Higher directed spaces and higher stratifications (a short note)

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Abstract

We define, using the geometric approach to higher category theory through weak complicial sets, a notion of higher directed space. Usual directed spaces define such higher directed spaces, but there are plenty of other examples associated to higher categories. We define the fundamental higher category of a higher directed space. Using the notion of higher directed space, we also define a higher analog of the notion of a poset-stratified space (replacing the poset by an arbitrary higher category), a higher analog of the exit path category and of a constructible stack on a stratified space.

Contents

1 Introduction

Many publications discuss the relationship between directed spaces and concurrency theory (see for example [FGH$^+$16]). An approach to directed algebraic topology was proposed by Grandis using directed paths in [Gra09]. In this setting, the fundamental categories are categories, or $(\infty, 1)$-categories. By getting inspiration from the so-called directed homotopy hypothesis, we propose here a new notion of higher directed space whose fundamental categories are given by weak simplicial ω-categories, in the sense of Street [Str87], Verity [Ver06a], Ozornova and Rovelli [OR20]. Classical directed spaces give examples of such structures, but there are many more examples coming from higher categories. We then proceed by defining the notion of a higher stratification (by replacing the posets of poset-stratifications by higher directed spaces). Classical stratifications define higher stratifications but there are also other interesting examples coming from directed geometry and higher category theory. We finish by defining the notions of exit path higher category and of a constructible stack of higher categories on a higher stratified space.

2 Classical directed spaces

We start by recalling Grandis' definition of a directed space from [Gra09].

Definition 1. A directed space is given by the datum of a pair (X, D) of a topological space together with a set $D \subset \mathrm{Hom}_{\mathrm{TOP}}([0,1], X)$ of directed paths containing the constant paths and stable by increasing reparametrization and composition. A morphism of directed spaces is a continuous map that sends directed paths to directed paths. We denote DTOP the category of directed spaces defined this way.

Example 1. Let $(X, \sigma : \mathbb{R} \times X \to X)$ be a continuous dynamical system. Then if we define its directed paths to be given by expressions of the form $t \mapsto \sigma(i(t), x)$ for $i : [0,1] \to \mathbb{R}$ an increasing continuous map and $x \in X$ a point, we get a classical directed space $(X, D(\sigma))$.

To a directed space, one associates an ∞-category called the fundamental ∞-category using the following receipe.

First, if X is a topological space, we denote $\Pi_\infty(X)$ the associated singular simplicial complex (often denoted $S(X)$ in the litterature), given by

$$\Pi_\infty(X)_n = \mathrm{Hom}_{\mathrm{TOP}}(|\Delta^n|, X).$$

Definition 2. One may equip each geometric simplex $|\Delta^n|$ with the structure of a directed space $(|\Delta^n|, D_n)$ by saying that a map $[0,1] \to |\Delta^n|$ is directed if it factors through the 1-skeleton (union of 1-faces), it has endpoints on 0-vertices, and it is increasing for the pre-order induced by the standard order on vertices.

We now recall the definition of the fundamental ∞-category associated to a classical directed space (found on the nlab contributive website).

Definition 3. Let (X, D) be a classical directed space. Define the sub-simplicial set $d(X, D)$ of $\Pi_\infty(X)$ given by morphisms of directed spaces

$$(|\Delta^n|, D_n) \to (X, D).$$

Then $d(X, D)$ is called the fundamental ∞-category of the directed space (X, D).

The fact that $d(X,D)$ is indeed a quasi-category follows from the fact that we may use the standard retracts of the topological horn inclusions, that are directed maps, to fill inner horns.

So from a directed space (X,D), we have defined a triple $(X, d(X,D), \varphi)$ composed of a topological space X, a quasi-category $d(X,D)$ and a morphism of quasi-categories $\varphi : d(X,D) \to \Pi_\infty(X)$. This will be the source of our adaptation of directed space to the setting of higher categories.

3 Higher directed spaces

We refer to [Ver06a], [Rie16] and [OR20] for the definition of the model categories in play in this section.

Just recall that Δ is the category of finite ordinals $[n] = [0 \leq \cdots \leq n]$ with increasing maps between them. Presheaves on Δ are called simplicial sets. To encode marked simplicial sets, one enhances Δ to a category $t\Delta$ obtained by adding some objects $[n]_t$ and (comarking) morphisms $[n] \to [n]_t$ for $n \geq 1$ fulfilling natural identities that mean that degenerate simplices should be marked. Presheaves on $t\Delta$ are called structurally marked simplicial sets. A presheaf X such that all $X([n]_t) \to X([n])$ are injective is called a marked simplicial set, or a stratified simplicial set. We denote $\mathrm{SET}^{t\Delta^{op}}$ the category of structurally marked simplicial sets.

There are model structures on structurally marked and marked simplicial sets whose cofibrations are monomorphisms and whose fibrant objects are respectivelly called saturated (weak) pre-complicial and saturated (weak) complicial sets and the associated localized $(\infty,1)$-categories are equivalent. An object in one of these $(\infty,1)$-categories represents an (∞,∞)-category, that we will also simply call an ω-category.

There is a sequence of adjoint functors

$$\mathrm{SET}^{t\Delta^{op}} \underset{v}{\overset{\sharp}{\leftrightarrows}} \mathrm{SET}^{\Delta^{op}} \underset{|\cdot|}{\overset{\Pi_\infty}{\leftrightarrows}} \mathrm{TOP}.$$

Definition 4. A higher directed space is a triple

$$(Y, dY, \varphi : dY \to \Pi_\infty(Y)^\sharp)$$

composed of a topological space Y, a structurally marked simplicial set dY and a morphism φ of structurally marked simplicial sets. We say that Y is directed by dY. We denote HDTOP the category of higher directed spaces, whose morphisms are pairs composed of a continuous morphism and a compatible marked morphism.

Recall that, by adjunction, the morphism φ is equivalent to a morphism $\varphi^v : v(dY) \to \Pi_\infty(Y)$, and to a morphism $\varphi^{|v|} : |v(dY)| \to Y$, where $v(dY)$ denotes the underlying simplicial set and $|-|$ denotes geometric realization.

We may in general be particularly interested in examples where φ is injective and dY is a saturated complicial set, encoding a sub-ω-category of the fundamental ∞-groupoid of Y.

Definition 5. Let X be a structurally marked simplicial. Then the natural map

$$X \to \Pi_\infty(|v(X)|)^\sharp$$

obtained by adjunction defines a higher directed space

$$|X| := (|v(dX)|, X, X \to \Pi_\infty(|v(X)|)^\sharp)$$

called the geometric realization of X. This defines a functor

$$|\cdot| : \mathrm{S{\scriptsize ET}}^{t\Delta^{op}} \to \mathrm{HDT{\scriptsize OP}}.$$

This construction applies in particular if X is a stratified saturated complicial set, representing a weak ω-category, in which case we get a map of weak ω-categories to the ∞-groupoid of $Y = |v(X)|$, represented by the saturated complicial set $\Pi_\infty(|v(X)|)^\sharp$. Moreover, if $(Y, dY, \varphi : dY \to \Pi_\infty(Y)^\sharp)$ is another higher directed space such that $dY = X$, the adjunction map $\varphi^{|v|} : |v(X)| = |v(dY)| \to Y$ defines a natural morphism of directed space

$$(\varphi^{|v|}, \mathrm{id}_X) : |X| \to (Y, X, \varphi : X \to \Pi_\infty(Y)^\sharp).$$

Example 2. Classical directed algebraic topology gives examples of higher directed spaces where dY is (the saturated complicial set associated to) a quasi-category. Recall from Section 2 that if (X, D) is a classical directed space, it has a fundamental ∞-category

$$d(X, D) \hookrightarrow \Pi_\infty(X)^\sharp$$

(the quasi-category $d(X, D)$ is equipped with its standard saturated marking, given by marking categorical equivalences in degree one, and marking everything in higher degree). This construction defines a functor

$$\begin{array}{ccc} \mathrm{DT{\scriptsize OP}} & \to & \mathrm{HDT{\scriptsize OP}} \\ (X, D) & \mapsto & (X, d(X, D), d(X, D) \to \Pi_\infty(X)^\sharp) \end{array}$$

from classical directed spaces to higher directed spaces.

Example 3. It follows from [Lur09], Appendix A.5, that posets also naturally define examples of higher directed spaces that are useful in the theory of stratified spaces (see Section 5). Suppose given a poset $(P, \leq)$. The set P may be equipped with the topology whose open sets are upward closed subsets. Then there is a natural map

$$\varphi : N(P, \leq) \to \Pi_\infty(P)$$

which carries a simplex $(a_0 \leq a_1 \leq \cdots \leq a_n)$ of $N(P, \leq)$ to the map $\sigma : |\Delta^n| \to P$ characterized by the formula

$$\sigma(t_0, \ldots, t_i, 0, \ldots, 0) = a_i$$

whenever $t_i > 0$. We may equip the nerve $N(P, \leq)$ with its natural saturated marked simplicial set structure (marked 1-vertices are isomorphisms, given by equalities $a = b$, and higher vertices are all marked). This defines a higher directed space

$$(P, N(P, \leq), \varphi : N(P, \leq) \to \Pi_\infty(P)^\sharp).$$

This construction defines a functor

$$\textsc{Posets} \to \text{HDT}\textsc{op}.$$

Proposition 1. *The category HDT*OP *has all small colimits.*

Proof. Let $(Y, dY, \varphi) : I \to \text{HDT}\textsc{op}$ be a small diagram of higher directed spaces. Let $X = \operatorname{colim}_i Y_i$ and $dX = \operatorname{colim}_i dY_i$ be the colimits of its first two components. Then we have

$$|v(dX)| = \operatorname*{colim}_i |v(dY_i)|$$

because $|v(\cdot)|$ is a left adjoint that commutes with colimits. We may thus define by functoriality a morphism $\operatorname{colim}_i \varphi_i : |v(dX)| \to X$, that corresponds by adjunction to a morphism $dX \to \Pi_\infty(X)^\sharp$, that defines a higher directed space that is the colimit of the given diagram. □

4 Fundamental higher categories of higher directed spaces

The natural functor

$$|\cdot| : \textsc{Set}^{t\Delta^{op}} \to \text{HDT}\textsc{op}$$

given by Definition 5 may be restricted to the standard marked simplices $t\Delta \hookrightarrow \textsc{Set}^{t\Delta^{op}}$. This way, we get a natural functor

$$|\Delta[\cdot]_{(t)}| : t\Delta \to \text{HDT}\textsc{op}.$$

and this defines a pair of functors

$$|\cdot| : \textsc{Set}^{t\Delta^{op}} \leftrightarrows \text{HDT}\textsc{op} : \Pi_\omega,$$

with

$$\Pi_\omega(Y, dY, \varphi)([n]_{(t)}) = \operatorname{Hom}_{\text{HDT}\textsc{op}}(|\Delta[n]_{(t)}|, (Y, dY, \varphi))$$

where $[n]_{(t)}$ denotes a generic object of $t\Delta$.

Definition 6. The saturated complicial homotopy class of $\Pi_\omega(Y, dY, \varphi)$ is called the fundamental higher category associated to the higher directed space (Y, dY, φ).

5 Higher analogs of stratifications

We recall some results on poset-stratifications from Lurie's book [Lur09], Appendix A.5 and A.6. If P is a poset, we also denote P the associated topological space, whose open subsets are given by upward closed subsets.

Definition 7. A stratified space is a triple (X, P, f) composed of a topological space X, a poset P, and of a continuous map $f : X \to P$.

Recall from Example 3 that P defines a higher directed space

$$(P, N(P, \leq), N(P, \leq) \to \Pi_\infty(P)^\sharp).$$

If $f : X \to P$ is a stratified space, the continuous map f induces, by functoriality, a natural map $\Pi_\infty(f) : \Pi_\infty(X) \to \Pi_\infty(P)$. We refer to loc. cit. for the definition of a conic P-stratification.

Theorem 1 (Lurie)**.** *The fiber product*

$$\Pi_\omega(X, P, f) := \Pi_\infty(X) \times_{\Pi_\infty(P)} N(P, \leq)$$

is (in the case of a conic stratification) an $(\infty, 1)$*-category called the exit path* $(\infty, 1)$*-category associated to the stratified space* (X, P, f)*.*

We propose here to adapt this construction by replacing P by an arbitrary topological space, $N(P, \leq)$ by an arbitrary higher category dP together with an arbitrary map $\varphi : dP \to \Pi_\infty(P)^\sharp$. This means that the datum of a poset $(P, \leq)$ is replaced by the datum of an arbitrary higher directed space.

Definition 8. A higher stratified space is a tuple (X, P, dP, φ, f) composed of a higher directed space $(P, dP, \varphi : dP \to \Pi_\infty(P)^\sharp)$, with dP a saturated complicial set, and of a topological space X, together with a continuous map $f : X \to P$. The exit path higher category of a higher stratified space (X, P, dP, φ, f) is the saturated complicial set given by the homotopy fiber product

$$\Pi_\omega(X, P, dP, \varphi, f) = \Pi_\infty(X)^\sharp \times^h_{\Pi_\infty(P)^\sharp} dP$$

for the saturated complicial model category structure.

Concretely, since all terms of the pullback diagram are fibrant (i.e., saturated complicial), the exit path higher category may be computed by a classical pullback diagram of the following shape:

$$\begin{array}{ccc} \Pi_\omega(X, P, dP, \varphi, f) & \longrightarrow & (\Pi_\infty(P)^\sharp)^I \\ \downarrow & & \downarrow \\ \Pi_\infty(X)^\sharp \times dP & \longrightarrow & \Pi_\infty(P)^\sharp \times \Pi_\infty(P)^\sharp \end{array}$$

Here, if S is a saturated complicial set, $S \overset{\sim}{\hookrightarrow} S^I \twoheadrightarrow S \times S$ denotes a path space (fibrant) object (factorization of the diagonal map $\Delta : S \to S \times S$ into a trivial cofibration and a fibration) for the saturated complicial model category structure. There is a natural morphism

$$\Pi_\omega(X, P, dP, \varphi, f) \to dP$$

of saturated complicial sets, that is in some sense analogous to the morphism

$$\Pi_\omega(X, f, P) \to N(P)$$

used by Haine in his study in [Hai23] of the stratified homotopy of poset-stratified spaces.

Example 4. Here are some examples of higher stratifications that arise naturally.

1. A classical stratification $f : X \to P$ clearly defines a higher stratification, using the higher directed space

   $$(P, N(P, \leq), N(P, \leq) \to \Pi_\infty(P)^\sharp)$$

   of Example 3. This is the motivating example for our generalization of the notion of stratification. In this situation, dP is an $(\infty, 1)$-category, since it is the nerve of a poset. In the case of a conic stratification, there is a natural map

   $$\Pi_\omega(X, P, f) \to \Pi_\omega(X, P, dP, \varphi, f).$$

   from the exit path $(\infty, 1)$-category to the exit path higher category given by the standard map between the fiber product and the homotopy fiber product.

2. If (P, D) is a classical directed space and $f : X \to P$ is a continuous map, we may use the higher directed space

   $$(P, d(P, D), d(P, D) \to \Pi_\infty(P)^\sharp)$$

   associated in Example 2 to (P, D) to get a higher stratification. Remark that even if $d(P, D)$ is an $(\infty, 1)$-category, this example is quite different from the previous one.

3. Let dP be a saturated complicial set and $|dP| = (|v(dP)|, dP, \varphi)$ be the associated higher directed space, given by Definition 5. Then any continuous map $f : X \to |v(dP)|$ defines a higher stratification on X. This gives examples where dP is an arbitrary (∞, ∞)-category so that the exit path higher category is also quite arbitrary.

Verity showed in [Ver06b] that there is a weak complicial set of small weak complicial sets. We may take the subcomplicial set ω' whose objects are small saturated weak complicial sets, and take the saturation of it, obtained by marking all categorical higher equivalences (to be clear, we take a fibrant replacement of ω' for the saturated complicial model structure). Denote ω the corresponding saturated weak complicial set (of small saturated weak complicial sets). We here get inspiration from the results of Lurie [Lur09], Appendix A.9, to give the following abstract definition.

Definition 9. Let $X = (X, P, dP, \varphi, f)$ be a higher stratified space. A constructible stack of higher categories on X is a morphism

$$\mathcal{F} : \Pi_\omega(X, P, dP, \varphi, f) \to \omega$$

of saturated complicial sets.

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