# The Radius of Convergence Function for First Order Differential Equations

Gilles CHRISTOL

ABSTRACT. We present an algorithm computing, for any first order differential equation $L$ over the affine line and any (Berkovich) point $t$ of this affine line, the $p$-adic radius of convergence $R_L(t)$ of the solutions of $L$ near $t$. We do explicit computations for the equation

$$L(f) \stackrel{\text{def}}{=} xf' - \pi(px^p + ax)f = 0 \qquad (\pi^{p-1} = -p). \tag{0.1}$$

where $a$ lies in some valued extension of $\mathbb{Q}_p$. For $a = -1$ and $t = 0$, a solution of $L$ near $t$ is the Dwork exponential $\exp(\pi x^p - \pi x)$. Among other important properties, it appears that the function $R_L(t)$ is entirely determined by its values on a finite subtree of the affine line.

The radius of convergence function has been shown to be a basic tool when studying $p$-adic differential equations. Notably, for first order differential equations, it gives the index of the underlying differential operator acting on various spaces of functions. However explicit computations are far from easy except in the few "trivial" cases where the "small radius theorem" and the logarithmic concavity are sufficient to conclude. In this paper we give an algorithm to compute the radius of convergence function for any first order $p$-adic differential equation defined on the affine line. It rests crucially on the proposition 2.15 that gives a criterion to decide whether the radius of convergence of a product is the smallest radius of convergence of the factors.

As a by-product, we prove a Baldassarri conjecture for first order differential equations without singularities in the affine line (corollary 3.3). Roughly speaking, this conjecture asserts that the radius of convergence function is entirely determined by its values on a finite sub-tree of the whole "quasi-polyhedron" structure made up by Berkovich points. Likely it should mean that the radius of convergence function is "definable" in the sense of [**10**].

As the computation becomes quickly very tedious, we achieve it only for the differential equation (0.1) which is the simplest but non-trivial case. This example as been already treated in [**1**] but here we present it following the general algorithm.

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The paragraph 1.1 sketches very shortly the Berkovich point... of view on the affine line, mainly to precise definitions. In view of further computations we also consider subsets of the affine line but this will not be used in this article. For the sake of self-containedness the first section contains also short overviews of both theory of Witt vectors and Artin-Hasse exponential.

The second section contains the basic tool : the radius of convergence of a product of so-called "Robba exponentials" is equal to the smallest radius of convergence of the factors. We gives two criterions under which the radius of convergence of a product of power series is equal to the smallest radius of convergence of the factors. Actually, we concentrate ourself on products of two power series, the general situation being deduced straightforwardly. Dwork [**7**] was the first to observe that a non-trivial product of two exponential series can have a radius of convergence strictly greater than the radius of convergence of each factor. Robba [**13**] did much better constructing exponential of polynomials whose radius of convergence is the greatest possible (namely such that the radius of convergence function of the underlying differential equation has only two slopes). Matsuda [**11**] and then Pulita [**12**] clarified the Robba construction and made it explicit. We think this beautiful theory deserves to be better known. So the second section contains a rather self-contained introduction to it pointing out the lot of underlying congruences. It also contains some "further" and deep properties that will not be directly used in the paper.

The third section presents the algorithm (actually two algorithms) and illustrates it by an example.

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## 1. More or less well known facts

**1.1. The quasi-polyhedron structure of the projective line.** Let $K$ be a complete valued extension of $\mathbb{Q}_p$. We will denote by $\overline{K}$ the residue field of $K$, by $K^{\mathrm{alg}}$ the algebraic closure of $K$ and by $\widehat{K}^{\mathrm{alg}}$ the completion of $K^{\mathrm{alg}}$. We will always assume the absolute value to be normalized by $|p| = p^{-1}$.

Our aim is to recall connections between points, Dwork's generic points and circular filters. These three points of view come from distinct mathematical schools but it is better to keep them all in mind.

In this paper we are only interested by points of the affine line over $K$, but let us consider, more generally, a subset $D$ of the affine line[1] built up from

$$\begin{array}{lll} & \text{"open" disks} & D(a,r) = \{x\,;\ |x-a| < r) \\ \text{or} & \text{"closed" disks} & D(a,r^+) = \{x\,;\ |x-a| \le r), \end{array}$$

with center $a$ in $K^{\mathrm{alg}}$ and radius $r > 0$, by a, possibly infinite, set of boolean operations. Then, for each extension $\Omega$ of $K$, the set $D(\Omega)$ of points of $D$ with value in $\Omega$ is well defined. For finite sets of boolean operations one gets affinoïd

[1] Using the change of variable $x \mapsto 1/(x-a)$ one could extend the theory to strict subset of the projective line. Then the complement of an open (closed) disk becomes a closed (open) disk with center at infinity

or analytic spaces [**2**]. For infinite sets of boolean operations one gets much more involved situations, namely infraconnected sets, [**4**], [**9**].

Let $\mathcal{R}(D)$ be the subring of $K(x)$ of rational fractions without pole in $D$, i.e. in $D(K^{\text{alg}})$, endowed with the topology of uniform convergence on $D$. Let $\mathcal{H}(D)$ be the completion of $\mathcal{R}(D)$ namely the ring of analytic elements on $D$. When $D$ is the affine line itself, one has $\mathcal{H}(D) = \mathcal{R}(D) = K[x]$.

Let $t$ be in some valued extension $\Omega$ of $K$. The distance $r(t)$ from $t$ to $\widehat{K}^{\text{alg}}$ is called the *radius* of $t$. When the distance from $t$ to $D(\widehat{K}^{\text{alg}})$ is also $r(t)$, then, for any $f$ in $\mathcal{H}(D)$, one can define $|f(t)|$ as the limit of $|f(x)|$ along the "circular filter" [**4**] made of subsets $\{x \in D(\widehat{K}^{\text{alg}})\,;\ |x-t| < r\}$ for $r > r(t)$. Under that condition

- $r(t) = 0$ if and only if $t$ belongs to $D(\widehat{K}^{\text{alg}})$. In that case $t$ will be called a *rigid point* of $D$ or a point of type (1).
- if $r(t) > 0$ then $t$ will be called a *Dwork generic point* of $D$.

A *Berkovich point* of $D$ is, by definition, a continuous multiplicative semi-norm on $\mathcal{H}(D)$. We denote by $\text{Mult}(D)$ the set of Berkovich points of $D$.

To a rigid or Dwork generic point $t$ of $D$ is associated the Berkovich point $|\cdot|_t$ defined by $|f|_t = |f(t)|$ for $f$ in $\mathcal{H}(D)$. It is noticeable that any Berkovich point can be obtained in this way (see [**2**] 1.2.2). More precisely, to a multiplicative semi-norm $|\cdot|$, we associate the field of fractions $\Omega$ of the (integral) ring $\mathcal{H}(D)/\{f\,;\ |f| = 0\}$. Then, it is easily checked that $|\cdot|_t = |\cdot|$ where $t$ is the image of the function $x$ (of $K(x)$) in $\Omega$. Actually, for "reasonable" $D$, $\{f\,;\ |f|_t = 0\} = \{0\}$ if and only if $t$ is a Dwork generic point. But there are "unreasonable" $D$, namely with $T$-filter, for which this is no longer true.

Actually one can limit himself to consider a "big enough" but fixed field $\Omega$. Big enough means algebraically closed, spherically complete, such that $|\Omega| = \mathbb{R}^{\geq 0}$ and with a residue field $\overline{\Omega}$ transcendental over $\overline{K}$. Moreover, as any continuous automorphism of $\Omega/K$ is an isometry, the map $t \mapsto |\cdot|_t$ factorizes through a map $D(\Omega)/\text{Gal}^{\text{cont}}(\Omega/K) \to \text{Mult}(D)$ which should be onto[2] In [**8**] 8, it is proved that points of $\Omega$ fixed by $\text{Gal}^{\text{cont}}(\Omega/K)$ are exactly those of $K$. The key point is that, for any generic point $t$ and $c$ in $\Omega$ such that $c < r(t)$ the group $\text{Gal}^{\text{cont}}(\Omega/K)$ contains an automorphism $\sigma$ such that $\sigma(t) = t + c$.

We can now classify generic points $t$ by the cardinality of the quotient set $\delta(t) = \{c \in \widehat{K}^{\text{alg}}\,;\ |t-c| = r(t)\}/\sim_{r(t)}$ where $c \sim_{r(t)} b$ means $|c-b| < r(t)$ (see [**2**] 1.4.4).

- If $\#(\delta(t)) \geq 2$, $t$ is said of type (2). Then $r(t)$ belongs to $\sqrt{|K^*|}$ and $\delta(t)$ is isomorphic to the algebraic closure of $\overline{K}$.
- If $\#(\delta(t)) = 1$, $t$ is said of type (3). Then $r(t)$ is not in $\sqrt{|K^*|}$.
- If $\delta(t) = \emptyset$, $t$ is said of type (4). This cannot happen when $\widehat{K}^{\text{alg}}$ is spherically complete.

To each Berkovich point of $D$ is associated a disk of radius $r(t)$ with center in $\widehat{K}^{\text{alg}}$ (an intersection of such disks when $t$ is of type (4)). Conversely, to each open disk $D(a,r)$ whose intersection with $D$ is not contained in $D(a,\rho)$ for $\rho < r$,

[2] I know no reference for such a result but it seems likely that this map is one to one. Actually, except when considering a tower of two generic points as in [**6**], I know no situation where a bigger $\Omega$ is needed.

is associated a (unique) Berkovich point $t_{a,r}$. We will extend that notation writing $t_{a,0} = a$ and $t_{a,\infty} = \infty$.

The set $\mathrm{Mult}(D)$ is endowed with a kind of "hairy" tree structure for which the Berkovich points $t_{a,r}$ for $r \leq \rho \leq R$, if all in $D$, make a path denoted by $[t_{a,r}, t_{a,R}]$. Then points of type (1) and (4) are among the leaves, and branching are among points of type (2). Such a structure is called "tree" in [**3**] and "quasi-polyhedron" in [**2**].

We will denote by $\mathfrak{A}$ the quasi-polyhedron $\mathrm{Mult}(\mathbb{A}_1)$ of the affine line.

Remark 1.1. For the "natural" topology on $\mathrm{Mult}(D)$ (namely the less fine one for which the maps $|\cdot| \mapsto |f|$ are continuous) the set of rigid points is a dense subset. In particular this topology is strictly less fine than the tree topology.

**1.2. Witt vectors.** We recall this well known theory for the sake of completeness but also to point out some special properties of Witt vectors over a $p$-adic ring. It contains a lot of congruences the simplest one being $(a+b)^p = a^p + b^p \pmod{p}$. A more complete presentation including almost all the following can be found in Boubaki commutative algebra chapter 9 (do not forget exercices !).

For $n \geq 0$, let :

$$W_n(X_0, \ldots, X_n) = \sum_{i=0}^{n} p^i X_i^{p^{n-i}} = X_0^{p^n} + pX_1^{p^{n-1}} + \cdots + p^n X_n.$$

so that $W_0 = X_0, \quad W_1 = X_0^p + X_1$ and

$$W_n(X_0, \ldots, X_n) = W_{n-1}(X_0^p, \ldots, X_{n-1}^p) + p^n X_n \tag{1.1}$$

When $A$ is a ring and $\mathbf{a} = (a_0,..., a_n,...)$ belongs to $W(A) = A^{\mathbb{N}}$, we set :

$$\mathcal{W}(\mathbf{a}) = \big(W_0(\mathbf{a}), \ldots, W_n(\mathbf{a}), \ldots\big) = \Big(W_0(a_0), \ldots, W_n(a_0, \ldots, a_n), \ldots\Big).$$

By definition of the ring of Witt vectors $W(A)$, the map $\mathcal{W}$ is a ring morphism from $W(A)$ to $A^{\mathbb{N}}$ endowed with the component-wise addition and multiplication.

The image of $\mathcal{W}$ can be characterized when $A$ is (a subring of) the ring of integers of some unramified extension of $\mathbb{Q}_p$. Such a characterization is not available for the ring of integers of ramified extensions. To bypass this difficulty, we will use the following trick : write $A = A_0[\pi]$ where $A_0$ is the ring of integers of an unramified extension of $\mathbb{Q}_p$ and get elements of $\mathcal{W}(A^{\mathbb{N}})$ using the specialisation $X \mapsto \pi$ from $A_0[X]$ to $A$.

Definition 1.2. A pNR ($p$-unramified) ring $A$ is a ring of characterisitic 0 endowed with a ring endomorphism $\tau : A \to A$ such that

$$(\forall a \in A) \qquad \tau(a) - a^p \in pA.$$

Remarks 1.3. Following facts are easily checked
**1)** $\mathbb{Z}$ is pNR for the identical endomorphism,
**2)** If $k$ is a field of characterisitic $p$, the Witt vectors ring $W(k)$ endowed with the Frobenius endomorphism is pNR,
**3)** If $A$ is pNR, then $A[x]$ (resp. $A[[x]]$, $A((x))$) is pNR when endowed with the endomorphism $\tau\big(\sum a_s x^s\big) = \sum \tau(a_s) x^{ps}$,
**4)** On the other hand, if $K$ is a ramified extension of $\mathbb{Q}_p$, then its ring of integers is not pNR. Indeed, let $\theta$ in $A$ and $n > 1$ such that $\theta^n = pa$ with $|a| = 1$. For $A$ to be pNR, we should have

$\star$ $|\tau(a) - a^p| \le |p| < |a^p|$ then $|\tau(a)| = |a^p| = 1$,
$\star$ $|\tau(\theta)^n| = |\tau(\theta^n)| = |p\tau(a)| = |p|$ then $|\tau(\theta)| = |p|^{1/n} = |\theta|$,
$\star$ $|\tau(\theta) - \theta^p| \le |p| < |p|^{1/n} = |\tau(\theta)|$ then $|\tau(\theta)| = |\theta^p| < |\theta|$.
Contradiction.

NOTATION 1.4. Let $P$ in $A[x]$ or in $x\,A[[x]]$. Whe define recursively

$$P^{\circ(0)} = x \quad , \qquad P^{\circ(n)}(x) = P\big(P^{\circ(n-1)}(x)\big).$$

Following results are classical and easily proved.

LEMMA 1.5. *Let $A$ be a ring and let $R$ in $A[x]$ and $P(x) = x^p + p\,R(x)$. For $a$ and $b$ in $A$ and $n$ in $\mathbb{N}$,*
*1) $b^n - a^n \in (b-a)A$,*
*2) $R(b) - R(a) \in (b-a)A$,*
*If moreover $b-a$ belongs to $p\,A$, then*
*3) $b^p - a^p \in p(b-a)A$,*
*4) $P(b) - P(a) \in p(b-a)A$,*
*5) $P^{\circ(n)}(b) - P^{\circ(n)}(a) \in p^n(b-a)A \subset p^{n+1}\,A$.*

PROPOSITION 1.6. *Let $A$ be a pNR ring. The map $\mathcal{W}$ is one to one and $(w_0, \dots, w_n, \dots)$ belongs to $\mathcal{W}(A^{\mathbb{N}})$ if and only if, for all $n$, $w_n - \tau(w_{n-1}) \in p^n A$.*

PROOF. For $\mathcal{W}(\mathbf{a}) = \mathcal{W}(\mathbf{b})$, we have $a_0 = W_0(a_0) = W_0(b_0) = b_0$ and, by (1.1)

$$\begin{aligned} p^n a_n &= W_n(\mathbf{a}) - W_n(a_0^p, \dots, a_{n-1}^p, 0) \\ &= W_n(\mathbf{b}) - W_n(b_0^p, \dots, b_{n-1}^p, 0) = p^n b_n, \end{aligned}$$

and $a_n = b_n$ because the characteristic of $A$ is 0.

For $a$ in $A$, $a^p - \tau(a) \in p\,A$. Then applying the statement 1.5-5 with $R = 0$ we get $a^{p^n} - \tau(a)^{p^{n-1}} \in p^n A$ whence :

$$W_n(a_0^p, \dots, a_n^p) - W_n\big(\tau(a_0), \dots, \tau(a_n)\big) = \sum p^k \big(a_k^{p^{n-k+1}} - \tau(a_k)^{p^{n-k}}\big) \in p^{n+1} A$$

Computing modulo $p^n A$, we get :

$$\begin{aligned} w_n &\equiv W_n(a_0, \dots, a_{n-1}, 0) = W_{n-1}(a_0^p, \dots, a_{n-1}^p) \\ &\equiv \tau\big(W_{n-1}(a_0, \dots, a_{n-1})\big) = \tau(w_{n-1}) \qquad \pmod{p^n A} \end{aligned}$$

In the other way, if, for all $n$, $w_n - \tau(w_{n-1}) \in p^n A$, we can construct recursively $(a_n)$ in $A^{\mathbb{N}}$ such that $(w_0, \dots, w_n, \dots) = \mathcal{W}(a_0, \dots)$ :
- for $n = 0$, we set $a_0 = W(a_0) = w_0$,
- let suppose $a_0, \dots a_{n-1}$ satisfying the property are given. The congruence

$$\begin{aligned} w_n - W_{n-1}(a_0^p, \dots, a_{n-1}^p) &\equiv w_n - \tau\big(W_{n-1}(a_0, \dots, a_{n-1})\big) \\ &= w_n - \tau(w_{n-1}) = 0 \qquad \pmod{p^n A}, \end{aligned}$$

shows $a_n$ do exist in $A$ such that

$$w_n = W_{n-1}(a_0^p, \dots, a_{n-1}^p) + p^n a_n = W_n(a_0, \dots, a_n). \qquad \square$$

From property 1.3-3, we know that $A = \mathbb{Z}[X_0, Y_0, \dots, X_n, Y_n, \dots]$ is a pNR ring. It is easy to deduce the following fundamental theorem in Witt vectors theory.

THEOREM 1.7. *Let $\Phi$ be in $\mathbb{Z}[X, Y]$. for each $n \ge 0$ there exists a unique polynomial $\varphi_n$ in $\mathbb{Z}[X_0, \dots, X_n, Y_0, \dots, Y_n]$ such that*

$$W_n\big(\varphi_0(X_0, Y_0), \dots, \varphi_n(X_0, \dots, X_n, Y_0, \dots, Y_n)\big) = \Phi\big(W_n(X_0, \dots, X_n), W_n(Y_0, \dots, Y_n)\big).$$

Following [**12**] we state an important consequence of the proposition 1.6. It could be rather easily generalized both by using a pNR ring $A$ instead of $\mathbb{Z}$ and by supposing that $R$ (resp $Q$) belongs to $A[[x]]$ (resp. $xA[[x]]$).

PROPOSITION 1.8. *Let $Q$ and $R$ be in $\mathbb{Z}[x]$ and set $P(x) = x^p + p\,R(x)$. Then*
**1)** $\Big(Q\big(P^{\circ(0)}(x)\big), \ldots, Q\big(P^{\circ(n)}(x)\big), \ldots\Big) \in \mathcal{W}\Big(W\big(\mathbb{Z}[x]\big)\Big)$,
**2)** *If $Q(0) = R(0) = 0$, then* $\Big(Q\big(P^{\circ(0)}(x)\big),\ldots, Q\big(P^{\circ(n)}(x)\big),\ldots\Big) \in \mathcal{W}\Big(W\big(x\,\mathbb{Z}[x]\big)\Big)$.

PROOF. **1)** The ring $\mathbb{Z}[x]$ is pNR for $\tau(H)(x) = H(x^p)$. As $P(x) - x^p = pR(x)$ belongs to $p\,\mathbb{Z}[x]$, by properties 1.5-2 and 1.5-5, we can compute modulo $p^n\,\mathbb{Z}[x]$

$$Q\big(P^{\circ(n)}(x)\big) = Q\Big(P^{\circ(n-1)}\big(P(x)\big)\Big) \equiv Q\Big(P^{\circ(n-1)}(x^p)\Big) = \tau\Big(Q\big(P^{\circ(n-1)}(x)\big)\Big)$$

and we conclude using proposition 1.6.

**2)** For $Q$ and $R$ in $x\,\mathbb{Z}[x]$, then, for $n \geq 0$, $Q\big(P^{\circ(n)}(x)\big)$ belongs to $x\,\mathbb{Z}[x]$. It only remains to check recursively from (1.1) that if $(w_0, \ldots) = \mathcal{W}(a_0, \ldots)$ for some polynomials $w_n \in x\,\mathbb{Z}[x]$ then the $a_n$ itself belong to $x\,\mathbb{Z}[x]$. □

REMARK 1.9. Proposition 1.8, with $Q(x) = x$, gives an $\mathbf{a} \in W\big(\mathbb{Z}[x]\big)$ such that $\mathcal{W}(\mathbf{a}) = \big(P^{\circ(0)}(x), \ldots, P^{\circ(n)}(x), \ldots\Big)$. Then

$$\mathcal{W}\big(Q(\mathbf{a})\big) = \Big(Q\big(P^{\circ(0)}(x)\big), \ldots, Q\big(P^{\circ(n)}(x)\big), \ldots\Big).$$

Actually the key point is that if $R(0) = Q(0) = 0$ then $Q\Big(W\big(x\,\mathbb{Z}[x]\big)\Big) \subset W\big(x\,\mathbb{Z}[x]\big)$. A direct proof is possible but rather painstaking.

**1.3. Artin-Hasse exponential.** The following theorem contains a lot of congruences, the simplest one being $(p-1)! = -1 \pmod{p}$.

THEOREM 1.10. *The formal power series*

$$\mathrm{E}(x) := \exp\Big(\sum_{h=0}^{\infty} p^{-h} x^{p^h}\Big) = \sum_{n=0}^{\infty} \frac{1}{n!}\Big(\sum_{h=0}^{\infty} p^{-h} x^{p^h}\Big)^n = \sum_{s=0}^{\infty} \alpha_s x^s$$

*has coefficients $\alpha_i$ in $\mathbb{Z}_p \cap \mathbb{Q}$.*

PROOF. Let set $n = dp^h$ with $d$ prime to $p$. One finds

$$-\log(1-x) = \sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{(d,p)=1} \frac{1}{d} \sum_{h=0}^{\infty} p^{-h} x^{dp^h}.$$

Moebius inversion formula (namely $\sum_{di=n} \mu(i) = 0$ pour $n \geq 2$) gives :

$$\begin{aligned} -\sum_{(i,p)=1} \frac{\mu(i)}{i} \log(1-x^i) &= \sum_{(i,p)=(d,p)=1} \frac{\mu(i)}{di} \sum_{h=0}^{\infty} p^{-h} x^{dip^h} \\ &= \sum_{(n,p)=1} \sum_{di=n} \frac{\mu(i)}{n} \sum_{h=0}^{\infty} p^{-h} x^{np^h} = \sum_{h=0}^{\infty} p^{-h} x^{p^h} \end{aligned}$$

so that

$$\mathrm{E}(x) = \prod_{(i,p)=1} (1-x^i)^{-\mu(i)/i}.$$

For $a \in \mathbb{Z}_p \cap \mathbb{Q}$, the power series

$$(1-x)^{-a} = \sum_s^{\infty} \frac{a(a+1)\cdots(a+s-1)}{s!} x^s$$

having coefficients in $\mathbb{Z}_p \cap \mathbb{Q}$, we are done. $\square$

We now put together results of paragraphs 1.2 and 1.3. Hence the following final proposition will contain all underlying congruences.

Proposition–definition 1.11. *Let $A$ be a ring and let $\boldsymbol{a}$ be in $W(A)$. The power series*

$$\mathrm{E}(\boldsymbol{a}, x) \stackrel{\mathrm{def}}{=} \exp\Big(\sum_{n=0}^{\infty} W_n(\boldsymbol{a})\, p^{-n} x^{p^n}\Big)$$

*has coefficients in $A[\mathbb{Z}_p \cap \mathbb{Q}]$. Moreover :*

**1)** $\mathrm{E}(\boldsymbol{a}+\boldsymbol{b}, x) = \mathrm{E}(\boldsymbol{a}, x)\ \mathrm{E}(\boldsymbol{b}, x)$*,*

**2)** *if $V(\boldsymbol{a}) = (0, a_0, a_1, \ldots)$ then $\mathrm{E}\big(V(\boldsymbol{a}), x\big) = \mathrm{E}(\boldsymbol{a}, x^p)$.*

Proof. One computes

$$\begin{aligned}\mathrm{E}(\mathbf{a}, x) &= \exp\Big(\sum_{n=0}^{\infty}\sum_{i=0}^{n} a_i^{p^{n-i}} p^{i-n} x^{p^n}\Big) = \exp\Big(\sum_{i=0}^{\infty}\sum_{h=0}^{\infty} a_i^{p^h} p^{-h} x^{p^{h+i}}\Big) \\ &= \exp\Big(\sum_{i=0}^{\infty}\sum_{h=0}^{\infty} p^{-h}\,(a_i\, x^{p^i})^{p^h}\Big) = \prod_{i=0}^{\infty} \mathrm{E}\big(a_i\, x^{p^i}\big).\end{aligned}$$

and concludes by theorem 1.10.

**1)** is a straightforward consequence of the sum definition in $W(A)$, namely $W_n(\mathbf{a}+\mathbf{b}) = W_n(\mathbf{a}) + W_n(\mathbf{b})$. **2)** is deduced from $W_n\big(V(\mathbf{a})\big) = p\, W_{n-1}(\mathbf{a})$. $\square$

Remark 1.12. One has $\mathrm{E}(\mathbf{1}, x) = \mathrm{E}(x)$ for $\mathbf{1} = (1, 0, \ldots)$.

## 2. First order differential equations

Let us call exponential series any power series with a polynomial logarithmic derivative i.e. solution of a first order differential equation over the affine line. In particular $\mathrm{E}(\mathbf{a}, x)$ is an exponential series if and only if $\mathbf{a}$ is *finite* namely if $W_n(\mathbf{a}) = 0$ for $n$ big enough. In view of studying first order differential equations, our next task will be to construct finite Witt vectors.

Using a trick, due to Chinellato [**5**] and developped by Pulita [**12**], we obtain a family $\{\boldsymbol{\pi}_m\}_{m \geq 0}$ of "finite" Witt vectors, namely whose images $\mathcal{W}(\boldsymbol{\pi}_{\Updownarrow})$ have only finitely many non zero components. The exponential series $e_{p^m,\varpi} = \mathrm{E}(\boldsymbol{\pi}_m, x)$ are called primitive Robba exponentials and enjoy beautiful properties. The exponential series $e_{p^m,\varpi}(\lambda x^d)$, for $\lambda \neq 0$ and $(d, p) = 1$, are called twisted Robba exponentials. The point is that there is enough twisted Robba exponentials to write any exponential series as a finite product of them. In fact, working along increasing or decreasing powers, there are two such decompositions. Among other these decompositions are interesting because, for both, the radius of convergence of the product is the minimum of the radius of convergence of the factors. We begin giving criterions for such a situation to happen.

### 2.1. Radius of convergence of a product.

Definition 2.1. For $b$ and $\{a_i\}_{\geq 0}$ in some valued extension of $\mathbb{Q}_p$, the function $f(x) = \sum_{i=0}^{\infty} a_i(x-b)^i$ will be called a *power series near the point* $b$. Its *radius of convergence* $\mathtt{RoC}(f) \stackrel{\text{def}}{=} \liminf |a_i|^{-1/i}$ is the biggest $R$ (in $[0,\infty]$) such that $f$ do converge in the disk $D(b,R)$ i.e. such that $\lim |a_i|\, r^i = 0$ for any $r < R$.

Remark 2.2. A power series $f$ near $b$ is actually a power series near each point $c$ of the disk $D\big(b, \mathtt{RoC}(f)\big)$. Hence one can compute the radius of convergence of $f$ viewing it as a power series near any $c$. But that radius of convergence is independent of the point $c$. So we do not precise the point $b$ in the notation $\mathtt{RoC}(f)$.

Proposition 2.3. *Let $f(x)$ and $g(x)$ two power series near the same $b$, then*

$$\mathtt{RoC}(fg) \geq \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}.$$

Proof. Let us set $f(x) = \sum_{i=0}^{\infty} a_i(x-b)^i$ and $g(x) = \sum_{i=0}^{\infty} b_i(x-b)^i$. Then

$$fg(x) = \sum_{i=0}^{\infty} c_i\,(x-b)^i \qquad \text{with} \qquad c_n = \sum_{i=0}^{n} a_i\, b_{n-i}.$$

For $\rho < \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}$, one has

$$|c_n|\,\rho^n \leq \max_{0\leq i\leq n} |a_i|\,\rho^i\,|b_{n-i}|\,\rho^{n-i} \leq \max_{0\leq n} |a_n|\,\rho^n \ \max_{0\leq n} |b_n|\,\rho^n.$$

So $\lim |c_n|\, r^n = 0$ for any $r < \rho$ i.e. for any $r < \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}$. □

The example $f(x) = 1 - x$ , $g(x) = \sum_{i=0}^{\infty} x^i$ , $fg(x) = 1$ shows that it can happen that $\mathtt{RoC}(fg) > \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}$ even when $f$ and $g$ have distinct radii of convergence.

2.1.1. *First criterion.* We will give two criterions assuring that $\mathtt{RoC}(fg) = \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}$. The first one (proposition 2.5) was largely used by Robba even if he did not insist upon it. As far as we know, the second one (proposition 2.7) has never been explicitly stated. Both criterions involve a product of two functions but they can straightforwardly be extended to products of $n$ functions.

Definition 2.4. We will call *exponential series near* $b$ any power series $f$ near $b$ whose logarithmic derivative belongs to $K[x]$[3].

Proposition 2.5 (First criterion). *Let $f$ and $g$ be exponential series near the same $b$ with distinct radii of convergence. Then* $\mathtt{RoC}(fg) = \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}$.

Proof. An exponential series $f$ near $b$ has no zero in its disk of convergence because the relation $f' = P\,f$ with $P$ in $K[x]$ implies recursively that such a zero should be of infinite order. Then $1/f$ is also an exponential series near $b$ and $\mathtt{RoC}(1/f) = \mathtt{RoC}(f)$.

Let us suppose $\mathtt{RoC}(g) < \mathtt{RoC}(f)$.

One has $\mathtt{RoC}(fg) \geq \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\} = \mathtt{RoC}(g)$ but also

$$\mathtt{RoC}(f) > \mathtt{RoC}(g) = \mathtt{RoC}(fg\,\frac{1}{f}) \geq \min\big\{\, \mathtt{RoC}(fg)\,;\ \mathtt{RoC}(f)\big\} = \mathtt{RoC}(fg)$$

Hence $\mathtt{RoC}(fg) = \mathtt{RoC}(g) = \min\big\{\, \mathtt{RoC}(f)\,;\ \mathtt{RoC}(g)\big\}$. □

[3]One could more generally suppose it belongs to $\mathcal{H}(D)$ that would only imply to limit the radius of convergence to the radius of the biggest open disk centered in $b$ and included in $D$.

2.1.2. *Second criterion.* Let $f$ be an exponential series near $b$ and let $P \stackrel{\text{def}}{=} f'/f \in K[x]$ be its logarithmic derivative. For any $t$ in some valued extension $\Omega$ of $K$, let $f_t$ be the unique power series near $t$ such that $f_t' = P\, f_t$ and $f_t(t) = 1$ and set $R_f(t) \stackrel{\text{def}}{=} \texttt{RoC}(f_t)$.

The Taylor's formula can be written

$$f_t(x) = \sum_{i=0}^{\infty} \frac{1}{i!} P_i(t)(x-t)^i \qquad \text{where } P_0 = 1 \text{ and } P_{i+1} = P_i' + P_i\, P. \tag{2.1}$$

and gives

$$R_f(t) = \liminf \left|\frac{1}{i!} P_i(t)\right|^{-1/i}.$$

Hence $R_f(t)$ depends only on the Berkovich point associated to $t$ and $R_f$ is actually a function from $\mathfrak{A}$ (the quasi-polyhedron of the affine line) to $\mathbb{R}^{\geq 0}$.

In general the formula (2.1) cannot be used to do explicit computations. However this is possible when $|P(t)|$ is big enough[4]

$$\text{If} \qquad |P(t)| > r(t)^{-1} \qquad \text{then} \qquad R_f(t) = p^{-1/(p-1)} |P(t)|^{-1}. \tag{2.2}$$

When interested by the restriction of $R_f$ on the path $[b, \infty]$, one chooses, for each $\rho > 0$, a Dwork generic point $t_{b,\rho}$ such that

$$|t_{b,\rho} - b| = \rho = r(t_{b,\rho})$$

(this is always possible). From formula 2.1, one deduces easily that the function $\rho \mapsto R_f(t_{b,\rho})$ is continuous and logarithmically concave. With more pain it can be proved that it is also logarithmically piecewise affine with integral slopes.

For $\rho < \texttt{RoC}(f)$, both $f$ and $f_{t_{b,\rho}}$ are exponential series near $t_{b,\rho}$ and have the same logarithmic derivative. Hence $f_{t_{b,\rho}} = f/f(t_{b,\rho})$ and $R_f(t_{b,\rho}) = \texttt{RoC}(f)$. By concavity, the function $\rho \mapsto R_f(t_{b,\rho})$, being constant on the interval $[0, \texttt{RoC}(f)]$, is strictly decreasing on the interval $[\texttt{RoC}(f), \infty)$.

Definition 2.6. Let $f$ be an exponential series. We call *first slope* of $f$ and denote by $\texttt{Slo}(f)$ the "logarithmic" right derivative of $R_f$ in $\texttt{RoC}(f)$, namely the right derivative of the function $\alpha \mapsto \log_p\big(R_f(t_{b,p^\alpha})\big)$ computed at $\alpha = \log_p\big(\texttt{RoC}(f)\big)$. It is the biggest logarithmic slope of the function $\rho \mapsto R_f(t_{b,\rho})$ on $[\texttt{RoC}(f), \infty)$.

The first slope is a non positive integer.

Proposition 2.7 (Second Criterion). *Let $f$ and $g$ be exponential series near $b$. If $\texttt{RoC}(g) = \texttt{RoC}(f)$ but $\texttt{Slo}(f) \neq \texttt{Slo}(g)$ then $\texttt{RoC}(fg) = \texttt{RoC}(g) = \texttt{RoC}(f)$.*

Proof. We proceed by contraposition.

Let suppose $\texttt{RoC}(fg) > \texttt{RoC}(f) = \texttt{RoC}(g)$ and choose $\rho$ in the non empty interval $\big(\texttt{RoC}(f) = \texttt{RoC}(g), \texttt{RoC}(fg)\big)$. The functions $fg$ and $f_{t_{b,\rho}} g_{t_{b,\rho}}$ are both non zero power series in $t_{b,\rho}$ and have the same logarithmic derivative, hence differ only by a multiplicative constant. Then $\texttt{RoC}(f_{t_{b,\rho}} g_{t_{b,\rho}}) = \texttt{RoC}(fg)$.

On the other side, one has $\texttt{RoC}(1/f_{t_{b,\rho}}) = \texttt{RoC}(f_{t_{b,\rho}}) < \rho \leq \texttt{RoC}(f_{t_{b,\rho}} g_{t_{b,\rho}})$. From proposition 2.5, this implies

$$\texttt{RoC}(g_{t_{b,\rho}}) = \texttt{RoC}(f_{t_{b,\rho}}\, g_{t_{b,\rho}} \frac{1}{f_{t_{b,\rho}}}) = \min\big\{\texttt{RoC}(f_{t_{b,\rho}}\, g_{t_{b,\rho}})\,;\ \texttt{RoC}(f_{t_{b,\rho}})\big\} = \texttt{RoC}(f_{t_{b,\rho}}).$$

[4] The computation giving the formula (2.2) had been made independantly by numerous people among which one used to cite only Young.

Hence $R_f(t_{b,\rho}) = R_g(t_{b,\rho})$ for $\rho$ in $\big(\mathtt{RoC}(f), \mathtt{RoC}(fg)\big)$ and $\mathtt{Slo}(f) = \mathtt{Slo}(g)$. $\square$

The radius of convergence does not behave in a simple way under the "$p$-ramification" $x = y^p$ (actually this is the starting point of the fruitful Frobenius theory for $p$-adic differential equations). Clearly this induces yet more difficulties when studying the behaviour of slopes under ramification. Hence, for the sake of simplicity, we will limit ourself to exponential series near 0. The singular behavior under the $p$-ramification is already obvious in that simple case.

Proposition 2.8. *Let $f$ be an exponential series near* 0 *and let $n > 0$ be an integer. If $n = d\,p^m$ with $(d,p) = 1$, then $g(x) \stackrel{\text{def}}{=} f(x^n)$ is an exponential series near* 0*, $\mathtt{RoC}(g) = \mathtt{RoC}(f)^{1/n}$ and $\mathtt{Slo}(g) = d\big(\mathtt{Slo}(f) - 1\big) + 1$.*

Proof. As $g$ is converging for $|x^n| < \mathtt{RoC}(f)$, then $\mathtt{RoC}(g) = \mathtt{RoC}(f)^{1/n}$.

The disk $D(t^n_{0,\rho}, \rho^n)$ do not contain any point of $K^{\text{alg}}$. Then $t^n_{0,\rho}$ is a Dwork generic point associated to the disk $D(0, \rho^n)$. So we can set $t_{0,\rho^n} = t^n_{0,\rho}$ [5].
Now, by formula (2.1), $f_{t_{0,\rho^n}}(x) = \sum_{i=0}^\infty \frac{1}{i!} P_i(t^n_{0,\rho})\,(x - t^n_{0,\rho})^i$ and

$$g_{t_{0,\rho}}(x) = f_{t_{0,\rho^n}}(x^n) = \sum_{i=0}^\infty \frac{1}{i!} P_i(t^n_{0,\rho})\,(x^n - t^n_{0,\rho})^i.$$

Our aim is now to compute $\mathtt{RoC}(g_{t_{0,\rho}})$ and we will use the binomial formula :

$$x^n - t^n_{0,\rho} = (x - t_{0,\rho} + t_{0,\rho})^n - t^n_{0,\rho} = \sum_{i=1}^n \binom{n}{i} t^{n-i}_{0,\rho}\,(x - t_{0,\rho})^i$$

By definition of the first slope, $R_f(t_{0,\rho}) = \mathtt{RoC}(f)^{1-\mathtt{Slo}(f)}\rho^{\mathtt{Slo}(f)} < \mathtt{RoC}(f)$ for $\rho$ bigger and close to $\mathtt{RoC}(f)$. In particular, $R_f(t_{0,\rho})$ is then smaller and close to $\mathtt{RoC}(f)$. On the other side, the smallest index $i$ for which the binomial number $\binom{n}{i}$ is not divisible by $p$ is $i = p^m$. Then, for $|x - t_{0,\rho}| < |t_{0,\rho}| = \rho$ big enough, the maximum in the following formula is reached for $i = p^m$

$$|x^n - t^n_{0,\rho}| = \max_{1\le i\le n} \left|\binom{n}{i} t^{n-i}_{0,\rho}\,(x - t_{0,\rho})^i\right| = |t_{0,\rho}|^{n-p^m}\,|x - t_{0,\rho}|^{p^m}. \tag{2.3}$$

Then, for $\rho^n > \mathtt{RoC}(f)$ not too big, we see that $g_{t_{0,\rho}}$ converges for

$$\rho^{n-p^m}\,|x - t_{0,\rho}|^{p^m} = |x^n - t^n_{0,\rho}| \le \mathtt{RoC}(f_{t_{0,\rho^n}}) = \mathtt{RoC}(f)^{1-\mathtt{Slo}(f)}\rho^{n\,\mathtt{Slo}(f)}.$$

From $n = dp^m$ we finally get

$$\begin{aligned} R_g(t_{0,\rho}) &= \mathtt{RoC}(g_{t_{0,\rho}}) = \rho^{1-d}\,\mathtt{RoC}(f)^{p^{-m}(1-\mathtt{Slo}(f))}\,\rho^{d\,\mathtt{Slo}(f)} \\ &= \mathtt{RoC}(f)^{p^{-m}(1-\mathtt{Slo}(f))}\,\rho^{d(\mathtt{Slo}(f)-1)+1}. \end{aligned}$$

Hence $\mathtt{Slo}(g) = d\big(\mathtt{Slo}(f) - 1\big) + 1$ $\big[$ from $\mathtt{RoC}(f) = \mathtt{RoC}(g)^n$, we indeed check

$$\lim_{\rho\to\mathtt{RoC}(g)^+} R_g(t_{0,\rho}) = \mathtt{RoC}(g)^{d(1-\mathtt{Slo}(f))+d(\mathtt{Slo}(f)-1)+1} = \mathtt{RoC}(g)\ \big].$$

$\square$

Remark 2.9. When $m = 0$, all slopes of $g$ and $f$ are related in the same way as the first ones. When $m > 0$, the maximum in formula 2.3 is reached for $i = p^{m'}$ with $m' \le m$. It remains possible but less easy to compute $R_g$ knowing $R_f$.

[5] Basically that only means $|g(t_{0,\rho})| \stackrel{\text{def}}{=} |f(t^n_{0,\rho})| = \sup_{|x|<\rho^n} |f(x)|$ for any $f$ in $K[x]$.

Example 2.10. Let $f(x) = \exp(\pi x)$ and $g(x) = \exp(-\pi x^p)$. Then $\texttt{RoC}(f) = 1$ and $\texttt{Slo}(f) = 0$. The proposition 2.8 asserts $\texttt{RoC}(g) = \texttt{RoC}(1/f) = 1$ and $\texttt{Slo}(g) = \texttt{Slo}(1/f) - 1 + 1 = 0$ (this is a particular case of theorem 2.15). Actually it is well known that $\texttt{RoC}(fg) > 1$ (this is a particular case of theorem 2.21).

### 2.2. Robba exponentials.

Definitions 2.11.

- A polynomial $P$ is said to be Lubin-Tate if $P(x) - x^p - p\,x \in px^2\mathbb{Z}[x]$.
- Given a Lubin-Tate polynomial $P$, a Tate generator $\varpi$ is a sequence $(\pi_m)$ of *non zero integers* in some extension $K$ of $\mathbb{Q}_p$ such that

$$P(\pi_0) = 0 \quad \text{and} \qquad P(\pi_m) = \pi_{m-1} \quad \text{for } m \geq 1$$

We will set $\pi_m = 0$ for $m < 0$ so that $P(\pi_m) = \pi_{m-1}$ for all $m$ in $\mathbb{Z}$.

Remark 2.12. Looking at the Newton polygon of $P$, one checks $|\pi_m| = |\pi|^{p^{-m}}$ (the possible non-integer roots of $P$ are not taken into account).

Proposition–definition 2.13 ([**12**]). *Let $\varpi$ be a Tate generator and let $m \geq 0$ be an integer. There is a (unique) Witt vector $\boldsymbol{\pi}_m \in W\big(\pi_m\,\mathbb{Z}[\pi_m]\big)$ such that $\big(\pi_m, \pi_{m-1}, ..., \pi_0, 0, ...\big) = \mathcal{W}(\boldsymbol{\pi}_m)$. In other words $W_i(\boldsymbol{\pi}_m) = \pi_{m-i}$ for all $i \geq 0$. Then, for $Q \in x\,\mathbb{Z}[x]$, $Q(\boldsymbol{\pi}_m) = (\lambda_0, ...)$ with $\lambda_i \in \pi_m\,\mathbb{Z}[\pi_m]$.*

Proof. Let $P$ be the Lubin-Tate polynomial corresponding to $\varpi$. By proposition 1.8, $\big(P^{\circ(0)}(x), \dots, P^{\circ(n)}(x), \dots\big) \in \mathcal{W}\Big(W\big(\mathbb{Z}[x]\big)\Big)$. Specializing $x$ into $\pi_m$, from $P^{\circ(i)}(\pi_m) = \pi_{m-i}$ for $i \geq 0$, we get $\big(\pi_m, ..., \pi_0, 0..\big) \in \mathcal{W}\Big(W\big(\mathbb{Z}[\pi_m]\big)\Big)$.

Now $\Big(Q\big(P^{\circ(0)}(x)\big), \dots, Q\big(P^{\circ(n)}(x)\big), \dots\Big) \in \mathcal{W}\Big(W\big(x\,\mathbb{Z}[x]\big)\Big)$ (proposition 1.8-2) and specializes into $\mathcal{W}\big(Q(\boldsymbol{\pi}_m)\big)$. Hence $Q(\boldsymbol{\pi}_m) \in W\big(\pi_m\,\mathbb{Z}[\pi_m]\big)$. □

Definition 2.14. Let $\varpi$ be a Tate generator.

(1) For $m \geq 0$, the (primitive) *Robba exponential* of order $p^m$ is the power series

$$e_{p^m,\varpi}(x) \stackrel{\text{def}}{=} \mathrm{E}(\boldsymbol{\pi}_m, x) = \exp\big(\pi_m\,x + \pi_{m-1}\,p^{-1}x^p + \cdots + \pi_0\,p^{-m}x^{p^m}\big).$$

(2) For $n = d\,p^m$ with $(d, p) = 1$ and $m \geq 0$, the (derived) *Robba exponential* of order $n$ is the power series $\quad e_{n,\varpi}(x) \stackrel{\text{def}}{=} e_{p^m,\varpi}(x^d)$.

Theorem 2.15. *Let $\varpi$ be a Tate generator and let $n \geq 1$ be an integer. If $n = dp^m$ with $(d, p) = 1$, the Robba exponential $e_{n,\varpi}(x)$ belongs to $\mathbb{Z}_p[\pi_m][[x]]$, $\texttt{RoC}(e_{n,\varpi}) = 1$ and $\texttt{Slo}(e_{n,\varpi}) = 1 - n$. More precisely, the function $\rho \mapsto R_{e_{n,\varpi}}(t_{0,\rho})$ is logarithmically affine on the interval $[1, \infty)$.*

Proof. As $\boldsymbol{\pi}_m$ belongs to $W\big(\mathbb{Z}[\pi_m]\big)$, from proposition 1.11, the power series $e_{p^m,\varpi}$ belongs to $\mathbb{Z}[\pi_m][[x]]$ and the same is true for $e_{n,\varpi}(x) = e_{p^m,\varpi}(x^d)$. As $|\pi_m| < 1$, its coefficients are $p$-adic integers. Hence $\texttt{RoC}(e_{n,\varpi}) \geq 1$.

Let $P(x) = \pi_m + \cdots + \pi_0 x^{dp^m - 1}$ be the logarithmic derivative of $e_{n,\varpi}$. For $|x| = \rho$ big enough, one has $|P(x)|_\rho = |\pi_0|\,\rho^{n-1} > \rho^{-1}$. Then formula 2.2 asserts

$$R_{e_{n,\varpi}}(t_{0,\rho}) = p^{-1/(p-1)}|P(t_{0,\rho})|^{-1} = \rho^{1-n}.$$

On the other side $R_{e_{n,\varpi}}(t_{0,\rho}) = \mathtt{RoC}(e_{n,\varpi}) \geq 1$ for $\rho < 1$ (actually for $\rho \leq 1$) and the function $R_{e_{n,\varpi}}$ is logarithmically concave. These three properties force to have

$$R_{e_{n,\varpi}}(t_{0,\rho}) = \begin{cases} \rho^{1-n} & \text{if } \rho \geq 1, \\ 1 & \text{if } \rho \leq 1. \end{cases}$$

In particular $\mathtt{RoC}(e_{n,\varpi}) = \lim_{\rho\to 0} R_{e_{n,\varpi}}(t_{0,\rho}) = 1$. □

Remark 2.16. Let $f$ be an exponential function near 0, let $\lambda \neq 0$ and let $g(x) = f(\lambda x)$. Then $R_g(t) = R_f(\lambda t)$ and the logarithmic graphs of the functions $\rho \mapsto R_g(t_{0,\rho})$ and $\rho \mapsto R_g(t_{0,\rho})$ differ only by a translation of vector $(\log_p|\lambda|, 0)$. In particular they have the same logarithmic slopes.

Proposition 2.17. *Let $f(x) = \prod_n e_{n,\varpi}(\lambda_n x)$ a finite product of "twisted Robba exponentials". Then* $\mathtt{RoC}(f) = \min_n \mathtt{RoC}\left(e_{n,\varpi}(\lambda_n x)\right) = \min_n \left|\lambda_n\right|^{-1}$.

Proof. Twisted Robba exponentials are exponential series near 0. Then, if the minimum radius of convergence is reached for a unique index $n$, the result is a consequence of the proposition 2.5. If not, by theorem 2.15 and remark 2.16 we know that twisted Robba exponentials have distinct slopes and the result is a consequence of the proposition 2.7. □

When $K$ is algebraically closed, looking recursively at the monomial of higher degree, it is easy to write any exponential of a polynomial as a finite product of twisted Robba exponentials. By looking recursively at the monomial of lower degree one can get a decomposition such as given in the proposition 2.18. In spite of appearances that second decomposition is more natural than the first. Indeed, proposition 2.20 will show it is connected with Witt vectors and, over all, it uses no root and hence does not require $K$ to be algebraically closed (see example 2.19).

Proposition 2.18. *Let $f(x) = \prod_{(d,p)=1} \prod_{i=0}^{m(d)} e_{dp^{m(d)-i},\varpi}(a_{i,d}\, x^{p^i})$ be a finite product. Then* $\mathtt{RoC}(f) = \min_{d,i} \mathtt{RoC}\left(e_{dp^{m(d)-i},\varpi}(a_{i,d}\, x^{p^i})\right) = \min_{d,i} \left|a_{i,d}\right|^{-p^{-i}}$.

Proof. The function $f(x)$ is a finite product of exponential series. By proposition 2.8 and remark 2.16 the slopes of factors

$$\mathtt{Slo}\left(e_{dp^{m(d)-i},\varpi}(a_{i,d}\, x^{p^i})\right) = \mathtt{Slo}(e_{dp^{m(d)-i},\varpi}) = 1 - dp^{m(d)-i}$$

are distinct. By proposition 2.7, its radius of convergence is the minimum of the radius of convergence of the factors. □

Example 2.19. For the function $f(x) = \exp(\pi_0 a\, x + \pi_0 b\, x^p)$ one gets the two decompositions :

$$\begin{aligned} f(x) &= e_{1,\varpi}\Big(\big(a - \frac{\pi_1}{\pi_0}(pb)^{1/p}\big)\, x\Big) \quad e_{p,\varpi}\Big((pb)^{1/p}\, x\Big) \\ &= e_{p,\varpi}\Big(\big(\frac{\pi_0}{\pi_1}a\big)\, x\Big) \quad e_{1,\varpi}\Big(\big(b - \frac{1}{p}\frac{\pi_0^p}{\pi_1^p}a^p\big)\, x^p\Big). \end{aligned}$$

Setting $\alpha = \dfrac{\pi_0}{p^{1/p}\pi_1}$, propositions 2.17 and 2.18 give two expressions for $\mathtt{RoC}(f)$ :

$$\mathtt{RoC}(f)^{-1} = \max\Big\{\left|a - \alpha^{-1}b^{1/p}\right|, \left|pb\right|^{1/p}\Big\} = \max\Big\{\left|b - \alpha^p a^p\right|^{1/p}, \left|p^{1/p}\alpha a\right|\Big\}.$$

To verify directly they are the same, it is useful to remark $|\alpha - 1| < 1$ (see lemma 3.4) and $\left|B^p - A^p\right|^{1/p} = |B - A|$ for $\left|p(B - A)\right| < |B|$ (with $A = .\alpha a$ and $B = b$).

**2.3. Further properties.** The proposition 1.11 shows $\mathtt{RoC}\left(\mathrm{E}(\mathbf{a},x)\right)\geq 1$ but in general do not give information about the exact value of this radius of convergence. In the proof of the proposition 1.11, we get a decomposition of $E(\mathbf{a},x)$ in a product that is finite when $\mathbf{a}$ is a finite Witt vector. However our criterions do not apply to this product. We will use an other decomposition for which the proposition 2.18 do apply.

PROPOSITION 2.20. *Let $\varpi$ be a Tate generator, let $m\geq 0$ be some integer, let $A$ be a ring containing $\mathbb{Z}[\pi_m]$ and let $\boldsymbol{a}=(a_0,\ldots)$ be a Witt vector in $W(A)$. Then*

$$\mathrm{E}(\boldsymbol{\pi}_m\,\boldsymbol{a},x)=\prod_{i=0}^{m} e_{p^{m-i},\varpi}(a_i\,x^{p^i})$$

*and* $\mathtt{RoC}\left(\mathrm{E}(\boldsymbol{\pi}_m\boldsymbol{a},x)\right)=\min_{0\leq i\leq m}|a_i|^{p^{-i}}\geq 1$. *In particular,* $\mathtt{RoC}\left(\mathrm{E}(\boldsymbol{\pi}_m\boldsymbol{a},x)\right)>1$ *if and only if* $|a_i|<1$ *for* $0\leq i\leq m$.

PROOF. By definition, $W_i(\boldsymbol{\pi}_m\mathbf{a})=\pi_{m-i}\,W_i(\mathbf{a})$ for $0\leq i$. Then

$$\begin{aligned}\mathrm{E}(\boldsymbol{\pi}_m\,\mathbf{a},x) &= \exp\Big(\sum_{i=0}^{m}\pi_{m-i}\big(\sum_{j=0}^{i}p^j a_j^{p^{i-j}}\big)\;p^{-i}\,x^{p^i}\Big)\\ &= \exp\Big(\sum_{j=0}^{m}\sum_{h=0}^{m-j}\pi_{m-h-j}a_j^{p^h}\;p^{-h}\,x^{p^{h+j}}\Big)=\prod_{j=0}^{m}e_{p^{m-j},\varpi}(a_j\,x^{p^j}).\end{aligned}$$

Proposition 2.18 do compute $\mathtt{RoC}\left(\mathrm{E}(\boldsymbol{\pi}_m\mathbf{a},x)\right)$. $\square$

A basic starting point in the Dwork's works is that $\mathtt{RoC}\left(\exp(\pi x-\pi x^p)\right)>1$. The following theorem is a generalization of this fact.

THEOREM 2.21 ([**12**] theorem 2.5). $\mathtt{RoC}\left(\dfrac{e_{m,\varpi}(x)}{e_{m,\varpi}(x^p)}\right)>1$.

PROOF. As $\pi_{-1}=0$, we get :

$$\begin{aligned}\frac{e_{m,\varpi}(x)}{e_{m,\varpi}(x^p)} &= \exp\left(\pi_m x+\pi_{m-1}\frac{x^p}{p}+\ldots+\pi_0\frac{x^{p^m}}{p^m}-\pi_m x^p-\ldots-\pi_0\frac{x^{p^{m+1}}}{p^m}\right)\\ &= \exp(p\pi_{m+1}\,x)\;\exp\Big((\pi_m-p\pi_{m+1})x+(\pi_{m-1}-p\pi_m)\frac{x^p}{p}+\ldots\\ &\qquad \ldots+(\pi_0-p\pi_1)\frac{x^{p^m}}{p^m}+(\pi_{-1}-p\pi_0)\,\frac{x^{p^{m+1}}}{p^{m+1}}\Big)\end{aligned}$$

Let set $Q(x)=\dfrac{1}{x}\big(P(x)-xp\big)=x^{p-1}+px\,R(x)$ in such a way that

$$\pi_{k-1}-p\pi_k=P(\pi_k)-p\pi_k=\pi_k Q(\pi_k)=W_{m+1-k}\big(\boldsymbol{\pi}_{m+1}Q(\boldsymbol{\pi}_{m+1})\big).$$

Then

$$\frac{e_{m,\varpi}(x)}{e_{m,\varpi}(x^p)}=\exp(p\pi_{m+1}\,x)\;\mathrm{E}\big(\boldsymbol{\pi}_{m+1}Q(\boldsymbol{\pi}_{m+1}),x\big)$$

But $Q$ belongs to $x\,\mathbb{Z}[x]$ and proposition 2.13 gives $Q(\boldsymbol{\pi}_m)=(\lambda_0,\ldots)$ with $|\lambda_i|<1$. From proposition 2.20, we know that the radius of convergence of the function $\mathrm{E}(\boldsymbol{\pi}_m\,Q(\boldsymbol{\pi}_m),x)$ is strictly greater than 1. As $|p\pi_{m+1}|<|\pi_0|=\mathtt{RoC}(\exp)$, the same is true for the function $\exp(p\pi_{m+1}\,x)$. $\square$

Remark 2.22. Let $\varpi$ and $\varpi'$ be two Tate generators. Using an argument similar to those of theorem 2.21 and setting $Q(x) = \dfrac{1}{x}\big(P(x) - P'(x)\big)$, one proves $\mathtt{RoC}\left(e_{m,\varpi}(x)/e_{m,\varpi'}(x)\right) > 1$ .

## 3. Computation of the radius of convergence for first order differential equations

**3.1. General algorithm.** Given a first order differential equation $L(f) = f' - Pf = 0$ with $P$ in $K[x]$[6], our aim is to compute the radius of convergence $R_L(t)$ of its solutions near a point $t$ of some extension $\Omega$ of $K$. As already seen, this radius of convergence depends only on the Berkovich point and can be theoretically computed from formula 2.1. But, except when the formula 2.2 is available, the computation of the limit is far from easy.

We will use a more direct way : we choose a Tate generator $\varpi$ and suppose that $K$ contains the $\pi_m$. We propose two algorithms

*First algorithm.*

(1) Compute $f_t$ the power series near $t$ such that $L(f_t) = 0$ and $f_t(t) = 1$,
(2) Write $f_t$ as a product $\prod e_{n,\varpi}\big(\lambda_n(t)\,(x-t)\big)$,
(3) Compute the functions $t \mapsto |\lambda_n(t)|$ on the quasi-polyhedron $\mathfrak{A}$.

Then by the proposition 2.17

$$R_L(t) \overset{\text{def}}{=} \mathtt{RoC}(f_t) = \min_n \big|\lambda_n(t)\big|^{-1}. \tag{3.1}$$

Unfortunately, as seen in example 2.19, the $\lambda_n$ are, in general, sums of roots of polynomials. To compute their absolute values can be rather painful.

*Second algorithm.*

(1) Compute $f_t$ the power series near $t$ such that $L(f_t) = 0$ and $f_t(t) = 1$,
(2) Write $f_t$ as a product $\prod_{(d,p)=1}\prod_{i=0}^{m(d)} e_{dp^{m(d)-i},\varpi}\big(a_{i,d}(t)\,(x-t)^{p^i}\big)$,
(3) Compute the functions $t \mapsto |a_{i,d}(t)|$ on the quasi-polyhedron $\mathfrak{A}$.

Then by the proposition 2.18

$$R_L(t) \overset{\text{def}}{=} \mathtt{RoC}(f_t) = \min_{i,d} \big|a_{i,d}(t)\big|^{-p^{-i}}. \tag{3.2}$$

It is easy to check that the $a_{i,d}$ are in $K[t]$. The computation of the functions $t \mapsto |a_{i,d}(t)|$ and hence of the functions $t \mapsto \big|a_{i,d}(t)\big|^{-p^{-i}}$ will be explained in proposition 3.2. To more easily state the result we need first a definition.

Definition 3.1. Given a quasi-polyhedron $\mathfrak{P}$ and a subtree $\mathfrak{T}$, a function $\phi$ defined on $\mathfrak{P}$ is said to be *entirely determined by* $\mathfrak{T}$ if for all point $t$ in $\mathfrak{P}$ one has $\phi(t) = \phi(\widetilde{t})$ where $\widetilde{t}$ is the unique point of $\mathfrak{T}$ connected with $t$ in $\mathfrak{P}$.

Proposition 3.2. *Given a polynomial $P(x) = a\prod(x-c_i)$ in $K[x]$, the function $|P| : t \mapsto |P(t)|$ is entirely determined by the subtree whose leaves are the roots $c_i$. More precisely, the function $|P|$ is logarithmically affine on each edge with a slope equal to the number (with multiplicity) of roots connected to it.*

[6]The same technique can be applied when $P$ is in $K(x)$, or more generally in $\mathcal{H}(D)$ for some subset of the affine line. But extra complications appear that are out the scope of this article : firstly Fuchsian factors can occur, secondly the radius of convergence function is no longer concave and can have positive slopes.

PROOF. Straightforward for $P(t) = (x - c)$ and easily generalized. □

COROLLARY 3.3. *For any first order differential equation $L(f) = f' - P f = 0$ with $P$ in $K[x]$, the radius of convergence function $R_L$ is entirely determined by a subtree of $\mathfrak{A}$.*

In paragraph 3.3, we will compute explicitly an example and further significant properties of the function radius of convergence will be given in the remarks 3.5.

**3.2. Dwork's Tate generator.** In view of explicit computations we have to choose a particular Tate generator. There is two natural choices :

- $P(x) = x^p + px$. In that case $\pi_0$ is a so called "Dwork's $\pi$" and the functions $e_{n,\varpi}$ have been constructed by Robba in [**13**] lemme 10.8 (see also [**14**] theorem 13.2.1). For that construction, Robba used a very clever but indirect process and was not able to specify where Taylor's coefficients lie.
- $P(x) = (x+1)^p - 1$, then $\pi_m = \zeta_m - 1$ with $\zeta_0^p = 1$ and $\zeta_m^p = \zeta_{m-1}$, in particular $\zeta_m$ is a $p^m$-th root of unity. The corresponding functions $e_{p^m,\varpi}$ have been considered by Matsuda [**11**].

Let us do the first choice. The components of a Tate generator are defined by

(1) $\pi_0 \stackrel{\text{def}}{=} \pi$ is a root of $\pi^{p-1} + p = 0$. In particular $|\pi| = p^{-1/(p-1)}$.
(2) recursively, for each $m \geq 1$, $\pi_m$ is a root of the equation $\pi_m^p + p\pi_m = \pi_{m-1}$. Looking at the Newton polygon of the polynomial $P$ one gets

$$|\pi_m| = |\pi_{m-1}|^{1/p} = p^{-1/p^m(p-1)} \qquad \text{hence} \qquad \left|\frac{\pi_m}{\pi_{m-1}}\right| = p^{p^{-m}}.$$

The following lemma specifies, in some sense, the value of $\pi_m$.

LEMMA 3.4. *Let $\zeta_m$ be such that $\zeta_m^{p^m} = p$. Then, for $m \geq 1$*

$$\left|\zeta_m \frac{\pi_m}{\pi_{m-1}} + 1\right| = p^{-(p-1)p^{-m-1}}.$$

PROOF. Set $b_m := \zeta_m \dfrac{\pi_m}{\pi_{m-1}} + 1$.

• For $m = 1$ one finds $\left(\zeta_1 \dfrac{\pi_1}{\pi}\right)^p = p\dfrac{\pi - p\pi_1}{-p\pi} = -1 + p\dfrac{\pi_1}{\pi}$.
Then $b_1$ is a root of the polynomial

$$(x-1)^p + 1 - p\zeta_1^{-1}(x-1) = x^p - px^{p-1} + \cdots + (p - p\zeta_1^{-1})x + p\zeta_1^{-1}.$$

Looking at the Newton polygon of this polynomial, as asserted one gets

$$|b_1| = |p\zeta_1^{-1}|^{1/p} = p^{(-1+1/p)/p} = p^{-(p-1)p^{-2}}$$

• For $m \geq 2$, let us first remark that the proposition is independent of the choice of $\zeta_m$. Actually if $\zeta'_m$ is another $p^m$-th root of $p$ then (remark $p^2 > (p-1)^2$)

$$\left|(\zeta_m - \zeta'_m)\frac{\pi_m}{\pi_{m-1}}\right| = p^{-p^{-m+1}/(p-1) - p^{-m}} p^{p^{-m}} < p^{-(p-1)p^{-m-1}}.$$

Supposing that the lemma is true at order $m-1$ and that $\zeta_m^p = \zeta_{m-1}$, one computes

$$\left(\zeta_m \frac{\pi_m}{\pi_{m-1}}\right)^p = \zeta_{m-1}\frac{\pi_{m-1} - p\pi_m}{\pi_{m-2} - p\pi_{m-1}} = \zeta_{m-1}\frac{\pi_{m-1}}{\pi_{m-2}}\left(1 - p\frac{\pi_m}{\pi_{m-1}}\right)\left(1 - p\frac{\pi_{m-1}}{\pi_{m-2}}\right)^{-1}.$$

So $b_m$ is a root of the equation

$$(x-1)^p = (b_{m-1} - 1)(1 - \alpha_m) = -1 + b_{m-1} - \alpha_m - b_{m-1}\alpha_m \tag{3.3}$$

where

$$\begin{aligned}
|\alpha_m| &= \left|1-\left(1-p\frac{\pi_m}{\pi_{m-1}}\right)\left(1+\sum_{i=1}^{\infty}\left(p\frac{\pi_{m-1}}{\pi_{m-2}}\right)^i\right)\right| \\
&\leq \max\left\{\left|p\frac{\pi_m}{\pi_{m-1}}\right|;\ \left|p\frac{\pi_{m-1}}{\pi_{m-2}}\right|\right\} = \max\left\{p^{-1+p^{-m}}\ ;\ p^{-1+p^{-m+1}}\right\} \\
&= p^{-1+p^{-m+1}} < p^{-(p-1)p^{-m}} = |b_{m-1}|
\end{aligned}$$

(indeed $-p^m+p \leq -p^2+p < -p+1$). Now, looking at the Newton polygon of the equation (3.3), as asserted one gets

$$|b_m| = |b_{m-1}|^{1/p} = p^{-(p-1)p^{-m}}. \qquad \square$$

**3.3. Explicit computations for the equations** (0.1)**.** When it happens, as in this example, that the first algorithm leads to *polynomial* coefficients $\lambda_n$, the formula 3.1 is more easy to handle with than the formula 3.2 (look at example 2.19 to convince yourself). So we will use the first algorithm. The interested reader can do computations using the second algorithm. He will notice that useless clusters of $p$ roots do appear in the process. They disappear when taking the $p$-th root of the absolute value of the corresponding polynomial.

Let $b_1$ and $\zeta_1$ be defined as in the lemma 3.4 and its proof. Moreover
1) we will set $\mathrm{Exp}\pi(x) \stackrel{\mathrm{def}}{=} \exp(\pi x) = e_{1,\varpi}(x)$.
2) the parameter $a$ of equation (0.1) being fixed, we will define $b$, $\beta$ and $\delta$ by

$$b = b_1 - 1 - a \quad , \qquad |b| = |p|^{\beta} = p^{-\beta} \quad , \qquad \delta = \frac{1-\beta}{p-1}.$$

In particular,

- if $a = 0$ then $\beta = 0$ and $\delta = \dfrac{1}{p-1}$
- if $a = -1$ then $\beta = \dfrac{p-1}{p^2}$ and $\delta = \dfrac{1}{p-1} - \dfrac{1}{p^2}$
- if $|b| \leq p^{-1/p}$, then $\beta \geq \dfrac{1}{p}$ and $\delta \geq \dfrac{1}{p}$ .

Let us now apply the general algorithm.

**(1)** write the solution of equation (0.1) taking the value 1 in $t$ :

$$f_t(x) = \mathrm{Exp}\pi(x^p + ax - t^p - at).$$

**(2)** write $f_t(x)$ as a product of twisted Robba exponentials

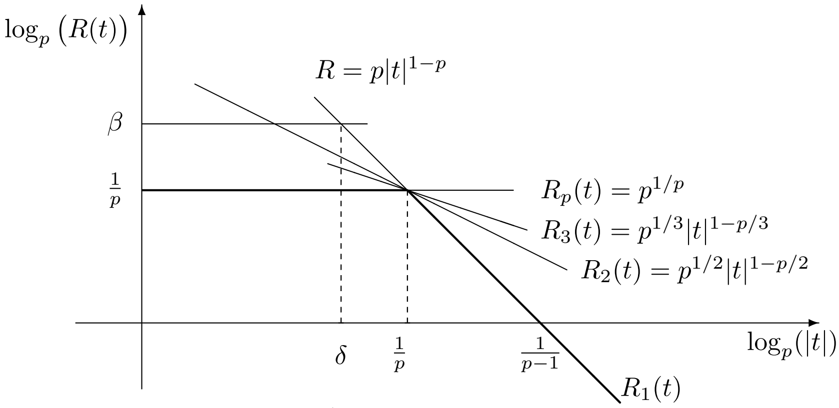
$$\begin{aligned}
f_t(x)) &= \mathrm{Exp}\pi\Big((x-t)^p + \sum_{n=1}^{p-1}\binom{p}{n}(x-t)^n t^{p-n} \quad + a(x-t)\Big) \\
&= \mathrm{Exp}\pi\big((x-t)^p + (b_1-1)(x-t)\big) \quad \prod_{n=2}^{p-1}\mathrm{Exp}\pi\Big(\binom{p}{n}t^{p-n}(x-t)^n\Big) \\
&\qquad \mathrm{Exp}\pi\big((-b_1+1+a+pt^{p-1})(x-t)\big) \\
&= e_{p,\varpi}\big(\zeta_1(x-t)\big) \prod_{n=2}^{p-1} e_{n,\varpi}\Big(\binom{p}{n}t^{p-n}(x-t)\Big) \quad e_{1,\varpi}\big((-b+pt^{p-1})(x-t)\big).
\end{aligned}$$

The proposition 2.17 says $R_L(t) \stackrel{\mathrm{def}}{=} \mathtt{RoC}(f_t) = \min_{1\leq n\leq p} R_n(t)$ with

$$R_p(t) = p^{1/p} \ , \quad R_n(t) = p^{1/n}|t|^{1-p/n} \ \text{ for } 2 \leq n < p \ , \quad R_1(t) = |-b+pt^{p-1}|^{-1}.$$

Only the case $n=1$ is non explicit and needs proposition 3.2 to be explicited. But already the simple $|-b+pt^{p-1}| = \max\{|b|, p|t|^{p-1}\}$ i.e. $R_1(t) = \min\{p^\beta, p|t|^{1-p}\}$ for $|t| \neq p^\delta$ (by definitions $p^\beta = p\left(p^\delta\right)^{1-p}$) is enough to get the (logarithmic) graph of the (continuous) function $R_L(t)$ on the subpath going from 0 to infinity (in other words viewing it as a function of $|t|$). The minimum of the $R_n(t)$ is then easily found using a picture. To draw it we have to distinguish two cases :

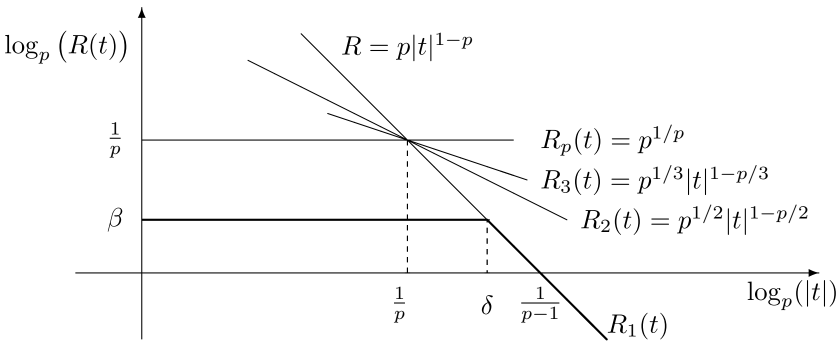
**A**.– $\beta \geq 1/p$ namely $\left|a - \zeta_1 \dfrac{\pi_1}{\pi}\right| \leq p^{-1/p}$.




Then $$R(t) = \begin{cases} p^{1/p} & \text{if } |t| \leq p^{1/p}, \\ p|t|^{1-p} & \text{if } |t| \geq p^{1/p}. \end{cases}$$

As the minimum is reached for $n=p$, there is no ambiguity for $|t| = p^\delta$.

**B**.– $\beta < 1/p$ namely $\left|a - \zeta_1 \dfrac{\pi_1}{\pi}\right| > p^{-1/p}$.

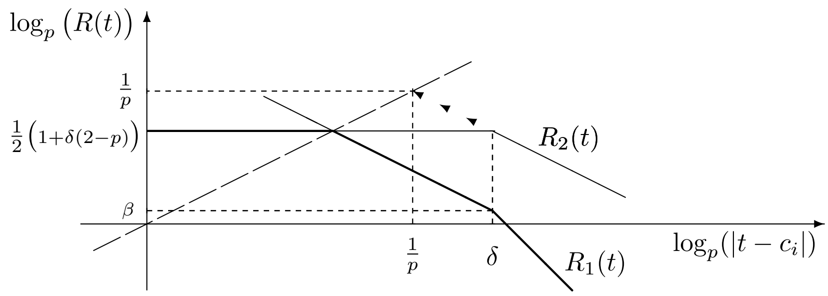



Then $$R(t) = \begin{cases} p^{\beta} & \text{if } |t| < p^{\delta}, \\ p|t|^{1-p} & \text{if } |t| > p^{\delta}. \end{cases}$$

To conclude when $|t| = p^\delta$, we have to know better the function $R_1(t)$, namely to use the third point of the algorithm.

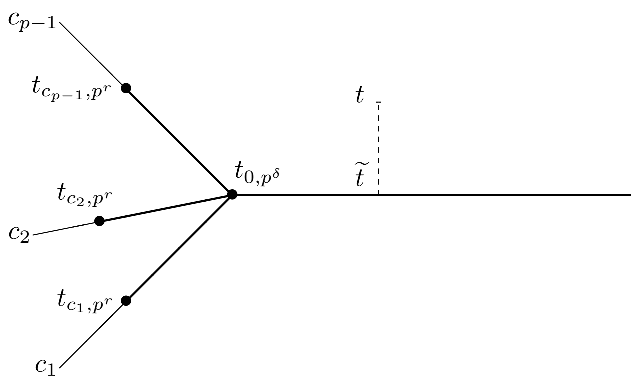
Let $c_i$ be the $p-1$ points such that $-b + pc_i^{p-1} = 0$. One cheks easily that $|c_i|^{p-1} = |p|^{\beta-1}$ i.e. $|c_i| = p^\delta$ and that the $p-1$ disks $D(c_i, p^\delta)$ are disjoints. The relation $R_1(t) = |-b+pt^{p-1}|^{-1} = \left|p\prod_{j=1}^{p-1}(t-c_j)\right|^{-1} = p\prod_{j=1}^{p-1}|t-c_j|^{-1}$ shows

• When $|t| = p^\delta$ but $t \notin \bigcup_i D(c_i, p^\delta)$ then $R_L(t) = R_1(t) = p^{1+\delta(1-p)} = p^\beta$.
• When $t \in D(c_i, p^\delta)$, for $j \neq i$, $|t - c_j| = p^\delta$ hence $R_1(t) = p^{1+\delta(2-p)}|t - c_i|^{-1}$.
We can now draw the picture giving the (logarithmic) graph of the (continuous) function $R(t)$ on the subpath going from $c_i$ to infinity (in other words viewing it as a function of $|t-c_i|$). For $|t-c_i| > p^\delta$, one has $|t| = |t-c_i|$ and the new graphs coincide with the former ones. Surprisingly the smallest function is $R_2(t) = p^{\big(1+\delta(2-p)\big)/2}$ for $|t - c_i|$ small enough. Only the graphs of $R_1(t)$ and $R_2(t)$ are drawn. There is an hidden subtility : the graphs of $R_1$ and $R_2$ meet on the line $R(t) = |t - c_i|$ confirming that $R_L(t)$ satisfies the property 3.5-2.




$$\text{Then} \qquad R_L(t) = \begin{cases} p^{\big(1+\delta(2-p)\big)/2} & \text{if } |t - c_i| \le p^{\frac{1}{2}\big(1-\delta(2-p)\big)}, \\ |t - c_i|^{-1}\, p^{1+\delta(2-p)} & \text{if } p^{\frac{1}{2}\big(1+\delta(2-p)\big)} \le |t - c_i| < p^\delta, \\ p|t - c_i|^{1-p} & \text{if } |t - c_i| > p^\delta. \end{cases}$$

Hence the function $R_L(t)$ is entirely determined by the following subtree $\mathfrak{T}$ of $\mathfrak{A}$ where we supposed the $c_i$ to be in $K$ and we set $r = \big(1 + \delta(2 - p)\big)/2$.




On the infinite edge the function $R_L(t) = p|t|^{1-p}$ has a (logarithmic) slope $1-p$ and on the finite edge (ending in $t_{c_i,p^r}$), the function $R_L(t) = |t - c_i|^{-1}\, p^{1-\delta(p-2)}$ has a (logarithmic) slope $-1$, confirming that $R_L(t)$ satisfies the property 3.5-2 (with our definition the slope in $t$ in direction of $c_i$ is $+1$).

Remarks 3.5.
• Computations remain true when replacing $b_1$ by any $b'$ such that $|b' - b_1| \le p^{1/p}$.
• The function $t \mapsto R_L(t)$ has two properties :

3.5.1. For any rigid point $c$ it is constant on the disk $D\big(c, R_L(c)\big)$. For instance the vertices of the tree $\mathfrak{T}$ are not in $c_i$ but in $t_{c_i,p^r}$.
3.5.2. Let $t$ be a point of type (2) of $\mathfrak{A}$ (for instance a vertex of $\mathfrak{T}$). For any rigid point $c$ such that $r(t) = |t-c|$, let us call (logarithmic) slope of the function $R_L$ at $t$ in direction of infinity (resp. of $c$) the right derivative (resp. the opposite of the left derivative) of the function $\alpha \mapsto \log_p \big(R_L(t_{c,p^\alpha})\big)$ at $\alpha = \log_p |t-c|$. Actually these slopes are integers, the slope in direction of infinity do not depend on the choice of $c$ and the slope in direction of $c$ depends only on the class of $c$ in $\delta(t)$. Then the sum of all slopes (indexed by $\delta(t) \cup \{\infty\}$) of $R_L$ at $t$ is 0 (almost all slopes at $t$ are 0, in particular non zero slopes are attached to edges of $\mathfrak{T}$).

The property 3.5.1 means that an analytic function has the same radius of convergence in each point of its disk of convergence.

The property 3.5.2 is much more subtle. It is a corollary of a deep Robba's theorem saying that the left (resp. right) derivative of the function $\alpha \mapsto \log_p \big(R_L(t_{c,p^\alpha})\big)$ at $\alpha = \log_p r$ is the index of the operator $L$ acting on the space of analytic functions in the disk $D(c, r)$ (resp. in the "closed" disk $D(c, r^+)$). One concludes easily observing that the closed disk is the disjoint union of open disks with the same radius. This property is often viewed as a kind of harmonicity.

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Université Pierre et Marie Curie, case 247, 4 place Jussieu, 75252 PARIS CEDEX 05

*E-mail address*: christol@math.jussieu.fr