ASPHERICITY STRUCTURES, SMOOTH FUNCTORS, AND FIBRATIONS

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ABSTRACT. The aim of this paper is to generalize Grothendieck's theory of smooth functors in order to include within this framework the theory of fibered categories. We obtain in particular a new characterization of fibered categories.

INTRODUCTION

In "Pursuing Stacks" [6], Grothendieck defines the notions of proper functors and smooth functors by analogy with the cohomological properties of proper morphisms and smooth morphisms between schemes. If

$$\mathcal{D} \; = \quad \begin{array}{ccc} X' & \xrightarrow{\;g\;} & X \\ {\scriptstyle f'}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Y' & \xrightarrow[\;h\;]{} & Y \end{array}$$

is a commutative square of schemes and M is an abelian étale sheaf on X, there is a canonical "base change" morphism

$$h^* Rf_*(M) \longrightarrow Rf'_*\, g^*(M) \quad .$$

Proper and smooth base change theorems state [1, lectures 12, 13 and 16] that if the square $\mathcal{D}$ is cartesian, then this morphism is an isomorphism in the following two cases:

(a) f is proper and M is of torsion;
(b) h is smooth, M is of torsion prime to the residual exponents of Y, and f is quasi-compact and quasi-separated.

We denote by $\mathcal{C}at$ the category of small categories and by $\mathcal{W}_\infty$ the class of arrows in $\mathcal{C}at$ whose image under the nerve functor is a simplicial weak equivalence, *i.e.* a morphism of simplicial sets whose topological realization is a homotopy equivalence. The localized category $\mathcal{W}_\infty^{-1}\mathcal{C}at$, obtained by formally inverting the arrows which belong to $\mathcal{W}_\infty$, is then equivalent to the homotopy category Hot of CW-complexes. For every small category I, define $\mathbb{D}(I)$ by

$$\mathbb{D}(I) = \big(\mathcal{W}_\infty^{I^\circ}\big)^{-1}\, \underline{\mathsf{Hom}}(I^\circ, \mathcal{C}at) \quad ,$$

where $\underline{\mathsf{Hom}}(I^\circ, \mathcal{C}at)$ denotes the category of presheaves on I with values in $\mathcal{C}at$, *i.e.* the contravariant functors from I to $\mathcal{C}at$. Let $\mathcal{W}_\infty^{I^\circ}$ be the class of natural transformations between functors from I° to $\mathcal{C}at$ which are componentwise in $\mathcal{W}_\infty$:

$$\mathcal{W}_\infty^{I^\circ} = \{\alpha \in \mathsf{Fl}(\underline{\mathsf{Hom}}(I^\circ, \mathcal{C}at)) \mid \alpha_i \in \mathcal{W}_\infty, i \in \mathsf{Ob}(I)\} \quad .$$

For every arrow $u : I \longrightarrow J$ of $\mathcal{C}at$, the inverse image functor

$$\underline{\mathsf{Hom}}(J^\circ, \mathcal{C}at) \longrightarrow \underline{\mathsf{Hom}}(I^\circ, \mathcal{C}at) \quad , \qquad F \longmapsto F \circ u^\circ \quad ,$$

defines a functor

$$u^* : \mathbb{D}(J) \longrightarrow \mathbb{D}(I) \quad .$$

By a result of Alex Heller [8], this functor has a left and a right adjoint

$$u_! : \mathbb{D}(I) \longrightarrow \mathbb{D}(J) \qquad \text{and} \qquad u_* : \mathbb{D}(I) \longrightarrow \mathbb{D}(J)$$

respectively.

Grothendieck defines the notion of proper (resp. smooth) functor as follows.

Definition. A functor between small categories $u : A \longrightarrow B$ (resp. $w : B' \longrightarrow B$) is said to be *proper* (resp. *smooth*) if for every cartesian square

$$\begin{array}{ccc} A' & \xrightarrow{v} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[w]{} & B \end{array} \qquad ,$$

the base change morphism

$$w^* u_* \longrightarrow u'_* v^*$$

(or, equivalently,

$$v_! u'^* \longrightarrow u^* w_! \quad ,$$

transpose of the previous morphism) is an isomorphism, and this property remains true after any base change.

Grothendieck obtains simple characterizations of proper functors and smooth functors, and he notices that his theory relies only on a small number of formal properties of $\mathcal{W}_\infty$, the most important one being Quillen's Theorem A [10]. This leads him to introduce the notion of basic localizer.

A *basic localizer* is a class $\mathcal{W}$ of arrows of $\mathcal{C}at$ satisfying the following properties.

Loc1 (Weak saturation.) *a*) Identities belong to the class $\mathcal{W}$.
b) If two out of the three arrows in a commutative triangle belong to $\mathcal{W}$, then so does the third.
c) If i is an arrow of $\mathcal{C}at$ which has a retraction r and if ir belongs to $\mathcal{W}$, then i belongs to $\mathcal{W}$.

Loc2 If A is a small category which has a final object, then the functor $A \longrightarrow e$ from A to the final category e belongs to $\mathcal{W}$.

Loc3 (Relative Quillen's Theorem A.) For every commutative triangle

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ & {\scriptstyle v}\searrow \quad \swarrow{\scriptstyle w} & \\ & C & \end{array}$$

of $\mathcal{C}at$, if for every object c of C, the functor $u/c : A/c \longrightarrow B/c$ (where $A/c = A \times_C (C/c)$ and $B/c = B \times_C (C/c)$ stand for the "comma categories" of objects over c) induced by u belongs to $\mathcal{W}$, then so does u.

Grothendieck conjectures in [6] that $\mathcal{W}_\infty$ is the smallest basic localizer. This conjecture is proved by D.-C. Cisinski in [5].

A small category A is said to be $\mathcal{W}$-aspheric if the morphism $A \longrightarrow e$ belongs to $\mathcal{W}$. A functor between small categories $A \longrightarrow B$ is said to be $\mathcal{W}$-aspheric if for every object b of B, the category A/b is $\mathcal{W}$-aspheric. It follows from Loc3 that a $\mathcal{W}$-aspheric functor belongs to $\mathcal{W}$. As in the case $\mathcal{W} = \mathcal{W}_\infty$, we define, for every small category I, the category $\mathbb{D}(I)$ by

$$\mathbb{D}(I) = \left(\mathcal{W}^{I^\circ}\right)^{-1} \underline{\mathsf{Hom}}(I^\circ, \mathcal{C}at) \quad ,$$

and for every arrow $u : I \longrightarrow J$ of $\mathcal{C}at$, we can show by elementary methods [9] that the inverse image functor $u^* : \mathbb{D}(J) \longrightarrow \mathbb{D}(I)$ has a left adjoint $u_! : \mathbb{D}(I) \longrightarrow \mathbb{D}(J)$.

We can therefore define the notion of $\mathcal{W}$-proper and $\mathcal{W}$-smooth functors as in the case of $\mathcal{W} = \mathcal{W}_\infty$, using base change morphisms related to $u_!$.

In fact, the theory developed in D.-C. Cisinski's thesis [4] shows also the existence of a right adjoint $u_* : \mathbb{D}(I) \longrightarrow \mathbb{D}(J)$, assuming a mild hypothesis on $\mathcal{W}$, namely that $\mathcal{W}$ is *accessible*, *i.e.* that it is generated by a *set* of arrows of $\mathcal{C}at$ (it is the smallest basic localizer containing a *set* of arrows of $\mathcal{C}at$).

Grothendieck obtains the following characterization of $\mathcal{W}$-smooth functors [7], [9].

Theorem. *Let $u : A \longrightarrow B$ be a morphism of $\mathcal{C}at$. The following conditions are equivalent:*

(a) *u is $\mathcal{W}$-smooth,* i.e. *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \xrightarrow{v} & A' & \longrightarrow & A \\ {\scriptstyle u''}\downarrow & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array} \quad ,$$

the base change morphism $u''_! v^* \longrightarrow w^* u'_!$ is an isomorphism;

(b) *for every object b of B, the inclusion*

$$A_b \longrightarrow b\backslash A \quad , \qquad a \longmapsto (a, 1_b : b \longrightarrow u(a) = b)$$

(where A_b denotes the fiber of A over b and $b\backslash A = A \times_B (b\backslash B)$ the "comma category" of objects under b) is a $\mathcal{W}$-aspheric functor.

(c) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \xrightarrow{v} & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array} \quad ,$$

if the functor w is $\mathcal{W}$-aspheric, then so is v.

Grothendieck is filled with wonder by the following fact that he deduces from this theorem:

Corollary. *A morphism $u : A \longrightarrow B$ of $\mathcal{C}at$ is $\mathcal{W}$-smooth if and only if the opposite functor $u^\circ : A^\circ \longrightarrow B^\circ$ is $\mathcal{W}$-proper.*

This result is a straightforward consequence of the characterization above, of the "dual" characterization of $\mathcal{W}$-proper morphisms, and of the fact that a functor between small categories belongs to $\mathcal{W}$ if and only if the opposite functor does [6], [7], [9].

Fibrations (morphisms $u : A \longrightarrow B$ of $\mathcal{C}at$ such that A is a fibered category over B) are important examples of $\mathcal{W}$-smooth functors. This follows from the fact that if $u : A \longrightarrow B$ is a fibration, then for every object b of B the inclusion

$$A_b \longrightarrow b\backslash A \quad , \qquad a \longmapsto (a, 1_b : b \longrightarrow u(a) = b)$$

has a right adjoint, and from the fact that a functor between small categories which has a right adjoint is a $\mathcal{W}$-aspheric functor.

The impetus for this work was the observation that the class of fibrations shares many formal properties with the class of $\mathcal{W}$-smooth functors and that the class of functors which have a right adjoint shares many properties with the class of $\mathcal{W}$-aspheric functors. Nevertheless, there is no basic localizer $\mathcal{W}$ such that the $\mathcal{W}$-smooth morphisms are exactly the fibrations, or such that the $\mathcal{W}$-aspheric morphisms are exactly the functors which have a right adjoint. Moreover, we notice that the notions of $\mathcal{W}$-aspheric, of $\mathcal{W}$-smooth and of $\mathcal{W}$-proper morphisms depend on the sole class of $\mathcal{W}$-aspheric categories. If two basic localizers have the same class of aspheric objects, then the corresponding notions of aspheric functors, smooth functors and proper functors are the same. So I tried to look for the minimal properties that a class of *objects* of $\mathcal{C}at$ should fulfil in order to give rise to a theory of smooth functors. This has led me, at the cost of breaking the symmetry of passing to the opposite category, to introduce the notion of right asphericity structure. Since this notion is not self-dual, one needs also to consider the dual notion of left asphericity structure, giving rise to a theory of proper functors.

In the first section, we define the notion of right asphericity structure, implying a notion of aspheric functor. We notice that there exists a minimal right asphericity structure, defined by the class of small categories which have a final object, the corresponding aspheric functors being exactly the functors between small categories which have a right adjoint. For every basic localizer, we define a right asphericity structure with the same aspheric functors as the given localizer.

The second and the third sections are logically independent. In the second one, we consider localizations of categories of functors with domain a small category and codomain $\mathcal{C}at$, with respect to natural transformations that are componentwise aspheric, and we prove the existence of left homotopical Kan extensions.

In the third one, we introduce the notion of smooth functor associated with a right asphericity structure. In this framework, it splits into two notions: smooth functors and weakly smooth functors. We give several equivalent characterizations for each of these two notions and we study their main properties. We show that the smooth functors with respect to the minimal right asphericity structure are exactly the fibrations, and that the weakly smooth functors with respect to this structure are exactly the prefibrations. This result provides a new characterization of fibered categories. Finally, we observe that, when the right asphericity structure is defined by a basic localizer, the notions of smooth functor and weakly smooth functor are equivalent and equivalent to the notion of smooth functor with respect to the basic localizer.

In the last section, we combine the results of the two previous ones. We prove a characterization of smooth functors (with respect to a right asphericity structure) in terms of base change morphisms. As in this paper we consider *covariant* functors with values in $\mathcal{C}at$, rather than presheaves, we obtain a characterization which is dual to Grothendieck's formulation.

Recently, Jonathan Chiche has generalized the notion of right asphericity structure, and some of the results of this paper, to the framework of 2-categories [3].

1. Right asphericity structures

1.1. A *right asphericity structure* is a class $\mathcal{A}$ of small categories satisfying the following two conditions.

- As1 Every small category which has a final object is in $\mathcal{A}$.
- As2 For every functor between small categories $u : A \longrightarrow B$, if B is in $\mathcal{A}$, and if for every object b of B, A/b is in $\mathcal{A}$, then A is also in $\mathcal{A}$.

Example 1.2. The class $\mathcal{A}$ of small categories which have a final object is a right asphericity structure. Indeed, condition As1 being fulfilled by definition, it is enough to check condition As2. Let B be a category with a final object b_0, and $u : A \longrightarrow B$ a morphism in $\mathcal{C}at$ such that for every object b of B, A/b has a final object. Then the category A, which is isomorphic to A/b_0, has a final object, hence the assertion. This right asphericity structure is the *minimal right asphericity structure.*

Example 1.3. Let $\mathcal{W}$ be a basic localizer. The class of $\mathcal{W}$-aspheric categories is a right asphericity structure.

In the sequel, we fix, once and for all, a right asphericity structure $\mathcal{A}$.

1.4. A small category C is said to be $\mathcal{A}$-*aspheric*, or more simply *aspheric*, if it belongs to $\mathcal{A}$. Using this definition, condition As1 states that a small category which has a final object is aspheric.

Proposition 1.5. *The product of two small aspheric categories is an aspheric category.*

Proof. Let A and B be two small aspheric categories. Let us show that their product $A \times B$ is aspheric too. Considering the first projection $A \times B \longrightarrow A$, it suffices by As2 to show that for every object a of A, the category $(A \times B)/a \simeq A/a \times B$ is aspheric. Considering the second projection $A/a \times B \longrightarrow B$, it suffices by As2 to show that for every object b of B, the category $(A/a \times B)/b \simeq A/a \times B/b$ is aspheric. Now, the latter has a final object, and the assertion follows from As1. □

1.6. A morphism $u : A \longrightarrow B$ is said to be $\mathcal{A}$-*aspheric*, or more simply *aspheric*, if for every object b of B, the category A/b is aspheric. In terms of this definition, condition As2 states that if the codomain of an aspheric morphism is an aspheric category, then its domain is aspheric, too. Notice that a small category A is aspheric if and only if the functor $A \longrightarrow e$ from A to the final category is an aspheric morphism. For every small category A, the identity functor $1_A : A \longrightarrow A$ is aspheric. Indeed, for every object a of A, the category A/a has a final object, and the assertion follows from condition As1.

Example 1.7. If $\mathcal{A}$ is the minimal right asphericity structure (*cf.* 1.2), then the aspheric morphisms are exactly the functors between small categories which have a right adjoint. Indeed, a functor between small categories $u : A \longrightarrow B$ has a right adjoint if and only if, for every object b of B, the category A/b has a final object.

Example 1.8. If $\mathcal{A}$ is the right asphericity structure associated with a basic localizer $\mathcal{W}$ (*cf.* 1.3), then the $\mathcal{A}$-aspheric morphisms are exactly the $\mathcal{W}$-aspheric functors.

Corollary 1.9. *Let $u : A \longrightarrow B$ and $u' : A' \longrightarrow B'$ be two aspheric morphisms of $\mathcal{C}at$. Then the functor $u \times u' : A \times A' \longrightarrow B \times B'$ is aspheric.*

Proof. We have to show that for every object (b, b') of $B \times B'$, the category $(A \times A')/(b, b')$ is aspheric. Now, $(A \times A')/(b, b')$ is canonically isomorphic to $(A/b) \times (A'/b')$ and by hypothesis A/b and A'/b' are aspheric. Therefore the assertion follows from proposition 1.5. □

Proposition 1.10. *Let*

$$\begin{array}{ccc} A & \xrightarrow{u} & B \\ & \searrow \quad \swarrow_{v} & \\ & C & \end{array}$$

be a commutative triangle in $\mathcal{C}at$. The morphism u is aspheric if and only if, for every object c of C, the morphism $u/c : A/c \longrightarrow B/c$, induced by u, is aspheric.

Proof. We check immediately that for every object c of C and for every object $(b, p : v(b) \longrightarrow c)$ of B/c, the category $(A/c)/(b, p)$ is canonically isomorphic to A/b. We deduce that if u is aspheric, then so is u/c. Conversely, assume that for every object c of C the morphism u/c is aspheric. Then, for every object b of B, the category $(A/v(b))/(b, 1_{v(b)}) \simeq A/b$ is aspheric, which proves that u is aspheric. □

Proposition 1.11. *Let $A \xrightarrow{u} B \xrightarrow{v} C$ be a pair of composable morphisms of $\mathcal{C}at$. If u and v are aspheric, then so is vu.*

Proof. For every object c of C, since v is aspheric, the category B/c is aspheric, and by the previous proposition, since u is aspheric, so is the functor $u/c : A/c \longrightarrow B/c$ induced by u. From this we deduce that the category A/c is aspheric (As2), which proves that vu is aspheric. □

Proposition 1.12. *Let $u : A \longrightarrow B$ be a morphism of $\mathcal{C}at$. If u has a right adjoint, then u is aspheric.*

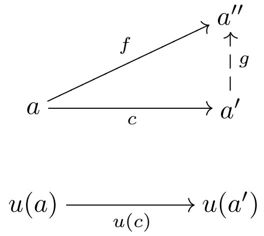
Proof. Let $v : B \longrightarrow A$ be a right adjoint to u. The functorial bijection

$$\mathsf{Hom}_B(u(a), b)) \simeq \mathsf{Hom}_A(a, v(b)), \quad a \in \mathsf{Ob}(A),\ b \in \mathsf{Ob}(B),$$

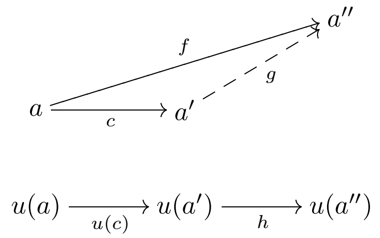
implies that, for every object b of B, the category A/b is isomorphic to the category $A/v(b)$, which has a final object. It follows that A/b is aspheric, hence the proposition. □

Corollary 1.13. *An equivalence between small categories is an aspheric functor.*

1.14. Let $u : A \longrightarrow B$ be a functor. We recall that the *fiber* of u over an object b of B is the subcategory (not full in general) A_b of A whose objects are the objects a of A such that $u(a) = b$, and whose morphisms are the arrows f of A such that $u(f) = 1_b$. An arrow $c : a \longrightarrow a'$ of A is called *cocartesian* (with respect to u, or over B) if for every morphism $f : a \longrightarrow a''$ of A such that $u(f) = u(c)$, there is a unique morphism $g : a' \longrightarrow a''$ of A such that $u(g) = 1_{u(a')}$ and $f = gc$.



The arrow c is called *hypercocartesian* (with respect to u, or over B) if for every morphism $f : a \longrightarrow a''$ of A and every morphism $h : u(a') \longrightarrow u(a'')$ of B such that $u(f) = h\,u(c)$, there is a unique morphism $g : a' \longrightarrow a''$ of A such that $u(g) = h$ and $f = gc$.



The functor u is called a *precofibration* if for every morphism $p : b \longrightarrow b'$ of B, and for every object a of A over b (*i.e.* $u(a) = b$), there is a cocartesian morphism $c : a \longrightarrow a'$ over p (*i.e.* $u(c) = p$). The functor u is called a *cofibration* if u is a precofibration, and if the class of cocartesian morphisms of A is stable under composition. It is easily checked that u is a cofibration if and only if for every morphism $p : b \longrightarrow b'$ of B and for every object a of A over b, there is a hypercocartesian morphism $c : a \longrightarrow a'$ over p.

Dually, a morphism of A is called *cartesian* (resp. *hypercartesian*) with respect to u, or over B, if the corresponding morphism of A° (the opposite category of A) is cocartesian (resp. hypercocartesian) with respect to $u^\circ : A^\circ \longrightarrow B^\circ$. The functor u is called a *prefibration* (resp. a *fibration*) if the functor u° is a precofibration (resp. a cofibration).

Lemma 1.15. *A functor $u : A \longrightarrow B$ is a precofibration if and only if for every object b of B, the canonical functor $A_b \longrightarrow A/b$, which sends an object a of the fiber A_b of u over b to the object $(a, 1_b)$ of A/b, has a left adjoint.*

The proof is left to the reader.

Proposition 1.16. *Let $u : A \longrightarrow B$ be a morphism of $\mathcal{C}at$, and assume that u is a precofibration and that for every object b of B, the fiber A_b of u over b is aspheric. Then u is aspheric.*

Proof. According to the previous lemma, for every object b of B, the functor

$$i_b : A_b \longrightarrow A/b$$
$$a \longmapsto (a, 1_b)$$

has a left adjoint

$$j_b : A/b \longrightarrow A_b \quad .$$

It follows from proposition 1.12 that the functor j_b is aspheric. Since A_b is aspheric, so is A/b (As2), hence u is aspheric. □

1.17. A morphism $u : A \longrightarrow B$ of $\mathcal{C}at$ is said to be *locally $\mathcal{A}$-aspheric*, or more simply *locally aspheric*, if for every object a of A, the morphism

$$A/a \longrightarrow B/b \quad , \qquad b = u(a) \quad ,$$

induced by u, is aspheric.

Examples 1.18. *a*) A fully faithful aspheric functor is locally aspheric.

b) For any small category C, the functor $C \longrightarrow e$ is locally aspheric.

Proposition 1.19. a) *A morphism $u : A \longrightarrow B$ of $\mathcal{C}at$ is locally aspheric if and only if, for every object b of B, the functor $u/b : A/b \longrightarrow B/b$, induced by u, is aspheric.*

b) *Let $A \xrightarrow{u} B \xrightarrow{v} C$ be a pair of composable morphisms of $\mathcal{C}at$. If u and v are locally aspheric, then so is vu.*

c) *Let $u : A \longrightarrow B$, $u' : A' \longrightarrow B'$ be two locally aspheric morphisms of $\mathcal{C}at$. Then the functor $u \times u' : A \times A' \longrightarrow B \times B'$ is locally aspheric.*

Proof. The first assertion is straightforward, the second one is a consequence of proposition 1.11, and the third one is a consequence of corollary 1.9. □

2. Homotopical Kan extensions

In this paragraph, we fix, once and for all, a right asphericity structure $\mathcal{A}$.

2.1. We denote by $\mathcal{A}sph_{\mathcal{A}}$, or simply $\mathcal{A}sph$, the class of aspheric morphisms of $\mathcal{C}at$, and we set

$$\mathsf{Hot} = \mathsf{Hot}_{\mathcal{A}} = \mathcal{A}sph^{-1}\mathcal{C}at \quad .$$

More generally, for every small category I, we denote by $\mathcal{A}sph_{\mathcal{A}}(I)$, or simply $\mathcal{A}sph(I)$, the class of morphisms of $\underline{\mathsf{Hom}}(I, \mathcal{C}at)$ which are componentwise aspheric, and we set

$$\mathsf{Hot}(I) = \mathsf{Hot}_{\mathcal{A}}(I) = \mathcal{A}sph(I)^{-1}\,\underline{\mathsf{Hom}}(I, \mathcal{C}at) \quad .$$

For every arrow $w : J \longrightarrow I$ of $\mathcal{C}at$, if we denote by

$$w^* : \underline{\mathsf{Hom}}(I, \mathcal{C}at) \longrightarrow \underline{\mathsf{Hom}}(J, \mathcal{C}at)$$

the inverse image functor, we have

$$w^*(\mathcal{A}sph(I)) \subset \mathcal{A}sph(J) \quad .$$

Therefore w^* induces a functor between the localized categories, denoted

$$w^* : \mathsf{Hot}(I) \longrightarrow \mathsf{Hot}(J) \quad ,$$

too, such that the following square is commutative:

$$\begin{array}{ccc}
\underline{\mathsf{Hom}}(I,\mathcal{C}at) & \xrightarrow{w^*} & \underline{\mathsf{Hom}}(J,\mathcal{C}at) \\
{\scriptstyle \gamma_I}\downarrow & & \downarrow{\scriptstyle \gamma_J} \\
\mathsf{Hot}(I) & \xrightarrow[w^*]{} & \mathsf{Hot}(J)
\end{array} \quad ,$$

where the vertical arrows are the canonical localization functors.

Example 2.2. If $\mathcal{A}$ is the minimal right asphericity structure (*cf.* 1.2), then Hot is the category $\overline{\mathcal{C}at}$ of small categories up to homotopy, *i.e.* the category with same objects as $\mathcal{C}at$, and such that for every pair of small categories A, B,

$$\mathsf{Hom}_{\overline{\mathcal{C}at}}(A,B) = \pi_0 \ \underline{\mathsf{Hom}}(A,B) \quad .$$

Indeed, let us denote by

$$\gamma : \mathcal{C}at \longrightarrow \mathsf{Hot} \quad , \qquad Q : \mathcal{C}at \longrightarrow \overline{\mathcal{C}at}$$

the canonical functors. By the universal properties of these functors, it is sufficient to show that:

a) the image under Q of an aspheric functor is a homotopism;
b) the images under γ of two homotopic functors are equal.

In order to show the first assertion, we notice that, since aspheric morphisms of $\mathcal{C}at$ with respect to the minimal right asphericity structure are exactly the functors between small categories which have a right adjoint (1.7), they are indeed homotopisms. In order to show the second assertion, it suffices to show that for every morphism $h : \Delta_1 \times A \longrightarrow B$ of $\mathcal{C}at$, where Δ_1 stands for the category $\{0 \longrightarrow 1\}$, we have $\gamma(h_0) = \gamma(h_1)$, where $h_\varepsilon = h(\partial_\varepsilon \times 1_A)$, $\varepsilon = 0, 1$, and $\partial_\varepsilon : e \longrightarrow \Delta_1$ is the morphism from the final category e to Δ_1 defined by the object ε of Δ_1. Now, if we denote by $p : \Delta_1 \times A \longrightarrow A$ the second projection, we have $p(\partial_0 \times 1_A) = 1_A = p(\partial_1 \times 1_A)$. Since the category Δ_1 has a final object, it follows from As1 and corollary 1.9 that the functor p is aspheric. Therefore $\gamma(p)$ is an isomorphism of Hot, and the equalities

$$\gamma(p)\gamma(\partial_0 \times 1_A) = \gamma(p(\partial_0 \times 1_A)) = \gamma(p(\partial_1 \times 1_A)) = \gamma(p)\gamma(\partial_1 \times 1_A)$$

imply the relation

$$\gamma(\partial_0 \times 1_A) = \gamma(\partial_1 \times 1_A) \quad ,$$

and hence we have

$$\gamma(h_0) = \gamma(h(\partial_0 \times 1_A)) = \gamma(h)\gamma(\partial_0 \times 1_A) = \gamma(h)\gamma(\partial_1 \times 1_A) = \gamma(h(\partial_1 \times 1_A)) = \gamma(h_1) \ .$$

Example 2.3. If $\mathcal{A}$ is the right asphericity structure associated with a basic localizer $\mathcal{W}$ (*cf.* 1.3), then it follows from the theory developed by D.-C. Cisinski [4] that

$$\mathsf{Hot} \simeq \mathcal{W}_{asph}^{-1}\,\mathcal{C}at \quad ,$$

where $\mathcal{W}_{asph}$ stands for the basic localizer generated by the arrows $A \longrightarrow e$ of $\mathcal{C}at$, for A a $\mathcal{W}$-aspheric category. Indeed, let us denote by

$$\mathcal{C}at \xrightarrow{\gamma} \mathsf{Hot} \quad , \qquad \mathcal{C}at \xrightarrow{\gamma'} \mathcal{W}_{asph}^{-1}\,\mathcal{C}at$$

the canonical functors. By their universal properties, it suffices to show that:

a) $\mathcal{A}sph \subset \mathcal{W}_{asph}$;
b) $\gamma(\mathcal{W}_{asph})$ is contained in the class of isomorphisms of Hot.

The assertion (*a*) is obvious. In order to prove (*b*), we first observe that every basic localizer being a filtered union of accessible ones, an easy argument shows that we can assume that the basic localizer is accessible. Thomason-Cisinski theory [4], [11] then states that there exists a proper closed model category structure on $\mathcal{C}at$ whose weak equivalences are the elements of $\mathcal{W}_{asph}$. This implies that $\mathcal{C}at$ has the structure of a category of fibrant objects [2], with weak equivalences the elements of $\mathcal{W}_{asph}$ and fibrations the arrows $u : A \longrightarrow B$ of $\mathcal{C}at$ such that for every diagram of cartesian squares

$$\begin{array}{ccccc} A'' & \xrightarrow{v} & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array} \quad ,$$

if w is in $\mathcal{W}_{asph}$, then so is v. Hence, by Ken Brown's lemma [2, I.1 Factorization lemma], in order to prove that the elements of $\gamma(\mathcal{W}_{asph})$ are isomorphisms of Hot, it suffices to show that if such a fibration is in $\mathcal{W}_{asph}$, then its image under γ is an isomorphism. Now, such an arrow is universally in $\mathcal{W}_{asph}$. It is therefore a $\mathcal{W}_{asph}$-aspheric morphism, and in particular $\mathcal{W}$-aspheric, which proves the assertion.

We can easily generalize this argument to show that for every small category I, the category $\mathsf{Hot}(I)$ is isomorphic to the localization of the category $\underline{\mathsf{Hom}}(I, \mathcal{C}at)$ by the arrows which are componentwise in $\mathcal{W}_{asph}$.

2.4. For every object I of $\mathcal{C}at$, we denote by $\mathcal{C}at/I$ the category of small categories over I, whose objects are the pairs $(A, A \longrightarrow I)$, consisting of a small category A and of a functor $A \longrightarrow I$, and whose morphisms are commutative triangles

$$\begin{array}{ccc} A & \longrightarrow & A' \\ & \searrow \quad \swarrow & \\ & I & \end{array} \quad .$$

We denote by $\mathcal{A}sph/I$ the class of arrows

$$\begin{array}{ccc} A & \longrightarrow & A' \\ & \searrow \quad \swarrow & \\ & I & \end{array}$$

of $\mathcal{C}at/I$ such that $A \longrightarrow A'$ is an aspheric functor. For every arrow $w : J \longrightarrow I$ of $\mathcal{C}at$, we denote by

$$\mathcal{C}at/w : \mathcal{C}at/J \longrightarrow \mathcal{C}at/I$$

the functor defined by

$$\begin{array}{ccc} A & & A \\ {\scriptstyle v}\downarrow & \longmapsto & {\scriptstyle wv}\downarrow \\ J & & I \end{array} \quad .$$

We have

$$(\mathcal{C}at/w)(\mathcal{A}sph/J) \subset \mathcal{A}sph/I \quad ,$$

and thus the functor $\mathcal{C}at/w$ induces a functor

$$\overline{\mathcal{C}at/w} : (\mathcal{A}sph/J)^{-1}(\mathcal{C}at/J) \longrightarrow (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I) \quad .$$

2.5. For every small category I, we define functors

$$\begin{array}{ccccccc} \mathcal{C}at/I & \xrightarrow{\Theta_I} & \underline{\mathsf{Hom}}(I, \mathcal{C}at) & , & \underline{\mathsf{Hom}}(I, \mathcal{C}at) & \xrightarrow{\Theta'_I} & \mathcal{C}at/I \\ (A,\, A \longrightarrow I) & \longmapsto & (i \longmapsto A/i) & , & F & \longmapsto & (\textstyle\int F,\, \int F \longrightarrow I) \quad , \end{array}$$

where $\int F$ stands for Grothendieck's construction of the cofibered category over I defined by the functor F, and $\int F \longrightarrow I$ is the canonical functor. We recall that the objects of the category $\int F$ are the pairs (i, a), where i is an object of I and a is an object of $F(i)$. A morphism of $\int F$ from (i, a) to (i', a') is a pair (k, f), where $k : i \longrightarrow i'$ is an arrow of I and $f : F(k)(a) \longrightarrow a'$ is an arrow of $F(i')$. The composition in $\int F$ is defined, for a pair of composable morphisms

$$(i,a) \xrightarrow{(k,f)} (i',a') \xrightarrow{(k',f')} (i'',a'') \quad ,$$

by the formula

$$(k',f') \circ (k,f) = (k'k, f' \cdot F(k')(f)) \quad .$$

The canonical functor $\int F \longrightarrow I$, which is a cofibration, is defined by

$$(i,a) \longmapsto i \quad , \qquad (k,f) \longmapsto k \quad .$$

For every object i of I, the fiber over i of the functor $\int F \longrightarrow I$ is canonically isomorphic to the category $F(i)$.

For every morphism $w : J \longrightarrow I$ of $\mathcal{C}at$, the functors

$$\mathcal{C}at/J \xrightarrow{\Theta_I \circ \mathcal{C}at/w} \underline{\mathsf{Hom}}(I,\mathcal{C}at) \quad , \qquad \underline{\mathsf{Hom}}(I,\mathcal{C}at) \xrightarrow{\Theta'_J \circ w^*} \mathcal{C}at/J$$

form an adjoint pair with counit and unit

$$\varepsilon : \Theta_I \circ \mathcal{C}at/w \circ \Theta'_J \circ w^* \longrightarrow 1_{\underline{\mathsf{Hom}}(I,\mathcal{C}at)} \ , \quad \eta : 1_{\mathcal{C}at/J} \longrightarrow \Theta'_J \circ w^* \circ \Theta_I \circ \mathcal{C}at/w$$

defined as follows.

a) Definition of ε. For every functor $F : I \longrightarrow \mathcal{C}at$, and every object i of I, we need to define a functor

$$\varepsilon_{F,i} : \left(\textstyle\int Fw\right) / \, i \longrightarrow F(i) \quad .$$

The objects of the category $(\int Fw)/i$ are the triples $(j, a, p : w(j) \longrightarrow i)$, where j is an object of J, a is an object of $Fw(j)$, and p is an arrow of I. A morphism of $(\int Fw)/i$ from $(j, a, p : w(j) \longrightarrow i)$ to $(j', a', p' : w(j') \longrightarrow i)$ is a pair (l, f), where $l : j \longrightarrow j'$ is an arrow of J and $f : Fw(l)(a) \longrightarrow a'$ is an arrow of $Fw(j')$ such that

$$\begin{array}{ccc} w(j) & \xrightarrow{w(l)} & w(j') \\ & {\scriptstyle p}\searrow \quad \swarrow{\scriptstyle p'} & \\ & i & \end{array} \qquad\qquad p = p'w(l) \quad .$$

We define the functor $\varepsilon_{F,i}$ by

$$\varepsilon_{F,i}(j, a, p : w(j) \longrightarrow i) = F(p)(a) \quad ,$$

$$\varepsilon_{F,i}(l,f) = F(p')(f) : F(p')Fw(l)(a) = F(p)(a) \longrightarrow F(p')(a') \quad .$$

We leave it to the reader to check compatibility with respect to composition and identities, as well as functoriality in i and in F.

b) Definition of η. For every object $(A, v : A \longrightarrow J)$ of $\mathcal{C}at/J$, we need to define a functor

$$\eta_{(A,v)} : A \longrightarrow \textstyle\int A/w(j) \quad ,$$

over J, where $\int A/w(j)$ denotes, by a slight abuse, the cofibered category over J defined by the functor

$$J \longrightarrow \mathcal{C}at \quad , \qquad j \longmapsto A/w(j) \quad .$$

The objects of $\int A/w(j)$ are the triples $(j, a, p : wv(a) \longrightarrow w(j))$, where j is an object of J, a is an object of A and p is an arrow of I. A morphism

$$(j, a, p : wv(a) \longrightarrow w(j)) \longrightarrow (j', a', p' : wv(a') \longrightarrow w(j'))$$

is a pair (l, f), where $l : j \longrightarrow j'$ is an arrow of J and $f : a \longrightarrow a'$ is an arrow of A such that the square

$$\begin{array}{ccc} wv(a) & \xrightarrow{wv(f)} & wv(a') \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle p'} \\ w(j) & \xrightarrow[w(l)]{} & w(j') \end{array}$$

is commutative. For every object a of A, we define

$$\eta_{(A,v)}(a) = \big(v(a), a, 1_{wv(a)} : wv(a) \longrightarrow wv(a)\big) \quad ,$$

and for every arrow $f : a \longrightarrow a'$ of A,

$$\eta_{(A,v)}(f) = \big(v(f), f\big) : \big(v(a), a, 1_{wv(a)}\big) \longrightarrow \big(v(a'), a', 1_{wv(a')}\big).$$

Compatibility with respect to composition and identities, as well as functoriality in (A, v), are easily checked.

We leave it to the reader to check the triangle identities

$$\big((\Theta'_J \circ w^*)\star\varepsilon\big)\big(\eta\star(\Theta'_J \circ w^*)\big) = 1_{\Theta'_J \circ w^*} \ , \ \ \big(\varepsilon\star(\Theta_I \circ \mathcal{C}at/w)\big)\big((\Theta_I \circ \mathcal{C}at/w)\star\eta\big) = 1_{\Theta_I \circ \mathcal{C}at/w}$$

$$\Theta'_J \circ w^* \xrightarrow{\eta\star(\Theta'_J \circ w^*)} \Theta'_J \circ w^* \circ \Theta_I \circ \mathcal{C}at/w \circ \Theta'_J \circ w^* \xrightarrow{(\Theta'_J \circ w^*)\star\varepsilon} \Theta'_J \circ w^*$$

$$\Theta_I \circ \mathcal{C}at/w \xrightarrow{(\Theta_I \circ \mathcal{C}at/w)\star\eta} \Theta_I \circ \mathcal{C}at/w \circ \Theta'_J \circ w^* \circ \Theta_I \circ \mathcal{C}at/w \xrightarrow{\varepsilon\star(\Theta_I \circ \mathcal{C}at/w)} \Theta_I \circ \mathcal{C}at/w$$

which prove that $(\Theta_I \circ \mathcal{C}at/w,\ \Theta'_J \circ w^*)$ is indeed an adjoint pair.

As a special case, for $J = I$ and $w = 1_I$, we get that (Θ_I, Θ'_I) is an adjoint pair.

Lemma 2.6. *Let I be a small category, $F, G : I \longrightarrow \mathcal{C}at$ be two functors and $u : F \longrightarrow G$ be a natural transformation. If, for every object i of I, the functor $u_i : F(i) \longrightarrow G(i)$ is aspheric, then the functor*

$$\textstyle\int u : \int F \longrightarrow \int G$$

is aspheric.

Proof. We define a functor

$$H : \textstyle\int G \longrightarrow \mathcal{C}at$$

as follows. For every object (i, b) of $\int G$, $i \in \mathsf{Ob}(I)$, $b \in \mathsf{Ob}(G(i))$, we set

$$H(i, b) = F(i)/b \quad ,$$

and for every morphism $(k, g) : (i, b) \longrightarrow (i', b')$ of $\int G$, where $k : i \longrightarrow i'$ is an arrow of I and $g : G(k)(b) \longrightarrow b'$ is an arrow of $G(i')$, we define

$$H(k, g) : F(i)/b \longrightarrow F(i')/b'$$

by

$$(a,\ p : u_i(a) \longrightarrow b) \longmapsto (F(k)(a),\ g \cdot G(k)(p) : u_{i'}F(k)(a) \longrightarrow b') \quad .$$

$$\begin{array}{ccc} u_{i'}F(k)(a) & \longrightarrow & b' \\ \| & & \nearrow{\scriptstyle g} \\ G(k)u_i(a) & & \\ & \searrow{\scriptstyle G(k)(p)} & \\ & G(k)(b) & \end{array}$$

Let us consider the category $\int H$. The objects of this category are the quadruples $(i,\, b,\, a,\, p\, :\, u_i(a) \longrightarrow b)$, $i \in \mathsf{Ob}(I)$, $b \in \mathsf{Ob}(G(i))$, $a \in \mathsf{Ob}(F(i))$, $p \in \mathsf{Fl}(G(i))$. A morphism from (i, b, a, p) to (i', b', a', p') is a triple (k, g, f)

$$k : i \longrightarrow i' \in \mathsf{Fl}(I)\ , \quad g : G(k)(b) \longrightarrow b' \in \mathsf{Fl}(G(i'))\ , \quad f : F(k)(a) \longrightarrow a' \in \mathsf{Fl}(F(i'))$$

such that the square

$$\begin{array}{ccc}
u_{i'}F(k)(a) & \xrightarrow{\ u_{i'}(f)\ } & u_{i'}(a') \\
\Vert & & \\
G(k)u_i(a) & & \Big\downarrow{\scriptstyle p'} \\
{\scriptstyle G(k)(p)}\Big\downarrow & & \\
G(k)(b) & \xrightarrow[\ g\ ]{} & b'
\end{array}$$

is commutative. For every pair of composable morphisms

$$(i,\, b,\, a,\, p) \xrightarrow{(k,\,g,\,f)} (i',\, b',\, a',\, p') \xrightarrow{(k',\,g',\,f')} (i'',\, b'',\, a'',\, p'') \quad ,$$

the composite morphism is defined by

$$(k',\, g',\, f') \circ (k,\, g,\, f) = (k'k,\, g' \cdot G(k')(g),\, f' \cdot F(k')(f)) \quad .$$

The canonical functor $\theta_H : \int H \longrightarrow \int G$ is defined by

$$(i, b, a, p) \longmapsto (i, b)\ ,\ \ (i, b, a, p) \in \mathsf{Ob}(\textstyle\int H)\ , \quad (k, g, f) \longmapsto (k, g)\ ,\ \ (k, g, f) \in \mathsf{Fl}(\textstyle\int H)\ .$$

We shall define a functor $S : \int F \longrightarrow \int H$ such that the triangle

$$\begin{array}{ccccc}
 & & \int H & & \\
 & {\scriptstyle S}\nearrow & & \searrow{\scriptstyle \theta_H} & \\
\int F & & \xrightarrow[\ \int u\ ]{} & & \int G
\end{array}$$

is commutative. For every object (i, a) of $\int F$, $i \in \mathsf{Ob}(I)$, $a \in \mathsf{Ob}(F(i))$, we set

$$S(i, a) = (i,\, u_i(a),\, a,\, 1_{u_i(a)}) \quad ,$$

and for every morphism $(k, f) : (i, a) \longrightarrow (i', a')$ of $\int F$, where $k : i \longrightarrow i'$ is an arrow of I and $f : F(k)(a) \longrightarrow a'$ is an arrow of $F(i')$,

$$S(k, f) = (k,\, u_{i'}(f),\, f) : (i,\, u_i(a),\, a,\, 1_{u_i(a)}) \longrightarrow (i',\, u_{i'}(a'),\, a',\, 1_{u_{i'}(a')}) \quad .$$

It is straightforward to check the compatibility of S with identities and composition, as well as to check the commutativity of the triangle above. We shall define a functor $R : \int H \longrightarrow \int F$ which will be a right adjoint to S as well as a retraction to it. For every object (i, b, a, p) of $\int H$, we set $R(i, b, a, p) = (i, a)$, and for every morphism $(k, g, f) : (i, b, a, p) \longrightarrow (i', b', a', p')$ of $\int H$, we set $R(k, g, f) = (k, f)$. Compatibility of R with identities and composition is obvious, and so is the equality $RS = 1_{\int F}$. We define a natural transformation $\varepsilon : SR \longrightarrow 1_{\int H}$ as follows. For every object $(i,\, b,\, a,\, p : u_i(a) \longrightarrow b)$ of $\int H$, we set

$$\varepsilon_{(i,b,a,p)} = (1_i, p, 1_a) : SR(i, b, a, p) = S(i, a) = (i,\, u_i(a),\, a,\, 1_{u_i(a)}) \longrightarrow (i, b, a, p) \quad ,$$

and we notice immediately that we have thus defined a morphism of $\int H$. For every morphism $(k,g,f) : (i,b,a,p) \longrightarrow (i',b',a',p')$ of $\int H$, the square

$$\begin{array}{ccc} SR(i,b,a,p) & \xrightarrow{\varepsilon_{(i,b,a,p)}} & (i,b,a,p) \\ {\scriptstyle SR(k,g,f)}\downarrow & & \downarrow{\scriptstyle (k,g,f)} \\ SR(i',b',a',p') & \xrightarrow{\varepsilon_{(i',b',a',p')}} & (i',b',a',p') \end{array}$$

is commutative, as expected. Indeed,

$$(k,g,f)\varepsilon_{(i,b,a,p)} = (k,g,f)(1_i,p,1_a) = (k, g\,G(k)(p), f) \quad ,$$

$$\varepsilon_{(i',b',a',p')}SR(k,g,f) = (1_{i'},p',1_{a'})S(k,f) = (1_{i'},p',1_{a'})(k,u_{i'}(f),f) = (k,p'u_{i'}(f),f)$$

and $p'u_{i'}(f) = g\,G(k)(p)$, since (k,g,f) is a morphism of $\int H$. From this we deduce that $\varepsilon : SR \longrightarrow 1_{\int H}$ is indeed a natural transformation. Finally, for every object (i,b,a,p) of $\int H$, we have

$$R(\varepsilon_{(i,b,a,p)}) = R(1_i,p,1_a) = (1_i,1_a) \quad ,$$

and for every object (i,a) of $\int F$, we have

$$\varepsilon_{S(i,a)} = \varepsilon_{(i,u_i(a),a,1_{u_i(a)})} = (1_i,1_{u_i(a)},1_a) \quad ,$$

which proves that

$$\varepsilon : SR \longrightarrow 1_{\int H} \qquad \text{and} \qquad 1_{1_{\int F}} : 1_{\int F} \longrightarrow RS = 1_{\int F}$$

satisfy the triangle identities and that R is a right adjoint to S. Therefore, it follows from proposition 1.12 that S is an aspheric functor. Since, for every object i of I, u_i is an aspheric functor by hypothesis, θ_H is a cofibration whose fibers are aspheric. From this we deduce that the functor θ_H is aspheric (proposition 1.16), and therefore that so is the composite morphism $\int u = \theta_H S$ (proposition 1.11), which finishes the proof. $\square$

Theorem 2.7. *For every small category I, we have*

$$\mathcal{A}sph/I = \Theta_I^{-1}(\mathcal{A}sph(I)) \quad , \qquad \Theta'_I(\mathcal{A}sph(I)) \subset \mathcal{A}sph/I$$

and the functors

$$\overline{\Theta}_I : (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I) \longrightarrow \mathcal{A}sph(I)^{-1}\,\underline{\mathsf{Hom}}(I,\mathcal{C}at) = \mathsf{Hot}(I)$$

and

$$\overline{\Theta'_I} : \mathsf{Hot}(I) = \mathcal{A}sph(I)^{-1}\,\underline{\mathsf{Hom}}(I,\mathcal{C}at) \longrightarrow (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I) \quad ,$$

induced by Θ_I and Θ'_I respectively, are equivalences of categories, quasi-inverse to each other.

Proof. The equality $\mathcal{A}sph/I = \Theta_I^{-1}(\mathcal{A}sph(I))$ follows from proposition 1.10, and the inclusion $\Theta'_I(\mathcal{A}sph(I)) \subset \mathcal{A}sph/I$ from the previous lemma. From this we deduce an adjoint pair of functors

$$\overline{\Theta}_I : (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I) \longrightarrow \mathsf{Hot}(I) \quad , \qquad \overline{\Theta'_I} : \mathsf{Hot}(I) \longrightarrow (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I)$$

the adjunction morphisms being induced by the adjunction morphisms

$$\varepsilon : \Theta_I\Theta'_I \longrightarrow 1_{\underline{\mathsf{Hom}}(I,\mathcal{C}at)} \quad , \qquad \eta : 1_{\mathcal{C}at/I} \longrightarrow \Theta'_I\Theta_I$$

(*cf.* 2.5). Therefore, it is sufficient to show that for every functor $F : I \longrightarrow \mathcal{C}at$, the natural transformation ε_F is in $\mathcal{A}sph(I)$ and that for every object $(A, v : A \longrightarrow I)$ of $\mathcal{C}at/I$, the morphism $\eta_{(A,v)}$ of $\mathcal{C}at/I$ is in $\mathcal{A}sph/I$.

a) By 2.5, for every object i of I, $\varepsilon_{F,i}$ is the morphism

$$\left(\int F\right) / \, i \longrightarrow F(i)$$

which sends an object $(j, a, p : j \longrightarrow i)$ of $\left(\int F\right) / \, i$, $j \in \mathsf{Ob}(I)$, $a \in \mathsf{Ob}(F(j))$, $p \in \mathsf{Fl}(I)$, to the object $F(p)(a)$ of $F(i)$. It is easy to check that this functor is a left adjoint to the inclusion functor

$$F(i) \longrightarrow \left(\int F\right) / \, i \quad , \qquad a \longmapsto (i, a, 1_i) \quad ,$$

(*cf.* lemma 1.15), which proves, by proposition 1.12, that it is aspheric. From this we deduce that ε_F is componentwise aspheric, which proves the assertion relative to ε_F.

b) By 2.5, the morphism $\eta_{(A,v)}$ is the inclusion

$$A \longrightarrow \int A/i \quad , \qquad a \longmapsto (v(a), a, 1_{v(a)}) \quad ,$$

where, by an abuse of notation, $\int A/i$ stands for the cofibered category over I defined by the functor

$$I \longrightarrow \mathcal{C}at \quad , \qquad i \longmapsto A/i \quad .$$

It is easy to check that this functor is a left adjoint to the functor

$$\int A/i \longrightarrow A \quad , \qquad (i, a, p : v(a) \longrightarrow i) \longmapsto a$$

(*cf.* proof of the lemma 2.6), which proves that it is aspheric (1.12) and finishes the proof of the theorem. □

2.8. For every arrow $w : J \longrightarrow I$ of $\mathcal{C}at$, we denote by

$$w_! : \underline{\mathsf{Hom}}(J, \mathcal{C}at) \longrightarrow \underline{\mathsf{Hom}}(I, \mathcal{C}at)$$

the composite functor $w_! = \Theta_I \circ \mathcal{C}at/w \circ \Theta'_J$.

$$\underline{\mathsf{Hom}}(J, \mathcal{C}at) \xrightarrow{\Theta'_J} \mathcal{C}at/J \xrightarrow{\mathcal{C}at/w} \mathcal{C}at/I \xrightarrow{\Theta_I} \underline{\mathsf{Hom}}(I, \mathcal{C}at)$$

Since, by theorem 2.7 and the considerations of paragraph 2.4, we have

$$\Theta'_J(\mathcal{A}sph(J)) \subset \mathcal{A}sph/J \, , \quad (\mathcal{C}at/w)(\mathcal{A}sph/J) \subset \mathcal{A}sph/I, \quad \Theta_I(\mathcal{A}sph/I) \subset \mathcal{A}sph(I) \, ,$$

this functor induces a functor between the localized categories, also denoted, by an abuse of notation, by

$$w_! : \mathsf{Hot}(J) \longrightarrow \mathsf{Hot}(I) \quad ,$$

which is the composite of the functors

$$\mathsf{Hot}(J) \xrightarrow{\overline{\Theta'_J}} (\mathcal{A}sph/J)^{-1}(\mathcal{C}at/J) \xrightarrow{\overline{\mathcal{C}at/w}} (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I) \xrightarrow{\overline{\Theta_I}} \mathsf{Hot}(I) \, ,$$

induced by Θ'_J, $\mathcal{C}at/w$ and Θ_I. We have a commutative square

$$\begin{array}{ccc} \underline{\mathsf{Hom}}(J, \mathcal{C}at) & \xrightarrow{w_!} & \underline{\mathsf{Hom}}(I, \mathcal{C}at) \\ {\scriptstyle \gamma_J}\downarrow & & \downarrow{\scriptstyle \gamma_I} \\ \mathsf{Hot}(J) & \xrightarrow[w_!]{} & \mathsf{Hot}(I) \end{array}$$

whose vertical arrows are the localization functors.

Theorem 2.9. *For every morphism $w : J \longrightarrow I$ of $\mathcal{C}at$, the functors*

$$w_! : \mathsf{Hot}(J) \longrightarrow \mathsf{Hot}(I) \quad , \qquad w^* : \mathsf{Hot}(I) \longrightarrow \mathsf{Hot}(J)$$

form an adjoint pair.

Proof. It follows from 2.1, 2.4, 2.5 and from theorem 2.7 that the pair of functors $(\overline{\Theta}_I \circ \overline{\mathcal{C}at/w}\,, \overline{\Theta'_J} \circ w^*)$

$$(\mathcal{A}sph/J)^{-1}(\mathcal{C}at/J) \xrightarrow{\overline{\mathcal{C}at/w}} (\mathcal{A}sph/I)^{-1}(\mathcal{C}at/I) \xrightarrow{\overline{\Theta}_I} \mathsf{Hot}(I)$$

$$\mathsf{Hot}(I) \xrightarrow{w^*} \mathsf{Hot}(J) \xrightarrow{\overline{\Theta'_J}} (\mathcal{A}sph/J)^{-1}(\mathcal{C}at/J)$$

is an adjoint pair. By theorem 2.7, the functors $\overline{\Theta}_J$ and $\overline{\Theta'_J}$ are equivalences of categories quasi-inverse to each other, therefore $w^* \simeq \overline{\Theta}_J \circ (\overline{\Theta'_J} \circ w^*)$ is a right adjoint to $(\overline{\Theta}_I \circ \overline{\mathcal{C}at/w}) \circ \overline{\Theta'_J} = w_!$, which proves the theorem. □

Remark 2.10. If $\mathcal{A}$ is the right asphericity structure associated with a basic localizer $\mathcal{W}$ (*cf.* 1.3), it follows from the theory developed by D.-C. Cisinski [4] that the functor

$$w_! : \mathsf{Hot}(J) \longrightarrow \mathsf{Hot}(I)$$

is canonically isomorphic to the left derived functor of the left adjoint of the inverse image functor

$$w^* : \underline{\mathsf{Hom}}(I, \mathcal{C}at) \longrightarrow \underline{\mathsf{Hom}}(J, \mathcal{C}at) \quad .$$

3. Smooth functors

In this paragraph, we fix, once and for all, a right asphericity structure $\mathcal{A}$.

Definition 3.1. A morphism $u : A \longrightarrow B$ of $\mathcal{C}at$ is said to be *weakly $\mathcal{A}$-smooth*, or more simply *weakly smooth*, if for every object b of B, the canonical morphism

$$j_b : A_b \longrightarrow b\backslash A \quad , \qquad a \longmapsto (a, 1_b : b \longrightarrow u(a)) \ , \quad a \in \mathsf{Ob}(A_b) \ ,$$

is aspheric.

Example 3.2. A prefibration is a weakly smooth morphism. Indeed, for every object b of B, the functor j_b has a right adjoint (dual of lemma 1.15), and therefore it is aspheric by proposition 1.12. If $\mathcal{A}$ is the minimal right asphericity structure (1.2), it follows from the characterization of aspheric functors for this structure (1.7) and from the dual of lemma 1.15 that the weakly smooth morphisms with respect to the minimal right asphericity structure are exactly the prefibrations.

Proposition 3.3. *Let $u : A \longrightarrow B$ be a morphism of $\mathcal{C}at$. The following conditions are equivalent:*

(a) *u is weakly smooth;*
(b) *for every object a of A, the fibers of the morphism*

$$A/a \longrightarrow B/b \quad , \qquad b = u(a) \quad ,$$

induced by u, are aspheric;
(c) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \longrightarrow & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ \Delta_0 & \longrightarrow & \Delta_1 & \longrightarrow & B \end{array} \quad ,$$

where $\Delta_0 \longrightarrow \Delta_1$ stands for the inclusion $\{0\} \hookrightarrow \{0 \longrightarrow 1\}$, the morphism $A'' \longrightarrow A'$ is aspheric;
(d) *for every arrow $g : b_0 \longrightarrow b_1$ of B, and every object a_1 of A_{b_1}, the category $A(a_1, g)$, the objects of which are the arrows $f : a \longrightarrow a_1$ of A whose target*

is a_1 *and which lift* g (i.e. $u(f) = g$), *and the morphisms of which are the commutative triangles in* A

$$\begin{array}{ccc} a & & \\ {\scriptstyle h}\downarrow & \overset{f}{\searrow} & \\ & & a_1 \\ & \underset{f'}{\nearrow} & \\ a' & & \end{array}$$

such that h *is a morphism of* A_{b_0} (i.e. $u(h) = 1_{b_0}$), *is aspheric.*

Proof. We leave it to the reader to check that for every arrow $g : b_0 \longrightarrow b_1$ of B, and every object a_1 of A over b_1, the category $A_{b_0}/(a_1, g)$ (defined by the functor $j_{b_0} : A_{b_0} \longrightarrow b_0\backslash A$ and the object $(a_1, g : b_0 \longrightarrow u(a_1) = b_1)$ of $b_0\backslash A$), as well as the fiber of $A/a_1 \longrightarrow B/b_1$ over the object (b_0, g) of B/b_1, are isomorphic to the category $A(a_1, g)$, which proves the equivalence of conditions (a), (b) and (d). Let us show the equivalence between (c) and (d). Observe that there is a one-to-one correspondence between arrows g of B, as in (d), and functors $\Delta_1 \longrightarrow B$, as in (c). Using the notations of (c), the inclusion $A'' \longrightarrow A'$ is, by definition, aspheric if and only if for every object a' of A', the category A''/a' is aspheric. If a' is an object of the fiber $A'_0 \simeq A''$ of A' over 0, this is true without any hypothesis on u, for in that case A''/a' has a final object. Therefore, it is sufficient to check this for a' in the fiber A'_1 of A' over 1. In that case, a' corresponds to an object a_1 of A_{b_1}, and we check that A''/a' is isomorphic to the category $A(a_1, g)$ of (d). This completes the proof. □

Corollary 3.4. *Weakly smooth morphisms are stable under base change,* i.e. *for every cartesian square in* $\mathcal{C}at$

$$\begin{array}{ccc} A' & \longrightarrow & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \longrightarrow & B \end{array} \quad ,$$

if u *is weakly smooth, then so is* u'.

Proof. The corollary follows from the previous proposition, since the condition (c) is stable under base change. □

Proposition 3.5. *Let* $u : A \longrightarrow B$ *be a morphism of* $\mathcal{C}at$*. The following conditions are equivalent:*

(a) u *is weakly smooth;*
(b) *for every object* a *of* A*, the morphism* $A/a \longrightarrow B/b$*,* $b = u(a)$*, induced by* u*, is weakly smooth.*

Proof. Assume that the functor u is weakly smooth, and let a be an object of A, and $b = u(a)$. By condition (b) of proposition 3.3, in order to prove that the functor $A/a \longrightarrow B/b$, induced by u, is weakly smooth, it is sufficient to show that for every object $(a', f : a' \longrightarrow a)$ of A/a, the fibers of the functor $(A/a)/(a', f) \longrightarrow (B/b)/(b', g)$, where $(b', g) = (u(a'), u(f))$, are aspheric. Now, this functor is canonically isomorphic to the functor $A/a' \longrightarrow B/b'$, induced by u, the fibers of which are aspheric, by condition (b) of proposition 3.3, hence the assertion. Conversely, assume that for every object a of A, the functor $A/a \longrightarrow B/b$, $b = u(a)$, induced by u, is weakly smooth. Let us show that, in that case, so is u. By condition (b) of proposition 3.3, the hypothesis that $A/a \longrightarrow B/b$ is weakly smooth implies that the fibers of the functor $(A/a)/(a, 1_a) \longrightarrow (B/b)/(b, 1_b)$ are aspheric. Now, the latter is canonically isomorphic to the functor $A/a \longrightarrow B/b$, induced by u, which proves that u is weakly smooth, by condition (b) of proposition 3.3. □

Definition 3.6. A morphism $u : A \longrightarrow B$ of $\mathcal{C}at$ is said to be $\mathcal{A}$-*smooth*, or more simply *smooth*, if for every diagram of cartesian squares

$$\begin{array}{ccccc} A'' & \xrightarrow{\ v\ } & A' & \longrightarrow & A \\ \big\downarrow & & \big\downarrow & & \big\downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array}\quad ,$$

if the morphism w is aspheric, then so is v.

Proposition 3.7. *The class of smooth morphisms is stable under composition and base change.*

Proof. It is a formal consequence of the definition. □

Proposition 3.8. *A smooth morphism is weakly smooth.*

Proof. It follows from condition (c) of proposition 3.3. □

Proposition 3.9. *A local isomorphism is a smooth morphism.*

Proof. Let $u : A \longrightarrow B$ be a local isomorphism, *i.e.* an arrow $u : A \longrightarrow B$ of $\mathcal{C}at$ such that, for every object a of A, the functor $A/a \longrightarrow B/b$, $b = u(a)$, induced by u, is an isomorphism, and consider the diagram of cartesian squares

$$\begin{array}{ccccc} A'' & \xrightarrow{\ v'\ } & A' & \xrightarrow{\ v\ } & A \\ {\scriptstyle u''}\big\downarrow & & \big\downarrow{\scriptstyle u'} & & \big\downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w']{} & B' & \xrightarrow[w]{} & B \end{array}\quad ,$$

where we assume that the morphism w' is aspheric. For every object a' of A', we deduce a diagram of cartesian squares

$$\begin{array}{ccccc} A''/a' & \longrightarrow & A'/a' & \longrightarrow & A/a \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ B''/b' & \longrightarrow & B'/b' & \longrightarrow & B/b \end{array}\quad ,$$

where $a = v(a')$, $b' = u'(a')$, $b = u(a) = w(b')$, the vertical arrows of which are isomorphisms. The category B''/b' being aspheric by hypothesis, so is A''/a', which proves the proposition. □

Proposition 3.10. *Let $u : A \longrightarrow B$ be a morphism of $\mathcal{C}at$. The following conditions are equivalent:*

(a) *u is smooth;*
(b) *for every object a of A, the morphism $A/a \longrightarrow B/b$, $b = u(a)$, induced by u, is smooth.*

Proof. Assume that u is smooth, and let a be an object of A, $b = u(a)$. Stability of smooth morphisms under base change (3.7) implies that the functor $A/b \longrightarrow B/b$, induced by u, is smooth. The canonical functor $A/a \longrightarrow A/b$ being a local isomorphism, it follows from the previous proposition that it is smooth. Stability of smooth morphisms under composition (3.7) thus implies that the composite $A/a \longrightarrow B/b$ is smooth.

Conversely, assume that for every object a of A, the functor $A/a \longrightarrow B/b$, $b = u(a)$, induced by u, is smooth, and consider a diagram of cartesian squares

$$\begin{CD} A'' @>{v}>> A' @>>> A \\ @VVV @VVV @VV{u}V \\ B'' @>>{w}> B' @>>> B \end{CD} \quad ,$$

where w is an aspheric functor. For every object a of A, we deduce a diagram of cartesian squares

$$\begin{CD} A''/a @>>> A'/a @>>> A/a \\ @VVV @VVV @VVV \\ B''/b @>>> B'/b @>>> B/b \end{CD} \quad ,$$

where $b = u(a)$. The functor w being aspheric, so is the morphism $B''/b \longrightarrow B'/b$ (1.10). The functor $A/a \longrightarrow B/b$ being smooth, it follows that $A''/a \longrightarrow A'/a$ is aspheric. Since this is true for every object a of A, it follows from proposition 1.10 that v is aspheric, which finishes the proof. □

Corollary 3.11. *Locally aspheric functors are stable under smooth base change,* i.e. *for every cartesian square in* $\mathcal{C}at$

$$\begin{CD} A' @>{v}>> A \\ @V{u'}VV @VV{u}V \\ B' @>>{w}> B \end{CD} \quad ,$$

if u is smooth and w is locally aspheric, then v is locally aspheric.

Proof. For every object a' of A', we have a cartesian square in $\mathcal{C}at$

$$\begin{CD} A'/a' @>>> A/a \\ @VVV @VVV \\ B'/b' @>>> B/b \end{CD} \quad ,$$

where $a = v(a')$, $b' = u'(a')$, and $b = u(a) = w(b')$. By the previous proposition, if u is smooth, then so is $A/a \longrightarrow B/b$, and if w is locally aspheric, then the functor $B'/b' \longrightarrow B/b$ is aspheric, which proves that $A'/a' \longrightarrow A/a$ is aspheric and finishes the proof. □

3.12. Let

$$\begin{CD} @. A \\ @. @VV{u}V \\ B' @>>{w}> B \end{CD}$$

be a diagram in $\mathcal{C}at$. We can form the cartesian square

$$\begin{CD} A' @>{v}>> A \\ @V{u'}VV @VV{u}V \\ B' @>>{w}> B \end{CD} \quad ,$$

where $A' = B' \times_B A$. We can also form the "2-square"

$$\begin{array}{ccc} A'_0 & \xrightarrow{v_0} & A \\ {\scriptstyle u'_0}\downarrow & \overset{\alpha}{\nearrow} & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[w]{} & B \end{array} \quad ,$$

where A'_0 is the comma category whose objects are triples

$$(b',\, a,\, g : w(b') \longrightarrow u(a)) \quad , \qquad b' \in \mathsf{Ob}(B') \ , \quad a \in \mathsf{Ob}(A) \ , \quad g \in \mathsf{Fl}(B) \ ,$$

a morphism from (b'_0, a_0, g_0) to (b'_1, a_1, g_1) being a pair (g', f), where $g' : b'_0 \longrightarrow b'_1$ is an arrow of B', and $f : a_0 \longrightarrow a_1$ is an arrow of A, such that the diagram

$$\begin{array}{ccc} w(b'_0) & \xrightarrow{w(g')} & w(b'_1) \\ {\scriptstyle g_0}\downarrow & & \downarrow{\scriptstyle g_1} \\ u(a_0) & \xrightarrow[u(f)]{} & u(a_1) \end{array}$$

is commutative. The functors u'_0, v_0 are defined by

$$\begin{aligned} &u'_0(b', a, g) = b' \ , \quad v_0(b', a, g) = a \ , \qquad (b', a, g) \in \mathsf{Ob}(A'_0) \ , \\ &u'_0(g', f) = g' \ , \qquad v_0(g', f) = f \ , \qquad (g', f) \in \mathsf{Fl}(A'_0) \ , \end{aligned}$$

and the natural transformation $\alpha : wu'_0 \longrightarrow uv_0$ is defined by

$$\alpha_{(b',a,g)} = g : wu'_0(b', a, g) = w(b') \longrightarrow u(a) = uv_0(b', a, g) \ , \quad (b', a, g) \in \mathsf{Ob}(A'_0) \quad .$$

We fix an object b'_0 of B', an object a_1 of A and a morphism $g : w(b'_0) \longrightarrow u(a_1)$. We set $b_0 = w(b'_0)$, $b_1 = u(a_1)$. We shall associate to these data three categories C_0, C_1, C_2, and leave it to the reader to check that they are isomorphic.

a) Definition of C_0. The triple (b'_0, a_1, g) is an object of A'_0, and there is a canonical functor $A' \longrightarrow A'_0$ which sends an object (b', a) of A', $b' \in \mathsf{Ob}(B')$, $a \in \mathsf{Ob}(A)$, $w(b') = u(a)$, to the object $(b', a, 1_{u(a)})$ of A'_0. The category C_0 is the category $A'/(b'_0, a_1, g)$.

b) Definition of C_1. The pair (b'_0, g) is an object of B'/b_1, and there is a cartesian square

$$\begin{array}{ccc} A'/a_1 & \longrightarrow & A/a_1 \\ \downarrow & & \downarrow \\ B'/b_1 & \longrightarrow & B/b_1 \end{array} \quad .$$

The category C_1 is the category $(A'/a_1)/(b'_0, g)$, defined by the left vertical arrow.

c) Definition of C_2. Let us consider the category $(B'/b'_0)^*$ obtained by adding a new final object to B'/b'_0, and the inclusion $B'/b'_0 \hookrightarrow (B'/b'_0)^*$. By the universal property of this construction, there is a unique morphism $(B'/b'_0)^* \longrightarrow B/b_1$ of $\mathcal{C}at$ such that the image of the final object of $(B'/b'_0)^*$ under this functor is the final object $(b_1, 1_{b_1})$ of B/b_1, and such that the square

$$\begin{array}{ccc} B'/b'_0 & \longrightarrow & B/b_0 \\ \big\downarrow & & \downarrow \\ (B'/b'_0)^* & \longrightarrow & B/b_1 \end{array}$$

is commutative, where the horizontal upper arrow is induced by w, and the vertical arrow on the right is defined by g. Let us consider the composite

$$(B'/b'_0)^* \longrightarrow B/b_1 \longrightarrow B$$

of this functor with the forgetful functor from B/b_1 to B, and let us form the diagram of cartesian squares

$$\begin{array}{ccccc} A'' & \longrightarrow & \overline{A'} & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B'/b'_0 & \longrightarrow & (B'/b'_0)^* & \longrightarrow & B \end{array} .$$

The category C_2 is the category A''/a'_1, where a'_1 is the object of $\overline{A'} = (B'/b'_0)^* \times_B A$ which is over the final object of $(B'/b'_0)^*$ and whose projection in A is a_1.

Theorem 3.13. *Let $u : A \longrightarrow B$ be an arrow of $\mathcal{C}at$. The following conditions are equivalent:*

(a) *u is smooth;*
(b) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \xrightarrow{v} & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array} ,$$

if the functor w is aspheric with respect to the minimal right asphericity structure, i.e. *if it has a right adjoint, then the morphism v is aspheric;*
(c) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \xrightarrow{v} & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array} ,$$

where B'' is a category which has a final object, B' is obtained by adding a new final object to B'', and w is the canonical inclusion, the morphism v is aspheric;
(d) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \longrightarrow & A' & \longrightarrow & A \\ {\scriptstyle u''}\downarrow & & \downarrow{\scriptstyle u'} & & \downarrow{\scriptstyle u} \\ B'' & \longrightarrow & B' & \longrightarrow & B \end{array} ,$$

and for every object a' of A', the morphism

$$A''/a' \longrightarrow B''/b' \quad , \qquad b' = u'(a') \quad ,$$

induced by u'', is aspheric.

Proof. The implications $(a) \Rightarrow (b) \Rightarrow (c)$ are clear. Let us show the implication $(c) \Rightarrow (d)$. As the condition (c) is stable under base change, it is sufficient to show that if

$$\begin{array}{ccc} A' & \xrightarrow{v} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[w]{} & B \end{array}$$

is a cartesian square, a_1 an object of A, $b_1 = u(a_1)$, and $(b'_0,\, g : w(b'_0) \longrightarrow b_1)$ an object of B'/b_1, then the category $(A'/a_1)/(b'_0, g)$ is aspheric. By 3.12, this category is isomorphic to the category A''/a_1', where

$$\begin{array}{ccccc} A'' & \longrightarrow & \overline{A'} & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow \\ B'/b_0' & \longrightarrow & (B'/b_0')^* & \longrightarrow & B \end{array}$$

is the diagram of cartesian squares considered in 3.12, (c), and a_1' is the object of $\overline{A'}$ over the final object of $(B'/b_0')^*$ whose image in A is the object a_1. Now, by condition (c), the functor $A'' \longrightarrow \overline{A'}$ is aspheric. From this we deduce that the category A''/a_1' is aspheric, therefore so is the category $(A'/a_1)/(b_0', g)$.

It remains to show the implication $(d) \Rightarrow (a)$. As the condition (d) is stable under base change, it is sufficient to show that if

$$\begin{array}{ccc} A' & \xrightarrow{v} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[w]{} & B \end{array}$$

is a cartesian square, where w is an aspheric morphism, then v is aspheric, too. Let a be an object of A. We need to show that the category A'/a is aspheric. Let $b = u(a)$. By condition (d), the morphism $A'/a \longrightarrow B'/b$, induced by u', is aspheric. By hypothesis, the functor w is aspheric, hence the category B'/b is aspheric, and therefore so is A'/a, which concludes the proof. □

Corollary 3.14. *A smooth functor is locally aspheric.*

Proof. It is an immediate consequence of condition (d) of the previous theorem. □

Proposition 3.15. *Let $u : A \longrightarrow B$ be a morphism of $\mathcal{C}at$. The following conditions are equivalent:*

(a) *u is a fibration;*
(b) *u is smooth with respect to the minimal right asphericity structure;*
(c) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \xrightarrow{v} & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ B'' & \xrightarrow[w]{} & B' & \longrightarrow & B \end{array} \quad ,$$

if the functor w has a right adjoint, then so has v;

(d) *for every diagram of cartesian squares*

$$\begin{array}{ccccc} A'' & \longrightarrow & A' & \longrightarrow & A \\ \downarrow & & \downarrow & & \downarrow{\scriptstyle u} \\ \Delta_1 & \longrightarrow & \Delta_2 & \longrightarrow & B \end{array} \quad ,$$

where $\Delta_1 \longrightarrow \Delta_2$ stands for the inclusion $\{0 \longrightarrow 1\} \hookrightarrow \{0 \longrightarrow 1 \longrightarrow 2\}$, the functor $A'' \longrightarrow A'$ has a right adjoint.

Proof. Since aspheric functors with respect to the minimal right asphericity structure are exactly those which have a right adjoint (1.7), equivalence between conditions (b) and (c) is tautological. As the implication $(c) \Rightarrow (d)$ is clear, it is sufficient to prove the implications $(a) \Rightarrow (b)$ and $(d) \Rightarrow (a)$.

Since fibrations are stable under base change, in order to show the implication $(a) \Rightarrow (b)$ it is sufficient to show that for every cartesian square

$$\begin{array}{ccc} A' = B' \times_B A & \longrightarrow & A \\ \big\downarrow & & \big\downarrow{\scriptstyle u} \\ B' & \xrightarrow[w]{} & B \end{array} \qquad ,$$

if u is a fibration, and if for every object b of B, the category B'/b has a final object, then for every object a of A, the category A'/a has a final object.

Accordingly, let a_1 be an object of A, b_1 its image in B, and $(b'_0,\, g : w(b'_0) \longrightarrow b_1)$ a final object of B'/b_1. Since u is a fibration, there exists a hypercartesian morphism $k : a_0 \longrightarrow a_1$ of A such that $u(k) = g$. We shall show that $\big((b'_0, a_0),\, k : a_0 \longrightarrow a_1\big)$ is a final object of A'/a_1. Let $\big((b', a),\, f : a \longrightarrow a_1\big)$ be an object of A'/a_1, $w(b') = u(a)$. We have to show that there exists an arrow of A'/a_1 whose domain is $\big((b', a), f\big)$ and whose codomain is $\big((b'_0, a_0), k\big)$, and that it is unique. In other words, we have to show that there exists a unique pair (g', h), where $g' : b' \longrightarrow b'_0$ is an arrow of B' and $h : a \longrightarrow a_0$ is an arrow of A, such that $w(g') = u(h)$ and $f = kh$. For such a pair, we have $u(f) = u(k)u(h) = gw(g')$, which means that

$$g' : \big(b',\, u(f) : w'(b') \longrightarrow b_1\big) \longrightarrow \big(b'_0,\, g : w(b'_0) \longrightarrow b_1\big)$$

is an arrow of B'/b_1. Now, the hypothesis that (b'_0, g) is a final object of B'/b_1 implies that such a g' exists and is unique. Since the morphism k is hypercartesian, there exists a unique arrow $h : a \longrightarrow a_0$ of A such that $u(h) = w(g')$ and $f = kh$, which proves the assertion.

It remains to show the implication $(d) \Rightarrow (a)$. Let $\Delta_2 \longrightarrow B$ be a functor defined by a pair of composable arrows

$$b_0 \xrightarrow{g_0} b_1 \xrightarrow{g_1} b_2$$

of B. Let us form the diagram of cartesian squares which appears in the statement of condition (d). This condition means that for every object a'_2 of A' whose image a_2 in A is over b_2, the category A''/a'_2 has a final object. Let us describe this category. The set of objects of A''/a'_2 can be canonically identified with the disjoint sum $A''_0 \amalg A''_1$, where

$$A''_0 = \{(a_0, f_0) \mid a_0 \in \mathsf{Ob}(A),\, u(a_0) = b_0,\, f_0 : a_0 \longrightarrow a_2 \in \mathsf{Fl}(A),\, u(f_0) = g_1 g_0\} \quad ,$$

$$A''_1 = \{(a_1, f_1) \mid a_1 \in \mathsf{Ob}(A),\, u(a_1) = b_1,\, f_1 : a_1 \longrightarrow a_2 \in \mathsf{Fl}(A),\, u(f_1) = g_1\} \quad ,$$

and for every (a_0, f_0), $(a'_0, f'_0) \in A''_0$, (a_1, f_1), $(a'_1, f'_1) \in A''_1$, we have

$$\mathsf{Hom}_{A''/a'_2}((a_0, f_0), (a'_0, f'_0)) = \{g \mid g : a_0 \longrightarrow a'_0 \in \mathsf{Fl}(A),\, u(g) = 1_{b_0},\, f_0 = f'_0 g\} \quad ,$$

$$\mathsf{Hom}_{A''/a'_2}((a_0, f_0), (a_1, f_1)) = \{g \mid g : a_0 \longrightarrow a_1 \in \mathsf{Fl}(A),\, u(g) = g_0,\, f_0 = f_1 g\} \quad ,$$

$$\mathsf{Hom}_{A''/a'_2}((a_1, f_1), (a_0, f_0)) = \varnothing \quad ,$$

$$\mathsf{Hom}_{A''/a'_2}((a_1, f_1), (a'_1, f'_1)) = \{g \mid g : a_1 \longrightarrow a'_1 \in \mathsf{Fl}(A),\, u(g) = 1_{b_1},\, f_1 = f'_1 g\} \quad .$$

Since the category A''/a'_2 has a final object, it is non-empty. The special case $b_0 = b_1$ and $g_0 = 1_{b_1}$ then shows that for every arrow $b_1 \longrightarrow b_2$ of B and for every object a_2 of A over b_2, there exists a morphism $a_1 \longrightarrow a_2$ of A over $b_1 \longrightarrow b_2$. Coming back to the general case ($b_0 \xrightarrow{g_0} b_1 \xrightarrow{g_1} b_2$ being arbitrary), this implies that A''_0 and A''_1 are non-empty sets. Since there is no morphism in A''/a'_2 from an object of A''_1 to an object of A''_0, we deduce that the final object (a_1, f_1) of A''/a'_2 belongs to A''_1. This implies that (a_1, f_1) is also a final object of the full subcategory of A''/a'_2 whose objects are the objects of A''_1. This means exactly that f_1 is a cartesian morphism over g_1 and shows that f_1 is determined by the sole g_1 and does not depend on g_0. As g_1 is an arbitrary arrow of B and a_2 an arbitrary object of the fiber of A over the target of g_1, this implies already that u is a prefibration. Now, (a_1, f_1) is a final object of A''/a'_2. Therefore, for every object (a_0, f_0) of A''/a'_2 which belongs to the set A''_0, $f_0 : a_0 \longrightarrow a_2 \in \mathsf{Fl}(A)$, $u(f_0) = g_1 g_0$, there exists a unique morphism $g : (a_0, f_0) \longrightarrow (a_1, f_1)$ of A''/a'_2. In other words, there is a unique arrow $g : a_0 \longrightarrow a_1$ of A such that $f_1 g = f_0$ and $u(g) = g_0$. Since g_0 is an arbitrary arrow of B whose codomain is the domain of g_1, and since f_1 does not depend on g_0, it follows that the morphism f_1 is hypercartesian, which finishes the proof. □

Corollary 3.16. *Fibrations are smooth functors (with respect to any right asphericity structure).*

Proof. The corollary follows from condition (c) of the previous proposition, condition (b) of theorem 3.13, and from proposition 1.12. □

Example 3.17. If $\mathcal{A}$ is the right asphericity structure associated with a basic localizer $\mathcal{W}$ (example 1.3), Grothendieck's characterization of $\mathcal{W}$-smooth functors and theorem 3.13 imply that, if u is a morphism of $\mathcal{C}at$, the following conditions are equivalent:

- (a) u is $\mathcal{W}$-smooth;
- (b) u is weakly $\mathcal{A}$-smooth;
- (c) u is $\mathcal{A}$-smooth.

On the other hand, since there exist prefibrations which are not fibrations, proposition 3.15 and example 3.2 show that if $\mathcal{A}$ is the minimal right asphericity structure, then the class of $\mathcal{A}$-smooth functors is a proper subclass of the class of weakly $\mathcal{A}$-smooth functors.

4. Smooth functors and base change morphisms

Lemma 4.1. *Let $w : J \longrightarrow I$ be a morphism of $\mathcal{C}at$ and $F : I \longrightarrow \mathcal{C}at$ be a functor. We then have a cartesian square*

$$\begin{array}{ccc} \int Fw & \xrightarrow{\widetilde{w}} & \int F \\ {\scriptstyle \theta_{Fw}}\downarrow & & \downarrow{\scriptstyle \theta_F} \\ J & \xrightarrow[w]{} & I \end{array} \quad ,$$

where θ_{Fw} and θ_F stand for the cofibrations associated with the functors Fw and F respectively, and $\widetilde{w}$ stands for the functor $(j, a) \longmapsto (w(j), a)$, $(j, a) \in \mathsf{Ob}(\int Fw)$, induced by w.

Proof. The lemma follows from an easy verification which is left to the reader. □

4.2. Let

$$\mathcal{D} \; = \quad \begin{array}{ccc} A' & \xrightarrow{w} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[v]{} & B \end{array}$$

be a cartesian square of $\mathcal{C}at$. For every functor $F : A \longrightarrow \mathcal{C}at$, we deduce a composite cartesian square

$$\begin{array}{ccc} \int Fw & \xrightarrow{\widetilde{w}} & \int F \\ {\scriptstyle \theta_{Fw}}\downarrow & & \downarrow{\scriptstyle \theta_F} \\ A' & \xrightarrow{w} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[v]{} & B \end{array}$$

(*cf.* 4.1). For every object b' of B', the functor $\widetilde{w}$ induces a functor

$$(\textstyle\int Fw)/b' \longrightarrow (\textstyle\int F)/v(b') \quad ,$$

and we notice that

$$(\textstyle\int Fw)/b' = (u'_!w^*(F))(b') \qquad \text{and} \qquad (\textstyle\int F)/v(b') = (v^*u_!(F))(b')$$

(*cf.* 2.8). We deduce a morphism

$$\kappa_{\mathcal{D}} : u'_!w^* \longrightarrow v^*u_!$$

of $\underline{\mathsf{Hom}}\big(\underline{\mathsf{Hom}}(A, \mathcal{C}at), \underline{\mathsf{Hom}}(B', \mathcal{C}at)\big)$, called *base change morphism associated with the square* $\mathcal{D}$.

Proposition 4.3. *Let*

$$\mathcal{D} = \quad \begin{array}{ccc} A' & \xrightarrow{w} & A \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle u} \\ B' & \xrightarrow[v]{} & B \end{array}$$

be a cartesian square of $\mathcal{C}at$*, where* v *is a smooth functor. Then the base change morphism* $\kappa_{\mathcal{D}} : u'_!w^* \longrightarrow v^*u_!$ *is componentwise aspheric. In other words, for every functor* $F : A \longrightarrow \mathcal{C}at$ *and for every object* b' *of* B'*, the morphism*

$$\kappa_{\mathcal{D},F}(b') : (u'_!w^*(F))(b') \longrightarrow (v^*u_!(F))(b')$$

is aspheric.

Proof. Let $F : A \longrightarrow \mathcal{C}at$ be a functor, and consider the cartesian square

$$\begin{array}{ccc} \int Fw & \xrightarrow{\widetilde{w}} & \int F \\ {\scriptstyle u'\theta_{Fw}}\downarrow & & \downarrow{\scriptstyle u\theta_F} \\ B' & \xrightarrow[v]{} & B \end{array} \quad .$$

Since v is a smooth functor, it follows from condition (d) of theorem 3.13 that for every object b' of B', the functor $\kappa_{\mathcal{D},F}(b') : (\int Fw)/b' \longrightarrow (\int F)/v(b')$, induced by $\widetilde{w}$, is aspheric, which proves the proposition. □

Theorem 4.4. *Let* $u : A \longrightarrow B$ *be a morphism of* $\mathcal{C}at$*. The following conditions are equivalent :*

(a) *u is smooth;*
(b) *for every diagram of cartesian squares in* $\mathcal{C}at$

$$\begin{array}{ccc} A'' & \longrightarrow & B'' \\ \downarrow & & \downarrow \\ A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow[u]{} & B \end{array} \quad ,$$

the base change morphism associated with the upper square is componentwise an aspheric functor.

Proof. Implication $(a) \Rightarrow (b)$ follows from proposition 4.3 and from stability of smooth morphisms under base change (proposition 3.7). Conversely, we notice that condition (b), applied to the constant functor $B'' \longrightarrow \mathcal{C}at$ with value the final category e, implies that for every object a' of A', if we denote its image in B' by b', the morphism $A''/a' \longrightarrow B''/b'$ is aspheric. We deduce that u fulfils condition (d) of theorem 3.13, which proves the theorem. □

Remark 4.5. Let

$$\mathcal{D} = \begin{array}{ccc} A' & \xrightarrow{w} & A \\ {\scriptstyle u'}\big\downarrow & & \big\downarrow{\scriptstyle u} \\ B' & \xrightarrow[v]{} & B \end{array}$$

be a cartesian square of $\mathcal{C}at$, and

$$\begin{array}{ccc} \underline{\mathsf{Hom}}(A, \mathcal{C}at) & \xrightarrow{w^*} & \underline{\mathsf{Hom}}(A', \mathcal{C}at) \\ {\scriptstyle u_!}\big\downarrow & \not\swarrow_{\kappa_{\mathcal{D}}} & \big\downarrow{\scriptstyle u'_!} \\ \underline{\mathsf{Hom}}(B, \mathcal{C}at) & \xrightarrow[v^*]{} & \underline{\mathsf{Hom}}(B', \mathcal{C}at) \end{array} \qquad u'_! w^* \xrightarrow{\kappa_{\mathcal{D}}} v^* u_!$$

the base change morphism (4.2). One checks easily that the natural transformation

$$\begin{array}{ccc} \mathsf{Hot}(A) & \xrightarrow{w^*} & \mathsf{Hot}(A') \\ {\scriptstyle u_!}\big\downarrow & \not\swarrow_{\overline{\kappa}_{\mathcal{D}}} & \big\downarrow{\scriptstyle u'_!} \\ \mathsf{Hot}(B) & \xrightarrow[v^*]{} & \mathsf{Hot}(B') \end{array} \qquad u'_! w^* \xrightarrow{\overline{\kappa}_{\mathcal{D}}} v^* u_! \quad ,$$

induced by $\kappa_{\mathcal{D}}$ by localization, is the "base change morphism" formally defined by the adjunctions (*cf.* theorem 2.9): the natural transformation $\overline{\kappa}_{\mathcal{D}}$ is the composite

$$u'_! w^* \xrightarrow{u'_! w^* \star \eta} u'_! w^* u^* u_! = u'_! u'^* v^* u_! \xrightarrow{\varepsilon' \star v^* u_!} v^* u_! \quad ,$$

where

$$\eta : 1_{\mathsf{Hot}(A)} \longrightarrow u^* u_! \quad , \qquad \varepsilon' : u'_! u'^* \longrightarrow 1_{\mathsf{Hot}(B')}$$

denote the adjunction morphisms. Proposition 4.3 implies that if the functor v is smooth, then the natural transformation $\overline{\kappa}_{\mathcal{D}}$ is an isomorphism.

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