The Jacquet-Langlands correspondence and the arithmetic Riemann-Roch theorem for pointed curves

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Abstract.- We show how the Jacquet-Langlands correspondence and the arithmetic Riemann-Roch theorem for pointed curves, relate the arithmetic self-intersection numbers of the sheaves of modular forms—with their Petersson norms—on modular and Shimura curves: these are equal modulo \( \sum_{l \in S} Q \log l \), where \( S \) is a controlled set of primes. These quantities were previously considered by Bost and Kuhn (modular curve case) and Kudla-Rapoport-Yang and Maillot-Roessler (Shimura curve case). By the work of Maillot and Roessler, our result settles a question raised by Soulé.

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1 Introduction

This article concerns the sheaves of modular forms on modular and Shimura curves, from the perspective of arithmetic intersection theory. In either case,
the line bundle of modular forms has a natural hermitian structure, the so-called $L^2$ or Petersson metric. Its arithmetic self-intersection number is an invariant of interest. In the case of modular curves, it has been computed independently by Bost [1] and Künn [30, Cor. 6.2]. Their methods take advantage of the presence of cusps. For instance, both works appeal to the Kronecker limit formula [30, Prop. 5.2]. Nevertheless, the $L^2$ metric is singular near the cusps: an extension of the arithmetic intersection theory of Gillet and Soulé [16] is required [2], [30]. On the other hand, the analogous computation has been accomplished for Shimura curves by Kudla-Rapoport-Yang [29, Thm. 0.5] and Maillot-Roessler [31, Conj. 2.1, Prop. 2.3, Thm. 4.3], following different techniques. We emphasize that the formalism of Gillet-Soulé does apply to the Shimura curve situation: the $L^2$ metric is smooth. Especially, the arithmetic Lefschetz fixed point formula of Kühler and Roessler [27] applies. This is the main issue in the approach of Maillot-Roessler. The two numbers, for both the modular and Shimura curves, essentially agree and involve the quantity $\zeta'(-1)/\zeta(-1)$, the logarithmic derivative of the Riemann zeta function evaluated at $-1$. However, the natures of the proofs are different and the two results remained unrelated so far.

Our purpose is to provide an explanation for the coincidence of the alluded arithmetic self-intersection numbers, without evaluating them (Theorem 1.1). This enterprise is thus independent of the literature just reviewed. The first observation is that these numbers fit into our arithmetic Riemann-Roch type formula for pointed stable curves [14, Thm. A]. This allows a reinterpretation in terms of the Selberg zeta functions and the Faltings’ heights of the curves. Then the Jacquet-Langlands correspondence for $GL_2$ (Theorem 2.5) intervenes to relate the Selberg zeta functions of the modular and Shimura curves (Theorem 3.7), as well as their Faltings’ heights (Theorem 4.1). The obvious outcome is that the computations by Bost and Künn on the one hand, and Kudla-Rapoport-Yang and Maillot-Roessler on the other hand, are equivalent. It seems worth pointing out the following facts:

i. the work of Maillot-Roessler and Theorem 1.1 lead to reinterpret the theorem of Bost and Künn solely in terms of the arithmetic Lefschetz fixed point formula and the arithmetic Riemann-Roch theorem for pointed stable curves. This settles a question raised by Soulé [37, Sec. 7].

ii. the formula of Bost and Künn and Theorem 1.1 give a direct proof of the result by Kudla-Rapoport-Yang and Maillot-Roessler, without appealing to the arithmetic Lefschetz fixed point formula. We also notice that the method of Kudla-Rapoport-Yang is much more involved (but much more

\footnote{There is an equality modulo $\sum_{l \in S} \mathbb{Q} \log l$, for a suitable set of primes $S$.}
far-reaching).\footnote{Kudla-Rapoport-Yang actually deal with algebraic stacks and compute the exact contribution of all the finite places, including those of bad reduction.}

From a conceptual point of view, the present report suggests the combination of the theory of automorphic forms and the arithmetic intersection theory of Gillet-Soulé \cite{16}–\cite{18} and Burgos-Kramer-Kühn \cite{3}–\cite{4}, to establish arakelovian variants of the Grothendieck-Riemann-Roch theorem for certain Shimura varieties of non-compact type, via the Jacquet-Langlands correspondence. Also, the article indirectly indicates the interest of evaluating certain arithmetic intersection numbers in terms of the Selberg trace formula, as well as exploring the interaction with other known cases of Langlands functoriality. The author hopes to develop these ideas in the future.

Before the statement of the main theorem, we briefly introduce the involved relevant objects.

**Shimura curves.** Let $M \geq 4$ be an integer and $D$ an indefinite quaternion algebra over $\mathbb{Q}$, whose discriminant $d$ is relatively prime to $M$. Assume $d = pq$ is the product of two different prime numbers. Fix a maximal order $\mathcal{O}_D$ in $D$. Let $X_1^D(M) \to \text{Spec} \mathbb{Z}[1/dM]$ be the scheme representing the algebraic stack of abelian schemes $(A/T, i, \alpha)$ of relative dimension 2, together with an injective unitary ring homomorphism $i : \mathcal{O}_D \to \text{End}_T(A)$ and a $\Gamma_0^1(M)$-level structure $\alpha$. The morphism $X_1^D(M) \to \text{Spec} \mathbb{Z}[1/dM]$ is proper, smooth, with geometrically irreducible fibers of dimension 1 \cite[Thm. 2.1, Prop. 2.4]{6}. Let $\pi_D : A \to X_1^D(M)$ be the universal abelian scheme. If $\Omega^1_{A/X_1^D(M)}$ is the sheaf of relative differentials, we define $\\omega_{X_1^D(M)} = \text{det} \pi_D^* \Omega^1_{A/X_1^D(M)}$. The invertible sheaf $\\omega_{X_1^D(M)}$ comes equipped with a natural $L^2$ metric $\| \cdot \|_{L^2}$. If $x \in X_1^D(M)(\mathbb{C})$ and $\theta \in \text{det} H^0(A_x, \Omega^1_{A_x})$, then

$$\|\theta\|^2_{L^2, x} = \frac{1}{(2\pi)^2} \int_{A_x(\mathbb{C})} |\theta \wedge \bar{\theta}|.$$ 

The couple $\\omega_{X_1^D(M), L^2} := (\\omega_{X_1^D(M)}, \| \cdot \|_{L^2})$ is a smooth hermitian line bundle in the sense of Arakelov geometry. There is a canonical isomorphism \cite[Lemma 7]{10}, \cite[Prop. 3.2]{28}

$$\kappa^D : \omega_{X_1^D(M)} \sim \omega_{X_1^D(M)/\mathbb{Z}[1/dM]}.$$ 

Let $\| \cdot \|_P$ be the hermitian metric on $\\omega_{X_1^D(M)/\mathbb{Z}[1/dM]}$ built up from $\| \cdot \|_{L^2}$ and $\kappa^D$. Consider a uniformization $X_1^D(M)(\mathbb{C}) \simeq \Gamma \backslash \mathbb{H}$, where $\mathbb{H}$ is the upper half-plane and $\Gamma \subset \text{PSL}_2(\mathbb{R})$ a torsion free fuchsian subgroup. The coordinate $\tau \in \mathbb{H}$...
\(\mathbb{H}\) serves as a local analytic coordinate on \(X_1(M)(\mathbb{C})\). Then the expression of the metric \(\parallel \cdot \parallel_P\) is \(\parallel d\tau \parallel_P = 4\pi \text{ Im } \tau\) [28, Chap. III, Sec. 3, Eq. 3.6].\(^4\)

At the intermediate steps we will need the Shimura curves \(X_0^P(M) \to \text{Spec } \mathbb{Z}[1/dM]\). These are coarse moduli schemes for the algebraic stacks of false elliptic curves \((A/T, i, \alpha)\) with a level structure \(\alpha\) of type \(\Gamma_0^P(M)\) [6, Sec. 2] (see also Proposition 5.4 below). The morphism \(X_0^P(M) \to \text{Spec } \mathbb{Z}[1/dM]\) is smooth, with geometrically connected fibers of dimension 1.

**Modular curves.** Let \(M \geq 5\) be an integer and \(X_1(M) \to \text{Spec } \mathbb{Z}[1/M]\) the scheme representing the algebraic stack of generalized elliptic curves \((E/T, \alpha)\), together with a \(\Gamma_1(M)\)-level structure \(\alpha\) [8, Def. 2.4.1, Rmk. 4.2.2]. Let \(Y_1(M) \subset X_1(M)\) be the open subscheme parametrizing elliptic curves. The morphism \(X_1(M) \to \text{Spec } \mathbb{Z}[1/M]\) is proper, smooth with geometrically connected fibers of dimension 1 [8, Thm. 3.2.7, Thm. 3.3.1, Thm. 4.2.1]. Let \(\pi : E \to X_1(M)\) be the universal generalized elliptic curve. Define \(\omega_{X_1(M)} = \pi_* \omega_{E/X_1(M)}\). For every \(x \in Y_1(M)(\mathbb{C})\) and \(\theta \in H^0(E_x, \Omega^1_{E_x})\), the rule

\[
\parallel \theta \parallel^2_{L^2,x} = \frac{i}{2\pi} \int_{E_x(\mathbb{C})} \theta \wedge \overline{\theta}
\]

defines a hermitian metric on \(\omega_{X_1(M),x}\). The collection of such norms constitute the \(L^2\) metric \(\parallel \cdot \parallel_{L^2}\) on \(\omega_{X_1(M)}|_{Y_1(M)}\). It is a smooth hermitian metric on \(Y_1(M)(\mathbb{C})\), and its asymptotic behavior near the cusps \((X_1(M) \setminus Y_1(M))(\mathbb{C}))\) can be determined. The metric \(\parallel \cdot \parallel_{L^2}\) is pre-log-log in the sense of Burgos-Kramer-Kühn [3, Sec. 7.3.2]–[4, Thm. 5.3], [30, Prop. 4.9, Par. 4.14]. We write \(\omega_{X_1(M),L^2}\) to refer to \((\omega_{X_1(M)}, \parallel \cdot \parallel_{L^2})\). There is a canonical Kodaira-Spencer isomorphism [26, Thm. 10.13.11]

\[
\kappa : \omega_{X_1(M)}^{\otimes 2} \iso \omega_{X_1(M)/\mathbb{Z}[1/M]}(\text{cusps}),
\]

where \(\text{cusps}\) denotes the reduced divisor of cusps \((X_1(M) \setminus Y_1(M))^{\text{red}}\) [26, Chap. 8, Sec. 8.6, Eq. 8.6.3.2]. Let \(\parallel \cdot \parallel_P\) denote the pre-log-log hermitian metric on \(\omega_{X_1(M)/\mathbb{Z}[1/M]}(\text{cusps})\) transported from \(\parallel \cdot \parallel_{L^2}\) via \(\kappa\). Uniformize the curve \(Y_1(M)(\mathbb{C})\) as \(\Gamma \backslash \mathbb{H}\), for some torsion free fuchsian subgroup \(\Gamma \subset \text{PSL}_2(\mathbb{R})\). The coordinate \(\tau \in \mathbb{H}\) provides a local parameter for \(Y_1(M)(\mathbb{C})\). The local expression of \(\parallel \cdot \parallel_P\) is again \(\parallel d\tau \parallel_P = 4\pi \text{ Im } \tau\) [30, Sec. 4.14].

We will also make use of the modular curves \(X_0^P(M) \to \text{Spec } \mathbb{Z}[1/M]\). These are coarse moduli schemes for generalized elliptic curves \((E/T, C)\) with a cyclic subgroup \(C\) of order \(M\), intersecting all the components of all

\(^4\)The reader will note that in loc. cit. a factor 2 is missing.
the geometric fibers of $E^{\text{sm}}/T$. The morphism $X_0(M) \to \text{Spec} \mathbb{Z}[1/M]$ is smooth with geometrically connected fibers of dimension 1.

**Theorem 1.1.** Let $M \geq 5$ be an integer and $D$ an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $d$ prime to $M$. Assume $d = pq$ is a product of two different prime numbers. Then the equality of normalized arithmetic self-intersection numbers

$$\frac{(\omega_{X^D_0(M),L}^2)}{\deg \omega_{X^D_0(M)}} = \frac{(\omega_{X_1(M),L}^{\otimes 2})^2}{\deg \omega_{X_1(M)}^{\otimes 2}}$$

holds in the group $\mathbb{R}/\sum_{l|3p(p^2-1),q(q^2-1),M} \mathbb{Q} \log l$.

**Remark 1.2.**

i. In the statement of the theorem, as well as throughout this article, we write $\deg L := \deg L_C$ for a line bundle $L$ over an arithmetic surface $X \to \text{Spec} \mathbb{Z}[1/M]$.

ii. The restriction $d = pq$ has been included for the sake of simplicity. The general case is in reach with the methods of this article, but it seems a tedious task to work out the details (see Remark 3.8 and the proof of Proposition 4.3).

iii. The indeterminacy in $\mathbb{Q} \log 3$ is due to the usage of the curves $X^D_0(M)$, $X_0(M)$ during the intermediate computations. They intervene to simplify the manipulations with scattering matrices and Selberg zeta functions (Section 3), as well as in comparisons of Faltings’ heights (Section 4).

iv. The indeterminacy in $\mathbb{Q} \log l$, $l | p(p^2-1)q(q^2-1)$, reflects the lack of control of the reduction of certain isogenies—between Jacobians of modular and Shimura curves—at Eisenstein primes (Section 4).

The article is structured as follows. Section 2 reviews the Jacquet-Langlands correspondence for automorphic representations arising from Maass cusp forms. This is a well-known topic for the experts, but the author could not find a reference were all the required details are worked out. In view of the importance of the correspondence for our purposes, it seems worth including a complete account. In Section 3, we combine the spectral correspondence derived from Jacquet-Langlands and the expression of the Selberg zeta function as a regularized determinant, proven by Efrat [11]–[12]. The outcome is a comparison of the Selberg zeta functions for modular curves $X_0(M)$ and Shimura curves $X^D_0(M)$. Section 4 carries out an analogous project to compare the heights of Jacobians of modular and Shimura curves. The work of Prasanna [34] provided the base for these computations. After several reduction steps, Section 5 establishes Theorem 1.1.
2 The Jacquet-Langlands correspondence for Maass forms

2.1 Notations

The following notations will prevail throughout this section. Let $M$ be a positive integer and $B$ an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $d$ prime to $M$. The case $d = 1$ is allowed, corresponding to $B = \mathbb{M}_2 \mathbb{Q}$. Otherwise, $d$ is a product of an even number of distinct primes [40, Thm. 3.1]. Let $\mathbb{A}$ (resp. $\mathbb{A}_f$) denote the ring of adèles of $\mathbb{Q}$ (resp. finite adèles). For every place $v \mid d$, $B_v := B \otimes_{\mathbb{Q}} \mathbb{Q}_v$ is a division algebra. If $v \nmid d$ (including $v = \infty$), $B_v$ is isomorphic to $\mathbb{M}_2 \mathbb{Q}_v$. We then fix an algebra isomorphism

$$
\gamma_v : \mathbb{M}_2 \mathbb{Q}_v \sim \rightarrow B_v.
$$

Observe that $\gamma_v$ induces a group isomorphism $\text{GL}_2 \mathbb{Q}_v \rightarrow B_v^\times$. By means of $\{\gamma_v\}$, one can give a sense to the restricted products defining $B(\mathbb{A})$ and $B^\times(\mathbb{A}_f)$. We introduce a compact open subgroup of $B^\times(\mathbb{A}_f)$. For every finite place $v$ of $\mathbb{Q}$ set

$$
K_0^1(M)_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathbb{Z}_v) \ | \ c \equiv 0 \mod M \right\}.
$$

Then define

$$
K_0^B(M_v) = \prod_{v \mid d} K_v^1 \times \prod_{v \nmid \infty} \gamma_v(K_0^1(M)_v) \subset B^\times(\mathbb{A}_f).
$$

Here, for $v \mid d$, $K_v^1$ denotes the group of invertible elements in the maximal order of $B_v$.\footnote{A quaternion algebra over a non-archimedian local field has a unique maximal order [40, Lemma 1.5].}

$$
K_v^1 = \left\{ x \in B_v^\times \ | \ N_v(x) \mid_v = 1 \right\} \quad (N_v \text{ is the reduced norm of } B_v).
$$

Finally we put

$$
\Gamma_0^B(M) = \gamma_\infty^{-1}(B^\times(\mathbb{Q}) \cap (K_0^B(M) \times B^\times(\mathbb{R}))) \subset \text{SL}_2(\mathbb{R}).\footnote{The isomorphism is unique up to conjugation.}
$$

This is an arithmetic subgroup of $\text{SL}_2(\mathbb{R})$, acting on the upper half-plane $\mathbb{H}$ by Möbius transformations. Accordingly there is a notion of Maass cusp form for $\Gamma_0^B(M)$. Later we will focus on automorphic representations of the Hecke algebra of $B^\times(\mathbb{A})$ generated by Maass cusp forms of weight 0.
2.2 Ramification and the Jacquet-Langlands correspondence

**Ramification.** We proceed to introduce the notion of ramification and conductor for certain irreducible admissible representations of (the Hecke algebra of) $B^\times(A)$. The reader is referred to [7] and [15, Chap. 4., Sec. B, Par. 3, pp. 71–74] for an elaboration on these concepts in the local case $GL_2(Q_p)$ and for infinite dimensional irreducible admissible representations.

**Lemma 2.1.** Let $F$ be a non-archimedian local field and $D$ a non-split quaternion algebra over $F$, with reduced norm $N$. Denote by $K^1$ the subgroup of invertible elements in the maximal order of $D$. Let $\rho : D^\times \to GL(E)$ be an irreducible admissible representation.

i. Suppose the restriction of $\rho$ to $K^1$ is the trivial representation. Then there exists an unramified quasi-character $\chi : F^\times \to \mathbb{C}^\times$ such that $\rho = \chi \circ N$.

ii. If moreover $\rho$ has trivial central character, then $\chi$ is a quadratic character, i.e. takes values in $\{\pm 1\}$.

**Proof.** Observe that $E$ is finite dimensional because $D^\times$ is compact modulo its center. Denote by $D^1$ the commutator subgroup of $D^\times$. By [25, Lemma 4.1], there is a commutative diagram of exact sequences of topological groups

\[
\begin{align*}
1 &\rightarrow D^1 \rightarrow K^1 \rightarrow \mathcal{O}_F^\times \rightarrow 1 \\
1 &\rightarrow D^1 \rightarrow D^\times \rightarrow F^\times \rightarrow 1.
\end{align*}
\]

Hence the reduced norm $N : D^\times \to F^\times$ induces an isomorphism $K^1 \backslash D^\times \simeq \mathcal{O}_F^\times \backslash F^\times$. Under i, it follows that $\rho$ is one dimensional and of the form $\chi \circ N$, for some unramified (i.e. trivial on $\mathcal{O}_F^\times$) quasi-character $\chi$ of $F^\times$. Finally, if $\rho$ has trivial central character, then $\chi$ is trivial on $N(F^\times) = (F^\times)^2$. Hence $\chi$ takes values in $\{\pm 1\}$.  

**Definition 2.2.** With the assumptions of Lemma 2.1 i, we say that $\rho$ is **unramified.**

**Definition 2.3.** Let $\pi = \otimes_{v} \pi_v$ be an irreducible admissible representation of $B^\times(A)$. Assume that for every $v \mid d$, $\pi_v$ is unramified and that for every

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8See [25, Chap. 9, 14] and [15, Chap. 10] for an account of admissible and automorphic representations of $B^\times(A)$.

9This notion is stronger than the conventional one, namely that $\rho \mid_{K^1}$ contains the identity representation. Nevertheless, we only deal with representations $\rho$ for which $\rho \mid_{K^1}$ is the identity.
finite place $v \nmid d$, $\pi_v$ is infinite dimensional. Then we define the conductor of $\pi$ to be the positive integer

$$c(\pi) = \prod_{v \mid \infty, v \mid d} c(\pi_v).$$

Here $c(\pi_v)$ denotes the conductor of $\pi_v$ [7, Thm. 1], [15, Thm. 4.24].

**Remark 2.4.** For an irreducible admissible representation $\pi$, a decomposition $\pi = \bigotimes v \pi_v$ exists, where the local factors $\pi_v$ are unique up to equivalence and unramified for almost every finite place $v$ [25, Prop. 9.1]. In particular, in Definition 2.3, $c(\pi_v) = 1$ for almost all $v$ and $c(\pi)$ is a well defined positive integer.

**Jacquet-Langlands correspondence.** We next state the Jacquet-Langlands correspondence. We do so under the assumptions to be encountered later. We assume $d > 1$ to avoid any trivial assertion.

**Theorem 2.5** (Jacquet-Langlands). i. There is an injective correspondence which associates to an irreducible automorphic representation $\pi'$ of $B^\times(\mathbb{A})$, an irreducible automorphic cuspidal representation $\pi = \pi(\pi')$ of $GL_2(\mathbb{A})$, provided $\pi'$ is not one dimensional. The image consists of those irreducible automorphic cuspidal representations $\pi$ with square-integrable local factors $\pi_v$, $v \nmid d$.

ii. For every place $v \nmid d$, $\pi_v \simeq \pi'_v$.

iii. If for some $v \mid d$, $\pi'_v$ is unramified and of the form $\chi_v \circ N_v$, for some quasi-character $\chi_v : \mathbb{Q}_v^\times \to \mathbb{C}^\times$, then $\pi_v$ is the special representation $\sigma(\chi_v \cdot |v|^{1/2}, \chi_v \cdot |v|^{-1/2})$.

iv. The correspondence is a bijection between the set of irreducible automorphic representations $\pi'$ of $B^\times(\mathbb{A})$, with trivial central character, unramified at primes $v \mid d$, infinite dimensional at finite places $v \nmid d$, and conductor $c$ prime to $d$, with the set of irreducible automorphic cuspidal representations $\pi$ of $GL_2(\mathbb{A})$, with trivial central character, infinite dimensional at places $v \nmid d$ and conductor $cd$.

**Proof.** The proof of i-iii is the content of [25, Thms. 4.2, 14.4, 15.1, 16.1] and [15, Thms. 7.6, 10.1, 10.2, 10.5]. We comment on iv.

Let $\pi'$ be an automorphic representation of $B^\times(\mathbb{A})$ with trivial central character, unramified at primes $v \mid d$ and conductor $c$. By Lemma 2.1, for every $v \mid d$, $\pi'_v$ is of the form $\chi_v \circ N_v$, for some unramified quadratic character of $\mathbb{Q}_v^\times$. According to iii, $\pi_v = \sigma(\chi_v \cdot |v|^{1/2}, \chi_v \cdot |v|^{-1/2})$. Since $\chi_v^2 = 1$, $\pi_v$
has trivial central character. By [15, Rmk. 4.25], \( \pi_v \) has conductor \( p \) (the characteristic of the residue field of \( v \)), because \( \chi_v \) is unramified. On the other hand, \( \pi_v = \pi'_v \) for \( v \nmid d \). We conclude that \( \pi \) has conductor \( cd \).

Let \( \pi \) be an irreducible automorphic cuspidal representation of \( GL_2(\mathbb{A}) \), with trivial central character and conductor \( cd \). We claim that for every \( v \mid d \), the local factor \( \pi_v \) has the form \( \sigma(\chi_v \cdot |v|^{1/2}, \chi_v \cdot |v|^{-1/2}) \), where \( \chi_v \) is an unramified quadratic character of \( \mathbb{Q}^\times_v \). Recall that \( d \) is a square-free integer. Therefore, for \( v \mid d \), \( \pi_v \) has conductor \( p \) (the characteristic of the residue field of \( v \)). By [15, Rmk. 4.25], \( \pi_v \) is either a principal series \( \pi(\mu_1, \mu_2) \) or a special representation \( \sigma(\chi_v \cdot |v|^{1/2}, \chi_v \cdot |v|^{-1/2}) \). We discard the first possibility. Observe that the central character of \( \pi(\mu_1, \mu_2) \) is the product of quasi-characters \( \mu_1 \mu_2 \) of \( \mathbb{Q}^\times_v \). The assumption of trivial central character yields \( \mu_1 = \mu_2^{-1} \). Looking at loc. cit. we find

\[
p = c(\pi(\mu_1, \mu_2)) = (\text{conductor of } \mu_1)(\text{conductor of } \mu_2)
= (\text{conductor of } \mu_1)^2.
\]

This is inconsistent and we conclude \( \pi_v = \sigma(\chi_v \cdot |v|^{1/2}, \chi_v \cdot |v|^{-1/2}) \), thus proving the claim. Again by [15, Rmk. 4.25], we see that \( \chi_v \) has to be unramified. Finally, for \( \pi_v \) to have trivial central character, it must happen \( \chi_v^2 = 1 \). Namely, \( \chi_v \) is a quadratic character. We finish by \( i-iii \).

### 2.3 Maass cusp forms and passage from the classical to the automorphic formalism

Let \( f \) be a Maass cusp form of weight 0 for \( \Gamma_0^B(M) \). We build up a function \( \phi_f : B^\times(\mathbb{A}) \to \mathbb{C} \) from \( f \). By the strong approximation theorem [40, Chap. III, Thm. 4.3] we can write

\[
B^\times(\mathbb{A}) = B^\times(\mathbb{Q}) \cdot (K_0^B(M) \times B^\times(\mathbb{R}))^+.
\]

If \( g = g_0kg_\infty \in B^\times(\mathbb{A}) \) is a decomposition according to \( (2.1) \), then we define \( \phi_f(g) = f(g_\infty i) \). Observe that \( g_\infty \) acts on \( \mathbb{H} \) via the identification \( \gamma_\infty \). The definition of \( \phi_f(g) \) does not depend on the particular choice of decomposition of \( g \). We summarize the features of \( \phi_f \):

- \( i \). \( \phi_f \in L^2_0(Z^\times(\mathbb{A})B^\times(\mathbb{Q})B^\times(\mathbb{A})) \), where \( Z \) denotes the center of \( B \);
- \( ii \). for every \( g \in B^\times(\mathbb{A}) \), \( r(\theta) \in SO(2, \mathbb{R}) \) and \( k \in K_0^B(M) \), we have \( \phi_f(gr(\theta)k) = \phi_f(g) \);
- \( iii \). if \( \Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}) \) and \( \Delta f = \lambda f \), then \( \phi_f \) is an eigenvector of eigenvalue \( \lambda \) for the Casimir element of \( B^\times(\mathbb{R}) \).
We denote by $V^B(M, \lambda)$ the vector space of functions satisfying properties \textit{i-iii} above. It is known to be a finite dimensional Hilbert space \cite[Thm. 4.7]{20}, with the hermitian structure induced from $L^2_0(Z^\times(\mathbb{A})B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}))$. Every $\phi \in V^B(M, \lambda)$ uniquely arises as a $\phi_f$, with $f$ a Maass cusp form for $\Gamma^B_0(M)$ and with eigenvalue $\lambda$ for $\Delta$.

If $m \mid M$ and $d \in B^\times(\mathbb{A})$ satisfies $d^{-1}K_0^B(M)d \subset K_0^B(m)$, then there is a linear map

$$A_{m,d} : V^B(m, \lambda) \longrightarrow V^B(M, \lambda)$$

$$\phi \longmapsto (\phi_d : g \mapsto \phi(gd)).$$

The Maass cusp form $f$ is a \textit{newform} if $\phi_f$ is orthogonal to the space generated by the images of the distinct operators $A_{m,d}$, $m \mid M$, $m \neq M$. More generally, if $M = M_0M_1$, we say that $f$ is $M_0$-\textit{new} if $\phi_f$ is orthogonal to the space generated by the images of the operators $A_{m,d}$ with $M_1 | m | M$ and $m \neq M$.

We define $V^B(M, \lambda)^{M_0-\text{new}}$ to be the subspace of $V^B(M, \lambda)$ generated by the $\phi_f$, with $f$ being $M_0$-new. The orthogonal complement of $V^B(M, \lambda)^{M_0-\text{new}}$ in $V^B(M, \lambda)$ is denoted by $V^B(M, \lambda)^{M_0-\text{old}}$. The space $V^B(M, \lambda)^{\text{new}}$ is spanned by functions $A_{m,d}\phi$, where $\phi \in V^B(m, \lambda)^{\text{new}}$ and $M_1 | m | M$.

Now suppose that $f$ is a Maass cusp newform for $\Gamma^B_0(M)$, of eigenvalue $\lambda$ for $\Delta$, and a simultaneous eigenfunction of the Hecke operators $T_p$, $p \nmid dM$.\footnote{Once the family $\{\gamma_v\}$ chosen, the Hecke operators are defined as in the modular case. See \cite[Chap. 5, Sec. B, p. 88]{15} for details.} Then the right translates of $\phi_f$ in $L^2_0(Z^\times(\mathbb{A})B^\times(\mathbb{Q})\backslash B^\times(\mathbb{A}))$ span an irreducible automorphic cuspidal representation $\pi_f$ of the Hecke algebra of $B^\times(\mathbb{A})$. The proof goes as in \cite[Chap. 5, Sec. C]{15} and \cite[Chap. 3, Par. 3.6, Thm. 3.6.1]{5}. The representation $\pi_f$ decomposes as a product of local irreducible admissible representations: $\pi_f = \otimes_v \pi_{f,v}$ \cite[Prop. 9.1]{25}.

**Lemma 2.6.** Assume $M$ is square-free. Let $v$ denote a finite place of $\mathbb{Q}$.

\textit{i.} If $v \mid d$, then $\pi_{f,v}$ is unramified.

\textit{ii.} If $v \nmid d$, then $\pi_{f,v}$ is infinite dimensional. If $v \nmid dM$, then $c(\pi_{f,v}) = 1$.

If $v \mid M$, then $c(\pi_{f,v}) = p$ (the residual characteristic of $v$).

As a result, the conductor of $\pi$ is well defined and $c(\pi) = M$.

**Proof.** The first item is clear. Indeed, for every $v \mid d$, $\phi_f$ is right invariant under $K_0^B(M)_v = K_v^1$.

We claim that for $v \nmid dM$, $\pi_{f,v}$ is a principal series representation and hence infinite dimensional. Indeed, this is seen combining that $f$ is a simultaneous eigenvector of the Hecke operators $T_p$, $p \nmid dM$, applying \cite[Thm. 4.23]{15} and \cite[Thm. 4.6.3, Prop. 4.6.5 and proof of Thm. 3.6.1]{5}. Moreover $c(\pi_{f,v}) = 1$ because $\phi_f$ is right invariant under $K_0^B(M)_v = \gamma_v(\text{GL}_2(\mathbb{Z}v))$. 

\begin{thebibliography}{99}

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\end{thebibliography}
Let $v \mid M$. Assume $\pi_{f,v}$ is finite dimensional. Then it is one dimensional and of the form $\chi \circ \det$ for some quasi-character $\chi : \mathbb{Q}_v^\times \rightarrow \mathbb{C}^\times$ [25, Prop. 2.7]. Because $\phi_f$ is invariant under $K_0^B(M)$, $\pi_{f,v}$ is trivial on $K_0^B(M_v)$. Namely, $\chi \mid _{\det K_0^B(M_v)} = 1$. We observe that $\det K_0^B(M_v) = \det \text{GL}_2(\mathbb{Z}_v) = \mathbb{Z}_v^\times$. Hence $\pi_{f,v}$ is trivial on $\text{GL}_2(\mathbb{Z}_v)$ and $\phi_f$ is invariant under the action of $\gamma_v(\text{GL}_2(\mathbb{Z}_v))$. This is in contradiction with the newform assumption for $f$. We infer that $\pi_{f,v}$ is infinite dimensional. The conductor of $\pi_{f,v}$ is thus defined. It remains to check that $c(\pi_{f,v}) = p$ (the residual characteristic of $v$). Because $\phi_f$ is invariant under $K_0^B(M_v)$, $\pi_{f,v}$ is trivial on $K_0^B(M_v)$. Namely, $\chi \mid _{\det K_0^B(M_v)} = 1$. We observe that $\det K_0^B(M_v) = \det \text{GL}_2(\mathbb{Z}_v) = \mathbb{Z}_v^\times$. Hence $\pi_{f,v}$ is trivial on $\text{GL}_2(\mathbb{Z}_v)$ and $\phi_f$ is invariant under the action of $\gamma_v(\text{GL}_2(\mathbb{Z}_v))$. The conductor of $\pi_{f,v}$ is thus defined. It remains to check that $c(\pi_{f,v}) = p$ (the residual characteristic of $v$). Because $\phi_f$ is invariant under $K_0^B(M_v)$ and $M$ is square-free, we see that $c(\pi_{f,v}) \mid p$. By [15, Rmk. 4.25], $\pi_{f,v}$ is a principal series representation or a special representation. As in the proof of Theorem 2.5 iv, the first option is excluded because $\pi_{f,v}$ has trivial central character. Therefore $\pi_{f,v}$ is a special representation of conductor at most $p$. Again by [15, Rmk. 4.25], we see that the conductor $c(\pi_{f,v})$ is exactly $p$. The proof is complete.

**Remark 2.7.** The lemma suggests we apply Theorem 2.5 iv to the representation $\pi_f$.

**Theorem 2.8.** Let $M$ be a square-free integer and $D$ an indefinite quaternion algebra over $\mathbb{Q}$, of discriminant $d > 1$ prime to $M$. Let $\lambda$ be a positive real number. Then there is a Hecke equivariant isomorphism

$$V^D(M, \lambda) \sim \rightarrow V^{M_2 \mathbb{Q}}(dM, \lambda)^{d-new}.$$

As a consequence, $\dim V^D(M, \lambda) = \dim V^{M_2 \mathbb{Q}}(dM, \lambda)^{d-new}$.

**Proof.** To begin with, observe that the Hecke operators preserve new and old spaces. To prove the proposition, it will be enough to show that for every $m \mid M$, there is a Hecke equivariant isomorphism

$$V^D(m, \lambda)^{new} \sim \rightarrow V^{M_2 \mathbb{Q}}(dm, \lambda)^{new}.$$

The Hecke operators on Maass cusp forms commute and are normal. This is seen as in [15, Chap. 5, Sec. B]. Moreover they commute with the hyperbolic Laplacian. Therefore $V^D(m, \lambda)^{new}$ and $V^{M_2 \mathbb{Q}}(dm, \lambda)^{new}$ admit bases of newforms which are simultaneous eigenfunctions of the Hecke operators and the Casimir element.

First of all we notice that for $B = D$ or $M_2 \mathbb{Q}$ and $\phi_f \in V^B(m, \lambda)^{new}$, the attached representation $\pi_f$ determines $f$ up to a non-zero scalar. We adopt the convention $d = 1$ in the matrix algebra case. We may assume $f \neq 0$. Let $H \subset L_0^2(\mathbb{A}^\times(B) \backslash B^\times(\mathbb{A}))$ be the space on which $\pi_f$ acts. Let us consider the space $E = H \cap V^B(m, \lambda)$. We observe that $\phi_f \in E$. We claim $E$ is one dimensional. Indeed, by Lemma 2.6, $\pi_{f,v}$ is unramified at every
place $v \mid d$, hence one dimensional. At every other finite place, $\pi_{f,v}$ is infinite dimensional. The conductor of $\pi_f$ is well defined and equals $m$. Finally, the trivial representation of $\text{SO}(2,\mathbb{R})$ occurs with multiplicity one in $\pi_{f,\infty}$ [15, Chap. 4, Sec. A]. These facts together with the definitions of conductor and $V^B(m,\lambda)$ yield the claim.

Suppose $\phi_f \in V^D(m,\lambda)_{\text{new}} \setminus \{0\}$ and $f$ is a simultaneous eigenfunction of the Hecke operators. Consider $\pi_f$ the attached representation of $D \times (A \backslash \text{GL}_2(\mathbb{A}))$. According to Lemma 2.6 and Theorem 2.5 iv, to $\pi_f$ corresponds an irreducible automorphic cuspidal representation $\tilde{\pi}_f$ of $\text{GL}_2(A)$, with trivial central character and well defined conductor $dm$. Also, remark that $\tilde{\pi}_{f,\infty} \simeq \pi_{f,\infty}$ and in particular the trivial representation of $\text{SO}(2,\mathbb{R})$ appears with multiplicity one in $\tilde{\pi}_{f,\infty}$. Let $H \subseteq \mathcal{L}_0^\infty(Z^\infty(\mathbb{A}) \backslash \text{GL}_2(\mathbb{Q}) \backslash \text{GL}_2(\mathbb{A}))$ be the space on which $\tilde{\pi}_f$ acts. By the preceding remarks and the definition of conductor, the space of functions in $H$ right invariant under $\mathcal{K}M_2\mathbb{Q}(dm,\lambda)$ is one dimensional. Let $\phi_{\tilde{f}}$ be a basis of this space. The right translates of $\phi_f$ must generate $\tilde{\pi}_f$ (irreducibility). Because $\tilde{\pi}_{f,v} \simeq \pi_{f,v}$ for every finite place $v \mid dm$. The association $\phi_f \mapsto \phi_{\tilde{f}}$ sends a basis of $V^D(m,\lambda)_{\text{new}}$ to linearly independent functions in $V^M_{\text{new}}(m,\lambda)_{\text{new}}$, by the strong multiplicity one theorem in the quaternion algebra case.\footnote{Strong multiplicity one for irreducible automorphic representations of $D^\infty(\mathbb{A})$ is itself a consequence of strong multiplicity one for irreducible automorphic cuspidal representations of $\text{GL}_2(\mathbb{A})$ [15, Thm. 5.14], [5, Thm. 3.3.6] and the Jacquet-Langlands correspondence, Theorem 2.5 above.} This way we construct a Hecke equivariant monomorphism of finite dimensional vector spaces $V^D(m,\lambda)_{\text{new}} \rightarrow V^M_{\text{new}}(m,\lambda)_{\text{new}}$. This ends the proof of the theorem.\hfill\Box

3 Selberg zeta functions of modular and Shimura curves

3.1 Notations

In this section, we fix a square-free positive integer $M$ with a prime factor $N \equiv 11 \mod 12$, and $D$ an indefinite quaternion algebra over $\mathbb{Q}$ of discriminant $d > 1$ prime to $M$. Let also $d'$ denote a divisor of $d$. Construct groups $\Gamma_D^0(M)$ and $\Gamma_0(d'M) = \Gamma_{0\mathbb{Q}}^M(d'M)$ as in §2.1. One can identify $\Gamma_D^0(M)$.
and $\Gamma_0(d'M)$ with their images in $\text{PSL}_2(\mathbb{R})$. These are fuchsian groups and thus act discontinuously on the upper half plane $\mathbb{H}$. Because $N \equiv 11 \mod 12$, they act without fixed points on $\mathbb{H}$ and are torsion free [40, Chap. IV, Sec. 3, Par. A]. Therefore the quotient spaces $X_0^D(M)^{an} = \Gamma_0^D(M)\backslash \mathbb{H}$ and $Y_0(d'M)^{an} = \Gamma_0(d'M)\backslash \mathbb{H}$ have a structure of Riemann surface, such that the respective coverings by $\mathbb{H}$ possess no branch points. While $X_0^D(M)^{an}$ is compact, $Y_0(d'M)^{an}$ is not. $Y_0(d'M)^{an}$ is compactified into a Riemann surface $X_0^0(d'M)$ by adding the orbits of $\mathbb{P}^1(\mathbb{Q})$, i.e. the cusps.

Equip the upper half plane $\mathbb{H}$ with its unique complete hyperbolic riemannian metric of constant curvature $-1$, whose riemannian tensor is $ds^2 = (dx^2 + dy^2)/y^2$. The scalar laplacian is $\Delta = -y^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$. These objects are invariant under the action of $\text{PSL}_2(\mathbb{R})$, so that they descend to $X_0^D(M)^{an}$ and $Y_0(d'M)^{an}$. We use the same notations $ds^2$ and $\Delta$ to refer to the descended objects. Because the coverings by $\mathbb{H}$ have no branch points, $ds^2$ and $\Delta$ agree with the intrinsic complete hyperbolic metric of curvature $-1$ and the associated scalar laplacian on $X_0^D(M)^{an}$ or $Y_0(d'M)^{an}$.

The Selberg zeta functions of $X_0^D(M)^{an}$ and $Y_0(d'M)^{an}$ are defined in terms of the length spectrum of $ds^2$. If $R$ denotes either of these surfaces, then

$$Z(R,s) = \prod_{\gamma} \prod_{k=0}^{\infty} (1 - e^{-(s+k)l(\gamma)})^2, \quad \text{Re } s > 1.$$  

The outer product runs over the closed primitive non oriented geodesics of $R$, and $l(\gamma)$ stands for the length of the geodesic $\gamma$. The product is absolutely convergent for $\text{Re } s > 1$, and admits a meromorphic continuation to the whole complex plane, with a simple zero at $s = 1$ [22, Chap. 10, Sec. 5], [20, Sec. 10.8].

### 3.2 Selberg zeta functions and determinants of laplacians

For the sake of simplicity, we now assume $d = pq$, where $p, q$ are different prime numbers, so that $d' \in \{1, p, q, pq\}$. Let $\lambda \geq 0$ be any real number. Consider the spaces $V^D(M, \lambda)$ and $V^{M_2(\mathbb{Q})}(d'M, \lambda)$ introduced in §2.3. Recall these are finite dimensional complex vector spaces.

**Notation 3.1.** We define

$$m^D(0) = m_{d'}(0) = 1$$

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12If $I$ is the identity matrix, we observe that $-I \notin \Gamma_0^D(M), \Gamma_0(d'M)$. This is clear because for $v \mid d$, $\gamma_v : M_2(\mathbb{Q}_v) \to D(\mathbb{Q}_v)$ is an algebra isomorphism, hence $\gamma_v(-I) = -1$. 

and
\[ m^D(\lambda) = \dim V^D(M, \lambda) \geq 0, \quad m_{d'}(\lambda) = \dim V^{M_{2Q}}(d'M, \lambda) \geq 0 \quad \text{if} \quad \lambda > 0. \]

**Remark 3.2.**

i. It is known that \( m^D(\lambda) \) and \( m_{d'}(\lambda) \) vanish except on a discrete subset of \( \mathbb{R}_+ \): the discrete spectrum of the laplacian \( \Delta \) on \( X_0^D(M)_{an} \) and \( Y_0(d'M)_{an} \), respectively [20, Thm. 4.7]. Then \( m^D(\lambda) \) (resp. \( m_{d'}(\lambda) \)) is the multiplicity of \( \lambda \) as an eigenvalue of \( \Delta \) on \( X_0^D(M)_{an} \) (resp. \( Y_0(d'M)_{an} \)).

ii. In the Shimura curve case, \( \lambda = 0 \) is an eigenvalue of multiplicity one by compactness and connectedness. In the modular curve case, \( \lambda = 0 \) is in the residual spectrum, known to be trivial and simple –i.e. multiplicity one– for congruence subgroups [20, Sec. 11.2, Thm 11.3].

**Lemma 3.3.** For every real number \( \lambda \geq 0 \), we have the relation
\[ m^D(\lambda) = m_{pq}(\lambda) + 4m_1(\lambda) - 2m_p(\lambda) - 2m_q(\lambda). \]

**Proof.** The relation is clear for \( \lambda = 0 \). For \( \lambda > 0 \) the lemma is a formal application of Lemma 2.6, [7, Cor. to the proof, p. 306], Theorem 2.8 and the definition of \( pq \)-new forms §2.3.

**Notation 3.4** ([11]–[12]). Let \( s, w \in \mathbb{C} \) with \( \text{Re} s > 1 \) and \( \text{Re} w > 1 \). We define the spectral zeta functions
\[ \zeta^D(w, s) = \sum_{\lambda \geq 0} m^D(\lambda) \sum_{\lambda = \sigma(1-s)} (\lambda - s(1-s))^{-w} \]
and
\[ \zeta_{d'}(w, s) = \sum_{\lambda \geq 0} m_{d'}(\lambda) \sum_{\lambda = \sigma(1-s)} (\lambda - s(1-s))^{-w} \]
\[ - \frac{1}{2\pi} \int_{-\infty}^{\infty} \phi_{d'}(1/2 + ir)((1/4 + r^2) - s(1-s))^{-w} dr. \]

Here \( \phi_{d'}(s) \) denotes the determinant of the scattering matrix \( \Phi_{d'}(s) \) for the group \( \Gamma_0(d'M) \) [22, Chap. 11, Prop. 4.8] (see [20, Sec. 6.3] for the general theory of scattering matrices and functional equations of Eisenstein series).

**Remark 3.5.** The sums involved in the definitions of \( \zeta^D(w, s) \) and \( \zeta_{d'}(w, s) \) make sense. Indeed, \( m^D(\lambda) \) and \( m_{d'}(\lambda) \) vanish outside a discrete subset of \( \mathbb{R}_+ \) and we assume \( \text{Re} s > 1 \). Hence, for \( \text{Re} w > 1 \), the sums are absolutely convergent, uniformly convergent on compact subsets of \( \text{Re} w > 1 \), and the summations do not depend on the order.
Lemma 3.6. For $\text{Re } w > 1$ and $\text{Re } s > 1$, we have the equality

\[(3.1) \quad \zeta^D(w, s) = \zeta_{pq}(w, s) + 4\zeta_1(w, s) - 2\zeta_p(w, s) - 2\zeta_q(w, s).\]

**Proof.** By Lemma 3.3, the corresponding relation for the terms $\sum_{\lambda \geq 0}$ is straightforward. It remains to show that the integrals contribute to 0 on the right hand side of (3.1). For this it will be enough to show

$$
\phi_{pq}(s) = \phi_p(s)^2 \phi_q(s)^2 \phi_1(s)^{-1}.
$$

This is an elementary application of [22, Chap. 11, Lemma 4.7 (v), Prop. 4.8].

Theorem 3.7. The following relation between meromorphic functions holds:

\[
Z(X_0^D(M)^{an}, s) = \frac{Z(Y_0(pqM)^{an}, s)Z(Y_0(M)^{an}, s)^4}{Z(Y_0(pM)^{an}, s)^2Z(Y_0(qM)^{an}, s)^2}.
\]

**Proof.** Because the Selberg zeta functions are meromorphic, we may restrict to $s \in \mathbb{R}, s > 1$. In [11, Sec. 2], it is shown that $\zeta^D(w, s), \zeta_d(w, s)$ extend to meromorphic functions in $w \in \mathbb{C}$, regular at $w = 0$. Loc. cit. and [12] establish the identity

\[
\exp\left(-\frac{\partial}{\partial w} \zeta_d(w, s) \big|_{w=0}\right) = Z(Y_0(d' M), s)^2 \psi_{d'}(s),
\]

\[
\psi_{d'}(s) := Z_{\infty d'}(s)^2 \Gamma(s + 1/2)^{-2h_{d'}}
\]

\[
\cdot (2s-1)^{A_{d'}} e^{B_{d'}(2s-1)^2 + C_{d'}(2s-1)+ E_{d'}}.
\]

The terms $Z_{\infty d'}(s), h_{d'}, A_{d'}, B_{d'}, C_{d'}$ and $E_{d'}$ will be made explicit below. An analogous expression holds for $\zeta^D(w, s)$. Together with Lemma 3.6, we derive

\[
(3.2) \quad Z(X_0^D(M), s)^2 = \varphi(s) \left( \frac{Z(Y_0(pqM), s)Z(Y_0(M), s)^4}{Z(Y_0(pM), s)^2Z(Y_0(qM), s)^2} \right)^2,
\]

where we wrote $\varphi(s) = \psi_{pq}(s)\psi_1(s)^4\psi_q(s)^{-2}\psi_q(s)^{-2}\psi^D(s)^{-1}$. We claim that $\varphi(s) = 1$. For this we quote from [11, Thm., p. 445]:

\[
Z_{\infty d'}(s) = \left(2\pi s \frac{\Gamma_2(s)}{\Gamma(s)} \right)^{-\chi(Y_0(d'M)^{an})} (\chi = \text{Euler characteristic},
\]

\[
h_{d'} = \text{number of cusps of } X_0(d'M)^{an}, \quad A_{d'} = h_{d'} - \text{tr } \Phi_{d'}(\frac{1}{2}),
\]
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\[ B_{d'} = \frac{\chi(Y_0(d'M)^{an})}{2}, \]

\[ C_{d'} = -h_{d'} \log 2, \]

\[ E_{d'} = -2\chi(Y_0(d'M)^{an})(2\zeta'(-1) - \log \sqrt{2\pi}) + 2h_{d'} \log \sqrt{2\pi} - A_{d'} \log 2. \]

An analogous expression holds for \( X_0^D(M)^{an} \). The recipe [20, Eq. (2.14), p. 45] for the number of cusps readily implies \( h_{pq} = 2h_p = 2h_q = 4h_1 \). This yields

\[ \chi(Y_0(pqM)^{an}) + 4\chi(Y_0(M)^{an}) - 2\chi(Y_0(pM)^{an}) - 2\chi(Y_0(qM)^{an}) = -2g + 2, \]

with

\[ g = g(X_0(pqM)^{an}) + 4g(X_0(M)^{an}) - 2g(X_0(pM)^{an}) - 2g(X_0(qM)^{an}), \]

and \( g(R) \) stands for the genus of a surface \( R \). As an outcome of the proofs of propositions 4.2–4.3 below, \( g = g(X_0^D(M)^{an}) \).\(^\text{13}\) Hence \( \chi(X_0^D(M)^{an}) = -2g + 2 \). Also, from [22, Chap. 11, Lemma 4.7 (iv)] results without difficulty the identity \( \text{tr} \Phi_{pq}^{\frac{1}{2}} = 2\text{tr} \Phi_p^{\frac{1}{2}} + 2\text{tr} \Phi_q^{\frac{1}{2}} - 4\text{tr} \Phi_1^{\frac{1}{2}} \). All in all, we get \( \varphi(s) = 1 \), as was to be shown. To end, for \( s > 1 \) the Selberg zeta functions are positive [22, Chap. 10, Prop. 5.2]. We come up with the conclusion by (3.2).

\[ \square \]

Remark 3.8. With the appropriate notations, the statements of this section carry over to any \( d > 1 \). For instance, Theorem 3.7 generalizes to

\[ Z(X_0^D(M)^{an}, s) = \prod_{d'|d} Z(X_0(d'M), s)^{\mu(d')2^{\nu(d/d')}}, \]

where \( \mu(d') \) is the Möbius function and \( \nu(d/d') \) stands for the number of prime factors of \( d/d' \). This generalizes [33, Thm. 9] to Eichler orders of “type” \( \Gamma_0^D(M) \).

4 Faltings’ heights of jacobians of modular and Shimura curves

Fix \( M \) a square-free positive integer and \( D \) an indefinite quaternion algebra over \( \mathbb{Q} \) of discriminant \( d \) prime to \( M \). Assume furthermore \( d = pq \), for two different prime numbers \( p, q \). Consider the curves \( X_0^D(M) \to \text{Spec} \mathbb{Z}[1/dM] \)

\(^{13}\)This is a consequence of the Jacquet-Langlands correspondence for holomorphic cusp forms of weight 2.
and $X_0(d'M) \to \text{Spec } \mathbb{Z}[1/d'M]$, $d' | d$. Denote by $J^D_0(M)$, $J_0(d'M)$ the corresponding jacobians. If $Q := \prod_{l | pqM} l(l^2 - 1)$, then we define the group $\mathbb{R}_{D,M} = \mathbb{R}/ \sum_{l | Q} \mathbb{Q} \log l$. The Faltings’ heights of the jacobians $J^D_0(M)$, $J_0(d'M)$ are well defined in $\mathbb{R}_{D,M}$.

The aim of this section is to provide a proof of the next theorem.

**Theorem 4.1.** The following equality of Faltings’ heights of jacobians holds in the group $\mathbb{R}_{D,M}$:

$$h_F(J^D_0(M)) = h_F(J_0(pqM)) + 4h_F(J_0(M)) - 2h_F(J_0(pM)) - 2h_F(J_0(qM)).$$

The theorem is the conjunction of Proposition 4.2 and Proposition 4.3 below.

Let us denote by $J_0(pqM)^{pq\text{-new}}$ the $pq$-new quotient of $J_0(pqM)$ [24, Sec. 2]. This is an abelian scheme over $\mathbb{Z}[1/pqM]$.

**Proposition 4.2** (Prasanna). The identity

$$h_F(J^D_0(M)) = h_F(J_0(pqM)^{pq\text{-new}})$$

holds in $\mathbb{R}_{D,M}$.

**Proof.** The argument follows the discussion of [34, pp. 962–963], that we briefly review. By the Jacquet-Langlands correspondence [25, Thm. 16.1], [36], [24, Thm 2.3, Cor. 2.4] and Faltings’ isogeny theorem [13, Cor. 2], there exists a non canonical Hecke equivariant isogeny $\varphi : J_0(pqM)^{pq\text{-new}} \to J^D_0(M)_Q$ [24, Prop. 2.2, Cor. 2.4]. Let $K = (\ker \varphi)(\mathbb{Q})$, $l \nmid Q$ a prime and $K_l$ the $l$-primary part of $K$. Then $K_l$ is a $\mathbb{T}_{pqM}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module of finite cardinality, where $\mathbb{T}_{pqM}$ is the Hecke algebra on $J_0(pqM)$. On the one hand, an irreducible subquotient\(^{14}\) $U$ of $K_l$ can be realized as an irreducible subquotient of $J_0(pqM)[m](\overline{\mathbb{Q}})$, for some maximal ideal $m$ of $\mathbb{T}_{pqM}$ lying over $l$ [32, Sec. 14, p.112]. On the other hand, $l \nmid Q$ ensures that $m$ is not an Eisenstein prime [38, Thm. 5.1] and $U$ is automatically isomorphic to $J_0(pqM)[m](\overline{\mathbb{Q}})$ [36, Thm. 5.2 (b)]. Since the Gal(\overline{\mathbb{Q}}/\mathbb{Q}) representation $J_0(pqM)[m](\overline{\mathbb{Q}})$ is autodual and $J_0(pqM)$ has good reduction at $l$, [34, Lemma 5.5] applies. Together with [13, Lemma 5], this completes the proof. \(\square\)

\(^{14}\)By irreducible subquotient we mean the quotient of two consecutive factors in a Jordan-Hölder filtration. These filtrations trivially exist for modules of finite cardinality, such as $K_l$. 

Proposition 4.3. The equality of Faltings’ heights of abelian schemes

\[ h_F(J_0(pqM)^{pq-new}) = h_F(J_0(pqM)) + 4h_F(J_0(M)) - 2h_F(J_0(pM)) - 2h_F(J_0(qM)) \]

holds in \( \mathbb{R}_{D,M} \).

The proof of the proposition requires some preliminary considerations.

Notation 4.4. Let \( M \) be a positive integer and \( p \) a prime number, prime to \( M \). The degeneracy maps \( \alpha_p^M : X_0(pM) \to X_0(M) \) and \( \beta_p^M : X_0(pM) \to X_0(M) \) are the morphisms of \( \mathbb{Z}[1/pM] \)-schemes defined, after passing to the coarse moduli schemes, by the assignments

\[ \alpha_p^M : (E, C) \mapsto (c(E), C[M]), \quad \beta_p^M : (E, C) \mapsto (E/C[p], C/C[p]). \]

Here \( E \) is a generalized elliptic curve over a \( \mathbb{Z}[1/pM] \) scheme and \( C \) is a finite flat cyclic subgroup of order \( pM \) of \( E \) sm, intersecting all the components of the geometric fibers of \( E \) sm; \( c(E) \) is the contraction of \( E \) away from \( C[p] \) [9, Chap. IV, Sec. 1]. We also define a degeneracy map between jacobians

\[ \iota_p^M : J_0(M)^2 \to J_0(pM) \]

\[ (x, y) \mapsto (\alpha_p^M)^*x + (\beta_p^M)^*y. \]

If \( p, q \) are different prime numbers, prime to \( M \), then we may construct

\[ \iota^M = \iota_p^M(\iota_q^M, \iota_p^M) : J_0(M)^4 \to J_0(pM)^2 \to J_0(pqM). \]

From the definitions of \( \alpha^\bullet, \beta^\bullet \), one checks with ease that interchanging the roles of \( p \) and \( q \) leads to the same morphism, i.e. \( \iota^M = \iota_p^M(\iota_q^M, \iota_p^M) \). To lighten notations, we will skip the reference to the superscripts of the degeneracy maps. Finally, the Atkin-Lehner involution \( w_p : X_0(pM) \to X_0(pM) \) arises from the correspondence

\[ (E, C) \mapsto (E/C[p], (C + E[p])/C[p]). \]

The identities \( w_p\alpha_p = \beta_p \) and \( w_p\beta_p = \alpha_p \) follow directly from the modular descriptions, and entail \( w_pt_p(x, y) = t_p(y, x) \).

Lemma 4.5. i. The degeneracy maps \( t_p \) and \( \iota \) are isogenies onto their images.

ii. Let \( A \) be an abelian subscheme of \( J_0(M)^2 \). Then \( t_p |_A \) is an isogeny onto its image. An analogous property is true for \( \iota \).
iii. Let $A$ be an abelian subscheme of $J_0(M)^2$, invariant under the diagonal action of $T_M$ on $J_0(M)^2$. Then the image $\iota_p(A)$ is invariant under the action of $T_{pM}$. An analogous property is true for $\iota$.

iv. Let $K$ be the kernel of the degeneracy map $\iota_p : J_0(M)^2 \to J_0(pM)$. Then $K(\overline{\mathbb{Q}})$ is a $T_M[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $J_0(M)^2(\overline{\mathbb{Q}})$, of finite cardinality.

v. Let $K$ be a $T_{pM}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $J_0(pM)^2(\overline{\mathbb{Q}})$. Then the subgroup $(\iota_p, \iota_p)^{-1}(K)$ is a $T_M[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $J_0(M)^4(\overline{\mathbb{Q}})$.

vi. Let $K$ be the kernel of the degeneracy map $\iota : J_0(M)^4 \to J_0(pqM)$. Then $K(\overline{\mathbb{Q}})$ is a $T_M[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $J_0(M)^4(\overline{\mathbb{Q}})$.

Proof. i. By definition of $\iota$, it suffices to see that $\iota_p$ is an isogeny onto its image. First of all, observe that the image of $\iota_p$ is an abelian subscheme of $J_0(pM)$; call it $A$. By [23, Cor. 1.6.2], the morphism $\iota_p : J_0(M)^2 \to A_{\mathbb{Q}}$ is an isogeny. Then there exist an integer $m$ and an isogeny $j_{\mathbb{Q}} : A_{\mathbb{Q}} \to J_0(M)^2_{\mathbb{Q}}$ such that $j_{\mathbb{Q}} \circ \iota_p = [m]$, the multiplication by $m$ map. By the universal property of Néron models, $j_{\mathbb{Q}}$ uniquely extends to a morphism $j : A \to J_0(M)^2$ of abelian schemes over $\mathbb{Z}[1/pM]$ and $j \circ \iota_p = [m]$. It then follows that $\iota_p$ is an isogeny.

ii. Let $A$ be an abelian subscheme of $J_0(M)^2$. By i, $\iota_p$ is finite. Hence $\iota_p(A)$ is an abelian subscheme of $J_0(pM)$ of the same relative dimension as $A$. In particular, $\iota_p : A_{\mathbb{Q}} \to \iota_p(A)_{\mathbb{Q}}$ is an isogeny. We conclude as in the proof of the first item i.

iii. We begin with $\iota_p$. Only the invariance under $T_p$ is not obvious. It is enough to appeal to the formula

\begin{equation}
T_p\iota_p(x, y) = (-y, px + T_p y)
\end{equation}

(see [36, Rmk. 3.9]). The claim is then clear. The corresponding property for $\iota$ is handled in a similar manner.

iv. Firstly, $K(\overline{\mathbb{Q}})$ is a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-submodule of $J_0(M)^2(\overline{\mathbb{Q}})$, of finite cardinality, because $\iota_p$ is a finite morphism of abelian schemes over $\mathbb{Z}[1/pM]$. We need to check the invariance under $T_M$. The only non-trivial issue is the invariance under $T_p$. Due to the identity $w_{p, T_p}(x, y) = (y, x)$, $K(\overline{\mathbb{Q}})$ is closed under $w_p$. From this fact and equation (4.1), we see that if $(x, y) \in K(\overline{\mathbb{Q}})$, then the elements $((p - 1)y, (p - 1)x), (-y, px + T_p y), (py + T_p x, -x)$ lie in $K(\overline{\mathbb{Q}})$ as well. Adding the last two elements, we find that $((p - 1)y + T_p x, (p - 1)x + T_p y)$ is also in $K(\overline{\mathbb{Q}})$. It follows $(T_p x, T_p y) \in K(\overline{\mathbb{Q}})$.

v. The proof goes as for iv.

vi. Combine iv-v. \hfill \Box

\textsuperscript{15}This amounts to an easy computation with $q$-expansions.
Lemma 4.6. Let $m$ be a positive integer, $A$ an abelian subscheme of $J_0(M)^m$ and $\varphi : A \to B$ an isogeny of abelian schemes with kernel $K$. Suppose $K(\overline{\mathbb{Q}})$ is a $T_m[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $J_0(M)^m(\overline{\mathbb{Q}})$. Then $h_F(A) = h_F(B)$ in $\mathbb{R}_{D,M}$.

Proof. Let $l$ be a prime number, $l \nmid Q$. The $l$-primary part $K_l$ of $K(\overline{\mathbb{Q}})$ is a $T_M[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module of finite cardinality. We claim that any irreducible subquotient of $K_l$ is isomorphic to $J_0(M)[m](\overline{\mathbb{Q}})$, for some maximal ideal $m$ of $\mathbb{T}_M$ lying above $l$. For this, notice that the projection $p_i$ of $J_0(M)^m$ onto its $i$-th factor is a Hecke equivariant morphism. Accordingly, $p_i(K_l)$ is a finite $l$-primary $T_M[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-module. Furthermore, $K_l$ becomes a $T_M[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $\prod_{i=1}^m p_i(K_l)$ through the morphism $(p_1, \ldots, p_m)$. Consequently, every irreducible subquotient of $K_l$ is an irreducible subquotient of some $p_j(K_l)$. By [32, Sec. 14, p. 112], every irreducible subquotient of $p_j(K_l)$ is an irreducible subquotient of $J_0(M)[m](\overline{\mathbb{Q}})$, for some maximal ideal $m$ of $\mathbb{T}_M$ lying over $l$. We come up with the claim because $l$ is not an Eisenstein prime, namely $J_0(M)[m](\overline{\mathbb{Q}})$ is irreducible. We conclude as in Proposition 4.2, i.e. by the autoduality of the Galois representation $J_0(M)[m](\overline{\mathbb{Q}})$, [34, Lemma 5.5] and [13, Lemma 5].

We are now in position to establish Proposition 4.3.

Proof of Proposition 4.3. We split the proof into several steps. Below, all the schemes are defined over $S = \text{Spec} \mathbb{Z}[1/pqM]$.

Step 1. Denote by $A$ and $B$ the images of the degeneracy maps $\iota_p : J_0(M)^2 \to J_0(pM)$ and $\iota_q : J_0(M)^2 \to J_0(qM)$. Define the quotients $Q_A = J_0(pM)/A$ and $Q_B = J_0(qM)/B$. By autoduality of the jacobians, there are closed immersions of abelian schemes

$$Q_A' \hookrightarrow J_0(pM), \quad Q_B' \hookrightarrow J_0(qM).$$

Here $Q_A'$ and $Q_B'$ denote the dual abelian schemes of $Q_A$ and $Q_B$. Observe that $Q_A'$ and $Q_B'$ are Hecke invariant. Consider the degeneracy maps $\iota_q : J_0(pM)^2 \to J_0(pqM)$ and $\iota_p : J_0(qM)^2 \to J_0(pqM)$. Finally, put $\widehat{A} = \iota_q(Q_A'^2)$ and $\widehat{B} = \iota_p(Q_B'^2)$. Then, by lemmas 4.5–4.6, there is a chain of equalities in $\mathbb{R}_{D,M}$

\begin{align}
(4.2) & \quad h_F(\widehat{A}) = h_F(Q_A'^2) = 2h_F(Q_A) \\
(4.3) & \quad = 2h_F(J_0(pM)) - 2h_F(A) \\
(4.4) & \quad = 2h_F(J_0(pM)) - 4h_F(J_0(M)).
\end{align}
In (4.2) we followed [35]. In (4.3) we applied [39, Prop. 3.3] and observed that $\log 2$ has zero image in $\mathbb{R}_{D,M}$. The corresponding formulas for $\tilde{B}$ are also true.

**Step 2.** Define $C = \tilde{A} + \tilde{B}$, the image of $\tilde{A} \times \tilde{B}$ under the addition morphism $J_0(pqM)^2 \to J_0(pqM)$. The map $\tilde{A} \times \tilde{B} \to C$ is a $T_{pqM}$-equivariant isogeny. Indeed, Lemma 4.5 iii ensures that $\tilde{A}$ and $\tilde{B}$ are Hecke invariant. The isogeny property is easily established over $\mathbb{C}$, by identification of the induced map on tangent spaces, via $q$-expansions. This implies the isogeny property over $\mathbb{Q}$, and then the same argument as for Lemma 4.5 i leads to the conclusion. The kernel of $\tilde{A} \times \tilde{B} \to C$ is isomorphic to $\tilde{A} \cap \tilde{B}$, embedded into $J_0(pqM)^2$ through the map $x \mapsto (x, -x)$. Because $K(\overline{\mathbb{Q}})$ is a $T_{pqM}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-submodule of $J_0(pqM)^2(\overline{\mathbb{Q}})$, Lemma 4.6 applies. We find

\begin{align}
(4.5) \quad h_F(C) &= h_F(\tilde{A}) + h_F(\tilde{B}) \\
(4.6) \quad &= 2h_F(J_0(pM)) + 2h_F(J_0(qM)) - 8h_F(J_0(M))
\end{align}

in $\mathbb{R}_{D,M}$. In the derivation of (4.6) we took into account (4.2)–(4.4) and the corresponding equations for $\tilde{B}$. If $Q_C = J_0(pqM)/C$, then in $\mathbb{R}_{D,M}$

\begin{align}
(4.7) \quad h_F(Q_C) &= h_F(J_0(pqM)) - h_F(C) \\
(4.8) \quad &= h_F(J_0(pqM)) + 8h_F(J_0(M)) \\
(4.9) \quad &- 2h_F(J_0(pM)) - 2h_F(J_0(qM)).
\end{align}

**Step 3.** Let $E$ be the image of the degeneracy map $\iota : J_0(M)^4 \to J_0(pqM)$. By Lemma 4.5 i–iii, $J_0(M)^4 \to E$ is an isogeny and $E$ is $T_{pqM}$-invariant. Let $\tilde{E}$ be the image of $E$ by the quotient map $J_0(pqM) \to Q_C$. One checks that $E \to \tilde{E}$ is an isogeny. If $K = \ker(E \to \tilde{E})$, then $K(\overline{\mathbb{Q}})$ is $T_{pqM}[\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})]$-invariant. For this, we notice that $E \to \tilde{E}$ is defined over $\mathbb{Z}[1/pqM]$, $K = C \cap E$ and use the $T_{pqM}$ invariance of $E$ and $C$. Lemma 4.5 vi and Lemma 4.6 then provide the relations in $\mathbb{R}_{D,M}$

\begin{align}
(4.10) \quad h_F(\tilde{E}) &= h_F(E) = 4h_F(J_0(M)).
\end{align}

**Step 4.** We remark that $J_0(pqM)^{pq\text{-new}} = Q_C/\tilde{E}$. Recalling that $\log 2$ has zero image in $\mathbb{R}_{D,M}$ and applying [39, Prop. 3.3], we infer

\begin{align}
(4.11) \quad h_F(J_0(pqM)^{pq\text{-new}}) &= h_F(Q_C) - h_F(\tilde{E})
\end{align}

in $\mathbb{R}_{D,M}$. The proposition results from (4.7)–(4.11).

**Remark 4.7.** The proofs of propositions 4.2–4.3 show that $g(X_0^G(M)) = g(X_0(pqM)) + 4g(X_0(M)) - 2g(X_0(pM)) - 2g(X_0(qM))$, where $g$ denotes the genus of a Riemann surface.
5 Proof of the main theorem

5.1 Preliminary reductions

Lemma 5.1. It is enough to prove the statement of Theorem 1.1 when \( M \) is prime, \( M \equiv 11 \mod 12 \) and computing in \( \mathbb{R} / \sum_{l|p(p^2 - 1)q(q^2 - 1)M(M^2 - 1)} \mathbb{Q} \log l \) instead of \( \mathbb{R} / \sum_{l|3(p^2 - 1)q(q^2 - 1)M} \).

Proof. Assume the statement of the theorem known for primes \( M \) of the form \( M \equiv 11 \mod 12 \), and computing in \( \mathbb{R} / \sum_{l|p(p^2 - 1)q(q^2 - 1)M(M^2 - 1)} \mathbb{Q} \log l \).

We must deduce the statement in its general form. Suppose first of all that \( M \geq 5 \) is an integer prime to \( pq \), without prime factors \( N \equiv 11 \mod 12 \) in \{11, 47, 83, 107\}. Chose distinct \( N_1, N_2 \in \{11, 47, 83, 107\} \) prime to \( pq \). The common prime factors of \( N_1(N_1^2 - 1) \) and \( N_2(N_2^2 - 1) \) are exactly 2 and 3. Observe that

\[
\text{Spec} \mathbb{Z}[\mathbb{1}/(p(p^2 - 1)q(q^2 - 1)N_1(N_1^2 - 1))] \\
\cup \text{Spec} \mathbb{Z}[\mathbb{1}/(p(p^2 - 1)q(q^2 - 1)N_2(N_2^2 - 1))]
\]

covers \( \text{Spec} \mathbb{Z}[\mathbb{1}/3p(p^2 - 1)q(q^2 - 1)] \). We conclude considering the étale covers

\[
X_1^{(D)}(MN_i) \quad X_1^{(D)}(M) \\
X_1^{(D)}(N_i) \quad X_1^{(D)}(M)
\]
defined over \( \text{Spec} \mathbb{Z}[\mathbb{1}/pqMN_i], i = 1, 2 \), and applying the functoriality properties of arithmetic intersection numbers and geometric degrees. Secondly, suppose that \( M \geq 5 \) is prime to \( pq \) and divisible by \( N_1 \in \{11, 47, 83, 107\} \). Chose \( N_2 \in \{11, 47, 83, 107\} \) prime to \( pqN_1 \). Then

\[
\text{Spec} \mathbb{Z}[\mathbb{1}/(p(p^2 - 1)q(q^2 - 1)N_1(N_1^2 - 1))] \\
\cup \text{Spec} \mathbb{Z}[\mathbb{1}/(p(p^2 - 1)q(q^2 - 1)N_2(N_2^2 - 1))]
\]

covers \( \text{Spec} \mathbb{Z}[\mathbb{1}/3p(p^2 - 1)q(q^2 - 1)] \). We finish by considering the étale covers

\[
X_1^{(D)}(N_1N_2) \quad X_1^{(D)}(M) \\
X_1^{(D)}(N_2) \quad X_1^{(D)}(N_1)
\]
each defined over the appropriate Zariski open subset of \( \text{Spec} \mathbb{Z} \). \( \square \)
Proposition 5.2. Let $N$ be a prime number with $N \equiv 11 \mod 12$. Let $M$ be a square-free positive integer divisible by $N$ and $S := \text{Spec} \mathbb{Z}[1/M]$. Then the canonical morphism of $S$-schemes $X_1(M) \rightarrow X_0(M)$ is étale.

Proof. Let $f$ denote $X_1(M) \rightarrow X_0(M)$. The relative curves $X_1(M)$ and $X_0(M)$ are smooth over $S$ and $f$ is finite surjective. Therefore $f$ is flat [19, Exp. I, Cor. 5.9], [21, Chap. III, Prop. 9.3]. It is enough to show that $f$ is unramified. We follow Mazur [32, Chap. II, Sec. 2].

Let $k$ be an algebraically closed field of characteristic prime to $M$. Let $(E, C)$ be a generalized elliptic curve together with a cyclic subgroup $C$ of order $M$, over $k$. Let $\text{Aut}_k(E, C)$ denote the stabilizer of $C$ in $\text{Aut}_k(E)$. We may prove $\text{Aut}_k(E, C)$ is at most $\{\pm 1\}$. If $E$ is a singular curve, then it is automatically a Néron polygon with $k$ sides [9, Prop. 1.15]. Write $M = \ln$. The $\Gamma_0(M)$ structure is the cyclic subgroup $C = \mu_l \times \mathbb{Z}/n\mathbb{Z}$ of $E^\text{sm} = \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$. Let $\varphi$ be an automorphism of $E$, compatible with the group structure on $E^\text{sm}$. The restriction of $\varphi$ to $\mathbb{G}_m \times \{0\}$ is either the identity or the inversion map $(z, 0) \mapsto (z^{-1}, 0)$. Assume the first case. One easily checks that the restriction of $\varphi$ to $\mathbb{G}_m \times \{a\}$ is of the form $\varphi(z, a) \mapsto (\lambda_a z, a)$, for some $\lambda_a \in \mu_l$ ($\varphi$ preserves $C$). Then $(1, 0) = \varphi((1, 0)) = \varphi((1, na)) = (\lambda_n^a, 0)$, so that $\lambda_a \in \mu_n$. Because $M$ is square-free, $\mu_l \cap \mu_n = 1$. Therefore $\varphi$ is the identity map. For the second case, introduce the involution $\iota$ of $E$ such that $\iota(z, a) = (z^{-1}, -a)$ for $(z, a) \in \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$. Clearly $\iota(C) = C$. The condition on $\varphi$ is $\iota \circ \varphi \mid_{\mathbb{G}_m \times \{0\}} = \text{id}$. We have just shown this implies $i \circ \varphi$ is the identity, hence $\varphi = \iota$. Let us now suppose $E$ is an elliptic curve. Notice that $C[N]$ is a cyclic subgroup of order $N$ of $C$ and $\text{Aut}_k(E, C) \subset \text{Aut}_k(E, C[N])$. It suffices to check $\text{Aut}_k(E, C[N]) \subset \{\pm 1\}$. This is the case treated by Mazur in loc. cit. The condition $N \equiv 11 \mod 12$ ensures the required property (see [32, Chap. II, Sec. 2, Table I]).

Recall the definition of the reduced divisor of cusps $\textbf{cusps}$ in $X_0(M)$ (resp. $X_1(M)$) [26, Chap. 8, Sec. 8.6, Eq. 8.6.3.2], i.e. $(X_0(M) \setminus Y_0(M))^\text{red}$, where $Y_0(M)$ is the open subscheme coarsely parametrizing elliptic curves. If $M$ is square-free and divisible by a prime $N \equiv 11 \mod 12$, then the hermitian metric $\|d\tau\|_P = 4\pi \text{Im} \tau$ on $\omega_E$ induces pre-log-log hermitian metrics on both $\omega_{X_1(M)/\mathbb{Z}[1/M]}(\textbf{cusps})$ and $\omega_{X_0(M)/\mathbb{Z}[1/M]}(\textbf{cusps})$, with singularities along the cusps cf. §3.1. We write $\omega_{X_i(M)/\mathbb{Z}[1/M]}(\textbf{cusps})_P$ to refer to these hermitian line bundles, $i = 0, 1$.

Corollary 5.3. Let $m, M$ be square-free positive integers with $m \mid M$ and both divisible by a prime number $N \equiv 11 \mod 12$. Write $S = \text{Spec} \mathbb{Z}[1/M]$. 


We have equalities of arithmetic intersection numbers

\[(5.1) \frac{(\omega_{X_1(M)/S}(\text{cusps})_P)^2}{\deg \omega_{X_1(M)/S}(\text{cusps})} = \frac{(\omega_{X_0(M)/S}(\text{cusps})_P)^2}{\deg \omega_{X_0(M)/S}(\text{cusps})}.\]

and

\[(5.2) \frac{(\omega_{X_0(M)/S}(\text{cusps})_P)^2}{\deg \omega_{X_0(M)/S}(\text{cusps})} = \frac{(\omega_{X_0(m)/S}(\text{cusps})_P)^2}{\deg \omega_{X_0(m)/S}(\text{cusps})}\]

in the group \(\mathbb{R}/\sum_{l|M} \mathbb{Q} \log l\).

**Proof.** For the first equality (5.1), let \(f : X_1(M) \to X_0(M)\) be the canonical morphism. By the proposition, the morphism \(f\) is étale. We have an isomorphism

\[f^*(\omega_{X_0(M)/S}(\text{cusps})) \simeq \omega_{X_1(M)/S}(\text{cusps}).\]

This isomorphism becomes an isometry for the metrics \(\| \cdot \|_P\), at the archimedean places. The corollary follows by the functoriality properties of the arithmetic intersection numbers and geometric degrees of line bundles.

The relation (5.2) follows from (5.1) and the functoriality properties of arithmetic intersection numbers, once we observe the morphism \(X_1(M) \to X_1(m)\) is étale, too. \(\square\)

**Proposition 5.4.** Let \(M\) be a positive integer, with a prime factor \(N \equiv 11 \mod 12\). Let \(D\) be an indefinite quaternion algebra over \(\mathbb{Q}\) of discriminant \(d\) prime to \(M\). Define \(S = \text{Spec} \mathbb{Z}[1/dM]\). Then the natural morphism of \(S\)-schemes \(X^D_1(M) \to X^D_0(M)\) is étale.

**Proof.** The relative curves \(X^D_1(M)\) and \(X^D_0(M)\) are smooth over \(S\) and \(f : X^D_1(M) \to X^D_0(M)\) is finite surjective. Therefore \(f\) is flat. It remains to show that \(f\) is unramified. The proof is adapted from Buzzard [6, Lemma 2.2].

Let \(k\) be an algebraically closed field of characteristic prime to \(dM\). Since \(X^D_1(M)\) is a fine moduli scheme, we may show that for a false elliptic curve with a \(\Gamma^0(M)\) structure over \(k\), \((A/k, i, \alpha)\), the automorphism group \(\text{Aut}_k(A, i, \alpha)\) is at most \(\{\pm 1\}\). Giving a \(\Gamma^0(M)\) structure \(\alpha\) on \((A/k, i)\) amounts to give an equivalence class of isomorphisms \(\beta : \mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z} \sim \to A[M]\), for the left action of the Borel subgroup

\[B = \left( \begin{array}{cc} * & * \\ 0 & * \end{array} \right) \subset \text{GL}_2(\mathbb{Z}/M\mathbb{Z}) \xrightarrow{\gamma} (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times,\]
on the set of full level $M$ structures on $(A/k, i)$. Let $\theta$ be an automorphism of $(A/k, i)$ preserving the class of $\beta$. Then

$$\theta \circ \beta = \beta \circ R_{g^x}$$

for some $g^x = \gamma \begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$, $a, c \in (\mathbb{Z}/M\mathbb{Z})^\times$ and $R_{g^x}: x \mapsto xg^x$ on $\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z}$. Put $\psi = \theta - 1$. According to Buzzard, we reduce to the case when $\psi$ is an isogeny. Consider the transpose isogenies $\theta^t, \psi^t: A \rightarrow A$. The following facts are shown in loc. cit.:

i. $\theta^t \theta = 1$ and therefore $\theta^t \beta = \beta R_{(g^x)^{-1}}$;

ii. $\psi^t \psi$ is the multiplication by an integer $n \geq 1$;

iii. $\theta + \theta^t$ is the multiplication by $m = 2 - n$, with $|m| \leq 2$, and therefore $1 \leq n \leq 4$;

iv. $\theta$ satisfies the equation $\theta^2 - m \theta + 1$ in $\text{End}_k(A)$.

We discuss on the possibilities for $\theta$, depending on $n \in \{1, \ldots, 4\}$.

Let $P = \gamma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z}$ and $Q = \beta(P)$. Combining the items i–iii we get

$$m \beta = \beta R_{g^x} + \beta R_{(g^x)^{-1}}.$$

Applied to $P$, this leads to

$$mQ = cQ + c^{-1}Q.$$

Therefore $(m - c - c^{-1})Q = 0$. Because $\beta$ is an isomorphism, we infer $m = c + c^{-1}$ in $\mathbb{Z}/M\mathbb{Z}$. We rewrite this equation as $c^2 + (n - 2)c + 1 = 0$ in $\mathbb{Z}/M\mathbb{Z}$. Projecting to $\mathbb{Z}/N\mathbb{Z}$, it will be enough to study the equation

$$c^2 + (n - 2)c + 1 = 0 \quad \text{in} \quad \mathbb{Z}/N\mathbb{Z}. \quad (5.3)$$

Case $n = 1$. Equation (5.3) reads $c^2 - c + 1 = 0$ in $\mathbb{Z}/N\mathbb{Z}$. Since $4 \in (\mathbb{Z}/N\mathbb{Z})^\times$, this is equivalent to

$$\begin{align*}
(2c - 1)^2 &= -3 \quad \text{in} \quad \mathbb{Z}/N\mathbb{Z}. \quad (5.4)
\end{align*}$$

\footnote{Under the identification $\gamma$, an element $g \in B$ becomes an element $g^x \in (\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z})^\times$. The left action of $g$ is obtained by composing on the right by the automorphism $R_{g^x}: x \mapsto xg^x$ of $\mathcal{O}_D \otimes \mathbb{Z}/M\mathbb{Z}$. Namely, by sending a full level $M$ structure $\beta$ to the full level $M$ structure $\beta \circ R_{g^x}$.}

\footnote{For an isogeny $h: A \rightarrow A$, the transpose $h^t$ is constructed from the dual isogeny $h^\vee: A^\vee \rightarrow A^\vee$ and the principal polarization on $A$ obtained after fixing $t \in \mathcal{O}_D$ with $t^2 = -d$. See [6, Sec. 1] for details.}
Observe that (5.4) has no solution in \( \mathbb{Z}/NZ \). Indeed, because \( N \equiv 11 \) mod 12, \(-1\) is not a square modulo \( N \). Therefore \((\frac{N}{N}) = -1\). On the other hand, \( N \equiv 2 \) mod 3, and so \((\frac{N}{3}) = -1\). By the quadratic reciprocity law \((\frac{3}{N}) = -(1)^{(N-1)/2}\). The congruence \((N - 1)/2 \equiv 5 \) mod 6 implies \((\frac{N}{5}) = 1\). We conclude \((\frac{N}{N}) = -1\), so that \(-3\) is not a square modulo \( N \).

Case \( n = 2 \). Equation (5.3) reduces to \( c^2 = -1 \) in \( \mathbb{Z}/NZ \). This is impossible because \( N \equiv 11 \) mod 12.

Case \( n = 3 \). Now (5.3) becomes \( c^2 + c + 1 = 0 \) in \( \mathbb{Z}/NZ \). In particular \( c \neq 1 \), since otherwise \( N = 3 \). This is excluded by the condition \( N \equiv 11 \) mod 12. The equation is thus equivalent to \( c^3 = 1 \) in \( \mathbb{Z}/NZ \). A nontrivial solution to this equation would imply \( N \equiv 1 \) mod 3, in contradiction with \( N \equiv 11 \) mod 12.

Case \( n = 4 \). By the property iv recalled above, the morphism \( \phi := \theta + 1 \) satisfies \( \phi^2 = 0 \). Define \( E \) to be \((\ker \phi)^{\text{red}}\). Then \( E \) is an abelian subvariety of \( A \). Observe that \( E \neq 0 \) since \( \phi^2 = 0 \). Suppose \( E \) has dimension 1. Namely, \( E \) is an elliptic curve. The action of \( \mathcal{O}_D \) on \( A \) induces an action by endomorphisms on \( E \): a map of unitary rings \( \mathcal{O}_D \to \text{End}_k(E) \). Tensoring by \( \mathbb{Q} \) we find a morphism of unitary rings \( D \to \text{End}_k(E) \otimes \mathbb{Q} \). But \( \text{End}_k(E) \otimes \mathbb{Q} \) is either \( \mathbb{Q} \), or an imaginary quadratic extension of \( \mathbb{Q} \) or a definite quaternion algebra over \( \mathbb{Q} \). Such maps don’t exist. We conclude \( E = A \), \( \theta = -1 \) and \( \text{Aut}_k(A,i,\alpha) = \{\pm 1\} \). \( \square \)

By the proposition, if \( M \) is divisible by a prime \( N \equiv 11 \) mod 12, both \( \omega_{X^D_1(M)/S} \) and \( \omega_{X^D_0(M)/S} \) come equipped with smooth hermitian metrics denoted \( \| \cdot \|_P \), coming from \( \|d\tau\|_P = 4\pi \text{Im} \tau \) on \( \omega_{\mathcal{H}} \) cf. §3.1. Let \( \omega_{X^D_1(M)/S,P} \) denote the corresponding hermitian line bundles.

**Corollary 5.5.** Let \( M \) be a positive integer, with a prime factor \( N \equiv 11 \) mod 12. Let \( D \) be an indefinite quaternion algebra over \( \mathbb{Q} \) of discriminant \( d \) prime to \( M \). Write \( S = \text{Spec} \mathbb{Z}[1/dM] \). Then we have an equality of normalized arithmetic intersection numbers

\[
\frac{(\omega_{X^D_1(M)/S,P})^2}{\deg \omega_{X^D_1(M)/S}} = \frac{(\omega_{X^D_0(M)/S,P})^2}{\deg \omega_{X^D_0(M)/S}} \equiv \sum_{l | dM} \mathbb{Q} \log l.
\]

**Proof.** The canonical map \( f : X^D_1(M) \to X^D_0(M) \) is étale and there is an isomorphism

\[
f^*\omega_{X^D_1(M)/S} \simeq \omega_{X^D_0(M)/S}.
\]

The isomorphism is an isometry for the metrics \( \| \cdot \|_P \), at the archimedean place. This proves the statement. \( \square \)
To conclude the preliminary reductions, we remark that the isometries
\[ \omega_{X^p(M),L^2} \sim \omega_{X^p(M)/S,P} \quad \text{and} \quad \omega_{X_1(M),L^2} \sim \omega_{X_1(M)/S(\text{cusps})_P} \]
explained in the introduction, together with Lemma 5.1, Corollary 5.3 and Corollary 5.5, reduce Theorem 1.1 to the validity of the following statement.

**Theorem 5.6.** Let \( N \) be a prime number, \( N \equiv 11 \mod 12 \), and \( D \) an indefinite quaternion algebra over \( \mathbb{Q} \), of discriminant \( d \) prime to \( N \). Assume \( d = pq \) is the product of two prime numbers. Let \( S = \text{Spec} \mathbb{Z}[1/pq] \). Then the equality of normalized arithmetic intersection numbers
\[ \left( \frac{\omega_{X^p(N)/S,P}}{\deg \omega_{X^p(N)/S}} \right)^2 = \left( \frac{\omega_{X_0(N)/S(\text{cusps})_P}}{\deg \omega_{X_0(N)/S(\text{cusps})}} \right)^2 \]
holds in the group \( \mathbb{R}/ \sum_{l \mid (p^2-1)q(q^2-1)N(N^2-1)} \mathbb{Q} \log l \).

### 5.2 Proof of Theorem 5.6

We are now prepared to establish Theorem 5.6, and therefore Theorem 1.1. The proof is an application of the arithmetic Riemann-Roch formula for pointed stable curves [14, Thm. A] and the results in the previous sections. The reader is referred to loc. cit. for the statement of the formula and precise definitions. Especially, the arithmetic \( \psi \) line bundle, endowed with the Wolpert metric [14, Sec. 1, Def. 2.1], has still not been used in the present article and is to be reviewed.

Let \( M \) be a square-free positive integer divisible by a prime \( N \equiv 11 \mod 12 \). The cusps of \( X_0(M) \) over \( S = \text{Spec} \mathbb{Z}[1/M] \) are given by disjoints sections \( \sigma_1, \ldots, \sigma_n : S \to X_0(M) \).\(^{18}\) The line bundle \( \psi := \otimes_j \sigma_j^* \omega_{X_0(M)/S} \) is endowed with the –tensor product– Wolpert metric. We write \( \psi_W = (\psi, \| \cdot \|_W) \).

**Proposition 5.7.** Let \( M \) be a square-free positive integer, with a prime factor \( N \equiv 11 \mod 12 \). We have
\[ \text{deg} \psi_W = 0 \quad \text{in} \quad \mathbb{R}/ \sum_{l \mid M} \mathbb{Q} \log l. \]

**Proof.** Introduce \( S' = \text{Spec} \mathbb{Z}[\zeta_M, 1/M] \), where \( \zeta_M \) is a primitive \( M \)-th root of unity. Let \( X_0(M)_{S'} \) denote the base change of \( X_0(M) \) to \( S' \), and \( \sigma_j' \),

\(^{18}\)The sections \( S \to X_0(M) \) are induced by the standard \( m \)-sided Néron polygons over \( S \), with \( m \mid M \), together with their natural \( \Gamma_0(M) \)-level structures.
We proceed to identify \( \tau \). The morphism \( f' : X_1(M)_{S'} \to X_0(M)_{S'} \) is étale and the cusps of \( X_1(M)_{S'} \) are given by sections \( S' \to X_1(M)_{S'} \) [26, Thm. 10.9.1, Thm. 10.9.4]. The isomorphism \( f'^*(\omega_{X_0(N)_{S'/S'}}(\text{cusps})) \simeq \omega_{X_1(N)_{S'/S'}}(\text{cusps}) \) is an isometry for the metric structures \( \| \cdot \|_p \). Therefore, if \( \tau \) is any cusp section of \( X_1(M)_{S'} \) lying over a \( \sigma'_j \), we have an isometry

\[
(\tau^* \omega_{X_1(M)_{S'/S'}}, \| \cdot \|_W) \simeq (\sigma'_j^* \omega_{X_0(M)_{S'/S'}}, \| \cdot \|_W).
\]

By the functoriality compatibilities of the arithmetic degree, we reduce to show \( \deg(\tau^* \omega_{X_1(M)_{S'/S'}}, \| \cdot \|_W) = 0 \) in \( \mathbb{R}/\sum l \mathbb{Q} \log l \). We next work out this case.

Let \( \tau_1, \ldots, \tau_m \) be the cusp sections of \( X_1(M)_{S'} \). For every integer \( k \geq 0 \), there is a canonical adjunction isomorphism

\[
(5.5) \quad \tau_1^* (\omega_{X_1(M)_{S'/S'}}^{(k+1)}(k \tau_1 + \ldots + k \tau_m)) \simeq \tau_1^* \omega_{X_1(M)_{S'/S'}}.
\]

We endow \( \tau_1^* (\omega_{X_1(M)_{S'/S'}}^{(k+1)}(k \tau_1 + \ldots + k \tau_m)) \) with the hermitian structure such that (5.5) becomes an isometry. We still denote this metric by \( \| \cdot \|_W \). Since the line bundle \( \omega_{X_1(M)_{S'/S'}}(\text{cusps}) \) is relatively ample, for \( k \) large enough there exists

\[
\theta \in H^0(X_1(M)_{S'}, \omega_{X_1(M)_{S'/S'}}^{(k+1)}(k \text{ cusps}))
\]

\[
\theta \notin H^0(X_1(M)_{S'}, \omega_{X_1(M)_{S'/S'}}^{(k+1)}((k-1) \tau_1 + k \tau_2 + \ldots + k \tau_m)).
\]

Equation (5.5) entails the identity

\[
\overline{\deg}(\tau_1^* \omega_{X_1(M)_{S'/S'}}, \| \cdot \|_W) = \overline{\deg}([\text{div}(\tau_1^* \theta), -\log \| \tau_1^* \theta \|_W^2])
\]

We proceed to identify \( \tau_1^* \theta \) and \( \| \tau_1^* \theta \|_W \). The completion of \( X_1(M)_{S'} \) along the section \( \tau_1 \) is isomorphic to the formal scheme \( \text{Spf}(\mathbb{Z}[\zeta_M, 1/M][[q]]) \), for some indeterminate \( q \) [26, Thm 10.9.1, Thm 19.9.4]. The pull-back of the invertible sheaf \( \omega_{X_1(M)_{S'/S'}}^{(k+1)}(\text{cusps}) \) to \( \text{Spf}(\mathbb{Z}[\zeta_M, 1/M][[q]]) \) is trivialized by the section \( (dq)^{(k+1)}/q^k \). The pull-back of \( \theta \) can be developed as

\[
\sum_{n \geq 0} a_n q^n \frac{(dq)^{(k+1)}/q^k}{q^n} \in \mathbb{Z}[\zeta_M, 1/M][[q]], \quad a_0 \neq 0.
\]

Then \( \tau_1^* \theta \) gets identified with the non-trivial global section \( a_0 \) of the trivial
line bundle $\mathcal{O}_{\mathbb{Z}[\zeta_M, 1/M]}$. By the very definition of the Wolpert metric\(^{19}\)

$$\log \|\tau_1^*\theta\|_W = \sum_{\sigma: \mathbb{Q}(\zeta_M) \hookrightarrow \mathbb{C}} \log |\sigma a_0|.$$ 

We conclude by the product formula:

$$\hat{\deg}[(\text{div}(\tau_1^*\theta), -\log \|\tau_1^*\theta\|_W^2)] = \sum_{\nu \nmid \infty, \nu \nmid M} \log |a_0|_\nu - \sum_{\sigma: \mathbb{Q}(\zeta_M) \hookrightarrow \mathbb{C}} \log |\sigma a_0| = 0$$

in $\mathbb{R}/\sum_{\nu \mid M} \mathbb{Q} \log \nu$. \(\square\)

**Proof of Theorem 5.6.** Consider the smooth relative curves $\pi^D: \mathcal{X}_0^D(N) \to S$ and $\pi_{d'}: \mathcal{X}_0(d'N) \to S$, $d' \mid d$, with $S = \text{Spec} \mathbb{Z}[1/dN]$. By the arithmetic Riemann-Roch formula for pointed stable curves [14, Thm. A], there are equalities in $\widehat{\text{CH}}^1(S)$

\[12\hat{c}_1(\lambda(\omega_{\mathcal{X}_0^D(N)/S})_Q) = \pi_*^D(\hat{c}_1(\omega_{\mathcal{X}_0^D(N)/S, \text{hyp}})^2) + \hat{c}_1(\mathcal{O}(C(g^D, 0))),\]

\[12\hat{c}_1(\lambda(\omega_{\mathcal{X}_0(d'N)/S})_Q) = \pi_{d'}(\hat{c}_1(\omega_{\mathcal{X}_0(d'N)/S}(\text{cusps})_{\text{hyp}})^2) + \hat{c}_1(\mathcal{O}(C(g_{d'}, h_{d'}))).\]

We clarify the notations:

- $\|\cdot\|_Q$ is a Quillen type metric [14, Def. 2].
- The subscript hyp stands for the hyperbolic metric structure. We only need to know $\|\cdot\|_{\text{hyp}}^2 = e^{-\alpha} \|\cdot\|_P^2$, for some universal constant $\alpha$. Then we have

\[\pi_*^D(\hat{c}_1(\mathcal{L}_{\text{hyp}})^2) = \pi_*^D(\hat{c}_1(\mathcal{L}_P)^2) + [0, \alpha \deg \mathcal{L}],\]

where $\mathcal{L}$ denotes either $\omega_{\mathcal{X}_0^D(N)/S}$ or $\omega_{\mathcal{X}_0(d'N)/S}(\text{cusps})$.

- The symbols $g^D$ and $g_{d'}$ denote the genus of $\mathcal{X}_0^D(N)$ and $\mathcal{X}_0(d'N)$, respectively, and $h_{d'}$ the number of cusps of the latter.

\(^{19}\)The compatibility of the algebraic and the transcendental–Fourier–developments follows from the Kodaira-Spencer isomorphism $\omega_{\mathcal{X}_1(M)_{\mathbb{C}}, \text{simp}} \cong \omega_{\mathcal{X}_1(M)_{\mathbb{C}}, \text{sing}}(\text{cusps})[26, \text{Thm 10.13.11}],$ the theory of the Tate curve [9, Chap. VII, Sec.1] and the comparison isomorphisms in [9, Chap. VII, Sec. 4].
• \( \mathcal{O}(C(g, n)) \) is the trivial line bundle on \( S \), endowed with the metric \( C(g, n)|·| \), where \( |·| \) is the absolute value on \( \mathbb{C} \) and

\[
C(g, n) = \exp \left[ (2g - 2 + n) \left( \frac{\zeta'(1)}{\zeta(1)} + \frac{1}{2} \right) \right].
\]

By construction, the hermitian determinant line bundles \( \lambda(\omega_{X/S})_Q \), \( X = X_0^D(N) \) or \( X_0(d'N) \) satisfy

\[
\hat{\deg} \lambda(\omega_{X/S})_Q = \frac{1}{2} \log Z'(Y, 1) + h_F(Jac(X)) + \frac{1}{2} \log E(g, n) - \frac{1}{2} \log(2g - 2 + n).
\]

(5.9)

Here \( (g, n) = (g^D, 0) \) and \( Y = X_0^D(N) \) if \( X = X_0^D(N) \), while \( (g, n) = (g_{d'}, h_{d'}) \) and \( Y = Y_0(d'N) \) if \( X = X_0(d'N) \). The constant \( E(g, n) \) is defined by

\[
E(g, n) = 2^{(g+n-2)/3} \cdot \exp \left[ (2g - 2 + n) \left( \frac{\zeta'(1)}{4} + \frac{1}{2} \log(2\pi) \right) \right].
\]

As in the proof of Theorem 3.7, one checks the relations

\[
\deg \omega_{X_0^D(N)/S} = \deg \omega_{X_0(pqN)/S(\text{cusp})} + 4 \deg \omega_{X_0(N)/S(\text{cusp})} - 2 \deg \omega_{X_0(pN)/S(\text{cusp})} - 2 \deg \omega_{X_0(qN)/S(\text{cusp})}
\]

(5.10)

and

\[
\log C(g^D, 0) = \log C(g_{pq}, h_{pq}) + 4 \log C(g_1, h_1) - 2 \log C(g_p, h_p) - 2 \log C(g_q, h_q), \quad \text{resp. for } E(g, n).
\]

(5.11)

Because for \( d' \mid d \) the morphism \( Y_0(d'N) \) is unramified\(^20\) of degree \( \prod_{l \mid d'/d}(l + 1) \) [20, Chap. 2, Eq. 2.11], one also has

\[
\deg \omega_{X_0(pqN)/S(\text{cusp})} = (p + 1) \deg \omega_{X_0(qN)/S(\text{cusp})}
\]

(5.12)

An easy manipulation with (5.10) and (5.12) yields

\[
\frac{(2g_{pq} - 2 + h_{pq})(2g_1 - 2 + h_1)^4}{(2g_p - 2 + h_p)^2(2g_q - 2 + h_q)^2} = \frac{2g_D - 2}{(p^2 - 1)(q^2 - 1)}.
\]

\(^{20}\)Recall this is ensured by the condition \( N \equiv 11 \) mod 12.
We now apply the arithmetic degree map $\widehat{\deg} : \widehat{\text{CH}}^1(S) \to \mathbb{R}/\sum_{l \mid pqN} \mathbb{Q}\log l$ to (5.6)–(5.7), and take into account the equations (5.8)–(5.11), (5.13) together with Theorem 3.7, Theorem 4.1 and Proposition 5.7. We infer an equality

\[(5.14) \quad (\omega_{X_0(pqN)/S, P})^2 = (\omega_{X_0(pqN)/S}(\text{cusps})_P)^2 + 4(\omega_{X_0(N)/S}(\text{cusps})_P)^2 - 2(\omega_{X_0(pN)/S}(\text{cusps})_P)^2 - 2(\omega_{X_0(qN)/S}(\text{cusps})_P)^2 \]

in the group $\mathbb{R}/\sum_{l \mid p^2-1,q^2-1} \mathbb{Q}\log l$. Corollary 5.3 and equation (5.10) lead to the conclusion. \qed

References


