SHARP COHOMOLOGY

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Abstract. [The general plan is to associate to an algebraic scheme X which is defined over $k = \mathbb{C}$, the complex numbers, a "sharp" singular cohomology $H^*_\sharp(X)$. That is a "formal Hodge structure" containing, in the underlying algebraic structure, a formal group over $k = \mathbb{C}$ which is an extension of ordinary singular cohomology mixed Hodge structure, *i.e.*, its étale part is $H^*(X_{\rm an}, \mathbb{Z})$. There will be "sharp" versions of De Rham (over k of zero characteristic) and crystalline (in positive characteristics) as well. Following Grothendieck strategy to construct a cohomology I'm taking care of the H^1 first *via* Laumon 1-motives and "sharp" realizations.]

Some references are linked browsing `http://www.math.unipd.it/~barbieri`

References

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Geography

[7], [2]: Laumon introduced a generalization of Deligne's 1-motives over a field k of characteristic zero by considering $M := [F \xrightarrow{u} G]$ where F is a formal group and G is a connected algebraic group. The main idea of [2] is to use Laumon's idea as a "model" for *formal* Hodge theory, providing the basic definition of *formal* (mixed) Hodge structures (of level ≤ 1). It is done by showing that the Hodge realization of Deligne's 1-motives extends to a realization $T_\oint$ from Laumon's 1-motives over $k = \mathbb{C}$ to formal Hodge structures (of level ≤ 1) yielding an equivalence of categories.

[1],[4],[9]: In [1] (*cf.* also the covering letter to Beilinson [2]) the general program is drafted. A full exposition regarding formal mixed Hodge structures with arbitrary Hodge numbers and improvements will be provided within [9]. The enriched Hodge structures of [4] are "special" formal Hodge structures.

[5],[6],[8],[10]: For an algebraic variety X the algebraic construction of Laumon 1-motives $\mathrm{Pic}_a^+(X)$ and $\mathrm{Alb}_a^+(X)$ along with their Cartier duals $\mathrm{Alb}_a^-(X)$ and $\mathrm{Pic}_a^-(X)$ is provided (in [8] for open algebraic varieties with good compactification). The associated étale 1-motives are $\mathrm{Pic}^+(X)$, $\mathrm{Alb}^+(X)$, $\mathrm{Alb}^-(X)$ and $\mathrm{Pic}^-(X)$ previously constructed (jointly with V. Srinivas, *cf.* [1]). Accordingly, these $\mathrm{Pic}_a^+(X)$ and $\mathrm{Alb}_a^+(X)$ are providing an algebraic definition of $H^1_\sharp(X)(1)$ and $H^{2n-1}_\sharp(X)(n)$ for $n = \dim X$.

[3]: The *sharp* (universal) extension of a Laumon 1-motive (with torsion) over a field of characteristic zero is provided. The sharp de Rham realization is the Lie-algebra, *e.g.*, providing $H^1_{\sharp-DR}(X)(1)$ when applied to $\mathrm{Pic}_a^+(X)$. Over the complex numbers a sharp de Rham comparison theorem in the category of formal Hodge structures is obtained.

Dictionary

MHS: is Deligne's category of (graded polarizable, if needed) $\mathbb{Z}$-mixed Hodge structures $(H_\mathbb{Z}, W_*, F^*_{Hodge})$

VSP: is the category of diagrams V given by

$$\cdots = V_k = V_{k-1} \to \cdots \to V_{k-h} \to 0 \to 0 \cdots$$

composable linear mappings of finite dimensional $\mathbb{C}$-vector spaces

EHS: is Bloch-Srinivas category of enriched mixed Hodge structures (E, V) given by commutative diagrams

$$\begin{array}{ccccc}
H_\mathbb{C} & \twoheadrightarrow & H_\mathbb{C}/F^{k-1}_{Hodge} & \twoheadrightarrow \cdots \twoheadrightarrow & H_\mathbb{C}/F^{k-h}_{Hodge} \\
\uparrow & & \uparrow & & \uparrow \\
E & \longrightarrow & V_{k-1} & \to \cdots \longrightarrow & V_{k-h} \\
\uparrow & & & & \\
H_\mathbb{C} & & & &
\end{array}$$

for $(H_{\mathbb{Z}}, W_*, F^*_{Hodge})$ with $0 = F^k_{Hodge} \subseteq \cdots \subseteq F^{k-h-1}_{Hodge} = H_{\mathbb{C}}$ and a splitting $E = H_{\mathbb{C}} \times E_{\dagger}$

FHS: is the category of formal Hodge structures. An object $(H, V) \in$ FHS is given by $H = H_{\mathbb{Z}} \times H^0$ a formal group over $\mathbb{C}$ such that $H_{\mathbb{Z}} = H_{\text{ét}}$ is the underlying group of $(H_{\mathbb{Z}}, W_*, F^*_{Hodge}) \in$ MHS, an object V of VSP and $V^0 \subseteq V$ a subobject, an augmentation map $v : H \to V$ and a $\mathbb{C}$-isomorphism $\sigma : H_{\mathbb{C}}/F_{Hodge} \xrightarrow{\simeq} V/V^0$ such that

$$\begin{array}{ccc} H_{\mathbb{Z}} & \xrightarrow{v_{\mathbb{Z}}} & V \\ c \downarrow & & \downarrow pr \\ H_{\mathbb{C}}/F_{Hodge} & \xrightarrow{\sigma} & V/V^0 \end{array}$$

commutes. Morphisms of formal Hodge structures are pairs of compatible maps. An object of FHS yields a commutative diagram

$$\begin{array}{ccccc} & & H_{\mathbb{C}}/F^{k-1}_{Hodge} & \twoheadrightarrow \cdots \twoheadrightarrow & H_{\mathbb{C}}/F^{k-h}_{Hodge} \\ & & \uparrow & & \uparrow \\ H & \longrightarrow & V_{k-1} & \to \cdots \longrightarrow & V_{k-h} \\ \uparrow & & & & \\ H_{\mathbb{Z}} & & & & \end{array}$$

with surjective vertical arrows.

$\text{FHS}_{\text{ét}}$: is the full subcategory of FHS of étale structures, *i.e.*, for $(H, V) \in$ FHS let $(H, V)_{\text{ét}} := (H_{\mathbb{Z}}, V/V^0) \cong (H_{\mathbb{Z}}, H_{\mathbb{C}}/F_{Hodge})$ and say that (H, V) is étale if $(H, V)_{\text{ét}} = (H, V)$.

FHS^0**:** are the connected structures, *i.e.*, if $(H, V)_{\text{ét}} = 0$. For example, here $(0, V^0)$ is a connected substructure of (H, V). Denote $(H, V)_{\times} := (H, V/V^0)$ and note that we have a *canonical* extension

$$0 \to (H, V)_{\text{ét}} \to (H, V)_{\times} \to (H^0, 0) \to 0$$

Here we also have that (H^0, V) is a connected structure associated to any (H, V) but it is not a substructure, in general.

FHS^s**:** are the special structures, *i.e.*, say that (H, V) is special if $(H^0, V^0) := (H, V)^0$ is a substructure of (H, V) or, equivalently, $(H, V)_{\text{ét}}$ is a quotient of (H, V), so that we have an extension

$$0 \to (H, V)^0 \to (H, V) \to (H, V)_{\text{ét}} \to 0$$

in this case.

$\text{FHS}_{\text{ét}} =$ MHS: the equivalence of $\text{FHS}_{\text{ét}}$ with the category MHS is *via* "the étale forgetful functor" $(H, V) \mapsto H_{\text{ét}}$

$\text{FHS}^0 =$ VSP: the equivalence of FHS^0 with the (augmented) category VSP is *via*

$$(H, V) \mapsto \text{Lie}\,(H^0) \to V_{k-1} \to \cdots \longrightarrow V_{k-h}$$

$\mathrm{FHS}^s = \mathrm{EHS}$: the category FHS^s is equivalent to the category EHS *via* $(H, V) \mapsto (H_{\mathbb{C}} \times \mathrm{Lie}\,(H^0), V)$ since $\mathrm{Hom}\,(H, V) = \mathrm{Hom}\,(H_{\mathbb{C}} \times \mathrm{Lie}\,(H^0), V)$ and, conversely, given (E, V) from the splitting $E = H_{\mathbb{C}} \times E_{\dagger}$ we get the formal group $H = H_{\mathbb{Z}} \times \widehat{E_{\dagger}}$ such that $\widehat{E_{\dagger}}$ is mapped to V^0_{k-1}.

$\mathrm{VSP} \subset \mathrm{FHS}$: is embedded as a Serre subcategory of FHS with a left inverse and quotient MHS

$\mathrm{MHS} \subset \mathrm{FHS}$: fully embedded in FHS has also a left inverse "the étale forgetful functor"

$\mathrm{EHS} \subset \mathrm{FHS}$: is the largest subcategory of FHS such that MHS into FHS has a left adjoint and VSP into FHS has a right adjoint (according with the presentation of an enriched or special structure as an extension of the étale part by the connected one).

We then have the following picture:

$$\begin{array}{ccccc} & & \mathrm{FHS} & & \\ & & \uparrow & & \\ & & \mathrm{EHS} & & \\ & \nearrow & & \nwarrow & \\ \mathrm{MHS} & & & & \mathrm{VSP} \end{array}$$

along with the corresponding left inverses from FHS and restricting to left/right adjoints from EHS. The relation with Laumon 1-motives is that Deligne's Hodge realization of 1-motives can be extended to a fully-faithful functor

$$T_{\oint} : \text{Laumon's 1-motives} \longrightarrow \mathrm{FHS}$$

yielding the equivalences

$$\begin{array}{ccc} \text{Deligne's 1-motives} & \xrightarrow{T_{Hodge}} & \mathrm{MHS}_1^{\mathrm{fr}} \\ \uparrow\downarrow & & \uparrow\downarrow \\ \text{Laumon's 1-motives} & \xrightarrow{T_{\oint}} & \mathrm{FHS}_1^{\mathrm{fr}} \end{array}$$

where the category $\mathrm{FHS}_1^{\mathrm{fr}}$ is given by (H, V) where $H_{\text{ét}}$ is free, carry on a mixed Hodge structures of level ≤ 1 and V is just a single $\mathbb{C}$-vector space. Further

$$T_{\oint}([F \xrightarrow{u} G]) := (T_{\oint}(F), \mathrm{Lie}\,(G))$$

where $T_{\oint}(F)_{\text{ét}}$ is the underlying abelian group to $T_{Hodge}([F \xrightarrow{u} G])$ and $T_{\oint}(F)^0 = F^0$. Under this equivalence Cartier duality corresponds to a canonical involution. Note that such (graded polarizable of level ≤ 1) special or enriched structures are not compatible with Cartier duality. Observe that a Laumon 1-motive $M = [F \xrightarrow{u} G]$ with u mapping F^0 to $V(G)$ (= the maximal additive subgroup of G) $\iff$ the realization $T_{\oint}(M) \in \mathrm{FHS}_1^{\mathrm{fr}}$ is special. Then note that the Cartier dual of $M = [\widehat{A} \to A]$ for an abelian variety A is the universal $\mathbb{G}_a$-extension $\mathrm{Pic}^{\,0,\natural}(A)$ of the dual $\mathrm{Pic}^{\,0}(A)$.

Preview

We then want to associate to an algebraic $\mathbb{C}$-scheme X a formal Hodge structure called "sharp" cohomology $H^r_\sharp(X)(s) = (H, V)$ such that $H_{\acute{e}t} = H^r(X_{\rm an}, \mathbb{Z}(s))$ is Deligne's mixed Hodge structure on singular cohomology of the associated analytic space (Tate twisted by s). There will be "sharp" versions of De Rham and crystalline as well. Taking care of the H^1 first *via* Laumon 1-motives and "sharp" realizations means that, we should obtain $H^1_\sharp(X)$, $H^1_{\sharp-DR}(X)$, $H^1_{\sharp-crys}(X)$ *via* a Laumon 1-motive $\mathrm{Pic}^+_a(X)$ (and, dually, the H_1 by its Cartier dual $\mathrm{Alb}^-_a(X)$). Similarly $H^{2n-1}_\sharp(X)(n)$, $H^{2n-1}_{\sharp-DR}(X)(n)$, $H^{2n-1}_{\sharp-crys}(X)(n)$ for $n = \dim X$ *via* a Laumon 1-motive $\mathrm{Alb}^+_a(X)$ (and, dually, the H_{2n-1} by its Cartier dual $\mathrm{Pic}^-_a(X)$). If X is proper $\mathrm{Pic}^+_a(X) := \mathrm{Pic}^0(X)$ and

$$H^1_\sharp(X)(1) := (H^1(X_{\rm an}, \mathbb{Z}(1)), H^1(X, \mathcal{O}_X)) = T_\oint(\mathrm{Pic}^+_a(X))$$

In general, for X a proper $\mathbb{C}$-scheme, it is not difficult to see that

$$H^r_\sharp(X)(s) := (H^r(X_{\rm an}, \mathbb{Z}(s)), \mathbb{H}^r(X, \Omega_X^{\star\leq r-s}))$$

belongs to FHS. We get it from the following diagram

$$\begin{array}{ccccc} & & \mathbb{H}^r(X_\bullet, \Omega_{X_\bullet}^{\star\leq r-s}) & \twoheadrightarrow \cdots \twoheadrightarrow & H^r(X_\bullet, \mathcal{O}_{X_\bullet}) \\ & & \uparrow & & \uparrow \\ H^r(X_{\rm an}, \mathbb{Z}(s)) & \longrightarrow & \mathbb{H}^r(X, \Omega_X^{\star\leq r-s}) & \to \cdots \longrightarrow & H^r(X, \mathcal{O}_X) \end{array}$$

where $X_\bullet \to X$ is a smooth hypercovering and $\mathbb{H}^r(X_\bullet, \Omega_{X_\bullet}^{\star\leq k-1}) \cong H^r(X_{\rm an}, \mathbb{C})/F^k_{Hodge}$. Actually, for $r = 2n-1$ and $s = n = \dim X$ we also have that

$$H^{2n-1}_\sharp(X)(n) := (H^{2n-1}(X_{\rm an}, \mathbb{Z}(n)), H^{2n-1}(X, \Omega_X^{\star\leq n-1})) = T_\oint(\mathrm{Alb}^+_a(X))$$

where $\mathrm{Alb}^+_a(X)$ is the Esnault-Srinivas-Viehweg Albanese. However, one should also be able to see that the largest 1-motivic part of $H^{1+i}_\sharp(X)(1)$ can be algebraically defined *via* Laumon 1-motives $\mathrm{Pic}^+_a(X, i)$ for $i \geq 0$ (generalizing Deligne's conjecture on 1-motives, etc.). If X is not proper, for example, if X is smooth then

$$H^1_\sharp(X)(1) := (H^1(X_{\rm an}, \mathbb{Z}(1)) \times \ker H^1(\overline{X}, \mathcal{O}_{\overline{X}}) \to H^1(X, \mathcal{O}_X), H^1(\overline{X}, \mathcal{O}_{\overline{X}})) = T_\oint(\mathrm{Pic}^+_a(X))$$

where $\overline{X}$ is a smooth compactification with normal crossing Y and $X = \overline{X} - Y$. The $\mathrm{Pic}^+_a(X)$ is given by $[F \to \mathrm{Pic}^0(\overline{X})]$, the étale part $F_{\acute{e}t}$ of the formal group F is given by $\mathrm{Div}^0_Y(\overline{X})$ algebraically equivalent to zero divisors on $\overline{X}$ supported on Y and F^0 has Lie algebra $\mathrm{H}^1_Y(\overline{X}, \mathcal{O}_{\overline{X}})$ modulo the image of $\mathrm{H}^0(X, \mathcal{O}_X)$, *i.e.*, the Cartier dual $\mathrm{Alb}^-_a(X)$ is the maximal Faltings-Wüstholz additive extension of the Serre's Albanese semi-abelian variety of X.

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