Some recent results on generalized analytic torsion classes

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Abstract.
In this presentation we give a brief account of an axiomatic approach to generalized analytic torsion classes for arbitrary projective morphisms of smooth complex varieties and of the classification of the theories that can be constructed using our axioms. This note summarizes the main results in Burgos Gil, J. I. and Liţcanu, R.: Singular Bott-Chern classes and the arithmetic Grothendieck-Riemann-Roch theorem for closed immersions, Doc. Math. 15 (2010), 73–176 and Burgos Gil, J. I., Freixas i Montplet, G. and Liţcanu, R.: Generalized Holomorphic Analytic Torsion, arXiv:1011.3702.

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1. INTRODUCTION

This paper is a brief survey on recent work giving an axiomatic characterization for generalized holomorphic analytic torsion classes. The ultimate aim of this work is to extend the arithmetic Riemann-Roch theorem to projective morphisms that are not smooth at the Archimedean place.

The arithmetic Grothendieck-Riemann-Roch theorem in degree one [16] uses, as a fundamental ingredient, the analytic torsion, or more precisely its holomorphic version, defined by Ray and Singer ([21], [22]). In order to establish an arithmetic Grothendieck-Riemann-Roch theorem for all degrees, it was necessary to generalize the analytic torsion and define higher analytic torsion classes [7]. Gillet and Soulé [16] noticed that, once a suitable theory of higher analytic torsion classes satisfying certain properties were developed, then the arithmetic Riemann-Roch theorem would follow. Always assuming that the morphisms considered are smooth on the Archimedean fibre.

The Grothendieck-Riemann-Roch theorem gives a formula that relates direct images and characteristic classes for vector bundles. If the vector bundles are endowed with hermitian metrics, then the Chern-Weil theory associates to each hermitian vector bundle...
a family of closed characteristic forms that represent the characteristic classes of the vector bundle. In general the formula provided by the Grothendieck-Riemann-Roch theorem is not valid for the characteristic forms. The analytic torsion classes measure the failure of an exact Grothendieck-Riemann-Roch theorem for projective submersions at the level of characteristic forms.

The generalized analytic torsion classes that we describe in this paper are meant to measure the failure of an exact Grothendieck-Riemann-Roch theorem for arbitrary projective morphisms. In fact, these classes are meant to give a functorial definition of push-forward maps at the level of the arithmetic $K$-groups associated to arithmetic varieties, and the arithmetic Grothendieck-Riemann-Roch theorem naturally follows. We distinguish two special cases, that in fact provide the general situation via a compatibility theorem: the closed immersions and the projective spaces.

Before describing these two cases, we briefly recall the role of the Bott-Chern classes. The characteristic classes associated to vector bundles are compatible with exact sequences. But this is not true for the characteristic forms. The Bott-Chern classes measure the lack of compatibility of the characteristic forms with exact sequences. More precisely, let $X$ be a complex manifold and let $\varphi$ be a symmetric power series in $r$ variables with real coefficients. Let $\mathcal{E} = (E, h)$ be a rank $r$ holomorphic vector bundle endowed with a hermitian metric. Using Chern-Weil theory, we can associate to $\mathcal{E}$ a differential form $\varphi(\mathcal{E}) = \varphi(-K)$, where $K$ is the curvature tensor of $\mathcal{E}$ viewed as a matrix of 2-forms. The differential form $\varphi(\mathcal{E})$ is closed and is a sum of components of bidegree $(p, p)$ for $p \geq 0$.

If

$$\begin{array}{rcl}
\xi : 0 & \longrightarrow & \mathcal{E}' \\
& \longrightarrow & \mathcal{E} \\
& \longrightarrow & \mathcal{E}'' \\
& \longrightarrow & 0
\end{array}$$


is a short exact sequence of holomorphic vector bundles endowed with hermitian metrics, then the differential forms $\varphi(\mathcal{E})$ and $\varphi(\mathcal{E}' \oplus \mathcal{E}'')$ may be different, but they represent the same cohomology class.

A Bott-Chern form associated to $\xi$ is a solution of the differential equation

$$-2\partial \bar{\partial} \varphi(\xi) = \varphi(\mathcal{E}' \oplus \mathcal{E}'') - \varphi(\mathcal{E})$$

(1.1)

obtained in a functorial way. The class of a Bott-Chern form modulo the image of $\partial$ and $\bar{\partial}$ is called a Bott-Chern class and is denoted by $\tilde{\varphi}(\xi)$.

There are three ways of defining the Bott-Chern classes. The first one is the original definition of Bott and Chern [10] and is based on a deformation (parametrized by a real variable) between the connection associated to $\mathcal{E}$ and the connection associated to $\mathcal{E}' \oplus \mathcal{E}''$. Note that both vector bundles are isomorphic as $C^\infty$ bundles but not as holomorphic bundles. A second possibility, due to Bismut, Gillet and Soulé [5], is based on the theory of superconnections. Finally, one can define the Bott-Chern classes axiomatically: they are characterized by three properties

(i) The differential equation (1.1).
(ii) Functoriality (i.e. compatibility with pull-backs via holomorphic maps).
(iii) The vanishing of the Bott-Chern class of a orthogonally split exact sequence.
The definition of Bott-Chern classes can be generalized to any bounded exact sequence of hermitian vector bundles (see [14, §2] for details).

We return now to the generalized analytic torsion classes. In the case of a closed immersion the classes we are interested in are called singular Bott-Chern classes. They will be described in section 3 and the details can be found in [14]. Then we shall characterize the analytic torsion classes for the projective spaces (see [12] for details).

We point out that, unlike the classical situation of the Bott-Chern classes, the analogues of the axioms (i) to (iii) do not characterize uniquely the singular Bott-Chern classes or the analytic torsion classes. Consequently there are various non-equivalent theories that are characterized by these axioms. They are classified by an arbitrary characteristic class and by a set of real numbers respectively. We also emphasize that one has to enlarge the category of hermitian vector bundles for making these definitions possible. In the case of closed immersions, because the direct image of a vector bundle is only a coherent sheaf, we define hermitian coherent sheaves (following Zha [23]) and hermitian embedded vector bundles. When we work with the projection of a projective bundle, we have to endow objects of \( D^b(X) \) with hermitian structures, as the higher derived functors of the direct image do not vanish.

Since each projective morphism is the composition of a closed immersion followed by the projection of a projective bundle, we can combine the theories for closed immersions and projective spaces in a global theory of generalized analytic torsion classes. This is done by imposing a compatibility equation similar to Bismut-Lebeau compatibility formula ([8], [3]) for the diagonal immersion \( \Delta: \mathbb{P}_\mathbb{C}^n \to \mathbb{P}_\mathbb{C}^n \times \mathbb{P}_\mathbb{C}^n, \ n \geq 0. \) The last section of the paper is devoted to the compatibility theorem.

The next section summarizes basic facts about Deligne complexes, that constitute the natural context where singular Bott-Chern classes and analytic torsion classes are defined. We also give some definitions and properties related to transversal morphisms and wave front sets.

## 2. PRELIMINARIES

For the convenience of the reader we summarize in this section some basic facts about the Deligne complexes. For more details the reader is referred to [11] and [13].

**Definition 2.1.** A *Dolbeault complex* \( A = (A^\bullet_R, d_A) \) is a graded complex of real vector spaces, which is bounded from below and equipped with a bigrading on \( A^\bullet = A^\bullet \otimes_{\mathbb{R}} \mathbb{C} \), i.e.,

\[
A^n_C = \bigoplus_{p+q=n} A^{p,q},
\]

satisfying the following properties:

(i) The differential \( d_A \) can be decomposed as the sum \( d_A = \partial + \bar{\partial} \) of operators \( \partial \) of type \((1,0)\), respectively \( \bar{\partial} \) of type \((0,1)\).

(ii) It satisfies the symmetry property \( A^{p,q} = \bar{A}^{q,p} \), where \( \bar{\cdot} \) denotes complex conjugation.
The basic example of Dolbeault complex is the complex of differential forms on a smooth variety $X$ over $\mathbb{C}$, denoted $E^*(X)_\mathbb{R}$.

Following [13, §5.2], to a Dolbeault complex one assigns a Deligne complex denoted $\mathcal{D}^\ast(A, \ast)$. The differential of the Deligne complex is denoted by $d_{\mathcal{D}} : \mathcal{D}^n(A, p) \to \mathcal{D}^{n+1}(A, p)$.

When $A$ is a Dolbeault algebra, that is, $A$ is a graded commutative real differential algebra and the product is compatible with the bigrading, then $\mathcal{D}^\ast(A, \ast)$ has a product

$$ \bullet : \mathcal{D}^n(A, p) \otimes \mathcal{D}^m(A, q) \longrightarrow \mathcal{D}^{n+m}(A, p+q) $$

that is graded commutative with respect to the first degree, it is associative up to homotopy and satisfies the Leibnitz rule.

The Deligne algebra of differential forms on $X$ is defined to be

$$ \mathcal{D}^\ast(X, \ast) := \mathcal{D}^\ast(E^*(X)_\mathbb{R}, \ast). $$

Recall that, if $X$ is equi-dimensional of dimension $d$, there is a natural trace map $\int : H^{2d}_c(X, \mathbb{R}(d)) \to \mathbb{R}$ given by

$$ \omega \mapsto -\frac{1}{(2\pi i)^d} \int_X \omega. $$

To take this trace map into account the Dolbeault complex of currents is constructed as follows. Denote by $E^c_\ast(X)_\mathbb{R}$ the space of differential forms with compact support. Then $D_{p,q}(X)$ is the topological dual of $E^{p,q}_c(X)$ and $D_n(X)_\mathbb{R}$ is the topological dual of $E^n_c(X)_\mathbb{R}$. In this complex the differential is given by

$$ dT(\eta) = (-1)^n T(d\eta) $$

for $T \in D_n(X)_\mathbb{R}$. For $X$ equi-dimensional of dimension $d$ we write

$$ D^{p,q}(X) = D_{d-p,d-q}(X), \quad D^n(X)_\mathbb{R} = (2\pi i)^{-d} D_{2d-n}(X). $$

With these definitions, $D^\ast(X)_\mathbb{R}$ is a Dolbeault complex and it is a Dolbeault module over $E^\ast(X)_\mathbb{R}$. We will denote

$$ \mathcal{D}^\ast_d(X, \ast) := \mathcal{D}^\ast(D^\ast(X)_\mathbb{R}, \ast). $$

for the Deligne complex of currents on $X$.

We denote $\mathbb{D}$ the base ring for Deligne cohomology (the ring obtained by considering the Deligne cohomology for the manifold Spec $\mathbb{C}$; see [14, §1]).

Observe that the trace map above defines an element

$$ \delta_X \in \mathcal{D}^0_\mathbb{D}(X, 0). $$

More generally, if $Y \subset X$ is a subvariety of pure codimension $p$, then the current integration along $Y$, denoted $\delta_Y \in \mathcal{D}^{2,p}_\mathbb{D}(X, p)$ is given by

$$ \delta_Y(\omega) = \frac{1}{(2\pi i)^{d-p}} \int_Y \omega. $$
If $\omega$ is a locally integrable differential form, we associate to it a current

$$[\omega](\eta) = \frac{1}{(2\pi i)^{\dim X}} \int_X \eta \wedge \omega.$$  

This map is injective and allow us to identify $\mathcal{D}^*(X,\ast)$ with a subcomplex of $\mathcal{D}^*_D(X,\ast)$. For instance, when in a formula there appear sums of currents and differential forms, we will tacitly assume that the differential forms are converted into currents by this map.

A particularly important current is $W_1 \in \mathcal{D}_D^1(\mathbb{P}^1, 1)$ given by

$$W_1 = \left[ -\frac{1}{2} \log ||t||^2 \right]. \quad (2.2)$$

With the above convention, this means that

$$W_1(\eta) = \frac{1}{2\pi i} \int_{\mathbb{P}^1} \left( -\frac{1}{2} \log ||t||^2 \right) \eta.$$ \quad (2.3)

By the Poincaré-Lelong equation

$$d_\mathcal{D} W_1 = \delta_\infty - \delta_0. \quad (2.4)$$

We also denote by $1_1 \in \mathcal{D}$ the element represented by the constant function 1 of $\mathcal{D}^1(\text{Spec} \mathbb{C}, 1) = \mathbb{R}$.

Note also that, if $f: X \to Y$ is a proper morphism of smooth complex varieties of relative dimension $e$, then there are direct image morphisms

$$f_*: \mathcal{D}^n(X, p) \longrightarrow \mathcal{D}^{n-2e}(X, p - e).$$

If $f$ is smooth, the direct image of differential forms is defined by, first converting them into currents and then applying the above direct image of currents. If $f$ is a smooth morphism of relative dimension $e$ we can convert them back into differential forms. This procedure gives us $1/(2\pi i)^e$ times the usual integration along the fiber.

We shall use the notations and definitions of [14]. In particular, we write

$$\mathcal{D}^n(X, p) = \mathcal{D}(X, p) / d_{\mathcal{D}} \mathcal{D}^{n-1}(X, p),$$

$$\mathcal{D}^n_D(X, p) = \mathcal{D}^n_D(X, p) / d_{\mathcal{D}} \mathcal{D}^{n-1}_D(X, p).$$

**Definition 2.5.** An additive genus in Deligne cohomology is a characteristic class $\phi$ for vector bundles on any rank in the sense of [14, Def. 1.5] that satisfies the equation

$$\phi(E_1 \oplus E_2) = \phi(E_1) + \phi(E_2). \quad (2.6)$$

A consequence of [14, Theorem 1.8] is that there is a bijection between the set of additive genus in Deligne cohomology and the set of power series in one variable $\mathbb{D}[[x]]$. An additive genus is called real if the corresponding power series belong to $\mathbb{R}[[x]]$. 

A current $\eta$ can be viewed as a generalized section of a vector bundle and, as such, has a wave front set that is denoted by $\text{WF}(\eta)$. The theory of wave front sets of distributions is developed in [18, Chapter VIII]. The wave front set of $\eta$ is a closed conical subset of the cotangent bundle of $X$ minus the zero section $T^*X_0 = T^*X \setminus \{0\}$. This set describes the points and directions of the singularities of $\eta$ and it allows us to define certain products and inverse images of currents.

Moreover, if $S \subset T^*X_0$ is a closed conical subset of the cotangent bundle of $X$ with the zero section removed, we will denote by $(D_\mathcal{D}(X,S,\cdot), d\omega)$ the Deligne complex of currents on $X$ whose wave front set is contained in $S$. We will denote by $D^*(X,S)$ its complex of global sections.

For instance, if we denote by $N^*_Y$ the conormal bundle to $Y$, then

$$\delta_Y \in D^2_{\mathcal{D}}(X,N^*_Y,p).$$

Let $f: X \rightarrow Y$ be a morphism of complex manifolds. The set of normal directions of $f$ is

$$N_f = \{(f(x), v) \in T^*Y_0 \mid df(x)^t v = 0\}.$$

This set measures the singularities of $f$. For instance, if $f$ is a smooth map then $N_f = \emptyset$ whereas, if $f$ is a closed immersion, $N_f$ is the conormal bundle of $f(X)$. Let $S \subset T^*Y_0$ be a closed conical subset. We will say that $f$ is transverse to $S$ if $N_f \cap S = \emptyset$. We will denote

$$f^*S = \{(x, df(x)^t v) \in T^*X_0 \mid (f(x), v) \in S\}.$$

We refer to [18, Chapter VIII] and [14, §4] for other definitions and results that are necessary for proving our results, but do not appear explicitly in this presentation. We state here only the Poincaré lemma for currents with fixed wave front set, a result that may be of independent interest. As usual, we will denote by $F$ the Hodge filtration of the Dolbeault complex $\mathcal{D}_{X,S}$.

**Theorem 2.7 (Poincaré lemma).** [14, Theorem 4.5] The natural morphism

$$\iota: (E^*(X), F) \rightarrow (D^*(X,S), F)$$

is a filtered quasi-isomorphism.

**Definition 2.8.** Let $f: X \rightarrow Y$ and $g: Z \rightarrow Y$ be morphisms of smooth complex varieties. We say that $f$ and $g$ are transverse if

$$N_f \cap N_g = \emptyset,$$

where $N_f$ and $N_g$ are the sets of normal directions to $f$ and $g$ respectively.

It is easily seen that, if $f$ is a closed immersion, this definition of transverse morphisms agrees with the definition given in [17, IV-17.13].

If $f$ and $g$ are transverse, then the cartesian product $X \times Y$ is smooth.
3. SINGULAR BOTT-CHERN CLASSES

A particular theory of singular Bott-Chern classes was constructed in [6]. In this section we summarize some of the results presented and proved in detail in [14], where we give an axiomatic characterization of singular Bott-Chern classes an a classification of the theories that satisfy the axioms.

As we have mentioned before, we need to define a way of putting hermitian metrics on arbitrary coherent sheaves. This definition is due to Zha [23], although still unpublished.

**Definition 3.1.** A **metrized coherent sheaf** $\mathcal{F}$ on $X$ is a pair $(\mathcal{F}, E_\ast \to \mathcal{F})$ where $\mathcal{F}$ is a coherent sheaf on $X$ and

$$0 \to E_n \to E_{n-1} \to \cdots \to E_0 \to \mathcal{F} \to 0$$

is a finite resolution by hermitian vector bundles of the coherent sheaf $\mathcal{F}$. This resolution is also called the metric of $\mathcal{F}$.

If $E$ is a hermitian vector bundle, we will also denote by $E$ the metrized coherent sheaf $(E, E \xrightarrow{id} E)$.

Note that the coherent sheaf 0 may have non-trivial metrics. In fact, any exact sequence of hermitian vector bundles

$$0 \to A_n \to \cdots \to A_0 \to 0 \to 0$$

can be seen as a metric on 0. It will be denoted $\mathcal{O}_X$. A metric on 0 is said to be **orthogonally split** if the exact sequence is orthogonally split.

A morphism of metrized coherent sheaves $\mathcal{F}_1 \to \mathcal{F}_2$ is just a morphism of sheaves $\mathcal{F}_1 \to \mathcal{F}_2$. A sequence of metrized coherent sheaves

$$\mathcal{E}: \cdots \longrightarrow \mathcal{F}_{n+1} \longrightarrow \mathcal{F}_n \longrightarrow \mathcal{F}_{n-1} \longrightarrow \cdots$$

is said to be exact if it is exact as a sequence of coherent sheaves.

**Definition 3.2.** Let $\mathcal{F} = (\mathcal{F}, E_\ast \to \mathcal{F})$ be a metrized coherent sheaf. Then the **Chern character form** associated to $\mathcal{F}$ is given by

$$\text{ch}(\mathcal{F}) = \sum_i (-1)^i \text{ch}(E_i).$$

One can also extended the theory of Bott-Chern classes to metrized coherent sheaves, in order to measure the lack of compatibility between the metrics in exact sequences (see [14 §2], [23]). They satisfy similar properties as the classes defined for vector bundles.

Now we can define the objects that give rise to singular Bott-Chern classes.

**Definition 3.3.** [14 Definition 6.4] Let $i: Y \longrightarrow X$ be a closed immersion of complex manifolds. Let $N$ be the normal bundle of $Y$ and let $h_N$ be a hermitian metric on $N$. We denote $N = (N, h_N)$. Let $r_N$ be the rank of $N$, that agrees with the codimension of $Y$ in $X$. Let $\mathcal{F} = (F, h_F)$ be a hermitian vector bundle on $Y$ of rank $r_F$. Let $E_\ast \to i_* F$ be a metric on the coherent sheaf $i_* F$. The four-tuple

$$\xi = (i, N, F, E_\ast).$$

(3.4)
is called a hermitian embedded vector bundle. The number $r_F$ will be called the rank of $\xi$ and the number $r_N$ will be called the codimension of $\xi$.

By convention, any exact complex of hermitian vector bundles on $X$ will be considered a hermitian embedded vector bundle of any rank and codimension.

**Definition 3.5.** A singular Bott-Chern class for a hermitian embedded vector bundle $\xi$ is a class $\tilde{\eta} \in \bigoplus_p \overline{\mathcal{D}}^{2p-1}(X, p)$ such that

$$d_{\mathcal{D}} \eta = \sum_{i=0}^{n} (-1)^i [\mathrm{ch}(E_i)] - i_*([\mathrm{Td}^{-1}(N) \mathrm{ch}(F)])$$

(3.6)

for any current $\eta \in \tilde{\eta}$.

The existence of this class is guaranteed by the Grothendieck-Riemann-Roch theorem, which implies that the two currents in the right hand side of equation (3.6) are cohomologous.

In each singular Bott-Chern class we can find a representative with controlled singularities ([14], Proposition 6.8). This allows us to define inverse images of singular Bott-Chern classes for maps satisfying a transversality condition by respect to the immersion $i$.

**Definition 3.7.** ([14], Definition 6.9] Let $r_F$ and $r_N$ be two integers. A theory of singular Bott-Chern classes of rank $r_F$ and codimension $r_N$ is an assignment which, to each hermitian embedded vector bundle $\xi = (i: Y \longrightarrow X, N, F, E_*)$ of rank $r_F$ and codimension $r_N$, assigns a class of currents $T(\xi) \in \bigoplus_p \overline{\mathcal{D}}^{2p-1}(X, p)$ satisfying the following properties

(i) (Differential equation) The following equality holds

$$d_{\mathcal{D}} T(\xi) = \sum_{i} (-1)^i [\mathrm{ch}(E_i)] - i_*([\mathrm{Td}^{-1}(N) \mathrm{ch}(F)])$$

(3.8)

(ii) (Functoriality) For every morphism $f: X' \longrightarrow X$ of complex manifolds that is transverse to $i$, then

$$f^* T(\xi) = T(f^* \xi).$$

(iii) (Normalization) Let $\overline{A} = (A_* , g_*)$ be a non-negatively graded orthogonally split complex of vector bundles. Write $\xi \oplus \overline{A} = (i: Y \longrightarrow X, N, F, E_* \oplus \overline{A}_*)$. Then

$$T(\xi) = T(\xi \oplus \overline{A}).$$

Moreover, if $X = \mathrm{Spec} \mathbb{C}$ is one point, $Y = \emptyset$ and $E_* = 0$, then

$$T(\xi) = 0.$$

A theory of singular Bott-Chern classes is an assignment as before, for all positive integers $r_F$ and $r_N$. When the inclusion $i$ and the bundles $F$ and $N$ are clear from the context, we will denote $T(\xi)$ by $T(E_*)$. 
Unlike the classical situation of Bott-Chern classes, these axioms do not uniquely characterize the singular Bott-Chern classes. In order to classify the various non-equivalent theories of singular Bott-Chern classes we have to look closer to the case when $Y$ is the zero section of a completed vector bundle. Let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles over $Y$. We denote $P = \mathbb{P}(\overline{N} \oplus \mathbb{C})$, the projective bundle of lines in $\overline{N} \oplus \mathcal{O}_Y$. Let $s: Y \to P$ denote the zero section and let $\pi_P: P \to Y$ denote the projection. On $P$ there is a tautological short exact sequence

$$0 \to \mathcal{O}(-1) \to \pi_P^*(\overline{N} \oplus \mathbb{C}) \to Q \to 0.$$  \hfill (3.9)

We obtain a morphism $Q^\vee \to \mathcal{O}_P$ that induces a long exact sequence

$$0 \to \bigwedge^n Q^\vee \to \ldots \to \bigwedge^1 Q^\vee \to \mathcal{O}_P \to s_* \mathcal{O}_Y \to 0.$$  

We also obtain the exact sequence

$$0 \to \bigwedge^n Q^\vee \otimes \pi_P^*F \to \ldots \to \bigwedge^1 Q^\vee \otimes \pi_P^*F \to \pi_P^*F \to s_* F \to 0.$$  \hfill (3.10)

**Definition 3.11.** The Koszul resolution of $s_*(F)$ is the resolution (3.10). The complex

$$0 \to \bigwedge^n Q^\vee \otimes \pi_P^*F \to \ldots \to \bigwedge^1 Q^\vee \otimes \pi_P^*F \to \pi_P^*F \to 0$$

will be denoted by $K(F,N)$. When $N$ is a hermitian vector bundle, the exact sequence (3.9) induces a hermitian metric on $Q$. If, moreover, $\overline{F}$ is also a hermitian vector bundle, all the vector bundles that appear in the Koszul resolution have an induced hermitian metric. We will denote by $K(\overline{F},\overline{N})$ the corresponding complex of hermitian vector bundles.

With these notations we have:

**Theorem 3.12.** [14, Theorems 6.16 and 7.1] Let $r_F$ and $r_N$ be two positive integers.

(i) Let $T$ be a theory of singular Bott-Chern classes of rank $r_F$ and codimension $r_N$. Let $Y$ be a complex manifold and let $\overline{F}$ and $\overline{N}$ be hermitian vector bundles of rank $r_F$ and $r_N$ respectively. Then the current $(\pi_P)_*(T(K(F,N)))$ is closed. Moreover the cohomology class that it represents does not depend on the metric of $N$ and $F$ and determines a characteristic class for pairs of vector bundles of rank $r_F$ and $r_N$. We denote this class by $C_{T,F,N}$.

(ii) Let $C$ be a characteristic class for pairs of vector bundles of rank $r_F$ and $r_N$ with values in

$$\bigoplus_p H^{2p-1}_{\mathcal{O}}(\underline{\mathbb{R}}(p)).$$

Then there exists a unique theory $T_C$ of singular Bott-Chern classes of rank $r_F$ and codimension $r_N$ such that $C_{T_C} = C$. 


Proof. We give hereafter a short idea of the proof (the reader is referred to [14] for
details).

Proof of (i). For the particular choice of singular Bott-Chern classes defined in [6] this
result is proven in loc. cit.. In general, the statement follows from the properties of the
Bott-Chern classes and of their direct images.

Proof of (ii). Let $\xi = (i: Y \longrightarrow X, N, F, E_*)$ be as in definition 3.7. The main tool of
the proof is the deformation of resolutions that is the following statement:

There exists a complex manifold $W = W(E_*)$, called the Grassmannian graph con-
struction, with a birational map $\pi: W \longrightarrow X \times \mathbb{P}^1$ and a complex of vector bundles,
$\text{tr}_1(E_*)$, over $W$ such that

(i) The map $\pi$ is an isomorphism away from $Y \times \{\infty\}$. The restriction of $\text{tr}_1(E_*)$ to
$X \times (\mathbb{P}^1 \setminus \{\infty\})$ is isomorphic to $(p_X \circ \pi)^*E_*$ restricted to $X \times (\mathbb{P}^1 \setminus \{\infty\})$ (where $p_X$ is the projection on the first factor). Moreover, If $\tilde{X}$ is the Zariski closure of $U \times \{\infty\}$
inside $W$, the restriction of $\text{tr}_1(E_*)$ to $\tilde{X}$ is split acyclic.

(ii) When $Y$ is non-empty and $F$ is a non-zero vector bundle over $Y$, then $W(E_*)$ agrees
with $W_{Y/X}$, the deformation to the normal cone of $Y$. The preimage of $X \times \{\infty\}$
can be written as $\tilde{X} \cup P$, where $P = \mathbb{P}(N_{Y/X} \oplus \mathbb{C})$ is a projective bundle over $Y$.
Moreover, there is an exact sequence of resolutions on $P$

$$
0 \longrightarrow A_* \longrightarrow \text{tr}_1(E_*) \mid_P \longrightarrow K(F, N_{Y/X}) \longrightarrow 0,
$$

where $A_*$ is split acyclic and $K(F, N_{Y/X})$ is the Koszul resolution.

(iii) Let $f: X' \longrightarrow X$ be a morphism of complex manifolds and assume that we are in
one of the following cases:

(a) The map $f$ is smooth.

(b) The map $f$ is arbitrary and $E_*$ is acyclic.

(c) $f$ is transverse to $Y$.

Then $E'_* := f^*(E_*)$ is exact over $f^{-1}(U)$,

$$
W' := W(E'_*) = W \times X',
$$

with $f_W: W' \longrightarrow W$ the induced map, and we have $f_W^*(\text{tr}_1(E_*)_*) = \text{tr}_1(f^*(E_*)_*)$.

(iv) If the vector bundles $E_i$ are provided with hermitian metrics, then one can choose a
hermitian metric on $\text{tr}_1(E_*)_*$ such that its restriction to $X \times \{0\}$ is isometric to $E_*$
and the restriction to $U \times \{\infty\}$ is orthogonally split. We will denote by $\text{tr}_1(E'_*)$ the
complex $\text{tr}_1(E_*)_*$ with such a choice of hermitian metrics. Moreover, this choice of
metrics can be made functorial.

For more details on the deformation of resolutions see [1 II.1], [6 §4 (c)], [16 §1] and [14 §5].
For each $k$ we will denote by $\eta_k$ the exact sequence of hermitian vector bundles

$$0 \longrightarrow A_k \longrightarrow \text{tr}_1(E^*_k)|_P \longrightarrow K(F, N)_k \longrightarrow 0. \quad (3.13)$$

With these notations we define

$$T_C(\xi) = -(p_W)_* \left( \sum_k (-1)^k W_1 \cdot \text{ch}(\text{tr}_1(E^*_k)) \right)$$

$$- \sum_k (-1)^k (p_P)_* \left[ \text{ch}(\eta^*_k) \right] + i_* C(F, N), \quad (3.14)$$

where $p_W : W \rightarrow X$ and $p_P : P \rightarrow X$ are the natural maps.

One can prove that this definition does not depend on the choice of the metric of $\text{tr}_1(E^*)$ or the metric of $A^*$, that $T_C$ satisfies the properties of definition 3.7 and that the characteristic class $C_{T_C}$ agrees with $C$. Moreover, the formula (3.14) implies that the singular Bott-Chern class is characterized by the properties of definition 3.7 and the characteristic class $C_T$, hence the uniqueness.

The next definition summarizes some important properties of the singular Bott-Chern classes.

**Definition 3.15.** Let $T$ be a theory of singular Bott-Chern classes.

(i) We say that $T$ is compatible with the projection formula if, for every hermitian embedded vector bundle $\xi = (i: Y \rightarrow X, N, F, E_*)$, and every hermitian vector bundle $G$ on $X$, if we denote $\xi \otimes G = (i: Y \rightarrow X, N, F \otimes i^* G, E_* \otimes G)$, the equation

$$T(\xi \otimes G) = T(\xi) \cdot \text{ch}(G)$$

holds.

(ii) A theory of singular Bott-Chern classes $T$ is called additive if for any closed embedding of complex manifolds $i: Y \hookrightarrow X$ and any hermitian embedded vector bundles $\xi_1 = (i, N, F_1, E_{1,*}), \xi_2 = (i, N, F_2, E_{2,*})$ the equation

$$T(\xi_1 \oplus \xi_2) = T(\xi_1) + T(\xi_2)$$

is satisfied.

(iii) Let $i_{Y/M} = i_{X/M} \circ i_{Y/X}$ be a composition of closed immersions of complex manifolds. Assume that the normal bundles $N_{Y/X}$, $N_{X/M}$ and $N_{Y/M}$ are provided with hermitian metrics. We will denote by $\mathcal{E}$ the exact sequence

$$\mathcal{E}: 0 \rightarrow N_{Y/X} \rightarrow N_{Y/M} \rightarrow i_{Y/X}^* N_{X/M} \rightarrow 0. \quad (3.16)$$

Let $F$ be a hermitian vector bundle over $Y$, let $E_* \longrightarrow (i_{Y/X})_* F$ be a resolution by hermitian vector bundles. Let $E^{'*}_k$ be a complex of complexes of vector bundles over $M$, such that, for each $k \geq 0$, $E^{'*}_{k,*} \longrightarrow (i_{X/M})_* E_k$ is a resolution, and these
resolutions are compatible with the direct image of $\mathcal{E}_* \to (i_Y/X)_*F$. It follows that we have a resolution $\text{Tot}(\mathcal{E}_{*,*}) \to (i_{Y/M})_*F$ of $(i_{Y/M})_*F$ by hermitian vector bundles. We denote $\mathcal{E}_Y \to X$, $\mathcal{E}_X \to M$, $\mathcal{E}_X \to M,k$ the hermitian embedded vector bundles obtained using these resolutions. We will say that $T$ is transitive if the equation

$$T(\mathcal{E}_Y \to M) = \sum_k (-1)^k T(\mathcal{E}_X \to M,k) + (i_{X/M})_* (T(\mathcal{E}_Y \to X) \bullet \text{Td}^{-1}(\mathcal{N}_X/M))$$

$$+ (i_{Y/M})_* \text{ch}(F) \bullet \text{Td}^{-1}(\mathcal{E}_N)$$

(3.17)

holds.

Let $X$ be a smooth complex variety and let $N$ be a hermitian vector bundle of rank $r$. We denote by $P = \mathbb{P}(N \oplus 1)$ the projective bundle obtained by completing $N$. Let $\pi_P: P \to X$ be the projection and let $s: X \to P$ be the zero section. Notice that $N$ can be identified with the normal bundle to $X$ on $P$. On $P$ we have a tautological quotient vector bundle with an induced metric $Q$. For each hermitian vector bundle $\mathcal{F}$ on $X$ we have the Koszul resolution $K(F,N)$ of $s_*F$. We denote by $K(\mathcal{F},\mathcal{N})$ the Koszul resolution with the induced metrics. Let $\xi = (s, N, F, K(F,N))$.

Definition 3.18. Let $T$ be a theory of singular Bott-Chern classes. We say that $T$ is homogeneous if, for every pair of hermitian vector bundles $\mathcal{N}$ and $F$ with $\text{rk} \mathcal{N} = r$, there exists a homogeneous class of bidegree $(2r - 1, r)$ in the Deligne complex

$$\tilde{e}(\mathcal{F},\mathcal{N}) \in \tilde{\mathcal{D}}_D^{2r-1}(P, N, r)$$

such that

$$T(\xi) \bullet \text{Td}(\mathcal{Q}) = \tilde{e}(\mathcal{F},\mathcal{N}) \bullet \text{ch}(\pi_P^*F).$$

(3.19)

The next theorem summarizes some of the results in [14] identifying the homogeneous theories $T$:

Theorem 3.20. [14 §9]

(i) There is a unique homogeneous theory of singular Bott-Chern classes, that we denote $T^h$. This theory is compatible with the projection formula, additive and transitive. Moreover, it coincides with the theory of Bismut-Gillet-Soulé [6] and with that of Zha [23] (once suitably normalized).

(ii) Let $T$ be any transitive theory of singular Bott-Chern classes, that is compatible with the projection formula. Then there is a unique real additive genus $S_T$ such that, for any hermitian embedded vector bundle $\xi$, we have

$$T(\xi) - T^h(\xi) = f_*[\text{ch}(\mathcal{F}) \bullet \text{Td}(\mathcal{N})^{-1} \bullet S_T(N) \bullet 1_1].$$

(3.21)

(iii) Conversely, any real additive genus $S$ defines, by means of equation (3.21), a unique transitive theory of singular Bott-Chern classes, that is compatible with the projection formula and additive.
4. ANALYTIC TORSION CLASSES FOR PROJECTIVE SPACES

In this section and the next one we summarize some recent results that are presented in detail in [12]. This section is mainly devoted to the case of the projective spaces but before that we explain how to introduce hermitian metrics in the bounded derived category of coherent sheaves. A language that will be useful in both sections.

When working with general projective morphisms of smooth complex varieties, the relative tangent complex is not a vector bundle but a complex. Moreover, the direct images and higher direct images of a vector bundle are not in general locally free. Finally, when considering the composition of two morphism, one has to deal with resolutions of resolutions, that lead to cumbersome notation. All these issues are easily solved using hermitian structures on the bounded derived category of coherent sheaves, and this is the point of view that we follow in [12]. Moreover, the use of hermitian structures on the derived category furnishes us a useful formalism to explore the properties of analytic torsion. Finally, the extension of secondary characteristic classes to the derived category is interesting in its own right.

Let \( X \) be a smooth complex algebraic variety over \( \mathbb{C} \), namely a reduced and separated scheme of finite type over \( \mathbb{C} \). We denote by \( \mathcal{V}^b(X) \) the exact category of bounded complexes of algebraic vector bundles on \( X \). Then \( \mathcal{V}^b(X) \) is defined as the category of pairs \( E = (E, h) \), where \( E \in \text{Ob} \mathcal{V}^b(X) \) and \( h \) is a smooth hermitian metric on the complex of analytic vector bundle \( E^{an} \). The complex \( E \) will be called the underlying complex of \( E \). We denote by \( \text{Coh}(X) \) the abelian category of coherent sheaves on \( X \), by \( \mathcal{C}^b(X) \) the abelian category of bounded cochain complexes of coherent sheaves on \( X \) and by \( \mathcal{D}^b(X) \) its bounded derived category. The objects of \( \mathcal{D}^b(X) \) are complexes of quasi-coherent sheaves with bounded coherent cohomology. The symbol \( \sim \) means either quasi-isomorphism in \( \mathcal{C}^b(X) \) or \( \mathcal{V}^b(X) \), or isomorphism in \( \mathcal{D}^b(X) \).

In [12] we define and characterize the notion of meager complex. Roughly speaking, a meager complex is a bounded acyclic complex of hermitian vector bundles whose Bott-Chern classes vanish for structural reasons. We also introduce the concept of tight morphism and tight equivalence relation between bounded complexes of hermitian vector bundles. We refer to loc. cit. for the exact definitions, computation rules and properties. In particular we deduce that the submonoid of acyclic complexes modulo meager complexes has a natural structure of abelian group, that we denote \( \text{KA}(X) \), and which is a universal abelian group for additive secondary characteristic classes. With these tools at hand, we can define hermitian structures on objects of \( \mathcal{D}^b(X) \). A hermitian metric on an object \( \mathcal{F} \) of \( \mathcal{D}^b(X) \) consists in choosing a bounded complex of hermitian vector bundles \( \overline{E} \) and a quasi-isomorphism \( \overline{E} \sim \mathcal{F} \). We introduce an equivalence relation on the set of hermitian metrics on \( \mathcal{F} \) (two equivalent hermitian metrics are said to fit tight). Then a hermitian structure on \( \mathcal{F} \) is a set of equivalence classes of hermitian metrics on \( \mathcal{F} \). The objects of the category \( \mathcal{D}^b(X) \) are objects of \( \mathcal{D}^b(X) \) together with a hermitian structure, and the morphisms are just morphisms in \( \mathcal{D}^b(X) \). The group \( \text{KA}(X) \) naturally acts on the fibers of the forgetful functor \( \mathcal{D}^b(X) \to \mathcal{D}^b(X) \), freely and transitively. We also construct secondary characteristic classes for isomorphisms and
distinguished triangles in $\mathcal{D}^b(X)$.

**Definition 4.1.** Let $f : X \to Y$ be a morphism of smooth complex varieties. The *tangent complex* of $f$ is the complex

$$T_f : 0 \to TX \xrightarrow{df} f^*TY \to 0$$

where $TX$ is placed in degree 0 and $f^*TY$ is placed in degree 1. It defines an object, also denoted $T_f \in \text{Ob} \ D^b(X)$. A *relative hermitian structure* on $f$ is the choice of an object $T_f \in D^b(X)$ over $T_f$.

We mention here the following situations of special interest:

- suppose $f : X \hookrightarrow Y$ is a closed immersion. Let $N_{X/Y}[-1]$ be the normal bundle to $X$ in $Y$, considered as a complex concentrated in degree 1. By definition, there is a natural quasi-isomorphism $p : T_f \sim N_{X/Y}[-1]$ in $C^b(X)$, and hence an isomorphism $p^{-1} : N_{X/Y}[-1] \to T_f$ in $D^b(X)$. Therefore, a hermitian metric $h$ on the vector bundle $N_{X/Y}$ naturally induces a hermitian structure $p^{-1} : (N_{X/Y}[-1], h) \to T_f$ on $T_f$.

- suppose $f : X \to Y$ is a smooth morphism. Let $T_{X/Y}$ be the relative tangent bundle on $X$, considered as a complex concentrated in degree 0. By definition, there is a natural quasi-isomorphism $\iota : T_{X/Y} \sim T_f$ in $C^b(X)$. Any choice of a hermitian metric $h$ on $T_{X/Y}$ naturally induces a hermitian structure $\iota : (T_{X/Y}, h) \to T_f$.

In what follows we shall place ourselves in the category $\overline{\text{Sm}}^\ast / \mathbb{C}$. The objects of this category are smooth complex varieties. The morphisms are pairs $\overline{f} = (f, \overline{T}_f)$ formed by a projective morphism of smooth complex varieties $f$, together with a hermitian structure on the tangent complex $T_f$. Moreover we denote

$$\overline{f}_!(\omega) := f_!(\omega \bullet \text{Td}(T_f)),$$

where $\bullet$ denotes the product in the Deligne complex and $\text{Td}$ is the Todd class.

We define now the objects to which we associate analytic torsion classes.

**Definition 4.2.** [12, Definition 3.3] Let $f : X \to Y$ be a projective morphism of smooth complex varieties and $\overline{f} \in \text{Hom}_{\overline{\text{Sm}}^\ast / \mathbb{C}}(X, Y)$ an arrow over $f$. Let $\overline{\mathcal{F}} \in \text{Ob} \ D^b(X)$ and let $\overline{f}_*\overline{\mathcal{F}} \in \text{Ob} \ D^b(Y)$ be an object over $f_*\mathcal{F}$. The triple $\overline{\xi} = (\overline{f}, \overline{\mathcal{F}}, \overline{f}_*\overline{\mathcal{F}})$ will be called a *relative metrized complex*. When $f$ is a closed immersion we will also call it an *embedded metrized complex*. When $\overline{\mathcal{F}}$ and $f_*\overline{\mathcal{F}}$ are clear from the context we will denote the relative metrized complex $\overline{\xi}$ by the arrow $\overline{f}$.

The embedded metrized complexes clearly generalize the hermitian embedded vector bundles. All the results from the previous section can be extended for embedded metrized complexes.

For the remaining of this section we restrict ourselves to the case of projective spaces.
Definition 4.3 ([12]). Let $X$ be a smooth complex variety and $\pi : \mathbb{P}^n_X \to X$ the projection. An analytic torsion class for the relative hermitian complex $\xi = (\pi, F, \pi_* F)$ is a class $\tilde{\eta} \in \bigoplus_p \mathcal{D}^{2p-1}(X, p)$ such that

$$d_{\mathcal{D}} \eta = \text{ch}(\pi_* F) - \pi^\flat \left[ \text{ch}(F) \right]$$

(4.4)

for any differential form $\eta \in \tilde{\eta}$. We will denote $d_{\mathcal{D}} \tilde{\eta} = d_{\mathcal{D}} \eta$.

The existence of this class is guaranteed by the Grothendieck-Riemann-Roch theorem, which implies that the two currents at the right hand side of equation (4.4) are cohomologous. Since the map $\pi$ is smooth, the analytic torsion class is the class of a smooth form.

As in the case of closed immersions, the analytic torsion class will not be uniquely defined, and we may choose between various theories of analytic torsion classes. Before stating the precise definition, we introduce some natural relative hermitian complexes for the case $X = \text{Spec} \mathbb{C}$, $\pi : \mathbb{P}^n_{\mathbb{C}} \to \text{Spec} \mathbb{C}$ being the trivial projective bundle.

We endow the trivial sheaf with the trivial metric and $\mathcal{O}(1)$ with the Fubini-Study metric. Using the short exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}^n_{\mathbb{C}}} \to \mathcal{O}(1)^{n+1} \to T_\pi \to 0.$$  

(4.5)

we obtain a quotient hermitian structure on the tangent bundle $T_\pi$. We denote the resulting hermitian vector bundle by $T_{\pi}^{FS}$ and call it the Fubini-Study metric of $T_\pi$.

We endow the invertible sheaves $\mathcal{O}(k)$ with the $k$-th tensor power of the Fubini-Study metric on $\mathcal{O}(1)$. We refer to them by $\mathcal{O}(k)$. Moreover, the complexes $\pi_* \mathcal{O}(k)$ can be endowed with natural hermitian structures (see [12], §3.4] for details). We denote, for every integer $k$

$$\xi_n(k) = (\pi_{\text{FS}}, \mathcal{O}(k), \pi_* \mathcal{O}(k)).$$

(4.6)

If $X$ is a smooth complex variety, we employ the same notation $\xi_n(k)$ to refer to the pull-back relative hermitian complex.

Let $\mathcal{F}$ be a metrized coherent sheaf on $X$. Then

$$\xi_n(k) \otimes \mathcal{F} := (\pi_{\text{FS}}, \mathcal{O}(k), \pi_* \mathcal{O}(k)).$$

Definition 4.7. ([12] Definition 3.49] Let $n$ be a non-negative integer. A theory of analytic torsion classes for projective spaces of dimension $n$ is an assignment that, to each relative metrized complex

$$\xi = (\pi : \mathbb{P}^n_X \to X, \mathcal{F}, \pi_* \mathcal{F})$$

of relative dimension $n$, assigns a class of differential forms

$$T(\xi) \in \bigoplus_p \mathcal{D}^{2p-1}(X, p),$$

satisfying the following properties.
(i) (Differential equation) The relation
\[ d_{\mathcal{O}} T(\bar{\xi}) = \text{ch}(\pi_\ast \mathcal{F}) - \pi_\ast [\text{ch}(\mathcal{F})] \] 
holds.

(ii) (Functoriality) Given a morphism \( f : Y \to X \), we form the cartesian diagram
\[
\begin{array}{ccc}
\mathbb{P}^n_Y & \xrightarrow{f'} & \mathbb{P}^n_X \\
\pi_Y \downarrow & & \downarrow \pi_X \\
Y & \xrightarrow{f} & X.
\end{array}
\]
Then the equality
\[ T(f^\ast \bar{\xi}) = f^\ast T(\bar{\xi}) \]
holds.

(iii) (Additivity and normalization) If \( \bar{\xi}_1 \) and \( \bar{\xi}_2 \) are relative metrized complexes on \( X \), then
\[ T(\bar{\xi}_1 \oplus \bar{\xi}_2) = T(\bar{\xi}_1) + T(\bar{\xi}_2). \]

(iv) (Projection formula) For any hermitian vector bundle \( \mathcal{G} \) on \( X \), and an integer \(-n \leq k \leq 0\), the equality
\[ T(\bar{\xi}_n(k) \otimes \mathcal{G}) = T(\bar{\xi}_n(k)) \cdot \text{ch}(\mathcal{G}). \]
holds.

A theory of analytic torsion classes for projective spaces is an assignment as before, for all non-negative integers \( n \).

**Definition 4.9.** Let \( T \) be a theory of analytic torsion classes for projective spaces of dimension \( n \). Fix \( X = \text{Spec} \mathbb{C} \). The characteristic numbers of \( T \) are
\[ t_{n,k}(T) := T(\bar{\xi}_n(k)) \in \tilde{\mathcal{D}}^1(\text{Spec} \mathbb{C}, 1) = \mathbb{R}, \quad k \in \mathbb{Z}. \] 
(4.10)
The characteristic numbers \( t_{n,k}(T) \), \(-n \leq k \leq 0\) will be called the main characteristic numbers of \( T \).

The main result concerning the analytic torsion classes for projective spaces is:

**Theorem 4.11.** [12, Theorem 3.53] Let \( n \) be a non-negative integer and let \( t = (t_{n,k})_{k=-n, \ldots, 0} \) be a family of arbitrary real numbers. Then there exists a unique theory \( T_t \) of analytic torsion classes for projective spaces of dimension \( n \), such that \( t_{n,k}(T_t) = t_{n,k} \).

**Proof (sketch).** The proof of the uniqueness is based on the following facts:

- we derive some anomaly formulas that imply that, if the value of \( T(\pi, \mathcal{F}, \pi_\ast \mathcal{F}) \) is fixed for a particular choice of hermitian structures on \( \pi, \mathcal{F} \) and \( \pi_\ast \mathcal{F} \) then the value of \( T(\pi', \mathcal{F}', \pi_\ast \mathcal{F}') \) for any other choice of hermitian structures is also fixed;
– if we know the value of $T(\pi, F, \pi_* F)$, for $F$ in a generating class of $D^b(P^n)$, then $T$ is determined;

– the characteristic numbers determine the values of $T(\xi(k) \otimes F), k = -n, \ldots, 0$

– the objects of the form $F(k), k = -n, \ldots, 0$ form a generating class of $D^b(P^n)$.

The existence is based on explicit formulas. We can derive an explicit inductive formula for all the characteristic numbers once we have fixed $n + 1$ consecutive characteristic numbers, in particular the main ones. We then obtain a general formula for $T(\xi)$.

5. THE COMPATIBILITY THEOREM AND GENERALIZED ANALYTIC TORSION CLASSES

In order to define generalized analytic torsion classes for arbitrary projective morphisms of smooth complex varieties, we need to study the compatibility between singular Bott-Chern classes for closed immersions and analytic torsion classes for projective spaces. It turns out that, once the compatibility between the diagonal embedding of $P^n$ into $P^n \times P^n$ and the second projection of $P^n \times P^n$ onto $P^n$ is established, then the other compatibilities follow. Essentially this observation can be traced back to [9].

Let $n$ be a non-negative integer and $P^n$ the projective space of lines in $\mathbb{C}^{n+1}$. We will denote by $V$ the trivial vector bundle of fiber $\mathbb{C}^{n+1}$ over any base.

We consider the following diagram

\[
\begin{array}{ccc}
P^n & \xrightarrow{\Delta} & P^n \times P^n \\
\downarrow{id} & & \downarrow{p_1} \\
\downarrow{p_2} & & \downarrow{\pi} \\
\downarrow{\pi_1} & & \downarrow{\text{Spec } \mathbb{C}} \\
\end{array}
\]

On $P^n$ we have the tautological short exact sequence

\[0 \rightarrow \mathcal{O}(-1) \rightarrow V \rightarrow Q \rightarrow 0\]

that induces on $P^n \times P^n$ the exact sequence

\[0 \rightarrow p_2^* \mathcal{O}(-1) \rightarrow V \rightarrow p_2^* Q \rightarrow 0\]

We also have an injection

\[p_1^* \mathcal{O}(-1) \hookrightarrow V\]

that provides, by composition,

\[p_1^* \mathcal{O}(-1) \rightarrow p_2^* Q,\]

hence a section of $p_2^* Q \otimes p_1^* \mathcal{O}(1)$. The image of the diagonal is the zero locus of this section. Moreover, the sequence

\[0 \rightarrow \Lambda^n(p_2^* Q^\vee) \otimes p_1^* \mathcal{O}_{P^n}(-n) \rightarrow \ldots \]

\[\ldots \rightarrow \Lambda^1(p_2^* Q^\vee) \otimes p_1^* \mathcal{O}_{P^n}(-1) \rightarrow \mathcal{O}_{P^n \times P^n} \rightarrow \Delta_* \mathcal{O}_{P^n} \rightarrow 0 \quad (5.1)\]
We place ourselves next in the category $\overline{\text{Sm}}_{s}/\mathbb{C}$. On $\mathcal{P}_{\mathbb{P}^{n}}$ and $\mathcal{P}_{\mathbb{P}^{n}\times\mathbb{P}^{n}}$ we consider the Fubini-Study metrics. We denote by $\overline{\Delta}$ and $\overline{\mathcal{P}}_{2}$ the morphisms of $\overline{\text{Sm}}_{s}/\mathbb{C}$ determined by these metrics. Moreover we endow the tangent complex of the identity map with the trivial hermitian structure: $T_{\text{id}_{\mathbb{P}^{n}}} = 0$. We have that $\overline{\mathcal{P}}_{2}\circ\overline{\Delta} = \text{id}_{\mathbb{P}^{n}}$.

The Fubini-Study metric on $\mathcal{O}(-1)$ and the metric induced by the tautological exact sequence on $\mathbb{Q}$, induce a metric on the Koszul complex that we denote $\overline{K}(\Delta)$. This is a hermitian structure on $\Delta_{s}\mathcal{O}_{\mathbb{P}^{n}}$.

Finally on $\mathcal{O}_{\mathbb{P}^{n}}$ we consider the trivial metric. This is a hermitian structure on $(p_{2})_{s}\overline{K}(\Delta)$.

Fix a real additive genus $S$ and denote by $T_{S}$ the theory of singular Bott-Chern classes for closed immersions that is compatible with the projection formula and transitive, associated to $S$, given by Theorem 3.20. Moreover, fix a family of real numbers $t = \{t_{n,k} \mid n \geq 0, -n \leq k \leq 0\}$ and denote $T_{t}$ the theory of generalized analytic torsion classes for projective spaces (Theorem 4.11).

Compatible singular Bott-Chern classes for closed immersions and analytic torsion classes for projective spaces should combine to provide analytic torsion classes for arbitrary projective morphisms, and these classes should be transitive. In particular the transitivity condition for the composition $\text{id}_{\mathbb{P}^{n}} = p_{2} \circ \Delta$ is

$$0 = T(\text{id}_{\mathbb{P}^{n}}, \overline{\mathcal{P}}_{\mathbb{P}^{n}}, \mathcal{O}_{\mathbb{P}^{n}}) = T_{t}(\overline{\mathcal{P}}_{2}, \overline{K}(\Delta), \mathcal{O}_{\mathbb{P}^{n}}) + (p_{2})_{s}(T_{S}(\overline{\Delta}, \mathcal{O}_{\mathbb{P}^{n}}, \overline{K}(\Delta))).$$

**Definition 5.2.** [12] Definition 3.86] The theories of singular Bott Chern classes $T_{S}$ and analytic torsion classes $T_{t}$ are called compatible if the following formula holds:

$$T_{t}(\overline{\mathcal{P}}_{2}, \overline{K}(\Delta), \mathcal{O}_{\mathbb{P}^{n}}) + (p_{2})_{s}(T_{S}(\overline{\Delta}, \mathcal{O}_{\mathbb{P}^{n}}, \overline{K}(\Delta))) = 0. \quad (5.3)$$

**Theorem 5.4.** [12] Theorem 3.88] Let $S$ be a real additive genus. Then there exists a unique family of real numbers $t = \{t_{n,k} \mid n \geq 0, -n \leq k \leq 0\}$ such that the theories of analytic torsion classes $T_{S}$ and $T_{t}$ are compatible. The theory $T_{t}$ will also be denoted $T_{S}$.

**Proof (sketch).** First we make equation (5.3) more explicit. We have (see [12] for details)

$$T_{t}(\overline{\mathcal{P}}_{2}, \overline{K}(\Delta), \mathcal{O}_{\mathbb{P}^{n}}) = \sum_{i=0}^{n} (-1)^{i}t_{n,-i}\text{ch}(\Lambda^{i}\mathcal{Q}^{\vee}).$$

Then we solve the equation

$$\sum_{i=0}^{n} (-1)^{i}t_{n,-i}\text{ch}(\Lambda^{i}\mathcal{Q}^{\vee}) = -(p_{2})_{s}(T_{S}(\overline{\Delta}, \mathcal{O}_{\mathbb{P}^{n}}, \overline{K}(\Delta))) \cdot \text{Td}(\overline{\mathcal{P}}_{2}). \quad (5.5)$$

As both sides are closed and defined up to boundaries, this is an equation in cohomology classes. Since the elements $\text{ch}(\Lambda^{i}\mathcal{Q}^{\vee}), i = 0, \ldots, n$ form a basis of the cohomology, this equation has a unique solution, hence the theorem is proved.

**Remark 5.6.** (i) Given a theory $T$ of analytic torsion classes for projective spaces, obtained from an arbitrary choice of characteristic numbers, in general, it does not exist an additive genus such that the associated theory of singular Bott-Chern classes is compatible with $T$.\hfill \Box
(ii) By definition, compatible analytic torsion classes for closed immersions and projective spaces satisfy a compatibility condition for the trivial vector bundle and the diagonal embedding. When adding the functoriality and the projection formula, we obtain compatibility relations for arbitrary sections of the trivial projective bundle and arbitrary objects, or for pull-back diagrams (Propositions 3.91, 3.98 in [12]).

We can now extend the definition of singular Bott-Chern classes for closed immersions and analytic torsion classes for smooth morphisms, to arbitrary projective morphisms of smooth complex varieties. Our construction is based on the construction of analytic torsion classes by Zha in [23].

**Definition 5.7.** [12, Definition 3.108] A theory of generalized analytic torsion classes is an assignment that, to each morphism \( f: X \rightarrow Y \) in \( \text{Sm}_{/\mathbb{C}} \) and each relative metrized complex

\[
\overline{\xi} = (\overline{f}, \overline{F}; f_*\overline{F}),
\]

assigns a class of currents

\[
T(\overline{\xi}) \in \bigoplus_{p=1}^{n+1} \mathcal{D}^{2p-1}(Y, N_f, p)
\]

satisfying the following properties:

(i) (Differential equation) The following equality holds

\[
d_{\mathcal{D}} \eta = \text{ch}(f_*\overline{F}) - f^!\left[\text{ch}(\overline{F})\right]
\]

for any current \( \eta \in T(\overline{\xi}). \)

(ii) (Functoriality) For every morphism \( g: Y' \rightarrow Y \) that is transverse to \( f \), the equation

\[
g^*T(\overline{\xi}) = T(g^*\overline{\xi})
\]

holds.

(iii) (Additivity and normalization) If \( \overline{\xi}_1 \) and \( \overline{\xi}_2 \) are relative metrized complexes on \( X \), then

\[
T(\overline{\xi}_1 \oplus \overline{\xi}_2) = T(\overline{\xi}_1) + T(\overline{\xi}_2).
\]

(iv) (Projection formula) If \( \overline{\xi} \) is a relative metrized complex, \( \overline{F} \) an object of \( D^b(Y) \) then

\[
T(\overline{\xi} \otimes \overline{F}) = T(\overline{\xi}) \cdot \text{ch}(\overline{F}).
\]

(v) (Transitivity) If \( \overline{f}: X \rightarrow Y \) and \( \overline{g}: Y \rightarrow Z \) are morphisms in \( \text{Sm}_{/\mathbb{C}} \) and \( (\overline{f}, \overline{F}; f_*\overline{F}) \) and \( (\overline{g}, \overline{f}_*\overline{F}; (g \circ f)_*\overline{F}) \) are relative metrized complexes, then

\[
T(\overline{g} \circ \overline{f}) = T(\overline{g}) + \overline{g}_!(T(\overline{f})).
\]

(5.9)
The following theorems are in the spirit of theorems \[3.12\], \[3.20\], \[4.11\] and \[5.4\].

**Theorem 5.10.** \([12, \text{Theorem 3.114}]\) Let \(S\) be a real additive genus. Then there exists a unique theory of analytic torsion classes that agrees with \(T_S\) when restricted to the class of closed immersions. We will denote such theory by \(T_S\). In particular, there is a unique theory of generalized analytic torsion classes that agrees with \(T^h\) when restricted to the class of closed immersions. This theory will be called homogeneous. Moreover, if \(T\) is a theory of generalized analytic torsion classes, then there exists a real additive genus \(S\) such that \(T = T_S\).

**Proof (sketch).** The transitivity axiom implies that \(T\) is determined by its values for closed immersions and projective spaces. Theorem \[5.4\] implies then the uniqueness.

For proving the existence, let \(T_S\) be the theory of analytic torsion classes for closed immersions and projective spaces determined by \(S\). Let \(\bar{f} : X \to Y\) be a morphism in \(\text{Sm}_{//\mathbb{C}}\), and let \(\bar{\xi} = (\bar{f}, \bar{F}, \bar{f}_* \bar{F})\) be a relative metrized complex. Since \(f\) is assumed to be projective, there is a factorization \(f = \pi \circ \iota\), where \(\iota : X \to \mathbb{P}^n_Y\) is a closed immersion and \(\pi : \mathbb{P}^n_Y \to Y\) is the projection. Choose auxiliary hermitian structures on \(\iota, \pi\) and \(\iota_* \bar{F}\).

Then we define

\[
T_S(\bar{\xi}) = T_S(\bar{\pi}) + \bar{\pi}_*(T_S(\bar{\iota})) + \bar{\iota}_* \left[ \text{ch}(\bar{F}) \bullet \bar{\text{Td}}_m(\bar{f}, \bar{\pi} \circ \bar{\iota}) \right] \quad (5.11)
\]

We then prove that this definition does not depend on the choice of hermitian structures on \(\iota, \pi\) and \(\iota_* \bar{F}\), on the factorization of \(f\) and that \(T_S\) satisfies the properties of a theory of analytic torsion classes.

**Theorem 5.12.** \([12, \text{Theorem 3.121}]\)

(i) Let \(T\) be a theory of generalized analytic torsion classes. Then there is a unique real additive genus \(S\) such that, for any relative metrized complex \(\bar{\xi} := (\bar{f}, \bar{F}, \bar{f}_* \bar{F})\), we have

\[
T(\bar{\xi}) - T^h(\bar{\xi}) = f_* [\text{ch}(\bar{F}) \bullet \text{Td}(T_f) \bullet S(T_f) \bullet 1]. \quad (5.13)
\]

(ii) Conversely, any real additive genus \(S\) defines, by means of equation \((5.13)\), a unique theory of generalized analytic torsion classes \(T_S\).

In \([12]\) we present several applications of the axiomatic characterization of analytic torsion classes. As a main application, we showed that the classes of the analytic torsion forms of Bismut-Köhler \([7]\) arise as the restriction to Kähler fibrations of the theory of generalized analytic torsion classes associated to the \(R\)-genus (modulo a normalization factor). Therefore, we have extended Bismut-Köhler analytic torsion classes to arbitrary projective morphisms in the algebraic category. This will allow us to extend in a forthcoming paper the arithmetic Grothendieck-Riemann-Roch theorem to arbitrary projective morphisms. As other applications, we obtained new proofs of two previously known results about analytic torsion (we reproved and generalized the theorems of Berthomieu-Bismut \([2]\) and Ma \([19, 20]\) on the compatibility of analytic torsion with the composition of submersions, and we reproved a weak form of the theorem of Bismut-Bost on the singularity of the Quillen metric for degenerating families of curves, whose singular fibers have at most ordinary double points \([4]\)). Moreover, we defined and studied the
dual theory $T^\vee$ to a given theory $T$ of generalized analytic torsion classes, we characterized self-dual theories (i.e. $T^\vee = T$) in terms of the coefficients of the attached real additive genus and, as an outcome, we obtained a conceptual explanation of the vanishing of the even coefficients of the $R$-genus of Gillet and Soulé [15];

REFERENCES