Model categories

1. Let $C$ be a category and $A 	o B$, $X 	o Y$ two arrows of $C$. We will say that $i$ has the left lifting property with respect to $p$ or that $p$ has the right lifting property with respect to $i$ if for every commutative (solid) square

\[
\begin{array}{ccc}
A & \xrightarrow{a} & X \\
\downarrow{i} & & \downarrow{p} \\
B & \xrightarrow{b} & Y
\end{array}
\]

there exists a lifting $b : B \to X$ such that $\alpha = b i$ and $\beta = p b$.

i.e. if the map

\[
\text{Hom}(B, X) \longrightarrow \text{Hom}(A, X) \times \frac{\text{Hom}(B, Y)}{\text{Hom}(A, Y)}
\]

is surjective. If $K$ is a class of arrows of $C$ we denote by $\bar{L}(K)$ (resp. $\bar{R}(K)$) the class of arrows of $C$ satisfying the left (resp. right) lifting property with respect to all arrows in $K$.

The following definition is due to Quillen.

A closed model category is a category $C$ endowed with three classes of arrows $W$, $\text{Cof}$, $\text{Fib}$ satisfying the following conditions:

\[\text{CM1}\] the category $C$ has finite limits and finite colimits;

\[\text{CM2}\] the class of arrows $W$ satisfy the two out of three property: in a commutative triangle of $C$, if two of the three arrows are in $W$, so is the third.
\textbf{CM3} \ W, \ \text{Cof}, \ \text{Fib} \ are \ stable \ under \ retracts.

\textbf{CM4} \ \text{Cof} = \ell(\text{Fib} \cap \text{W}), \ \text{Fib} = \ell(\text{Cof} \cap \text{W})

\textbf{CM5} \ \text{Every \ arrow} \ f \ \text{in} \ C \ \text{can be decomposed as} \ f = p \circ j \ \text{with} \ p \in \text{Fib}, \ j \in \text{Cof} \cap \text{W}, \ q \in \text{Fib} \cap \text{W}, \ q \in \text{Cof}

Arrows \ in \ W \ are \ called \ weak \ equivalences, \ arrows \ in \ \text{Cof}, \ \text{cofibrations}, \ in \ \text{Cof} \cap \text{W} \ \text{trivial cofibrations}, \ in \ \text{Fib}, \ \text{fibrations}, \ in \ \text{Fib} \cap \text{W} \ \text{trivial fibrations}. \ \text{If \ we \ denote} \ f \ \text{by} \ \phi \ \text{(resp.} \ \ast) \ \text{an \ initial} \ (\text{resp.} \ \text{final}) \ \text{object \ of} \ C \ (\text{CM1}), \ \text{an \ object} \ X \ \text{of} \ C \ \text{is \ called \ cofibrant} \ (\text{resp.} \ \text{fibrant}) \ \text{if} \ \phi \rightarrow X \ \text{is \ a \ cofibration} \ (\text{resp.} \ X \rightarrow \ast \ \text{a} \ \text{fibration})

\text{It \ can \ be \ "easily" \ proved \ that \ in \ a \ closed \ model \ category \ the \ following \ relations \ hold:}

\begin{align*}
\text{Cof} = \ell(\text{Fib} \cap \text{W}), & \quad \text{Fib} = \ell(\text{Cof} \cap \text{W}) \\
\text{Cof} \cap \text{W} = \ell(\text{Fib}), & \quad \text{Fib} \cap \text{W} = \ell(\text{Cof}) \\
\text{W} = (\text{Fib} \cap \text{W}) \circ (\text{Cof} \cap \text{W}) \quad \text{(an \ arrow} \ \text{of} \ C \ \text{is \ in} \ W \ \text{if \ and \ only \ if} \ \text{it \ can \ be \ written \ as} \ f = q \circ j \ \text{with} \ q \in \text{Fib} \cap \text{W} \ \text{and} \ j \in \text{Cof} \cap \text{W})
\end{align*}

\text{In \ particular, \ any} \ \text{two \ out \ of \ the \ three \ classes} \ W, \ \text{Cof}, \ \text{Fib} \ \text{determine \ the \ third \ one}
**Examples**

1) \( C = \text{Top} = \text{category of topological spaces and continuous maps} \)

\( W = \) ordinary weak equivalences

\( \text{Fib} = \text{Serre fibrations} = \mathcal{R}(\mathcal{J}) \)

\( \mathcal{J} = \{ [0, 1]^m \to [0, 1]^{m+1} \mid m > 0 \} \)

\( (x_1, \ldots, x_m) \to (x_1, x_1, \ldots, x_m) \)

\( \text{Cof} = \mathcal{L}(\text{Fib} \cap \mathcal{W}) \)

2) \( C = \text{SSets} = \text{category of simplicial sets} \)

\( \text{Cof} = \text{monomorphisms} \)

\( \text{Fib} = \text{Kan fibrations} \)

\( W = \mathcal{R}(\text{Cof}) \cap \mathcal{L}(\text{Fib}) \) (= maps of simplicial sets whose topological realization is an homotopy equivalence of CW-complexes)

3) \( C = \text{C}(\mathcal{A}) = \text{category of unbounded cochain complexes of } \mathcal{A} \) \( \mathcal{A} \) being a Grothendieck category, i.e. a cocomplete abelian category, with exact filtered colimits and a generator

\( W = \text{quasi-isomorphisms} \)

\( \text{Cof} = \text{monomorphisms} \)

\( \text{Fib} = \mathcal{R}(\text{Cof} \cap \mathcal{W}) \)

4) many others ...
2. The following "Kan Brown" lemma is very useful.

**Lemma** Let \((C, W, \text{Cof}, \text{Fib})\) be a closed model category, \((C', W')\) a localizer such that \(W'\) satisfies the two out of three property, and \(F : C \to C'\) a functor carrying trivial cofibrations between cofibrant objects of \(C\) to arrows in \(W'\). Then \(F\) carries all weak equivalences between cofibrant objects of \(C\) to arrows in \(W'\).

**Theorem (Existence of derived functors)** Let \((C, W, \text{Cof}, \text{Fib})\) be a closed model category, \(D\) a category and \(F : C \to D\) a functor carrying trivial cofibrations between cofibrant objects of \(C\) to isomorphisms of \(D\). Then the functor \(F\) admits an absolute left derived functor.

**Sketch of the construction of the derived functor** (This can be skipped and go directly to corollary page 9.)

For every object \(X\) of \(C\) choose by CM5 a decomposition

\[
\phi \to LX \xrightarrow{\sim} X
\]

of \(\phi \to X\) to a cofibration followed by a trivial fibration \(p_X\). The object \(LX\) is then cofibrant. For every arrow \(f : X \to Y\) of \(C\) choose by CM4 a lifting \(Lf\)

\[
\phi \xrightarrow{\sim} LX \xrightarrow{Lf} LY \quad p_Y \cdot Lf = f \cdot p_X
\]
Define a functor \( C \xrightarrow{F} D \) by

\[
\begin{align*}
X & \mapsto F(LX) \\
\phi & \mapsto F(L\phi)
\end{align*}
\]

One has to verify

\[
\begin{align*}
F(L1_X) &= 1_{F(LX)} \\
F(L(g\phi)) &= F(Lg)F(L\phi) \\
X & \xrightarrow{\phi} Y \xrightarrow{g} Z
\end{align*}
\]

Therefore to conclude in both cases it's enough to prove

Lemma Let \( A \xrightarrow{u} B \xrightarrow{p} T \) a diagram in \( C \) such that \( pu = pv \), \( p \) is a trivial fibration and \( A \) cofibrant. Then \( F(u) = F(v) \)
Proof. Choose by CM5 a decomposition

$$A \sqcup A \xrightarrow{(\bar{A}, f')} \bar{A} \xrightarrow{s} A$$

of the codiagonal $A \sqcup A \rightarrow A$ to a cofibration followed by a trivial fibration. As $A$ and $\bar{A}$ are cofibrant and $s$ a weak equivalence the Ken Brown lemma implies that $F(s)$ is an isomorphism and as $s d^0 = 1_A = s d^1$ this implies that $F(d^0) = F(d^1)$. By CM4 there is a lifting $\lambda$

$$A \sqcup A \xrightarrow{(\bar{\lambda}, \lambda')} \beta \quad \text{so that} \quad \bar{\lambda} = F d^0, \quad \lambda' = F d^1$$

which implies

$$F(\bar{\lambda}) = F(\lambda') F(d^0) = F(\lambda') F(d^1) = F(\lambda') \square$$

The commutativity of the square $(\ast)$ and CM2 imply that if $f : X \rightarrow Y$ is a weak equivalence of $C$, $Lf : L X \rightarrow L Y$ is also a weak equivalence, and as $L X$ and $L Y$ are cofibrant the Ken Brown lemma implies that $LF(f) = F(Lf)$ is an isomorphism. So the functor $LF : C \rightarrow D$ induces a functor $LF : W^{-1}C \rightarrow D$ such that $LF = LF \circ p$

$$\begin{align*}
\rho & : \quad C \\
\downarrow & \quad \\
Ho C := W^{-1}C & \xrightarrow{LF} D
\end{align*}$$
The commutativity of the squares (1) implies that if we define \( \alpha \) by \( \alpha_x = F(p_x) \) then \( \alpha : LF \circ p \rightarrow F \) is a natural transformation.

Let us prove that \((LF, \alpha)\) is a left derived functor of \(F\). First observe that by the Ker Brown Lemma if \(X\) is cofibrant, then \( \alpha_x = F(p_x) \) is an isomorphism.

Let \( \gamma : G \circ p \rightarrow F \) be a natural transformation where \( G : W^{-1}C \rightarrow D \) is a functor. By the precise universal property of the localization, in order to prove \((LF, \alpha)\) is a left derived functor, it is enough to prove that there exists a unique natural transformation \( \delta : G \circ p \rightarrow LF \circ p \) so that the triangle

\[
\begin{array}{ccc}
G \circ p & \xrightarrow{\gamma} & F \\
\delta & \downarrow \alpha_x & \\
LF \circ p & \end{array}
\]

commutes.

Unicity comes from the following commutative prism and the fact that \( \alpha_x \) and \( G \circ p (p_x) \) are invertible.
It remains to prove that if a derived function is absolute, then the universality of the arrow exists. (and the universality of the arrow)
the functor $F' = H^1F : C \rightarrow D'$ satisfies the hypotheses of the theorem, and if we apply the same construction (with the same choice of $LX$ and $LF$), we obtain a derived functor $(LF', \alpha')$ of $F'$ such that $LF' = H^1LF$ and $\alpha' = \alpha$.

Corollary. Let $(C, W, Cof, F, L)$ be a closed model category, $(C', W')$ a localizer such that $W'$ satisfies the two out of three property, and $F : C \rightarrow C'$ a functor carrying trivial cofibrations between cofibrant objects of $C$ to arrows in $W'$. Then the functor $F$ admits an absolute total left derived functor $(LF, \alpha)$.
Theorem (Composition of derived functors). Let 
\((C, W_c, \text{Cof}_c, F_0^c)\), 
\((C', W'_c, \text{Cof}'_c, F_0'^c)\), and 
\((C'' W''_c, \text{Cof}''_c, F_0''^c)\) 
be three closed model categories, \(F : C \to C'\), \(F' : C' \to C''\) 
two functors, and suppose that \(F\) and \(F'\) carries 
trivial cofibrations between cofibrant objects to 
weak equivalences and that \(F\) carries cofibrant 
objects to cofibrant objects. If \((L_0 F, \alpha)\), \((L_0 F', \alpha')\) 
are total left derived functors respectively of 
\(F_0, F'_0\) (which exist and are absolute by the exist-
ence theorem), then \((L_0 F' \circ L_0 F, (\alpha' \circ \alpha))\) is 
\[ \begin{array}{ccc} 
C & \xrightarrow{F} & C' \\
\downarrow \alpha & & \downarrow \alpha' \\
W'' C & \xrightarrow{L_0 F'} & W''' C'' 
\end{array} \] 
an absolute total left derived functor of \(F' \circ F\).
Theorem (Adjunction) Let \((C, W, \text{Cof}, \text{Fib})\) and 
\((C', W', \text{Cof}', \text{Fib}')\) two closed model categories, 
\[ F : C \rightarrow C', \quad G : C' \rightarrow C \]
a pair of adjoint functors, and suppose that \(F\) (resp. \(G\)) carries trivial cofibrations (resp. trivial fibrations) between cofibrant (resp. fibrant) objects to weak equivalences. If 
\[ (L_F, R_F) \quad \text{and} \quad (R_G, L_G) \]
is a total left (resp. right) derived functor of \(F\) (resp. \(G\)) (which exists and is absolute by the existence theorem (resp. the dual existence theorem)), then \(L_F, R_F\)
is a pair of adjoint functors.

This theorem is now just an application of the abstract adjunction theorem. It is much stronger than its usual formulation in terms of Quillen's adjunctions (a notion that was introduced by Quillen).

A Quillen's adjunction is a pair of adjoint functors between closed model categories satisfying the equivalent conditions of the following proposition (and if \((F, G)\) is a Quillen adjunction, \(F\) is called a left \text{Quillen} functor and \(G\) a right \text{Quillen} functor).
Proposition. Let \((F, G)\) be an adjoint pair of functors between closed model categories. The following conditions are equivalent:

i) \(F\) carries trivial cofibrations to trivial cofibrations and \(G\) carries trivial fibrations to trivial fibrations.

ii) \(F\) carries trivial cofibrations to trivial cofibrations and cofibrations to cofibrations.

iii) \(G\) carries trivial fibrations to trivial fibrations and fibrations to fibrations.

iv) \(F\) carries cofibrations to cofibrations and \(G\) carries fibrations to fibrations.

Corollary (of the theorem). Let \((F, G)\) a Quillen adjunction between closed model categories. Then \(LF\) and \(RG\) exist, are absolute and \((LF, RG)\) is a pair of adjoint functors.

A Quillen's adjunction is called a Quillen's equivalence of the equivalent conditions of the following proposition are satisfied.

Proposition. Let \((F: C \to C', G: C' \to C)\) a Quillen adjunction. The following conditions are equivalent:

i) \(LF\) is an equivalence of categories;

ii) \(RG\) is an equivalence of categories;

iii) for every cofibrant object \(X\) of \(C\) and every fibrant object \(X'\) of \(C'\) a map \(FX \to X'\) of \(C'\) is a weak equivalence if and only if the associated map \(X \to GX'\) of \(C\) is a weak equivalence.
A model category \((\mathcal{C}, W, \text{Cof}, \text{Fib})\) is called \(I\)-projective, where \(I\) is a small category, if \(\mathcal{C}\) is endowed with pointwise weak equivalences, pointwise fibrations, and cofibrations defined by the left lifting property with respect to \(I\)-trivial fibrations. This structure is called the \(I\)-projective model structure on \(\mathcal{C}\).

This category is called \(I\)-projective if it is \(I\)-projective for every small category \(I\). The notions of \(I\)-injective and \(I\)-injective closed model category are defined dually.

**Theorem** Let \((\mathcal{C}, W, \text{Cof}, \text{Fib})\) a projective model and stable by arbitrary small products, category. If the category \(\mathcal{C}\) is cocomplete, then the localizer \((\mathcal{C}, W)\) is cocomplete.

The proof goes through several lemmas. In what follows, fix a closed model category \((\mathcal{C}, W, \text{Cof}, \text{Fib})\) such that \(\mathcal{C}\) is cocomplete.

**Lemma 1** If \(\mathcal{C}\) is \(I\)-projective and \(\mathcal{C}^I\) endowed with the projective model structure, then

\[
\begin{align*}
\mathcal{C}^I & \xrightarrow{\lim_I} \mathcal{C}, \\
\mathcal{C} & \xrightarrow{\Delta I} \mathcal{C}^I
\end{align*}
\]

is a Quillen adjunction, In particular, the homotopy colimit functor 

\[
\lim^I : W^I\mathcal{C}^I \rightarrow W^I\mathcal{C}
\]

exists and is the total left derived functor of 

\[
\lim^I (\text{R\text{hom}}^I) \simeq \text{L\text{hom}}^I
\]
Proof: As weak equivalences and fibrations are defined pointwise in \( C^I \), the functor \( \Delta^I \) carries fibrations (resp. trivial fibrations) of \( C \) to fibrations (resp. trivial fibrations) of \( C^I \).

More generally:

Lemma 2: Let \( u: I \to J \) a functor between small categories, \( I \) if \( C \) is \( I \) and \( J \)-projective and \( C_I \), \( C_J \) endowed with the projective model structures, then

\[
\begin{array}{ccc}
C_I & \xrightarrow{u^*} & C_J \\
\downarrow{u} & & \downarrow{u} \\
C & \to & C
\end{array}
\]

is a Quillen adjunction. In particular, the homotopy left Kan extension functor \( u_{!R}: W^I \to C_I \to W^J \to C_J \) exists and is the total left derived functor of \( u \) (\( u_{!R} = L u \)).

The proof is similar to the proof of Lemma 1.

Lemma 3: Let \( u: I \to J \) be a local isomorphism between small categories (which means that for every object \( i \) of \( I \) the functor \( I/i \to J/u(i) \) induced by \( u \) is an isomorphism, or equivalently that \( u \) is a Grothendieck fibration with discrete fibers). If \( C \) admits arbitrary small products, if \( C_I \) and \( C_J \) are endowed with the projective model structure, then \( u^* \) is a left Quillen functor.
Proof. As $u^*$ carries weak equivalences to weak equivalences, it's enough to prove that $u^*$ carries cofibrations to cofibrations. As $u$ is a left Quillen functor, it can be easily verified that for every object $j$ of $J$ the functor

$$
\begin{align*}
(\text{id}, d_0 \to d_1) & \mapsto (j, u(d_0) \to u(d_1)) \\
(\text{id}, u(d_0) \to u(d_1)) & \mapsto (u(j), u(d_0) \to u(d_1))
\end{align*}
$$

is an isomorphism. Remark that $(\text{id}, d_0 \to d_1)$ is an initial object of $d_1 \times I$. So, as $C$ admits arbitrary small products, the right Kan extension function $f_\ast : C \to C^J$ exists and is defined by

$$
u_!(F)(j) = \lim_{\leftarrow \in \mathcal{I}} F(j) \cong \lim_{\leftarrow \in \mathcal{I}} u(j) \cong \prod_{u(d_0) \to u(d_1)} F(d_0)
$$

As trivial fibrations are stable by products, $u^*$ carries trivial fibrations to trivial fibrations and as a consequence $u^*$ carries cofibrations to cofibrations.

Proof of the Theorem. Suppose now that $C$ is cocomplete, projective and stable by small products. By Lemma 1 or 3 the localizer $(C, u)$ admits homotopy colimits and more generally homotopy left Kan extensions. It remains to prove that every functor $u : I \to J$ between small categories, any $F : I \to C$ and any object $j$ of $J$ the canonical map

$$\nu_!(F)(j) \leftarrow \text{holim}_{I/J} (F | (I/j))$$
is an isomorphism in \( W^{-1}C \), i.e. that
\[
\overline{f}^* \cong u_! \xrightarrow{\sim} \text{colim} \xrightarrow{\sim} \overline{g}^*
\]  

(1)

where \( \overline{f} : \mathcal{E} \to \mathcal{F} \) denotes also the functor from the point category \( \mathcal{E} \) to \( \mathcal{F} \) defined by the object \( f \) and \( \overline{g} : I/f \to I \) the forgetful functor \( \underline{u}(i, 0, g) \to i \).

By the classical formula for left Kan extension functors we have a canonical isomorphism
\[
(\ast) \quad \overline{f}^* u_! \cong \text{colim} \xrightarrow{\sim} \overline{g}^*
\]

Observe that \( \overline{f}^* \cong L \overline{f}^* \) and that by

Lemma 2, \( u_! \) is a left Quillen functor, so the pair \( \overline{f}^*, u_! \) satisfy the conditions of the theorem of composition of total left derived functors (it can be proved that \( L \overline{f}^* \) is also a left Quillen functor, but this is not needed here), as \( \overline{f}^* \) carries weak equivalence to weak equivalence.

and
\[
\overline{f}^* u_! \cong L \overline{f}^* \cong L \text{colim} \xrightarrow{\sim} \overline{g}^*
\]

On the other hand, \( \overline{g}^* \) is a local isomorphism and by Lemmas 2 and 3, \( \text{colim} \xrightarrow{\sim} I/f \) and \( \overline{g}^* \) are left Quillen functors, so the composition theorem applies again and
\[
\text{colim} \xrightarrow{\sim} \overline{g}^* \cong L \text{colim} \xrightarrow{\sim} \overline{g}^* \cong L (\text{colim} I/f)
\]

and (\ast) implies the theorem.

Remark. In order to prove that (1) is an isomorphism, the only "projectivity" condition used is that \( C \subseteq I \) and \( I/f \) - projective.
Lemma 4 Let \((C,W,C^f,F_i^g)\) a closed model category stable under arbitrary small products and \(I\) a small category. Suppose that \(C^I\) is endowed with a closed model structure such that weak equivalences are defined pointwise and such that pointwise fibrations are fibrations. Then for every object \(i \in I\), if we denote also \(i : i \rightarrow I\) the functor from the point category \(A \rightarrow I\), I defined by the object \(i\), the functor \(i^*\) is a left Quillen functor. In particular any cofibration in \(C^I\) is a pointwise cofibration (But the converse is not true).

Proof As \(C\) is stable under arbitrary small products the right Kan extension functor \(\delta^*\) and right adjoint of \(\delta^*\) exists and is defined by

\[(\delta^*X)(c) = \lim_{c \rightarrow c^1} \pi \rightarrow \text{Hom}(c,c^1) \times (\ast)\]

As \(\delta^*\) carries weak equivalences to weak equivalences, it is enough to prove that \(\delta^*\) carries cofibrations to cofibrations, or equivalently that \(\delta^*\) carries trivial fibrations to trivial fibrations. This is a consequence of the formula \((\ast)\) and the stability of trivial fibrations by products.
Theorem. Every cofibrantly generated closed model category is projectively exponential.

We recall the definition of a cofibrantly generated closed model category.

Let $\kappa$ an infinite cardinal, a set $E$ (resp. a category $I$) is called $\alpha$-small if $\text{card}(E) < \alpha$ (resp. if $I$ is small and $\text{card} \text{ Ar } I < \alpha$). An infinite cardinal $\alpha$ is called regular if $\alpha$-small colimits (indexed by $\alpha$-small categories) of $\alpha$-small sets are $\alpha$-small. Let $\kappa$ a regular cardinal and $A$ a category. The category $A$ is called $\alpha$-filtered if for every category $I$, any functor $F : I \to A$ has a cocone in $A$ (there exists an object $X$ in $A$ and a natural transformation $F \Rightarrow X$, where $X$ denotes also the constant functor $I \to A$ of value $X$). Let $C$ a category stable under small $\alpha$-filtered colimits (indexed by a small $\alpha$-filtered category). An object $X$ of $C$ is called $\alpha$-presentable if the functor

$$\text{Hom}_C(X, ?) : C \to \text{Sets}, \quad T \mapsto \text{Hom}_C(X, T)$$

preserves small $\alpha$-filtered colimits. A object $X$ of a category $C$ is called presentable if there exists a regular cardinal $\kappa$ such that $C$ is stable under small $\alpha$-filtered colimits and $X$ is $\alpha$-presentable.
Let $C$ be a cocomplete category and $I$ be a set of arrows of $C$. We say that $I$ allows the small object argument if the domains of arrows in $I$ are presentable.

The small object argument implies that every arrow $f$ in $C$ can be written as

$$f = q_i, \quad q \in r(I), \quad i \in \text{cell}(I)$$

where $\text{cell}(I)$ denotes the class of arrows in $C$ which are transfinite compositions of pushouts of arrows in $I$.

An easy formal lemma, known as the "retract lemma," implies then that $r(I)$ is the class of retracts of arrows in $\text{cell}(I)$.

We will say that a closed model category $(C, W, \text{Cof}, \text{Fib})$ is cofibrantly generated if $C$ is cocomplete and if there exist sets $I, J$, allowing the small object arguments such that

$$\text{Fib} = r(I) \quad \text{and} \quad \text{Fib} \wedge W = r(J)$$

or equivalently

$$\text{Cof} = \text{Fr}(J) \quad \text{and} \quad \text{Cof} \wedge W = \text{Fib}(J)$$

We will say that $I$ (resp. $J$) generates the cofibrations (resp. the trivial cofibrations) and that $(I, J)$ generates the cofibrantly generated closed model category $C$.

To prove the theorem on page 18, we need the following Lemma.
Lemma (Trans.) Let $(C, W, 	ext{Cof}, F, i)$ be cofibrantly generated closed model category, generated by a pair $(F, i)$ and let

$$F : C \rightarrow C', \quad G : C' \rightarrow C$$

a pair of adjoint functors. Suppose:

i) $C'$ is complete and cocomplete;

ii) the sets $F(i)$ and $F(i)$ allows the small object argument.

iii) $\mathcal{G}(\text{re}(F(i))) = W'$

Define

$$W' = G^{-1}(W), \quad F(i)^1 = G^{-1}(F(i)), \quad \text{Cof}' = \mathcal{L}(F(i)^1 \cap W').$$

Then $(C', W', \text{Cof}', F(i)^1)$ is a cofibrantly generated closed model category generated by $(F(i), F(i))$.

Proof. Axioms CM1, CM2, CM3 are clear, and 1/2 of CM4 is by definition. Observe that

$$F(i)^1 = G^{-1}(F(i)) = G^{-1}(\mathcal{G}(i)) = \mathcal{G}(F(i))$$

and

$$F(i)^1 \cap W' = G^{-1}(F(i) \cap W') = G^{-1}(\mathcal{G}(i)) = \mathcal{G}(F(i))$$

In particular

$$\text{Cof}' = \mathcal{L}(\mathcal{G}(F(i)))$$

and

$$\mathcal{L}(\mathcal{G}(F(i))) = \mathcal{L}(F(i)^1) \subseteq \mathcal{L}(F(i)^1 \cap W') = \text{Cof}'$$

As the condition (iii) implies that $\mathcal{G}(F(\mathcal{i})) \subseteq W'$, we have

$$\mathcal{G}(F(i)) \subseteq \text{Cof}' \cap W'$$

Let us prove the other inclusion. Let $i \in \text{Cof}' \cap W'$. 


As $F(J)$ allows the small object argument, factorize

$$i = q j \quad j \in \text{Cof}(F(J)), \quad q \in r(F(J)) = F(J)$$

As $i$ and $j$ are in $W$, so is $q$, so

$$q \in F(J) \cap W'$$

and $i$ has the left lifting property with respect to $q$. The retract lemma implies that $i$ is a retract of $j$ and so $i \in \text{Cof}(F(J))$

We have proved that

$$\text{Cof}(W) = \text{Cof}(F(J)) = \text{Cof}(F(J))$$

and

$$\text{Cof}(W) = \text{Cof}(F(J)) = \text{Cof}(F(J))$$

and this ends the proof as CMS is now a consequence of the fact that $F(J)$ and $F(J)$ allow the small object argument.

We can now prove a more precise version of the theorem on page 18:

**Proposition.** Let $(C, W, \text{Cof}, F(J))$ be a cofibrantly generated closed model category generated by the pair $(I, J)$, and $I$ a small category. Then $(C, W', \text{Cof}, F(J))$, where $W' = (\text{Cof}(F(J)))$ is the class of pointwise weak equivalences (resp. fibrations) and $\text{Cof}(F(J))$...
is a cofibrantly generated closed model category
generated by the pair \((\mathcal{I}_I, \mathcal{J}_I)\) where
\[ \mathcal{I}_I = \bigcup_{i \in \mathcal{I}} i(\mathcal{I}) \quad \text{and} \quad \mathcal{J}_I = \bigcup_{i \in \mathcal{I}} i(\mathcal{J}) \]
where \(i_i : C \to C^I\) is the left adjoint of
the functor \(i^* : C^I \to C, \quad X \mapsto X(i)\).

**Proof** Observe that the natural projection
\[ \text{Hom}_{C^I}(i_! X, Y) = \text{Hom}_{C}(X, Y(i)) \]
implies that if \(X\) is a presentable object of \(C\)
then \(i_! X\) is a presentable object of \(C^I\), so that
\(\mathcal{I}_I\) and \(\mathcal{J}_I\) allow the small object argument.

The particular case where \(I\) is a discrete category
is an easy exercise left to the reader, and
gives simply the product model structure where
cofibrations are also defined pointwise.

The general case follows from the discrete
case by the preceding Lemma. Let \(I_0\) denote
the discrete category whose objects are the objects
are the objects of \(I\) and \(p : I_0 \to I\) the
inclusion. In order to apply the preceding
lemma and conclude the only thing one has
to verify is the condition (\(c0\)) of this lemma for
the adjunction
\[ C^I \xrightarrow{R^I} C, \quad C \xrightarrow{L^I} C^I. \]
Let \( i \) be an object of \( C \) and \( i_0 : e \rightarrow i_0 \), \( i : e \rightarrow i \) the functors defined by the object \( i \).

We have to prove that

\[ \mathbb{R}^* \mathcal{L} \mathcal{R} \mathbb{L} \, i_0 \, i_0 \left( \mathcal{S} \right) \subseteq W_I, \]

which simply means that

\[ \mathcal{L} \mathcal{R} \, i_1 \left( \mathcal{S} \right) \subseteq W_I. \]

In order to prove this, it is enough to prove the stronger condition that \( \mathcal{L} \mathcal{R} \, i_1 \left( \mathcal{S} \right) \) is contained in the class of pointwise trivial cofibrations. As this class is stable under pushout, transfinite composition and retractions, it is enough (by the small object argument) to prove that \( \mathcal{L} \mathcal{R} \, i_1 \left( \mathcal{S} \right) \) is contained in this class. But an explicit calculation shows that for every object \( X \) of \( C \)

\[ (i_1 X)(i') = \prod_{\text{Hom}(i', i')} X \quad (i' \in \mathcal{I}), \]

and as trivial cofibrations of \( C \) are stable under small coproducts, this finishes the proof.
Reedy categories

A Reedy category is a triple \((I, I^+, I^-)\), where \(I\) is a small category and \(I^+, I^-\) are two (non-full) subcategories of \(I\) satisfying the following conditions:

\[ R1 \] Every arrow \( f \) of \( I \) can be uniquely decomposed as \( f = f^+ \circ f^- \) with \( f^+ \) (resp. \( f^- \)) in \( I^+ \) (resp. \( I^- \)).

\[ R2 \] There exists a function \( J \) from the set of objects of \( I \) to a well ordered set such that if \( \alpha : i \to i' \) is a non-identity map of \( I^+ \) (resp. \( I^- \)) then \( J(i') > J(i) \) (resp. \( J(i') < J(i) \)).

Elementary properties

(a) The subcategories \( I^+ \) and \( I^- \) have same set of objects as \( I \).

(b) The intersection of \( I^+ \) and \( I^- \) is reduced to the identities of \( I \).

(c) Every split monomorphism is in \( I^+ \); every split epimorphism is in \( I^- \).

(d) The only isomorphisms of \( I \) are identities.

Examples
1) The category \( \Delta \) of simplices whose objects are the non-empty ordered sets
\[
[n] = \{ 0 < 1 < \cdots < n \}, \quad n \geq 0,
\]
and arrows the non-decreasing maps, endowed with the subcategories \( \Delta^+ \) and \( \Delta^- \) where \( \Delta^+ \) (resp. \( \Delta^- \)) is the subcategory of \( \Delta \) with same
objects and whose arrows are the monomorphisms (resp. epimorphisms) of \( \mathcal{A} \), is a Reedy category.

2) If \((I, I_+, I_-)\) is a Reedy category, then \((I^0, I^-_0, I^+_0)\) is a Reedy category.

3) If \((I, I_+, I_-)\) and \((J, J_+, J_-)\) are two Reedy categories, then \((I \times J, I_+ \times J_+, I_- \times J_-)\) is a Reedy category.

4) If \((I, I_+, I_-)\) is a Reedy category and \(X\) a presheaf on \(I\), then \((I/X, I^+_+/X, I^-/-X)\) is a Reedy category.

A direct category \(I\) is a small category \(I\) such that \((I, I, I_0)\), where \(I_0\) is the discrete subcategory of \(I\) with some objects as \(I\) and only identities as morphisms, is a Reedy category. This means simply that there exists a function from the set of objects of \(I\) to a well ordered set such that if \(j : i \to i'\) is a non-identity arrow of \(I\), then \(j(i) \prec j(i')\). The notion of inverse category is defined dually.

If \((I, I_+, I_-)\) is a Reedy category, then \(I_+\) is a direct category and \(I_-\) an inverse category.

If \(I\) is a finite category, the following conditions are equivalent:

a) \(I\) is a direct category;

b) \(I\) is an inverse category;

c) the free category generated by the graph of non-identity arrows of \(I\) is finite;

d) the nerve of \(I\) is a finite simplicial set (having a finite set of non-degenerate simplices)
Fix a Reedy category \((I, I_+, I_-)\).

For an object \(i\) of \(I\) denote \(I_i^+\) (resp. \(I_i^-\)) the full subcategory of \(I_{i^+}/\partial_i^+\) (resp. \(I_i^+ \setminus I\)) with set of objects \(\mathcal{O} \mathcal{E}(I_i^+/\partial_i^+ \setminus I) \ominus \{(0_-, i)\}\) (resp. \(\mathcal{O} \mathcal{E}(I_i^+ \setminus I) \ominus \{(0, i)\}\)).

Let \(C\) be a cocomplete category. For \(i\) an object of \(I\), define the \(\textit{catching object} \) functor

\[
L_i : C^I \longrightarrow C
\]

by

\[
L_i X = \operatorname{colim}_{(i^+ \ni \partial_i^+ \ni i)} X(i')
\]

and the \(\textit{catching object natural transformation} \)

\[
L_i \longrightarrow i^*
\]

by the universal property of colimits

\[
L_i X = \operatorname{colim}_{(0, i^+ \ni \partial_i^+ \ni i)} X(i') \longrightarrow X(i) = i^*(X)
\]

applied to the cocone

\[
\left( X(i') \longrightarrow X(i) \right)_{i' \ni i \in \text{Ar}(I_i^+ \setminus I)}
\]

Dually, if \(C\) is a complete category, define the \(\textit{matching object} \) functor

\[
M_i : C^I \longrightarrow C, \quad \partial_i^+ \ni i
\]
By
\[
M_i X \leftarrow \lim_{\delta \to \delta'} X(\delta')
\]
and the matching object natural transformation
\[
i^* \to M_i^*
\]

\[
i^* X \equiv X(i) \to \lim_{\delta \to \delta'} X(\delta')
\]

Fix now a complete and cocomplete closed model category \((C, W, \text{Cof}, \text{Fib})\). A Reedy cofibration (resp. a Reedy fibration) in \(C^I\) is a map \(X \to Y\) of \(C^I\) such that for every object \(i\) of \(I\) the map

\[
\begin{array}{ccc}
X(i) & \to & Y(i) \\
\downarrow & & \downarrow \\
L_i X & \to & L_i Y
\end{array}
\]

(resp.

\[
\begin{array}{ccc}
X(i) & \to & Y(i) \\
\downarrow & & \downarrow \\
X \to Y
\end{array}
\] deduced from the
commutative square

\[
\begin{array}{ccc}
X(i) & \to & M_i X \\
\downarrow & & \downarrow \\
Y(i) & \to & M_i Y
\end{array}
\]

is a cofibration (resp. a fibration) of \(C\).

A Reedy weak equivalence in \(C^I\) is a pointwise
weak equivalence.

Theorem. The category \(C^I\) endowed with Reedy
weak equivalences, Reedy cofibrations and Reedy
fibrations is a closed model category.
The closed model category structure on $C^I$ of the preceding theorem is known as the Reedy model structure.

Some properties of the Reedy model structure:

a) An arrow $X \rightarrow Y$ of $C^I$ is a trivial cofibration (resp. a trivial fibration) if and only if for every object $i$ of $I$ the canonical map

$$X(i) \rightarrowtail L_i Y \rightarrow Y(i) \quad (\text{resp. } X(i) \rightarrowtail Y(i) \times_{M_i Y} M_i X)$$

is a trivial cofibration (resp. a trivial fibration).

b) Reedy cofibrations, trivial cofibrations, fibrations, trivial fibrations are respectively positive cofibrations, trivial cofibrations, fibrations, trivial fibrations the converse being false in general.

c) If $X \rightarrow Y$ is an arrow in $C^I$ which is a Reedy cofibration (resp. trivial cofibration) then for every object $i$ of $I$, $L_i X \rightarrow L_i Y$ is a cofibration (resp. trivial cofibration). Dually if $X \rightarrow Y$ is a Reedy fibration (resp. trivial fibration) then $M_i X \rightarrow M_i Y$ is a fibration (resp. trivial fibration).
A Reedy category is called left connected if for every object $i$ of $I$ the category $\mathcal{E} I$ is connected (possibly empty).

Proposition. Let $(I, I^+, I^-)$ a left connected Reedy category and $(\mathcal{C}, W, \mathcal{C}_{af}, R, \Omega)$ a complete and cocomplete closed model category. If we endow $\mathcal{C}^I$ with the Reedy model structure then the pair of functors

$$C \xrightarrow{\lim} \mathcal{C} \quad \quad C \xrightarrow{\Delta_I} \mathcal{C}^I$$

$\lim_I$ is a Quillen adjunction and in particular $\lim_I$ has a total left derived functor

$$\mathcal{L} \lim_I \cong \hom_{\mathcal{I}}$$

Proof. Observe that if $x \to y$ is an arrow in $\mathcal{C}$ and $i$ an object of $I$, then the category $\mathcal{E} I$ is connected, the morphism

$$\Delta_i(x) \mathcal{E} I \to \Delta_i(y) \mathcal{E} I$$

is either an isomorphism of final objects (if $\mathcal{E} I$ is empty) or isomorphic to the arrow $x \to y$, and the canonical map

$$\Delta_i(x) \to \lim \mathcal{E} I$$

is either isomorphic to $x \to y$ or an isomorphism.
So if $X \rightarrow Y$ is a fibration (resp. trivial fibration) of $C$, then $\Delta_+(X) \rightarrow \Delta_+(Y)$ is a Reedy fibration (resp. trivial fibration) of $C$. 

The notion of right connected Reedy category is defined dually.

If $I$ is a direct category, by definition $(I, I^+, I^-)$, where $I^+ = I$ and $I^- = I^0$, a discrete category with some objects as $I$, is a Reedy category, and every object $o$ of $I$,

If $I$ is empty. In this case we have even better.

For every model category $C$ and any morphism $X \rightarrow Y$ of $C^I$, $\text{Mor}_I(X, Y)$ is an (co)smash of final objects and the canonical map

$X(o) \rightarrow Y(o) \times \text{Mor}_I(Y, X)$

is isomorphic to the map $X(o) \rightarrow Y(o)$. So in this case Reedy fibrations are exactly pointwise fibrations, so:

**Proposition** Let $I$ a direct category and $(C, W, C_{bf}, C_{f})$ a closed model category. If $C$ is cocomplete then $C$ is projectively $I$-exponential.