{Mod}(\mathcal{D}^{op}) \\ \mathcal{M} & \mapsto & \mathcal{M}\otimes \mathrm{Ber}_M \end{array}$$

between left and right $\mathcal{D}$-module. One can use this equivalence to transpose functorial operations on $\mathcal{D}$-modules to operations on $\mathcal{D}^{op}$-modules (being carefully with derived functors, that get a decallage by n). In particulary, one gets a natural tensor structure on $\textsc{Mod}(\mathcal{D}^{op})$ that we will denote $\otimes^!$. The above equivalence is then an equivalence

$$(\textsc{Mod}(\mathcal{D}), \otimes, \mathcal{O}) \cong (\textsc{Mod}(\mathcal{D}^{op}), \otimes^!, \Omega^n).$$

Definition 8.3. Let $(X_i, \mathcal{L}_i)_{i\in I}$ be a family of pairs given by a variety X_i and a $\mathcal{D}$-module $\mathcal{L}_i$ on X_i. The exterior tensor product of the above family is the $\mathcal{D}$-module on $\prod_i X_i$ defined by

$$\boxtimes \mathcal{L}_i := \otimes_i p_i^* X_i$$

where $p_{i_0} : \prod_i X_i \to X_{i_0}$ is the natural projection.

Definition 8.4. Let $(\mathcal{L}_i)_{i\in I}$ be a family of $\mathcal{D}^{op}$-modules and $\mathcal{M}$ be a $\mathcal{D}^{op}$-module. We define the *-operation space

$$P_I^*(\{\mathcal{L}_i\}, \mathcal{M}) := \mathrm{Hom}(\boxtimes \mathcal{L}_i, \Delta_*^{(I)} \mathcal{M})$$

where $\Delta^{(I)} : M \to M^I$ is the diagonal embedding.

The main interest of *-operations is that they induce usual operations in de Rham cohomology. Since de Rham cohomology is the main tool of local functional calculus, we will make a systematic use of these operations.

Proposition 22. *The central de Rham cohomology functor* $h : \textsc{Mod}(\mathcal{D}^{op}) \to \textsc{Mod}(\mathbb{R}_M)$ *given by* $h(\mathcal{M}) := \mathcal{M} \otimes_{\mathcal{D}} \mathcal{O}$ *induces a natural map*

$$h : P_I^*(\{\mathcal{L}_i\}, \mathcal{M}) \to \mathrm{Hom}(\otimes_i h(\mathcal{L}_i), h(\mathcal{M}))$$

from *-operations to multilinear operations.

One can define natural composition maps of pseudo-tensor operations.

Remark that there exists a natural internal homomorphism object for the *-operations P_I^*.

Proposition 23. *Let* $\mathcal{H}om^*(\mathcal{M}, \mathcal{N}) := \mathcal{H}om_{\mathcal{D}}(\mathcal{M}, \mathcal{N} \otimes^! \mathcal{D})$ *and* $\mathrm{ev} \in P_2^*(\{\mathcal{M}, \mathcal{H}om^*(\mathcal{M}, \mathcal{N})\}, \mathcal{N})$ *be the evaluation map. The map induced by* ev *on *-operations*

$$P_I^*(\{\mathcal{L}_i\}, \mathcal{H}om^*(\mathcal{M}, \mathcal{N})) \to P_{I \coprod \{1\}}(\{\mathcal{L}_i, \mathcal{M}\}, \mathcal{N})$$

is a bijection.

There is a natural local composition

$$\mathcal{H}om^*(\mathcal{M}, \mathcal{N}) \boxtimes \mathcal{H}om^*(\mathcal{N}, \mathcal{L}) \to \Delta_* \mathcal{H}om^*(\mathcal{M}, \mathcal{L})$$

of internal homomorphisms and in particular a natural local bracket

$$[.,.] : \mathcal{E}nd^*(\mathcal{M}) \boxtimes \mathcal{E}nd^*(\mathcal{M}) \to \Delta_* \mathcal{E}nd^*(\mathcal{M})$$

between *-endomorphisms of a given $\mathcal{D}$-module.

8.2 Differential algebras and non-linear PDEs

8.2.1 $\mathcal{D}$-algebras

We will use systematically in this section the language of differential calculus in symmetric monoidal categories introduced in section 2.5.1, specialized to the situation of the symmetric monoidal category

$$(\textsc{Mod}(\mathcal{D}_M), \otimes_{\mathcal{O}_M})$$

of left $\mathcal{D}_M$-module on a given (sometimes super-)variety M.

Recall that if $P \in \mathbb{Z}[X]$ is a polynomial, one can study the solution space

$$\mathrm{Sol}_{P=0}(A) = \{x \in A,\ P(x) = 0\}$$

of P with values in any commutative unitary ring. Indeed, in any such ring, one has a sum, a multiplication, a zero and a unit that fulfil the necessary compatibilities to be able to write down the polynomial. This solution space is representable, meaning that there is a functorial isomorphism

$$\mathrm{Sol}_{P=0}(.) \cong \mathrm{Hom}_{\textsc{Rings}_{cu}}(\mathbb{Z}[X]/(P), .).$$

This shows that the solution space of an equation essentially determine the equation itself. Remark that the polynomial P lives in the free algebra $\mathbb{Z}[X]$ on the given variable that was used to write it.

Suppose now given the bundle $\pi : C = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$ of smooth varieties. We would like to study an algebraic non-linear PDE

$$F(t, \partial_t^i x) = 0$$

that applies to sections $x \in \Gamma(M, C)$, that are functions $x : \mathbb{R} \to \mathbb{R}$. It is given by a polynomial $F(t, x_i) \in \mathbb{R}[t, \{x_i\}_{i \geq 0}]$. The solution space of such an equation can be studied with values in any $\mathcal{O}$-algebra $\mathcal{A}$ equiped with an action of the differentiation ∂_t (that fulfils a Leibniz rule for multiplication), the basic example being given by the algebra $\mathrm{Jet}(\mathcal{O}_C) := \mathbb{R}[t, \{x_i\}_{i \geq 0}]$ above with the action $\partial_t x_i = x_{i+1}$. The solution space of the given PDE is then given by the functor

$$\mathrm{Sol}_{F=0}(A) := \{x \in A,\ P(t, \partial_t^i x) = 0\}$$

defined on all $\mathcal{O}_C$-algebras equiped with an action of ∂_t. To be more precise, we define the general notion of a $\mathcal{D}$-algebra.

Definition 8.5. Let M be a variety. A $\mathcal{D}_M$-algebra is an algebra $\mathcal{A}$ in the monoidal category of $\mathcal{D}_M$-modules. More precisely, it is an $\mathcal{O}_M$-algebra equiped with an action

$$\Theta_M \times \mathcal{A} \to \mathcal{A}$$

of vector fields on M such that the product in $\mathcal{A}$ fulfils the Leibniz rule

$$\partial(fg) = \partial(f)g + f\partial(g).$$

Theorem 35. *Let M be a variety and $\mathcal{O} := \mathcal{O}_M$. The forgetful functor*

$$F : \textsc{Alg}_{\mathcal{D}} \to \textsc{Alg}_{\mathcal{O}}$$

has an adjoint (free $\mathcal{D}$-algebra on a given $\mathcal{O}$-algebra)

$$\mathrm{Jet} : \textsc{Alg}_{\mathcal{O}} \to \textsc{Alg}_{\mathcal{D}}$$

called the (infinite) jet functor. It fulfils the universal property that for every $\mathcal{D}$-algebra $\mathcal{B}$, the natural map

$$\mathrm{Hom}_{\textsc{Alg}_{\mathcal{O}}}(R, \mathcal{B}) \cong \mathrm{Hom}_{\textsc{Alg}_{\mathcal{D}}}(\mathrm{Jet}(R), \mathcal{B})$$

induced by the inclusion $R \subset \mathrm{Jet}(R)$ is a bijection.

Proof. One defines $\mathrm{Jet}(R)$ as the quotient of $\mathrm{Sym}(\mathcal{D} \otimes R)$ by the ideal generated by the elements

$$\partial(1 \otimes r_1.1 \otimes r_2 - 1 \otimes r_1 r_2) \in \mathrm{Sym}^2(\mathcal{D} \otimes R) + \mathcal{D} \otimes R)$$

and

$$\partial(1 \otimes 1_R - 1) \in \mathcal{D} \otimes R + \mathcal{O},$$

for $\partial \in \mathcal{D}$. $\square$

Using the Jet functor, one can show that the solution space of the non-linear PDE

$$F(t, \partial_t^i x) = 0$$

of the above example is representable, meaning that there is a natural isomorphism of functors on $\mathcal{D}$-algebras

$$\mathrm{Sol}_{F=0}(.) \cong \mathrm{Hom}_{\textsc{Alg}_{\mathcal{D}}}(\mathrm{Jet}(\mathcal{O}_C)/(F), .)$$

where (F) denotes the $\mathcal{D}$-ideal generated by F. This shows that the Jet functor plays the role of the polynomial algebra in the differential algebraic setting. If $\pi : C \to M$ is a bundle, we define

$$\mathrm{Jet}(C) := \mathrm{Spec}(\mathrm{Jet}(\mathcal{O}_C)).$$

To study the smooth version of this construction, one can replace usual algebras by smoothly closed geometric algebras in the above setting and algebraic spectrum by real spectrum. One then gets usual smooth jet spaces, that are given by smooth closures of algebraic smooth jet spaces.

If $C = \mathbb{R}^{n+m} \to \mathbb{R}^n = M$ is a trivial bundle of dimension $m+n$ over M of dimension n, with algebra of coordinates $\mathcal{O}_C := \mathbb{R}[\underline{t}, \underline{x}]$ for $\underline{t} = \{t^i\}_{i=1,\dots,n}$ and $\underline{x} = \{x^j\}_{j=1,\dots,m}$ given in multi-index notation, the Jet algebra is

$$\mathrm{Jet}(\mathcal{O}_C) := \mathbb{R}[\underline{t}, \underline{x}_\alpha]$$

where $\alpha \in \mathbb{N}^m$ is a multi-index representing the derivation index. The $\mathcal{D}$-module structure is given by making $\frac{\partial}{\partial t^i}$ act through the total derivative

$$D_i := \frac{\partial}{\partial t^i} + \sum_{\alpha,k} x^k_{i\alpha} \frac{\partial}{\partial u^k_\alpha}$$

where $i\alpha$ denotes the multi-index α increased by one at the i-th coordinate. For example, if $C = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$, one gets

$$D_1 = \frac{\partial}{\partial t} + x_1 \frac{\partial}{\partial x} + x_2 \frac{\partial}{\partial x_1} + \dots.$$

Definition 8.6. Let $\pi : C \to M$ be a bundle. A non-linear PDE on the space $\Gamma(M, C)$ of sections of π is given by a quotient $\mathcal{D}_M$-algebra

$$p : \mathrm{Jet}(\mathcal{O}_C) \twoheadrightarrow \mathcal{A}$$

of the jet algebra of the $\mathcal{O}_M$-algebra $\mathcal{O}_C$. The non-local space of solutions of the non-linear PDE $(\mathcal{A}, p)$ is the subspace of $\underline{\Gamma}(M, C)$ given by

$$\underline{\mathrm{Sol}}_{(\mathcal{A},p)} := \{x \in \underline{\Gamma}(M, C) |\ (j_\infty x)^* L = 0 \text{ for all } L \in \mathrm{Ker}(p)\}$$

where $(j_\infty x)^* : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{O}_M$ is (dual to) the Jet of x. Equivalently, $x \in \underline{\mathrm{Sol}}_{(\mathcal{A},p)}$ if and only if there is a natural factorization

$$\begin{tikzcd} \mathrm{Jet}(\mathcal{O}_C) \arrow[r, "(j_\infty x)^*"] \arrow[dr, "p"'] & \mathcal{O}_M \\ & \mathcal{A} \arrow[u] \end{tikzcd}$$

of the jet of x through p.

8.2.2 Local functionals

The natural functional invariant associated to a given $\mathcal{D}$-algebra $\mathcal{A}$ is given by the de Rham complex

$$\mathrm{DR}(\mathcal{A}) := (I_{*,M} \otimes_{\mathcal{O}} \mathcal{D}[n]) \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{A}$$

of its underlying $\mathcal{D}$-module with coefficient in the universal complex of integral forms $I_{*,M} \otimes_{\mathcal{O}} \mathcal{D}[n]$, and its cohomology $h^*(\mathrm{DR}(\mathcal{A}))$. We will denote

$$h(\mathcal{A}) := h^0(\mathrm{DR}(\mathcal{A})) = \mathrm{Ber}_M \otimes_{\mathcal{D}} \mathcal{A}$$

where Ber_M here denotes the Berezinian object (and not only the complex concentrated in degree 0). If M is a usual variety, one gets

$$\mathrm{DR}(\mathcal{A}) = \Omega^n_M \overset{\mathbb{L}}{\otimes}_{\mathcal{D}} \mathcal{A} \qquad \text{and} \qquad h(\mathcal{A}) = \Omega^n_M \otimes_{\mathcal{D}} \mathcal{A}.$$

The de Rham cohomology is given by the cohomology of the complex

$$\mathrm{DR}(\mathcal{A}) = I_{*,M}[n] \otimes_{\mathcal{O}_M} \mathcal{A},$$

which gives

$$\mathrm{DR}(\mathcal{A}) = \wedge^* \Omega^1_M[n] \otimes_{\mathcal{O}_M} \mathcal{A}$$

in the non super case. If $\mathcal{A}$ is a jet algebra of section of a bundle $\pi : C \to M$ with basis a classical manifold, the de Rham complex identifies with a sub-complex of the usual de Rham complex of $\wedge^* \Omega^1_{\mathcal{A}/\mathbb{R}}$ of $\mathcal{A}$ viewed as a usual ring.

Definition 8.7. Let M is a super-variety of dimension $p|q$. For every smooth simplex Δ_n, we denote $\Delta_{n|q}$ the super-simplex obtained by adjoining q odd variables to Δ_n. The singular homology of M with compact support is defined as the homology $H_{*,c}(M)$ of the simplicial set

$$\mathrm{Hom}(\Delta_{\bullet|q}, M)$$

of super-simplices with compact support condition on the heart and non-degeneracy condition on odd variables.

Proposition 24. *Let $\pi : C \to M$ be a bundle and $\mathcal{A}$ be the $\mathcal{D}_M$-algebra $\mathrm{Jet}(\mathcal{O}_C)$. There is a natural integration pairing*

$$\begin{array}{ccc} H_{*,c}(M) \times h^{*-n}(\mathrm{DR}(\mathcal{A})) & \to & \mathrm{Hom}(\underline{\Gamma}(M,C), \mathbb{R}) \\ (\Sigma, \omega) & \mapsto & [x \mapsto \int_\Sigma (j_\infty x)^* \omega] \end{array}$$

where $j_\infty x : M \to J^\infty C$ is the taylor series of a given section x. If $p : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{A}$ is a given PDE on $\Gamma(M, C)$ one also gets an integration pairing

$$\begin{array}{ccc} H_{*,c}(M) \times h^{*-n}(\mathrm{DR}(\mathcal{A})) & \to & \mathrm{Hom}(\underline{\mathrm{Sol}}_{(\mathcal{A},p)}, \mathbb{R}) \\ (\Sigma, \omega) & \mapsto & S_{\Sigma,\omega} : [x \mapsto \int_\Sigma (j_\infty x)^* \omega]. \end{array}$$

Remark that the values of the above pairing are given by partially defined functions, with a domain of definition given by Lebesgue's domination condition to make $t \mapsto \int_\Sigma (j_\infty x_t)^* \omega$ a smooth function of t if x_t is a parametrized trajectory.

Definition 8.8. A functional $S_{\Sigma,\omega} : \underline{\Gamma}(M, C) \to \mathbb{R}$ or $S_{\Sigma,\omega} : \underline{\mathrm{Sol}}_{(\mathcal{A},p)} \to \mathbb{R}$ obtained by the above constructed pairing is called a local functional.

8.2.3 Differential forms and vector fields on $\mathcal{D}$-spaces

Working as in section 2.5.1 in the symmetric monoidal category of left $\mathcal{D}$-modules, one defines the inner derivations $\underline{\mathrm{Der}}(\mathcal{A}, \mathcal{M}) \subset \underline{\mathrm{Hom}}(\mathcal{A}, \mathcal{M})$ from a given $\mathcal{D}$-algebra $\mathcal{A}$ to an $\mathcal{A}[\mathcal{D}]$-module $\mathcal{M}$ to be given by usual derivations

$$D : \mathcal{A} \to \mathcal{M}$$

that are moreover $\mathcal{D}$-linear. One shows that the derivation functor is representable by a universal $\mathcal{A}[\mathcal{D}]$-derivation

$$d : \mathcal{A} \to \Omega^1_{\mathcal{A}}.$$

Remark that the natural $\mathcal{O}$-linear map

$$\Omega^1_{\mathcal{A}/\mathcal{O}} \to \Omega^1_{\mathcal{A}}$$

is an isomorphism of $\mathcal{O}$-modules.

The $\mathcal{D}$-module structure on $\Omega^1_{\mathcal{A}}$ can be seen as an Ehresman connection, i.e., a section of the natural projection

$$\Omega^1_{\mathcal{A}/\mathbb{R}} \to \Omega^1_{\mathcal{A}/M}.$$

In the case of the jet space algebra $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ for $C = \mathbb{R}^{n+m} \to \mathbb{R}^n = M$, a basis of $\Omega^1_{\mathcal{A}/M}$ compatible with this section is given by the Cartan forms

$$\theta^i_\alpha = dx^i_\alpha - \sum_{j=1}^n x^i_{j\alpha} dt^j.$$

The de Rham differential $d : \mathcal{A} \to \Omega^1_{\mathcal{A}}$ in the $\mathcal{D}$-algebra setting (often denoted d^V in the litterature), can then be computed by expressing the usual de Rham differential $d : \mathcal{A} \to \Omega^1_{\mathcal{A}/M}$ in the basis of Cartan forms.

If $\Omega^1_{\mathcal{A}}$ is of finite $\mathcal{A}[\mathcal{D}]$-presentation, one defines the $\mathcal{A}[\mathcal{D}^{op}]$-module of vector fields by

$$\Theta_{\mathcal{A}} := \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\Omega^1_{\mathcal{A}}, \mathcal{A}[\mathcal{D}]).$$

The conceptual explanation of the above definition is the following.

Proposition 25. *The $\mathcal{A}[\mathcal{D}^{op}]$-module $\Theta_{\mathcal{A}}$ is equiped with a local Lie bracket*

$$[.,.] : \Theta_{\mathcal{A}} \boxtimes \Theta_{\mathcal{A}} \to \Delta_* \Theta_{\mathcal{A}}$$

and a local operation

$$\Theta_{\mathcal{A}} \boxtimes \mathcal{A} \to \Delta_* \mathcal{A}$$

on $\mathcal{A}$.

Proof. The bracket and the action are induced by the inclusion

$$\Theta_{\mathcal{A}} \subset \mathcal{E}nd^*(\mathcal{A}, \mathcal{A})$$

that is given by the fact that $\Theta_{\mathcal{A}}$ is identified with the space of internal *-derivations (one could also say local derivations) on $\mathcal{A}$. One can then apply proposition 23. We refer the reader to Beilinson-Drinfeld, 1.4.16 for this last fact. □

Proposition 26. *Let $\pi : C \to M$ be a bundle and $\pi_\infty : \mathrm{Jet}(C) \to C$ be the natural projection and $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$. The natural morphism of $\mathcal{A}[\mathcal{D}^{op}]$-modules*

$$\Theta_{\mathcal{A}} \to (\pi_\infty^* \Theta_{C/M}) \otimes_{\mathcal{A}} \mathcal{A}[\mathcal{D}^{op}]$$

is an isomorphism. The image $\ni_\chi$ of a vertical derivation $\chi = f \frac{\partial}{\partial u}$ of $\pi_\infty^* \Theta_{C/M}$ in $\Theta_{\mathcal{A}}$ is called the evolutionary vector field associated to the vertical vector field χ.

Proof. By duality, this follows from the fact that the natural morphism

$$\mathcal{A}[\mathcal{D}] \otimes_{\mathcal{A}} \pi^* \Omega^1_{C/M} \to \Omega^1_{\mathcal{A}/M}$$

is an isomorphism in the case $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$. □

In the above example $\pi : C = \mathbb{R} \times \mathbb{R} \to \mathbb{R} = M$, the relative vector fields $\Theta_{C/M}$ are generated by $\frac{\partial}{\partial x}$ where t is the coordinate in M and (t, x) is the coordinate in C. The pull-back bundle $\pi_\infty^* \Theta_{C/M}$ has the same basis as an $\mathcal{A}$-module. The natural map

$$\pi_\infty^* \Theta_{C/M} \to \Theta_{\mathcal{A}}$$

is given by sending a derivation of the form $f \frac{\partial}{\partial u}$ with $f \in \mathcal{A}$ to

$$\ni_f := \sum_n (D_t^n f) \frac{\partial}{\partial x_n}$$

where $D_t := \frac{\partial}{\partial t} + \sum_n u_{n+1} \frac{\partial}{\partial u_n}$ is the natural action of the standard derivation in Θ_M on functions in $\mathcal{A}$ given by its $\mathcal{D}$-algebra structure. More generally, if $\pi : C = \mathbb{R}^{n+m} \to \mathbb{R}^n$, the evolutionary vector field in $\Theta_{\mathcal{A}}$ associated to a vertical derivation $\chi = f \frac{\partial}{\partial u^i}$ is given by

$$\ni_f^i = \sum_\alpha (D_i^\alpha f) \frac{\partial}{\partial x_\alpha}.$$

The relation of these vector fields with vector fields on the space of histories is the following. Let $h(\Theta_{\mathcal{A}}) = \Theta_{\mathcal{A}} \otimes_{\mathcal{D}^{op}} \mathcal{O} = \pi^* \Theta_{C/M}$.

Proposition 27. *Let $\pi : C \to M$ be a bundle and $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$. There is a natural map*

$$h(\Theta_{\mathcal{A}}) \to \mathrm{Hom}(\underline{\Gamma}(M, C), T\underline{\Gamma}(M, C))$$

so that vector fields on the jet $\mathcal{D}$-space induce vector fields on the space of histories (sections of the fiber bundle).

Proof. We first restrict the situation to the target of the map $\Theta_{\mathcal{A}} \to \pi^* \Theta_{C/M}$. Let $v = f \frac{\partial}{\partial u}$ be a local element in $\pi^* \Theta_{C/M}$. Let φ be a field in $\underline{\Gamma}(M, C)(\mathbb{R}) = \Gamma(M, C)$. It corresponds to a map $\varphi^* : \mathcal{O}_C \to \mathcal{O}_M$ given by $\varphi^*(h) = h \circ \varphi$. Recall that

$$T\underline{\Gamma}(M, C)(\mathbb{R}) = \underline{\Gamma}(M, C)(\mathbb{R}[\epsilon]/(\epsilon^2))$$

is the space of maps

$$V : \mathcal{O}_C \to \mathcal{O}_M[\epsilon]/(\epsilon^2)$$

that induce retractions of the natural injection $\mathcal{O}_M[\epsilon]/(\epsilon^2) \to \mathcal{O}_C[\epsilon]/(\epsilon^2)$. We define such a retraction by the formula

$$V_v(h) := h \circ \varphi + \left((f \circ j_\infty \varphi).(\frac{\partial h}{\partial u} \circ \varphi) \right) .\epsilon$$

which makes sense since its coefficients are functions on M. One can easily write down these formulas with a field φ_t parametrized by the spectrum of a ring $\mathrm{Spec}(A)$. $\square$

In the case of a more general sub-scheme of the jet space, one has to use the universal linearization of the given differential operator to describe the tangent $\mathcal{D}$-bundle $\Theta_{\mathcal{A}}$.

Definition 8.9. Let V and W be two smooth M-schemes. Let π and π' be the corresponding structural maps for jet schemes. A non-linear differential operator from V to W is a morphism of $\mathcal{D}$-schemes

$$\Delta : \mathrm{Jet}(V) \to \mathrm{Jet}(W)$$

(or, which is the same by the universal property of jet schemes, a morphism of M-schemes $\Delta_0 : \mathrm{Jet}(V) \to W$). The tangent map to Δ is a morphism of $\mathcal{A}_{Jet(V)}[\mathcal{D}_X]$-modules

$$D\Delta : \Theta_{\mathrm{Jet}(V)} \to \Delta^*\Theta_{\mathrm{Jet}(W)}$$

and its middle de Rham cohomology

$$\ell_\Delta : h(D\Delta) : \pi^*\Theta_{V/X} \to \Delta^*(\pi')^*\Theta_{W/M} = \pi^*\Theta_{W/M}$$

is called the universal linearization of Δ.

In local coordinates, if $\chi = f\frac{\partial}{\partial u} \in \pi^*\Theta_{V/M}$, we have

$$\ell_\Delta(\chi) = \partial\!\!\!/_\chi(\Delta) := \sum_\alpha \frac{\partial \Delta}{\partial u_\alpha} D_\alpha \chi.$$

Proposition 28. *Let $\pi : C \to M$ be a bundle. Let $\pi_\infty : \mathrm{Jet}(C) \to C$ be the natural projection and $\mathcal{A}$ be the $\mathcal{D}$-algebra of functions on $\mathrm{Jet}(C)$. Let $i : Y \hookrightarrow \mathrm{Jet}(C)$ be a smooth closed sub-$\mathcal{D}$-scheme defined by a $\mathcal{D}$-ideal $\mathcal{I} \subset \mathcal{A}$. Suppose further that $\mathcal{I}$ is differentially generated by the equations $F_i = 0$, for $F_i \in \mathcal{A} = \mathrm{Hom}_{\mathcal{D}}(\mathrm{Jet}(C), \mathrm{Jet}(\mathbb{A}^1_M))$, i.e.,*

$$\mathcal{I} = \{\psi \in \mathcal{A} |\, \psi = \sum_{i,\alpha} \psi_{i,\alpha} D_\alpha(F_i),\ \psi_{i,\alpha} \in \mathcal{A}\}.$$

The $\mathcal{O}_Y[\mathcal{D}]$-module of vector fields Θ_Y on Y is the submodule of $i^*\Theta_{\mathrm{Jet}(C)}$ generated by (the restriction of) evolutionary vector fields $\partial\!\!\!/_\chi$, for $\chi \in \pi^*\Theta_{C/M}$ such that

$$i^*(\ell_{F_i}(\chi)) = 0,$$

i.e. by homogeneous solutions of the universal linearization of the differential operators F_i on the $\mathcal{D}$-subvariety Y.

In local coordinates, if $\chi = f\frac{\partial}{\partial u}$, we get the equations

$$\sum_{\alpha} \frac{\partial F_i}{\partial u_\alpha}(D_\alpha f).$$

These equations allowed people (see [BCD$^+$99] for references) to compute explicitely the higher symmetries of non-linear equations.

Example 8.1. Consider the example, treated in [KK00], p80, of the Burgers equation $\mathcal{B}$, whose ideal is differentially generated by

$$u_t = u_{xx} + uu_x.$$

The total derivatives are given by

$$\begin{aligned} D_x &= \tfrac{\partial}{\partial x} + \textstyle\sum_{k=0}^{\infty} u_{i+1}\tfrac{\partial}{\partial u_i}, \\ D_t &= \tfrac{\partial}{\partial t} + \textstyle\sum_{k=0}^{\infty} u_{i+1} D_x^i(u_2 + u_0 u_1)\tfrac{\partial}{\partial u_i}. \end{aligned}$$

The operator of universal linearization for the equation is given by

$$\ell_{\mathcal{B}} = D_t - u_1 - u_0 D_x - D_x^2,$$

and an evolutionary vector field

$$\partial\!\!\!\!\ni_\varphi = \sum_{i=1}^{\infty} D_x^i(\varphi)\frac{\partial}{\partial u_i}$$

is a symmetry for $\mathcal{B}$ is and only if the function $\varphi(x, t, u_0, \ldots, u_k)$ satisfies the equation

$$\ell_{\mathcal{B}}(\varphi) = 0.$$

A clever computation gives that the Lie algebra of symmetries is simple and generated by the evolutionary vector fields of the following generating functions

- $\varphi_1^0 = u_1$,
- $\varphi_2^2 = t^2 u_2 (t^2 u_0 + tx) u_1 + t u_0 + x$,
- $\varphi_0^3 = u_3 + \frac{3}{2} u_0 u_2 + \frac{3}{2} u_0^2 + \frac{3}{4} u_0^2 u_1$.

8.2.4 Derived solution spaces

One can use the general methods of homotopy theory presented in chapter 4 to derive the solution space of a given non-linear partial differential equation and study derived differential calculus on this space, given by working directly with the de Rham complex $\mathrm{DR}(\mathcal{A})$ and the variational bicomplex $\mathrm{DR}(\Omega^*_{\mathcal{A}})$ up to homotopy.

Let $\pi : C \to M$ be a bundle, $\mathcal{A} := \mathrm{Jet}(\mathcal{O}_C)$ and $p : \mathcal{A} \to \mathcal{B}$ be a non-linear PDE on $\Gamma(M, C)$. One defines the derived solution space of $(\mathcal{B}, p)$ as the derived functor

$$\mathbb{R}\mathrm{Sol}_{(\mathcal{B},p)}(.) := \mathbb{R}\mathcal{H}om_{dg-\mathcal{D}-\textsc{Alg}}(\mathcal{B}, .) : \mathrm{Ho}(\textsc{Alg}_{dg,\mathcal{D}}) \to \mathrm{Ho}(\mathrm{S}\textsc{Sets}).$$

To study differential forms on the derived solution space of $\mathcal{B}$, one has to work with the cotangent complex (see Illusie [Ill71] for a general notion and for references to Quillen's original work on this). The idea is quite simple: one would like to derive the differential form functor

$$\Omega^1 : \textsc{Alg}_{\mathcal{A}} \to \textsc{Mod}(\mathcal{A}).$$

This is given by extending this functor to differential graded $\mathcal{A}$-algebras and deriving it, so that the cotangent complex is

$$^{\mathbb{L}}\Omega^1 : \mathrm{Ho}(\textsc{Alg}_{dg,\mathcal{A}}) \to \mathrm{Ho}(\textsc{Mod}_{dg}(\mathcal{A})).$$

One can see the exterior algebra $\wedge^*\Omega^1_{\mathcal{B}}$ of a given $\mathcal{D}$-algebra as the symmetric algebra $\mathrm{Sym}_{dg}(\Omega^1_{\mathcal{B}}[1])$ on the graded module $\Omega^1_{\mathcal{B}}[1]$ in odd degree. The derived version of the construction of de complex of differential forms is thus given by the derived symmetric functor

$$\mathbb{R}\mathrm{Sym} : \mathrm{Ho}(\textsc{Mod}_{dg}(\mathcal{A})) \to \mathrm{Ho}(\textsc{Alg}_{dg,\mathrm{Jet}(\mathcal{A})})$$

applied to the cotangent complex $^{\mathbb{L}}\Omega^1_{\mathcal{B}}[1]$, so that one gets

$$^{\mathbb{R}\mathbb{L}}\Omega^*_{\mathcal{B}} := \mathbb{R}\mathrm{Sym}(^{\mathbb{L}}\Omega^1_{\mathcal{B}}[1]).$$

If $\mathcal{M}$ is an $\mathcal{A}[\mathcal{D}]$-dg-module, we denote

$$\mathbb{D}(\mathcal{M}) := \mathbb{R}\mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathcal{M}, \mathcal{A}[\mathcal{D}])$$

and

$$\mathrm{DR}(\mathcal{M}) := \mathrm{DR}_{\mathcal{A}\otimes_{\mathcal{O}}\mathrm{Ber}_M}(\mathcal{M}) := (\mathcal{A} \otimes_{\mathcal{O}} \mathrm{Ber}_M) \overset{\mathbb{L}}{\otimes}_{\mathcal{A}} \mathcal{M}.$$

The derived analog of the notion of vector field defined in previous section is given by

$$\mathbb{R}\Theta_{\mathcal{A}} := \mathbb{D}(\Omega^1_{\mathcal{A}}) = \mathbb{R}\mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\Omega^1_{\mathcal{A}}, \mathcal{A}[\mathcal{D}]).$$

Proposition 29. *Suppose that $\Omega^1_{\mathcal{A}}$ is of finite $\mathcal{A}[\mathcal{D}]$-presentation. There is a canonical isomorphism*

$$\mathrm{DR}(\Omega^1_{\mathcal{A}}) \cong \mathbb{R}\mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathrm{Ber}^*_M \otimes \mathbb{R}\Theta_{\mathcal{A}}, \mathcal{A}).$$

The above isomorphism is called the insertion isomorphism. It is a homotopical $\mathcal{D}$-analog of the usual insertion morphism

$$\begin{array}{rccc} i: & \Omega^1_M & \to & \mathrm{Hom}_{\mathcal{O}}(\Theta_M, \mathcal{O}) \\ & \omega & \mapsto & [X \mapsto i_X\omega] \end{array}$$

8.3 The covariant phase space

8.3.1 The classical covariant phase space

Definition 8.10. Let $\pi : C \to M$ be a bundle, $H \subset \Gamma(M, C)$ be a subspace and $S : H \to \mathbb{R}$ be an action functional. The covariant phase space (also called the space of trajectories) for the lagrangian variational problem determined by the above data is the space

$$T = \{x \in H |\ d_x S = 0\}.$$

A lagrangian variational problem is called local if $H \subset \Gamma(M, C)$ is determined by a partial differential equation $p : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{A}$ on $\Gamma(M, C)$ and if $S \in h(\mathcal{A}) := \mathrm{Ber}_M \otimes_{\mathcal{D}} \mathcal{A}$ is an element of zeroth de Rham cohomology of the corresponding $\mathcal{D}$-algebra $\mathcal{A}$.

We will be interested in local variational problems for which the space H of histories is given by nice boundary conditions that imply that the covariant phase space is given by the solution space of a simple partial differential equation derived from the given action $S \in h(\mathcal{A})$.

We refer to Beilinson-Drinfeld [BD04], 2.3.20 for the proof of the following proposition by non-derived methods.

Proposition 30. *Suppose that $\mathcal{A}$ is a smooth $\mathcal{D}$-algebra on M. There is a natural "insertion" isomorphism*

$$h(\Omega^1_{\mathcal{A}}) \cong \mathrm{Hom}_{\mathcal{A}[\mathcal{D}]}(\mathrm{Ber}^*_M \otimes_{\mathcal{O}} \Theta_{\mathcal{A}}, \mathcal{A}).$$

Proof. This is obtained by applying the central cohomology functor $h(\_) := \mathrm{Ber}_M \otimes_{\mathcal{D}} \_$ to the isomorphism of proposition 29. □

Let $S \in h(\mathcal{A})$ be a local action functional. Applying the h functor to the universal derivation

$$d : \mathcal{A} \to \Omega^1_{\mathcal{A}},$$

one gets a morphism

$$h(d) : h(\mathcal{A}) \to h(\Omega^1_{\mathcal{A}})$$

and using the insertion isomorphism, one gets from $h(d)(S)$ a morphism of $\mathcal{A}[\mathcal{D}^{op}]$-module

$$i_{dS} : \mathrm{Ber}^*_M \otimes_{\mathcal{O}} \Theta_{\mathcal{A}} \to \mathcal{A}.$$

Definition 8.11. The Euler-Lagrange ideal $\mathcal{I}_S \subset \mathcal{A}$ is defined as the image of the insertion morphism $i_{dS} : \mathrm{Ber}^*_M \otimes_{\mathcal{O}} \Theta_{\mathcal{A}} \to \mathcal{A}$. The Euler-Lagrange equation on $\Gamma(M, C)$ is defined as the natural projection map

$$p : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{A}/\mathcal{I}_S.$$

If we work with a trivial bundle $C = \mathbb{R}^{n+m} \to \mathbb{R}^n = M$, denoting $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ and $S = [L\omega] \in h(\mathcal{A})$ for $L \in \mathcal{A}$ and $\omega \in \mathrm{Ber}_M$, the vector field in $\Theta_{\mathcal{A}}$ associated to the vertical derivation $\frac{\partial}{\partial x^i}$ is sent to the Euler-Lagrange function

$$E_i(L) = \sum_{\alpha} (-1)^{|\alpha|} D_\alpha \left(\frac{\partial L}{\partial x^i_\alpha} \right)$$

on jet space.

Definition 8.12. A local field theory (M, C, π, H, S) is said to have convenient histories if the space of trajectories

$$T = \{x \in H |\ d_x S = 0\}$$

is identified with the solution space in $\Gamma(M, C)$ of the Euler-Lagrange partial differential equation

$$p : \mathrm{Jet}(\mathcal{O}_C) \to \mathcal{A}/\mathcal{I}_S.$$

We will work mainly with local field theories with convenient space of histories. This assumption means essentially that if one computes the space of trajectories for the action functional $S : H \to \mathbb{R}$, all the boundary terms that appear in integration by part (stokes formula) annihilate because of the boundary conditions, so that one just gets the Euler-Lagrange equation, given locally by

$$T = \{x \in H |\ E_i(L) \circ j_\infty x = 0 : M \to \mathbb{R}\}.$$

Theorem 36. *Suppose that the Euler-Lagrange $\mathcal{D}$-algebra $\mathcal{A}/\mathcal{I}_S$ is smooth. There is a natural cohomology class $\omega \in h^{-1}(\Omega^2_{\mathcal{A}/\mathcal{I}_S})$ on the covariant phase space equation of a local field theory with convenient histories.*

Proof. Let $i_{dS}:\mathrm{Ber}^*_M\otimes\Theta_{\mathcal{A}}\to\mathcal{A}$ be the insertion map associated to $dS\in h(\Omega^1_{\mathcal{A}})$. The restriction of i_{dS} to $\mathcal{A}/\mathcal{I}_S$ is the zero map

$$i_{dS}:\mathrm{Ber}^*\otimes\Theta_{\mathcal{A}/\mathcal{I}_S}\to\mathcal{A}/\mathcal{I}_S.$$

Since $\mathcal{A}/\mathcal{I}_S$ is smooth, this map also identifies with the cohomology class of $d\bar{S}$ in $h^*(\Omega^1_{\mathcal{A}/\mathcal{I}_S})$ where $\bar{S}$ is now considered as leaving in the image by h of the $\mathcal{D}$-algebra $\mathcal{A}/\mathcal{I}_S$. Now since this class is zero in

$$h(\mathcal{A}/\mathcal{I}_S)=h^0(\mathrm{DR}(\mathcal{A}/\mathcal{I}_S)),$$

there exists a so-called local legendre form

$$\theta\in\Omega^{n-1}_M\otimes_{\mathcal{O}}\Omega^1_{\mathcal{A}/\mathcal{I}_S}.$$

Let $\omega:=d\theta\in\Omega^{n-1}_M\otimes_{\mathcal{O}}\Omega^2_{\mathcal{A}/\mathcal{I}_S}$. The form ω is closed for the differential induced by the de Rham differential $\bar{d}$ on M, because $\bar{d}d=d\bar{d}$, so that it defines a cohomology class

$$\omega\in h^{-1}(\Omega^2_{\mathcal{A}/\mathcal{I}_S}).$$

□

8.3.2 Gauge symmetries

We keep the notations of the previous section and refer to Vitagliano's paper [Vit08] for the following definition of Gauge symmetries and to Fulp-Lada-Stasheff [FLS02] for explicit coordinate computations.

Definition 8.13. Let $(\mathcal{A},S)$ be a local lagrangian field theory on a given bundle $\pi:C\to M$. A Noether symmetry of S is a class $X\in h(\Theta_{\mathcal{A}})$ such that

$$X.S=0.$$

We denote $\mathrm{Sym}(\mathcal{A},S)$ the space of Noether symmetries.

Proposition 31. *The symmetries of a local lagrangian field theory identify with the symmetries of the corresponding Euler-Lagrange equation, meaning that the natural map*

$$\mathrm{Sym}(\mathcal{A},S)\to h(\Theta_{\mathcal{A}/\mathcal{I}_S})$$

is an isomorphism of Lie algebras.

We only give here the local version of the definition of Gauge symmetries that can be found in Fulp-Lada-Stasheff [FLS02]. For a more abstract definition, one can look at [Vit08].

Definition 8.14. Let $C \to M$ be a trivial fiber bundle, $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$ and $S \in h(\mathcal{A})$ be a local action functional. A gauge symmetry of S is defined as a linear map

$$g : \mathcal{A} \to \mathrm{Sym}(\mathcal{A}, S)$$

from the space of local functionals to the space of Noether symmetries given locally by a formula of the form

$$g(\epsilon) = \sum_{\alpha,i} R^{i,\alpha}(D_\alpha \epsilon) \frac{\partial}{\partial x^i}.$$

The space of Gauge symmetries is denoted $\mathfrak{g}_S$.

Proposition 32. *The restriction of the space of Gauge symmetries to $\mathcal{A}/\mathcal{I}_S$ is a Lie algebra.*

Definition 8.15. Let $(\mathcal{A}, S)$ be a local lagrangian field theory on a given bundle $\pi : C \to M$. The space of Noether identities is defined as $\mathcal{A}[\mathcal{D}]$-module $\mathcal{N}_S$ given by the kernel of the insertion map

$$i_{dS} : \Theta_{\mathcal{A}} \to \mathcal{A}.$$

By definition, we have two exact sequences of $\mathcal{A}[\mathcal{D}]$-modules

$$0 \to \mathcal{N}_S \to \Theta_{\mathcal{A}} \to \mathcal{I}_S \to 0$$

and

$$0 \to \mathcal{I}_S \xrightarrow{i_{dS}} \mathcal{A} \to \mathcal{A}/\mathcal{I}_S \to 0.$$

The following theorem shows that one can actually use the Noether identities as a simple definition of the notion of gauge symmetry.

Theorem 37 (Noether)**.** *There is a natural isomorphism*

$$\mathfrak{g}_S \cong \mathcal{N}_S$$

between gauge symmetries and Noether identities.

Proof. This can be found in [FLS02]. $\square$

8.3.3 The local derived covariant phase space

In all this section, we omit the twist between left and right $\mathcal{D}$-modules given by tensoring by Ber_M, for notation simplicity. All $\mathcal{A}$-modules in play are considered as $\mathcal{A}$-modules in the tensor category of $\mathcal{D}$-modules, i.e., $\mathcal{A}[\mathcal{D}]$-modules. We refer to Henneaux and Teitelboim's book [HT92], chapter 17 for a physical presentation of these matters (that is independent of chapters 1-16 of loc. cit.) and to Beilinson-Drinfeld [BD04], chapter 1, for a more abstract presentation.

Remark that the construction of this section can be done in every situation where the insertion map associated to the differential of a function makes sense. One has to check if this works for non-local functionals.

Let $(\mathcal{A}, S)$ be a local lagrangian field theory on the space of sections of a bundle $\pi : C \to M$. Let

$$i_{dS} : \Theta_{\mathcal{A}} \to \mathcal{A}$$

be the corrresponding insertion map, $\mathcal{N}_S$ its kernel and $\mathcal{I}_S$ its image.

Definition 8.16. The stationnary surface (also called critical space) for $(\mathcal{A}, S)$ is defined as the $\mathcal{A}$-algebra $\mathcal{A}/\mathcal{I}_S$ of solutions of the Euler-Lagrange equation. A cofibrant resolution

$$\mathcal{P}_S \xrightarrow{\sim} \mathcal{A}/\mathcal{I}_S$$

by a dg-$\mathcal{A}$-algebra is called a derived stationnary surface for $(\mathcal{A}, S)$.

We suppose that $\mathcal{A}$ is smooth (for example the jet algebra $\mathrm{Jet}(\mathcal{O}_C)$). Using the exact sequence

$$0 \to \mathcal{N}_S \to \Theta_{\mathcal{A}} \to \mathcal{I}_S \to 0,$$

and the fact that $\Theta_{\mathcal{A}}$ is projective, one shows that $\mathcal{N}_S$ is projective if and only if $\mathcal{I}_S$ is projective. This shows that $\mathcal{A}/\mathcal{I}_S$ is regular (up to extension from $\mathbb{R}$ to $\mathbb{C}$, this means smooth) if and only if $\mathcal{N}_S$ is a projective $\mathcal{A}[\mathcal{D}]$-module. The natural local Lie bracket

$$[.,.] : \Theta_{\mathcal{A}} \boxtimes \Theta_{\mathcal{A}} \to \Delta_* \Theta_{\mathcal{A}}$$

induces a local Poisson Nijenhuis bracket on the algebra

$$\mathrm{Sym}_{\mathcal{A}-dg}(\mathcal{C}(i_{dS})) := \wedge^* \Theta_{\mathcal{A}}$$

where

$$\mathcal{C}(i_{dS}) := [\underset{-1}{\Theta_{\mathcal{A}}} \xrightarrow{i_{dS}} \underset{0}{\mathcal{A}}]$$

is the cone of the map of complexes in degree zero defined by i_{dS}.

Definition 8.17. The local-dg-poisson algebra $(\wedge^*\Theta_{\mathcal{A}}, d_{i_{dS}})$ is called the pre-derived critical space of $(\mathcal{A}, S)$.

Remark that the pre-derived critical space has cohomology $\mathcal{A}/\mathcal{I}_S$ in degree 0. It has nonzero cohomology in higher degree if and only if $\mathcal{N}_S \neq 0$. If $\mathcal{N}_S = 0$, the pre-derived critical space is a Koszul resolution of the regular ideal $\mathcal{I}_S$, so that the natural projection map

$$\wedge^*\Theta_{\mathcal{A}} \xrightarrow{\sim} \mathcal{A}/\mathcal{I}_S$$

is a quasi-isomorphims of dg-$\mathcal{A}$-algebra that gives a cofibrant replacement of $\mathcal{A}/\mathcal{I}_S$.

If $\mathcal{A}/\mathcal{I}_S$ is smooth, or equivalently $\mathcal{N}_S$ is projective of finite rank, one has to replace the pre-derived critical space by the dg-algebra

$$\mathrm{Sym}_{\mathcal{A}-dg}([\underset{-2}{\mathcal{N}_S} \to \underset{-1}{\Theta_{\mathcal{A}}} \xrightarrow{i_{dS}} \underset{0}{\mathcal{A}}]).$$

This new algebra is a cofibrant resolution of $\mathcal{A}/\mathcal{I}_S$ called a Koszul-Tate resolution. We thus get the following.

Proposition 33. *If $\mathcal{N}_S$ is $\mathcal{A}[\mathcal{D}]$-projective, the critical space $\mathcal{A}/\mathcal{I}_S$ is smooth. We then have:*

- *If $\mathcal{N}_S = 0$ (the theory is said to have no gauge symmetries), the pre-derived critical space $(\wedge^*\Theta_{\mathcal{A}}, d_{i_{dS}})$ is a derived critical space for $(\mathcal{A}, S)$. It is equiped with a local Poisson bracket.*

- *More generally, if $\mathcal{N}$ is only $\mathcal{A}[\mathcal{D}]$-projective (the theory is said to have irreducible gauge symmetries), the algebra*

  $$\mathrm{Sym}_{\mathcal{A}-dg}([\underset{-2}{\mathcal{N}_S} \to \underset{-1}{\Theta_{\mathcal{A}}} \xrightarrow{i_{dS}} \underset{0}{\mathcal{A}}]).$$

  is a derived critical space for $(\mathcal{A}, S)$.

Remark that the local bracket on $\Theta_{\mathcal{A}}$ does not always induce a local bracket on $\mathcal{N}_S$. If it does, one says that the gauge symmetries close off-shell. In this situation, one can extend the Poisson-Nijenhuis bracket on $\wedge^*\Theta_{\mathcal{A}}$ to the derived critical space

$$\mathrm{Sym}_{\mathcal{A}-dg}([\underset{-2}{\mathcal{N}_S} \to \underset{-1}{\Theta_{\mathcal{A}}} \xrightarrow{i_{dS}} \underset{0}{\mathcal{A}}]).$$

for $(\mathcal{A}, \mathcal{I}_S)$.

We now turn to the case of more general gauge theories, that have reducible gauge symmetries that do not necessarily close off-shell (meaning that $\mathcal{N}_S$ is not projective and does not have an induced local Lie bracket). We will define a particular derived stationnary surface associated to the insertion map $i_{dS} : \Theta_{\mathcal{A}} \to \mathcal{A}$: choose a projective resolution $\mathcal{R}$ of the $\mathcal{A}$-module $\mathcal{N}_S$ of Noether identities and define

$$\mathcal{M}_S := [\mathcal{R} \to \underset{-1}{\Theta_{\mathcal{A}}} \to \underset{0}{\mathcal{A}}]$$

and

$$\mathcal{P}_S := \mathrm{Sym}_{\mathcal{A}-dg}(\mathcal{M}_S).$$

This is a cofibrant resolution of $\mathcal{A}/\mathcal{I}_S$ as a dg-$\mathcal{A}$-algebra, i.e., a derived critical space. Such a cofibrant resolution is also called the Koszul-Tate resolution in the physics litterature. The generators of the various terms in $\mathcal{P}_S$ as $\mathcal{A}[\mathcal{D}]$-modules have particular names:

- generators of $\Theta_{\mathcal{A}}$ are called the antifields,
- generators of $\mathcal{R}_0$ (which identify with generators of $\mathcal{N}_S$) are called antighosts.

Now let

$$\mathcal{S} := \mathbb{R}\mathcal{H}om^*_{\mathcal{A}}(\mathcal{N}_S, \mathcal{A}) = \mathbb{R}\mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathcal{N}_S, \mathcal{A}[\mathcal{D}]).$$

One has a quasi-isomorphism

$$\mathcal{S} \xrightarrow{\sim} \mathcal{H}om_{\mathcal{A}[\mathcal{D}]}(\mathcal{R}, \mathcal{A}[\mathcal{D}]).$$

We suppose for simplicity that each component of $\mathcal{R}$ is a finitely presented $\mathcal{A}[\mathcal{D}]$-module. There is a natural local duality pairing

$$\langle ., . \rangle : \mathcal{R} \overset{\mathbb{L}}{\boxtimes} \mathcal{S} = \mathcal{N}_S \overset{\mathbb{L}}{\boxtimes} \mathbb{R}\mathcal{H}om^*(\mathcal{N}_S, \mathcal{A}) \to \mathbb{R}\Delta_* \mathcal{A}.$$

The generators of the various terms in $\mathcal{S}$ have particular names in the physics litterature:

- generators of $\mathcal{S}_0$ (that identify with generators of the dual of $\mathcal{N}_S$) are called ghosts and are related to gauge symmetries,
- generators of $\mathcal{S}_1$ are called ghosts of ghosts,
- and so on...

The above local duality pairing relates ghosts to antifields, ghosts of ghosts to antighosts, and so on. Remark also that the Schouten-Nijenhuis bracket on $\wedge^*\Theta_{\mathcal{A}}$ relates fields coordinates in $\mathcal{A}$ to antifield coordinates in $\Theta_{\mathcal{A}}$.

Define the BRST-BV algebra as the bidifferential bi-graded $\mathcal{A}$-algebra

$$\mathcal{A}_S^{\bullet,\bullet} := \mathrm{Sym}_{\mathcal{A}-loc-dg}(\mathcal{S}) \otimes \mathcal{P}_S.$$

We now follow a similar road as the finite dimensional BRST construction of section 7.6. There is a natural identification of bigraded algebras

$$\mathcal{A}_S^{\bullet,\bullet} \cong \mathrm{Sym}(\mathcal{C}(i_{dS})) \otimes \mathrm{Sym}(\mathcal{N}_S \oplus \mathbb{R}\mathcal{H}om^*(\mathcal{N}_S, \mathcal{A})).$$

The natural duality pairing $\langle ., . \rangle$ allows one to construct a local Clifford algebra

$$\mathrm{Cliff}(\mathcal{N}_S \oplus \mathbb{R}\mathcal{H}om^*(\mathcal{N}_S, \mathcal{A}), \langle ., . \rangle)$$

whose graded algebra is

$$\mathrm{Sym}(\mathcal{N}_S \oplus \mathbb{R}\mathcal{H}om^*(\mathcal{N}_S, \mathcal{A})).$$

The supercommutator in the Clifford algebra induces a natural local bracket on the graded symmetric algebra. We denote $\mathcal{A}_S^\bullet := \mathrm{Tot}(\mathcal{A}_S^{\bullet,\bullet})$ the total complex and D its differential, called the BRST differential.

Theorem 38. *The local pairing between $\mathcal{R}$ and $\mathcal{S}$ and the Schouten-Nijenhuis bracket on $\wedge^*\Theta_{\mathcal{A}}$ induce a local Poisson bracket on the BRST algebra. The $(0,0)$-th cohomology of the BRST algebra $\mathcal{A}_S^{\bullet,\bullet}$ identifies with the algebra $(\mathcal{A}/\mathcal{I}_S)^{\mathfrak{g}_S}$ of gauge invariant functions on the stationnary surface. There exists an element $S_{cm} \in \mathcal{A}_S^\bullet$, called the BRST action, such that*

$$\{S_{cm}, .\} = D$$

where D is the BRST differential.

Proof. We refer to Henneaux and Teitelboim [HT92], chapter XVII, for a proof of this theorem expressed in a coordinate fashion and to Kostant-Sternberg [KS87] for a mathematical proof without coordinates in a simplified instance of the finite dimensional setting. □

Since $D^2 = 0$, the BRST action fulfils the so called classical master equation

$$\{S_{cm}, S_{cm}\} = 0.$$

The quantization of the given local field theory starts at this point by a functional integral quantization of the pair

$$(\mathcal{A}^\bullet, S_{cm}).$$

8.3.4 The non-local derived critical space

Let $\pi : C \to M$ be a bundle, $H \subset \underline{\Gamma}(M, C)$ a subspace and $S : H \to \mathbb{R}$ an action functional. If LEGOS is any of categories of legos considered in this course, the contravariant notion of differential form on H is defined, so that one has always a well defined differential $dS \in \Omega^1_H$. Let $\mathcal{O}_H = \Omega^0_H$ be the space of real valued functions on H.

Definition 8.18. An H-object F with value in a given category C is the datum, for each points $f : U \to H$ of H with values in a lego U, of an object f^*F of C, so that this association is compatible with morphisms of points.

For example, the space of differential forms Ω^*_H is an H-object with values in $\mathrm{ALG}_{s,\mathbb{R}}$. One can define an $\mathcal{O}_H$-module $\mathcal{M}$ (resp. $\mathcal{O}_H$-algebra $\mathcal{A}$) as an H-object with values in $\mathrm{VECT}_{\mathbb{R}}$ (resp. $\mathrm{ALG}_{\mathbb{R}}$) that is equiped with an action of $\mathcal{O}_H$ (resp. a morphism of rings $\mathcal{O}_H \to \mathcal{A}$).

There is a natural $\mathcal{O}_H$-module structure on Ω^1_H that makes $d : \mathcal{O}_H \to \Omega^1_H$ the universal derivation with values in $\mathcal{O}_H$-modules. Let $\Theta_H := \mathrm{Hom}_{\mathcal{O}_H}(\Omega^1_H, \mathcal{O}_H)$ be the linear dual of Ω^1_H. There is a natural insertion map

$$i_{dS} : \Theta_H \to \mathcal{O}_H.$$

There is a natural isomorphism

$$\Theta_H \cong \mathcal{D}er(\mathcal{O}_H, \mathcal{O}_H)$$

induced by the composition with the universal derivation $d : \mathcal{O}_H \to \Omega^1_H$.

Definition 8.19. The ideal of critical functions $\mathcal{I}_S \subset \mathcal{O}_H$ is defined as the image of i_{dS}. The module $\mathcal{N} \subset \Theta_{\mathcal{A}}$ of Noether relations is defined as its kernel. The algebraic critical phase space is the H-algebra $\mathcal{O}_H/\mathcal{I}_S$. A derived critical phase space is a cofibrant resolution of $\mathcal{O}_H/\mathcal{I}_S$ in the category of dg-$\mathcal{O}_H$-algebras.

One can consider as before the exact sequence

$$0 \to \mathcal{N} \to \Theta_H \xrightarrow{i_{dS}} \mathcal{O}_H \to \mathcal{O}_H/\mathcal{I}_S \to 0$$

but there is no reason in general for Θ_H or even Ω^1_H to be a projective $\mathcal{O}_H$-module. This makes the non-local theory quite formal and prevents us from constructing easily a non-local analog of the BV dg-algebra. Such a construction would involve the choice of a projective resolution Θ'_H of Θ_H, the homotopical extension of its bracket to this resolution, and a projective resolution of $\mathcal{N}$. One can then probably combine these resolutions to get

a homotopical non-local analog of the BV algebra, in a similar way as in the last section, but this construction is highly non-effective since, contrary to the local case, one does not have a local basis of Θ_H.

Another take on this problem can be given if $(\mathcal{A}, S)$ is a local field theory: one can define the corresponding local BV dg-algebra as the realization $\mathcal{O}^{\bullet,\bullet}_{H,loc}$ in terms of functional calculus on H of the algebra $h(\mathcal{A}^{\bullet,\bullet})$ of central de Rham cohomology of the BV dg-algebra defined in the last section. This is most probably the approach we will use for the rest of these notes.

Chapter 9

Variational problems of experimental classical physics

We now define the basic lagrangian variational problems of experimental physics and explain a bit their physical interpretation.

9.1 Newtonian mechanics

In Newtonian mechanics, trajectories are given by smooth maps

$$x : [0,1] \to \mathbb{R}^3,$$

which are the same as section of the fiber bundle

$$\boxed{\pi : C = \mathbb{R}^3 \times [0,1] \to [0,1] = M.}$$

The trajectories represent one material points moving in $\mathbb{R}^3$. The space of histories is given by fixing a pairs of starting and ending points for trajectories $\{x_0, x_1\}$, i.e.,

$$H = \{x \in \Gamma(M, C),\ x(0) = x_0, x(1) = x_1\}.$$

If $\langle , \rangle$ is the standard metric on $\mathbb{R}^3$ and $V : \mathbb{R}^3 \to \mathbb{R}$ is a given "potential" function, one defines the action functional by

$$S(x) = \int_M \frac{1}{2} m \|\partial_t x\|^2 - V(x(t)) dt.$$

The tangent space to H is given by smooth functions $\vec{h} : \mathbb{R} \to \mathbb{R}^3$ that fulfil $\vec{h}(0) = \vec{h}(1) = 0$. Defining

$$d_x S(\vec{h}) := \lim_{\epsilon \to 0} \frac{S(x + \epsilon h) - S(x)}{\epsilon},$$

one gets

$$d_x S(\vec{h}) = \int_M \langle m\partial_t x, \partial_t \vec{h} \rangle - \langle d_x V(x), \vec{h} \rangle dt$$

and by integrating by parts using that $h(0) = h(1) = 0$, finally,

$$d_x S(\vec{h}) = \int_M \langle m\partial_t^2 x - d_x V(x), \vec{h} \rangle dt$$

The space of physical trajectories is thus the space of maps $x : [0, 1] \to \mathbb{R}^3$ such that

$$m\partial_t^2 x = -V'(x).$$

This is the standard law of newtonian mechanics. For example, if $V = 0$, the physical trajectories are those with constant speed, which corresponds to free galilean inertial bodies. Their trajectories are also the geodesics in $\mathbb{R}^3$, i.e., lines.

This setting can be easily generalized to a configuration space $C = [0, 1] \times X$ where X is a variety equiped with a euclidean metric

$$g : TX \times_X TX \to \mathbb{R}_X.$$

If $V : X \to \mathbb{R}$ is a smooth “potential” function, the action functional on $x : M \to C$ is given by

$$S(x) = \int_M \frac{1}{2} m.x^* g(Dx, Dx) - x^* V dt$$

where $Dx : TM \to x^*TX$ is the differential and $x^*g : x^*TX \times x^*TX \to \mathbb{R}_M$ is the induced metric. Remark that we suppose here fixed the constant vector field $1 : [0, 1] \to T[0, 1] = \mathbb{R} \times [0, 1]$. This generalization has to be done if one works with constrained trajectories, for example with a pendulum.

Remark that the free action functional above can be replaced by

$$S(x) = \int_M \frac{1}{2} \sqrt{m} \|\partial_t x\| dt,$$

and this gives the same drawings in C but there is then a natural (gauge) symmetry of S by reparametrizations, i.e., diffeomorphisms of M. This means that the trajectories are not necessarily of constant speed. However, they correspond to the smallest path in C between two given points for the given metric on C, also called the geodesics.

9.2 Relativistic mechanics

Special relativity is based on the following simple lagrangian variational problem, that one can take as axiom of this theory. The space of configurations for trajectories is not usual space $\mathbb{R}^3$, but spacetime $\mathbb{R}^{3,1} := \mathbb{R}^4$, equiped with the Minkowski quadratic form

$$g(t,x) = -c^2t^2 + \|x\|^2.$$

The space of parameters for trajectories is $M = [0,1]$, and its variable is called the proper time of the given particle and denoted τ to make it different of the usual time of Newtonian mechanics t, which is one of the coordinates in spacetime. The fiber bundle in play is thus given by

$$\boxed{\pi : C = \mathbb{R}^{3,1} \times [0,1] \to [0,1] = M.}$$

The action functional is exactly the same as the action of Newtonian mechanics, except that the quadratic form used to write is is Minkowski's metric g. It is given on a section $x : M \to C$ by

$$S(x) = \int_M \frac{1}{2} m^2c^2 g(\partial_\tau x, \partial_\tau x) - V(x(\tau))$$

for $V : \mathbb{R}^{3,1} \to \mathbb{R}$ a given potential function.

The equations of motion are given as before by

$$m.\partial_\tau^2 x = -V'(x),$$

but they are now valid only if x is in the light cone (domain where the Minkowski quadratic form is positive)

$$g(t,x) \geq 0.$$

One can think of the free action functional

$$S(x) = \int_M \frac{1}{2} mg(\partial_t x, \partial_t x)$$

as measuring the (square of) Minkowski lenght of the given trajectory, which is called proper time by physicists. Minimizing this action functional then means making your proper time the smallest possible to go from one point to another point in spacetime. This means that you are going on a line. The free trajectories (without potential) are here of constant speed.

In physics, one usually replaces the above (free) action functional by

$$S(x) = \int_M \frac{1}{2} mc \sqrt{g(\partial_\tau x, \partial_\tau x)}.$$

It has a (gauge) symmetry by reparametrization of $[0,1]$, i.e., by diffeomorphisms of $[0,1]$ that fix 0 and 1. This means that the free trajectories are following geodesics (smallest proper time, i.e., length of the path in Minkowski's metric), but they can have varying speed. It is in some sense problematic to allow varying speed because the true theory of accelerated particles is not special relativity but general relativity: special relativity only treats of inertial frames (with no acceleration). However, for a general relativistic particle, which can be accelerated, this is the correct version of the action functional.

One has nothing more to know to understand special relativity from a mathematical viewpoint. It is important to have a take on it because it is the basis of quantum field theory.

9.3 Electromagnetism

A simple physical example of a lagrangian variational problem whose parameter space for trajectories is not a segment, as this was the case for Newtonian and special relativistic mechanics, is given by the field theory on spacetime underlying maxwell's theory of electromagnetism. We refer to Landau and Lifchitz's book [LL66] for more details and to Derdzinski's book [Der92] for the covariant formulation.

9.3.1 Flat space formulation

Let M be an oriented 4-dimensional manifold. A Minkowski metric on M is a symmetric non-degenerate bilinear form

$$g : TM \times_M TM \to \mathbb{R}_M$$

that is locally of signature $(3,1)$, i.e., isomorphic to the Minkowski metric. We can see g as a bundle isomorphism $g : TM \xrightarrow{\sim} T^*M$, that can be extended to a bundle isomorphism

$$g : \wedge^* TM \to \wedge^* T^* M.$$

There is a natural operation of contraction of multivectors by the volume form

$$i_\bullet \omega_{\mathrm{vol}} : \wedge^* TM \xrightarrow{\sim} \wedge^* T^* M$$

and the Hodge $*$-operator is defined by

$$* := i_\bullet \mathrm{vol} \circ g^{-1} : \wedge^* T^* M \to \wedge^* T^* M.$$

If $d : \Omega^i \to \Omega^{i+1}$ is the de Rham differential, we denote $d^* : \Omega^3 \to \Omega^2$ the adjoint of d for the given metric g given by $g(d^*\omega, \nu) = g(\omega, d\nu)$. The bundle $\pi : C \to M$ underlying classical electromagnetism in a flat space is the bundle

$$\boxed{\pi : C = T^* M \to M}$$

of differential one forms on M. Let $J \in \Omega^3_M$ be a fixed 3-form (called the charge-current density or the source) that fulfils

$$dJ = 0.$$

The lagrangian density is the map

$$L_J : J^1 T^* M \to \wedge^4 T^* M$$

given by

$$L_J(A, dA) = \frac{1}{2} dA \wedge *dA + A \wedge J.$$

It gives the action functional $S : \Gamma(M, C) \to \mathbb{R}$ given by

$$S_J(A) = \int_M L_J(A, dA).$$

Its variable A is called the electromagnetic potential and the derivative $F = dA$ is called the electromagnetic field. If one decomposes the spacetime manifold M in a product $T \times N$ of time and space, one can define the electric field $E_A \in \Omega^1(N)$ and magnetic field $B_A \in \Omega^2(N)$ by

$$F = B_A + dt \wedge E_A.$$

One then gets the lagrangian density

$$L(A, dA) = \frac{1}{2} \left(\frac{|E_A|^2}{c^2} - |B_A|^2 \right)$$

if the source J is zero. Minimizing the corresponding action functional means that variations of the electric field induce variation of the magnetic field and vice versa, which is the induction law of electromagnetism.

Proposition 34. *The equations of motion for the electromagnetic field lagrangian $S_J(A)$ are given by*

$$d * dA = J.$$

Proof. As explained in chapter 8, one can compute the equations of motion from the lagrangian density in terms of differential forms by

$$\frac{\partial L}{\partial A} - d\left(\frac{\partial L}{\partial(dA)}\right) = 0.$$

Recall that the lagrangian density is

$$L_J(A, dA) = \frac{1}{2}dA \wedge *dA + A \wedge J.$$

On has $\frac{\partial L}{\partial A} = J$ and $\frac{\partial L}{\partial(dA)} = *dA$ because of the following equalities, that are true for $\epsilon^2 = 0$:

$$\begin{array}{rcc} (A_1 + \epsilon A_1') \wedge *(A_1 + \epsilon A_1') - A_1 \wedge *A_1 & = & \epsilon(A_1' \wedge *A_1) + A_1 \wedge *A_1'), \\ & = & \epsilon(- * A_1 \wedge A_1' + A_1 \wedge *A_1'), \\ & = & 2\epsilon(A_1 \wedge *A_1'). \end{array}$$

This gives

$$d * dA = J.$$

□

The second condition is equivalent to

$$\Box A = J$$

where $\Box := d * d$ is the d'Alembertian (or wave) operator. On Minkowski's space, this identifies with $d^*d + dd^*$, which gives in local coordinates

$$\Box = -\frac{1}{c^2}\partial_t^2 + \partial_x^2.$$

The difference between this approach and the covariant one is given by a Riemann curvature term, that here annihilates.

In classical physics, $F = dA$ is the object of interest and the solution to $\Box A = J$ is not unique because for any function α, one also has $\Box(A + d\alpha) = 0$ since $d^2 = 0$. One can however fix a section of the projection $A \mapsto F := dA$ (called a Gauge fixing) by setting

for example $\sum_\mu A_\mu = 0$. This implies the unicity of the solution A to the equation of motion $\Box A = J$.

In quantum physics, the electromagnetic potential itself can be thought as the wave function of the photon light particle.

If one wants to study the motion of a charged particle in a fixed given electromagnetic potential $A \in \Omega^1_X$ on a given Lorentzian manifold (X, g), one uses the action functional on relativistic trajectories $x : [0, 1] \to X$ defined by

$$S(x) = \int_M \frac{1}{2} mc \sqrt{g(\partial_\tau x, \partial_\tau x)}$$

and add to it a so called coulomb potential term of the form

$$\int_M \frac{e}{c} x^* A$$

where the constant e called the charge of the particle that can be of positive or negative sign.

9.3.2 Generally covariant formulation

Let M be an oriented manifold of dimension n equiped with a non-degenerate metric $g : TM \times_M TM \to \mathbb{R}_M$ (the corresponding volume form is denoted ω_g), and a connection ∇ on the tangent bundle given on sections by a map

$$\nabla : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1_M$$

that fulfils Leibniz's rule. The bundle $\pi : C \to M$ underlying generalized electromagnetism is one of the following linear bundles:

$$\mathbb{R}_M,\ \wedge^n T^*M,\ T^*M,$$

(the case of interest for electromagnetism is $C = T^*M$ but other cases can be useful in particle physics). We will now write down the lagrangian density for a generalized electromagnetism action functional

$$S : \Gamma(M, C) \to \mathbb{R}.$$

Since the bundle C is linear, we will denote it F. The metric and connection on TM induces a metric $g : F \times_M F \to \mathbb{R}_M$ and a connection

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_M.$$

Definition 9.1. Fix an element $J \in \mathcal{F}$ called the source. The generalized electromagnetism lagrangian (F, g, ∇) with source J sends a section $A \in \mathcal{F}$ to the differential form of maximal degree

$$L_J(A, \nabla A) := g(\nabla A, \nabla A)\omega_g + g(J, A)\omega_g$$

where g is here extended naturally to $\mathcal{F} \otimes \Omega^1_M$.

The usual electromagnetic lagrangian (in not necessarily flat space) is obtained by using for F the cotangent bundle T^*M.

Definition 9.2. The d'Alembertian operator of the pair (g, ∇) is the operator $\Box : \mathcal{F} \to \mathcal{F}$ defined by contracting the second order derivative with the metric

$$\Box := \mathrm{Tr}\left((\mathrm{id}_{\mathcal{F}\otimes\Omega^1} \otimes g^{-1}) \circ \nabla_1 \circ \nabla\right).$$

Let us explain this in details. The inverse of the metric gives an isomorphism $g^{-1} : \Omega^1_M \xrightarrow{\sim} \mathcal{T}_M$ and thus an isomorphism

$$\mathrm{id} \otimes g^{-1} : \Omega^1 \otimes \Omega^1_M \to \Omega^1 \otimes \mathcal{T}_M \cong \mathrm{End}(\mathcal{T}_M).$$

If we denote

$$\nabla_1 : \mathcal{F} \otimes \Omega^1 \to Fc \otimes \Omega^1 \otimes \Omega^1$$

the extension $\nabla_1 := \nabla \otimes \mathrm{id} + \mathrm{id} \otimes \nabla$ of ∇ to differential 2-forms, one gets by composition

$$\mathcal{F} \xrightarrow{\nabla} \mathcal{F} \otimes \Omega^1 \xrightarrow{\nabla_1} \mathcal{F} \otimes \Omega^1 \otimes \Omega^1 \xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathfrak{g}^{-1}} \mathcal{F} \otimes \mathrm{End}(\mathcal{T})$$

a map $\mathcal{F} \to \mathcal{F} \otimes \mathrm{End}(\mathcal{T})$ of which we can compute the trace to get the d'Alembertian.

If we suppose that $F = T^*M$, $\nabla g = 0$ and ∇ is torsion free, meaning that

$$\nabla_x y - \nabla_y x = [x, y],$$

we get as equations of motion for the above lagrangian the equation

$$\Box A = J$$

that is the wave equation on the given metric manifold with connection.

9.3.3 Gauge theoretic formulation

Let M be an oriented manifold of dimension n equiped with a non-degenerate metric $g : TM \times_M TM \to \mathbb{R}_M$ (the corresponding volume form is denoted ω_g) and a connection $\nabla : \mathcal{T}_M \to \mathcal{T}_M \otimes \Omega^1_M$ (that is the Levi-Civita connection for example). Let $G := \mathrm{SU}(1) \subset \mathbb{C}^*$ be the group of complex numbers of norm 1 and consider a principal G-bundle P. Recall that the Lie algebra $\mathfrak{g}$ of G is identified with $2i\pi\mathbb{R}$ by the exponential map. It as a natural S^1 -equivariant inner product $\langle ., . \rangle$. The variable of the gauge theory formulation of electromagnetism is a principal G-connection A on P, i.e., an equivariant differential form

$$A \in \Omega^1(P, \mathfrak{g})^G,$$

meaning that the bundle underlying this theory is

$$\boxed{\pi : C = (T^*P \times \mathfrak{g})/G \to M.}$$

The curvature of A is defined as the $\mathfrak{g}$-valued 2 form

$$F := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g})^G.$$

Remark that the choice of a section $s : M \to P$ gives an identification

$$s^* : \Omega^1(P, \mathfrak{g})^G \to \Omega^1(M, \mathfrak{g}) = \Omega^1(M)$$

of the space of G-equivariant principal connections on P with differential forms on M by pull-back. This relates this gauge theory formulation with the covariant formulation of subsection 9.3.2.

The Yang-Mills action for electromagnetism is given by

$$S(A) := \int_M \langle F \wedge *F \rangle + A \wedge *J.$$

The corresponding equation is

$$d_A * F = *J,$$

where d_A is the combination of the gauge derivative and the covariant derivative.

9.4 General relativity

This section is mainly inspired by the excellent old book of Landau and Lifchitz [LL66].

9.4.1 The Einstein-Hilbert-Palatini action

We here give a version of general relativity that is due (in another language) to Palatini. It is the one used by people studying quantization of gravity, and it is better adapted to the introduction of fermionic ("matter") fields in general relativity. It is called the first order formulation of relativity because the equations of motion are of first order, contrary to the classical Einstein-Hilbert approach that gives an equation of order two.

Let M be an oriented and compact 4-dimensional manifold. A Minkowski metric on M is a symmetric non-degenerate bilinear forms

$$g : TM \times_M TM \to \mathbb{R}_M$$

that is locally of signature $(3, 1)$, i.e., isomorphic to the Minkowski metric. We will denote

$$\mathrm{Sym}^2_{mink}(T^*M) \to M$$

the bundle whose sections are metrics of this kind. It is a sub-bundle of the linear bundle $\mathrm{Sym}^2(T^*M) \to M$ of all bilinear forms. Recall that a connection on a linear bundle $F \to M$ is given by a covariant derivation

$$\nabla : \mathcal{F} \to \Omega^1_M \otimes \mathcal{F}$$

on the space $\mathcal{F} = \Gamma(M, F)$ of section of F, i.e., an $\mathbb{R}$-linear map that fulfils

$$\nabla(gf) = dg \otimes f + g\nabla(f).$$

We will consider the space $\mathrm{Con}(F)$ of (parametrized families of) connections on M whose points with values in a smoothly closed algebra A are

$$\mathrm{Con}(F)(A) := \left\{ (x, \nabla) \,\middle|\, \begin{array}{l} x : \mathrm{Spec}(A) \to M \\ \nabla : \mathcal{F} \otimes_{\mathcal{C}^\infty(M)} A \to \Omega^1_M \otimes (\mathcal{F} \otimes_{\mathcal{C}^\infty(M)} \mathcal{A}) \end{array}, \nabla \text{ is an } A\text{-linear connection} \right\}.$$

There is a natural projection

$$\pi : \mathrm{Con}(F) \to M$$

that sends the pair (x, ∇) to x. There is also a natural action of sections A of $\Omega^1_M \otimes_{\mathcal{O}} \mathrm{End}(\mathcal{F})$ on the space $\mathcal{C}on(F)$ of connections on F given by

$$\nabla \mapsto \nabla + A$$

that makes it a principal homogeneous space. In the case $M = \mathbb{R}^4$ and $F = TM$, this means that every connection can be written as

$$\nabla = d + A$$

where d is the de Rham differential extended to the trivial bundle $TM = \mathbb{R}^4 \times \mathbb{R}^4$ and A is a 1-form on M with values in $\mathrm{End}(\mathbb{R}^4)$.

The bundle $\pi : C \to M$ that is at the heart of the lagrangian formulation of general relativity is the bundle

$$\boxed{\pi : C = \mathrm{Con}_{tf}(TM) \times_M \mathrm{Sym}^2_{mink}(T^*M) \to M}$$

whose sections are pairs (∇, g) with

- $\nabla : \mathcal{T}_M \to \Omega^1_M \otimes \mathcal{T}_M$ a connection on the tangent bundle (often called an affine connection) that is supposed to be torsion free, i.e., such that

  $$\nabla_x y - \nabla_y x = [x, y],$$

  and

- $g : TM \times_M TM \to \mathbb{R}_M$ a metric of Minkowski's type.

The Riemann curvature tensor of the connection is given by the composition

$$R_\nabla := \nabla_1 \circ \nabla : \mathcal{T} \to \mathcal{T} \otimes \Omega^2.$$

It measures the local difference between ∇ and the de Rham trivial connection on a trivial bundle. Contrary to the original connection, this is a $\mathcal{C}^\infty(M)$-linear operator that can be seen as a tensor

$$R_\nabla \in \mathrm{End}(\mathcal{T}) \otimes \mathrm{Sym}^2 \Omega^1_M.$$

The Ricci curvature is simply the trace

$$\mathrm{Ric}(\nabla) := -\mathrm{Tr}(R_\nabla) \in \mathrm{Sym}^2 \Omega^1_M.$$

One can also interpret it as a morphism

$$\mathrm{Ric} : \mathcal{T} \to \Omega^1_M.$$

The metric g, being non-degenerate, is an isomorphism $g : \mathcal{T} \to \Omega^1$ and its inverse g^{-1} is a morphism $g^{-1} : \Omega^1 \to \mathcal{T}$. One can then define the scalar curvature of the pair (∇, g) as the function on M defined by

$$R(\nabla, g) := \mathrm{Tr}(g^{-1} \circ \mathrm{Ric}).$$

The given metric on M gives a fundamental class $d\mu_g$ and one defines the Einstein-Hilbert action functional $S : \Gamma(M, C) \to \mathbb{R}$ by

$$S(\nabla, g) := \int_M R(\nabla, g) d\mu_g.$$

One sometimes adds a constant Λ to get an action

$$S(\nabla, g) := \int_M [R(\nabla, g) - \Lambda] d\mu_g$$

that takes into accound the "expansion of universe" experiment (red shift in old/far-away starlight).

The space of histories H is equal to $\Gamma(M, C)$.

Proposition 35. *The space $T \subset H$ of physical trajectories for the Einstein-Hilbert action functional, defined by*

$$T := \{(\nabla, g) \in H,\ d_{(\nabla,g)} S = 0\}$$

is given by the space of solutions of the following partial differential equations

∇ *is the Levi-Civita connection for g:*

$$\nabla g = 0,$$

and

Einstein's equations:

$$\mathrm{Ric}(\nabla) - \tfrac{1}{2} R(\nabla, g) g = 0.$$

Proof. □

9.4.2 Moving frames and the Cartan formalism

The main advantage of the Cartan approach is that it allows to combine general relativity with fermionic matter particles, and that is is also the base of the present work on quantum gravity and on the covariant formulation of supersymmetric gravity theory. We use subsection 2.4 for background material on connections and subsection **??** for background on Cartan geometry. We juste recall that a Cartan geometry is given by a space whose geometry is given by pasting infinitesimally some classical, so called Klein geometric spaces, of the form G/H for $H \subset G$ two Lie groups. This can be nicely explained by the example of a sphere pasted infinitesimally on a space through tangent spaces, or, as in Wise's article [Wis06], by a Hamster ball moving on a given space:

We just recall the definition of a Cartan connection.

Definition 9.3. Let M be a variety, $H \subset G$ be two groups. A Cartan connection on M is the data of

1. a principal G-bundle Q on M,
2. a principal G-connection A on Q,
3. a section $s : M \to E$ of the associated bundle $E = Q \times_G G/H$ with fibers G/H,

such that the pullback $e = s^*A \circ ds : TM \to VE$, called the moving frame (vielbein), for $A : TE \to VE$ the associated connection, is a linear isomorphism of bundles.

The role of the section s here is to "break the G/H symmetry". It is equivalent to the choice of a principal H-subbundle $P \subset Q$.

In the cases of interest for this section, E is a linear bundle and the section s is simply the zero section that breaks the translation symmetry (action of $G/H = V$ on the sections of the vector bundle)

In general relativity, we are mostly interested in the case of a four dimensional space-time M, with the group H being the Lorentz group $O(3,1)$, G being the Poincaré group $\mathbb{R}^4 \rtimes O(3,1)$. There is a canonical GL_4 principal bundle on the variety M given by the bundle

$$P_{TM} := \mathrm{Isom}(TM, \mathbb{R}^4_M)$$

of frames in the tangent bundle TM. From a given metric g on TM, one can define a principal $O(3,1)$ bundle by considering

$$P := PO_{(TM,g)} := \mathrm{Isom}_{isom}((TM,g),(\mathbb{R}^4_M, g_{lorentz}))$$

of isometries between (TM, g) and the trivial bundle with its standard Lorentzian metric

$$g(t,x) = -c^2t^2 + \|x\|^2.$$

One then defines the underlying principal G-bundle of the Cartan connection by

$$Q = P \times_H G := PO_{(TM,g)} \times_{O(3,1)} (\mathbb{R}^4 \rtimes O(3,1)),$$

and the G/H bundle is given by

$$E = Q \times_G (G/H) = P \times_H (G/H) := PO_{(TM,g)} \underset{O(3,1)}{\times} \mathbb{R}^4.$$

The bundle E is a vector bundle equiped with a natural metric $g : E \times_M E \to E$ and the induced connection $A : TE \to VE$ being G-equivariant is also linear, so that it corresponds to a usual Koszul connection on E.

In fact, since E is linear, the natural projection $VE \to E$ is an isomorphism. This gives an isomorphism of TM with E, which is actually the identity in this particular case because E was constructed from (TM, g).

We thus find back from the cartan connection $(Q, A, s : M \to E)$ on E the data of a Koszul connection ∇ and a metric on TM, which are the fields of the Einstein-Hilbert-Palatini formulation of gravity. The curvature of the G-equivariant Ehresmann connection A is directly related to the Riemann curvature tensor through this relation.

If we suppose the first data $(Q, P, s : M \to E)$ of the Cartan connection fixed, the bundle underlying the Cartan-Palatini action is

$$\boxed{\pi : C = \mathrm{Con}_{\mathrm{Cartan}}(Q, P, s) \to M}$$

whose sections are principal G-connections $A \in \Omega^1(Q, TQ)^G$ on Q fulfilling Cartan's condition. Remark that C is a subbundle of $(T^*Q \times_Q TQ)/G$.

Remark that the G-connection A on the principal G-bundle Q is equivalent to an equivariant $\mathfrak{g}$-valued differential form

$$A : TQ \to \mathfrak{g},$$

and its restriction to $P \subset Q$ gives an H-equivariant differential form

$$A : TP \to \mathfrak{g}.$$

This is the original notion of Cartan connection form. One can decompose $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}/\mathfrak{h}$ in an H-equivariant way (the Cartan geometry is called reductive). In fact, $\mathfrak{h}$ is the special orthogonal Lie algebra and $\mathfrak{g}/\mathfrak{h} = \mathbb{R}^4$. The Cartan connection form thus can be decomposed in

$$A = \omega + e$$

for $\omega \in \Omega^1(P, \mathfrak{h})$ and $e \in \Omega^1(P, \mathfrak{g}/\mathfrak{h})$. In this particular case, e can also be seen as

$$e \in \Omega^1(M, \underline{\mathfrak{g}/\mathfrak{h}})$$

for $\underline{\mathfrak{g}/\mathfrak{h}}$ the H-bundle associated to $\mathfrak{g}/\mathfrak{h}$. This form is called the vielbein by cartan. It gives an isomorphism

$$e : TM \to \underline{\mathfrak{g}/\mathfrak{h}}.$$

In terms of these coordinates, the Palatini action

$$S_{pal} : \Gamma(M, C) \to \mathbb{R}$$

is given by

$$S_{pal}(\omega, e) = \int_M *(e \wedge e \wedge R)$$

where

- $R \in \Omega^2(P, \mathfrak{h})$ is the curvature of $\omega : TP \to \mathfrak{h}$,
- we use the isomorphism $\mathfrak{h} = \mathfrak{so}(3,1) \cong \wedge^2 \mathbb{R}^{3,1} = \wedge^2 \mathfrak{g}/\mathfrak{h}$ to identify R with an element of $\Omega^2(P, \wedge^2 \underline{\mathfrak{g}/\mathfrak{h}})$,
- the wedge product acts on both space-time indices and internal Lorentz indices and
- $* : \Omega^4_M(\wedge^4 \underline{\mathfrak{g}/\mathfrak{h}}) \to \Omega^4_M$ is the Hodge $*$-operator.

The equations of motions for this action (obtained by deriving with respect to ω and e) are

$$\begin{cases} e \wedge R &= 0 \\ d_\omega e &= 0. \end{cases}$$

The first equation is Einstein's equation. If one adds a spinorial action functional by setting

$$S_{pal}(\omega, e) = S_{spin}(\omega, e, \psi),$$

with ψ a section of some spinor bundle, one gets $d_\omega e = T$, where T is the spinor energy impulsion tensor. For antisymmetry reason, this tensor T could never be equal to the classical Einstein equation

$$\mathrm{Ric}(\nabla) - R_{scal}(\nabla, g).g.$$

Remark that, up to a simple change of variable, this Cartan action for M of dimension 3 and a Riemannian metric is equivalent to the Chern-Simons action (same equations of motion), that is a topological field theory (the equations do not depend on the metric and are diffeomorphism invariants), to be studied in the chapter on actions of mathematical physics. Its quantum partition function gives important topological invariants like for example the Jones polynomial for varieties of dimension 3.

As explained before, one of the main motivations for the Cartan formalism is to treat the case of fermionic variables in general relativity. So let $PSpin$ be a principal $\mathrm{Spin}(3,1)$-bundle on M, $H = \mathrm{Spin}(3,1)$, $V \cong \mathbb{R}^4$ be its orthogonal representation and $G = V \rtimes$

Spin(3, 1). Remark that the representation of Spin(3, 1) on V is not faithful. We consider the fiber bundle

$$Q = PSpin \underset{\mathrm{Spin}(3,1)}{\times} G$$

over M and a principal G-connection A on Q. The zero section s of the vector bundle

$$\underline{V} = Q \times_G (G/H) = PSpin \times_{\mathrm{Spin}(3,1)} V$$

gives the principal H-subbundle $PSpin \subset Q$. The pull-back of the induced connection gives an isomorphism

$$TM \to \underline{V}$$

of the tangent bundle with $\underline{V}$ which is naturally equiped with a metric (induced by the Minkowski metric on $V \cong \mathbb{R}^4$) and a connection (induced by the given connection A). We thus get a metric g and a connection ∇ on TM. Moreover, if S is the standard spin representation of Spin(3, 1), the G-connection form A induced a connection on the linear spin bundle

$$\underline{S} := PSpin \underset{\mathrm{Spin}(3,1)}{\times} S.$$

Remark that, in the construction above, one could use any principal G-bundle Q. This means that one could replace the bundle

$$\pi : C = \mathrm{Con}_{\mathrm{Cartan}}(Q, P, s) \to M$$

by the categorical bundle

$$\boxed{\pi : C = \textsc{BunCon}_{G,\mathrm{Cartan}}(M) \to M}$$

whose sections are triples (Q, A, s) of a principal G-bundle, Q, a principal G-connection A on Q and a section $s : M \to E = Q \times_G G/H$ of the associated bundle fulfilling Cartan's condition. This bundle is actually a Stack, meaning that it is not a usual space

$$C : \textsc{Alg}_{\mathbb{R}} \to \textsc{Sets}$$

but a homotopical space

$$C : \textsc{Alg}_{\mathbb{R}} \to \mathrm{S}\textsc{Sets}$$

that encodes the groupoid of isomorphisms between the above data. The differential geometry of such space can only be dealed with in the setting of derived geometry, that is given by extending the homotopical space to

$$C : \mathrm{S}\textsc{Alg}_{\mathbb{R}} \to \mathrm{S}\textsc{Sets}.$$

This shows that even in classical general relativity, the tools of homotopical geometry are mandatory to properly understand the differential geometric properties of the space of gravitational fields.

9.4.3 A black hole solution with simple applications

The most important solution of Einstein's equations for experimental matters was found by Schwarzschild in 1916. With some symmetry condition on the background variety M of general relativity, one can explicitely compute particular solutions of the general relativity equations. We refer to Landau-Lifchitz [LL66], 12.100, Besse [Bes08], chapter III section F and Sachs-Wu [SW77], example 1.4.2 for a detailed study of the Schwarzschild solution, that we call a black hole solution because it is used to formalize black holes. This solution is also the one used to explain the bending of light by a massive star: if you wait for the moon to make a solar eclypse, and look at a star that is far away behind the sun, the light of the star will be bended by the sun's gravitational field. The bending angle can be computed using Schwarzschild's solution. One can also use this model to explain Mercury's perihelion precession.

We here give a description of the Schwarzschild solution by first describing the background variety, and then giving the metric. This is not very satisfactory from the physicist viewpoint because the geometry of spacetime should be dynamically obtained from Einstein's equations and some additional physical hypothesis. The physical hypothesis here is that

1. spacetime is static, i.e., there is a non-zero timelike Killing vector field X on M (i.e., a vector field such that $L_X g = 0$),
2. spacetime is spherically symmetric, i.e., there is an action of SO(3) on M whose orbits are either points or spacelike hypersurfaces.

We now construct the most simple static and spherically symmetric spacetime. Let S^2 be the unit two sphere, equiped with its metric h induced by the embedding $S^2 \subset \mathbb{R}^3$ and its fundamental form ω. Let $\mu \in]0, +\infty[$ be given. Consider the space

$$M = (]0, \mu[\cup]\mu, +\infty[) \times S^2 \times \mathbb{R}$$

with coordinates (r, θ, t) (θ being the vector of $\mathbb{R}^3$ representing a point in S^2). One must -not- think of t as being a time coordinate in all of M. Remark that $(1 - \frac{2\mu}{r})$ is a smooth function from M to $]-\infty, 0[\cup]0, 1[$. Denote by $p : M \to S^2$ the natural projection. The Schwarzschild metric on M is defined from the metric h on S^2 by

$$g := \left(1 - \frac{2\mu}{r}\right)^{-1} d^2r + r^2 p^* h - \left(1 - \frac{2\mu}{r}\right) d^2t.$$

The Schwarzschild spacetime is the subspace N of M defined by

$$N =]\mu, +\infty[\times S^2 \times \mathbb{R},$$

with coordinates (r, θ, t). The complement of N in M is identified with

$$B = S^2 \times]0, \mu[\times \mathbb{R}.$$

If $r_0 >> 8\pi\mu$, one can interpret the open submanifold of N defined by $r > r_0$ as an excellent history of the exterior of a star of radius r_0 and active mass $8\pi\mu$. The interior of the star is not modelled by any submanifold of (M, g).

This model is used to explain Mercury's perihelion precession. The space around the sun is modelized by a Schwarzschild spacetime (N, g). Consider the earth turning around the sun, given by a geodesic

$$x : \mathbb{R} \to N$$

lying in a totally geodesic submanifold $]\mu, +\infty[\times S^1 \times \mathbb{R}$ where S^1 is the S^2 component of $\mathbb{R}$. If on supposes that the radius r is constant, meaning that the orbit is circular, we get (see [Bes08], III.G) Keppler's law: the orbit of the planet around a star of mass M takes proper time

$$T = \frac{4\pi^2}{M} R$$

where R is the distance between the planet and the center of the star. Using almost circular geodesics $x : \mathbb{R} \to N$, one gets (see [Bes08], III.H) that the period of a Mercury (of mass m) turning around the sun (of mass M) is different of the Newtonian one by a factor

$$\left(1 - \frac{6m}{M}\right)^{-1/2}.$$

This is the most famous experimental prediction of general relativity theory.

The study of null geodesics (light rays) in N is used to explain the bending of light coming from a far away star behind the sun to the earth (see [Bes08], III.J).

The subspace B of M is called the Schwarzschild black hole of active mass $8\pi\mu$. Remark that the vector field $(\partial/\partial r)_{|B}$ is timelike, so that an observer close to B in N will tend to B, and he will experiment an infinite curvature of spacetime when arriving to the boundary of B. We advise the reader not to try this.

9.4.4 The big bang solution

We refer to Landau-Lifchitz [LL66], Chapter 14, for a very nice account of this theory. This solution, called the Friedmann-Lemaitre-Robertson-Walker solution, or the standard model of cosmology, is a solution that can be obtained by assuming that spacetime M is

1. homogeneous, i.e., of the form $M = N \times I$ with I an interval in $\mathbb{R}$ and (N, h) a simply connected Riemannian space,

2. and isotropic, meaning that (N, h) is of constant curvature.

We refer to Wolf's book [Wol84] for a complete study of spaces of constant scalar curvature. If $p : M \to N$ is the projection, the lorentzian metric on M is then supposed to be of the form

$$g = -c^2 dt^2 + a(t)^2 p^* h.$$

The main examples of Riemannian spaces (N, h) like the one above are S^3 (positive curvature, defined by the equation $\|x\|^2 = a^2$ in euclidean $\mathbb{R}^4$), $\mathbb{R}^3$ (zero curvature) and $\mathbb{H}^3$ (negative curvature, defined as the positive cone in $\mathbb{R}^{3,1}$). The above metric gives a theoretical explanation of the expansion of universe, which is experimentally explained by remarking that the spectrum of light of far away stars is shifted to the red.

One can also give examples that have a Klein geometric description as homogeneous spaces under a group. This allows to add geometrically a cosmological constant in the lagrangian, by using the Cartan formalism (see [Wis06]).

- The de Sitter space, universal covering of $\mathrm{SO}(4, 1)/\mathrm{SO}(3, 1)$, is with positive cosmological constant,
- The Minkowski space $\mathbb{R}^{3,1} \rtimes \mathrm{SO}(3, 1)/\mathrm{SO}(3, 1)$ has zero cosmological constant.
- The anti de Sitter space, universal covering of $\mathrm{SO}(3, 2)/\mathrm{SO}(3, 1)$ has negative cosmological constant.

Recent fine measurements of the red shift show that the expansion of universe is accelerating. The standard model of cosmology with cosmological constant is able to explain these two experimental facts by using a FLRW spacetime with positive cosmological constant.

Chapter 10

Variational problems of experimental quantum physics

The following variational problems are not interesting if they are not used to make a quantum theory of the corresponding fields. They can't be used in the setting of classical physics. However, their extremal trajectories are often called classical fields. This is because in the very special case of electromagnetism, the classical trajectories correspond to the motion of an electromagnetic wave in space.

10.1 The Klein-Gordon lagrangian

10.1.1 The classical particle and the Klein-Gordon operator

One can see the Klein-Gordon operator as the canonical quantization of the homological symplectic reduction of the classical particle phase space. This relation is explained in Polchinski's book [Pol05], p129.

Let X be an ordinary manifold equiped with a metric g and $M = [0, 1]$. The bundle underlying the classical particle is the bundle

$$\boxed{\pi : C = X \times M \to M}$$

of spaces. Its sections correspond to maps $x : [0, 1] \to X$. The free particle action functional is given by

$$S(x) = \int_{[0,1]} \frac{1}{2}\sqrt{x^* g(Dx, Dx)}.$$

The space of initial conditions (so called phase space of the corresponding hamiltonian system) for the corresponding Euler-Lagrange equation is the cotangent bundle T^*X, with algebra of polynomial functions

$$\mathcal{O}_{T^*X} = \mathrm{Sym}^*_{\mathcal{O}_X}(\Theta_X)$$

given by the symmetric algebra on the $\mathcal{O}_X$-module $\Theta_X = \Gamma(X, TX)$ of vector fields on X.

There is a canonical two form $\omega \in \Omega^2_{T^*X}$ that gives a symplectic structure

$$\omega : \Theta_{T^*X} \otimes \Theta_{T^*X} \to \mathcal{O}_{T^*X}.$$

The Weyl algebra $\mathcal{W}_{T^*X}$ is given by the universal property

$$\mathrm{Hom}_{\mathrm{ALG}(\mathcal{O}_{T^*X})}(\mathcal{W}_{T^*X}, B) \cong \{j \in \mathrm{Hom}_{\mathrm{MOD}(\mathcal{O}_{T^*X})}(\Theta_{T^*X}, B) | j(v).j(w) - j(w).j(v) = \omega(v,w).1_B\}.$$

It is defined as a quotient of the tensor algebra $T_{\mathcal{O}_{T^*X}}(\Theta_{T^*X})$. This gives a natural filtration and the corresponding graded algebra is

$$\mathrm{gr}^F \mathcal{W}_{T^*X} \cong \mathrm{Sym}^*_{\mathcal{O}_{T^*X}}(\Theta_{T^*X}) = \mathcal{O}_{T^*(T^*X)}.$$

Remark that the algebra $\mathcal{D}_{T^*X}$ of differential operators on T^*X also has a filtration whose graded algebra is

$$\mathrm{gr}^F \mathcal{D}_{T^*X} = \mathcal{O}_{T^*(T^*X)}.$$

The algebra of differential operators $\mathcal{D}_X$ is in some sense a generalization of the Weyl algebra to non-symplectic manifolds, since one always has that its graded algebra

$$\mathrm{gr}^F \mathcal{D}_X \cong \mathrm{Sym}^*_{\mathcal{O}_X}(\Theta_X) = \mathcal{O}_{T^*X}$$

is isomorphic to the algebra of functions on T^*X. The action of $\mathcal{D}_X$ on $L^2(X)$ is thought by physicists as the canonical quantization of the algebra $\mathcal{O}_{T^*X}$ of functions on phase space.

Because of the symmetries of the lagrangian of the free particle by reparametrization, the complete explanation of the relation between the classical particle and the Klein-Gordon operator can only be done in the setting of the BRST formalism for fixing the gauge symmetry. This is explained in Polchinski's book [Pol05], page 129.

10.1.2 The Klein-Gordon lagrangian

The Klein-Gordon lagrangian is a slight generalization of the lagrangian of electromagnetism, that includes a mass for the field in play. We here use the book of Derdzinski [Der92]. Let (M, g) be a metric variety, equiped with a connection $\nabla : \Omega^1 \to \Omega^1 \otimes \Omega^1$. The bundle $\pi : C \to M$ underlying the Klein-Gordon lagrangian is one of the following linear bundles

1. $\mathbb{R}_M$: the corresponding particle is called of spin 0 (because the action of the orthogonal group $O(g)$ on it is trivial) and of parity 1. Its classical state is thus represented simply by a real valued function on M.

2. $\wedge^n T^*M$: the corresponding particle is also of spin 0 but of parity -1.

3. T^*M: the corresponding particle is of spin 1 (the standard representation of $O(g)$). The electromagnetic particles, called Photons, are of this kind.

The metric and connection on TM can be extended to C and to $C \otimes T^*M$. We will now define the Klein-Gordon action functional

$$S : \Gamma(M, C) \to \mathbb{R}$$

by its lagrangian density.

Definition 10.1. The Klein-Gordon lagrangian of mass m is given on sections $\varphi \in \Gamma(M, C)$ by

$$L(\varphi) = -\frac{1}{2}\left[g(\nabla\varphi, \nabla\varphi) + \frac{m^2c^2}{\hbar^2} g(\varphi, \varphi)\right]\omega_g.$$

The equation of motion for this lagrangian is given by

$$\left(\Box - \frac{m^2c^2}{\hbar^2}\right)\varphi = 0,$$

where the d'Alembertian $\Box$ was introduced in the subsection 9.3.2 on the covariant formulation of electromagnetism.

10.2 The Dirac lagrangian

We refer to section 3.4 for details on spinors and the Clifford algebra.

10.2.1 fermionic particles and the Dirac operator

The Dirac operator can be seen as the canonical quantization of the fermionic particle. This relation is explained in Polchinski's book [Pol98] on superstring theory.

Let X be an ordinary manifold equiped with a metric g and $M = \mathbb{R}^{0|1}$. The bundle underlying the fermionic particle is the bundle

$$\boxed{\pi : C = X \times M \to M}$$

of superspaces. Its sections correspond to maps $x : \mathbb{R}^{0|1} \to X$.

The fermionic particle action functional is given by

$$S(x) = \int_{\mathbb{R}^{0|1}} \frac{1}{2} x^* g(Dx, Dx).$$

Remark that the superspace $\underline{\Gamma}(M, C)$ of such maps is identified with the odd tangent bundle $T[1]X$ of X, whose super-functions are given by the super-algebra

$$\mathcal{O}_{T[1]X} = \wedge^* \Omega^1_X$$

of differential forms. This space $T[1]X$ plays the role of a phase (i.e. initial condition) space for the fermionic particle and superfunctions on it are classical observables for this system. If one fixes a metric g on X (for example if X is usual spacetime), one can think (as explained in section 3.4 about Clifford algebras) of $\Gamma(X, \mathrm{Cliff}(TX, g))$ as a canonical quantization of the algebra $\mathcal{O}_{T[1]X}$, and the spinor bundle S on which it acts as the state space for the canonical quantization of the fermionic particle. This means that fermions in the setting of field theory are obtained by canonically quantizing the fermionic particle.

10.2.2 Generally covariant formulation

Let (M, g, ∇) be an oriented pseudo-metric manifold with a connection and let $S \subset \mathrm{Cliff}(TM_{\mathbb{C}}, g)$ be an irreducible subrepresentation of the action of the complexified clifford algebra bundle on itself over (M, g). We suppose that S is of minimal dimension (equal to the dimension of a column of a fiber). It is then called a Dirac spinor bundle. The connection ∇ is supposed to extend to S (this is the case for the Leci-civita connection). Denote $\mathcal{S}$ the space of sections of S. The Clifford multiplication induces a natural map

$$c : \Omega^1_{M,\mathbb{C}} \otimes \mathcal{S} \to \mathcal{S}.$$

The Dirac operator is defined by

$$\not{D} = c \circ \nabla : \mathcal{S} \to \mathcal{S}.$$

Suppose that there exists a real spinor bundle S whose extension of scalars to $\mathbb{C}$ gives the one used above. The Clifford multiplication gives a real Dirac operator

$$\not{D} : \mathcal{S} \to \mathcal{S}.$$

There are natural pairings

$$\tilde{\Gamma} : S \times_M S \to TM$$

or equivalently

$$\tilde{\Gamma} : S \times_M (S \times_M T^*M) \to \mathbb{R}_M,$$

and

$$\epsilon : S \times_M S \to \mathbb{R}_M.$$

If $m \in \mathbb{R}$ is a fixed number, we define the Dirac lagrangian acting on sections of the super-bundle

$$\boxed{\pi : C = \Pi S \to M}$$

by

$$L(\psi) = \left(\frac{1}{2}\psi \not{D}\psi - \frac{1}{2}\psi m \psi\right) d^n x,$$

where the kinetic term is given by

$$\psi \not{D} \psi := \tilde{\Gamma}(\psi, \nabla\psi)$$

and the mass term is given by

$$\psi m \psi := m\epsilon(\psi, \psi).$$

Let us describe the above construction in the case of $M = \mathbb{R}^{3,1}$ and g is Minkowski's metric, following Dedzinski's book [Der92], section 1.3. Let (V, q) be the corresponding quadratic real vector space. In this case, we have

$$\mathrm{Cliff}(V, q) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{M}_{4,\mathbb{C}} \text{ and } \mathrm{Cliff}^0(V, q) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{M}_{2,\mathbb{C}},$$

the spinor group is $\mathrm{Spin}(3,1) = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathrm{SL}_{2,\mathbb{C}}$ and its real spinor representation is $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$. An element of S is called a Weyl spinor and an element of $S_{\mathbb{C}}$ is called a Dirac spinor. One has a canonical symplectic form ω on S given by

$$\omega(x, y) = x_1 y_2 - x_2 y_1.$$

Now consider the vector space

$$V' := \{f \in \mathrm{End}_{\mathbb{R}}(S),\ f(\lambda x) = \bar{\lambda} f(x),\ \omega(., f.) \text{ is hermitian}\}.$$

Remark that V' is a real vector space of dimension 4 with a lorentz form q', such that

$$fg + gf = q'(f, g).\mathrm{id}_S.$$

There is an identification between (V, q) and (V', q') and an identification

$$S \otimes \bar{S} \cong \mathrm{Res}_{\mathbb{C}/\mathbb{R}} V_{\mathbb{C}}.$$

One defined $\epsilon : S \otimes S \to \mathbb{R}$ by

$$\epsilon = \mathrm{Im}(\omega).$$

The Dirac lagrangian can then be written

$$L(\psi) = \frac{1}{2}\mathrm{Im}\,\omega(\psi, (\not{D} + m)\psi).$$

10.2.3 The Dirac lagrangian in Cartan formalism

One can also formulate this in a more gauge theoretical viewpoint by using Cartan's formalism. Let (V, q) be a quadratic space, $H = \mathrm{Spin}(V, q)$ and $G = V \rtimes \mathrm{Spin}(V, q)$. We choose a section of the exact sequence

$$0 \to V \to G \to H \to 1,$$

i.e., an embedding $H \subset G$. Let Q be a principal G-bundle equiped with a principal G-connection and $s : M \to P \times_G G/H$ be a reduction of the structure group of P to H, i.e., a sub-principal H-bundle P of Q. The hypothesis for doing Cartan geometry is that the pull-back of the connection A along $P \subset Q$ gives an isomorphism

$$e : TM \to \underline{V}$$

between the tangent bundle and the fake tangent bundle

$$\underline{V} := P \times_{\mathrm{SO}(V,q)} V.$$

If S is a spinorial representation of $\mathrm{Spin}(V, q)$, the pairings

$$\tilde{\Gamma} : S \otimes S \to V$$

and

$$\epsilon : S \otimes S \to \mathbb{R},$$

being $\mathrm{SO}(V, q)$-equivariant, induce natural pairings on the associated bundles $\underline{S}$ and $\underline{V}$. The connection A induces a connection on both of them.

In this setting, the underlying bundle of the Dirac lagrangian is the odd super-bundle

$$\boxed{\pi : C := \Pi\underline{S} \to M}$$

and the Dirac lagrangian is defined as before by

$$L(\psi) = \left(\frac{1}{2}\psi\not{D}_A\psi - \frac{1}{2}\psi m\psi\right) d^n x.$$

10.3 Classical Yang-Mills theory

We already presented the gauge theoretical version of electromagnetism in subsection 9.3.3.

Let M be a variety and G be a connected lie group. Suppose given a principal G-bundle P on M The bundle underlying Yang-Mills theory is the bundle

$$\boxed{\pi : C = \mathrm{Con}_G(P) \to M}$$

of principal G-connections on P. Recall that such a connection is given by a G-equivariant $\mathfrak{g}$-valued 1-form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

and that its curvature is the G-equivariant $\mathfrak{g}$-valued 2-form

$$F_A := dA + [A \wedge A] \in \Omega^2(P, \mathfrak{g})^G.$$

We suppose given a bi-invariant inner product $\langle ., . \rangle$ on $\mathfrak{g}$. The pure Yang-Mills action is given by the formula

$$S(A) := -\int_M \frac{1}{2} \langle F_A \wedge *F_A \rangle.$$

The principal G-connection A induces a covariant derivation

$$d_A : \Omega^*(P, \mathfrak{g}) \to \Omega^*(P, \mathfrak{g}).$$

The equations of motion are given by

$$d_A * F_A = 0.$$

The choice of a section $s : M \to P$, called a gauge fixing, gives an isomorphism

$$s^* : \Omega^1(P, \mathfrak{g})^G \to \Omega^1(M, \mathfrak{g})$$

between G-principal connections on P and Lie algebra valued differential forms on M. However, such a choice is not always possible and we will see later that the obstruction to this choice plays an important role in quantization of Yang-Mills gauge theories.

Remark that there is no reason to choose a particular principal bundle so that the bundle underlying Yang-Mills theory could be the moduli space

$$\pi : C = \mathrm{B{\small UN}Con}_G(M) \to M$$

of pairs (P, A) composed of a principal bundle P on M and a principal connection. This moduli space is not a usual space

$$C : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SETS}$$

but a simplicial, groupoid valued, space

$$C : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SSETS}.$$

This shows that the natural setting of Yang-Mills gauge theory is homotopical geometry.

10.4 Classical matter and interaction particles

One can easily combine the Klein-Gordon, Dirac and Yang-Mills action functionals. We will do this in the Cartan formalism that is practically easier to use with fermions. The generally relativistic version can be found by supposing that the Cartan connection is the Levi-Civita connection. For a very complete description of the classical standard model, one can use the book of Derdzinski [Der92].

Let (V, q) be an even dimensional quadratic space and M be a manifold of the same dimension.

Suppose given a Lie group K with Lie algebra $\mathfrak{h}$, and a bi-invariant scalar product $\langle ., . \rangle$. Let R be a principal K-bundle on M. We keep the notations of 10.2.3 for the Dirac lagrangian in Cartan formalism, so that Q is a principal $G = V \rtimes \mathrm{Spin}(V, q)$-bundle and P is a principal $\mathrm{Spin}(V, q)$ bundle.

Suppose given a representation S of $K \times \mathrm{Spin}(V, q)$ such that the two components commute. We associate to S the bundle

$$\underline{S} := (R \times P) \times_{K \times \mathrm{Spin}(V,q)} S.$$

If B is a Cartan connection on (P, Q) (that is supposed to be fixed here) and A is a principal K-connection on R, one gets a covariant derivation

$$d_{(A,B)} : \mathcal{S} \to \mathcal{S} \otimes \Omega^1_M$$

on the space $\mathcal{S}$ of sections of $\underline{S}$, that can be combined with the Clifford multiplication map

$$\mathcal{S} \otimes \Omega^1_M \to \mathcal{S}$$

to yield the covariant Dirac operator

$$\not{D}_{(A,B)} : \mathcal{S} \to \mathcal{S}.$$

In the physicists language, the Cartan connection (which is in practice the combination of the metric and the Leci-Civita connection on spacetime) is called the fixed background, the principal K-connection A on R is called the gauge field and a section ψ of $\Pi\underline{S} \to M$ is called an interacting spinor field.

The fiber bundle of Yang-Mills theory in a fixed gravitationnal background and with interaction is the bundle

$$\boxed{\pi : C = \mathrm{Con}_K(R) \times_M \Pi\underline{S} \to M}$$

whose sections are pairs (A, ψ) of a principal K-connection on the gauge bundle R and of a section ψ of the super-bundle $\Pi\underline{S}$ of fermionic interacting particles.

The Yang-Mills theory with fermionic matter lagrangian is then given by simply adding the Pure Yang-Mills and the Dirac lagrangian in the above setting

$$L(A, \psi) := -\frac{1}{2}\langle F_A \wedge *F_A\rangle + \psi \not{D}_A \psi + \psi m \psi.$$

Remark that the Dirac operator now depends on the gauge field A.

For example, in the case of electromagnetism, we can work with the representation given by complex spinors $S_{\mathbb{C}}$ equiped with the action of SU_1 by multiplication and the standard action of $\mathrm{Spin}(V, q)$. This gives the Quantum Electrodynamics lagrangian.

10.5 The standard model

We refer to the book of de Faria and de Melo for the description of the standard model representation. The Yang-Mills theory underlying the standard model of elementary particle has gauge group $K = U(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)$. For $i = 1, 2, 3$, denote $\det_i$ the determinant representations of these three groups on $\mathbb{C}$, V_i their standard representation on $\mathbb{C}^i$, $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$ the spinorial representation of $\mathrm{Spin}(3, 1)$ and V its standard representation. Let $G = K \times (V \rtimes \mathrm{Spin}(3, 1))$ be the full gauge group of gravity paired with matter. We denote $\bar{S}$ and $\bar{V}_i$ the complex conjugated representations.

The representation of G underlying the standard model is given by defining

1. The left-quark doublet representation

$$Q_L = V_1 \otimes V_2 \otimes V_3 \otimes S,$$

2. The Up-quark singlet representation

$$U_R = V_1 \otimes \det_2 \otimes \bar{V}_3 \otimes S,$$

3. The Down-quark singlet representation
$$D_R = V_1 \otimes \det_2 \otimes \bar{V}_3 \otimes S,$$
4. The left lepton doublet representation
$$L_L = V_1 \otimes V_2 \otimes \det_3 \otimes S,$$
5. The right-electron singlet representation
$$E_R = V_1 \otimes \det_2 \otimes \det_3 \otimes S,$$
6. The Higgs representation
$$H = V_1 \otimes V_2 \otimes \det_3 \otimes S.$$

and admitting the experimental fact that there are three generations of quarks and leptons, the full standard model representation for matter particles in interaction is
$$V_{sm} := (Q_L \oplus U_R \oplus D_R \oplus L_L \oplus E_R)^{\oplus 3} \oplus H.$$
The bundle underlying the standard model is thus the Yang-Mills bundle
$$\boxed{\pi : C = \mathrm{Con}_K(R) \times_M \Pi \underline{V}_{sm} \to M}$$
where $\underline{V}_{sm}$ is the bundle associated to a given principal bundle $R \times Q$ under
$$G = (\mathrm{U}(1) \times \mathrm{SU}(2) \times \mathrm{SU}(3)) \times (V \rtimes \mathrm{Spin}(3,1))$$
and to the representation V_{sm} by the associated bundle construction
$$\underline{V}_{sm} := ((R \times Q) \times V_{sm})/G.$$

10.6 The Higgs mechanism

Le mécanisme de Higgs (appelé aussi mécanisme de Englert-Brout-Higgs-Guralnik-Hagen-Kibble sur scholarpedia) est un procédé qui permet de donner une masse aux bosons de jauge des théories de Yang-Mills avec une brisure spontanée de symétrie. La méthode prend ses origines dans la théorie de la supraconductivité.

L'idée de départ de la brisure spontanée de symétrie est classique: même si une action classique S est invariante de Jauge sous un groupe de Lie G, ses extrémas peuvent être invariants sous un plus petit groupe H. Les générateurs de G qui ne sont pas dans H introduisent alors de nouvelles particules (bosons de Higgs) qui ont le bon goût, en pratique, de permettre de donner une masse aux champs d'interaction (bosons de jauge), qui a priori n'en ont pas.

Chapter 11

Variational problems of theoretical physics

We will now present, sometimes in a more sketchy fashion, some important models of theoretical/mathematical physics. We chose to only give here the superspace formulations of superfields, because of its evident mathematical elegance.

11.1 Kaluza-Klein's theory

The aim of Kaluza-Klein's theory is to combine general relativity on spacetime M and electromagnetism in only one "generally relativistic" theory on a $4+1$ dimensional space, locally given by a product $M \times S^1$.

Let M be a 4-dimensional variety equiped with a principal U(1)-bundle $p : P \to M$. Fix a $U(1)$-invariant metric g on $U(1)$, which is equivalent to a quadratic form $g_U : \mathfrak{u}(1) \times \mathfrak{u}(1) \to \mathbb{R}$ on its Lie algebra. Fix also a U(1)-invariant metric

$$g_P : TP \times TP \to \mathbb{R}_P$$

on P. The lengh of a fiber of P (isomorphic to $U(1)$) for this metric is denoted λ_P. The metric on P induces a Levi-Civita connexion on TP. This connection, being also an Ehresman connection, induces a splitting of the canonical exact sequence

$$0 \to VP \longrightarrow TP \xrightarrow{dp} p^*TM \to 0,$$

and thus a decomposition $TP = HP \oplus VP$ where $HP \cong p^*TM$ and $VP = \mathrm{Ker}(dp)$ is the space of vertical tangent vectors. Recall that a principal bundle being parallelizable,

there is a canonical trivialization of the vertical tangent bundle

$$VP \cong \mathfrak{u}(1)_P := \mathfrak{u}(1) \times P.$$

We suppose that

1. the restriction of the metric g_P on P to VP induces the given fixed metric on $U(1)$.

2. the restriction of the metric g_P to $HP \cong p^*TM$ is induced by pull-back of a lorentzian metric g on M.

The Kaluza-Klein theory has underlying fiber bundle the bundle

$$\boxed{\pi : C = \mathrm{Sym}^2_{KK}(T^*P) \to M}$$

of such metrics on P. The Kaluza-Klein action functional

$$S : \Gamma(M, C) \to \mathbb{R}$$

is given by

$$S(g_P) := \int_P R_{scal}(g_P)\mathrm{vol}_{g_P},$$

where R_{scal} is the scalar curvature.

The scalar curvature then decomposes in coordinates in

$$R_{scal}(g_P) = p^*(R_{scal}(g) - \frac{\Lambda^2}{2}|F|^2)$$

and if on integrates on the fibers, one gets the action

$$S(g_P) := \Lambda \int_M (R(g) - \frac{1}{\Lambda^2}|F|^2)\mathrm{vol}(g).$$

If g is fixed and the action varies with respect to the primitive A of F, one gets Maxwell's theory. If one fixes F and vary the action with respect to g, one gets Einstein's equation for the Einstein-Hilbert action with an electromagnetic term.

11.2 Bosonic gauge theory

We refer to Freed's lectures on supersymmetry [Fre99]. Let M be an oriented variety, G be a Lie group with Lie algebra $\mathfrak{g}$, and suppose given a G-invariant product $\langle .,. \rangle$ on $\mathfrak{g}$. Let (X, g) be a metric manifold on which G acts by isometries. Let $V : X \to \mathbb{R}$ be a G-invariant -so called- potential function. Suppose given a principal G-bundle P. The bundle underlying bosonic gauge theory is the bundle

$$\boxed{\pi : C = \mathrm{Con}_G(P) \times P \times_G X \to M}$$

whose sections are pairs (A, φ) of a principal G-connection on P and of a section φ of the associated bundle $P \times_G X$.

The action functional $S = \Gamma(M, C) \to \mathbb{R}$ is given by

$$S(A, \varphi) = \int_M -\frac{1}{2}|F_A|^2 + \frac{1}{2}|d_A\varphi|^2 - \varphi^* V$$

where d_A is the covariant derivative associated to A.

11.3 Poisson sigma model

We describe the Poisson sigma model because it inspired Kontsevich's construction in [Kon03] of a deformation quantization formula for general Poisson manifolds (see the article by Cattaneo and Felder [CF01] for more details on the poisson sigma model underlying Kontsevich's ideas on deformation quantization).

Let (X, θ) be a poisson manifold, i.e., a variety equiped with a bivector $\theta \in \Gamma(X, \wedge^2 TX)$. Let M be a smooth surface. The bundle underlying this variational problem is the bundle

$$\boxed{\pi : C = (X \times M) \times_X (T^*X \times M) \to M}$$

whose sections are pairs (x, η) composed of a map $x : M \to X$ and of a differential form $\eta \in \Omega^1(M, x^*T^*X)$ with values in the pull-back by x of the cotangent bundle on X.

The action functional is given by

$$S(x, \eta) = \int_M \langle \eta \wedge Dx \rangle + \frac{1}{2}\langle \eta \wedge x^*\theta(\eta) \rangle$$

where

- $Dx : TM \to x^*TX$ is the differential of x,

- θ is viewed as a morphism $\theta : T^*X \to TX$ so that $x^*\theta$ is a map $x^*\theta : x^*T^*X \to x^*TX$, and
- $\langle , \rangle : T^*X \times TX \to R_X$ is the standard duality pairing.

11.4 Chern-Simons field theory

Let M be a three dimensional manifold and G be a lie group. Suppose given an invariant pairing $\langle , \rangle$ on the Lie algebra $\mathfrak{g}$ of G and a principal G-bundle P. The bundle underlying this variational problem is the bundle

$$\boxed{\pi : C = \mathrm{Con}_G(P) \to M}$$

of principal G-connections on P. Such a connection is given by a G-invariant differential form

$$A \in \Omega^1(P, \mathfrak{g})^G$$

so that one has an identification

$$\mathrm{Con}_G(P) = (T^*P \times \mathfrak{g})/G.$$

The Chern-Simons action functional was introduced by Witten in [Wit89] to study topological invariants of three manifold from a quantum field theoretical viewpoint. One says that it is a topological action functional because it does not depend on a metric and it is diffeomorphism invariant. It is given by

$$S(A) = \int_M \mathrm{Tr}(A \wedge dA + \frac{1}{3} A \wedge A \wedge A).$$

One can show that this action is classically equivalent (meaning has same equations of motions) to the Cartan formalism version of euclidean general relativity in dimension 3. This shows that general relativity in dimension 3 is a purely topological theory, which is very different of the 4-dimensional case.

11.5 Supersymmetric particle

We refer to Freed lectures on supersymmetry [Fre99] for the two classical presentations (in component and superspacy) of the supersymmetric particle.

Let (X, g) be the Minkowski space. The super bundle underlying the supersymmetric particle is the bundle

$$\boxed{\pi : C = X \times \mathbb{R}^{1|1} \to \mathbb{R}^{1|1}}$$

whose sections are morphisms

$$\varphi : \mathbb{R}^{1|1} \to X.$$

We introduce on $\mathbb{R}^{1|1}$ the three vector fields

$$\partial_t,\ D = \partial_\theta - \theta\partial_t,\ \text{and}\ \tau_Q = \partial_\theta + \theta\partial_t.$$

The action functional for the supersymmetric particle is the integral

$$S(\varphi) = \int_{M^{1|1}} -\frac{1}{2}\langle D\varphi, \partial_t\varphi\rangle$$

with respect to the standard super-measure $|dt|d\theta$. The action is invariant with respect to ∂_t, D and τ_Q, so that it is called supersymmetric.

11.6 Superfields

We refer to [DF99], supersolutions.

Let M be usual Minkowski space, C a positive cone of time-like vectors in V, S a real representation of $\mathrm{Spin}(V)$, and

$$\Gamma : S^* \otimes S^* \to V$$

a pairing which is positive definite in the sense that $\Gamma(s^*, s^*) \in \bar{C}$ for $s^* \in S^*$ with $\Gamma(s^*, s^*) = 0$ only for $s^* = 0$. Associated to this, there is a morphism

$$\tilde{\Gamma} : S \otimes S \to V.$$

Definition 11.1. The super translation group is the supergroup with underlying space $V^s = V \times \Pi S^*$ and multiplication given by

$$(v_1, s_1^*)(v_2, s_2^*) = (v_1 + v_2 - \Gamma(s_1^*, s_2^*), s_1^* + s_2^*).$$

Its points with value in a super-commutative ring R is the even part of

$$(V + \Pi S^*) \otimes R.$$

The super Minkowski space is the underlying manifold of the super translation group. The super Poincaré group is the semi-direct product

$$P^s := V^s \rtimes \mathrm{Spin}(V).$$

The super Minkowski space is usually called the "N superspace" if S is the sum of N irreducible real spin representations of $\mathrm{Spin}(V)$.

The bundle underlying superfield gauge theory is the bundle

$$\boxed{\pi : C = \mathrm{Con}_K(R) \times X \to M^s}$$

with X an ordinary manifold, K a super-group and R a super-K-principal bundle over M^s.

We only give here two examples of lagrangian action functional on such a super-Minkowski space $M^{4|4}$ that is based on usual Minkowski space $\mathbb{R}^{3,1}$ with 4 super-symmetries (be careful, this notation could be misleading: it has nothing to do with super-affine space $\mathbb{A}^{4|4}$; the first 4 is for spacetime dimension 4 and the second 4 is for 4 supersymmetries, i.e., one has 4 copies of the standard real spin representation S of $\mathrm{Spin}(3,1)$).

First consider the case of Chiral superfield φ, that is a section of

$$\boxed{\pi : C = \mathbb{C} \times M^{4|4} \to M^{4|4},}$$

i.e., a supermap $\varphi : M^{4|4} \to \mathbb{C}$. We use the standard volume form on $S = \mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathbb{C}^2$ and on M^4 to get a supertranslation invariant volume form on $M^{4|4}$. The action functional is given by

$$S(\varphi) = \int_{M^{4|4}} \frac{1}{4}\bar{\varphi}\varphi.$$

Now let X be a K"ahler manifold, i.e.,

1. a holomorphic manifold X,

2. equiped with a symplectic form $\omega \in \Omega^2_X$,

such that

$$g(v, w) = \omega(v, Jw)$$

is a Riemannian metric.

We suppose given locally on X a K"ahler potential, i.e., a real valued function K on X such that the Kähler form is

$$\omega = i\partial\bar{\partial}K.$$

The lagrangian action functional applies to sections of the bundle

$$\boxed{\pi : C = X \times M^{4|4} \to M^{4|4},}$$

and is given by

$$S(\varphi) = \int_{M^{4|4}} \frac{1}{2}K(\bar{\varphi}, \varphi).$$

11.7 Bosonic strings

We carefully inform the reader that the action given here is not supposed to be quantized through a functional integral. There is at the time being no proper definition of an hypothetic lagrangian field theory (called M-theory) whose quantum field theoretic perturbative expansions would give the perturbative series defined *ad-hoc* by string theoretists to define quantum string propagation amplitudes.

We refer to Polchinski's book [Pol05] for a complete physical introduction to bosonic string theory. Let M be a two dimensional manifold and let

$$\mathrm{Sym}^2_{min}(T^*M) \to M$$

be the bundle of minkowski metrics on M Let (P, g_P) be a given space with a fixed metric. For example, this could be a Kaluza-Klein $U(1)$-fiber bundle (P, g_P) over usual spacetime. The fiber bundle underlying bosonic string theory on a fixed string topology M is the bundle

$$\boxed{\pi : C = P \times \mathrm{Sym}^2_{min}(T^*M) \to M}$$

whose sections are pairs (x, g) composed of a map $x : M \to P$ and a lorentzian metric on M.

The Polyakov action functional $S : \Gamma(M, C) \to \mathbb{R}$ for bosonic strings is given by the formula

$$S(x, g) = \int_M \mathrm{Tr}(g^{-1} \circ x^* g_P) d\mu_g,$$

where

- $g^{-1} : T^*M \to TM$ is the inverse of the metric map g,
- $x^* g_P : TM \to T^*M$ is the inverse image of $g_P : TP \to T^*P$,
- and the trace applies to $g^{-1} \circ x^* g_P \in \mathrm{End}(TM)$.

If one fixes x and varies g, the extremal points for this action are given by $g = x^* g_P$, but in a quantization process, it is important to allow "quantum" fluctuations of the metric g around these extremal points.

One can also add a potential function $V : P \to \mathbb{R}$ to the lagrangian to get more general bosonic string action functionals.

Remark that the euclidean version of this theory, with the bundle $\mathrm{Sym}^2_+(T^*M) \to M$ is directly related to the moduli space of curves of a given genus. Indeed, it is the moduli

space of curves equiped with a quadratic differential on them. If the curve is non-compact, the moduli spaces in play will be the moduli spaces

$$T^*\mathcal{M}_{g,n} \to \mathcal{M}_{g,n}$$

of quadratic differentials on curves of genus g with n marked points.

11.8 Superstrings

The same remark as in section 11.7 applies to this section. There are two way to define classical superstrings. We refer to the basic book [McM09] for a basic introduction to superstrings and to [Pol98] and [GSW88] for more specialized presentations.

In the Ramond-Neveu-Schwarz formalism, superstrings are a special case of superfields on a lorentzian curve with N supersymmetries $M^{2|N}$. For $N = 1$, for example, one considers the super-space extension $M^{2|1}$ of the surface M that appear in the description of the Bosonic string in section 11.7 with two fermionic variables that correspond to the real two dimensional real spinorial representation of $\mathrm{Spin}(1,1)$. The strings are then given by superfields with values in say a complex target manifold X with given local Kähler potential K as in section 11.6

$$x : M^{2|N} \to X.$$

So the fields of the theory are superfields on the supersymmetric extension of a given surface, meaning that the underlying bundle is

$$\boxed{\pi : C = X \times M^{2|N} \to M^{2|N}.}$$

In the Green-Schwarz formalism, superstrings are given by maps

$$x : M^2 \to X_s^N$$

from a surface to an N-super-Poincaré extension X_s^N of a given Lorentzian manifold X. In this case, the fields are classical fields on a surface but with values in a super-space, meaning that the underlying bundle is

$$\boxed{\pi : C = X_s^N \times M^2 \to M^2.}$$

In the two cases, the space of fields $\Gamma(M, C)$ is a super-space that can be quantized by a functional integral approach.

11.9 Supergravity

We refer to Lott's paper [Lot90] and to Egeileh's thesis [Ege07] for the superspace formulation of supergravity.

We only define the underlying bundle. We need the definition of a Cartan superconnection on $M^{4|4}$ for the pair $(H, G) = (\mathrm{Spin}(3,1), P^s)$ composed of usual spinor group and super-Poincaré group. It is given by

1. a principal G-bundle Q on $M^{4|4}$,
2. a principal H-sub-bundle $P \subset Q$ of Q,
3. a section $s : M \to Q \times_G G/H$ of the associated bundle,
4. a principal G-connection $A \in \Omega^1(Q, \mathfrak{g})$ on Q,

such that the pull-back s^*A induces an isomorphism

$$e_A : TM^{4|4} \cong Q \times_G V^s$$

called the Cartan coframe, are also sometimes the supervielbein. One has to suppose moreover that the torsion of the given Cartan connection fulfil some first order constraint, for which we refer the reader to Lott's paper, loc. cit.

The bundle underlying super-gravity theory is the bundle

$$\boxed{\pi : C = \mathrm{Con}_{\mathrm{Cartan},tc}(Q, P, s) \to M^{4|4} = M}$$

whose sections are Cartan connections with torsion constraints.

The super-gravity action functional is then essentially the same as the Cartan version of Einstein-Hilbert-Palatini action given in subsection 9.4.2.

Part III

Quantum trajectories and fields

Chapter 12

Quantum mechanics

For historical reasons, we present the first quantum theories as they were thought by Heisenberg, Schrödinger and von Neumann among others. This quantization methods are essentially restricted to systems without interactions, and the most mathematically developped theory based on them is the deformation quantization program that says that for any Poisson manifold (M, π), the Poisson bracket on $\mathcal{O}_M$ can be extended to an associative multiplication operation $*$ on $\mathcal{O}_M[[\hbar]]$ such that one has

$$f * g = fg + \hbar\{f, g\} + O(\hbar^2).$$

This general result, due to Kontsevich [Kon03], can be used in perturbative algebraic quantum field theory (see K. Keller's thesis [Kel07] for a survey and references).

The author's viewpoint is that the covariant methods, using functional integral, have better functorial properties and generalize better to systems with a variable number of interacting particles, with fermionic variables. A hike through the hamiltonian methods is however useful to better understand the physicists intuitions with quantum fields. The lazzy reader can pass directly to chapter 13.

We will use systematically spectral theory, refering to section 5.3 for some basic background.

12.1 Principles in von Neumann's approach

We recall the von Neumann presentation of Heisenberg/Schrödinger approach to quantum mechanics, that one can find in the old but very pedagogical reference [vN96].

Definition 12.1. A quantum mechanical (hamiltonian) system is a tuple

$$(\mathcal{H}, \mathcal{A}, I, \mathrm{Hist}, H : \mathcal{H} \to \mathcal{H})$$

made of the following data:

1. an interval I of $\mathbb{R}$ that represents the physical time parameter,

2. a Hilbert space $\mathcal{H}$ that plays the role of the phase space T^*X of classical hamiltonian systems, and whose norm 1 vectors are the physical states of the system.

3. A C*-algebra $\mathcal{A} \subset \mathcal{L}(H)$ of (not necessarily continuous) operators on $\mathcal{H}$ called the observables of the system.

4. two sets of applications $\mathrm{Hist}_{\mathcal{H}} \subset \mathrm{Hom}(I, \mathcal{H})$ and $\mathrm{Hist}_{\mathcal{A}} \subset \mathrm{Hom}(I, \mathcal{A})$ respectively called the space of possible evolutions for the states and observables of the system (analogous to the space of histories of hamiltonian mechanics).

5. An autoadjoint operator $H : \mathcal{H} \to \mathcal{H}$ called the Hamiltonian of the system.

If A is an observable, the elements of its spectrum $\lambda \in \mathrm{Sp}(A) \subset \mathbb{R}$ are the possible measures for the value of A, i.e., the real numbers that one can obtain by measuring A with a machine. The spectrum $\mathrm{Sp}(A)$ is thus the spectrum of possible measures for the value of A.

From a quantum mechanical system, one defines the space $\mathcal{T} \subset \mathrm{Hist}_{\mathcal{H}}$ of possible trajectories for the states of the system as the solutions of Schrödinger's equation

$$\varphi_{t_0} = \varphi \text{ and } \frac{\hbar}{2i\pi}\frac{\partial \varphi_t}{\partial t} = -H\varphi_t.$$

This translates more generally on the evolution of observables $A_. \in \mathrm{Hist}_{\mathcal{A}}$ of the system by

$$i\hbar\frac{\partial A}{\partial t} = [H, A],$$

which corresponds to the naïve quantization of the equation

$$\frac{\partial a}{\partial t} = \{H, a\}$$

of Hamiltonian mechanics obtained by replacing the Poisson bracket by commutators (see section 12.2).

One can consider that quantum mechanics is based on the following physical principles, that give the link with experiment:

Principle 4. 1. If $A \in \mathcal{A} \subset \mathcal{L}(\mathcal{H})$ is an observable (autoadjoint operator) of the physical system and $[b, c] \subset \mathbb{R}$ is an interval, the probability that one finds a number in the interval $[b, c]$ by measuring with a device the value of $A \in \mathcal{B}(\mathcal{H})$ on the system in the state $\varphi \in \mathcal{H}$ is

$$\mathbb{P}(A, [b, c]) := \sqrt{\langle \varphi, \mathbb{1}_{[b,c]}(A)\varphi \rangle}.$$

2. Two observables can be measured simultaneously if and only if they commute.

3. A measure of the observable A on a state φ changes the state φ in the state $A\varphi$.

4. A projector P is an observable that gives a true or false property for a state of the system.

If R and S are two observables that don't commute, on associates to them the operator P of projection on the kernel M of the commutator $[R, S]$. The projector then corresponds to the property that the two observables are measurable simultaneously. If one supposes that $M = \{0\}$, this means that R and S cannot be measured simultaneously. One can restrict to operators as the P and Q of quantum mechanics that fulfil the relation

$$[P, Q] = \frac{\hbar}{2i\pi}\mathrm{Id}.$$

One can then evaluate the dispersion of the measures of P and Q on a state φ by setting for $R = P, Q$,

$$d(R, \varphi) = \|R\varphi - \langle R\varphi, \varphi\rangle\varphi\|.$$

This shows the difference between the state $R\varphi$ obtained after the measure R and the state $\langle R\varphi, \varphi\rangle\varphi$ essentially equivalent to φ (the physical states being unitary). One then gets the Heisenberg uncertainty principle:

$$d(P, \varphi).d(Q, \varphi) \geq \frac{\hbar}{4\pi}$$

that means that the less the value of P on φ changes the state φ, the more the measure of the value of Q on φ change the state φ of the given system. This can also be translated by saying that the more your measure P with precision, the less you can measure Q with precision.

12.2 Canonical quantization of Heisenberg/Schrödinger

There is a strong analogy between quantum mechanical systems and classical hamiltonian systems, given by the following array:

	CLASSICAL	QUANTUM
System	$(P, \pi, I, \mathrm{Hist}, H)$	$(\mathcal{H}, \mathcal{A}, I, \mathrm{Hist}, H)$
States	$x \in P$	$x \in \mathcal{H},\ \|x\| = 1$
Observables	$a \in C^\infty(P)$	$a \in \mathcal{A}$
Bracket	$\{a, b\}$	$[a, b]$
Hamiltonian	$H \colon X \to \mathbb{R}$	$H : \mathcal{H} \to \mathcal{H}$ autoadjoint $\pi : \mathcal{A} \to \mathcal{L}(\mathcal{H})$
Time evolution	solution of the Hamiltonian vector field $\xi_H = \{H, .\}$	$\sigma : \mathbb{R} \to \mathrm{Aut}(\mathcal{A})$ $e^{itH}\pi(a)e^{-itH} = \pi(\sigma_t(a))$

If a classical hamiltonian system $(P = T^*X, \pi, I, \mathrm{Hist}, H)$ is given, one can define a quantum mechanical system $(\mathcal{H}, \mathcal{A}, I, \mathrm{Hist}, H)$ by the following method:

1. one sets $\mathcal{H} = L^2(X)$,
2. to every function $q \in \mathcal{C}^\infty(X)$ one associates the operator/observable $Q = m_q$ of multiplication by q in $\mathcal{H}$. The coordinate functions on $X = \mathbb{R}^n$ are denoted q_i.
3. one associates to a vector field $v \in \theta_X$ the operator $P_v(.) = \frac{\hbar}{2i\pi} d_V(.)$ of Lie derivative along the vector field v given by
$$d_v(f) = Df \circ v : X \to \mathbb{R}.$$
The derivations along coordinates are denoted P_i.
4. More generally, one would like to have a Lie algebra morphism
$$\varphi : (\mathcal{A}, \{.,.\}) \to (\mathrm{End}(\mathcal{H}), [.,.])$$
(where $\mathcal{A} \subset \mathcal{C}^\infty(T^*X)$ is the algebra of interesting observables that is supposed in particular to contain
   - the classical hamiltonian H,
   - the coordinate functions q_i, and
   - the derivatives with respect to the q_i, that one can also see as the applications $\dot{q}_i(x, \omega) = \omega(\dot{q}_{ix})$.

Such a Lie algebra morphism, called a quantization, would allow to quantize all the useful algebras on $P = T^*X$ and give back the above explicitely given constructions as particular cases, to compute the bracket between observable quantities by setting

$$[A, B] = \frac{\hbar}{i}\{a, b\}.$$

This hard problem is partially solved by the theory called deformation quantization (see [Kon03] for the main result).

5. In particular, one gets the quantum Hamiltonian by replaçing, when possible, the variables q_i and p_i on T^*X by the corresponding operators Q_i and P_i.

The probability amplitude that a state $f \in \mathcal{H}$ of the system is situated in a Borelian B of the space X is given by the value

$$\mathbb{A}_B(f) := \int_B |f|^2 d\mu.$$

One can thus also think of the quantum states in $\mathcal{H}$ as wave functions for some given particles.

12.3 Algebraic canonical quantization

We now turn on to the algebraic approach to canonical quantization, that was invented in the fermionic case by Dirac [Dir82], and for which we also refer to section 3.4 on Clifford algebras. This section can be seen as an algebraic explanation of the fact that "first" canonical quantization of particles give the free field equations of field theory, that we will use latter.

Finally, we remark that the quantization of the algebra of polynomial functions on the cotangent bundle $P = T^*X$ of a Riemannian variety (X, g), that is the phase space for a particle $x : [0, 1] \to X$ moving in X, can be given by the algebra of differential operators $\mathcal{D}_X$. This algebra acts on $L^2(X)$ and has a filtration F whose graded algebra is the algebra

$$\mathrm{gr}^F\mathcal{D}_X = \mathcal{O}_P$$

of functions on phase space space. Moreover, the bracket on $\mathcal{D}_X$ induces the Poisson bracket on $\mathcal{O}_P$. The quantum Hamiltonian can be thus seen as a differential operator $H \in \mathcal{D}_X$ acting on the state space $L^2(X)$.

In the case of a fermionic particle $x : \mathbb{R}^{0|1} \to X$, the phase space is the supervariety $P = T^*[1]X$ with coordinate algebra $\mathcal{O}_P = \wedge^*\Theta_X$. The Clifford algebra $\mathrm{Cliff}(\Theta_X, g)$ gives

a nice canonical quantization of $\mathcal{O}_P$ since it also has a filtration whose graded algebra is the algebra

$$\mathrm{gr}^F \mathrm{Cliff}(TX, g) = \wedge^* \Theta_X = \mathcal{O}_P$$

of functions on phase space. Moreover, the bracket on $\mathrm{Cliff}(\Theta_X, g)$ induces the so called Schouten-Nijenhuis super poisson bracket on $\mathcal{O}_P$. The quantum Hamiltonian is then an element $H \in \mathrm{Cliff}(\Theta_X, g)$ and it acts naturally on the complex spinor representation S of the Clifford algebra, that is the state space of the quantum fermionic particle.

12.4 Weyl quantization and pseudo-differential calculus

Let X be a variety equiped with a metric $g : TX \times TX \to \mathbb{R}_X$. One can think of (X, g) as a solution of Einstein's equations with g of Minkowski's signature. To perform the Weyl quantization, one needs a Fourier transform and a linear structure on spacetime. The easiest way to replace a curved spacetime by a flat one is to work in the tangent space of spacetime at a given point, which is the best linear approximation to spacetime. This remark is very important, since it shows that from a mathematical viewpoint,

it does not really makes sense to combine gravity and quantum mechanics: one of them lives on spacetime, and the other lives in the fiber of the tangent bundle of spacetime at a given point.

We denote (x, q) the local coordinates on TX and (x, p) the local coordinates on T^*X. We denote

$$\langle ., . \rangle : T^*X \times_X TX \to \mathbb{R}_X$$

the natural pairing given by

$$\langle (x, p), (x, q) \rangle = (x, p(q)).$$

One has a fiberwise Fourier transform

$$\mathcal{F} : \mathcal{S}(T^*X) \to \mathcal{S}(TX)$$

on the Schwartz spaces (functions with compact support on the base X and rapid decay along the fibers equiped with their linear Lebesgue measure) given by $f \in \mathcal{S}(T^*X)$ by

$$\mathcal{F}(f)(x, q) = \int_{T_x^*X} e^{i\langle p,q \rangle} f(x, p) dp.$$

If one gives a symbol $H(x,p,q) \in \mathcal{S}(T^*X \times_X TX)$ (for quantification, this will typically be the Hamiltonian, that depends only on p et q and not on x) and a function $f(x,q)$ of $\mathcal{S}(TX)$, one can associate to it the product $H.(f \circ \pi) \in \mathcal{S}(T^*X \times_X TX)$ where $\pi : T^*X \times_X TX \to TX$ is the natural projection.

Definition 12.2. We denote $L^2_{\infty,c}(TX)$ the space of functions on TX that are smooth with compact support on X and square integrable in the fibers of $TX \to X$. The operator $\hat{H}$ on $L^2_{\infty,c}(TX)$ associated to H is then given by the formula

$$(\hat{H}.f)(x,q) := \int_{T_xX} \int_{T_x^*X} e^{i\langle p, q-q_0\rangle} H(x,p,q).f(x,q_0) d\mu_g(q_0) d\mu_{g^*}(p).$$

This map

$$\begin{array}{ccc} \mathcal{S}(T^*X \times_X TX) & \to & \mathcal{L}(L^2_{\infty,c}(TX)) \\ H & \mapsto & \hat{H} \end{array}$$

is called Weyl quantization or infinitesimal pseudo-differential calculus.

The (non-canonical) relation with usual pseudo-differential calculus is the following: if $U \subset \mathbb{R}^n$ is an open and $h : T^*U \cong U \times (\mathbb{R}^n)^* \to \mathbb{R}$ is a symbol $h(x,p)$ on U, one associates to it the function

$$H : T^*U \times_U TU \cong U \times (\mathbb{R}^n)^* \times \mathbb{R}^n \to \mathbb{R}$$

by replacing x by q in $h(x,p)$, i.e., by the formula

$$H(x,p,q) := h(p,q)$$

obtained geometrically by identifying U with a subspace of the fiber of its tangent bundle $\mathbb{R}^n$, by the implicit function theorem (this space is a plane in $\mathbb{R}^n \times \mathbb{R}^n$). One gets an operator $\hat{H}$ on $L^2_{\infty,c}(TU)$, that is constant on U and thus gives an operator on the fibers, that are $L^2(\mathbb{R}^n)$.

In practice, one often supposes $X = \mathbb{R}^n$, which gives $TX = \mathbb{R}^n \times \mathbb{R}^n$ and one can then consider the functions on TX that depend only on q in the coordinates (x,q) on TX. The choice of this decomposition is clearly not canonical, but Weyl quantization is often described as the map

$$\begin{array}{ccc} \mathcal{S}(T^*X) & \to & \mathcal{L}(L^2(X)) \\ H & \mapsto & \hat{H}. \end{array}$$

With a convenient normalization of the Fourier transform, the Weyl quantization gives back the Heisenberg quantization described in section 12.2. Indeed, a linear coordinate is

a function $q_n : TX \to \mathbb{R}$ (coordinate in the fiber of TX) that one can compose with the projection $\pi_T : T^*X \times_X TX \to TX$, which gives by quantization an operator

$$Q_n := \widehat{q_n \circ \pi_T}$$

in $\mathcal{C}^\infty(X, \mathcal{L}(L^2_{\infty,c}(TX)))$ that is simply the multiplication by the coordinate q_n in the fibers.

Given a vector field $\vec{v}_n \in \Theta_X = \Gamma(X, TX)$ (derivation in the x_n coordinate direction on X, and not on TX, this time: it is at this point that the identification between coordinates on X and on TX, $x_n = q_n$ used by physicists, is non-canonical), one can see this as the application $p_{n,h} : T^*X \to \mathbb{C}$, given by $p_{n,h}(x,p) = \frac{h}{i}\langle p, \vec{v}_n\rangle$, and composable with the projection $\pi_{T^*} : T^*X \times_X TX \to T^*X$, that gives by quantization an operator

$$P_n := \widehat{p_{n,h} \circ \pi_{T^*}}$$

in $\mathcal{C}^\infty(X, \mathcal{L}(\mathcal{S}(TX)))$ that is nothing else than $P_n = \frac{1}{2i\pi}\frac{\partial}{\partial q_n}$.

12.5 Quantization of the Harmonic oscillator

The quantum harmonic oscillator is at the base of the free quantum field theory. Its resolution by the ladder method (creation and annihilation operators) is due to Dirac [Dir82]. We also refer to Wikipedia for this section.

The classical harmonic oscillator on $X = \mathbb{R}$ is given by the hamiltonian syste whose trajectories are maps from $\mathbb{R}$ into the Poisson variety $P = T^*X \cong \mathbb{R}\times\mathbb{R}$ (with coordinates (q,p) called position and impulsion) and whose Hamiltonian is the function $H : P \to \mathbb{R}$ given by

$$H(q,p) = \frac{1}{2m}(p^2 + m^2\omega^2 q^2).$$

Recall that the classical solution of the Hamilton equations are then given by

$$\left\{\begin{array}{rcl} \frac{\partial q}{\partial t} & = & \frac{\partial H}{\partial p} = \frac{p}{m} \\ \frac{\partial p}{\partial t} & = & -\frac{\partial H}{\partial q} = m\omega^2 q, \end{array}\right.$$

which gives the newton equations of motion

$$\frac{\partial^2 q}{\partial t^2} = \omega^2 q.$$

Canonical quantization of this system is obtained by working with the Hilbert space $\mathcal{H} = L^2(X)$ and replacing the position and impulsion by the operators $Q = m_q$ of multiplication by the function $q : X \to \mathbb{R}$ and $P = -i\hbar\partial_q$ of derivation with respect to q.

One then compute the spectrum of the Hamiltonian (whose elements are possible energy level for the system)

$$H(Q,P) = \frac{1}{2m}(P^2 + m^2\omega^2 Q^2),$$

which also means that one solves the eigenvalue equation

$$H\psi = E\psi.$$

To find the solutions of this eigenvalue equation, one uses, following Dirac's ladder method, two auxiliary operators

$$\begin{aligned} a &= \sqrt{\frac{m\omega}{2\hbar}}\left(Q + \frac{i}{m\omega}P\right) \\ a^\dagger &= \sqrt{\frac{m\omega}{2\hbar}}\left(Q - \frac{i}{m\omega}P\right) \end{aligned}$$

This gives the relations

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\left(a^\dagger + a\right) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}}\left(a^\dagger - a\right) \end{aligned}$$

Theorem 39. *One can solve the eigenvalue equation for the operator H by using the operators a and $a^\dagger$. More precisely,*

1. *one has $H = \hbar\omega\left(a^\dagger a + 1/2\right)$.*
2. *The spectrum of $a^\dagger a$ is the set $\mathbb{N}$ of natural numbers and the spectrum of H is $\hbar\omega(\mathbb{N} + 1/2)$.*
3. *If ψ_0 is nonzero and such that $a\psi_0 = 0$, we denote $\psi_n = (a^\dagger)^n\psi_0$ and $\psi_0 = |\emptyset\rangle$ is called the empty state. One then has*

$$H\psi_n = \hbar\omega(n + 1/2)\psi_n.$$

4. One has the commutation relations

$$\begin{aligned}[a, a^\dagger] &= 1 \\ [a^\dagger, H] &= \hbar\omega a^\dagger \\ [a, H] &= -\hbar\omega a\end{aligned}$$

Remark that the empty state ψ_0 has as energy (eigenvalue) $\frac{\hbar\omega}{2}$ and is given by the function (solution of the differential equation in q of first order $a\psi_0 = 0$)

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega}{2\hbar}x^2}.$$

12.6 Canonical quantization of the free scalar field

We refer to the Folland's excellent book [Fol08] for a very complete treatment of this issue.

The free scalair field of mass m is a function on Minkowski space $X = \mathbb{R}^{3,1}$, that is solution of the Klein-Gordon equation

$$(\Box + \frac{m^2c^2}{\hbar^2})\varphi = (\frac{1}{c^2}\partial_t^2 - \partial_x^2 + \frac{m^2c^2}{\hbar^2})\varphi = 0.$$

The partial Fourier transform on the variable x of $\varphi(t,x)$, denoted $(\mathcal{F}_x\varphi)(t,\xi)$ fulfils, at each point $\xi \in \mathbb{R}^3$) the differential equation of the classical Harmonic oscillator

$$(\frac{1}{c^2}\partial_t^2 + \xi^2 + \frac{m^2c^2}{\hbar^2})(\mathcal{F}_x\varphi)(t,\xi) = 0$$

whose classical hamiltonian is the function $h(q,p) = p^2 + \omega^2q^2$ with, for ξ fixed, $\omega^2 = \xi^2 + \frac{m^2c^4}{\hbar^2}$, $q = (\mathcal{F}_x\varphi)(t,\xi)$ and $p = \frac{\partial q}{\partial t}$.

The quantification of the free scalair field can be essentially done by replacing the classical harmonic oscillator by a quantum harmonic oscillator with Hamiltonian $H_\xi(Q,P) = P^2 + \omega^2Q^2$ acting on $L^2(\mathbb{R}^3)$. We remark that this hamiltonian depends on $\xi \in \mathbb{R}^3$ that is the Fourier dual variable to the position variable x in space.

There exists a nice way to formalize the above construction more canonically. One works in relativistic units, i.e., $c = \hbar = 1$. Let φ be a tempered distribution on $\mathbb{R}^4$ that solves the Klein-Gordon equation

$$(\Box + m^2)\varphi = 0.$$

Its Lorentz-covariant Fourier transform is the a tempered distribution that fulfils $(-p^2 + m^2)\hat{\varphi} = 0$ so that it is supported on the two fold hyperboloid

$$X_m = \{p \in \mathbb{R}^{3,1} | p^2 = m^2\}$$

called the mass hyperboloid. One can decompose X_m in two components X_m^+ and X_m^- corresponding to the sign of the first coordinate p_0 (Fourier dual to time). One puts on X_m^+ the normalized measure

$$d\lambda = \frac{d^3\mathbf{p}}{(2\pi)^3 \omega_{\mathbf{p}}}$$

with $\omega_{\mathbf{p}} = p_0 = \sqrt{|\mathbf{p}|^2 + m^2}$. One then considers the Hilbert space

$$\mathcal{H} = L^2(X_m^+, \lambda).$$

Definition 12.3. The total Fock space (resp. bosonic, resp. fermionic) on $\mathcal{H}$, denoted $\mathcal{F}(\mathcal{H})$ (resp. $\mathcal{F}_s(\mathcal{H})$, $\mathcal{F}_a(\mathcal{H})$) is the completed of the tensor algebra $\mathcal{F}^0(\mathcal{H})$ (resp. symmetric $\mathcal{F}_s^0(\mathcal{H}) := \mathrm{Sym}_{\mathbb{C}}(\mathcal{H})$, resp. exterior $\mathcal{F}_a^0(\mathcal{H}) := \wedge_{\mathbb{C}}\mathcal{H}$). The subspaces given by the algebraic direct sums $\mathcal{F}^0$, $\mathcal{F}_s^0$ et $\mathcal{F}_a^0$ are called finite particle subspaces.

One defines the number operator N on $\mathcal{F}^0(\mathcal{H})$ by

$$N = kI \text{ sur } \otimes^k \mathcal{H}.$$

For $v \in \mathcal{H}$, one defines the operators B and $B^\dagger$ on $\mathcal{F}^0(\mathcal{H})$ by

$$\begin{array}{rcl} B(v)(u_1 \otimes \cdots \otimes u_k) & = & \langle v | u_1 \rangle . u_2 \otimes u_k, \\ B(v)^\dagger (u_1 \otimes \cdots \otimes u_k) & = & v \otimes u_1 \otimes \cdots \otimes u_k, \end{array}$$

and one shows that these operators are adjoint for $v \in \mathcal{H}$ fixed. The operator $B(v)^\dagger$ does not preserve the symmetric subspace $\mathcal{F}_s^0(\mathcal{H})$ but $B(v)$ preserves it. One also defines $A(v) = B(v)\sqrt{N}$ and denote $A^\dagger(v)$ the adjoint of the operator $A(v)$ on the bosonic foch space $\mathcal{F}_s^0(\mathcal{H})$. One can also see it as the operator

$$A(v)^\dagger = P_s B(v)^\dagger \sqrt{N+1} P_s$$

with $P_s : \mathcal{F}^0(\mathcal{H}) \to \mathcal{F}^0(\mathcal{H})$ the projection on the bosonic subspace $\mathcal{F}_s^0(\mathcal{H})$.

One has the following commutation relations of operators on the bosonic Foch space: $[A(v), A(w)^\dagger] = \langle v | w \rangle I$ and $[A(v), A(w)] = [A(v)^\dagger, A(w)^\dagger] = 0$.

One has a natural map $R : \mathcal{S}(\mathbb{R}^4) \to \mathcal{H}$ defined by

$$Rf = \hat{f}_{|X_m^+}$$

with $\hat{f}(p) = \int e^{ip_\mu x^\mu} f(x) d^4x$ the Lorentz-covariant Fourier transform.

Definition 12.4. The neutral quantum free quantum scalar field of mass m is the distribution on $\mathbb{R}^4$ valued in operators on the bosonic Foch finite particle space $\mathcal{F}_s^0(\mathcal{H})$ given by

$$\Phi(f) = \frac{1}{\sqrt{2}}\left[A(Rf) + A(Rf)^\dagger\right].$$

Remark that Φ is a distribution solution of the Klein-Gordon equation, i.e., $\Phi((\Box + m^2)f) = 0$ for every $f \in \mathcal{S}(\mathbb{R}^4)$ because $(-p^2 + m^2)\hat{f} = 0$ on X_m^+.

For charged scalar field (i.e. with complex values), it is necessary to use the two Hilbert spaces $\mathcal{H}_+ = L^2(X_m^+)$ and $\mathcal{H}_- = L^2(X_m^-)$ and the anti-unitary operator $C : \mathcal{H}_\pm \to \mathcal{H}_\mp$ given by $Cu(p) = u(-p)^*$. One defines the creation and annihilation operators for $v \in \mathcal{H}_-$ on $\mathcal{H}_+$ by

$$B(v) = A(Cv) \text{ et } B(v)^\dagger = A(Cv)^\dagger.$$

Now here comes the trick. One considers as before the Fock space $\mathcal{F}_s^0(\mathcal{H}_+)$ and see it as defined over $\mathbb{R}$ by considering $\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\mathcal{F}_s^0(\mathcal{H}_+)$. The operators $B(v)$ and $B(v)^\dagger$ are $\mathbb{R}$-linear because the operator $C : \mathcal{H}_\pm \to \mathcal{H}_\mp$ is, and that's why one takes the scalar restriction.

One then extends the scalars from $\mathbb{R}$ to $\mathbb{R}$ by applying $.\otimes_\mathbb{R} \mathbb{C}$ to get a space that we will denote $\mathcal{F}_s^0$, and that can be decomposed as a tensor product of two Foch spaces isomorphic to $\mathcal{F}_s^0(\mathcal{H}_+)$, and exchanged by an automorphism induced by complex conjugation. One then defines, for $f \in \mathcal{S}(\mathbb{R}^4)$, two maps $R_\pm : \mathcal{S}(\mathbb{R}^4) \to \mathcal{H}_\pm$ by

$$R_\pm f = \hat{f}_{|X_m^\pm}.$$

Definition 12.5. The charged free scalar field of mass m is the data of the two distributions on $\mathbb{R}^4$ with values in operators on the Foch space $\mathcal{F}_s^0$ defined by

$$\begin{array}{rcl} \Phi(f) & = & \frac{1}{\sqrt{2}}\left[A(R_+f) + B(R_-f)^\dagger\right], \\ \Phi(f)^\dagger & = & \frac{1}{\sqrt{2}}\left[A(R_+f)^\dagger + B(R_-f)\right]. \end{array}$$

12.7 A word about quantization of non-free fields

To quantize non-free fields, one uses perturbative methods by expressing the Hamiltonian $H = F + \lambda V$ as a sum of a free (quadratic) part, that can be quantized freely as before and a potential λV, with λ a formal variable. One then uses the free state space (foch space) of the previous section and the action of some operators constructed from the free field using the potential. This method is purely perturbative (in that the parameter λ is purely formal). More on this can be found in Folland's book [Fol08].

We prefer to use functional integral methods, because

- they are closer to geometry and allow a uniform treatment of fermionic and bosonic variables, starting from classical lagrangian field theory, and
- they allow a very general treatment of gauge symmetries.

Chapter 13

Methods and mathematical difficulties of functional integrals

In this chapter, we authorize ourselves to use physicists notations

and some of their problematic computations,

describing where are the mathematical problems, so that one has a good idea of what needs to be done to solve them.

The author's viewpoint is that functional integrals (and particularly the Dyson-Schwinger approach to them) give the description of interacting quantum fields that is presently the closest one to a proper mathematical theory. The road to such a theory is tricky but paved by physicists, at least in each and all of the examples of physical relevance. It thus gives an interesting area of research for mathematicians. It seems impossible to explain quantum gauge fields to a mathematician without using this clumsy road. We will see later in the renormalization section 17 that one can get at the end to proper mathematical definitions in some interesting examples.

13.1 Functional derivatives

In this course, we used well defined functional derivatives on the space of histories H of a field theory $(\pi : C \to M, S : \Gamma(M, C) \to \mathbb{R})$ along a given vector field $\vec{v} : H \to TH$. This kind of functional derivatives are necessary to understand physicists' computations with fermionic variables.

We will now explain the physicists' approach to functional derivatives, even if it has more mathematical drawbacks. This will be necessary to understand properly the mathematical problems that appear in the manipulation of functional integrals.

Definition 13.1. Let $\mathcal{H}$ and $\mathcal{G}$ be two complete locally convex topological vector spaces, whose topology are given by a families of seminorms. If $F : \mathcal{H} \to \mathcal{G}$ is a functional, $f \in \mathcal{H}$ and $h \in \mathcal{H}$, one defines its gâteaux derivative, if it exists, by

$$\frac{\delta F}{\delta \vec{h}}(f) := \lim_{\epsilon \to 0} \frac{F(f + \epsilon.h) - F(f)}{\epsilon}.$$

One can apply this definition for example to functionals $F : \mathcal{H} \to \mathbb{R}$ on a topological space of fields $\mathcal{H} = \Gamma(M, C)$ of a bosonic field theory where $\pi : C = F \times M \to M$ is a trivial linear bundle on spacetime with fiber F, since it is enough to understand the main difficulties. In particular, one can work with $\mathcal{H} = \mathcal{C}_c^\infty(\mathbb{R}^n)$ or $\mathcal{H} = \mathcal{S}(\mathbb{R}^n)$ (equiped with their usual topologies).

One can also work with spaces of distributions. For example, the Gâteaux derivative of a functional $F : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}$ along Dirac's δ_x distribution (that one often finds in physics books), if it exists, defines a map

$$\frac{\delta}{\delta \vec{\delta}_x} : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R},$$

that associates for example to a locally integrable function φ a derivative

$$\frac{\delta F}{\delta \vec{\delta}_x}(\varphi) \in \mathbb{R}$$

of F at φ along the Dirac δ distribution, denoted $\frac{\delta F(\varphi)}{\delta \varphi(x)}$ by physicists. For example, if $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ is fixed, the corresponding functional $[\varphi_0](\varphi) := \int_{\mathbb{R}^n} \varphi_0(x)\varphi(x)dx$ on $\mathcal{S}(\mathbb{R}^n)$ can be extended to a functional of distributions $[\varphi_0] : \mathcal{S}'(\mathbb{R}^n) \to \mathbb{R}^n$ by setting

$$[\varphi_0](D) := D(\varphi_0).$$

One then has a well defined functional derivative for the functional $[\varphi_0]$ given by

$$\frac{\delta[\varphi_0]}{\delta \varphi(x)}(\varphi) := \frac{\delta[\varphi_0]}{\delta \vec{\delta}_x} = \varphi_0(x).$$

One can also consider the evaluation functional $F = ev_x : \mathcal{S}(\mathbb{R}^n) \to \mathbb{R}$ given by $f \mapsto f(x)$. One then has

$$\frac{\delta F}{\delta \vec{h}}(f) := \lim_{\epsilon \to 0} \frac{F(f + \epsilon.h) - F(f)}{\epsilon} = \lim_{\epsilon \to 0} \frac{f(x) + \epsilon.g(x) - f(x)}{\epsilon} = g(x)$$

for every $f \in \mathcal{S}(\mathbb{R}^n)$, so that the functional derivative of F at f exists and is

$$\frac{\delta F}{\delta \vec{h}}(f) = F(\vec{h}).$$

One here remarks that the functional derivative has nothing in common with the derivative of distributions. One also remarks that the function $F = ev_x$ does not extend to tempered distributions and that one can't compute its functional derivative along δ_x, because $F(\delta_x) = \mathrm{ev}_x(\delta_x)$ isn't well defined. This is one of the explanation of the infinities that appear in the physicists' computations of functional integrals, that impose the use of renormalization methods.

For the rest of this section, we will suppose that all our functionals extend to distributions and use the following definition of functional derivatives:

$$\frac{\delta F(f)}{\delta f(x)} := \frac{\delta F(f)}{\delta(\delta_x)} = \lim_{\epsilon \to 0} \frac{F(f + \epsilon\delta_x) - F(f)}{\epsilon}.$$

The most useful functional derivatives in field theory are given by the following proposition (that one can find in Folland's book [Fol08], page 269).

Proposition 36. *One has the following computations:*

Functional	*Formula*	*Functional derivative*
Integral	$F(f) = \int f(y)h(y)dy$	$\frac{\delta F(f)}{\delta f(x)} = h(x)$
Exponential integral	$\exp\left[\int f(y)h(y)dy\right]$	$\frac{\delta^J F(f)}{\delta f(x_1)\dots\delta f(x_J)} = h(x_1)\dots h(x_J)F(f)$
Quadratic integral	$F(f) = \int\int f(x)K(x,y)g(y)dxdy$	$\frac{\delta F(f)}{\delta f(u)} = 2\int K(u,z)f(z)dz$
Quadratic exponential	$F(f) = \exp\left(\int\int f(x)K(x,y)g(y)dxdy\right)$	$\frac{\delta F(f)}{\delta f(u)} = \left[2\int K(u,z)f(z)dz\right]F(f)$

Proof. We will only give the proof for the two first integrals. The functional $F(f) = \int f(y)h(y)dy$ can be extended from $\mathcal{S}(\mathbb{R})$ to distributions f on $\mathcal{S}(\mathbb{R})$ by $F(f) = f(h)$: this is simply the evaluation of distributions at the given function h. It is a linear functional in the distribution f. One can thus compute $F(f + \epsilon\delta_x) = F(f) + \epsilon F(\delta_x)$ and get

$$\frac{\delta F(f)}{\delta(\delta_x)} = \delta_x(h) = h(x).$$

Similarly, for $F(f) = \exp\left[\int f(y)h(y)dy\right]$, one can extend it to distributions by $F(f) = \exp(f(h))$, getting

$$\begin{aligned}
\frac{\delta F(f)}{\delta(\delta_x)} &= \lim_{\epsilon\to 0} \frac{\exp(f(h) + \epsilon\delta_x(h)) - \exp(f(h))}{\epsilon} \\
&= \lim_{\epsilon\to 0} \exp(f(h)).\frac{\exp(\epsilon\delta_x(h)) - 1}{\epsilon} \\
&= h(x)\exp(f(h)).
\end{aligned}$$

The other equalities are obtained by similar methods. □

13.2 Functional integrals

We recall here the main idea of Feynman's sum over histories approach to quantum physics. Let M be a spacetime with its Lorentz metric g, let $\pi : C = M \times F \to M$ be a trivial linear bundle with fiber F, and $S : \Gamma(M, C) \to \mathbb{R}$ be a local action functional. Let $\iota_i : \Sigma_i \hookrightarrow M$, $i = 0, 1$ be two spacelike hypersurfaces (that one can think as the space at times t_0 and t_1). If we suppose that the action functional is of order 1 (first order derivatives only), one can specify a space of histories by choosing two sections $\varphi_i \in \Gamma(\Sigma_i, \iota_i^* C)$, $i = 0, 1$ and saying that

$$H = H_{(\Sigma_0,\varphi_0),(\Sigma_1,\varphi_1)} := \{\varphi \in \Gamma(M, C),\ \varphi_{|\Sigma_0} = \varphi_0,\ \varphi_{|\Sigma_1} = \varphi_1\}.$$

We now suppose that $S(\varphi) = S_{free}(\varphi) + S_{int}(\varphi)$ is a sum of a free/quadratic part in the field, whose equations of motion give a linear partial differential equation

$$D\varphi = 0$$

and of an interaction part that depend only on the values of the field and not of its derivatives.

One can think of the value $\varphi_0 : \Sigma_0 \to \iota_0^* C$ as specifying the value of the classical free field $\tilde{\varphi}_0$ on the spacetime region before the time corresponding to Σ_0, which is the starting time for the collision experiment. The value of $\tilde{\varphi}_0$ is measured experimentaly in the preparation of the machine and its data is essentially equivalent to the datum of the pair (S_{free}, φ_0) composed of the free lagrangian and the initial value for the experiment. One gets $\tilde{\varphi}_0$ from (S_{free}, φ_0) by solving the backward in time cauchy problem for the hyperbolic linear partial differential equation of free motion

$$D\tilde{\varphi}_0 = 0,\ \varphi_{|\Sigma_0} = \varphi_0$$

with initial condition φ_0.

Similarly, the value $\varphi_1 : \Sigma_1 \to \iota_1^* C$ specifies the value of the classical free field $\tilde{\varphi}_1$ after the final time Σ_1 of the collision experiment, by solving the forward in time cauchy problem for the free equation of motion. It is also measured after the experiment by the apparatus.

To sum up, what you put in the machine is a free field and what you get after the experiment is also a free field.

The quantum process essentially takes place between Σ_0 and Σ_1, and is composed of collisions and various particle creations and annihilations. The information that one can gather about the probabilistic property of this quantum process with respect to a given functional $A : H \to \mathbb{R}$, called an observable, are all contained in Feynman's sum over history, also called functional integral:

$$\langle \varphi_1 | A | \varphi_0 \rangle = \frac{\int_H A(\varphi) e^{\frac{i}{\hbar} S(\varphi)} [d\varphi]}{\int_H e^{iS(\varphi)} [d\varphi]}.$$

One can understand this formal notation by taking a bounded box $B \subset M$ whose timelike boundaries are contained in the spacetime domain between Σ_0 and Σ_1, and that represent a room (or an apparatus' box) in which one does the experiment. We now suppose that $M = \mathbb{R}^{3,1}$, B is a 0-centered square and $\Lambda = \mathbb{Z}^{3,1} \subset \mathbb{R}^{3,1}$ is the standard lattice. An observable is simple enough if it can be restricted to the space

$$\mathrm{Hom}(\Lambda \cap B, F) = F^{\Lambda \cap B} \subset \Gamma(M, C)$$

of functions on the finite set of points in $\Lambda \cap B$ with values in the fiber F of the bundle $\pi : C \to M$. For simple enough observables, one can make sense of the above formula by using usual integration theory on the finite dimensional linear space $F^{\Lambda \cap B}$ with its Lebesgue measure.

The main problem of renormalization theory is that if one makes the size L of the box B tend to infinity and the step ℓ of the lattice Λ tend to 0, one does not get a well defined

limit. The whole job is to modify this limiting process to get well defined values for the above formal expression.

13.3 Schwinger's quantum variational principle

Another way of giving a mathematical meaning to the expectation value $\langle\varphi_1|A|\varphi_0\rangle$ is to consider the dual linear bundle $C^\dagger$ of C and to add to the action functional a source term

$$S_{source}(\varphi) = S(\varphi) + \int_M \langle J, \varphi\rangle(x) d^4x.$$

The aim is to make sense of the partition function

$$Z_A(J) = \langle\varphi_1|A|\varphi_0\rangle_J = \frac{\int_H A(\varphi) e^{\frac{i}{\hbar}S_{source}(\varphi)}[d\varphi]}{\int_H e^{iS(\varphi)}[d\varphi]},$$

whose value at $J = 0$ gives the sum over histories. The main advantage of this new formal expression is that it replaces the problem of defining an integral on the space of histories by the problem of defining the functional differential equation on the dual space whose solution is $Z_A(J)$. This equation is called the Dyson-Schwinger equation and will be used from now on to define implicitely the partition function, since this is usually how physicist using functional integral proceed (see for example Zinn-Justin's book [ZJ93], chapter 7 or Rivers' book [Riv90]).

The idea behind the Dyson-Schwinger approach can be explained quickly in the case of a scalar field theory $\pi : C = \mathbb{R} \times M \to M$ on spacetime. The space $\mathcal{O}_H$ of functions on H is equiped with a (partially defined) functional derivation $\frac{\delta}{\delta\varphi(x)}$ defined by

$$\frac{\delta F}{\delta\varphi(x)} = \lim_{t\to 0} \frac{F(\varphi + t\delta_x) - F(\varphi)}{t}.$$

The hypothetic functional integral $\int_H [d\varphi] : \mathcal{O}_H \to \mathbb{R}$ is supposed to fulfil the following formal equalities, that are inspired by usual properties of finite dimensional integrals and Fourier transforms:

$$\int_H [d\varphi] \frac{\delta}{\delta\varphi(x)} F(\varphi) = 0$$

and

$$\int_H [d\varphi] F(\varphi) G(\varphi) e^{\frac{i}{\hbar}J(\varphi)} = F\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] \int_H G(\varphi) e^{\frac{i}{\hbar}J(\varphi)} [d\varphi].$$

Here, the expression $F\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right]$ is a functional differential operator obtained from $F(\varphi)$ by replacing φ by $-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}$. Applying this to the (normalized) functional integral, one gets

$$\int_H \frac{\delta}{\delta\varphi(x)} e^{-\frac{i}{\hbar}[S(\varphi)+J(\varphi)]}[d\varphi] = 0,$$

and finally

$$\left(\frac{\delta S}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] + J(x)\right).Z(J) = 0$$

and more generally

$$\frac{\delta A}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right].Z(J) + \left(\frac{\delta S}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] + J(x)\right).Z_A(J) = 0.$$

13.4 Theories with gauge freedom: Zinn-Justin's equation

We refer to Henneaux and Teitelboim [HT92], chapters XVII and XVIII. The formalism of this section was first used by Zinn-Justin to prove the renormalizability of Yang-Mills, and independently by Batalin-Vilkovisky. It is called the Zinn-Justin equation, or the quantum BV formalism in the physics litterature.

Let $\pi : C \to M$ be a bundle and $S \in h(\mathcal{A})$ be a local action functional with $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$. In section 8.3.3, we explained the methods of homological symplectic reduction in local field theory. Roughly speeking, the result of this procedure is a dg-$\mathcal{D}$-algebra $(\mathcal{A}^\bullet, D)$ equiped with a local bracket

$$\{.,.\} : \mathcal{A}^\bullet \boxtimes \mathcal{A}^\bullet \to \Delta_*\mathcal{A}^\bullet$$

and a particular element $S_{cm} \in \mathcal{A}^\bullet$ called the classical master action, or also the BRST generator, such that

$$D = \{S_{cm}, .\}.$$

This construction works in the most general context of a local field theory on a superspace. The space of classical observables is then given by the algebra

$$H^0(\mathcal{A}^\bullet, D).$$

Roughly speaking, the canonical version of the BV quantization (operator formalism) is given by a quantization of the graded algebra $\mathcal{A}^\bullet$, denoted $\hat{\mathcal{A}}^\bullet$, that gives also a quantization of the BRST generator S_{cm} and thus a quantization $\hat{D}$ of the BRST differential. The algebra of quantum observables is then given by

$$H^0(\hat{\mathcal{A}}^\bullet, \hat{D}).$$

The functional integral quantization is a bit more involved. We denote by φ_i the fields and ghost variables and by φ_i^* the antifield variables. In physicists' notations, the duality between fields and antifields is given by

$$(\varphi^i, \varphi_i^*) = \delta_j^i.$$

This means that the antibracket is extended to the algebra $\mathcal{A}^\bullet$ by

$$(A, B) = \frac{\delta^R A}{\delta \varphi^i} \frac{\delta^L B}{\delta \varphi_i^*} - \frac{\delta^R A}{\delta \varphi_i^*} \frac{\delta^L B}{\delta \varphi^i}.$$

Here, the left and right functional derivatives are distributions defined by

$$\frac{d}{dt} F(\varphi + t\psi_\varphi) = \int_M \frac{\delta^L F}{\delta \varphi^a(x)} \psi_\varphi^a(x) dx = \int_M \psi_\varphi^a(x) \frac{\delta^R F}{\delta \varphi^a(x)} dx$$

for ψ_φ tangent vectors to the graded field variable φ.

One can not naively use the functional integral (Dyson-Schwinger solution)

$$\int [d\varphi d\varphi^*] \exp \frac{i}{\hbar} S_{cm}(\varphi, \varphi^*)$$

because this would give infinite values because of the gauge freedom. The idea is to replace this by an expression

$$\int [d\varphi d\varphi^*] \exp \frac{i}{\hbar} S_{qm}(\varphi, \frac{\delta \psi}{\delta \varphi})$$

for some given functional ψ of the fields variables called the gauge fixing. The classical master action has to be expanded to a quantum master action $S_{qm} = S_{cm} + \sum_{n \geq 1} \hbar^n S_n$ that fulfils the quantum master Zinn-Justin equation

$$i\hbar \Delta S_{qm} - \frac{1}{2}(S_{qm}, S_{qm}) = 0,$$

where $\Delta = \frac{\delta^R}{\delta\varphi^i}\frac{\delta^R}{\delta\varphi_i^*}$ is the super laplacian on $\mathcal{A}^\bullet$ (that is not well defined an needs to be regularized because it involves double functional derivatives). This equation guaranties that the above functional integral does not depend on the gauge fixing functional ψ. It is equivalent to the equation

$$\Delta \exp \frac{i}{\hbar} S_{qm} = 0.$$

If $A_0(\varphi^i) \in \mathcal{A}$ is a classical local gauge invariant function, a classical BRST extension is given by an element $A(\varphi, \varphi^*)$ of $\mathcal{A}^\bullet$ solution of $(A, S_{cm}) = 0$. A quantum BRST extension of A is given by an element $\alpha \in \mathcal{A}^\bullet[[\hbar]]$ of the form

$$\alpha = A + \sum_{n \geq 1} \hbar \alpha_i,$$

such that the quantum master observable equation

$$\Delta \alpha \exp \frac{i}{\hbar} S_{qm} = 0$$

is fulfilled, the quantum master action S_{qm} being already fixed. This quantum master observable equation is again equivalent to

$$i\hbar\Delta\alpha - (\alpha, S_{qm}) = 0.$$

The function $A_\psi(\varphi) = A(\varphi, \frac{\delta\psi}{\delta\varphi})$ is called the gauge fixed version of A. The expectation value of A_0 is finally given by

$$\langle A_0 \rangle = \int \alpha\left(\varphi, \frac{\delta\psi}{\delta\varphi}\right) e^{\frac{i}{\hbar} S_{qm}\left(\varphi, \frac{\delta\psi}{\delta\varphi}\right)} [d\varphi].$$

The main problem with this definition is that the BV laplacian does not make sense because it involves second order functional derivatives. One of course also gets into trouble when writing down the Dyson- Schwinger equation for this functional integral since it involves higher order functional derivatives.

13.5 Towards the mathematics of the Dyson-Schwinger equation

We now give a mathematical approach to the definition of the partition function, using the Dyson-Schwinger equation, that was derived formally “à la physicist” in the previous

section. One can think of this definition in analogy with the definition of $\sqrt{2}$ as a formal solution to the equation $x^2 - 2 = 0$. We don't have at present a formal analog of the algebra $\mathbb{Z}[\sqrt{2}] := \mathbb{Z}[x]/(x^2 - 2)$ that contains the universal solution to this problem, but at least, we have the analog of the equation. The main advantage of this approach, as already remarked by Schwinger in his original work of 1952, is that it allows us to treat fermionic and bosonic fields on equal footing. The use of space of histories from the functor of point viewpoint, that we did from the beginning, is mandatory to treat properly fermionic variables here, and justifies itself fully only now.

We now give a short account of this approach, using the following notation:

- $\pi : C \to M$ is a bundle,
- $H \subset \Gamma(M, C)$ is a subspace,
- $\mathcal{O}_H := \mathrm{Hom}(H, \mathbb{R})$ is the space of function on H,
- $\Theta_H := \Gamma(H, TH)$ is the space of vector fields on H,
- $S \in \mathcal{O}_H$ is an action functional.

Remark that a vector field $\vec{v} : H \to TH$ in Θ_H defines a natural map

$$\frac{\delta}{\delta \vec{v}} : \mathcal{O}_H \to \mathcal{O}_H$$

by associating to $F \in \mathcal{O}_H$ the functional $\frac{\delta F}{\delta \vec{v}} \in \mathcal{O}_H$ given by the diagram

$$\begin{tikzcd} TH \arrow[r, "DF"] & T\mathbb{R} \arrow[d, "p_2"] \\ H \arrow[u, "\vec{v}"] \arrow[r, "\frac{\delta F}{\delta \vec{v}}"] & \mathbb{R} \end{tikzcd},$$

where

$$p_2 : T\mathbb{R}(A) := \mathrm{Hom}(\mathbb{R}[x], A[\epsilon]/(\epsilon^2)) = A[\epsilon]/(\epsilon^2) \to \mathrm{Hom}(\mathbb{R}[y], A) =: \mathbb{R}(A) = A$$

is given by $p_2(a + \epsilon b) = b$. We implicitely supposed here that

- either we are working in a category LEGOS of graded or super-algebras whose soul is smoothly affine, like described in 2.3.2,
- or we work with algebraic graded algebras whose soul are usual real algebras.

We suppose moreover that C is a linear (or at least affine) graded bundle on M, so that $\Gamma(M,C)$ is a linear (or affine) super-space. Let $H^\dagger := \Gamma(M,C^\dagger)$ be its fiberwise twisted dual where $C^\dagger := \mathrm{Hom}_{\mathbb{R}_M}(C,\mathrm{Ber}_M)$. There is a natural duality pairing of linear spaces

$$\begin{array}{ccc} \Gamma(M,C)\times\Gamma(M,C^\dagger) & \to & \mathbb{R} \\ (x,J) & \mapsto & \langle J,x\rangle := \int_M J(x). \end{array}$$

The variable J in $H^\dagger := \Gamma(M,C^\dagger)$ is called the source. The above duality pairing induces a map

$$\begin{array}{ccc} H^\dagger & \to & \mathcal{O}_H \\ J & \mapsto & J:[x\mapsto\langle J,x\rangle]. \end{array}$$

that we will use to identify $J\in H^\dagger$ with a functional $J\in\mathcal{O}_H$.

Hypothesis 1. Suppose given

- a subspace $\Theta_H^{ad}\subset\Theta_H$ called the space of admissible vector fields on H,
- a class $\mathcal{O}_H^{ad}\subset\mathcal{O}_H$ of admissible functionals on H that contains the action functional with source $S_{source}=S+J$ and its functional derivatives $\frac{\delta S_{source}}{\delta\vec{v}}$ with respect to every vector field $\vec{v}:H\to TH$ in Θ_H^{ad}, for every fixed source $J\in H^\dagger$ and
- a map

  $$\begin{array}{cccc} Q: & \mathcal{O}_H^{ad} & \to & \mathrm{Hom}(\mathcal{O}_{H^\dagger},\mathcal{O}_{H^\dagger}) \\ & F & \mapsto & F\left[\frac{-i}{\hbar}D_J\right] \end{array}$$

  called the quantization map.

The triple $(\Theta_H^{ad},\mathcal{O}_H^{ad},Q)$ is called a quantization datum.

The map Q above can be seen as a functional version of the map

$$\begin{array}{cccc} Q: & \mathbb{R}[\underline{\mathrm{x}},\underline{\xi}] & \to & \mathcal{D}_{\mathbb{R}[\underline{\mathrm{x}}]} \\ & P(\underline{\mathrm{x}},\underline{\xi})=\sum a_\alpha(\underline{\mathrm{x}})\underline{\xi}^\alpha & \mapsto & P(D):=\sum a_\alpha\partial_{\underline{\mathrm{x}}}^\alpha \end{array}$$

that replaces the ξ variable by the universal derivation

$$\begin{array}{cccc} D: & \mathbb{R}[\underline{\mathrm{x}}] & \to & \mathrm{Hom}(\Theta_{\mathbb{R}[\underline{\mathrm{x}}]},\mathbb{R}[\underline{\mathrm{x}}]) \\ & f & \mapsto & [D\mapsto Df]. \end{array}$$

on $\mathbb{R}^n$, so that one can think of the formal sign D_J as a kind of universal derivation

$$\begin{array}{cccc} D_J: & \mathcal{O}_{H^\dagger} & \to & \mathrm{Hom}(\Theta_{H^\dagger},\mathcal{O}_{H^\dagger}) \\ & F & \mapsto & [\frac{\delta}{\delta\vec{j}}\mapsto\frac{\delta F}{\delta\vec{j}}]. \end{array}$$

This shows that the quantization map is similar to the classical hamiltonian Weyl quantization map from polynomial functions on the phase space T^*X to differential operators on the base X. From this point of view [1], the Hilbert space of quantum mechanics is replaced by the space $\mathcal{O}_{H^\dagger}$ of functions on the space of sources, and the quantization of classical observables in $\mathcal{O}_H$ gives differential operators on $\mathcal{O}_{H^\dagger}$.

Definition 13.2. A quantum partition function associated to the classical system

$$(\pi : C \to M, H, S)$$

and the quantization datum $(\Theta_H^{ad}, \mathcal{O}_H^{ad}, Q)$ is a function $Z \in \mathcal{O}_{H^\dagger}[[\hbar]]$ that is a solution of the Dyson-Schwinger equations

$$\forall \vec{v} \in \Theta_H^{ad}, \left(\frac{\delta S}{\delta \vec{v}} + \frac{\delta J}{\delta \vec{v}}\right)\left[-\frac{i}{\hbar}D_J\right].Z(J) = 0.$$

If $A \in \mathcal{O}_H$ is an observable such that $\frac{\delta A}{\delta \vec{v}} \in \mathcal{O}_H^{adm}$ is an admissible observable for every $\vec{v} \in \Theta_H^{adm}$, the partition function of A is the function $Z_A \in \mathcal{O}_{H^\dagger}[[\hbar]]$ that is a solution of the Dyson-Schwinger equation

$$\forall \vec{v} \in \Theta_H^{ad}, \left(\frac{\delta S}{\delta \vec{v}} + \frac{\delta J}{\delta \vec{v}}\right)\left[-\frac{i}{\hbar}D_J\right].Z_A(J) + \frac{\delta A}{\delta \vec{v}}\left[-\frac{i}{\hbar}D_J\right].Z(J) = 0.$$

The partition function does not define quantum fields (except for scalar fields) but quantum observables.

Remark that the expression $\frac{\delta J}{\delta \vec{v}}$ being the functional derivative of a linear functional, can be simplied to

$$\frac{\delta J}{\delta \vec{v}} = \int_M J(h_{\vec{v}}),$$

where $\vec{v}(x) = x + \epsilon h_{\vec{v}}$.

To relate the above definition to the physics litterature, one can think that Physicists use the functional derivative along the ill-defined tangent vector $\vec{v}$ given by the δ_t distribution $h_{\vec{v}} = \delta_t$ as a formal notation (which is only justified in the local case), as follows:

$$\left(\frac{\delta S}{\delta x(t)}\left[\frac{-i}{\hbar}\frac{\overleftarrow{\delta}}{\delta J(t)}\right] + J(t)\right).Z(J) = 0,$$

[1]Proposed to the author by Thiaggo Drummond at the view of the above definition.

the right arrow derivative being due to the given twist by the Berezinian in $H^\dagger$. The main problem with this notation is that it involves higher order functional derivatives

$$\frac{\overleftarrow{\delta}^2}{\delta J(t)^2},$$

that are not well defined on the space of local functionals because the evaluation functional $J \mapsto J(0)$ can not be evaluated at the δ_0 function.

Suppose we work with a local action funcional S that comes from a cohomology class $S \in h(\mathcal{A})$ for $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C)$. We will then use as admissible vector fields Θ_H^{adm} the local vector fields $\frac{\partial}{\partial u} \in \Theta_{C/M}$ that generate

$$\Theta_{\mathcal{A}} = (\pi_\infty^* \Theta_{C/M}) \otimes_{\mathcal{A}} \mathcal{A}[\mathcal{D}^{op}]$$

as an $\mathcal{A}[\mathcal{D}^{op}]$-module. These are given by cohomology classes in $h(\Theta_{\mathcal{A}})$. Their image through the local insertion map

$$i_{dS} : \Theta_{\mathcal{A}} \to \mathcal{A}$$

give exactly the Euler-Lagrange equations, as functions on the jet space $\mathrm{Jet}(C)$. If $\vec{v}$ is a local vector field on H induced by $v \in h(\Theta_{\mathcal{A}}) = \Theta_{\mathcal{A}} \otimes_{\mathcal{D}^{op}} \mathcal{O}$, and $\langle dS, v\rangle := h(i_{dS})(v) \in h(\mathcal{A})$ is the corresponding local functional, one has

$$\frac{\delta S}{\delta \vec{v}}(\varphi) = \int_M (j_\infty \varphi)^* \langle dS, v\rangle,$$

meaning that the functional derivative of a local action functional along a local vector field is identified with the evaluation of the corresponding local functional derivative (it is the aim of local functional calculus to formalize properly such a statement).

Chapter 14

The perturbative approach to functional integrals

14.1 Perturbative expansions in finite dimension

The main idea of perturbative methods in quantum field is to use explicit computations of gaussian integrals in finite dimension as definitions of the analogous gaussian integrals in field theory. We thus review perturbative computations of gaussian integrals in finite dimension.

14.1.1 Gaussian integrals with source

To understand the notations used by physicists, it is good to understand their normalizations. One can however use other normalization that the ones commonly used in physics to escape the infiniteness problem of normalization of gaussian integrals in infinite dimension. This is explained in details in Cartier and DeWitt-Morette's book [CDM06] and expanded to the case of fields in J. Lachapelle's thesis.

We essentially follow here chapter 1.2.2 of Zinn-Justin's book [ZJ05]. Let A be a positive definite symmetric matrix that induces a quadratic form q on $\mathbb{R}^d$ and $\Delta = A^{-1}$ its inverse (that we will later call the propagator), that induces a quadratic form on $(\mathbb{R}^d)^\vee$. For $b \in (\mathbb{R}^d)^\vee$ a linear form, one denotes

$$Z(A, b) = \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}q(x) + b.x\right) dx.$$

The change of variable $x = y + \Delta.b$ gives that this integral is

$$Z(A,b) = \exp\left(\frac{1}{2}w(b)\right)\int_{\mathbb{R}^d}\exp\left(-\frac{1}{2}q(y)\right) = \exp\left(\frac{1}{2}w(b)\right)Z(A,0).$$

The value of $Z(A,0)$ is obtained by diagonalizing from the dimension 1 value and gives

$$Z(A,0) = \frac{(2\pi)^{d/2}}{\sqrt{\det(A)}}.$$

One remarks that the limit as d goes to infinity of $Z(A,b)$ makes no sense but the limit of the quotient $\frac{Z(A,b)}{Z(A,0)}$ can have one, because one has the formula

$$\frac{Z(A,b)}{Z(A,0)} = \exp\left(\frac{1}{2}w(b)\right).$$

If F is a function on $\mathbb{R}^n$, its mean value is given by

$$\langle F\rangle := \frac{\int_{\mathbb{R}^n} F(x)e^{-\frac{1}{2}q(x)}dx}{Z(A,0)}.$$

Remark in particular that

$$\langle \exp(b.x)\rangle = \frac{Z(A,b)}{Z(A,0)}.$$

If we denote $b.x = \sum b_i.x_i$, we can find the mean values of monomials by developping $\exp(b.x)$ in power series:

$$\langle x_{k_1}\dots x_{k_r}\rangle = \left[\frac{\partial}{\partial b_{k_1}}\cdots\frac{\partial}{\partial b_{k_r}}\frac{Z(A,b)}{Z(A,0)}\right]\Bigg|_{b=0} = \left[\frac{\partial}{\partial b_{k_1}}\cdots\frac{\partial}{\partial b_{k_r}}\exp(\frac{1}{2}w(b))\right]\Bigg|_{b=0}.$$

Each partial derivative $\frac{\partial}{\partial b_{k_i}}$ can either take a linear factor b_i out, or differential one that is already present. The result is zero if there are the same number of these two operations, so that

$$\langle x_{k_1}\dots x_{k_r}\rangle = \sum_{\text{index pairings}} w_{i_{j_1},i_{j_2}}\dots w_{i_{j_{l-1}},i_{j_l}}$$

with $w_{i,j}$ the coefficients of the matrix w. This formula is called Wick's formula.

More generally, if F is a power series in the x_i's, one finds the identity

$$\langle F\rangle = \left[F\left(\frac{\partial}{\partial b}\right)\exp\left(\frac{1}{2}w(b)\right)\right]\Bigg|_{b=0}.$$

This is this mean value function that physicists generalize to the infinite dimension. The auxiliary linear form b is often denoted J and called the external source.

14.1.2 Perturbative gaussians

We refer to the introductory article [Phi01]. The aim of Feynman diagrams is to encode the combinatorics of Wick's formula for the value of mean values

$$\langle F\rangle = \left[F\left(\frac{\partial}{\partial b}\right)\exp\left(\frac{1}{2}w(b)\right)\right]\Bigg|_{b=0}$$

of functions for a gaussian measure.

To compute these mean values in the case where F is a monomial, one develops the exponential $\exp\left(\frac{1}{2}w(b)\right)$ in formal power series:

$$\exp\left(\frac{1}{2}w(b)\right) = \sum_n \frac{w(b)^n}{2^n n!} = \sum_n \frac{1}{2^n n!}\left(\sum_{i,j=1}^{d} w_{i,j}b_ib_j\right)^n.$$

The terms of this series are homogeneous in the b_i of degree $2n$. If one differentiates k times such a monomial, and evaluate it at 0, on gets 0, except if $k = 2n$. It is thus enough to compute the value of $2n$ differentiations on a monomial of the form $(\sum_{i,j=1}^{d} w_{i,j}b_ib_j)^n$.

The differentiation most frequently used in this computation is

$$\frac{\partial}{\partial b_k}\left(\frac{1}{2}\sum_{i,j=1}^{d} w_{i,j}b_ib_j\right) = \sum_{i=1}^{d} w_{i,j}b_i,$$

where one uses the symmetry of the matrix of quadratic form w, inverse of the quadratic form q. We denote $\partial_i = \frac{\partial}{\partial b_i}$.

Let's compute on examples:

1. The term of order $n = 1$ gives

$$\langle x_1x_2\rangle = \partial_2\partial_1\left(\frac{1}{2}\sum_{i,j=1}^{d} w_{i,j}b_ib_j\right) = \partial_2\left(\sum_j w_{1,j}b_j\right) = w_{1,2}$$

by using the symmetry of the matrix w, and

$$\langle x_1x_1\rangle = \partial_1\partial_1\left(\frac{1}{2}\sum_{i,j=1}^{d} w_{i,j}b_ib_j\right) = w_{1,1}.$$

2. The term of order $n = 2$ gives

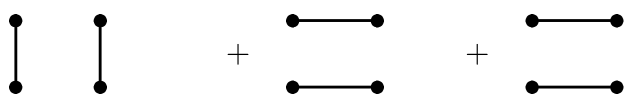
$$\begin{aligned}
\langle x_1 x_2 x_3 x_4 \rangle &= \partial_4 \partial_3 \partial_2 \partial_1 \left(\tfrac{1}{2^2 2!} \textstyle\sum_{i,j=1}^{d} w_{i,j} b_i b_j \right)^2 \\
&= \partial_4 \partial_3 \partial_2 \left(\tfrac{1}{2} (\textstyle\sum w_{i,j} b_i b_j)(\sum w_{1,j} b_j) \right) \\
&= \partial_4 \partial_3 \left[(\textstyle\sum w_{2,j} b_j)(\sum w_{1,j} b_j) + \tfrac{1}{2} (\sum w_{i,j} b_i b_j) w_{1,2} \right] \\
&= \partial_4 \left[w_{2,3} (\textstyle\sum w_{1,j} b_j) + w_{1,3} (\sum w_{2,j} b_j) + w_{1,2} (\sum w_{3,j} b_j)) \right] \\
&= w_{2,3} w_{1,4} + w_{2,4} w_{1,3} + w_{3,4} w_{1,2}.
\end{aligned}$$

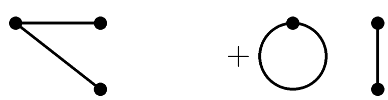
Similarly, one finds

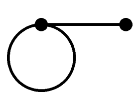
$$\begin{aligned}
\langle x_1 x_1 x_3 x_4 \rangle &= \partial_4 \partial_3 \partial_1 \partial_1 \left(\tfrac{1}{2^2 2!} \textstyle\sum_{i,j=1}^{d} w_{i,j} b_i b_j \right)^2 &&= 2 w_{1,4} w_{1,3} + w_{3,4} w_{1,1}, \\
\langle x_1 x_1 x_1 x_4 \rangle &= \partial_4 \partial_1 \partial_1 \partial_1 \left(\tfrac{1}{2^2 2!} \textstyle\sum_{i,j=1}^{d} w_{i,j} b_i b_j \right)^2 &&= 3 w_{1,4} w_{1,1}, \\
\langle x_1 x_1 x_4 x_4 \rangle &= \partial_4 \partial_4 \partial_1 \partial_1 \left(\tfrac{1}{2^2 2!} \textstyle\sum_{i,j=1}^{d} w_{i,j} b_i b_j \right)^2 &&= 2 w_{1,4} w_{1,4} + w_{4,4} w_{1,1}, \\
\langle x_1 x_1 x_1 x_1 \rangle &= \partial_1 \partial_1 \partial_1 \partial_1 \left(\tfrac{1}{2^2 2!} \textstyle\sum_{i,j=1}^{d} w_{i,j} b_i b_j \right)^2 &&= 3 w_{1,1} w_{1,1}.
\end{aligned}$$

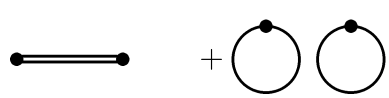
The combinatorics of these computations being quite complex, it is practical to represent each of the product that appear there by a graph, whose vertices correspond to indices of the coordinates x_i that appear in the functions whose mean values are being computed and each $w_{i,j}$ becomes an edge between the i and the j vertex.

One thus gets the diagrams:

$\langle x_1 x_2 x_3 x_4 \rangle =$


$\langle x_1 x_1 x_3 x_4 \rangle = 2.$


$\langle x_1 x_1 x_1 x_4 \rangle = 3.$


$\langle x_1 x_1 x_4 x_4 \rangle = 2.$


$\langle x_1 x_1 x_1 x_1 \rangle = 3.$


14.1.3 Finite dimensional Feynman's rule

We keep the notations of the preceeding sections and give a polynomial V in the coordinates $x_1, \ldots, x_d$. The integrals whose generalizations are interesting for physics are of the type

$$Z(A,V) := \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}q(x) + \hbar V(x)\right) dx,$$

that one can rewrite in the form

$$Z(A,V) = \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}q(x)\right)\left[\sum_n \frac{1}{n!}(\hbar V(x))^n\right] dx.$$

One can evaluate these integrals as before by using the formal formula

$$Z(A,V) = Z(A,0)\exp\left(\hbar V\left(\frac{\partial}{\partial b}\right)\right)\exp\left(\frac{1}{2}w(b)\right)\Big|_{b=0},$$

whose power series expansion can be computed by using the preceding computations by the n-points function. The combinatorics of these computations is encoded in diagrams whose vertices correspond to the coefficients $a_{i_1,\ldots,i_d}$ of the monomials of the polynomial $V = \sum_{i\in\mathbb{N}^d} a_i x^i$ and whose edges correspond to the coefficients $w_{i,j}$ of the quadratic forms that appears in the Wick formula for these monomials.

For example, one can consider a polynomial of the form $V(x) = \sum_{i,j,k} v_{ijk}x_ix_jx_k$ (situation similar to the famous "φ^3" toy model) and compute

$$\begin{aligned} Z(A,V) &= \int_{\mathbb{R}^d} \exp\left(-\tfrac{1}{2}q(x) + \hbar \textstyle\sum_{i,j,k} v_{ijk}x_ix_jx_k\right) dx \\ &= Z(A,0)\exp\left(\hbar \textstyle\sum_{i,j,k} v_{ijk}\partial_i\partial_j\partial_k\right)\exp\left(\tfrac{1}{2}w(b)\right)\Big|_{b=0}. \end{aligned}$$

Let's compute the terms of degree 2 in $\hbar$. This terms have 6 derivatives and their sum is

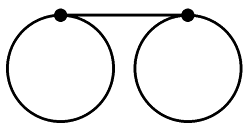
$$\sum_{i,j,k}\sum_{i',j',k'} v_{ijk}v_{i'j'k'}\partial_i\partial_j\partial_k\partial_{i'}\partial_{j'}\partial_{k'}\exp\left(\frac{1}{2}w(b)\right)\Big|_{b=0}.$$

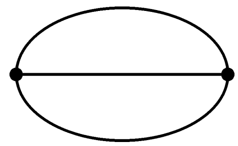
By Wick's theorem, one can rewrite this sum as

$$\sum_{i,j,k}\sum_{i',j',k'} w_{i_1,i_2}w_{i_3,i_4}w_{i_5,i_6}v_{ijk}v_{i'j'k'}$$

where the internal sum is taken on all pairings $\{(i_1,i_2),(i_3,i_4),(i_5,i_6)\}$ de $\{i,j,k,i',j',k'\}$.

One associates to these pairings the graphs whose trivalent vertices correspond to the factors v_{ijk} and whose edges correspond to the $w_{i,j}$. In this case, one gets exactly to distinct graphs called respectively the dumbbell and the theta:





One then has to number the edges at each vertices to encode the corresponding monomial.

14.2 Perturbative definition of functional integrals

We refer to Connes-Marcolli's book [CM08] for a smooth mathematical presentation of the perturbative expansions. The main idea about perturbative expansions is to define the functional integral by their expansions in terms of Feynman diagrams, using the analogy with finite dimensional gaussian integrals. In this section, as in chapter 13, we authorize ourselves to use physicists notations

and some of their problematic computations,

describing where are the mathematical problems, so that one has a good idea of what needs to be done to solve them.

One thus considers the action functional $S(\varphi)$ of a given system $\pi : C \to M$ without gauge symmetries as a perturbation of a free action functional $S_{free}(\varphi) = \int_M \langle \varphi, D\varphi \rangle$ that is quadratic in the fields with the linear equation $D\varphi = 0$ as Euler-Lagrange equation, by an interaction term $S_{int}(\varphi) = \int_M V(\varphi)$ with V a potential function. One writes

$$S(\varphi) = S_{free}(\varphi) + \lambda S_{int}(\varphi).$$

If Δ is the Schwarz kernel of D, also called the free propagator, one then defines

$$Z_{free}(J) = \int_H e^{\frac{i}{\hbar}(S_{free}(\varphi) + \int J(\varphi))}[d\varphi] := e^{-\frac{i}{\hbar} J \Delta J}.$$

If the kernel Δ is a smooth function, this free partition function fulfils the free Dyson-Schwinger equation

$$\left(\frac{\delta S_{free}}{\delta \varphi(x)} \left[-\frac{i}{\hbar} \frac{\delta}{\delta J} \right] + J(x) \right) . Z_{free}(J) = 0$$

that is perfectly well defined in this case because there are no higher order functional derivatives in play. If Δ is only a distribution, one can not apply the functional derivative to $Z_{free}(J)$ because it involves evaluating a distribution at the δ function. The solution of this problem is given by the regularization procedure for propagators.

One then formally writes

$$Z(J) = \int_H e^{\frac{i}{\hbar}(S(\varphi)+\int J(\varphi))}[d\varphi] = e^{\frac{i}{\hbar}S_{int}}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J}\right].Z_{free}(J)$$

and permute usual and functional integrals to finally define

$$\begin{aligned} Z(J) \quad := & \quad \textstyle\sum_{k=0}^{\infty} \frac{i^k}{k!\hbar^k} \underbrace{\int \dots \int}_{k \text{ terms}} \mathcal{L}_{int}^k \left[\frac{\delta}{\delta J}\right] e^{-\frac{i}{\hbar}J\Delta J} dx_1 \dots dx_k \\ = & \quad \textstyle\sum_{k=0}^{\infty} \frac{i^k}{k!} \int J(x_1)\dots J(x_k) G_k(x_1,\dots,x_k) dx_1 \dots dx_k. \end{aligned}$$

The terms that appear in this formal power series in $\hbar$ are integrals of some expressions that are ill defined because they involve the evaluation of some distributions at some δ functions. One can however check that this expression formally fulfils the Dyson-Schwinger equation

$$\left(\frac{\delta S}{\delta\varphi(x)}\left[-\frac{i}{\hbar}\frac{\delta}{\delta J(x)}\right] + J(x)\right).Z(J) = 0.$$

One then has to regularize, i.e., replace these distributions by usual families of functions that converge to them, compute everything with these regularized solutions and then pass to the distributional limit. The green functions G_k that appear in the perturbative expansion can be described (or actually defined) by analogy with the usual gaussian case as sums

$$G_k(x_1,\dots,x_k) = \sum_\Gamma \int \frac{V(\Gamma)(p_1,\dots,p_k)}{\sigma(\Gamma)} e^{i(x_1p_1+\dots+x_kp_k)} \prod_i \frac{dp_i}{(2\pi)^d},$$

indexed by Feynman graphs that are not necessarily connected. The problem is that the rational fractions $V(\Gamma)(p_1,\dots,p_k)$ are usually not integrable, so that one has to replace them by their renormalized values, that are obtained by adding to them some counterterms (obtained by adding local counter terms to the lagrangian), to make them convergent. This is the theory of renormalization.

Remark that it is also interesting to consider the connected partition function defined by the equality

$$Z(J) =: e^{\frac{i}{\hbar}W(J)}$$

since its perturbative expansion is described by connected Feynman graphs, and so combinatorially simpler. One actually uses the more simple 1PI partition function, also called the effective action $S_{eff}(\varphi)$. It is formally defined as a kind of Legendre transform by

$$S_{eff}(\varphi) = [J_E(\varphi) - W(J_E)]_{|J_E=J(\varphi)}$$

where J_E is chosen so that

$$\frac{\delta W}{\delta J}(J_E) = \varphi.$$

One can properly define it by taking the unrenormalized values

$$U(\Gamma)(\varphi) = \frac{1}{k!}\int_{\sum p_j=0} \hat{\varphi}(p_1)\dots\hat{\varphi}(p_k)U(\Gamma(p_1,\dots,p_k))\prod_i \frac{dp_i}{(2\pi)^d},$$

and defining

$$S_{eff}(\varphi) = S(\varphi) - \sum_{\Gamma\in 1\mathrm{PI}} \frac{U(\Gamma)(\varphi)}{\sigma(\Gamma)}.$$

In this perturbative setting, where one defines functional integrals by their Feynman graph expansions, one has the equality

$$\int F(\varphi)e^{\frac{i}{\hbar}S(\varphi)}[d\varphi] = \int_{\text{tree level}} F(\varphi)e^{\frac{i}{\hbar}S_{eff}(\varphi)}[d\varphi]$$

where the tree-level functional integral means that one only uses diagrams that are trees. This shows that the knowledge of the effective action is enough to determine all quantum observables.

Chapter 15

Connes-Kreimer-van Suijlekom view of renormalization

We follow here the Hopf algebra / Riemann-Hilbert description of renormalization, due to Connes and Kreimer (see [CK00] and [CK01]). This method is described in [CM08] and in [dM98] for theories without gauge symmetries.

We give here an overview of van Suijlekom's articles on this formalism (see [van08a] and [van08b]) for theories with gauge symmetries.

These constructions give a nice mathematical formulation of the BPHZ renormalization procedure in the dimensional regularization scheme, that is the main method for the renormalization of the standard model.

15.1 Diffeographism groups for gauge theories

We thank W. van Suijlekom for allowing us to include in this section some diagrams from his paper [van08a].

Suppose we are working with the classical BV action of a given gauge theory, given by an element S_{cm} in $h(\mathcal{A}^\bullet)$ for $\mathcal{A}^\bullet$ the BV dg-algebra of fields and antifields. As a graded $\mathcal{A}[\mathcal{D}]$-algebra, this algebra is generated by an even family

$$\Phi = \{\varphi^1, \ldots, \varphi^n, \varphi_1^*, \ldots, \varphi_n^*\}$$

of generators that will be called the fields. One can actually define a linear bundle $E^{tot} = E \oplus E^*$ over spacetime M with E and E^* both graded such that $\mathrm{Jet}(\mathcal{O}_{E^{tot}}) = \mathcal{A}^\bullet$ as a graded $\mathcal{A}[\mathcal{D}]$-algebra.

We suppose that S_{cm} is the cohomology class of a polynomial lagrangian density $\mathcal{L}$ (which is the case for Yang-Mills with matter).

We denote R the set of monomials in $\mathcal{L}$, and decompose it into the free massless terms R_E, called the set of edges, and the interaction and mass terms R_V, called the set of vertices. Each edge in R_E will be represented by a different type of line, called the propagator, that corresponds to the free massless motion of the corresponding particle. Each vertex in R_V will be denoted by a vertex whose entering edges are half lines of the same type as the line in R_E that corresponds to the given type of field.

The datum of the lagrangian density is thus equivalent to the datum of the pair $R = (R_E, R_V)$ of sets of colored edges and vertices and of a map

$$\iota : R_E \coprod R_V \to \mathcal{A}^\bullet.$$

Indeed, one can find back the lagrangian density by setting

$$\mathcal{L}_{free-massless} = \sum_{e \in R_E} \iota(e), \qquad \mathcal{L}_{int-mass}(\varphi) = \sum_{v \in R_V} \iota(v),$$

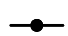
and finally

$$\mathcal{L}(\varphi) = \mathcal{L}_{free-massless} + \mathcal{L}_{int-mass}.$$

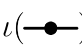
Example 15.1. The lagrangian density of quantum electrodynamics is a $U(1)$-gauge invariant Yang-Mills lagrangian (see section 10.3) of the form

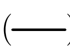
$$L(A, \psi) = -\frac{1}{2}\langle F_A \wedge *F_A \rangle + \bar{\psi} \not{D}_A \psi + \bar{\psi} m \psi$$

where A is the photon field given by an $i\mathbb{R} = \mathrm{Lie}(U(1))$-valued 1-form $A \in \Omega^1_M$ and a section $\psi : M \to S$ of the (here supposed to be) trivial spinor bundle $M \times \mathbb{C}^2$ on flat spacetime that represents the electron field. Once a basis for the space of connections is fixed, the A-dependent Dirac operator decomposes as a sum of a Dirac propagator $i\bar{\psi}\not{D}\psi$ and an interaction term $-e\bar{\psi}\gamma \circ A\psi$. The space of vertices and edges for this lagrangian are thus given by

$R_V = \{$  ,  $\}$; $R_E = \{$, $\}$.

The corresponding monomials in $\mathcal{A}^\bullet$ are

$\iota($$) = -e\bar{\psi}\gamma \circ A\psi$, $\iota($$) = -m\bar{\psi}\psi$,

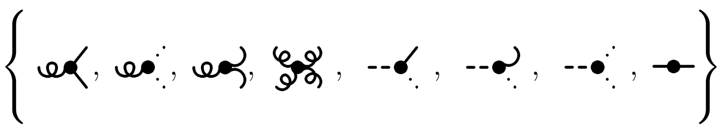
$\iota($$) = i\bar{\psi}\gamma \circ d\psi$, $\iota($$) = -dA * dA$.

with e and m the electric charge and mass of the electron (coupling constants), respectively.

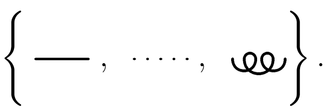
Example 15.2. Similarly, the lagrangian density of quantum chromodynamics SU(3)-gauge invariant Yang-Mills lagrangian (see section 10.3) of the form

$$L(A, \psi) = -\frac{1}{2}\langle F_A \wedge *F_A\rangle + \bar{\psi}\not{D}_A\psi + \bar{\psi}m\psi$$

where A is the quarks field, described by wiggly lines and ψ is the gluon field, described by straight lines. In addition, one has the ghost fields ω and $\bar{\omega}$, indicated by dotted lines, as well as BRST antifields K_ψ, K_A, K_ω and $K_{\bar{\omega}}$. Between the fields there are four interactions, three BRST-source terms, and a mass term for the quark. This leads to the following sets of vertices and edges,

$R_V =$ 

with the dashed lines representing the BRST-source terms, and

$R_E =$ 

Note that the dashed edges do not appear in R_E, i.e. the source terms do not propagate and in the following will not appear as internal edges of a Feynman graph.

Definition 15.1. A Feynman graph Γ is a pair composed of a set Γ_V of vertices each of which is an element in R_V and Γ_E of edges in R_E, and maps

$$\partial_0, \partial_1 : \Gamma_V \to \Gamma_E \cup \{1, 2, \ldots, N\},$$

that are compatible with the type of vertex and edge as parametrized by R_V and R_E, respectively. One excludes the case that ∂_0 and ∂_1 are both in $\{1, \ldots, N\}$. The set $\{1, \ldots, N\}$ labels the external lines, so that $\sum_j \mathrm{card}(\partial_j^{-1}(v)) = 1$ for all $v \in \{1, \ldots, N\}$. The set of external lines is $\Gamma_E^{ext} = \partial_0^{-1}(\{1, \ldots, N\}) \cup \partial_1^{-1}(\{1, \ldots, N\})$ and its complement Γ_E^{int} is the set of internal lines.

One sees the external lines Γ_E^{ext} as labeled $e_1, \ldots, e_N$ where $e_k = \partial_0^{-1}(k) \cup \partial_1^{-1}(k)$.

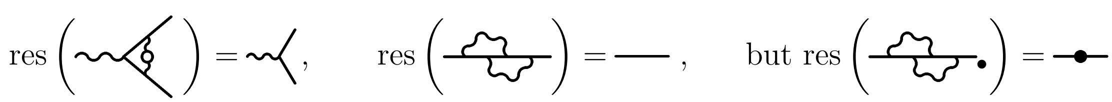
Definition 15.2. An automorphism of a Feynman graph Γ is given by as a pair of isomorphisms $g_V : \Gamma_V \to \Gamma_V$ and $g_E : \Gamma_E \to \Gamma_E$ that is the identity on Γ_E^{ext} that are compatible with the boundary maps, meaning that for all $e \in \Gamma_E$,

$$\cup_j g_V(\partial_j(e)) = \cup_j \partial_j(g_E(e)).$$

Moreover, we require g_V and g_E to respect the type of vertex/edge in the set R. The automorphism group $\mathrm{Aut}(\Gamma)$ of Γ consists of all such automorphisms; its order is called the symmetry factor of Γ and denoted $\mathrm{Sym}(\Gamma)$.

Definition 15.3. A Feynman graph is called a one-particle irreducible graphe (1PI) if it is not a tree and can not be disconnected by removal of a single edge. The residue $\mathrm{res}(\Gamma)$ of a Feynman graph Γ is obtained by collapsing all its internal edges and vertices to a point.

If a Feynman graph Γ has two external lines, both corresponding to the same field, we would like to distinguish between propagators and mass terms. In more mathematical terms, since we have vertices of valence two, we would like to indicate whether a graph with two external lines corresponds to such a vertex, or to an edge. A graph Γ with two external lines is dressed by a bullet when it corresponds to a vertex, i.e. we write $\Gamma_\bullet$. The above correspondence between Feynman graphs and vertices/edges is given by the residue. For example, we have:



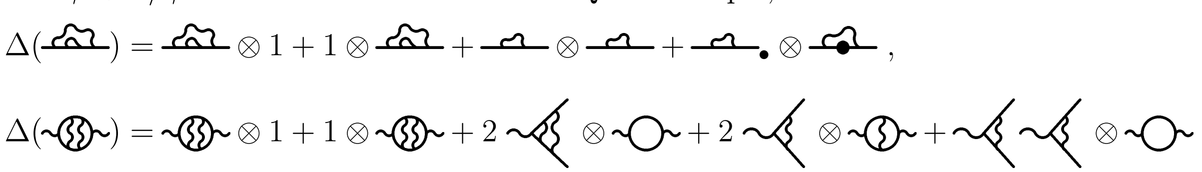
Definition 15.4. The diffeographism group scheme G is the affine group scheme whose algebra of functions A_G is the free commutative algebra generated by all 1PI graphs, with counit $\epsilon(\Gamma) = 0$ unless $\Gamma = \emptyset$, and $\epsilon(\emptyset) = 1$, coproduct

$$\Delta(\Gamma) = \Gamma \otimes 1 + 1 \otimes \Gamma + \sum_{\gamma \subsetneq \Gamma} \gamma \otimes \Gamma/\gamma,$$

and antipode given recursively by

$$S(\Gamma) = -\Gamma - \sum_{\gamma \subsetneq \Gamma} S(\gamma)\Gamma/\gamma.$$

In the above definition, the graph Γ/γ is obtained by contracting in Γ the connected components of the subgraph γ to the corresponding vertex/edge. If the connected component γ' of γ has two external lines, there are possibly two contributions corresponding to the valence two vertex and the edge; the sum involves the two terms $\gamma'_\bullet \otimes \Gamma/(\gamma' \to \bullet)$ and $\gamma' \otimes \Gamma/\gamma'$. To illustrate this with the QED example, one has



The Hopf algebra A_G is graded (meaning has a grading that respects the product and coproduct) by the loop number $L(\Gamma) := b_1(\Gamma)$ of a graph. This gives a decomposition

$$A_G = \oplus_{n\in\mathbb{N}} A_G^n.$$

It has also a multigrading indexed by the group $\mathbb{Z}^{(R_V)}$. For $R = R_E \coprod R_V$, one defined $m_{\Gamma,r}$ as the number of vertices/internal edges of type $r \in R$ appearing in Γ, and $n_{\gamma,r}$ the number of connected component of γ with residue r. For each $v \in R_V$, one defines the degree d_v by setting

$$d_v(\Gamma) = m_{\Gamma,v} - n_{\Gamma,v}.$$

The multidegree indexed by R_V, given by

$$d : A_G \to \mathbb{Z}^{(R_V)}$$

is compatible with the Hopf algebra structure. This gives a decomposition

$$A_G = \oplus_{\alpha\in\mathbb{Z}^{(R_V)}} A_G^\alpha.$$

For $\alpha \in \mathbb{Z}^{(R_V)}$, we denote $p_\alpha : A_G \to A_G^\alpha$ the corresponding projection.

15.2 The Riemann-Hilbert correspondence

Let $X = \mathbb{P}^1(\mathbb{C}) - \{x_0, \ldots, x_n\}$. A flat bundle on X is a locally free $\mathcal{O}_X$-module $\mathcal{F}$ with a connection

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega_X^1$$

whose curvature is zero, i.e., such that $\nabla_1 \circ \nabla = 0$. Equivalently, this is a $\mathcal{D}_X$-module that is also a locally free $\mathcal{O}_X$-module.

Given a flat bundle $(\mathcal{F}, \nabla)$ its space of solutions (also called horizontal sections) is the locally constant bundle (local systems) on X (locally isomorphic to $\mathbb{C}^n{}_X$ given by

$$\mathcal{F}^\nabla = \{f \in \mathcal{F},\ \nabla f = 0\}.$$

The functor

$$\begin{array}{rccc} \mathrm{Sol}: & \textsc{Flat Bundles} & \to & \textsc{Local systems} \\ & (\mathcal{F}, \nabla) & \to & \mathcal{F}^\nabla \end{array}$$

is fully faithful. Remark that a local system is equivalent to a representation of the fundamental group $\pi_1(X)$. One of Hilbert's problem is to show that it is essentially surjective,

meaning that to every representation V of $\pi_1(X)$, one can associate a differential equation whose corresponding local system is given by the local system

$$\underline{V} := V \underset{\pi_1(X)}{\times} \tilde{X}$$

where $\tilde{X} \to X$ is the universal covering of X. Results in this direction are called Riemann-Hilbert correspondences.

Remark that if $(\mathcal{F}, \nabla)$ is a local system, and if $\gamma : x \to y$ is a path in X (whose homotopy class is an element in the fundamental groupoid $\Pi(X)$), the analytic continuation of solutions of $(\mathcal{F}, \nabla)$ along γ gives a linear map, called the monodromy transformation

$$M_\gamma : \mathcal{F}_x^\nabla \to \mathcal{F}_y^\nabla$$

between the two finite dimensional $\mathbb{C}$-vector spaces give by the fibers of the solution space at the two given point. If

$$\mathrm{End}(\mathcal{F}^\nabla) \to X \times X$$

is the algebraic groupoid whose fiber over (x, y) is the space of linear morphisms $\mathrm{Hom}(\mathcal{F}_x^\nabla, \mathcal{F}_y^\nabla)$, there is a natural groupoid morphism

$$M : \Pi(X) \to \mathrm{End}(\mathcal{F}^\nabla)$$

given by the monodromy representation.

A theorem due to Cartier and independently by Malgrange is that the Zariski-Closure of the image of the monodromy representation M in the algebraic groupoid $\mathrm{End}(\mathcal{F}^\nabla)$ is equal to the differential galois groupoid. To explain this result, one has to define the differential Galois groupoid of $(\mathcal{F}, \nabla)$. It is the groupoid whose fiber over a point $(x, y) \in X \times X$ is the space

$$\mathrm{Isom}(\omega_x, \omega_y)$$

between the fiber functors

$$\omega_x, \omega_y : \langle(\mathcal{F}, \nabla)\rangle \to (\mathbb{C} - \textsc{Vector spaces}, \otimes)$$

given by

$$\omega_{x,y}(\mathcal{G}, \nabla) = \mathcal{G}_{x,y}^\nabla.$$

where $\langle(\mathcal{F}, \nabla)\rangle$ denotes the sub-monoidal category of the category of all flat bundles generated by the given flat bundle.

One can interpret the diffeographism groups of the Connes-Kreimer theory as some differential Galois groups associated to the defining equations of time-ordered exponentials

(iterated integrals). This relation between differential Galois theory and the Connes-Kreimer theory is due to Connes and Marcolli and explained in detail in [CM08].

We will now explain a particular version of the Riemann-Hilbert correspondence that is directly related to the renormalization procedure in the dimensional regularization scheme.

Let $\gamma : C \to G$ be a loop with values in an arbitrary complex Lie group G, defined on a smooth simple curve $C \subset \mathbb{P}^1(\mathbb{C})$. Let $C_\pm$ be the two complements of C in $\mathbb{P}^1(\mathbb{C})$, with $\infty \in C_-$. A Birkoff decomposition of γ is a factorization of the form

$$\gamma(z) = \gamma_-(z)^{-1}\gamma(z),\ (z \in C),$$

where $\gamma_\pm$ are (boundary values of) two holomorphic maps on $C_\pm$, respectively with values in G.

Definition 15.5. The value of $\gamma_+(z)$ at $z = 0$ is called the renormalized value of the given loop and the terme $\gamma_-(z)$ is called the counter-term associated to the given loop.

Theorem 40. *Let G be a group scheme whose hopf algebra is graded commutative (this means that G is pro-unipotent). Then any loop $\gamma : C \to G$ admist a Birkoff decomposition.*

15.3 Dimensional regularization

15.4 Connes-Kreimer's view of renormalization

We refer to [CM08] for a complete overview of the Connes-Kreimer theory and for a description of its relations with Riemann-Hilbert correspondence and motivic galois theory.

We consider here the effective action from section 14. It is given as a formal Legendre transform of the connected partition function that is itself given by the logarithm of the full partition function. One expresses this effective action in terms of the unrenormalized values of the Feynman graphs of the given theory.

If S is the original (gauge fixed) action functional, one writes

$$S_{eff}(\varphi) = S(\varphi) - \sum_{\Gamma \in 1\mathrm{PI}} \frac{U(\Gamma)(\varphi)}{\mathrm{Sym}(\Gamma)}$$

where

$$U(\Gamma)(\varphi) = \frac{1}{N!}\int_{\sum p_j = 0} \hat{\varphi(p_1)} \dots \hat{\varphi}(p_N) U(\Gamma(p_1, \dots, p_N)) \prod_j \frac{dp_j}{(2\pi)^d}.$$

where $U(\Gamma(p_1,\dots,p_N))$ actually denotes a multilinear distribution on the fields, called the bare value of the graph, and is specified by the Feynman rules of the theory. More precisely, one has

$$U(\Gamma(p_1,\dots,p_N)) = \int I_\Gamma(k_1,\dots,k_L,p_1,\dots,p_N)d^dk_1\dots d^dk_L$$

where $L = b_1(\Gamma)$ is the loop number of the graph, $k_i \in \mathbb{R}^d$ are the momentum variables assigned to the internal edges. The rational fraction $I_\Gamma(k_1,\dots,k_L,p_1,\dots,p_N)$ is obtained by applying the Feynman rules. Roughly speaking, this is done by assigning

- a fundamental solution (or its Fourier transform) of the linear differential operator that corresponds to each internal line (actually, one regularizes this solution to get a smooth function that approximate it),
- a momentum conservation rule to each vertex,
- a power of the coupling constant to each more than 3-valent vertex,
- a factor m^2 to each mass 2-point vertex,

The rational fraction is $I_\Gamma(k_1,\dots,k_L)$ is given by the the product of all these terms, except some distributional terms. It is supposed to be (when regularized) a smooth function. The problem is that if one sends the regularization parameters to 0, one gets divergencies. It is thus necessary to modify these bare values $U(\Gamma)$ to get meaningful limits at the end. This is the aim of the BPHZ renormalization procedure.

Now if G is the diffeographism group of a given field theory, the bare values $U(\Gamma)(z)$ associated to the Feynman graphs give algebra morphisms from A_G to $\mathbb{C}$ and thus point of G with values in $\mathbb{C}$. If C is a curve around $0 \in \mathbb{P}^1(\mathbb{C})$, one can define a loop in G by $\gamma(z)(\Gamma) := U(\Gamma)(z)$.

Applying the Birkoff decomposition theorem 40 to the above loop in the diffeographism group, one gets finite values from the dimensionally regularized bare Feynman amplitudes. Connes and Kreimer showed in [CK00] that these are exactly the values computed by physicists by the BPHZ renormalization procedure.

15.5 Zinn-Justin's equation and Slavnov-Taylor identities

Recall that we defined the BV laplacian, anti-bracket and quantum BV formalism in section 13.4.

The Zinn-Justin equation, also called the quantum master equation, is the equation that must be fulfilled by the deformed action functional S_{qm} associated to the classical master action S_{cm}, and the deformed observable A_{qm} associated to a classical observable $A_0 \in H^0(\mathcal{A}^\bullet)$, such that the corresponding functional integral with gauge fixing functional $\psi(\varphi)$

$$\langle A_0 \rangle = \int \alpha\left(\varphi, \frac{\delta\psi}{\delta\varphi}\right) e^{\frac{i}{\hbar} S_{qm}\left(\varphi, \frac{\delta\psi}{\delta\varphi}\right)} [d\varphi],$$

makes sense and does not depend of the choice of the extensions S_{qm} and A_{qm} and of the gauge fixing functional. This is the computation of the effective action associated to this functional integral that is achieved purely in terms of Feynman graphs in this section. The quantum master equation is usually expressed in terms of relations between the renormalized Green functions called the Slavnov-Taylor identities. The aim of this section is to explain, following van Suijlekom's work, how these identities are encoded in the Connes-Kreimer setting.

Let $\mathcal{A}_R := \mathbb{C}[[\lambda_{v_1}, \ldots, \lambda_{v_k}]] \otimes \mathcal{A}^\bullet$, $k = |R_V|$, be the algebra obtained from the BV $\mathcal{D}$-dg-algebra by extension of scalars to the coupling constants algebra. We now suppose that all the interaction (and mass) terms of the Lagrangian density $\mathcal{L}$ of the given theory are affected with a corresponding coefficient that is now a formal variable among the λ_{v_i}. Let $A_G^{Green} \subset A_G$ be the Hopf subalgebra generated by elements $p_\alpha(Y_v)$, $v \in R_V$ and $p_\alpha(G^e)$, $e \in R_E$, for $\alpha \in \mathbb{Z}^{R_V}$,

$$G^e = 1 - \sum_{\mathrm{res}(\Gamma)=e} \frac{\Gamma}{\mathrm{Sym}(\Gamma)}, G^v = 1 + \sum_{\mathrm{res}(\Gamma)=v} \frac{\Gamma}{\mathrm{Sym}(\Gamma)} \text{ and } Y_v := \frac{G^v}{\prod_\varphi} (G^\varphi)^{N_\varphi(v)/2}$$

and

$$G^{\varphi^i} = G^e,\ G^{\varphi_i^*} = (G^e)^{-1}.$$

We denote

$$G \longrightarrow\!\!\!\!\!\rightarrow G_{Green} = \mathrm{Spec}(A_G^{Green})$$

the corresponding quotient of the diffeomorphism group.

Theorem 41. *The algebra $\mathcal{A}_R$ is a comodule BV-algebra for the Hopf algebra A_G^{green}. The coaction $g : \mathcal{A}_R \to A_G^{Green} \otimes \mathcal{A}_R$ is given by ...*

The BV-ideal $\mathcal{I}_{BV}$ in $\mathcal{A}_R$ is generated by the antibracket (S, S) (that corresponds to the classical BV equation $(S, S) = 0$. There is a natural Hopf ideal in A_G^{Green} such that the corresponding subgroup of diffeographisms

$$G_{ST} \subset G_{Green}$$

acts on $\mathcal{A}_R/\mathcal{I}_{BV}$. The corresponding group is given by the sub-group of G that fixes the ideal $\mathcal{I}$ by the above coaction. This takes into account the Slavnov Taylor identities corresponding to the quantum master equation (here obtained by using only the classical master equation). To conclude geometrically, one has a diagram of diffeographism group actions on dg-$\mathcal{D}$-spaces

$$
\begin{tikzcd}
G_{ST} \arrow[r, tail] & G_{Green} & G \arrow[l, two heads] \\
\mathrm{Spec}(\mathcal{A}_R/\mathcal{I}_{BV}) \arrow[r, tail] \arrow[u, phantom, "\curvearrowleft" description] & \mathrm{Spec}(\mathcal{A}_R) \arrow[u, phantom, "\curvearrowleft" description]
\end{tikzcd}
$$

and G_{ST} is defined as the subgroup of G_{Green} that fixes the subspace of $\mathrm{Spec}(\mathcal{A}_R)$ that corresponds to the classical BV equation.

Chapter 16

Wilson's renormalization following Costello

We give here an overview of Costello's book [Cos10], that renormalizes perturbatively à la Wilson the euclidean Yang-Mills theories of order 1. This result is supposed to be physically equivalent to the renormalization of Yang-Mills by a triangular change of variable in the functional integral. Remark that Wilson's renormalization group method can be also applied in non-perturbative settings, where $\hbar$ is not a formal variable but a real number.

16.1 Energetic effective field theory

One usually considers that an effective quantum field theory is given by an effective action $S_{eff}(\Lambda, \varphi)$ that depends on a cut-off parameter Λ. Observables are then functionals

$$O : \mathcal{C}^\infty(M)_{\leq\Lambda} \to \mathbb{R}[[\hbar]]$$

on the space of fields (here smooth functions) whose expression in the basis of eigenvalues for the laplacian contain only eigenvalues smaller that the cut-off energy Λ. One clearly has an inclusion of observables at energy Λ' into observables at energy Λ for $\Lambda' \leq \Lambda$. The working hypothesis for Wilson's method is that the mean value of an observable O at energy Λ is given by the functional integral of O on the space $\mathcal{C}^\infty_{\leq\Lambda}$ (which is a usual integral since this space is finite dimensional)

$$\langle O \rangle := \int_{\mathcal{C}^\infty_{\leq\Lambda}} O(\varphi) e^{S_{eff}(\Lambda,\varphi)} [d\varphi]$$

where $S_{eff}(\Lambda, \varphi)$ is a functional called the effective action. The equality between mean values for $\Lambda' \leq \Lambda$ induces an equality

$$S_{eff}(\Lambda, \varphi_L) = \hbar \log \int_{\varphi_H \in \mathcal{C}^\infty(M)_{[\Lambda', \Lambda]}} \exp\left(\frac{1}{\hbar} S_{eff}(\Lambda, \varphi_L + \varphi_H)\right).$$

This equation is called the renormalization group equation. The relation of the effective action with the local action S (usual datum defining a classical field theory) is given by an ill-defined infinite dimensional functional integral

$$S_{eff}(\Lambda, \varphi_L) = \hbar \log \left(\int_{\varphi_H \in \mathcal{C}^\infty_{[\Lambda, \infty]}} \exp\left(\frac{1}{\hbar} S(\varphi_L + \varphi_H)\right)\right).$$

A drawback of this energetic approach is that it is global on the variety M because the eigenvalues of the laplacian give global information.

16.2 Effective propagators

The starting point is to remark that, if $\Delta + m^2$ is the linear operator for the equations of motion of the free euclidean scalar field, its inverse (free propagator) can be computed by using the heat kernel K_τ, that is the fundamental solution of the heat equation

$$\partial_\tau u(\tau, x) = \Delta u(\tau, x),$$

with Δ the laplacian (we here denote τ the time variable, since one think of it as the proper time of a euclidean particle). This amouts to describe $e^{-\tau(\Delta+m^2)}$ as the convolution by a function $K_\tau(x, y)$ (called the heat kernel) to obtain the propagator

$$P(x, y) = \frac{1}{\Delta + m^2} = \int_0^\infty e^{-\tau(\Delta+m^2)} := \int_0^\infty e^{-\tau m^2} K_\tau(x, y) d\tau.$$

The heat kernel can be interpreted as the mean value of a (Wiener) measure on the space of path on M, meaning that one can give a precise meaning to the equality

$$K_\tau(x, y) = \int_{f:[0,1] \to M, f(0)=x, f(1)=y} \exp\left(-\int_0^\tau \|df\|^2\right).$$

The propagator can then be interpreted as the probability for a particle to start at x and end at y along a given random path. This also permits to see Feynman graphs as random graphical paths followed by families of particles with prescribed interactions. It is also defined on a compact Riemannian manifold with the operator $\Delta_g + m^2$, where g is the metric.

Theorem 42. *The heat kernel has a small τ asymptotic expansion, in normal coordinates near a point (identify M with a small ball B), of the form*

$$K_\tau \equiv \tau^{-\dim(M)/2} e^{-\|x-y\|^2} \sum_{i \geq 0} t^i \Phi_i$$

with $\Phi_i \in \Gamma(B, M) \otimes \Gamma(B, M)$. More precisely, if K_τ^N is the finite sum appearing in the asymptotic expansion, one has

$$\|K_\tau - K_\tau^N\|_{C^l} = O(t^{N-\dim(M/2)-l}).$$

One can moreover define a weak notion of heat kernel for non-compact varieties, whose use do not pose existential problems.

The effective propagator of the theory is then given by a cut-off of the heat kernel integral

$$P_{\epsilon,L}(x, y) := \int_\epsilon^L e^{-\tau m^2} K_\tau(x, y) d\tau.$$

16.3 The renormalization group equation

Let $E \to M$ be a graded fiber bundle and $\mathcal{E} = \Gamma(M, E)$ be its space of sections (nuclear Fréchet space). We denote $\mathcal{E}^{\otimes n}$ the space

$$\mathcal{E}^{\otimes n} := \Gamma(M^n, E^{\boxtimes n}).$$

Definition 16.1. A (not necessarily local) functional is an element of the space $\mathcal{O}$ of formal power series on $\mathcal{E}$, given by

$$\mathcal{O} = \prod_{n \geq 0} \mathrm{Hom}(\mathcal{E}^{\otimes n}, \mathbb{R})^{S_n},$$

vwhere morphisms are supposed to be multilinear continuous maps. We denote $\mathcal{O}^+[[\hbar]] \subset \mathcal{O}_\mathcal{E}[[\hbar]]$ the space of functionals I that are at least cubic modulo $\hbar$. A local functional is an element of the space $\mathcal{O}_X^{loc}$ of functionals that can be written

$$\Phi = \sum_n \Phi_n$$

with $\Phi_n : \mathcal{E}^{\otimes n} \to \mathbb{R}$ of the form

$$\Phi_n(e_1, \ldots, e_n) = \sum_{j=1}^k \int_M (D_{1,j} e_1) \ldots (D_{n,j} e_n) d\mu$$

with the $D_{i,j} : \mathcal{E} \to \mathcal{C}^\infty_M$ some linear differential operators. One can identify these local functionals to global sections of the tensor product over $\mathcal{D}_M$ of the algebra of formal power series on jets by the maximal differential forms on M.

One has a natural algebra structure on $\mathcal{O}_X$.

Let $I \in \mathcal{O}^+[[\hbar]]$ be a functional and denote

$$I = \sum_{i,k} \hbar^i I_{i,k}$$

with $I_{i,k}$ homogenous of degree k in $\mathcal{O}$. We denote $\mathrm{Sym}^n\mathcal{E}$ the space of S_n-invariants in $\mathcal{E}^{\boxtimes n}$.

Definition 16.2. Let $P \in \mathrm{Sym}^2\mathcal{E}$ and $I \in \mathcal{O}^+[[\hbar]]$ be given. Let ∂_P be the contraction with P. The P-renormalization group image of I is given by the functional

$$W(P, I) = \hbar \log(\exp(\hbar\partial_P)\exp(I/\hbar)) \in \mathcal{O}^+[[\hbar]].$$

Example 16.1. If one starts with the standard local interaction functional $I \in \mathrm{Hom}(\mathcal{C}^\infty(M^3), \mathbb{R})$, given by

$$I = \int_M \varphi^3(x)dx = \int_{M^3} \varphi(x)\varphi(y)\varphi(z)\delta_{\Delta_{123}}dxdydz,$$

of the φ^3 theory (where $\Delta_{123} = M \subset M^3$ is the diagonal) and $P(x, y) \in \mathcal{C}^\infty(M \times M)$ is symmetric in x and y, the contraction of I by P is the distribution $\partial_P(I) \in \mathcal{C}^\infty(M)'$ defined as

$$\langle\partial_P(I), \varphi\rangle = \int_M P(x, x)\varphi(x)dx = \int_{M^3} P(x, y)\varphi(z)\delta_{\Delta_{123}}dxdydz.$$

16.4 Effective theory with interaction (parametrices)

Definition 16.3. Let D be a generalized laplacian on a graded bundle $E \to M$. A parametrix for the operator D is a distributional section $P \in \mathrm{Hom}(\mathcal{E}^{\otimes 2}, \mathbb{R})$ of $E \boxtimes E \to M \times M$, which is symmetric, smooth away from the diagonal, and such that

$$(D \otimes 1)P - K_0 \in \Gamma_{\mathcal{C}^\infty}(M \times M, E \boxtimes E),$$

where K_0 is the time zero heat operator (δ_M for the laplacian on flat space).

For example, if $P(0, L)$ is the heat kernel propagator, $P = P(0, L)$ is a parametrix. The difference $P - P'$ between two parametrices is a smooth function. Parametrices are partially ordered by their support.

Definition 16.4. A function $J \in \mathcal{O}(\mathcal{E})$ has smooth first derivative if the continuous linear map

$$\mathcal{E} \to \mathcal{O}(\mathcal{E}), \varphi \mapsto \frac{\partial J}{\partial \varphi}$$

extends to the space $\mathcal{E}'$ of distributional sections of $\mathcal{E}$.

Definition 16.5. A quantum field theory is a collection of functionals

$$I[P] \in \mathcal{O}^+(\mathcal{E})[[\hbar]]$$

indexed by parametrices, such that

1. (Renormalization group equation) If P, P' are parametrices, then

   $$W(P - P', I[P']) = I[P].$$

   This expression makes sense , because $P - P'$ is smooth in $\Gamma(M \times M, E \boxtimes E)$.

2. (locality) For any (i, k), the support $\mathrm{supp}(I_{i,k}[P]) \subset M^k$ can be made as close as we like to the diagonal by making the parametrix P small (for any small neighborhood U of the diagonal, there exists a parametrix such that the support is in U for all $P' \leq P$).

3. The functionals $I[P]$ all have smooth first derivative.

One denotes $\mathcal{T}^{(n)}$ the space of theories defined modulo $\hbar^{n+1}$.

If $I[L]$ is a family of effective interactions in the length formulation,

$$I[P] = W(P - P(0, L), I[L])$$

gives a quantum field theory in the parametrix sense.

Remark that the effective action functional is given by

$$S_{eff}[P] = P(\varphi \otimes \varphi) + I[P](\varphi).$$

16.5 The quantization theorem (parametrices)

The variations of periods are functions $f(t)$ on $]0,\infty[$ of the form

$$f(t) = \int_{\gamma_t} \omega_t$$

with ω a relative logarithmic differential form on a family of varieties $(X, D) \to U$ with normal crossing divisor defined on an open subset of the affine line $\mathbb{A}^1_{\mathbb{R}}$ that contains $]0,\infty[$. We denote $\mathcal{P} = \mathcal{P}(]0,\infty[)$ the set of period variations. These are smooth functions of t. We denote $\mathcal{P}_{\geq 0} \subset \mathcal{P}$ the sub-space of period variations that have a limit at 0.

Definition 16.6. A renormalization scheme is a choice of a complementary subspace (singular parts) $\mathcal{P}_{<0}$ to $\mathcal{P}_{\geq 0}$ in $\mathcal{P}$, i.e., of a direct sum decomposition

$$\mathcal{P} = \mathcal{P}_{\geq 0} \oplus \mathcal{P}_{<0}.$$

The singular part of a period variation for this scheme is its projection

$$\mathrm{sing} : \mathcal{P} \to \mathcal{P}_{<0}.$$

Theorem 43. *The space $\mathcal{T}^{(n+1)} \to \mathcal{T}^{(n)}$ is canonically equiped with a principal bundle structure under the group $\mathcal{O}_X^{loc}$ of local functionals. Moreover, $\mathcal{T}^{(0)}$ is canonically isomorphic to the space $\mathcal{O}^{+,loc}$ of local functionals that are at least cubic modulo $\hbar$.*

Proof. If $I[P]$ and $J[P]$ are two theories that coincide modulo $\hbar^{n+1}$, then

$$I_{0,*}[P] + \delta\hbar^{-(n+1)}(I[P] - J[P]) \in \mathcal{O}[\delta]/\delta^2$$

define a tangent vector to the space $\mathcal{T}^{(0)}$ of classical theories, that is canonically isomorphic to $\mathcal{O}^{+,loc}$. To prove the surjectivity, one needs to show that if

$$I[P] = \sum_{i,k} \hbar^i I_{i,k}[P] \in \mathcal{T}^{(n)}$$

is a theory defined modulo $\hbar^{n+1}$, there exist counter-terms

$$I^{CT}(\epsilon) = \sum_{i,k} \hbar^i I^{CT}_{i,k}(\epsilon) \in \mathcal{O}^{loc}[[\hbar]] \otimes \mathcal{P}_{<0}$$

such that

$$I[P] := \lim_{\epsilon \to 0} W(P - P(0,\epsilon), I - I^{CT}(\epsilon))$$

exists and defined a theory in $\mathcal{T}^{(n+1)}$. Recall that the renormalization group equation reads

$$W(P-P',I)=\hbar\log(\exp(\hbar\partial_{P-P'})\exp(I/\hbar))$$

for P and P' parametrices. One has by construction

$$W(P-P',W(P'-P",I[P"]))=W(P-P",I[P"])$$

and decomposing in homogeneous components in $\hbar$ and $\mathcal{E}$, one also has

$$W(P-P',I)_{i,k}=W(P-P',I_{<(i,k)})_{i,k}+I_{i,k}.$$

Suppose that the limit of

$$W_{<(i,k)}\left(P-P(0,\epsilon),I-\sum_{(r,s)<(i,k)}\hbar^r I^{CT}_{r,s}(\epsilon)\right)$$

as ϵ tends to 0 exists. Using the above relations, one have to set

$$I^{CT}_{(i,k)}(\epsilon):=\mathrm{sing}_\epsilon\left(W_{(i,k)}(P-P(0,\epsilon),I-\sum_{(r,s)<(i,k)}\hbar^r I^{CT}_{(r,s)}(\epsilon))\right).$$

One has to who that this expression does not depend on P. This can be proved in the case $P=P(0,L)$ by showing that derivative in L of what is inside the singular part is non-singular: it is given by graphs with vertices containing small degree expressions that are non-singular by induction and one just add a propagator $P(\epsilon,L)$ between them that is also non-singular. Some more work is needed to prove the locality axiom. $\square$

16.6 The quantum BV formalism

Definition 16.7. A free BV theory is given by

1. A $\mathbb{Z}$-graded vector bundle E on M, whose graded space of global sections is denoted $\mathcal{E}$.

2. An odd antisymmetric pairing of graded vector bundles

$$\langle,\rangle_{loc}:E\otimes E\to\mathrm{Dens}(M)[-1]$$

   that is supposed to be fiberwise nondegenerate.

3. A differential operator $Q : \mathcal{E} \to \mathcal{E}[1]$ of square zero that is anti-autoadjoint for the given pairing such that $H^*(\mathcal{E}, Q)$ is finite dimensional.

4. a solution $I \in \mathcal{O}^+_{loc}$ of the classical master equation
$$Q(I) + \frac{1}{2}\{I, I\} = 0.$$

A gauge fixation on a BV theory is the datum of an operator
$$Q^{GF} : \mathcal{E} \to \mathcal{E}[-1]$$
such that

1. Q^{GF} is odd, of square zero and autoadjoint,

2. the commutator
$$D := [Q, Q^{GF}]$$
is a (generalized) laplacian.

One thinks of Q^{GF} as a homological version of the usual gauge fixing, that gives a way to fix a lagrangian subspace in $\mathcal{E}$ on which the quadratic form $\int \varphi D \varphi$ is non-degenerate.

Definition 16.8. A parametrix Φ for a BV theory is a symmetric distributional section of $(E \boxtimes E)[1]$ on $M \times M$ closed under the differential $Q \otimes 1 + 1 \otimes Q$, with proper support and such that $(D \otimes 1)\Phi - K_0$ is a smooth section of $E \boxtimes E$. The propagator of a parametrix is
$$P(\Phi) = (Q^{GF} \otimes 1)\Phi$$
and its fake heat kernel is
$$K(\Phi) = K_0 - (D \otimes 1)\Phi.$$

The difference between two parametrix is smooth because it is in the kernel of the elliptic operator D.

The effective BV laplacian is defined as the insertion of $K(\Phi)$ by
$$\Delta_\Phi := -\partial_{K(\Phi)} : \mathcal{O} \to \mathcal{O}.$$
The ill-defined BV laplacian would be
$$\Delta = \lim_{L \to 0} \Delta_{P(0,L)}.$$

One defined a bracket $\{,\}_\Phi$ on $\mathcal{O}$ by the formula
$$\{\alpha, \beta\}_\Phi = \Delta_\Phi(\alpha\beta) - (\Delta_\Phi \alpha)\beta - (-1)^{|\alpha|}\alpha \Delta_\Phi \beta$$
that measures the obstruction for Δ_Φ to be a graded-derivation.

Definition 16.9. A functional $I[\Phi] \in \mathcal{O}^+[[\hbar]]$ satisfies the scale Φ quantum master equation if

$$QI[\Phi] + \{I[\Phi], I[\Phi]\}_\Phi + \hbar\Delta_\Phi I[\Phi] = 0.$$

One can also show that

$$[\partial_{P(\Psi)-P(\Phi)}, Q] = \partial_{K_\Psi} - \partial_{K_\Phi} = \Delta_\Phi - \Delta_\Psi$$

so that $\partial_{P(\Psi)-P(\Phi)}$ defines a homotopy operator between the BV laplacians at different scales. Remark that if $I[\Phi]$ satisfies the scale Φ quantum master equation, then

$$I[\Psi] = W(\Psi - \Phi, I[\Phi])$$

satisfies the scale Ψ quantum master equation.

Definition 16.10. A BV theory is the datum of a family $I[\Phi] \in \mathcal{O}[[\hbar]]$ such that

1. the collection defines a theory in the classical sense.
2. the collection fulfils the quantum master equation.

Theorem 44. *The obstruction to lift a solution to quantum master up to order n to a solution to quantum master up to order $n+1$ is given by a cohomology class in the classical local BV algebra equiped with the differential $\{S_{BV}, .\}$.*

Proof. Let $I[P]$ be a theory defined up to order $\hbar^{n+1}$ that fulfils quantum BV and $\tilde{I}[P]$ be a theory defined up to order $\hbar^{n+2}$. Let

$$O_{n+1}[P] := \hbar^{-(n+1)}(Q(\tilde{I} + \frac{1}{2}\{I, I\}_P + \hbar\Delta_P(\tilde{I})$$

be the obstruction for I to fulfil the QME. Let $\tilde{J}$ be another lifting and $J := \hbar^{-(n+1)}(\tilde{I} - \tilde{J})$. □

16.7 Effective BV theory (length version)

If one denotes $\mathcal{E}^!$ the sections of $E^\vee \otimes \mathrm{Dens}(M)$, the pairing defines

$$\mathcal{E}^! \to \mathcal{E}$$

that one can compose with the gauge fixing $Q^{GF} : \mathcal{E} \to \mathcal{E}$ to get $D' : \mathcal{E}^! \to \mathcal{E}$. The pair (D, D') defines a free theory in the usual sense to which one can apply the preceeding results.

The heat kernel $K_l \in \mathcal{E} \boxtimes \mathcal{E}$ for the operator e^{-lD}, allows to define the effective propagator by

$$P(\epsilon, L) = \int_\epsilon^L (Q^{GF} \otimes 1).K_l dl.$$

One associates to the heat kernel K_l the effective BV laplacian

$$\Delta_l = -\partial_{K_l} : \mathcal{O}(\mathcal{E}) \to \mathcal{O}(\mathcal{E}).$$

The BV laplacian, if it would exist, would be given by

$$\Delta := \lim_{l \to 0} \Delta_l.$$

Definition 16.11. An effective BV theory assocaited to a local BV theory I is an effective theory $\{I_L\}$ associated to the gauge fixed I_{BV} that moreover fulfils the quantum master equation

$$(Q + \hbar \Delta_L) e^{I/\hbar} = 0.$$

By the above theorem for theories without symmetries, one can allways associate an effective theory to the gauge fixed local theory (that has no symmetries anymore). However, there exist obstructions for this to be constructed and to also fulfil the quantum master equation More exactly, if one looks at the application

$$\mathcal{T}_{BV}^{n+1} \to \mathcal{T}_{BV}^n$$

between theories defines up to order $n+1$ in $\hbar$ and theories defined to order n, the obstruction to its lifting is of local cohomological nature, i.e., is given by a simplicial map

$$O_{n+1} : \mathcal{T}^n \to \mathcal{O}_{loc}$$

for families of theories parametrized by $\Omega^*(\Delta^*)$.

16.8 Effective theory with interaction (length version)

Definition 16.12. An effective quantum field theory with interaction is the datum of a family

$$I \in \mathcal{O}_X^+(\mathcal{C}^\infty_{]0,\infty[})[[\hbar]]$$

of functionals $I_L \in \mathcal{O}_X^+$, called effective interactions, such that

1. The renormalization group equation

$$I_{L'} = W(P(L, L'), I_L)$$

   is fulfilled.

2. Each $I_{i,k,L}$ has an asymptotic expansion for small L of the form

$$I_{i,k,L}(\varphi) \cong \sum \psi_r(\varphi) f_r(L)$$

   where $\psi_r \in \mathcal{O}_X^{loc}$ are local functionals.

We denote $\mathcal{T}^{(\infty)}(X)$ the space of theories and $\mathcal{T}^{(n)}(X)$ the space of theories defined modulo $\hbar^{n+1}$.

Recall that if f/g_1has a finite limit c_1, one says that f has principal part $c_1 g_1$. By induction, one defines an asymptotic expansion

$$f \cong \sum c_i g_i$$

as a series such that $f - \sum_{i<n} c_i g_i$ has as principal part $c_n g_n$. In fact, Costello uses as a definition that for all R, there exists an increasing sequence of integers tending to infinity (d_R) such that

$$\lim_{\epsilon \to 0} L^{d_R} \left(I_{i,k,L}(\varphi) - \sum_{r=0}^{R} \psi_r(\varphi) f_r(L) \right) = 0.$$

16.9 Feynman graph expansion of the renormalization group equation

Definition 16.13. A stable graph is a graph γ, with external edges, and for each vertex v, an element $g(v) \in \mathbb{N}$ called the genus of the vertex, with the property that any vertex of genus 0 is at least trivalent.

If $P \in \mathrm{Sym}^2 X$ (a propagator) and $I \in \mathcal{O}_X^+[[\hbar]]$ (an interaction), for every stable graph γ, one defines

$$w_\gamma(P, I) \in \mathcal{O}_X$$

by the following.

Let $T(\gamma)$ be the set of tails, $H(\gamma)$ be the set of half edges, $V(\gamma)$ the set of vertices and $E(\gamma)$ the set of internal edges. We denote $b_1(\gamma)$ the first betti number of the graph. The

tensor product of interactions $I_{i,k}$ at vertices of γ with valency k and genus i define an element of

$$\mathrm{Hom}(X^{\boxtimes H(\gamma)}, \mathbb{A}^1_{\mathbb{R}}).$$

One can define an element of

$$X^{\boxtimes 2E(\gamma)} \boxtimes X^{\boxtimes T(\gamma)} \cong X^{\boxtimes H(\gamma)}$$

by associating to each internal edge a propagator and to each tail a field $\varphi \in X$. The contraction of these two elements given an element

$$w_\gamma(P, I)(\varphi) \in \mathbb{A}^1_{\mathbb{R}}.$$

To say it differently, propagators give an element of $X^{\boxtimes 2E(\gamma)}$ that can be contracted to get

$$w_\gamma(P, I) \in \mathrm{Hom}(X^{\boxtimes T(\gamma)}, \mathbb{A}^1_{\mathbb{R}}).$$

One can thus define

$$W(P, I) = \sum_\gamma \frac{1}{|\mathrm{Aut}(\gamma)|} \hbar^{b_1(\gamma)} w_\gamma(P, I) \in \mathcal{O}_X^+[[\hbar]].$$

Example 16.2. Take a scalar φ^3 theory with

$$I = I_{3,0} = \int_{M^3} \varphi_1(x)\varphi_2(y)\varphi_3(z)\delta_\Delta(x, y, z)d\mu_3 := \int_M \varphi_1(x)\varphi_2(x)\varphi_3(x)d\mu,$$

where $\Delta : M \subset M^3$ is the diagonal. The effective propagator $P(\epsilon, L)$ for the laplacian is a distribution on M^2, that is given by a smooth function on M^2, leaving in $\mathcal{C}^\infty(M^2, \mathbb{R}^{\boxtimes 2}) = \mathcal{C}^\infty(M^2)$. One can also see it as an element of the tensor product $\mathcal{C}^\infty(M) \otimes \mathcal{C}^\infty(M)$, i.e., of $X^{\boxtimes 2}$. One associates to the vertex of valency 3 simply

$$I_{3,0} \in \mathrm{Hom}(X^{\boxtimes 3}, \mathbb{A}^1_{\mathbb{R}}).$$

If one has a graph with only one propagator and two vertices of valency 3 and genus 0, one associate to it the contraction of the tensor product

$$I_{3,0} \otimes I_{3,0} \in \mathrm{Hom}(X^{\boxtimes 6}, \mathbb{A}^1_{\mathbb{R}})$$

with propagator $P = P(\epsilon, L) \in X^{\boxtimes 2}$, that gives a morphism

$$w(P, \gamma) \in \mathrm{Hom}(X^{\boxtimes 4}, \mathbb{A}^1_{\mathbb{R}}).$$

The explicit expression for $w(P,\gamma)$ is given by

$$\begin{array}{rcl} w(P,\gamma)(\varphi_1,\dots,\varphi_6) & := & \int_{M^6}\varphi_1(x_1)\varphi_2(x_2)P(x_3,x_4)\varphi_5(x_5)\varphi_6(x_6)\delta_{\Delta_{123}}(x_i)\delta_{\Delta_{456}}(x_i)d\mu_6 \\ & := & \int_{M^2}\varphi_1(x_1)\varphi_2(x_1)P(x_1,x_2)\varphi_5(x_2)\varphi_6(x_2)d\mu_2, \end{array}$$

were $\Delta_{123} : M \subset M^6$ and $\Delta_{456} : M \subset M^6$ are the diagonal maps into the indexed subspaces. Remark that this new functional $w(P,\gamma)$ is not supported on the diagonal $\Delta_{123456} : M \subset M^6$, so that it is not anymore a local functional.

One can associate to $P \in \mathrm{Sym}^2 X$ a differential operator

$$\partial_P : \mathcal{O}_X \to \mathcal{O}_X$$

that gives a contraction in degree n. One then gets

$$W(P,I) = \hbar \log(\exp(\hbar\partial_P)\exp(I/\hbar)),$$

that is the usual equation for the definition of Feynman graphs.

16.10 The existence theorem

Theorem 45. *The space $\mathcal{T}^{(n+1)}(X) \to \mathcal{T}^{(n)}(X)$ is canonically equiped with a principal bundle structure under the group $\mathcal{O}_X^{loc}$ of local functionals. Moreover, $\mathcal{T}^{(0)}(X)$ is canonically isomorphic to the space $\mathcal{O}_X^{+,loc}$ of local functionals on X that are at least cubic modulo $\hbar$.*

The proof of this theorem (that can be stated independently of the renormalization scheme) is done by the proof (that needs a renormalization scheme) of the fact that each local interaction

$$I = \sum \hbar^i I_{i,k} \in \mathcal{O}_X^{loc}[[\hbar]]$$

allows to construct an effective field theory I_L in $\mathcal{T}^{(\infty)}(X)$. This is done by a lexicographic induction on the indices (i,k), by constructing counter-terms

$$I_{i,k}^{CT}(\epsilon) \in \mathcal{O}_X^{loc}(\mathcal{C}^\infty_{]0,\infty[}).$$

One uses for this a renormalization scheme.

The variations of periods are functions $f(t)$ on $]0,\infty[$ of the form

$$f(t) = \int_{\gamma_t} \omega_t$$

with ω a relative logarithmic differential form on a family of varieties $(X, D) \to U$ with normal crossing divisor defined on an open subset of the affine line $\mathbb{A}^1_{\mathbb{R}}$ that contains $]0, \infty[$. We denote $\mathcal{P}(]0, \infty[)$ the set of period variations. These are smooth functions of t. We denote $\mathcal{P}(]0, \infty[)_{\geq 0} \subset \mathcal{P}(]0, \infty[)$ the sub-space of period variations that have a limit at 0.

Definition 16.14. A renormalization scheme is a choice of a complementary subspace (singular parts) $\mathcal{P}(]0, \infty[)_{<0}$ à $\mathcal{P}(]0, \infty[)_{\geq 0}$ in $\mathcal{P}(]0, \infty[)$, i.e., of a direct sum decomposition

$$\mathcal{P}(]0, \infty[) = \mathcal{P}(]0, \infty[)_{\geq 0} \oplus \mathcal{P}(]0, \infty[)_{<0}.$$

The singular part of a period variation for this scheme is its projection on $\mathcal{P}(]0, \infty[)_{<0}$.

The homogeneous components of counter-terms are given by singular parts of the asymptotic expansion of the Feynman graphs

$$\mathrm{sing}_\epsilon w_\gamma(P(\epsilon, L), I).$$

If $I \in \mathcal{O}_X^{+,loc}[[\hbar]]$ is a local functional, one constructs an effective theory by renormalizing ultraviolet (small distances) by the renormalization group

$$I_L := W^R(P(0, L), I) := \lim_{\epsilon \to 0} W(P(\epsilon, L), I - I^{CT}(\epsilon)).$$

The counter-terms are chosen to make this limit exist.

Theorem 46. *There exists a series of local counter-terms*

$$I^{CT}_{i,k}(\epsilon) \in \mathcal{O}_{loc} \otimes \mathcal{P}(]0, \infty[)_{<0}$$

such that for all $L \in]0, \infty[$, the limit

$$I_L := W^R(P(0, L), I) := \lim_{\epsilon \to 0} W\left(P(\epsilon, L), I - \sum_{i,k} \hbar^i I^{CT}_{i,k}(\epsilon)\right)$$

exists.

Proof. One shows the existence of the series of counter-terms by a lexicographic induction on the indices (i, k). If $\Gamma_{i,k}$ denotes the set of stable graphs of genus i and with k external edges, we denote

$$W_{i,k}(P, I) = \sum_{\gamma \in \Gamma_{i,k}} w_\gamma(P(\epsilon, L), I) \text{ and } W_{<(i,k)}(P, I) = \sum_{(j,l)<(i,k)} \hbar^j W_{j,l}(P, I).$$

One has

$$W(P,I)=\sum \hbar^i W_{i,k}(P,I).$$

Recall that one has the equation

$$W(P,I)=\hbar\log(\exp(\hbar\partial_P)\exp(I/\hbar)),$$

that is the "usual" defining equation for Feynman graph expansions.

This equation implies two other fundamental identities, that are at the heart of effective constructions:

$$\begin{array}{rcl} W(P(L,L'),W(P(\epsilon,L),I)) & = & W(P(\epsilon,L'),I) \\ W_{i,k}(P(\epsilon,L),I) & = & W_{i,k}(P(\epsilon,L),I_{<(i,k)})+I_{i,k}. \end{array}$$

The induction hypothesis is that one can construct local functionals $I^{CT}_{r,s}(\epsilon)$ for $(r,s)<(i,k)$ such that

1. $I^{CT}_{r,s}(\epsilon)$ is a finite sum
$$I^{CT}_{r,s}(\epsilon)=\sum g_l(\epsilon)\Phi_l,$$
with g_l having only a finite pole at 0 and Φ_l local,

2. the limit
$$\lim_{\epsilon\to 0} W_{<(i,k)}\left(P(\epsilon,L), I-\sum_{(r,s)<(i,k)}\hbar^r I^{CT}_{r,s}(\epsilon)\right)$$
exists.

One then defines the new counter-term by the formula

$$I^{CT}_{i,k}(\epsilon):=\operatorname{sing}_\epsilon W_{i,k}\left(P(\epsilon,L), I-\sum_{(r,s)<(i,k)}\hbar^r I^{CT}_{r,s}(\epsilon)\right),$$

and apply the fundamental identities above to show the identity

$$W_{i,k}\left(P(\epsilon,L), I-\sum_{(r,s)<(i,k)}\hbar^r I^{CT}_{r,s}-\hbar^i I^{CT}_{i,k}(L,\epsilon)\right)=$$

$$W_{i,k}\left(P(\epsilon,L), I-\sum_{(r,s)<(i,k)}\hbar^r I^{CT}_{r,s}(\epsilon)\right)-\hbar^i I^{CT}_{i,k}(L,\epsilon),$$

which implies that the limit

$$\lim_{\epsilon \to 0} W_{\leq (i,k)} \left(P(\epsilon, L), I - \sum_{i,k} \hbar^i I_{i,k}^{CT}(\epsilon) - \hbar^i I_{i,k}^{CT}(L, \epsilon) \right)$$

exists. One here suppose that there is an asymptotic expansion at small ϵ so that one can take its singular part. □

16.11 The local renormalization group action

We now consider theories on $\mathbb{R}^n$, that is not compact. One thus has to restrict to functionals with some decreasing conditions at infinity for the homogeneous components.

Consider the operation

$$R_l : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$$

defined by

$$R_l(\varphi)(x) = l^{n/2-1} \varphi(lx).$$

If $I \in \mathcal{O}(\mathcal{S}(\mathbb{R}^n))$ is a funcitonal on $\mathcal{S}(\mathbb{R}^n)$, one defines

$$R_l^*(I)(\varphi) := I(R_{l^{-1}}(\varphi)).$$

Definition 16.15. The local renormalization group is the action

$$\mathcal{RG}_l : \mathcal{T}^{(\infty)} \to \mathcal{T}^{(\infty)}$$

one the space of theories defined by

$$\mathcal{RG}_l(\{I_L\}) := \{R_l^*(I_{l^2 L})\}.$$

One shows that the dependence on l is in $\mathbb{C}[\log l, l, l^{-1}]$.

Definition 16.16. A theory is called valuable if the terms of its effective action vary in $l^k (\log l)^r$ when $k \geq 0$ for $r \in \mathbb{Z}_{\geq 0}$. We denote $\mathcal{R}^{(\infty)}$ the space of valuable theories. It is called renormalizable if it is valuable and, term by term in $\hbar$, it has only a finite number of deformations, i.e., the tangent space $T_{\{I_L\}} \mathcal{R}^{(n)}$ is finite dimensional.

16.12 Application to pure Yang-Mills theory

In the Yang-Mills case of dimension 4, the graded space E is given by

$$\mathcal{E} = \Omega^0(M) \otimes \mathfrak{g}[1] \oplus \Omega^1(M) \otimes \mathfrak{g} \oplus \Omega^3(M) \otimes \mathfrak{g}[-1] \oplus \Omega^4(M) \otimes \mathfrak{g}[-2].$$

The elements of the four components of $\mathcal{E}$ are respectively called the ghosts, fields, antifields and antighosts and denoted X, A, $A^\vee$ and $X^\vee$.

The differential is given by

$$\Omega^0(M) \otimes \mathfrak{g} \xrightarrow{d} \Omega^1(M) \otimes \mathfrak{g} \xrightarrow{d*d} \Omega^3(M) \otimes \mathfrak{g} \xrightarrow{d} \Omega^4(M) \otimes \mathfrak{g}.$$

Recall that the Yang-Mills field is in $\Omega^1(M)\otimes\mathfrak{g}$ and the gauge symmetries are in $\Omega^0(M)\otimes\mathfrak{g}$.

The pairing on $\Omega^*(M) \otimes \mathfrak{g}$ is given by

$$\langle \omega_1 \otimes E_1, \omega_2 \otimes E_2 \rangle := \int_M \omega_1 \wedge \omega_2 (E_1, E_2)_{\mathfrak{g}}.$$

The Yang-Mills action functional is given by

$$S(A) = \frac{1}{2} \int_M \left[\langle F_A, F_A \rangle + \langle F_A, *F_A \rangle \right].$$

The first term is topological and the second is the usual Yang-Mills functional.

The solution of the classical master equation is obtained by adding to the lagrangian the ghost terms

$$\frac{1}{2} \langle [X, X], X^\vee \rangle$$

and the terms that give the action of the ghosts on fields

$$\langle dX + [X, A], A^\vee \rangle.$$

One has to use a first order formulation of Yang-Mills theory to have a gauge fixing (such that $D = [G, G^{GF}]$ is a laplacian and not in degree 4). This theory has two fields: one in $\Omega^1(M) \otimes \mathfrak{g}$ and the other autodual in $\Omega^2_+(M) \otimes \mathfrak{g}$.

The problem of proving the existence of an effective BV theory is then reduced to a computation of a cohomological obstruction. We refer to Costello's book for more details and for applications to renormalization of pure Yang-Mills in dimension 4.

Chapter 17

Renormalization (old version)

On peut considérer pour simplifier que la renormalisation est une méthode systématique pour:

1. associer à des intégrales de fractions rationnelles (diagrammes de Feynman) des valeurs finies (c'est la régularisation des divergences),

2. et ceci de manière compatible avec les relations entre les-dites fractions (qui sont encodées dans la structure de la fonction de partition), ce dernier point s'avérant le plus difficile à systématiser (c'est le coeur de la renormalisation).

17.1 Regularization

On se réfère ici à Folland [Fol08], section 7.1. Considérons l'intégrale divergente (φ étant une fonction lisse à support compact sur $\mathbb{R}$)

$$\int_0^\infty x^s \varphi(x) dx.$$

La régularisation est le processus d'associer à une telle intégrale infinie, une valeur finie.

17.1.1 Hadamard's finite part

Hadamard a donné une façon systématique d'extraire de cette intégrale divergente pour $s = -1$, un nombre fini appelée la partie finie, et notée

$$\langle \mathrm{pf} x_+^s, \varphi \rangle.$$

Cette méthode peut être adaptée à l'étude d'une intégrale divergente de la forme

$$\int_{\mathbb{R}^n} \frac{\varphi(x)}{P(x)} dx$$

avec $P(x)$ un polynôme à plusieurs variables et φ une fonction test, si on connaît suffisamment bien le lieu des zéros du polynôme P (théorème de résolution des singularités de Hironaka).

On écrit le développement de Taylor en zéro

$$\varphi(x) = \sum_{k<N} \varphi^{(k)}(0) \frac{x^k}{k!} + \varphi_N(x),$$

avec $Re(s) + N > 0$ et on retire à l'intégrande $x_+^s \varphi(x)$ la partie du développement de Taylor qui rend l'intégrale divergente, donnée par

$$I_N(\epsilon) \;=\; \int_\epsilon^\infty x^s \sum_{k<N} \varphi^{(k)}(0) \tfrac{x^k}{k!} dx \;.$$

La partie finie est alors donnée par

$$\langle \mathrm{pf} x_+^s, \varphi \rangle = \lim_{\epsilon \to 0} \left(\int_\epsilon^1 x^s \varphi(x) - I_N(\epsilon) \right).$$

Dans le cas où $s = -1$, il s'agit de la valeur principale de Cauchy, donnée par

$$\langle \mathrm{vp} |x|^{-1}, \varphi \rangle = \langle \mathrm{pf} |x|^{-1}, \varphi \rangle = \lim_{\epsilon \to 0} \int_{|x|>\epsilon} \frac{\varphi(x)}{x} dx.$$

17.1.2 Meromorphic continuation

Cette méthode de régularisation, qui généralise la valeur principale de Cauchy, peut être adaptée à l'étude d'une intégrale divergente de la forme

$$\int_{\mathbb{R}^n} \frac{\varphi(x)}{P(x)} dx$$

avec $P(x)$ un polynôme à plusieurs variables, par prolongement méromorphe de la distribution $\langle P(x)^s, . \rangle$ (grâce au polynôme de Sato-Berstein), en prenant le terme constant de la série de Laurent distributionnelle obtenue en $s = -1$. Ceci permet de démontrer l'existence d'une solution fondamentale à toute opérateur linéaire à coefficient constant,

par transformation de Fourier. C'est le théorème de Malgrange-Ehrenpreis (voir la section ??).

La fonction $|x|^s$ est localement intégrable sur $\mathbb{R}$ pour $Re(s) > -1$. Elle correspond donc à une distribution notée $\langle |x|^s, .\rangle$. L'équation fonctionnelle

$$\langle |x|^s, .\rangle = \frac{d}{dx}\left[\mathrm{sgn}(x).\langle \frac{|x|^{s+1}}{s+1}, .\rangle\right]$$

permet de prolonger méromorphiquement la distribution $\langle |x|^s, .\rangle$ au plan complexe par bandes successives.

17.2 Evaluation of individual Feynman graphs

17.2.1 The integrals of Feynman diagrams

On utilise à nouveau le livre de Folland [Fol08], section 7.1. On considère la solution fondamentale Δ_F de l'équation de Klein-Gordon pour un champ scalaire de masse m

$$(\Box + m^2)\Delta_F = \delta_0,$$

appelée propagateur de Feynman, et dont la transformée de Fourier est donnée par (la limite quand $\epsilon \to 0$ de)

$$\hat{\Delta}_F = \frac{1}{-q^2 + m^2 - i\epsilon}.$$

De manière similaire, on a les propagateurs de Dirac $\hat{\Delta}_D(q)$ pour un champ de Dirac de masse m, de jauge $\hat{\Delta}_G(q)$ pour un champ de jauge sans masse et de Proca $\hat{\Delta}_P(q)$ pour un champ de jauge de masse m donnés dans loc. cit.

On se donne un diagramme avec V sommets internes, I lignes internes et E lignes externes. On note $L = I - V + 1$ le nombre de boucles indépendantes. L'intégrale correspondant au diagramme est alors

$$C(p)\int \dots \int \prod_{sommets} \delta_s(p,q) \prod_{lignes\, internes} \hat{\Delta}_i(q_i) d^4 q_i.$$

C'est donc l'intégrale d'une fraction rationnelle sur $\mathbb{R}^I$, avec $\delta_s(p,q)$ l'annulation de la somme des moments au sommet donné s.

Remark 17.1. Dans la variable espace, cette intégrale correspond à une composition (par convolution) des opérateurs correspondants, qui sont les inverses d'opérateurs différentiels.

On peut donc la voire comme découlant du calcul de l'inverse d'un opérateur différentiel composé de plusieurs opérateurs différentiels de champs libres. C'est ce point de vue sur les diagrammes de Feynman qui à le plus de chance de se généraliser à des théories des champs sur des espaces courbés, puisqu'il ne nécessite pas l'utilisation de la transformée de Fourier, qui est une opération seulement disponible sur un espace plat.

Cette intégrale est généralement divergente à l'infini (hautes énergies, i.e., divergences ultraviolettes) mais aussi au voisinage des pôles des propagateurs (divergences infrarouges) si $\epsilon \to 0$ et la masse est nulle.

Après un changement de variable et la rotation de Wick (qui transforme les propagateurs Lorentzien en des propagateurs euclidiens, mais qu'on épargne au lecteur, voir [Fol08], pages 195-196), on obtient une intégrale de la forme

$$\int \cdots \int \prod_{i=1}^{I} \frac{P_i(q,p)}{|f_i(q,p)|^2 + m_i^2} d^{4L}q.$$

17.2.2 Power counting

Le comptage de puissance permet d'étudier la divergence des intégrales considérées dans la section précédente. On se place dans $\mathbb{R}^d$. Le degré superficiel de divergence est

$$D = \deg(R) + dL$$

avec R la fraction rationnelle à intégrer. On dit que le diagramme est superficiellement divergent si $\deg(R) + dL < 0$ (le passage en coordonnées polaires donne une intégrale clairement divergente).

L'exemple $R(x,y) = \frac{1}{(1+x^2)^2}$ sur $\mathbb{R}^2$ permet de voir qu'un diagramme superficiellement convergent n'est pas nécessairement convergent (pas de décroissance en y, ce qui se traduit par un sous-diagramme divergent). Le diagramme (i.e. l'intégrale) est convergente si et seulement si tous les sous-diagrammes sont superficiellement convergents.

17.2.3 Evaluation and regularization of Feynman diagrams

L'astuce de Feynman (ajout de variables) permet de ramener le dénominateur de l'intégrale

$$\int \cdots \int \prod_{i=1}^{I} \frac{P_i(q,p)}{|f_i(q,p)|^2 + m_i^2} d^{4L}q,$$

qui est un produit de formes quadratiques, au carré d'une expression linéaire en des variables supplémentaires.

La formule est

$$\frac{1}{c_1 \dots c_n} = (n-1)! \int_{[0,1]^n} \frac{\delta(1-\sum x_i)}{(\sum c_j x_j)^n} d^n x.$$

On obtient ainsi, par application, supposée possible, du théorème de fubini, l'intégrale

$$\int_{[0,1]^I} \left[\int_{\mathbb{R}^{4L}} \frac{P'(q', p, x)}{(|q'| + c^2(p,q))^{I+1}} d^{4L} \right] \delta(1 - \sum x_j) d^I x$$

avec c^2 une fonction quadratique en x et p. On évalue ensuite l'intégrale et on fait la rotation de Wick sur les moments externes (variables restantes) pour obtenir l'amplitude physique.

On utilise ensuite un changement de variable vers les coordonnées polaires, et la régularisation s'obtient par prolongement analytique en la variable de dimension d, essentiellement en utilisant la fonction Γ. Le développement de Taylor de la fonction obtenue en $(4-d)$ permet de calculer les quantités intéressantes pour la physique.

Exercices sheets for the course

Appendix A

Categories and universal properties

IMPA - Rio de Janeiro Master Course
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Categories and universal properties

Definition A.1. A category C is given by the following data:

1. a class $\mathcal{O}b(C)$ called the objects of C,
2. for each pair of objects X, Y, a set $\mathrm{Hom}(X, Y)$ called the set of morphisms,
3. for each object X a morphism $\mathrm{id}_X \in \mathrm{Hom}(X, X)$ called the identity,
4. for each triple of objects X, Y, Z, a composition law for morphisms

$$\circ : \mathrm{Hom}(X, Y) \times \mathrm{Hom}(Y, Z) \to \mathrm{Hom}(X, Z).$$

One supposes moreover that this composition law is associative, i.e., $f \circ (g \circ h) = (f \circ g) \circ h$ and that the identity is a unit, i.e., $f \circ \mathrm{id} = f$ et $\mathrm{id} \circ f = f$.

Definition A.2. A universal property [1] for an object Y of C is an explicit description (compatible to morphisms) of $\mathrm{Hom}(X, Y)$ (or $\mathrm{Hom}(Y, X)$) for every object X of C.

Example A.1. Here are some well known examples.

1. Sets whose objects are sets and morphisms are maps.
2. Grp whose objects are groups and morphisms are group morphisms.
3. Grab whose objects are abelian groups and morphisms are group morphisms.
4. Rings whose objets are commutative unitary rings and whose morphisms are ring morphisms.
5. Top whose objets are topological spaces and morphisms are continuous maps.

Principle 5. (Grothendieck)

[1] Every object has exactly two universal properties, but we will usually only write the simplest one.

Ce main interest in mathematics are not the mathematical objects,
but their relations
(i.e., morphisms).

Exercice 1. (Universal properties) What is the universal property of

1. the empty set?
2. the one point set?
3. $\mathbb{Z}$ as a group?
4. $\mathbb{Z}$ as a commutative unitary ring?
5. $\mathbb{Q}$ as a commutative unitary ring?
6. the zero ring?

To be more precise about universal properties, we need the notion of "morphism of categories".

Definition A.3. A (covariant) functor $F : C \to C'$ between two categories is given by the following data:

1. For each object X in C, an object $F(X)$ in C',
2. For each morphism $f : X \to Y$ in C, a morphism $F(f) : F(X) \to F(Y)$ in C'.

One supposes moreover that F is compatible with composition, i.e., $F(f \circ g) = F(f) \circ F(g)$, and with unit, i.e., $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$.

Definition A.4. A natural transformation φ between two functors $F : C \to C'$ and $G : C \to C'$ is given by the following data:

1. For each object X in C, a morphism $\varphi_X : F(X) \to G(X)$,

such that if $f : X \to Y$ is a morphism in C, $G(f) \circ \varphi_X = \varphi_Y \circ F(f)$.

We can now improve definition A.2 by the following.

Definition A.5. A universal property for an object Y of C is an explicit description of the functor $\mathrm{Hom}(X, .) : C \to \textsc{Sets}$ (or $\mathrm{Hom}(., X) : C \to \textsc{Sets}$).

The following triviality is at the heart of the understanding of what a universal property means.

Exercice 2. (Yoneda's lemma) Let $C^\vee$ be the "category" whose objects are functors $F : C \to \textsc{Sets}$ and whose morphisms are natural transformations.

1. Show that there is a natural bijection
$$\mathrm{Hom}_C(X, Y) \to \mathrm{Hom}_{C^\vee}(\mathrm{Hom}(X, .), \mathrm{Hom}(Y, .)).$$
2. Deduce from this the an object X is determined by $\mathrm{Hom}(X, .)$ uniquely up to a unique isomorphism.

Exercice 3. (Free objects) Let C be a category whose objects are described by finite sets equiped with additional structures (for example, $\textsc{Sets}$, $\textsc{Grp}$, $\textsc{Grab}$, $\textsc{Rings}$ or $\textsc{Top}$). Let X be a set. A free object of C on X is an object $L(X)$ of C such that for all object Z, one has a natural bijection
$$\mathrm{Hom}(L(X), Z) \cong \mathrm{Hom}_{\mathrm{Ens}}(X, Z).$$
Let X be a given set. Describe explicitely

1. the free group on X,
2. the free abelian group on X,
3. the free $\mathbb{R}$-module on X,
4. the free unitary commutative ring on X,
5. the free commutative unitary $\mathbb{C}$-algebra on X,
6. the free associative unitary $\mathbb{C}$-algebra on X.

Exercice 4. (Products and sums) The product (resp. the sum) of two objects X and Y is an object $X \times Y$ (resp. $X \coprod Y$, sometimes denoted $X \oplus Y$) such that for all object Z, there is a bijection natural in Z
$$\begin{gathered}\mathrm{Hom}(Z, X \times Y) \cong \mathrm{Hom}(Z, X) \times \mathrm{Hom}(Z, Y)\\ (\text{resp. } \mathrm{Hom}(X \textstyle\coprod Y, Z) \cong \mathrm{Hom}(X, Z) \times \mathrm{Hom}(Y, Z)).\end{gathered}$$
Explicitely describe

1. the sum and product of two sets,
2. the sum and product of two abelian groups, and then of two groups,

3. the sum and product of two unitary associative rings.

4. the sum and product of two unitary commutative rings.

Exercice 5. (Fibered products and amalgamed sums) The fibered product (resp. amalgamed sum) of two morphims $f : X \to S$ and $g : Y \to S$ (resp. $f : S \to X$ and $f : S \to Y$) is an object $X \times_S Y$ (resp. $X \coprod_S Y$, sometimes denoted $X \oplus_S Y$) such that for all object Z, there is a natural bijection

$$\begin{array}{rcl} \mathrm{Hom}(Z, X \times_S Y) & \cong & \{(h,k) \in \mathrm{Hom}(Z,X) \times \mathrm{Hom}(Z,Y) | f \circ h = g \circ k\} \\ \text{(resp. } \mathrm{Hom}(X \coprod_S Y, Z) & \cong & \{(h,k) \in \mathrm{Hom}(X,Z) \times \mathrm{Hom}(Y,Z) | h \circ f = k \circ g\}). \end{array}$$

1. Answer shortly the questions of the previous exercice with fibered products and amalgamed sums.

2. Let $a < b < c < d$ be three real numbers. Describe explicitely the sets

$$]a,c[\times_{]a,d[}]b,d[\qquad \text{and} \qquad]a,c[\coprod_{]b,c[}]b,d[.$$

3. Describe explicitely the abelian group

$$\mathbb{Z} \times_{\mathbb{Z}} \mathbb{Z}$$

where $f, g : \mathbb{Z} \to \mathbb{Z}$ are given by $f : n \mapsto 2n$ and $g : n \mapsto 3n$.

Exercice 6. (Projective limits) Let $(I, \leq)$ be a partially ordered set. A projective system indexed by I is a family

$$A_\bullet = ((A_i)_{i \in I}, (f_{i,j})_{i \leq j})$$

of objects and for each $i \leq j$, morphims $f_{i,j} : A_j \to A_i$ such that $f_{i,i} = \mathrm{id}_{A_i}$ and $f_{i,k} = f_{i,j} \circ f_{j,k}$ (such a data is a functor $A_\bullet : I \to C$ to the given category C). A projective limit for $A_\bullet$ is an object $\varprojlim_I A_\bullet$ such that for all object Z, one has a natural bijection

$$\mathrm{Hom}(Z, \varprojlim_I A_\bullet) \cong \varprojlim_I \mathrm{Hom}(A_i, Z)$$

where $\varprojlim_I \mathrm{Hom}(A_i, Z) \subset \prod_i \mathrm{Hom}(A_i, Z)$ denotes the families of morphisms h_i such that $f_{i,j} \circ h_j = h_i$. One defines inductive limits $\varinjlim A_\bullet$ in a similar way by interverting source and target of the morphims.

1. Show that product and fibered products are particular cases of this construction.

2. Describe the ring $\varprojlim_n \mathbb{C}[X]/(X^n)$.

3. Describe the ring $\mathbb{Z}_p := \varprojlim_n \mathbb{Z}/p^n\mathbb{Z}$.

Exercice 7. **(Localization)** Let A be a unitary commutative ring, $S \subset A$ a multiplicative subset (stable by multiplication and containing 1_A). The localization $A[S^{-1}]$ of A with respect to S is defined by the universal property

$$\mathrm{Hom}_{\mathrm{Rings}}(A[S^{-1}], B) = \{f \in \mathrm{Hom}_{\mathrm{Rings}}(A, B) | \forall s \in S,\ f(s) \in B^\times\},$$

where $B^\times$ is the set of invertible elements in the ring B.

1. Describe $\mathbb{Z}[1/2] := \mathbb{Z}[\{2^{\mathbb{Z}}\}^{-1}]$.

2. Is the morphism $\mathbb{Z} \to \mathbb{Z}[1/2]$ finite (i.e. is $\mathbb{Z}[1/2]$ a finitely generated $\mathbb{Z}$-module)? Of finite type (i.e. can $\mathbb{Z}[1/2]$ be described as a quotient of a polynomial ring over $\mathbb{Z}$ with a finite number of variables)?

3. Construct a morphism $\mathbb{Z}[1/2] \to \mathbb{Z}_3$ where $\mathbb{Z}_3$ are the 3-adiques integers defined in the previous exercice.

4. Does there exist a morphism $\mathbb{Z}[1/3] \to \mathbb{Z}_3$?

Appendix B

Differential geometry

IMPA - Rio de Janeiro — Master Course

Year 2009-2010

Differential geometry

Exercice 8. (Algebraic and smooth spaces) We denote $\mathrm{ALG}_{\mathbb{R}}$ the category of commutative unital algebras over $\mathbb{R}$. An algebra A is called smoothly closed if for every family $a_1, \ldots, a_n$ of elements of A and every smooth function $F : \mathbb{R}^n \to \mathbb{R}$, there exists $a \in A$ such that the function $\mathrm{Hom}(A, \mathbb{R}) \to \mathbb{R}$ defined by

$$F \circ (\mathrm{ev}_{a_1} \times \cdots \times \mathrm{ev}_{a_n}) = \mathrm{ev}_a.$$

We denote $\mathrm{ALG}_{sc,\mathbb{R}}$ the category of smoothly closed algebras. An algebra is called geometric if the the "Gelfand transform"

$$A \to \mathrm{Hom}_{\mathrm{SETS}}(\mathrm{Hom}_{\mathrm{ALG}_{\mathbb{R}}}(A, \mathbb{R}), \mathbb{R})$$

is injective. We denote $\mathrm{ALG}_{scg,\mathbb{R}}$ the category of geometric smoothly closed algebras.

1. Show that the affine space functor

$$\mathbb{A}^n : \mathrm{ALG}_{\mathbb{R}} \to \mathrm{SETS}$$

given by $\mathbb{A}^n(A) = A^n$ is representable (isomorphic to $\mathrm{Hom}(B, .)$ for a given algebra B.

2. What about the restrictions $\mathbb{A}^n : \mathrm{ALG}_{sc,\mathbb{R}} \to \mathrm{SETS}$ and $\mathbb{A}^n = \mathrm{ALG}_{scg,\mathbb{R}} \to \mathrm{SETS}$ of the affine space functor to the subcategories of smoothly closed and geometric smoothly closed algebras?

3. Let $F_1, \ldots, F_m \in \mathbb{R}[X_1, \ldots, X_n]$. Show that the space of solutions of $F_i = 0$ is representable.

4. Same question for the solution space of $F_1, \ldots, F_m \in \mathcal{C}^\infty(\mathbb{R}^n)$ with values in $\mathrm{ALG}_{scg,\mathbb{R}}$.

5. Show that the atlas definition of smooth morphisms is equivalent to the definition using spaces of points.

Exercice 9. (Affine super-spaces) Define the affine super-space as the functor

$$\mathbb{A}^{n|m} : \textsc{Alg}_{s,\mathbb{R}} \to \textsc{Sets}$$

from super-algebras to sets sending $A = A^0 \oplus A^1$ to $(A^0)^n \times (A^1)^m$.

1. Show that $\mathbb{A}^{n|m}$ is representable in $\textsc{Alg}_{s,\mathbb{R}}$.

2. Let $\textsc{Alg}_{sg,\mathbb{R}}$ be the subcategory of $\textsc{Alg}_{s,\mathbb{R}}$ given by super-algebras of the form

$$A = \mathcal{C}^\infty(S^0) \otimes_{\mathcal{C}^\infty(M)} \Gamma(M, \wedge^* S^1)$$

for $S = S^0 \oplus S^1 \to M$ a finite dimensional graded bundle on a smooth manifold M. Compute the restriction of $\mathbb{A}^{n|m}$ to this sub-category and show that it is also representable.

3. Describe morphisms $f : \mathbb{A}^{n|0} \to \mathbb{A}^{0|q}$ and $f : \mathbb{A}^{n|0} \to \mathbb{A}^{p,q}$.

4. Describe morphisms $f : \mathbb{A}^{0|1} \to \mathbb{A}^{n|0}$.

5. Describe the super-space of (internal) homomorphisms $\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, \mathbb{A}^{n|0})$ defined by

$$\underline{\mathrm{Hom}}(\mathbb{A}^{0|1}, \mathbb{A}^{n|0})(A) := \mathrm{Hom}(\mathbb{A}^{0|1} \times \mathrm{Spec}(A), \mathbb{A}^{n|0}).$$

6. Compute all morphisms of algebraic super-spaces

$$f : \mathbb{A}^{n|m} \to \mathbb{A}^{r|s}.$$

Appendix C

Groups and representations

IMPA - Rio de Janeiro — Master Course

Year 2009-2010

Groups and representations

Exercice 10. (Algebraic and Lie groups) We work with spaces defined on the category $\mathrm{Alg}_{\mathbb{R}}$ or $\mathrm{Alg}_{scg,\mathbb{R}}$ of real algebras or real smoothly closed geometric algebras. An algebraic (resp Lie) group space is a functor $G : \mathrm{Alg}_{\mathbb{R}} \to \mathrm{Grp}$ (resp. $G : \mathrm{Alg}_{scg,\mathbb{R}} \to \mathrm{Grp}$).

1. Show that the group space $\mathrm{GL}_n = \{(M, N) \in M_n^2,\ MN = NM = \mathrm{id}\}$ is an algebraic and a Lie group space.

2. Show that the map $(M, N) \mapsto M$ induces an isomorphism
$$\mathrm{GL}_n \cong \{M \in M_n,\ \det(M) \text{ invertible}\}.$$

3. Applying the local inversion theorem to the exponential map $\exp : M_n \to \mathrm{GL}_n$, show that GL_n is a smooth variety, i.e., a Lie group space covered by open subsets of $\mathbb{R}^{n^2}$.

Exercice 11. (Lie algebras) Recall that if $G : \mathrm{Alg}_{\mathbb{R}} \to \mathrm{Sets}$ is a group valued functor, one defines the space $\mathrm{Lie}(G)$ by
$$\mathrm{Lie}(G)(A) := \{g \in G(A[\epsilon]/(\epsilon^2)) |\, g = \mathrm{id} \mod \epsilon\}.$$

1. Show that the Lie algebra of GL_n is M_n.

2. Show that the operation $[.,.] : \mathrm{Lie}(\mathrm{GL}_n) \times \mathrm{Lie}(\mathrm{GL}_n) \to \mathrm{Lie}(\mathrm{GL}_n)$ given by
$$[h, k] = hkh^{-1}k^{-1}$$
identifies with the commutator of matrices.

3. Let $g \in \mathrm{GL}_n$ be fixed. Show that the operation $h \mapsto ghg^{-1}$ of g on M_n is the derivative of the operation $h \mapsto ghg^{-1}$ on GL_n.

4. Compute the Lie algebra of SL_2.

5. Compute the Lie algebra of SL_n.

6. Let (V, b) be a bilinear space and

$$\mathrm{Sim}(V, b) := \{g \in \mathrm{End}(V) | b(gv, gw) = b(v, w), \forall v, w \in V\}.$$

Show that $\mathrm{Sim}(V, b)$ can be extended to a group space.

7. Compute the Lie algebra of $\mathrm{Sim}(V, b)$.

8. Deduce from the above computation the description of the Lie algebra of the standard orthogonal and symplectic groups, that correspond to the bilinear forms ${}^t vAw$ for $A = I$ on $\mathbb{R}^n$ and $A = \left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$ on $\mathbb{R}^{2n}$ respectively.

Exercice 12. (Morphisms of algebraic groups)

1. Using the commutative diagram

$$\begin{tikzcd} \mathrm{GL}_1(\mathbb{R}[x, x^{-1}]) \arrow[r, "m^*"] \arrow[d, "f"'] & \mathrm{GL}_1(\mathbb{R}[x_1, x_2, x_1^{-1}, x_2^{-1}]) \arrow[d, "f"] \\ \mathrm{GL}_1(\mathbb{R}[x, x^{-1}]) \arrow[r, "m^*"] & \mathrm{GL}_1(\mathbb{R}[x_1, x_2, x_1^{-1}, x_2^{-1}]) \end{tikzcd}$$

induced by the multiplication morphism $m : \mathrm{GL}_1 \times \mathrm{GL}_1 \to \mathrm{GL}_1$, show that the image of $x \in \mathbb{R}[x, x^{-1}]^\times$ by f is a sum $\sum_{n \in \mathbb{Z}} a_n x^n$ with $a_n^2 = a_n$ for all n and $a_n a_m = 0$ if $n \neq m$.

2. Deduce from the above all the morphisms of group spaces $\mathrm{GL}_1 \to \mathrm{GL}_1$.

3. Let $\mathbb{G}_a(A) = A$ equiped with addition. Show that if $f : \mathbb{G}_a \to \mathbb{G}_a$ is a morphism, the image of the universal point $x \in \mathbb{R}[x]$ by the multiplication map fulfils

$$\sum_n a_n (x_1 + x_2)^n = \sum_n a_n x_1^n + \sum_m a_m x_2^m.$$

4. Deduce from the above that $\mathrm{End}(\mathbb{G}_a) = \mathbb{R}$.

Exercice 13. (Structure and representations of SL_2**)** Let $T = \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right)$ be the maximal torus of SL_2.

1. Compute the roots of SL_2, i.e., the non-trivial characters of T (morphisms $T \to \mathrm{GL}_1$) that occur in the adjoint representation of SL_2 on $\mathrm{Lie}(\mathrm{SL}_2)$.

2. For each root $r : T \to \mathrm{GL}_1$, construct a morphism

$$r : \mathbb{G}_a \to \mathrm{SL}_2.$$

3. Show that SL_2 is generated by the images of r and T. Describe their relations.

4. Let $\rho : \mathrm{SL}_2 \to \mathrm{GL}(V)$ be an irreducible representation of SL_2. By using the commutation relation between the roots and the generator a of T, compute the possible weights of T on V.

5. Show that every irreducible representation of SL_2 is isomorphic to $\mathrm{Sym}^n(V)$ with V the standard representation.

Exercice 14. (Super-algebraic groups) We work with superspaces defined on the category $\textsc{Alg}_{s,\mathbb{R}}$ of real super-algebras.

1. Show that one can define a super-group space structure on $\mathbb{R}^{1|1}$ by

$$(t_1, \theta_1).(t_2, \theta_2) = (t_1 + t_2 + \theta_1\theta_2, \theta_1 + \theta_2).$$

2. Recall that the free module of dimension $p|q$ on a super-algebra A is defined by

$$A^{p|q} := \mathbb{R}^{p|q} \otimes A \underset{\textsc{Vect}_{\mathbb{R}}}{=} (\mathbb{R}^p \otimes A^0 \oplus \mathbb{R}^q \otimes A^1)^0 \oplus (\mathbb{R}^q \otimes A^0 \oplus \mathbb{R}^p \otimes A^1)^1.$$

Show that the super space $\mathrm{GL}_{p|q}(A) = \mathrm{Aut}_{\textsc{Mod}(A)}(A^{p|q})$ is representable.

3. Show that the Berezinian gives a super-group morphism

$$\mathrm{GL}_{p|q} \to \mathrm{GL}_{1|0}.$$

Appendix D

Homotopical algebra

IMPA - Rio de Janeiro Master Course
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Homotopical algebra

Recall that the simplicial category Δ is the category whose objects are the finite ordered sets $[n] = [0, \ldots, n-1]$ and whose morphisms are increasing maps, and the category of simplicial sets SSets is the category of contravariant functors $\Delta^{op} \to$ Sets.

Exercice 15. (Cylinders and homotopies) The category Top of Hausdorf compactly generated topological spaces has an internal homomorphism object defined by $\underline{\mathrm{Hom}}(Y, X)$ to be the set of continuous maps between Y and X equiped with the topology generated by the subsets $V(U, K) = \{f : Y \to X,\ f(K) \subset U\}$ indexed by compacts $K \subset Y$ and opens $U \subset X$. If X is in Top, denote $\mathrm{Cyl}(X) = X \times [0, 1]$ and $\mathrm{Cocyl}(X) := \underline{\mathrm{Hom}}([0, 1], X)$

1. Show that for X and Y locally compact, the internal $\underline{\mathrm{Hom}}$ of Top fulfils its usual adjunction property with products, given by a functorial isomorphism
$$\mathrm{Hom}(Z \times X, Y) \cong \mathrm{Hom}(Z, \underline{\mathrm{Hom}}(X, Y)).$$
2. Deduce that Cyl and Cocyl are adjoint, meaning that
$$\mathrm{Hom}(\mathrm{Cyl}(X), Y) \cong \mathrm{Hom}(X, \mathrm{Cocyl}(Y)).$$
3. Show that Cyl is indeed a cylinder in Top with its standard model structure, meaning that $X \coprod X \to \mathrm{Cyl}(X)$ is a cofibration (closed embedding) and $\mathrm{Cyl}(X) \to X$ is a weak equivalence.
4. Show that $\mathrm{Cocyl}(X)$ indeed is a cocylinder in Top, meaning that $\mathrm{Cocyl}(X) \to X \times X$ is a fibration (lifting property with respect to $Y \times \{0\} \to Y \times [0, 1]$) and $X \to \mathrm{Cocyl}(X)$ is a weak equivalence (see [DK01], theorem 6.15).
5. Deduce that right and left homotopies coincide in Top.

Exercice 16. (Mapping cylinder and mapping cone) We refer to [DK01], 6.6 for this exercice. Let $f : X \to Y$ be a morphism in Top. Define the mapping path space P_f of f as the fiber product

$$\begin{tikzcd} P_f \arrow[r] \arrow[d] & \mathrm{Cocyl}(Y) \arrow[d, "p_0"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

and the mapping path fibration $p : P_f \to Y$ by $p(x, \alpha) = \alpha(1)$.

1. Show that there exists a homotopy equivalence $h : X \to P_f$ such that the diagram

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & P_f \arrow[dl, "p"] \\ Y & \end{tikzcd}$$

commutes.

2. Show that the map $p : P_f \to Y$ is a fibration.

3. Show that if $f : X \to Y$ is a fibration, then h is a relative (or fiber) homotopy equivalence on Y.

4. We now work in Top_* (see [DK01], 6.11 for this question). Let $E \to B$ be a fibration with fiber F. Let Z be the fiber of the mapping path fibration of $F \to E$. Show that Z is homotopy equivalent to the loop space $\Omega B := \underline{\text{Hom}}_{\text{Top}_*}(S^1, B)$.

Exercice 17. (Simplicial sets)

1. Prove the same statements as those in exercice 15 for the cylinder $\text{Cyl}(X) := X \times \Delta^1$ and cocylinder $\text{Cocyl}(X) := \underline{\text{Hom}}(\Delta^1, X)$ of simplicial sets.

2. Show that the geometric realisation and the singular simplex functors are adjoint.

3. Characterize fibrant simplicial spaces and show that if Y is a topological space, the singular simplex $S(Y)$ is fibrant.

Exercice 18. (Fibrations) Let X and Y be two topological spaces.

1. Show that a projection map $p : X \times Y \to X$ is a fibration.

2. Deduce that a fiber bundle $p : B \to X$ (locally isomorphic on X to a projection) is a fibration.

3. Show that if X is path connected, two fibers of a fibration $p : F \to X$ are homotopy equivalent.

Exercice 19. (Groupoids and simplicial spaces) A (small) groupoid is a category whose objects form a set X_0 and whose morphisms are all invertible.

1. Show that a groupoid can be described by a tuple $(X_1, X_0, s, t, \epsilon, m)$ composed of two sets X_1 and X_0, equiped with source, target, unit, and composition maps:

$$\begin{tikzcd} X_1 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & X_0 \arrow[l, bend right, "\epsilon"'] \end{tikzcd}, \qquad X_1 \underset{s,X_0,t}{\times} X_1 \xrightarrow{m} X_1,$$

and describe the condition on these data to give a groupoid.

2. Show that the families of n composable arrows can be described by the set

$$X_n := \underbrace{X_1 \underset{s,X_0,t}{\times} \dots \underset{s,X_0,t}{\times} X_1}_{n \text{ times}}.$$

3. Show the the map $[n] \mapsto X_n$ can be extended to a simplicial set, i.e., to a functor

$$X : \Delta^{op} \to \textsc{Sets}.$$

4. Describe the condition for a simplicial set $X : \Delta^{op} \to \textsc{Sets}$ to come from a groupoid by the above construction.

5. Describe the simplicial space BG associated to a group G, viewed as a groupoid with one object

$$\begin{tikzcd} G \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \{.\} \arrow[l, bend right, "\epsilon"'] \end{tikzcd}, \qquad G \times G \xrightarrow{m} G.$$

6. Show that BG is cofibrant.

Appendix E

Functional calculus

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Functional calculus

Exercice 20. (Calculus of variation in mechanics) Let $\pi : C = \mathbb{R} \times [0,1] \to [0,1] = M$ be the trivial bundle with base coordinate t and fiber coordinate x, whose sections are trajectories of a particle in a line (i.e., functions $\mathbb{R} \to \mathbb{R}$). We use polynomial functions $\mathcal{O}_M = \mathbb{R}[t]$ and $\mathcal{O}_C = \mathbb{R}[t,x]$. Let $L(t,x_i) \in \mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t,x_i]$ be a lagrangian density and $H \subset \Gamma(M,C)$ be the subspace of trajectories that start and end at two fixed points x_0 and x_1 with fixed derivates up to the order of derivative that appear in L. We denote

$$S(x) = \int_0^1 L(t, x(t), \partial_t x(t), \dots) dt$$

the corresponding action functional.

1. Compute the tangent space to $\Gamma(M,C)$ and show that the tangent space to H has fiber at $x \in H$ the space

$$T_x H = \{h \in \Gamma(M,C),\ h^{(k)}(0) = h^{(k)}(1) = 0 \text{ for all } k \leq \mathrm{order}(L)\}.$$

2. Let

$$D_1 := \frac{\partial}{\partial t} + \sum_{n=0}^{\infty} x_{n+1} \frac{\partial}{\partial x_n}$$

be the total derivative operator on the algebra $\mathcal{A} = \mathrm{Jet}(\mathcal{O}_C) = \mathbb{R}[t, x_i]$. Show that D_1 is indeed a derivation on $\mathcal{A}$.

3. Give an explicit description of the differential forms and vector fields of the jet $\mathcal{D}$-space $\mathrm{Jet}(C)$.

4. Show that for every $L \in \mathcal{A}$ and $x \in H$,

$$\int_0^1 (D_1 L)(t, \partial_t^i x(t)) dt = \left[L(t, \partial_t^i x(t))\right]_0^1$$

and deduce from that the variational version of integration by parts

$$\int_0^1 (F.D_1 G)(t, \partial_t^i x(t)) dt = -\int_0^1 (D_1 F.G)(t, \partial_t^i x(t)) + \left[(FG)(t, \partial_t^i x)\right]_0^1.$$

5. We denote $\underline{x} = (x, x_1, \dots)$ and $\underline{h} = (h, h_1, \dots)$ two universal coordinates on jet space. Let $L(t, \underline{x}) = \sum_i \sum_\alpha a_{i,\alpha} t^i \underline{x}^\alpha$ be a general lagrangian and suppose that $|\alpha| > 0$ for every term in L. Show that
$$(\underline{x} + \epsilon \underline{h})^\alpha := \prod_i (x_i + \epsilon h_i)^{\alpha_i} = \prod_i (x_i^{\alpha_i} + \epsilon \alpha_i x_i^{\alpha_i - 1} h_i) = \underline{x}^\alpha + \epsilon \sum_j \alpha_j \underline{x}^{\alpha - \underline{j}} \underline{h}^{\underline{j}}$$
for $\epsilon^2 = 0$ and $\underline{j}$ the multi-index nontrivial only in degree j.

6. Deduce that
$$L(\underline{x} + \epsilon \underline{h}) = L(\underline{x}) + \epsilon \sum_j \frac{\partial L}{\partial x_j} . D_j h.$$

7. By using integration by part on the above expression for L, deduce that
$$S(x + \epsilon h) - S(x) = \epsilon B(x, h) + \epsilon \int_0^1 \left[\sum_j (-1)^j D_j (\frac{\partial L}{\partial x_j})(x, \partial_t^i x(t)) \right] h dt,$$
where B is a function, called the boundary term, given by
$$B(x, h) = \sum_j \sum_{k=1}^{j} (-1)^{k+1} \left[(\frac{\partial L}{\partial x_j})(t, \partial_t^i x(t)) \partial_t^k h(t) \right]_0^1 .$$

8. Describe the space of critical points of $S : \Gamma(M, C) \to \mathbb{R}$.

9. Describe locally (PDE) the space of trajectories $T = \{x \in H |\ d_x S = 0\}$.

10. Describe locally more explicitely the space of trajectories of the lagrangian of standard Newtonian mechanics $L(t, x_i) = x_1^2 - V(x_0)$ for $V : \mathbb{R} \to \mathbb{R}$ a given function.

11. Describe the local critical space of free Newtonian mechanics ($V = 0$), and its canonical symplectic structure.

12. Described the derived critical phase space of Newtonian mechanics.

Exercice 21. (Gauge symmetries in classical mechanics) Describe the trajectories and Noether relations of the lagrangian
$$L(t, x_i) = \sqrt{x_1^2}$$
of classical mechanics invariant by reparametrization.

Exercice 22. (Electromagnetism) Describe the critical space and derived critical space of local electromagnetism (expressed in terms of the electromagnetic potential $A \in \Omega_M^1$ as a differential form on spacetime M).

Appendix F

Distributions and partial differential equations

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Distributions and partial differential equations

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