Some Remarks on Kasparov Theory

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INTRODUCTION

This work consists of the simplification of two central points of Kasparov's paper [4]: the homotopy invariance of the "Ext" bifunctor [4, Sect. 6, Theorem 1] and the associativity of the Kasparov (intersection) product [4, Sect. 4, Theorem 4].

The proof of homotopy invariance given here, is based upon the following remark: the Kasparov product may be defined for the groups where only "operatorial homotopy" is allowed (Theorem 12). When looking at it carefully, one sees that this proof is very similar to the one given by Kasparov in [4], but, I believe that it may seem more conceptual.

The associativity of the Kasparov product is seen through the notion of connexion introduced in [2, Appendix A]. Both of these proofs use in a crucial manner the technical part of Kasparov's work [4, Sect. 3], especially Theorem 4 therein.

NOTATION

All gradings are $\mathbb{Z}/2$ gradings. All tensor products are graded and minimal (spacial) tensor products. All commutators are graded commutators ($[a, b] = ab - (-1)^{\partial a \cdot b} ba$, where $\partial a$ is the degree of $a$). The reference for graded Hilbert $C^*$-modules, endomorphisms, compact endomorphisms of Hilbert $C^*$-modules, graded tensor products, etc, is [4, Sects. 1, 2] (see also [3]). As in [4] if $\mathcal{E}$ is a graded Hilbert $C^*$-module $\mathcal{L}(\mathcal{E})$ (resp. $\mathcal{K}(\mathcal{E})$) denotes the graded $C^*$-algebra of endomorphisms (resp. compact endomorphisms) of $\mathcal{E}$.

If $x$ is an element of a graded $C^*$-algebra, or a graded Hilbert $C^*$-module, $x = x^{(0)} + x^{(1)}$ represents its decomposition in the even and odd parts. The grading operator (resp. the grading automorphism) of a graded $C^*$-module (resp. $C^*$-algebra) is the operator $x^{(0)} + x^{(1)} \mapsto x^{(0)} - x^{(1)}$. If $A$ is a graded $C^*$-algebra, $\tilde{A}$ denotes the graded $C^*$-algebra with an added unit of degree 0.
Let us begin by recalling Kasparov's Theorem 4 of [4, Sect. 3], in the form it will be used several times here. Let $B$ be a graded C*-algebra and $\mathcal{E}$ a countably generated graded Hilbert C*-module over $B$.

**Theorem 1** [4, Sect. 3, Theorem 4]. Let $E_1, E_2 \subset \mathcal{L}(\mathcal{E})$ be graded subalgebras $\mathcal{F} \subset \mathcal{L}(\mathcal{E})$ a graded vector subspace. Assume that:

(i) $E_1$ has a countable approximate unit and $\mathcal{H}(\mathcal{E}) \subset E_1$,

(ii) $E_1, E_2$ are separable,

(iii) $E_1 \cdot E_2 \subset \mathcal{H}(\mathcal{E})$, $[\mathcal{F}, E_1] \subset E_1$.

Then there exist $M, N$ of degree 0 (for the grading) such that

$$M + N = 1, \quad M \geq 0, \quad N \geq 0,$$

$$M \cdot E_1 \subset \mathcal{H}(\mathcal{E}), \quad N \cdot E_2 \subset \mathcal{H}(\mathcal{E}), \quad [\mathcal{F}, M] \subset \mathcal{H}(\mathcal{E}).$$

(Notice that $\mathcal{E}$ being countably generated, $\mathcal{H}(\mathcal{E})$ has a countable approximate unit.)

Let us also recall the basic definitions and notations of the $KK$ groups [4, Sect. 4, Definitions 1-3].

**Definition 2.** (1) Let $A, B$ be graded C*-algebras. An $A, B$ bimodule is a countably generated graded Hilbert C*-module $\mathcal{E}$ over $B$ acted upon by $A$ through a grading preserving * homomorphism $A \to \mathcal{L}(\mathcal{E})$.

(2) A Kasparov $A, B$ bimodule is a pair $(\mathcal{E}, F)$, where $\mathcal{E}$ is an $A, B$ bimodule, $F \in \mathcal{L}(\mathcal{E})$ is of degree 1 and satisfies

$$\forall a \in A, \quad [a, F] \in \mathcal{H}(\mathcal{E}), \quad a(F^2 - 1) \in \mathcal{H}(\mathcal{E}), \quad a(F - F^*) \in \mathcal{H}(\mathcal{E}).$$

$\mathcal{E}(A, B)$ denotes the set of all Kasparov $A, B$ bimodules.

(3) A Kasparov $A, B$ bimodule $(\mathcal{E}, F)$ will be said to be degenerate iff

$$[a, F] = a(F^2 - 1) = a(F - F^*) = 0, \quad \forall a \in A. \quad \mathcal{E}(A, B)$ \text{ denotes the set of degenerate Kasparov } A, B \text{ bimodules.}

(4) An operatorial homotopy through Kasparov bimodules, is a homotopy $(\mathcal{E}_t, F_t)$, where $t \to F_t$ is norm continuous.

(5) An element of $\mathcal{E}(A,B \otimes C([0, 1]))$ is given by a family $(\mathcal{E}_t, F_t) \in \mathcal{E}(A, B)$ which will be called a homotopy between $(\mathcal{E}_0, F_0)$ and $(\mathcal{E}_1, F_1)$.

(6) The addition of two Kasparov bimodules $(\mathcal{E}_1, F_1)$ $(\mathcal{E}_2, F_2)$ is defined by $(\mathcal{E}_1, F_1) \oplus (\mathcal{E}_2, F_2) = (\mathcal{E}_1 \oplus \mathcal{E}_2, F_1 \oplus F_2)$.

(7) The set $KK(A, B)$ is defined as the quotient of $\mathcal{E}(A, B)$ by the equivalence relation given by homotopy.
(8) The set $\tilde{\mathcal{K}}K(A, B)$ is defined as the quotient of $\mathcal{E}(A, B)$ by the equivalence relation generated by addition of elements of $\mathcal{D}(A, B)$ and operatorial homotopy.

Remark 3. (a) If $(\mathcal{E}, F) \in \mathcal{D}(A, B)$ then $(\mathcal{E}, F)$ is homotopic to the Kasparov bimodule $(0, 0)$, because $(\mathcal{E} \otimes C_0([0, 1]), F \otimes 1) \in \mathcal{E}(A, B \otimes C([0, 1]))$.

(b) The definition given here of $KK(A, B)$ is different from the one given in [4, Sect. 4 Definition 3]. However, using the stabilization theorem [3, Theorem 2] these two definitions coincide when $B$ has a countable approximate unit.

One has

**Proposition 4** [4, Sect. 4, Theorem 1]. $KK(A, B)$ and $\tilde{\mathcal{K}}K(A, B)$ are abelian groups when equipped with addition as in Definition 2(6). $KK(A, B)$ is a quotient of $\tilde{\mathcal{K}}K(A, B)$.

**Proof:** The second assertion follows from Remark 3(a). For the first one, we just recall that $-(\mathcal{E}, F) = (\mathcal{E}, -UFU^{-1})$, where $-\mathcal{E}$ is the same Hilbert $B$ module with opposite grading, $U \in \mathcal{U}(\mathcal{E}, -\mathcal{E})$ is the identity, and the action of $A$ is given by $aU\xi = U(\alpha(a)\xi)$, where $\alpha$ is the grading automorphism of $A$. Then

$$
\begin{bmatrix}
\cos \theta \cdot F & \sin \theta U^{-1} \\
\sin \theta \cdot U & -\cos \theta UFU^{-1}
\end{bmatrix}, \quad \theta \in [0, \pi/2]
$$

defines an operatorial homotopy joining $(\mathcal{E}, F) \oplus (-(\mathcal{E}, F))$ to a degenerate element.

We may notice that if $(\mathcal{E}, F)$ and $(\mathcal{E}', F')$ are unitarily equivalent [4, Sect. 4, Definition 2], then $(\mathcal{E}, F) \oplus (-(\mathcal{E}', F'))$ is operatorially homotopic to a degenerate element. Hence the classes of $(\mathcal{E}, F)$ and $(\mathcal{E}', F')$ in $\tilde{\mathcal{K}}K$ and $KK$ coincide.

**Functorial Property 5.** (1) Let $f: A_2 \to A_1$ be a homomorphism of graded $C^*$-algebras. Let $(\mathcal{E}, F)$ be a Kasparov $A_1$, $B$ bimodule. Then $\mathcal{E}$ may be looked at as an $A_1$, $B$ bimodule $f^*\mathcal{E}$ through the $A_2$ action $A_2 \to A_1 \to \mathcal{E}(\mathcal{E})$. This defines a map $f^*: \mathcal{E}(A_1, B) \to \mathcal{E}(A_2, B), f^*(\mathcal{E}, F) = (f^*\mathcal{E}, F)$.

(2) Let $g: B_1 \to B_2$ be a homomorphism of graded $C^*$-algebras, and $(\mathcal{E}, F) \in \mathcal{E}(A, B_1)$. Then put $g_*(\mathcal{E}, F) = (\mathcal{E} \otimes_{B_1} B_2, F \otimes 1)$ [4, Sects. 2, 8]. This defines a map $g_*: \mathcal{E}(A, B_1) \to \mathcal{E}(A, B_2)$. (Note that as $\mathcal{E}$ is countably generated, the same holds for $\mathcal{E} \otimes_{B_1} B_2$).
(3) Both of these maps pass to the quotients \( KK \) and \( \widetilde{KK} \). We keep the notations \( f^* \), \( g_* \) for these quotient homomorphisms.

(4) For a graded \( C^* \)-algebra \( D \) with a countable approximate unit one defines the map \( \tau_D : \mathcal{E}(A, B) \to \mathcal{E}(A \otimes D, B \otimes D) \) by putting \( \tau_D(\mathcal{E}, F) = (\mathcal{E} \otimes D, F \otimes 1) \).

Again this map passes to the quotients \( KK \) and \( \widetilde{KK} \), and gives homomorphisms still called \( \tau_D \). One has obviously \([4, \text{Sect. 4, Theorem 3}])\).

**Proposition 6.** The bifunctor \( KK(A, B) \) is homotopy invariant in both entries.

The homotopy invariance theorem given here is in fact the equality \( KK = \widetilde{KK} \).

**Lemma 7.** Let \((\mathcal{E}, F)\) be a Kasparov \( A, B \) bimodule. Let \( f : D_1 \to D_2 \) be a homomorphism of graded \( C^* \)-algebras.

(a) If \( D_1 \) and \( D_2 \) are unital, and \( f(1) = 1 \), then

\[
f^*(\tau_{D_1}(\mathcal{E}, F)) = f^*(\tau_{D_2}(\mathcal{E}, F))
\]

(b) In general, this equality holds in \( KK(A \otimes D_1, B \otimes D_2) \).

**Proof:** (a) is obvious.

(b) Put \( J = D_1 \otimes_{D_1} D_2 \); it is the right ideal in \( D_2 \) generated by \( f(D_1) \).

Then \( f^*(\tau_{D_1}(\mathcal{E}, F)) = (\mathcal{E} \otimes_C J, F \otimes 1) \) and \( f^*(\tau_{D_2}(\mathcal{E}, F)) = (\mathcal{E} \otimes_C D_2, F \otimes 1) \).

Let \( \mathcal{E}' \) be the \( D_1, D_2 \otimes C([0, 1]) \) bimodule \( \mathcal{E}' \subseteq D_2 \otimes C([0, 1]) \), \( \mathcal{E}' = \{ f : [0, 1] \to D_2 \mid f(1) \in J \} \). Then \((\mathcal{E} \otimes_C \mathcal{E}', F \otimes 1) \in (A \otimes D_1, B \otimes D_2 \otimes C([0, 1]))\) realizes a homotopy between \( f^*(\tau_{D_2}(\mathcal{E}, F)) \) and \( f^*(\tau_{D_1}(\mathcal{E}, F)) \).

**Definition 8** [2, Appendix, Definition A.1]. Let \( \mathcal{E}_2 \) be a \( D, B \) bimodule. Let \( \mathcal{E}_1 \) be a Hilbert \( D \) module. Put \( \mathcal{E} = \mathcal{E}_1 \otimes_D \mathcal{E}_2 \). Let \( F_2 \in \mathcal{L}(\mathcal{E}_2) \). An element \( F \in \mathcal{L}(\mathcal{E}) \) is said to be an \( F_2 \) connexion for \( \mathcal{E}_1 \) iff \( \forall \xi \in \mathcal{E}_1 \),

\[
[T_1, F_2 \otimes F] \in \mathcal{L}(\mathcal{E}_2 \otimes \mathcal{E}), \quad \text{where} \quad T_1 = \begin{bmatrix} 0 & T^*_1 \\ T_1 & 0 \end{bmatrix} \in \mathcal{L}(\mathcal{E}_2 \otimes \mathcal{E}),
\]

\( T_1 \subseteq \mathcal{L}(\mathcal{E}_2, \mathcal{E}) \) being defined by \( T_1(\eta) = \xi \otimes \eta \in \mathcal{E} \).

Let us gather some easy results about connexions in

**Proposition 9** [2, Appendix, Proposition A.2]. (a) If \( F_2 \) satisfies
If \( F \) is an \( F_2 \) connexion, then \( F^* \) (resp. \( F^{(0)}, F^{(1)} \)) is an \( F_2^* \) (resp. \( F_2^{(0)}, F_2^{(1)} \)) connexion.

(c) If \( F \) is an \( F_2 \) connexion and \( F' \) is an \( F'_2 \) connexion, then \( F + F' \) (resp. \( FF' \)) is an \( F_2 + F'_2 \) (resp. \( F_2 \cdot F'_2 \)) connexion.

(d) The space of 0 connexions for \( \mathcal{E}_1 \) is \( \Omega = \{ T \in \mathcal{L}(\mathcal{E}) \mid \forall x \in \mathcal{H}(\mathcal{E}_1) \otimes 1, T x \in \mathcal{H}(\mathcal{E}), x T \in \mathcal{H}(\mathcal{E}) \} \).

(e) If \( F \) is an \( F_2 \) connexion for some \( F_2 \), then \( |F, x| \in \mathcal{H}(\mathcal{E}) \). \( \forall x \in \mathcal{H}(\mathcal{E}_1) \otimes 1 \).

(f) If \( |F_2, d| = 0, \forall d \in D, 1 \otimes F_2 \) makes sense in \( \mathcal{L}(\mathcal{E}) \), and \( 1 \otimes F_2 \) is an \( F_2 \) connexion. Moreover for any \( T \in \mathcal{L}(\mathcal{E}) \), \( |T \otimes 1, 1 \otimes F_2| = 0 \).

(g) If \( F_2 \in \mathcal{L}(\mathcal{E}_2) \) and \( F \in \mathcal{L}(\mathcal{E}) \) are normal and \( F \) is an \( F_2 \) connexion, \( f(F) \) is an \( f(F_2) \) connexion (\( f : \mathcal{C} \to \mathcal{C} \) continuous).

(h) If \( (\mathcal{E}_2, F_2) \) is a Kasparov \( D, B \) bimodule and \( \mathcal{E}_1 \) is countably generated, and if \( F \) is an \( F_2 \) connexion of degree \( 1 \), then \( (\mathcal{E}, F) \) is a Kasparov \( \mathcal{R}(\mathcal{E}_1) \otimes 1, B \) bimodule.

(i) Assume \( \mathcal{E}_2 \) is a \( D, E \) bimodule and \( \mathcal{E}_3 \) is an \( E, B \) bimodule. Let \( F \in \mathcal{L}(\mathcal{E}_1 \otimes_D \mathcal{E}_2 \otimes_F \mathcal{E}_3), F_2 \in \mathcal{L}(\mathcal{E}_2 \otimes_F \mathcal{E}_3), F_3 \in \mathcal{L}(\mathcal{E}_3) \).

Recall that the proof of (a) is an easy consequence of the stabilization theorem [2, Theorem 2]: If \( \mathcal{E}_1 = P \mathcal{H}_D^0 \), where \( P \in \mathcal{L}(\mathcal{H}_D) \) is a degree 0 projection, then \( \mathcal{E}_1 \otimes \mathcal{E}_2 = (P \otimes 1) \cdot (\mathcal{H}_D \otimes \mathcal{E}_2) = (P \otimes 1)(\mathcal{H}_D \otimes \mathcal{E}_2) \). Then \( (P \otimes 1)(1 \otimes F_2)(P \otimes 1) \) is an \( F_2 \) connexion (the grassmann connexion). The other statements are obvious.

**Definition 10** [2, Appendix, Theorem A.3]. Let \( A, B, D \) be graded \( \mathcal{C}^* \)-algebras \( (\mathcal{E}_1, F_1) \in \mathcal{E}(A, D), (\mathcal{E}_2, F_2) \in \mathcal{E}(D, B) \). Call \( \mathcal{E} \) the \( A, B \) bimodule \( \mathcal{E}_1 \otimes_D \mathcal{E}_2 \). The pair \( (\mathcal{E}, F), F \in \mathcal{L}(\mathcal{E}) \) is called a Kasparov product of \( (\mathcal{E}_1, F_1) \) by \( (\mathcal{E}_2, F_2) \) (one writes \( F \in F_1 \#_D F_2 \)) if and only if

(a) \( (\mathcal{E}, F) \) is a Kasparov \( A, B \) bimodule (Definition 2(2)).

(b) \( F \) is an \( F_2 \) connexion.

(c) \( \forall a \in A, a[F_1 \otimes 1, F] a^* \geq 0 \) modulo \( \mathcal{H}(\mathcal{E}) \).

Note that \( \mathcal{E}_1, \mathcal{E}_2 \) being countably generated, \( \mathcal{E} \) is countably generated.

We will need

**Lemma 11.** Let \( \mathcal{E} \) be an \( A, B \) bimodule. Let \( F, F' \in \mathcal{L}(\mathcal{E}) \) be such that
(\mathcal{E}, F) \in \mathcal{E}(A, B), \ (\mathcal{E}, F') \in \mathcal{E}(A, B), \text{ and } \forall a \in A, \ a[F, F'] a^* \geq 0 \text{ modulo } \mathcal{H}(\mathcal{E}). \text{ Then } (\mathcal{E}, F) \text{ and } (\mathcal{E}, F') \text{ are operatorially homotopic.}

Proof: Let \mathcal{A} be the subalgebra of \mathcal{L}(\mathcal{E})

\mathcal{A} = \{ T \in \mathcal{L}(\mathcal{E})/ [T, a] \in \mathcal{H}(\mathcal{E}), \forall a \in A \}

and \mathcal{J} be the ideal of \mathcal{A}

\mathcal{J} = \{ T \in \mathcal{A}/ Ta \in \mathcal{H}(\mathcal{E}), \forall a \in A \}.

Then \[ F, F' \] \subset \mathcal{A} and is positive modulo \mathcal{J}. Write \[ F, F' = P + K, \] where \[ P \in \mathcal{A}, \ P \geq 0, \text{ and } K \in \mathcal{J}; \] \[ P \text{ and } K \text{ being of degree } 0. \] Note that as \( F^2 - 1 \text{ and } F'^2 - 1 \in \mathcal{J}, \) \[ [F, P] \text{ and } [F', P] \in \mathcal{J}. \] Put \[ F_t = (1 + \cos t \cdot \sin t)P^{-1/2} (\cos tF + \sin tF') \ (t \in [0, \pi/2]). \] Then \[ F_t \in \mathcal{A}, \ F_t - F_t^* \in \mathcal{J}, \text{ and } F_t^2 - 1 \in \mathcal{J}. \] Hence \( (\mathcal{E}, F_t) \) realizes the desired operatorial homotopy.

Note that in the above proof it is enough to assume that \[ |F, F'| \geq \lambda \text{ modulo } \mathcal{J}, \] where \( \lambda \in \mathbb{R}, \lambda > -2. \)

Theorem 12 [4, Sect. 4, Theorem 4; 2, Theorem A.3]. Assume \( A \) is separable, \( (\mathcal{E}_1, F_1) \) is a Kasparov \( A, D \) bimodule, \( (\mathcal{E}_2, F_2) \) is a Kasparov \( D, B \) bimodule.

(a) There exists a Kasparov product \( (\mathcal{E}, F) \) of \( (\mathcal{E}_1, F_1) \) by \( (\mathcal{E}_2, F_2) \) unique up to operatorial homotopy.

(b) The map \( (\mathcal{E}_1, F_1), (\mathcal{E}_2, F_2) \rightarrow (\mathcal{E}, F) \) passes to the quotients, and defines maps \( KK(A, D) \otimes KK(D, B) \rightarrow KK(A, B) \) and \( KK(A, D) \otimes KK(D, B) \rightarrow KK(A, B). \) The quotient maps are noted \( \otimes_D, (x, y) \rightarrow x \otimes_D y. \)

Proof: (a) Existence. Let \( G \in \mathcal{L}(\mathcal{E}), \) of degree 1, be an \( F_2 \) connexion (Proposition 9(a)). Put \( E_1 = \mathcal{H}(\mathcal{E}_1) \otimes 1 + \mathcal{H}(\mathcal{E}) \) and let \( E_2 \) be the subalgebra of \( \mathcal{L}(\mathcal{E}) \) generated by \( (G^2 - 1), [G, a] \ (a \in A), \ G - G^*, \) \( [G, F_1 \otimes 1]. \) Let \( \mathcal{F} \) be the vector space spanned by \( \{ F \otimes 1, G, A \}. \) One checks that Theorem 1 applies to give \( M \) and \( N \) such that

\[ M + N = 1, \ \ ME_1 \subset \mathcal{H}(\mathcal{E}), \ \ NE_2 \subset \mathcal{H}(\mathcal{E}), \ \ [M, \mathcal{F}] \subset \mathcal{H}(\mathcal{E}). \]

Then put \( F = M^{1/2}(F_1 \otimes 1) + N^{1/2}G. \) Then \( (\mathcal{E}, F) \) is obviously seen to be a Kasparov \( A, B \) bimodule. Note that \( M \) is a zero-connexion (Proposition 9(d)). As \( [F_1 \otimes 1, M] \in \mathcal{H}(\mathcal{E}), \ M^{1/2}(F_1 \otimes 1) \) is also a zero-connexion. And, hence (Proposition 9(c) and (g)) \( F \) is an \( F_2 \) connexion. Finally \( [F_1 \otimes 1, F] = M^{1/2}[F_1 \otimes 1, F_1 \otimes 1] \mod \mathcal{H}(\mathcal{E}) \) and hence \( a[F_1 \otimes 1, F] a^* = 2aM^{1/2}(F_1^2 \otimes 1)a^* \mod \mathcal{H}(\mathcal{E}) = 2aM^{1/2}a^* \mod \mathcal{H}(\mathcal{E}). \) Thus \( F \in F_1 \not\sim_D F_2. \)
Uniqueness. Let \( F, F' \in F_1 \not\#_D F_2 \). Put \( E_1 = \mathcal{H}(\mathcal{F}_1) \otimes 1 + \mathcal{H}(\mathcal{F}) \) and \( E_2 \) is the subalgebra generated by \( [F_1 \otimes 1, F], [F_1 \otimes 1, F'], F - F' \). Then take \( M, N \) satisfying the conclusion of Theorem 1 and put \( F'' = M^{1/2}(F_1 \otimes 1) + N^{1/2}F \). One has \((\mathcal{F}, F'') \in \mathcal{E}(A, B)\) and \( \forall a \in A, a[F_1, F'']a^* \geq 0 \mod \mathcal{H}(\mathcal{F}) \) and \( a[F', F'']a^* \geq 0 \mod \mathcal{H}(\mathcal{F}) \). The conclusion follows by Lemma 11.

(b) Let \((\mathcal{F}_1, F_1) \in \mathcal{E}(A, D \otimes C([0, 1]))\) be a homotopy. Let \((\mathcal{F}, F)\) be a Kasparov product of \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_2, F_2)\). Then \((\mathcal{F}, F)\) realizes a homotopy between a Kasparov product of \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_2, F_2)\) and a Kasparov product of \((\mathcal{F}_0, F_0')\) by \((\mathcal{F}_2, F_2)\). In the same way, if \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_2, F_2)\) realizes a homotopy between a Kasparov product of \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_0, F_0')\) and a Kasparov product of \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_2, F_2)\). This carries over the KK case.

If \((\mathcal{F}_1, F_1)\) is degenerate, then \((\mathcal{F}, F_1 \otimes 1) \in \mathcal{E}(A, B)\) and is operatorially homotopic to any Kasparov product of \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_2, F_2)\) by Lemma 11. If \((\mathcal{F}_2, F_2)\) is degenerate, then \( 1 \otimes F_2 \) has an obvious meaning in \( \mathcal{E}(\mathcal{F}_1 \otimes D, \mathcal{F}_2) \), and satisfies the conditions to be a (degenerate) Kasparov product of \((\mathcal{F}_1, F_1)\) by \((\mathcal{F}_2, F_2)\) (Proposition 9(f)).

Let \((\mathcal{F}_i, F_i')\) \((i = 1, 2)\) be operatorial homotopies. Let \( G'\), \( t \in [0, 1]\), be a norm continuous family such that each \( G'\) is an \( F_i'\) connexion for \( \mathcal{F}_i \). (Such a family exists; take, e.g., the “grassmann connexions” \( G' = (P \otimes D, 1) \otimes (F_i') (P \otimes D, 1) \) for some trivialization \( \mathcal{F}_i = P \otimes D \).) Put \( E_1 = \mathcal{H}(\mathcal{F}_1) \otimes 1 + \mathcal{H}(\mathcal{F}) \) \((\mathcal{F}_1 \otimes \mathcal{F})\). \( E_2 \) is the \( C^*\)-algebra generated by \( [G', A], (G'_t - 1), F_i - G_i^*, [F_i' \otimes 1, G'_i], t \in [0, 1] \). \( \mathcal{F} \) is the subspace generated by \( A, F_1 \otimes 1, G'_t, t \in [0, 1] \). One checks that Theorem 1 applies.

Let \( M, N \) satisfy the conclusions of this theorem. Then \((\mathcal{F}, M^{1/2}(F_1 \otimes 1) + N^{1/2}G')\) is the desired operatorial homotopy. 

**Proposition 13** (Functoriality of the Kasparov product). (a) If \( A_1 \) and \( A_2 \) are separable, \( f: A_2 \to A_1 \) is a homomorphism \( x_1 \in KK(A_1, D) \), \( x_2 \in KK(D, B) \) (or \( KK \)), then \( f^*(x_1) \otimes D x_2 = f^*(x_1) \otimes D f_2 \).

(b) If \( h: D_1 \to D_2 \) is a homomorphism \( x_1 \in KK(A, D_1) \), \( x_2 \in KK(D_2, B) \) (or \( KK \)), then \( h^*(x_1) \otimes D_1 h^*(x_2) \).

(c) If \( g: B_1 \to B_2 \) is a homomorphism \( x_1 \in KK(A, D) \), \( x_2 \in KK(D, B_1) \) (or \( KK \)), then \( g^*(x_1) \otimes D_2 g^*(x_2) \).

**Proof.** (a) If \((\mathcal{F}_1, F_1) \in \mathcal{E}(A_1, D)\) and \((\mathcal{F}_2, F_2) \in \mathcal{E}(D, B)\), then one has \( f^*(F_1 \not\# D F_2) \subset f^*(F_1) \not\# D f_2 \).

(b) If \((\mathcal{F}_1, F_1) \in \mathcal{E}(A, D_1)\), \((\mathcal{F}_2, F_2) \in \mathcal{E}(D_2, B)\), then one has \( h^*(F_1) \not\# D_1 F_2 = F_1 \not\# D_1 h^*(F_2) \).
Let $A_1, A_2$ be separable. Let $x_1 \in KK(A_1, B_1 \hat{\otimes} D)$ (resp. $\widetilde{KK}(A_1, B_1 \hat{\otimes} D)$) and $x_2 \in KK(D \hat{\otimes} A_2, B_2)$ (resp. $\widetilde{KK}(D \hat{\otimes} A_2, B_2)$). Consider the product $	au_{A_2}(x_1) \otimes_{B_1 \otimes D \hat{\otimes} A_2} i^*(\tau_{B_1}(x_2))$ (i: $B_1 \rightarrow \tilde{B}_1$ is inclusion).

One sees that it gives in fact an element of $KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$ (resp. $\widetilde{KK}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$). This, because, if $(\mathcal{E}, F)$ is a Kasparov bimodule which defines this product, one has

$$\forall \xi, \eta \in \mathcal{E}, \quad \langle \xi, \eta \rangle \in B_1 \hat{\otimes} B_2 \subset \tilde{B}_1 \hat{\otimes} B_2.$$

**Definition 15.** The Kasparov product $x_1 \otimes_{\mathcal{D}} x_2$ is defined as the element of $KK(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$ (resp. $\widetilde{KK}(A_1 \hat{\otimes} A_2, B_1 \hat{\otimes} B_2)$) corresponding to the Kasparov product $\tau_{A_2}(x_1) \otimes_{B_1 \otimes D \hat{\otimes} A_2} i^*(\tau_{B_1}(x_2)).$

**Remark 16.** (a) If $B_1$ has a countable approximate unit, $\tau_{B_1}(x_2)$ makes sense and one has $x_1 \otimes_{\mathcal{D}} x_2 = \tau_{A_2}(x_1) \otimes_{B_1 \otimes D \hat{\otimes} A_2} \tau_{B_1}(x_1).$

(b) More generally, one may take, instead of $\tilde{B}_1$ any algebra with countable approximate unit in which $B_1$ is an ideal. The result will not change.

**Proposition 16** (Part of [4, Theorem 4.4]). The product $x_1 \otimes_{\mathcal{D}} x_2$ is bilinear, contravariantly functorial in $A_1$, and covariantly functorial in $B_1$ and $B_2$. Moreover, it is contravariantly functorial in $A_2$ in $KK$ and with respect to unital maps in $\widetilde{KK}$. If $h: D_1 \rightarrow D_2$, $x_1 \in KK(A_1, B_1 \hat{\otimes} D_1)$, $x_2 \in KK(D_2 \hat{\otimes} A_2, B_2)$ (or in the corresponding $\widetilde{KK}$) $h_*(x_1) \otimes_{D_2} x_2 = x_1 \otimes_{D_1} h^*(x_2)$.

**Proof.** Follows obviously from Proposition 13. The $A_2$ functoriality uses Lemma 7.

Let $1 \in \widetilde{KK}(\mathbb{C}, \mathbb{C}) (=KK(\mathbb{C}, \mathbb{C}))$ be given by the $\mathbb{C}, \mathbb{C}$ bimodule $\mathbb{C}$, trivially (zero) graded and the zero operator.

**Proposition 17** [4, Sect. 4, Theorem 5]. Let $A$ be separable, $x \in KK(A, B)$ or $\widetilde{KK}(A, B)$. Then $x \otimes_{\mathbb{C}} 1 = x$. If $A$ is unital, then $1 \otimes_{\mathbb{C}} x = x$. In general, this equality holds in $KK(A, B)$.

**Proof.** That $x \otimes_{\mathbb{C}} 1 = x$ is obvious. Assume $A$ is unital. Let $(\mathcal{E}, F)$ be a Kasparov $A, B$ bimodule, and let $P \in \mathcal{L}(\mathcal{E})$ be the image of $1 \in A$. The product $1 \otimes_{\mathbb{C}} (\mathcal{E}, F)$ is given by $(P\mathcal{E}, PFP)$. But $(\mathcal{E}, F)$ is operatorially homotopic to $(\mathcal{E}, PFP) = (P\mathcal{E}, PFP) + ((1 - P) \mathcal{E}, 0)$ (this last term being degenerate). Let $(\mathcal{E}, F) \in \mathcal{E}(A, B)$ ($A$ non unital). One may extend the action on $\mathcal{E}$ to $\tilde{A}$ sending $1 \in \tilde{A}$ to $1 \in \mathcal{L}(\mathcal{E})$. (This does not in general give a Kasparov $\tilde{A}, B$ bimodule!)
Let \( \mathcal{E}' \) be the \( A \otimes \mathcal{A} \otimes \mathcal{C}([0,1]) \) bimodule, \( \mathcal{E}' = \{ f: [0,1] \to \mathcal{A}, f(1) \in A \} \). Put \( \mathcal{F} = \mathcal{E}' \otimes_A \mathcal{A} \otimes \mathcal{C}([0,1]) \). Let \( F \in \mathcal{L}(\mathcal{E}) \) be an \( F \otimes 1 \) connexion for \( \mathcal{E}' \). Then \( (\mathcal{E}, F) \in \mathcal{E}(A, B \otimes \mathcal{C}([0,1])) \) and \( (\mathcal{E}'^0, F^0) \) is a Kaparov product \( 1 \otimes (\mathcal{E}, F) \) and \( (\mathcal{E}'^0, F^0) \) is operatorially homotopic to \( (\mathcal{E}, F) \).

Now let \( f_t: C([0,1]) \to \mathbb{C} \) be evaluation at time \( t \).

**Lemma 18.** One has \( f_t^*(1) = f_t^*(1) \) in \( \mathcal{K}\mathcal{K}(C([0,1]), \mathbb{C}) \).

We do not want to get into the proof. It is a consequence of "homotopy invariance" in the abelian case, for the Ext functor \( [1, \text{Theorem 2.14}] \) see, also, \([4, \text{Sect. 6, Theorem 1, beginning of proof}] \).

**Theorem 19 (Homotopy invariance) \([4, \text{Sect. 6, Theorem 1}] \).** Assume \( A \) is separable. Then, the map \( \mathcal{K}\mathcal{K}(A, B) \to \mathcal{K}\mathcal{K}(A, B) \) is an isomorphism.

**Proof.** Let \( (\mathcal{E}', F') \in (A, B \otimes \mathcal{C}([0,1])) \) be a homotopy. Let \( x \) be its class in \( \mathcal{K}\mathcal{K}(A, B \otimes \mathcal{C}([0,1])) \). Let \( x_t \) be the class of \( (\mathcal{E}', F') \) in \( \mathcal{K}\mathcal{K}(A, B) \). One has

\[
x_0 = f_0^*(x) = f_0^*(x) \otimes 1 \quad \text{(Proposition 17)}
\]

\[
x = x \otimes_{\mathcal{C}([0,1])} f_1^*(1) \quad \text{(Proposition 16)}
\]

\[
x = x \otimes_{\mathcal{C}([0,1])} f_1^*(1) \quad \text{(Lemma 18)}
\]

Let us recall

**Proposition 20** \([4, \text{Sect. 4, Theorem 4}] \). The Kasparov product satisfies the following commutation relations with the functor \( \tau_D \):

\( a \) \( \tau_{D_1}(x_1) \otimes_{D_1 \otimes D_2} \tau_{D_2}(x_2) = x_1 \otimes_{D_1} x_2 \quad (x \in \mathcal{K}(K(A_1, B_1 \otimes D_2 \otimes D_3), y \in \mathcal{K}(D \otimes D_2 \otimes A_2, B_2); A_1, A_2, D_1 \text{ separable, } D_1 \text{ with countable approximate unit}) \).

\( b \) \( \tau_{D_1}(x_1 \otimes_{D_1} x_2) = \tau_{D_1}(x_1) \otimes_{D_1} \tau_{D_1}(x_2) \quad (x \in \mathcal{K}(K(A_1, B_1 \otimes D), x \in \mathcal{K}(K(D \otimes A_2, B_2); A_1, A_2, D_1 \text{ separable}).

**Proof.** It follows from Definition 15 and Remark 16.

Let us now pass to associativity:

**Theorem 21 (Part of \([4, \text{Sect. 4, Theorem 4}] \).** Let \( A_1, A_2, A_3, D_1 \) be separable and let \( x_1 \in \mathcal{K}(K(A_1, B_1 \otimes D_1)), x_2 \in \mathcal{K}(K(D_1 \otimes A_2, B_2 \otimes D_3), x_3 \in \mathcal{K}(D_2 \otimes A_3, B_3). \) Then \( (x_1 \otimes_{D_1} x_2) \otimes_{D_2} x_3 = x_1 \otimes_{D_1} (x_2 \otimes_{D_2} x_3) \).
Proof. Put $A = A_1 \otimes A_2 \otimes A_3$, $D = B_1 \otimes D_1 \otimes A_2 \otimes A_3$, $E = B_1 \otimes B_2 \otimes D_2 \otimes A_3$, $B = B_1 \otimes B_2 \otimes B_3$. Replace the $x_i$'s by Kasparov bimodules representing them. Let $i_1 : B_1 \to \tilde{B}_1$, $i_2 : B_2 \to \tilde{B}_2$ be the inclusions and put

$$(\mathcal{E}_1, F_1) = \tau_{A_3 \otimes A_1}(x_1) \in \mathbb{E}(A, D),$$

$$(\mathcal{E}_2, F_2) = i_1^* \tau_{B_1 \otimes A_1}(x_2) \in \mathbb{E}(D, D),$$

$$(\mathcal{E}_3, F_3) = i_2^* \tau_{B_2 \otimes A_3}(x_3) \in \mathbb{E}(E, B).$$

Due to Proposition 20, Theorem 21 is a consequence of

**Lemma 22.** Let $A$ be separable $(\mathcal{E}_1, F_1) \in \mathbb{E}(A, D)$, $(\mathcal{E}_2, F_2) \in \mathbb{E}(D, E)$, $(\mathcal{E}_3, F_3) \in \mathbb{E}(E, B)$, and take $G \in F \#_D F'$ and $H \in G \#_E F_3$. Assume that $F_2 \#_E F_3$ is nonempty and $G_2 \subset F_2 \#_E F_3$. Take then $F \in F_2 \#_E G_2$. Then $(\mathcal{E}_1 \otimes_D \mathcal{E}_2 \otimes_E \mathcal{E}_3, F)$ and $(\mathcal{E}_1 \otimes_D \mathcal{E}_2 \otimes_E \mathcal{E}_3, H)$ are operatorially homotopic.

Notice that $G_1$ and $H$ are unique up to norm homotopy by Theorem 12, but $G_2$ and hence $F$ are not apriori unique. Notice also that for the proof of Theorem 21, we know that $F_2 \#_E F_3 \neq \emptyset$ because $(\mathcal{E}_2, F_2)$ and $(\mathcal{E}_3, F_3)$ are of the form $\tau_{\mathcal{E}_1}$, of something, for which Theorem 12 applies.

**Proof of Lemma.** Put $\mathcal{E}'_1 = \mathcal{E}_1 \otimes_D \mathcal{E}_2$, $\mathcal{E}'_2 = \mathcal{E}_2 \otimes_E \mathcal{E}_3$, $\mathcal{E}' = \mathcal{E}_1 \otimes_D \mathcal{E}'_2 = \mathcal{E}_1 \otimes_E \mathcal{E}_3$. As $G_1$ is an $F_2$ connexion for $\mathcal{E}_1$ and $F$ is a $G_2$ connexion, $[G_1 \otimes 1, F]$ is an $[F_2 \otimes 1, G_2]$ connexion for $\mathcal{E}_1$. To see this, write

$$\tilde{F} = F + G_2 \in \mathcal{L}(\mathcal{E} \oplus \mathcal{E}_2), \quad \tilde{G} = G_1 + F_2 \in \mathcal{L}(\mathcal{E}'_1 \oplus \mathcal{E}_2).$$

Then $\forall \xi \in \mathcal{E}_1$, $[[\tilde{F}, \tilde{G} \otimes 1], \tilde{T}_1] = [[\tilde{F}, \tilde{T}_1], \tilde{G} \otimes 1] - [[\tilde{T}_1, \tilde{G} \otimes 1], \tilde{F}]$. The first term belongs to $\mathcal{K}(\mathcal{E}'_1 \oplus \mathcal{E}_2)$ because $F$ is a $G_2$ connexion. Also $[[\tilde{T}_1, \tilde{G} \otimes 1], \tilde{F}] \in \mathcal{K}(\mathcal{E} \oplus \mathcal{E}_2)$ because $\tilde{F}$ is an $F_3$ connexion for $(\mathcal{E}_1 \oplus \mathcal{E}_2)$ (Proposition 9(i) and (e)). Hence $[G_1 \otimes 1, F]$ is a 0 connexion for $\mathcal{E}_1$ (Proposition 9(i)) and $[G_1 \otimes 1, F] - \text{Re}[G_1 \otimes 1, F]^*$ is a 0 connexion for $\mathcal{E}_1$, (Proposition 9(g)).

Put $F_1 = \mathcal{K}(\mathcal{E}) + \mathcal{K}(\mathcal{E}_1) \otimes_D 1 + \mathcal{K}(\mathcal{E}_2) \otimes_D 1 \subset \mathcal{L}(\mathcal{E})$. $E_2$ = subalgebra of $\mathcal{L}(\mathcal{E})$ generated by $[G_1 \otimes 1, F] - \text{Re}[G_1 \otimes 1, F]^*$. $\mathcal{F}_1 = \otimes D 1, G \otimes 1, A$.

We may apply Theorem 1. Let $M, N$ satisfy the conclusions of Theorem 1. Put $F' = M^{1/2}(F_1 \otimes 1) + N^{1/2}F$. Then $(\mathcal{E}, F') \in \mathbb{E}(A, B)$ and $\forall a \in A, a[F, F']a^* \geq 0 \text{ mod } \mathcal{K}(\mathcal{E})$. Hence $(\mathcal{E}, F)$ and $(\mathcal{E}, F')$ are operatorially homotopic (Lemma 11). On the other hand, $\forall a \in A, a[F', G_1 \otimes 1]a^* \geq 0 \text{ mod } \mathcal{K}(\mathcal{E})$ and $F'$ is an $F_3$ connexion for $\mathcal{E}_1$. Hence $F' \in G_1 \#_D F_3$.

**Remark 23.** Let $(\mathcal{E}_1, F_1) \in \mathbb{E}(A_1, B_1)$, $(\mathcal{E}_2, F_2) \in \mathbb{E}(A_2, B_2)$, where $A_1, A_2$ are separable. Let $\mathcal{E}$ be the $A_1 \otimes A_2$, $B_1 \otimes B_2$ bimodule $\mathcal{E}_1 \otimes \mathcal{E}_2$. Using [4, Sect. 3, Theorem 3], one finds $M, N \in \mathcal{L}(\mathcal{E})$ of degree 0, $M \geq 0, N \geq 0$,
$M + N = 1$, such that $M(\mathcal{H}(F_1) \otimes 1) \subseteq \mathcal{H}(F)$, $N(1 \otimes \mathcal{H}(F_2) \subseteq \mathcal{H}(F)$, and $[M, A] \subseteq \mathcal{H}(F)$, $[M, F_1 \otimes 1] \subseteq \mathcal{H}(F)$, $[M, 1 \otimes F_2] \subseteq \mathcal{H}(F)$. Then $M^{1/2}(F_1 \otimes 1) + N^{1/2}(1 \otimes F_2)$ is a Kasparov product of $(\mathcal{H}, F_1)$ by $(\mathcal{H}, F_2)$ and also a Kasparov product of $(\mathcal{H}, F_2)$ by $(\mathcal{H}, F_1)$.

**References**

2. A. Connes and G. Skandalis, The longitudinal index theorem for foliations, preprint I.H.E.S.