# TOPOLOGICAL QUANTUM FIELD THEORY AND THE NIELSEN-THURSTON CLASSIFICATION OF $M(0,4)$

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Abstract. We show that the Nielsen-Thurston classification of mapping classes of the sphere with four marked points is determined by the quantum $SU(n)$ representations, for any fixed $n \geq 2$. In the Pseudo-Anosov case we also show that the stretching factor is a limit of eigenvalues of (non-unitary) $SU(2)$-TQFT representation matrices. It follows that at big enough levels, Pseudo-Anosov mapping classes are represented by matrices of infinite order.

Contents



## 1. Introduction

Quantum invariants of 3-manifolds and the associated 'Topological Quantum Field Theories' (TQFT) give rise to finite-dimensional representations of mapping class groups of surfaces. It is an obvious quest to seek a relationship between these representations and the Nielsen-Thurston theory of mapping classes.

The representations can be constructed using a number of different techniques. (See also [M2] for a survey.) On the combinatorial side one has the constructions of Reshetikhin and Turaev [RT1] and [RT2] based on the theory of quantum groups, and more generally by Turaev based on the theory of modular categories. Using skein theory there are the constructions of Blanchet, Habegger, Masbaum and Vogel [BHMV] based on the Kaufmann bracket and [Bl] using the HOMFLY-polynomial.

In the geometric approaches the representations come from (projectively) flat bundles over Teichmüller spaces. The construction of these bundles with their projectively flat connections was first given from the point of view of conformal field theory by Tsuchiya, Ueno and Yamada [TUY], [U1] and [U2] using the theory of representations of affine Lie algebras. In [AU1], [AU2] and [AU3] (in preparation), the first and last author are currently working to establish that these representations are isomorphic to the ones constructed by the skein methods of [BHMV]. Using the method of geometric quantization of moduli spaces of flat connections one gets yet another construction, which is due to Axelrod, Della Pietra and Witten [ADW] and Hitchin [H].[1] In fact Teichmüller space naturally parametrizes Kähler structures on the moduli space of flat connections and the fiber of the bundle over a point in Teichmüller space is the geometric quantization of the moduli space with respect to the corresponding Kähler structure.

Motivated by the above quest, the first author analyzed in [A2] (and in [A1]) the behavior of the family of Kähler structures on the moduli space near the Thurston boundary of Teichmüller space and found many new polarizations on the moduli spaces in this way. The relevance of this being that the Thurston boundary of Teichmüller space has the following two fundamental properties, namely it compactifies Teichmüller spaces to a topological closed ball and the action of the mapping class group extends to a continuous action on the compactification, so therefore any mapping class has a fixed point in Thurston's compactification of Teichmüller space. Points on the boundary are represented by projective classes of measured foliations on the surface and in fact by the Nielsen-Thurston classification, a mapping class which is neither finite order nor reducible, is a Pseudo-Anosov mapping class, that is, it preserves two transverse foliations. These two foliations support unique (up to scale) transverse measures and the Pseudo-Anosov mapping class scales one by the so-called stretching factor and the other by one over the stretching factor.

This leaves us with the following two very natural questions (see also problem 8.11 formulated by the second author in Ohtsuki's problem list [Oh]):

**Question 1.1.**
*(1) Can one determine the Nielsen-Thurston classification of mapping classes using quantum representations?* [†]
*(2) If so, can one compute the stretching factor for Pseudo-Anosov mapping classes from the quantum representations?*

The main result of this paper is to give a positive answer to these questions in the case of the mapping class group $M(0,4)$ of a surface of genus zero with four marked points.

**Theorem 1.2.** *For every $n \geq 2$, the Nielsen-Thurston classification of mapping classes in $M(0,4)$ are determined by the quantum $SU(n)$ representations. Moreover in the Pseudo-Anosov case, these representations also determine the stretching factor.*

This will be proved in Section 4, using the geometric approach to the representations. We also note an analog of the asymptotic faithfulness result of [A3] (see also [FWW]) in this case (Corollary 4.3). In Section 5, we discuss the skein-theoretical approach (although for simplicity in the $SU(2)$ case only) and obtain the following sharpening of our result:

**Corollary 1.3.** *A mapping class $\phi \in M(0,4)$ is Pseudo-Anosov if and only if its $SU(2)$-TQFT representation matrix has infinite order for large enough level $k$. Moreover, if $\phi$ is Pseudo-Anosov with stretching factor $\lambda(\phi)$, then for $k$ big enough*

[1] That these bundles are isomorphic to the ones constructed from conformal field theory is due to Laszlo [La1].

[†] *Note added in proof:* For an affirmative answer to Question 1.1.(1) see [A4].

*the $SU(2)$-TQFT matrix $\rho_k^{(S)}(\phi)$ has a unique eigenvalue $\lambda_k$ such that $|\lambda_k| > 1$, and*

$$\lim_{k\to\infty} |\lambda_k| = \lambda(\phi) \ .$$

Here, we must choose the roots of unity in the definition of the $SU(2)$-TQFT appropriately (in the second statement only). We remark that for this choice, the TQFT is non-unitary (for $k$ big enough). See Section 5 for details.

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## 2. A motivating example, train tracks, and open questions.

Let $\Sigma = \Sigma_{0,4}$ be a surface of genus zero with four marked points $p_1, p_2, p_3, p_4$. We denote its mapping class group $M(0,4)$ simply by $M$. Thus $M$ is the group of path components of the group of orientation preserving self-homeomorphisms of $\Sigma$ which map the set of marked points to itself.

By blowing up each marked point to a boundary circle, we obtain a compact surface $\Sigma_b$ with 4 boundary components. We can also think of $M$ as the isotopy classes of orientation preserving self-homeomorphisms of $\Sigma_b$ where neither the homeomorphisms nor the isotopies are required to be the identity on the boundary. This point of view is the natural one for describing the measured foliations associated to pseudo-Anosov mapping classes (see *e.g.* [FLP]).

We have the following well known presentation of $M$ in terms of the standard generators of the braid group $B_4$ (see Fig. 1 below).

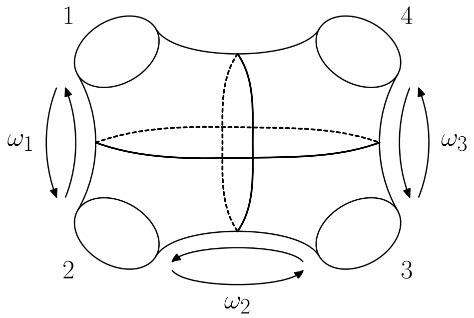



Figure 1. The braid generators $\omega_i$ and the Penner system.

**Theorem 2.1.** [B, Theorem 4.5] *The mapping class group $M = M(0,4)$ has a presentation with generators $\omega_1, \omega_2, \omega_3$ and relations*

$$\begin{aligned} \omega_1\omega_3 &= \omega_3\omega_1 \\ \omega_1\omega_2\omega_1 &= \omega_2\omega_1\omega_2 \\ \omega_2\omega_3\omega_2 &= \omega_3\omega_2\omega_3 \\ \omega_1\omega_2\omega_3^2\omega_2\omega_1 &= 1 \\ (\omega_1\omega_2\omega_3)^4 &= 1 \end{aligned}$$

**Example.** In [P] Penner gave a method for constructing Pseudo-Anosov mapping classes from certain systems of curves. We consider the system of two curves illustrated in Fig. 1 each of which cuts $\Sigma$ into two pairs of pants. By Penner's construction this system gives the train track $\tau$ on $\Sigma$ illustrated in Fig. 2.

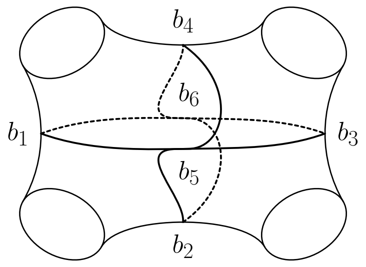



FIGURE 2. The train track $\tau$.

This train track has 6 branches and we label them $b_1$ to $b_6$ as indicated in Fig. 2. A transverse measure $\mu$ on the train track $\tau$ is an assignment $\mu(b_i) = \mu_i$, of non negative real numbers to the branches of $\tau$, which satisfies the switch conditions. Hence we see that $\mu_5$ and $\mu_6$ are determined by $\mu_1, \dots, \mu_4$ and that they have to satisfy

$$\mu_1 + \mu_2 = \mu_3 + \mu_4.$$

The mapping classes $\omega_1^{-1}$, $\omega_2$ and $\omega_3^{-1}$ preserve $\tau$ and give the following incidence matrices

$$I(\omega_1^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \ I(\omega_2) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \ I(\omega_3^{-1}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

Suppose now $\phi \in M$ is represented by any positive word $w$ in $\omega_1^{-1}, \omega_2$ and $\omega_3^{-1}$, which contains at least one $\omega_2$ and one $\omega_1^{-1}$ or one $\omega_3^{-1}$. Then $\phi$ preserves the train track $\tau$ and its incidence matrix $I(\phi) = I(w)$ is Perron-Frobenius, so that all such $\phi \in M$ are Pseudo-Anosov. [2] In particular, $I(\phi)$ has a unique maximal eigenvalue $\lambda > 1$ which is the stretching factor of $\phi$.

Now consider the representation matrices of $\phi$ in TQFT. As we will prove, $\lambda$ is a limit of (absolute values of) eigenvalues of such matrices. We can therefore ask:

**Question 2.2.** *What is the relation between (any of) the constructions of TQFT representations and measured train tracks?*

At the time of this writing, we don't have a good answer to this question. Our proofs exploit, instead, the fact that the Nielsen-Thurston classification of $M(0, 4)$ is in some sense homological; for example, the stretching factor of $\phi$ can be computed from the action of $\phi$ on the homology of the double cover of the sphere branched over the four marked points. (See Section 3.)

[2] In fact any Pseudo-Anosov mapping class in $M$ is conjugate to one of this form modulo the translation subgroup $N$ (see Section 3). This follows from Matsuoka's classification of Pseudo-Anosov conjugacy classes in the braid group on 3 strands [Mat].

While appropriate for $M(0,4)$, this reasoning does not generalize to higher genus, as the stretching factor of a Pseudo-Anosov mapping class in general cannot be computed by homological means. This leads us to mention the following related questions which have puzzled us for some time.

**Question 2.3.** *1. Can one make sense of factorizing quantum representations along a measured train track?*

*2. Can one define the space of conformal blocks on a surface equipped with a measured foliation as opposed to a conformal structure?*

*3. Does a generic measured foliation determine a Lagrangian foliation on the moduli space of flat connections?*

*4. What is the dynamics of these flat bundles over Teichmüller space with respect to the flow on Teichmüller space induced by a measured foliation?*

We end this section with the following

**Conjecture 2.4.** *A Pseudo-Anosov mapping class (on a surface of arbitrary genus) is represented in TQFT by matrices of infinite order, except for finitely many values of the level $k$.*

## 3. The Nielsen-Thurston classification of mapping classes in $M(0,4)$.

In this section, we review the Nielsen-Thurston classification of mapping classes in $M = M(0,4)$ and remark that it is determined by the "homology representation" $h : M \to \mathrm{PSL}_2(\mathbb{Z})$.

Let $\pi : \widetilde{\Sigma} \to \Sigma$ be the double branched cover of the sphere $\Sigma$ ramified at the four marked points. We can choose an identification of $\widetilde{\Sigma}$ with the 2-torus $T = \mathbb{R}^2/\mathbb{Z}^2$ such that the covering transformation is minus the identity acting on the torus. Recall that the mapping class group of the torus, $M(1,0)$, is $\mathrm{SL}_2(\mathbb{Z})$. We can describe the mapping class group $M = M(0,4)$ in a similar fashion.

Given a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ and a vector $v \in (\frac{1}{2}\mathbb{Z})^2 \subset \mathbb{R}^2$, the transformation

$$x \mapsto Ax + v$$

of $\mathbb{R}^2$ defines a diffeomorphism of $T = \mathbb{R}^2/\mathbb{Z}^2$ which commutes with $-\mathrm{Id}$. We denote by $\phi_{A,v}$ the induced diffeomorphism of $\Sigma = T/(-\mathrm{Id})$.

**Theorem 3.1.** *All mapping classes in $M$ are represented by diffeomorphisms of the form $\phi_{A,v}$.*

*Proof.* It suffices to check that the braid generators $\omega_i$ (see Theorem 2.1) can be given in this way. Indeed, we can represent them as

$$\omega_i = \phi_{A_i,v_i} \qquad (i = 1,2,3)$$

where

$$A_1 = A_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},\ A_2 = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix},\ v_1 = \begin{pmatrix} 1/2 \\ 0 \end{pmatrix},\ v_2 = \begin{pmatrix} 0 \\ 1/2 \end{pmatrix},\ v_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

□

**Remark 3.2.** In this description, the four ramification points $p_1, p_2, p_3, p_4$ are represented by the points in $\mathbb{R}^2$ with coordinates $\binom{1/2}{0}, \binom{0}{0}, \binom{0}{1/2}, \binom{1/2}{1/2}$, respectively.

Of course, $A$ and $v$ are not uniquely determined by $\phi_{A,v}$. Analyzing the ambiguity one obtains the following

**Corollary 3.3.** *We have a short exact sequence*

$$1 \to N \to M \xrightarrow{h} \mathrm{PSL}_2(\mathbb{Z}) \to 1 \ ,$$

*where* $h(\phi_{A,v}) = \pm A \in \mathrm{PSL}_2(\mathbb{Z})$*, and*

$$N \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \ .$$

We call $h$ the *homology representation* of $M$, because $\pm A$ is already determined by the action on the homology of the torus $T$. We also call $N$ the *translation subgroup* of $M$.

*Proof of Corollary 3.3.* Let $M'$ be the group of diffeomorphisms (not considered up to isotopy) of the form $\phi_{A,v}$. It is easy to see that the corollary holds with $M'$ in place of $M$. It remains to see that $M' = M$. In other words, we must show that two elements of $M'$ are isotopic if and only if they coincide. This can be checked by looking at the action on the homology of $T$ (recall this is given by $h$) and how the four ramification points are permuted. □

**Remark 3.4.** It is clear by elementary topological considerations that any mapping class in $M$ lifts to a self-homeomorphism of $T$, well-defined up to the covering transformation. As the mapping class group of the torus is $\mathrm{SL}_2(\mathbb{Z})$, this provides an alternative definition of $h : M \to \mathrm{PSL}_2(\mathbb{Z})$ which does not invoke Theorem 3.1.

**Remark 3.5.** One can check that $M$ is a semi-direct product of $\mathrm{PSL}_2(\mathbb{Z})$ with $N$, the action of $\mathrm{PSL}_2(\mathbb{Z})$ on $N$ being the natural one *via* the homomorphism $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{Z}/2\mathbb{Z})$. Birman [B, section 5.4] discusses a similar semi-direct product decomposition starting from the presentation of $M$ given in Theorem 2.1.

We recall that a matrix $A \in \mathrm{SL}_2(\mathbb{Z})$ (or an element $\pm A \in \mathrm{PSL}_2(\mathbb{Z})$) is called *Anosov* if and only if

$$|\,\mathrm{Tr}(A)| > 2 \ .$$

In that case, the matrix $A$ has real eigenvalues (w.l.o.g.) $\lambda > 1$ and $\lambda^{-1}$, and for any choice of $v \in (\frac{1}{2}\mathbb{Z})^2$, the associated mapping class $\phi_{A,v}$ is pseudo-Anosov with stretching factor $\lambda$. We briefly recall the construction. The action of $A$ on $T = \mathbb{R}^2/\mathbb{Z}^2$ preserves two linear foliations with irrational slope, which are also preserved by the translation by $v$. The induced foliations on $\Sigma$ have singularities at the four ramification points, but give rise to the right kind of measured foliation on the blown up surface with boundary, $\Sigma_b$. (Compare [FLP, p. 217].[3] )

Conversely, if $\phi = \phi_{A,v}$ is pseudo-Anosov, it is easy to see that $h(\phi) = \pm A$ must be Anosov. The following lemma is now almost obvious.

**Lemma 3.6.** *A mapping class* $\phi \in M$ *is finite order, reducible or Pseudo-Anosov if and only if* $h(\phi) \in \mathrm{PSL}_2(\mathbb{Z})$ *respectively is finite order, reducible or Anosov. Moreover, for a pseudo-Anosov mapping class* $\phi$ *we have that the stretching factor* $\lambda(\phi)$ *is* $|\lambda|$*, where* $\pm\lambda$ *is the (real) eigenvalue of the matrix* $\pm h(\phi)$ *such that* $|\lambda| > 1$.

*Proof.* First observe that $\phi \in M$ is finite order if and only if $h(\phi)$ is, since $N = \ker(h)$ is finite. The preceding discussion shows that $\phi$ is pseudo-Anosov if and only if $h(\phi)$ is Anosov. By exclusion according to the Nielsen-Thurston classification of diffeomorphisms (see e.g. [FLP]), the lemma follows. □

[3] But notice that contrary to [FLP] we allow permutations of the ramification points.

## 4. The quantum $SU(n)$ representations determine the homology representation

Choose a projective tangent vector at each of the marked points $p_i \in \Sigma$. Let $\widetilde{M}$ be the group of components of the group of orientation preserving self-diffeomorphisms of $\Sigma$ which map the set consisting of the four chosen projective tangent vectors to itself. Then $\widetilde{M}$ is a quotient of the ribbon braid group on four strands $RB_4$. We have of course also a surjection $\widetilde{M} \to M$ whose kernel is generated by the Dehn-twists around the four marked points. Let $\sigma_1, \sigma_2$ and $\sigma_3$ be the standard (i.e. no twists in the bands that make up the $\sigma_i$) lifts to $RB_4$ of $\omega_1, \omega_2$ respectively $\omega_3$. Pick a fifth point with a projective tangent vector in the complement of the four marked points on $\Sigma$. Denote this point $\infty$ and let $\Sigma'$ be the resulting surface with these five marked points. Then we see $RB_4$ as the subgroup of the ribbon mapping class group of $\Sigma'$ which fixes $\infty$.

Let $k \geq 2$ be an integer and consider the $SU(n)$-TQFT at level k. The label set of this theory is the set of level $k$ integrable highest weight representations of the affine Lie algebra associated to the Lie algebra of $SU(n)$, which is indexed by the set of Young diagrams with less than $n$ rows and less than or equal to $k$ columns. There is also an involution $\dagger$ of this set, which takes the dual representation. - This means in particular that if we label the marked points of $\Sigma'$ by such diagrams, we get a finite dimensional vector space associated to this surface. If we choose the same label on the first four marked points, then $RB_4$ acts on this vector space. We now choose the label $\square$ at four marked points of $\Sigma$ and $\boxplus^{\dagger}$ (the dual of $\boxplus$) at $\infty$. For $SU(2)$ this diagram is not in the label set, but in this case it represents the trivial representation. Let $V_{n,k}$ denote the resulting vector space and $\tilde{\rho}_{n,k}$ the resulting representation of $RB_4$. For the construction of $V_{n,k}$ and $\tilde{\rho}_{n,k}$ in terms of conformal blocks we refer to [TUY], [U2] and [K]. For the skein theory construction we refer to [BHMV] and [B]. In this section we will focus on the geometric definition.

Let $q = e^{2\pi i/(k+n)}$.

As described in [K] (and [KT] for the $SU(2)$-case) there is the path basis for $V_{n,k}$ indexed by Young tableaux on $\boxplus$ . There are exactly two such tableaux, namely

$$\begin{array}{|c|c|}\hline 3 & 4 \\ \hline 1 & 2 \\ \hline \end{array} \quad \text{and} \quad \begin{array}{|c|c|}\hline 2 & 4 \\ \hline 1 & 3 \\ \hline \end{array}\,.$$

Scale the vector corresponding to the first tableau by $B^{1/2}$ and the other vector by $B^{-1/2}$, where $B = \sqrt{q^3 + q^2 + q}$. From the explicit formulae in [K] one then gets in this rescaled path basis that

$$\tilde{\rho}_{n,k}(\sigma_1) = \tilde{\rho}_{n,k}(\sigma_3) = q^{-\frac{n+1}{2n}} \begin{pmatrix} q & 0 \\ 0 & -1 \end{pmatrix}$$

$$\tilde{\rho}_{n,k}(\sigma_2) = q^{-\frac{n+1}{2n}} \frac{1}{1+q} \begin{pmatrix} -1 & q^3+q^2+q \\ 1 & q^2 \end{pmatrix}.$$

Moreover, each of the boundary parallel Dehn-twists (around each of the first four points) acts by $q^{\frac{n+1}{2n}}$ times the identity.

We will see (by the computation below) that $\tilde{\rho}_{n,k}$ is actually a projective representation of our original mapping class group $M$. Here we use the following definition of a projective representation of a group given by generators and relations: it is an assignment of matrices to the generators such that the relations are satisfied up to scalar multiples of the identity matrix.

**Remark 4.1.** In the case $n = 2$ we know *a priori* that $\rho_{2,k}$ defines at least a projective representation of $\widetilde{M}$, and, hence, one of $M$ (since the kernel of the projection $\widetilde{M} \to M$ acts by scalar matrices). This is because the label at $\infty$ is trivial for $SU(2)$.

Let us now relate $\tilde{\rho}_{n,k}$ to the homology representation $h$. First, we conjugate $\tilde{\rho}_{n,k}$ by

$$C = \begin{pmatrix} 1 & q^2+q+1 \\ 0 & q+1 \end{pmatrix}$$

to get upper and lower triangular matrices, as follows:

$$C\tilde{\rho}_{n,k}(\sigma_1)C^{-1} = C\tilde{\rho}_{n,k}(\sigma_3)C^{-1} = q^{-\frac{n+1}{2n}} \begin{pmatrix} q & -(q^2+q+1) \\ 0 & -1 \end{pmatrix}$$

$$C\tilde{\rho}_{n,k}(\sigma_2)C^{-1} = q^{-\frac{n+1}{2n}} \begin{pmatrix} q & 0 \\ 1 & -1 \end{pmatrix}.$$

Motivated by this, we consider the following matrices $A, B$ with coefficients in the ring of Laurent polynomials $\mathbb{Z}[s, s^{-1}]$:

$$A = s^{-1} \begin{pmatrix} s^2 & -(s^4+s^2+1) \\ 0 & -1 \end{pmatrix}, \qquad B = s^{-1} \begin{pmatrix} s^2 & 0 \\ 1 & -1 \end{pmatrix}.$$

Then one can check that $ABA = BAB$ and $(AB)^3 = 1$, so that

$$\rho(\omega_1) = \rho(\omega_3) = A, \qquad \rho(\omega_2) = B$$

defines a linear representation $\rho : M \to \mathrm{GL}_2(\mathbb{Z}[s, s^{-1}])$. We now define

$$\rho_{n,k}(\omega_i) = q^{1/2n} C\tilde{\rho}_{n,k}(\sigma_i)C^{-1} ,$$

then, by construction, we have

$$\rho_{n,k}(\omega_i) = \rho(\omega_i)|_{s=q^{1/2}} .$$

Thus $\rho_{n,k}$ is a linear representation of $M$. It is also clearly equivalent, as a projective representation, to the original $\tilde{\rho}_{n,k}$ . Moreover, since Laurent polynomials are determined by their values at infinitely many points, the rescaled TQFT representations $\rho_{n,k}$ determine $\rho$ uniquely and *vice versa.* (Note that it is enough to consider them for a fixed $n$.) We therefore call $\rho$ the universal quantum representation of $M$.

It remains to relate $\rho$ to the homology representation $h$. For this, we evaluate at $s = i$:

$$\rho(\omega_1)|_{s=i} = \rho(\omega_3)|_{s=i} = i \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \qquad \rho(\omega_2)|_{s=i} = i \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}.$$

Thus, we see that $\rho$ evaluated at $s = i$ is, up to powers of $i$, equal to the homology representation $h : M \to \mathrm{PSL}_2(\mathbb{Z})$. Another way to state this is that $\rho$ evaluated at $s = i$ followed by the projection $\mathrm{GL}_2(\mathbb{C}) \to \mathrm{PSL}_2(\mathbb{C})$ is the same as $h$ followed by the inclusion $\mathrm{PSL}_2(\mathbb{Z}) \to \mathrm{PSL}_2(\mathbb{C})$.

Since $h$ determines the Nielsen-Thurston classification (Lemma 3.6), we have proved:

**Theorem 4.2.** *The Nielsen-Thurston classification of a mapping class $\phi \in M = M(0,4)$ is determined, for every $n \geq 2$, by the collection of quantum representations $\rho_{n,k}(\phi)$ $(k \geq 1)$. Moreover, in the Pseudo-Anosov case, these representations also determine the stretching factor.*

An immediate consequence is that for all $n$, we have

$$\bigcap_{k=1}^{\infty} \ker(\rho_{n,k}) = \ker \rho = \ker h = N\ .$$

We can strengthen this statement by considering the associated projective representations, which we will now consider as homomorphisms $M \to \mathrm{PSL}_2(\mathbb{C})$ and denote by $\rho^P_{n,k}$.

**Corollary 4.3.** *For all $n \geq 2$, we have that*

$$\bigcap_{k=1}^{\infty} \ker(\rho^P_{n,k}) = N.$$

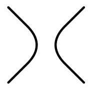
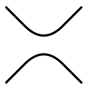
This is the appropriate analog of the asymptotic faithfulness theorem of the first author in [A3] (see also Freedman, Walker, and Wang [FWW] for a proof in the skein-theoretical context).

*Proof.* If $\phi$ is in the kernel of all $\rho^P_{n,k}$, then the off-diagonal entries of all matrices $\rho_{n,k}(\phi)$ are zero. It follows that the same is true for $\rho(\phi)$, hence also for $h(\phi)$, proving that $h(\phi) = 1$. □

## 5. The homology representation as a limit of quantum representations

In the preceding section, we have exhibited the homology representation $h$ as the evaluation at $s = i$ of the universal quantum representation $\rho$ with coefficients in $\mathbb{Z}[s, s^{-1}]$. Observe that the evaluation $s = i$ corresponds to $q = -1$. If we consider the skein-theoretical construction of $\rho_{n,k}$ instead of the construction via conformal blocks, we can take for $q$ not just $e^{2\pi i/(k+n)}$ but any root of unity of order $k+n$. In particular, we can choose a sequence of roots of unity which converges to $-1$ as the level goes to infinity, and thereby exhibit the homology representation $h$ as a limit of the quantum representations. This has interesting applications (see Theorem 5.3 and Corollaries 5.5, 5.6, and 5.8). For simplicity, we restrict to the $SU(2)$-case, using the skein-theoretical approach to the $SU(2)$-TQFT of [BHMV].

The skein-theoretical analog of the representation $V_{2,k}$ is the Kauffman bracket skein module of a 3-ball with four marked points colored 1 on the boundary. It can be defined over the Laurent polynomial ring $\mathbb{Z}[A, A^{-1}]$. If one then specializes $A$ to be a primitive $(4k+8)$-th root of unity, one gets exactly the $SU(2)$-TQFT vector space at level $k$. It has a standard basis given by the tangles

$h_0 =$  $v_0 =$ 

As usual, the lines in this figure stand for bands (or ribbons), and the four boundary points must also be thought of as “banded” points (*i.e.,* small intervals). (This corresponds to the projective tangent vectors used in the geometric construction.)

We represent a generator $\sigma_i$ of $\widetilde{M}$ by the corresponding obvious half-twist of the 3-ball, corrected by small half-twists near the two banded points permuted by $\sigma_i$ so as to preserve the band structure at these points. We can arrange the correction so that $\sigma_1$ and $\sigma_3$ leave the tangle $h_0$ fixed, and $\sigma_2$ leaves the tangle $v_0$ fixed. This gives a representation of $\widetilde{M}$ which we denote by $\tilde{\rho}^{(S)}$ (where the $S$ stands for *skein* *theory*). It will be convenient to write our matrices in the basis $(A^{-1}h_0, Av_0)$. Then a Kauffman bracket calculation gives the following matrices:

$$\tilde{\rho}^{(S)}(\sigma_1) = \tilde{\rho}^{(S)}(\sigma_3) = \begin{pmatrix} 1 & -A^4 \\ 0 & -A^4 \end{pmatrix} \ , \quad \tilde{\rho}^{(S)}(\sigma_2) = \begin{pmatrix} -A^4 & 0 \\ -1 & 1 \end{pmatrix} .$$

We omit the details, as the calculation is similar to the one in [M1], where the second author already computed the matrix for the mapping class $\sigma_1^{-2}\sigma_2^2$ (and showed that this matrix has infinite order in the $SU(2)$-TQFT representations at level $k$ except if $k = 1, 2, 4$, or $8$).

This $\tilde{\rho}^{(S)}$ is only a projective representation of $M$, but if we define

$$\rho^{(S)}(\omega_i) = A^{-2}\tilde{\rho}^{(S)}(\sigma_i) \ ,$$

we get a linear representation of $M$. Moreover, $\rho^{(S)}$ evaluated at $A^2 = -i$ coincides with $\rho$ evaluated at $s = i$, where $\rho$ is the universal quantum representation we considered in the geometric context. Recall that this specialization is essentially the homology representation $h$.

Let us now use this to exhibit $h$ as a limit of $SU(2)$-TQFT representations. We need the following simple lemma.

**Lemma 5.1.** *For every integer $r \geq 3$ there exists an integer $\ell = \ell(r)$ such that* $\gcd(l, 4r) = 1$ *and* $|2\ell - r| \leq 4$.

*Proof.* Take $\ell = (r+2)/2$ (respectively $(r+1)/2$, $(r+4)/2$, $(r-1)/2$) according to whether $r$ is congruent to 0 (respectively $1, 2, 3$) modulo 4. $\square$

Let $A_k = e^{-2\pi i\ell/(4k+8)}$ where $\ell = \ell(k+2)$. This is a primitive $(4k+8)$-th root of unity so by the construction of [BHMV] the specialization $A = A_k$ gives rise to a TQFT.

**Remark 5.2.** For this choice of root of unity (and big enough $k$), this TQFT is not unitary. In other words, the natural non-degenerate hermitian form on the TQFT vector spaces is (in general) indefinite. Compare [BHMV, Remark 4.12].

Let us denote by $\rho_k^{(S)}$ the corresponding representation of $M$ as described above. Everything has been done so that $A_k^2$ converges to $-i$ as $k \to \infty$. Thus for every mapping class $\phi \in M$, the limit of the matrices $\rho_k^{(S)}(\phi)$ exists and is equal to a power of $i$ times $h(\phi)$. This proves the following

**Theorem 5.3.** *For every mapping class $\phi \in M = M(0, 4)$, we have*

$$\lim_{k\to\infty} |\operatorname{Tr}\rho_k^{(S)}(\phi)| = |\operatorname{Tr} h(\phi)| \ .$$

**Remark 5.4.** We may replace $\rho_k^{(S)}$ by the original projective representation $\tilde{\rho}_k^{(S)}$ in this theorem, as long as we agree that $\tilde{\rho}_k^{(S)}(\phi)$ means any matrix obtained by writing $\phi$ as a word in the generators and applying $\tilde{\rho}_k^{(S)}$ letter by letter to this word. This is because the only projective ambiguities of $\tilde{\rho}_k^{(S)}$ are roots of unity, so that one has $|\operatorname{Tr}\tilde{\rho}_k^{(S)}(\phi)| = |\operatorname{Tr}\rho_k^{(S)}(\phi)|$.

The following corollary generalizes the result of [M1].

**Corollary 5.5.** *Let $\phi \in M = M(0,4)$ be Pseudo-Anosov. Then no power of the matrix $\rho_k^{(S)}(\phi)$ is a multiple of the identity matrix, except possibly for finitely many values of $k$.*

*Proof.* Observe that the matrices $\rho_k^{(S)}(\omega_i)$ have determinant $-1$. Thus if a power of the matrix $\rho_k^{(S)}(\phi)$ is a multiple of the identity, then that multiple is $\pm 1$ or $\pm i$. Therefore a further power of $\rho_k^{(S)}(\phi)$ is the identity, hence the eigenvalues of $\rho_k^{(S)}(\phi)$ must be roots of unity and thus $|\operatorname{Tr}\rho_k^{(S)}(\phi)| \leq 2$. But for $k$ big enough this is impossible by our theorem, since $\phi$ Pseudo-Anosov implies $|\operatorname{Tr} h(\phi)| \geq 3$. □

**Corollary 5.6.** *A mapping class $\phi \in M = M(0,4)$ is Pseudo-Anosov if and only if its $SU(2)$ TQFT representation matrix $\rho_k^{(S)}(\phi)$ has infinite order for large enough level $k$.*

*Proof.* It suffices to show that reducible mapping classes are represented by matrices of finite order. This is easily checked, as every reducible mapping class is a conjugate of a power of $\omega_1$ times an element of the translation subgroup $N \cong (\mathbb{Z}/2\mathbb{Z})^2$ (this follows from the description in Corollary 3.3). □

**Remark 5.7.** Corollaries 5.5 and 5.6 are algebraic statements, and therefore true also for the level $k$ TQFT where $A$ is an arbitrary primitive $(4k+8)$-th root of unity.

In a similar vein, we have the following corollary (here we assume that $A$ is the root $A_k$ as defined above).

**Corollary 5.8.** *Let $\phi \in M = M(0,4)$ be Pseudo-Anosov with stretching factor $\lambda(\phi)$. Then for $k$ big enough $\rho_k^{(S)}(\phi)$ has a unique eigenvalue $\lambda_k$ such that $|\lambda_k| > 1$, and*

$$\lim_{k\to\infty} |\lambda_k| = \lambda(\phi) \ .$$

*Proof.* The eigenvalues of $\rho_k^{(S)}(\phi)$ converge to a power of $i$ times those of $h(\phi)$. Since $|\operatorname{Tr} h(\phi)| = \lambda(\phi) + \lambda(\phi)^{-1}$, the result follows. □

Note that we cannot obtain such a result from a unitary TQFT (such as the one we get if we put $A = e^{2\pi i/(4k+8)}$) because there the eigenvalues of the matrices $\rho_k^{(S)}(\phi)$ would have unit norm.

**Remark 5.9.** A similar story exists for the mapping class group $M(1,1)$ [M3].

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