THE KONTSEVICH INTEGRAL AND MILNOR'S INVARIANTS

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Abstract. A formula for computing the Milnor (concordance) invariants from the Kontsevich integral is obtained.

The reduced Kontsevich integral (with values in the quotient by all loop diagrams) is shown to be the universal concordance invariant of finite type.

Some applications are discussed.

Contents

Introduction

In [M1, M2], Milnor defined invariants of links, known as the Milnor $\overline{\mu}$ (mu-bar) invariants. In fact, these invariants are not universally defined, *i.e.*, if the Milnor invariants of order less than n do not vanish, then those of order n are either not defined, or at best, they have indeterminacies. On the other hand, finite-type (or Vassiliev) invariants [Va, Gu, Ko, B2] are universally defined invariants (*i.e.*, for all links). Thus the naively posed question "Are Milnor's invariants of finite type?" is not well-posed.

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In [HL1], the notion of string link was introduced, together with the philosophy that Milnor's invariants are actually invariants of string links. Indeterminacies are determined precisely by the indeterminacy of representing a link as the closure of a string link. This philosophy led to the classification of links up to (link) homotopy.

From this point of view, the well-posed question "Are Milnor's invariants of string links of finite type?" was answered affirmatively by D. Bar-Natan in [B1] and by X. S. Lin in [L]. (See also remarks 11.3 and 12.8.)

Since the Milnor invariants are derived from (expansions of) the longitudes of a link, and since the longitudes determine (and are determined by) the Artin representation of a string link (each generator is conjugated by its longitude), one may (in the philosophy of [HL1]) *define* the 'Universal' Milnor invariant of a string link to be its Artin representation. This point of view was continued in [HL2], where the true indeterminacy of Milnor's concordance $\overline{\mu}$ invariants was determined.

In the present paper we address the relationship between Milnor's concordance invariants (of string links), *i.e.*, the Artin representation, and the Kontsevich integral. Since finite type invariants are defined dually to the Vassiliev filtration, and since the Kontsevich integral is the universal finite type invariant, they should be calculable directly from the Kontsevich integral.

One of the main results of this paper is that this is possible. Specifically, in Theorem 6.1, we show the following. First, we prove that the filtration on string links induced by Milnor's invariants coincides with what we call the Z^t-filtration, that is, the filtration induced by a certain quotient Z^t of the Kontsevich integral, obtained by killing all diagrams containing loops. (The t in Z^t stands for *tree* diagrams, which generate the quotient.) And second, we give a combinatorial formula to compute the first non-vanishing Milnor invariants of a string link σ from the first non-vanishing term of $Z^t(\sigma) - 1$. Although the corresponding result in the case of Milnor's homotopy invariants is nearly a tautology, and certainly suggested by [B1], this formula does not seem to have been discovered elsewhere.

The combinatorial formula given in Theorem 6.1 generalizes to a formula (see Theorem 12.7) which expresses the full Magnus expansion of the longitudes in terms of the Kontsevich integral. (This formula grew out of a discussion with D. Bar-Natan in January 1997.) In principle, this formula allows one to calculate all the Milnor invariants, not just the first non-vanishing ones, although its explicit form for the higher order invariants depends on a certain unipotent automorphism (see remark 12.8).

The idea that Milnor's invariants are related to tree diagrams had been around before (the second-named author learned about it from D. Bar-Natan and L. Rozansky in summer 1995). From the physics viewpoint, a connection between Milnor invariants and Feynman diagrams which are trees seems to have appeared for the first time in L. Rozansky's work [R]. However, it is remarkable that already before that, one had the 'Bing doubling' construction of T. Cochran [Co2], realizing 'realizable' Milnor invariants, starting from trees.

Since Milnor's invariants are concordance invariants, the above suggests that the reduced Kontsevich integral Z^t might be the universal finite type concordance invariant. We show that this is indeed the case. While for string links this may be deduced from the above relationship between Z^t and Milnor's invariants (see remark 14.3), taking this approach for general tangles would require a generalization of the structure theorem of [HL2]. There is, however, a direct proof which we give in Theorems 14.1 and 15.1. This proof relies on the combinatorial computation of the Kontsevich integral [Ca, B3, LM1, LM2, KT]. For this we were particularly influenced by T. Le and J. Murakami's point of view.

A consequence of this is that there are no rational finite type concordance invariants of string links other than Milnor's invariants and their products (corollary 6.4). (We understand that this result, which was originally conjectured by X. S. Lin, has also been obtained by T. Stanford and D. Thurston [ST].) This generalizes a result of K. Y. Ng [Ng], who showed that there are no non-trivial rational finite type concordance invariants of knots. In particular, finite type concordance invariants miss the 'transfinite' concordance invariants of J. Y. Le Dimet [D] and T. Cochran and K. Orr [CO].

The Kontsevich expansion Z takes values in rational vector spaces generated by diagrams and it is an open problem to determine its image on links or string links. This image is *a priori* not contained in the lattice generated by diagrams (since Z has rational coefficients). Our formula allows us to determine explicitly the lattice given by the possible values of the first non-vanishing term of the reduced Kontsevich integral Z^t. See remark 8.2 for the answer.

Another application of theorem 6.1 is that the Kontsevich integral of a link with vanishing Milnor invariants can be expressed as a sum of treeless diagrams.

There are several known constructions of links with given Milnor invariants [O, HL2, Co1, Co2]. One of the byproducts of our considerations about Milnor invariants is a construction of string links whose first non-vanishing term in its Kontsevich expansion is an arbitrarily given integer linear combination of connected diagrams. This 'realization' result is obtained using a procedure, pioneered by T. Stanford [S], which modifies a link by a pure braid. In particular (at least rationally), this construction also produces links with given Milnor invariants.

In the appendix, as an application of our methods, we give a new and different proof of the fact, shown recently by D. Bar-Natan [B4], that the Vassiliev filtration of the group ring of the pure braid group (that is, the filtration induced by the Vassiliev filtration on string links), coincides with the 'horizontal' Vassiliev filtration.

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1. Review of basic concepts and notation.

Let I denote the unit interval $[0, 1]$ and let $\partial \mathrm{I} = \{0, 1\}$ be its boundary.

By *tangle*, we mean a (smooth) proper embedding $T\colon X \to \mathbb{C} \times \mathrm{I}$, where X is a (smooth) compact 1-manifold with boundary, ∂X, such that ∂X is embedded in $\mathbb{R} \times \partial \mathrm{I}$. By the *type* of the tangle, we mean the underlying manifold X, together with the embedding of ∂X. It is customary to often confuse T with the image $T(X)$, but in general, it will be useful to keep X at our disposal.

For $i = 0, 1$, we set $\partial_i X = T^{-1}(\mathbb{C} \times \{i\})$. One could impose the condition that $T(\partial_i X)$ be a standard set of points, but this is not necessary, as only the order of these points is actually important (see section 3).

One may also wish to think in terms of *tangle diagrams*, *i.e.*, (properly) immersed curves in $\mathbb{R} \times \mathrm{I}$, together with over and under crossing data. One obtains a tangle diagram by generic projection. See figure 1 for an example.

If $X = \coprod_{j=1}^{j=l} S^1$, the tangle is called an l-component *link*. A 1-component link is also called a *knot*.

Fix a collection of points $p_1 < \ldots < p_l$ in $\mathbb{R} \subset \mathbb{C}$. By the *trivial* l-component string link, we mean the tangle $1_l \colon \coprod_{j=1}^{j=l} \mathrm{I}_j \to \mathbb{C} \times \mathrm{I}$, where $1_l(t) = (p_j, t), t \in \mathrm{I}_j$. By an *$l$-component string link*, we mean any tangle of the same type (see above). We shall often use the notation σ for string links.

Note that by definition, the components of links and string links are ordered.

A tangle is a *braid* if the tangent vectors always have a non-zero component in the vertical direction. (We consider I as the vertical direction and $\mathbb{C}$ as a horizontal plane.) A braid which is a string link is called a *pure* braid.

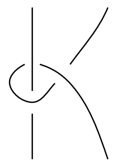


Figure 1. A string link which is not isotopic to a pure braid.

Let $T \colon X \to \mathbb{C} \times \mathrm{I}$ be a tangle for which $T(\partial_0 X) = T(\partial_1 X)$, under the identification of $\mathbb{C} \times \{0\}$ with $\mathbb{C} \times \{1\}$. One defines the *closure* of T to be the link defined on the space obtained from X by identifying $\partial_0 X$ with $\partial_1 X$, by identifying $\mathbb{C} \times \{0\}$ with $\mathbb{C} \times \{1\}$ to obtain $\mathbb{C} \times S^1$ and embedding this in the standard way in $\mathbb{C} \times \mathrm{I}$.

A *concordance* between tangles T_0 and T_1 of the same type is an embedding $H \colon X \times \mathrm{I} \to \mathbb{C} \times \mathrm{I} \times \mathrm{I}$, such that $H(x, i) = (T_i(x), i)$, and such that for $x \in \partial X$, $H(x, t) = (T_0(x), t)$, for all $t \in \mathrm{I}$. A concordance is an *isotopy* precisely when the t-level is preserved (*i.e.*, $H(x, t) = (T_t(x), t)$, where for $0 \le t \le 1, T_t \colon X \to \mathbb{C} \times \mathrm{I}$). An isotopy β_t, $0 \le t \le 1$, of braids is said to be *horizontal* if for all t, β_t is a braid.

By a *homotopy* between tangles (sometimes referred to as a *link homotopy*, but we won't use this terminology here), one means a homotopy rel. boundary in the usual sense, subject to the condition that distinct components always have disjoint images. In other words, one allows strands of the *same* component to cross one another (during a generic regular homotopy through immersions), but one does not allow strands of different components to cross one another.

By a *framed tangle*, we mean a tangle equipped with a non-vanishing normal vector field, such that the restriction to the boundary is the restriction of a fixed unit vector field normal to $\mathbb{R}$ in $\mathbb{C}$. Isotopy and concordance of framed tangles are defined in the obvious way. Two framed tangles will be considered to be homotopic if their underlying tangles are homotopic.

Note. The terminology 'framing' is slightly abusive here, as the non-vanishing normal vector field does not quite determine a trivialization of the normal bundle, unless X is oriented (in which case a second section may be determined using the orientation of $\mathbb{C}\times \mathrm{I}$). Note that for a knot, an isotopy class of framing may be specified by an integer, known as the self-linking number or writhe.

Convention. All tangles, links, and string links, will be considered to be *framed*. A concordance of tangles will also be assumed to be framed. Pure braids will be equipped with the zero framing (*i.e.*, the framing inducing the zero framing on the closure of each component).

The set of isotopy classes of framed tangles form a category $\mathcal{T}$, where the objects are finite sets of points in $\mathbb{R}$ (equipped with the standard framing), and composition is given by (vertical) stacking. Composition[1] will be denoted by $(T_1, T_2) \mapsto T_1T_2$. (Our convention is to put the tangle T_2 above the tangle T_1. The domain of the tangle T_1T_2 will be denoted X_1X_2. It is obtained from the disjoint union of X_1 and X_2 by identifying $\partial_1 X_1$ with $\partial_0 X_2$.) The tangle category has a monoidal (or tensorial) structure[2] given by the juxtaposition of tangles (*i.e.*, one puts one tangle to the right of the other), denoted by $(T_1, T_2) \mapsto T_1 \times T_2$. The set of isotopy classes of l-component framed string links will be denoted by $SL(l)$. Note that string links need not have inverses, so that $SL(l)$ is not a group but only a monoid.

By Artin's theorem [A], two pure braids which are isotopic as string links are already isotopic as braids, *i.e.*, via a horizontal isotopy. Thus isotopy classes of l-component pure braids form a subgroup $PB(l)$ of $SL(l)$. The inverse of a pure braid σ is $\overline{\sigma} = R \circ \sigma \circ r$, where R is the reflection through a horizontal plane (*i.e.*, $R\colon \mathbb{C}\times \mathrm{I} \to \mathbb{C}\times \mathrm{I}$ is given by $R(x,t) = (x, 1-t)$), and r is reflection at the source (*i.e.*, the self-map of $\coprod_{j=1}^{j=l} \mathrm{I}_j$ which sends t_j to $1-t_j$).

Concordance classes of tangles also form a monoidal category $\mathcal{CT}$. The set of framed concordance classes of l-component framed string links will be denoted by $CSL(l)$. It is well-known that $CSL(l)$ is a group. The inverse of a string link σ up to concordance is again represented by $\overline{\sigma}$ (see for example [D]). See figure 10 in section 14 for an example.

Similarly, homotopy classes of tangles form a monoidal category $\mathcal{HT}$. Homotopy classes of l-component string links form a group $HSL(l)$, with $\overline{\sigma}$ again inverse to σ. It is well-known that any string link is homotopic to a pure braid (see [HL1]), and that concordance of string links implies homotopy [Gi, Go] (see also [H]), so that $HSL(l)$ is a quotient of $PB(l)$ and of $CSL(l)$. (Not every string link is concordant to a pure braid.)

Finally, since isotopy of tangles implies concordance, and concordance implies homotopy, we have forgetful functors, $\mathcal{T} \to \mathcal{CT}$ and $\mathcal{CT} \to \mathcal{HT}$.

Note. In [HL2], the notation $SL(l)$ and $CSL(l)$ is used for the unframed version of our $SL(l)$ and $CSL(l)$.

The Vassiliev filtration. Let X be a tangle type, that is, a 1-manifold whose boundary ∂X is embedded in $\mathbb{R}\times\partial\mathrm{I}$. We denote by $\mathbb{Q}\mathcal{T}(X)$ the $\mathbb{Q}$ vector space with basis the set $\mathcal{T}(X)$ of framed embeddings of X in $\mathbb{C}\times\mathrm{I}$, extending the embedding on the boundary, up to isotopy rel. boundary. The analogous vector spaces with isotopy replaced with concordance or homotopy are denoted by $\mathbb{Q}\mathcal{CT}(X)$ and $\mathbb{Q}\mathcal{HT}(X)$.

[1] Two tangles T_1, T_2 are composable, if the map T_1 restricted to $\partial_1 X_1$ agrees with the map T_2 restricted to $\partial_0 X_2$, considered as taking values in $\mathbb{C}$.

[2] See [M] for generalities about monoidal categories.

We define the Vassiliev filtration to be the filtration of the linearized category $\mathbb{Q}\mathcal{T}$ by powers of the augmentation ideal, that is, the ideal generated by differences of tangles of the same type. In other words, one has that $\mathbb{Q}\mathcal{T}(X)_1$ is generated by differences $T_1 - T_2$, where T_2 is obtained from T_1 by a crossing change or a change of framing.[3] The n-th term of the filtration is generated by compositions of n elements of filtration 1. Thus, the filtration is multiplicative with respect to the juxtaposition and composition of tangles.

We denote the n-th term of the Vassiliev filtration on $\mathbb{Q}\mathcal{T}(X)$ by $\mathbb{Q}\mathcal{T}(X)_n$. The induced filtrations on the vector spaces $\mathbb{Q}\mathcal{CT}(X)$ and $\mathbb{Q}\mathcal{HT}(X)$ and on the algebras $\mathbb{Q}PB(l)$, $\mathbb{Q}CSL(l)$, and $\mathbb{Q}HSL(l)$, will again be called the Vassiliev filtrations and denoted by $\mathbb{Q}\mathcal{CT}(X)_n$, $\mathbb{Q}\mathcal{HT}(X)_n$, and so on.

2. The vector space $A(X)$.

We refer the reader to [B2, B1] (see also [Vo2]) for definitions of the $\mathbb{Q}$-vector space $A(X)$ of chord diagrams, modulo 4T relations, on the 1-manifold X. Elements of $A(X)$ may also be represented by linear combinations of vertex-oriented[4] diagrams $X \cup \Gamma$ (occasionally referred to as Feynman diagrams on X), subject to the AS (antisymmetry) and IHX (or Jacobi) relations.[5] (Here Γ is a uni-trivalent graph, whose univalent vertices lie in the interior of X. Each component of Γ is required to have a univalent vertex.) It is customary to refer to components of X as *solid*, and to components of Γ as *dashed.*

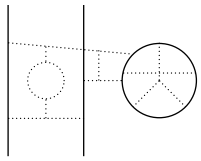


Figure 2. A diagram $X \cup \Gamma$ of degree 9. Vertex orientations are those induced from the orientation of the plane. Four-valent vertices are an artifact of the diagrammatic representation and should be ignored.

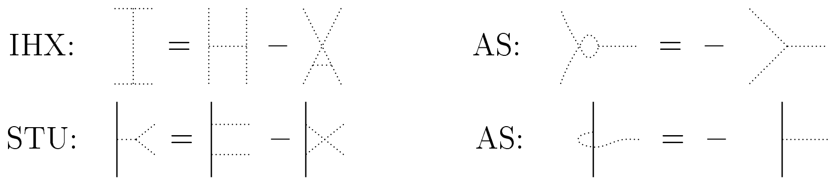


Figure 3. The IHX and AS relations. The IHX relation involving vertices on X is commonly referred to as the STU relation.

The space $A(X)$ is graded by the degree, where the degree of a diagram is half the number of vertices of Γ. The degree n part of $A(X)$ will be denoted by $A_n(X)$.

[3]This differs from the definition in [KT], in that we include $+1$ framing changes and not just crossing changes, which yield only $+2$ framing changes. Thus we avoid the $\mathbb{Z}/2$-residue of [KT] and the Vassiliev filtration is the one induced from the Kontsevich integral, see 3.3 below.

[4]An orientation of a 3-valent vertex is an ordering of the half-edges at that vertex, up to cyclic permutation.

[5]As pointed out by P. Vogel, one need not require X to be oriented. We require instead a vertex orientation of the vertices of Γ on X, which are considered as trivalent vertices of $X \cup \Gamma$, and the AS relations are also required to hold at these vertices. See figure 3.

By abuse of notation, we denote the graded completion (*i.e.*, replace the direct sum of the $A_n(X)$ with the direct product) of $A(X)$ again by $A(X)$.

If $i\colon X \to Y$ is an embedding, there is an induced map $i_*\colon A(X) \to A(Y)$.

Throughout this paper we set $A(l) = A(\coprod_{j=1}^{j=l} \mathrm{I}_j)$. Composition (*i.e.*, stacking one diagram vertically over the other) makes this into an algebra. It is known (see [B2]) that $A(S^1)$ is isomorphic to $A(\mathrm{I}) = A(1)$ and thus is also an algebra.

For all X, the vector space $A(X)$ has a natural cocommutative comultiplication Δ defined as in [B2, p. 439]. This makes $A(l)$ into a Hopf algebra. (See [MM].)

Notation. We let $P(l)$ denote the space of primitive elements of $A(l)$, and $P_n(l)$ its degree n part. (An element ξ is called primitive if $\Delta(\xi) = 1 \otimes \xi + \xi \otimes 1$.) One has [B2] that $P(l)$ is the graded subspace of $A(l)$ generated by diagrams with *connected* dashed graphs.

3. The universal invariant $Z(T)$.

The original Kontsevich integral [Ko, B2] associates to every (unframed) knot K in S^3, an invariant, which is an element of the quotient of $A(S^1)$ by the ideal generated by the chord diagram having a single chord. Although this construction can be extended to tangles, we find it convenient to use a combinatorial version of it, based ultimately on seminal work of Drinfel'd [D1, D2]. This combinatorial version has been studied by Le and Murakami [LM1, LM2], Bar-Natan [B3], Cartier [Ca], Kassel and Turaev [KT]. This approach replaces tangles by q-tangles (called non-associative tangles in [B3]).

The category of q-tangles is defined as follows. A *q-structure* on an ordered set is a bracketing; for example, the bracketings of the set $\{1, 2, 3, 4\}$ are $(((12)3)4), ((12)(34)), ((1(23))4), (1(2(34))), (1((23)4))$. (See [LM1, LM2, B3, KT] for a formal definition.) A *q-tangle* is a tangle equipped with a q-structure on $\partial_i X$, for $i = 0, 1$. Framed q-tangles again are the morphisms of a monoidal category, the objects of which are objects of the framed tangle category but equipped with a q-structure. For example, the identity string link with two different q-structures at the top and at the bottom is an isomorphism (and not an identity) in this category. Composition of q-tangles is again given by stacking in the obvious way[6], and the monoidal product is again given by juxtaposition, where, by way of example, the bracketing of the juxtaposition of $(1(23))$ and (45) is $((1(23))(45))$. By abuse of notation, we again denote this category by $\mathcal{T}$.

By the *q-type* of the q-tangle, we mean the type of the tangle, together with the q-structure on $\partial_i X$, $i = 0, 1$. Note that the set of q-tangles of a given q-type is naturally in bijection with the set of tangles of the underlying type, so that the forgetful functor from the q-tangle category to the tangle category is a bijection on morphisms. In particular, the set of endomorphisms of the identity object (represented by the empty set of points) of the framed q-tangle category is just the set of isotopy classes of framed links.

To a framed q-tangle $T\colon X \to \mathbb{C} \times \mathrm{I}$, the combinatorial version of the Kontsevich integral associates an element $Z(T)$,[7] which lies in (the graded completion of) $A(X)$. It is an isotopy invariant of the framed q-tangle. (In fact, one may allow ambient isotopies of $\mathbb{C} \times \mathrm{I}$ which preserve $\mathbb{R} \times \partial\mathrm{I}$ set-wise, which is why only the order of the points in $\partial_i X$ is important.)

[6] Two q-tangles T_1, T_2 are composable if the underlying tangles are composable and if the q-structures on $\partial_1 X_1$ and $\partial_0 X_2$ coincide.

[7] T. Le and J. Murakami use the notation $Z_f(T)$ for their preliminary version of the Kontsevich integral, and write $\widehat{Z}_f(T)$ for their final invariant. We shall not use this notation in the present paper.

One has that $Z(T)$ is multiplicative (*i.e.*, $Z(T_1T_2) = Z(T_1)Z(T_2)$) and also monoidal (*i.e.*, $Z(T_1 \times T_2) = Z(T_1) \otimes Z(T_2)$). This latter property is one advantage of using q-tangles, as it allows one to compute the Kontsevich integral of a link locally, by cutting it into elementary morphisms in the q-tangle category.

3.1. Remarks. (i) It is sometimes useful to view A as a category with the same objects as the tangle category $\mathcal{T}$, and with morphisms given by linear combinations of diagrams on 1-manifolds (*i.e.*, tangle types). Then A is a monoidal category (with monoidal structure induced by juxtaposition), and Z is a monoidal functor from the category of q-tangles to A.[8]

(ii) The following terminology is sometimes useful. To every object of the tangle category, one associates an object of the q-tangle category by giving it the so-called *canonical* q-structure. Here, the canonical q-structure of an ordered set is the one obtained by successively taking the bracketing of the predecessors of an element with the element. For example, $(((12)3)4)$ is the canonical bracketing of the ordered set $\{1, 2, 3, 4\}$.

(iii) The functor Z depends on the choice of a certain element Φ of $A(3)$ called the *associator*. This is the image under Z of the identity 3-component string link with the canonical q-structure at the bottom and the other q-structure at the top. It is chosen so as to satisfy certain properties (*e.g.*, to ensure isotopy invariance). By Drinfeld's work, we can (and do) choose an associator whose non-zero terms all lie in even degrees.[9] This ensures properties 3.2.6 and 3.2.7 below.

(iv) Unless stated otherwise, in this paper string links will be equipped with the same q-structure at the top and at the bottom. For a string link σ, the first (*i.e.*, lowest degree) non-vanishing term of $Z(\sigma) - 1$ does not depend on the q-structure.

Notation. In this paper, the expression $O(n)$ will be used to denote terms of degree greater than or equal to n.

3.2. Some properties of $Z(T)$.

1. One has $Z(T) = 1 + O(1)$, where we denote by $1 \in A_0(X) = \mathbb{Q}$ the diagram $X \cup \Gamma$ with empty dashed graph Γ. Note that if σ is an l-component string link, $Z(\sigma)$ lies in the algebra $A(l)$, and the element $1 \in A_0(l)$ is the identity element of that algebra.
2. Let $\sigma_1 \colon \mathrm{I}_1 \amalg \mathrm{I}_2 \to \mathbb{C} \times \mathrm{I}$ denote the generator of the braid group on 2 strands. Then $Z(\sigma_1) = \exp(\xi_{1,2}/2)$, where $\xi_{1,2} \in A(\mathrm{I}_1 \amalg \mathrm{I}_2)$ is the diagram consisting of a single chord having one vertex on I_1 and the other on I_2 (with the 'same' vertex orientation for both vertices).
3. Let $\cap$ and $\cup$ be the 1-component framed tangles associated to the obvious tangle diagrams suggested by their symbols. Then $Z(\cup)$ and $Z(\cap)$ (when viewed in $A(1)$) are equal to an element usually called $\nu^{1/2}$. The value of ν can be computed from the associator by the formula $\nu = i_*(\Phi)$, where $i_* \colon A(3) \to A(1)$ is the map induced by the (order preserving) inclusion i of 3 intervals in a single interval which is orientation preserving on the first and third and is orientation reversing on the second. In particular, one has $\nu = 1 + O(2)$ (since $\Phi = 1 + O(2)$).
4. If the tangle T' is obtained from the tangle T by removing a component, then $Z(T')$ is obtained from $Z(T)$ by setting all diagrams having a vertex on that component

[8] There is, however, more structure here. The functor Z respects the partition of morphisms in $\mathcal{T}$ into q-types X, in the sense that for all $T \in \mathcal{T}(X)$, $Z(T)$ lies in $A(X)$.

[9] Such an associator is called 'good' in [LM2].

equal to zero (considering all other diagrams to lie on the complement of the removed component).

5. A key fact which we will use is that $Z(T)$ is always a *group-like* element (*i.e.*, one has $\Delta(Z(T)) = Z(T) \otimes Z(T)$). (See [LM2, thm. 5.1].) Consequently,[10] if σ is a string link, then $Z(\sigma) = \exp(\xi)$, for ξ a *primitive* element of $A(l)$. Note that in particular, the first non-vanishing term in the expansion of $Z(\sigma) - 1$ is therefore primitive, and hence a linear combination of diagrams with connected dashed graphs.
6. (Doubling Formula) One has that

$$Z(D_i(T)) = D_i(Z(T))$$

where $D_i(T)$ denotes the double of T along the i-th component, C_i, say, and D_i of a diagram is the sum over all lifts of vertices on C_i, to vertices on the 2 components over C_i. (See [LM2, thm. 4.2 and commentary].) See figure 4.

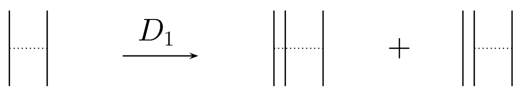


Figure 4. The doubling operation on the first strand.

7. Let σ be a string link with l components. In section 1, we defined $\overline{\sigma}$, which is a concordance inverse of σ. By [LM2, prop. 3.1], one has

$$Z(\overline{\sigma}) = R(Z(\sigma))$$

where R is the (anti-)involution of $A(l)$ defined as follows. If $\xi \in A_n(l)$, then $R(\xi) = (-1)^n r_*(\xi)$, where r denotes the self-map of $\coprod_{j=1}^{j=l} \mathrm{I}_j$ defined by $t_j \mapsto 1 - t_j$.

Remark 3.3. The invariant $Z(T)$ is the *universal* finite type invariant of framed q-tangles. By this, one means that there is an exact sequence,

$$0 \to \mathbb{Q}\mathcal{T}(X)_{n+1} \to \mathbb{Q}\mathcal{T}(X) \xrightarrow{Z_{\leq n}} A_{\leq n}(X) \to 0, \tag{1}$$

where $A_{\leq n}(X)$ is the space of chord diagrams on X of degree $\leq n$, and $Z_{\leq n}$ is the composite of Z with the projection onto $A_{\leq n}(X)$. Thus Z induces an isomorphism from the completion of $\mathbb{Q}\mathcal{T}(X)$, with respect to the Vassiliev filtration, to the graded completion of $A(X)$.[11]

[10] It is an exercise to show that in a (graded completed) cocommutative Hopf algebra, the exponential of a primitive element is group-like and vice versa.

[11] The proof of the exactness of (1) is essentially the same as the proof given in [B2] for (unframed) knots. It comes down to showing that the natural map $A_n(X) \to \mathbb{Q}\mathcal{T}(X)_n/\mathbb{Q}\mathcal{T}(X)_{n+1}$ (defined as in [B2]) is surjective. Note that this map isn't surjective in the combinatorial framed version of Kassel-Turaev [KT], who have a $\mathbb{Z}/2$ residue due to their different definition of the Vassiliev filtration. With our definition, the map is indeed surjective, which can be seen using the equality $t - t^{-1} = 2(t-1) + (t-1)(t^{-1}-1)$, where t denotes a twist of framing. This shows that a $+1$ framing change is equivalent, modulo higher filtration, to a crossing change yielding a $+2$ framing change.

4. The Hopfian monoidal category quotients A^h and A^t.

We say that $\overline{A}$ is a monoidal category quotient of A, if for all 1-manifolds X, $\overline{A}(X)$ is a graded quotient of $A(X)$ and the product $A(X_1) \otimes A(X_2) \to A(X_1X_2)$ and monoidal product $A(X_1) \otimes A(X_2) \to A(X_1 \amalg X_2)$ pass to maps on quotients. If in addition, the comultiplication $A(X) \to A(X) \otimes A(X)$ passes to maps on quotients, the monoidal category quotient is referred to as Hopfian. We denote by $\overline{Z}(T)$, the image of $Z(T)$ in $\overline{A}(X)$ (where $T\colon X \to \mathbb{C} \times \mathrm{I}$).

In [B1], the Hopfian monoidal category quotient A^h of A was defined to be the quotient by the relations generated by chord diagrams containing a chord between the same component. It follows (see [B1]) that any trivalent diagram containing a dashed component which is either not simply connected, or which has two edges on the same solid component, is also a relation. We denote $\overline{Z}$ by Z^h.

The superscript h here stands for 'homotopy'. Indeed, D. Bar-Natan [B1] has shown that the graded vector space associated to $\mathbb{Q}HSL(l)$ with respect to the Vassiliev filtration is precisely $A^h(l)$. Moreover, it is easy to see that Z^h is the universal finite type invariant of tangles up to (link) homotopy. In other words, one has the analogue of the exact sequence of 3.3, with $\mathbb{Q}\mathcal{T}(X)$ replaced by $\mathbb{Q}\mathcal{HT}(X)$, and $Z_{\leq n}$ replaced by $Z^h_{\leq n}$, so that Z^h induces the Vassiliev filtration on $\mathbb{Q}\mathcal{HT}(X)$.

Similarly, let $\overline{A} = A^t$ (and let $\overline{Z}$ be denoted by Z^t), where all trivalent diagrams containing a non-simply connected dashed component are considered as relations. A^t is also a Hopfian monoidal category quotient of A. The superscript t stands for 'tree'; this is because tree diagrams on $\coprod_{j=1}^{j=l} \mathrm{I}_j$ generate the subspace $P^t(l)$ of primitive elements in $A^t(l)$.

In section 14, we will prove that Z^t is the universal finite type *concordance* invariant. Note that A^h is a quotient of A^t, so that Z^h factors through Z^t. This coincides nicely on the level of finite type invariants with the fact that concordant tangles are homotopic.

Chinese characters.[12] There is a natural isomorphism of $A(l)$ with $B(l)$, the $\mathbb{Q}$ algebra of *Chinese characters*, *i.e.*, uni-trivalent vertex-oriented (dashed) graphs, modulo AS and IHX relations, whose univalent vertices are labelled by elements of the set $\{1, \ldots, l\}$. (See [B2, B1].) This restricts to an isomorphism between $P(l)$ and $C(l)$, the space of *connected* Chinese characters. The multiplication in the algebra $B(l)$ is given by disjoint union, so that $B(l)$ is the commutative polynomial algebra on $C(l)$. The isomorphism of $A(l)$ with $B(l)$ is comultiplicative but not multiplicative.[13]

The isomorphism between $A(l)$ and $B(l)$ passes to an isomorphism between $A^h(l)$ and $B^h(l)$, where $B^h(l)$ is the commutative polynomial algebra on the space $C^h(l)$ of labelled Chinese characters which are trees with *distinct* labels. The proof given in [B1] also shows that it passes to an isomorphism between $A^t(l)$ and $B^t(l)$, where $B^t(l)$ is the commutative polynomial algebra on the space $C^t(l)$ of labelled Chinese characters which are trees.

[12] The name "Chinese characters" for this kind of diagrams was introduced by D. Bar-Natan in [B2, B1]. Other names for them include "web graphs/diagrams" or "unitrivalent graphs/diagrams".

[13] The correct identification of $B(l)$ with $A(l)$ sends the Chinese character to the *average* of all the ways of attaching the Chinese character (along the univalent vertices) to the strings (*i.e.*, we divide the sum of all the ways by the total number of ways). This factor is missing in [B1] and, as pointed out by T. Le, without the factor, the map is not comultiplicative.

Remark 4.1. We will use the fact that $Z^t(\cup)$ and $Z^t(\cap)$ are equal to 1 in $A^t(1)$. This is seen as follows. As stated in 3.2.3, one has $Z(\cup) = (\cap) = \nu^{1/2}$, where $\nu = 1 + O(2)$. Thus $\nu = \exp(\mu)$, with μ primitive of degree ≥ 2 and hence zero in $A^t(1)$, since the only primitives in $A^t(1)$ are in degree 1.

5. Review of Milnor's μ invariants.

For a group G, let G_n denote the lower central series of G, *i.e.*, $G = G_1$ and inductively $G_{n+1} = [G, G_n]$ (where for sets $A, B \subset G$, $[A, B]$ denotes the group generated by all commutators $[a, b] = aba^{-1}b^{-1}$, $a \in A$, $b \in B$).

The Artin representation and the longitudes. Let $F(l)$ be the free group on $x_1, \dots, x_l$. We will also denote the image of x_j in the lower central series quotient $F(l)/F(l)_n$ again by x_j. Let $Aut_0(F(l)/F(l)_n)$ be the group of automorphisms of $F(l)/F(l)_n$, which send each generator x_j to a conjugate of itself, and which leave their product $x_1 x_2 \cdots x_l$ fixed.

The n-th *Artin representation* is the homomorphism

$$\mathcal{A}_n \colon SL(l) \to Aut_0(F(l)/F(l)_{n+1})$$

defined as follows. Let σ be a string link with l components. Consider the inclusions i_0 and i_1 of the complement of the l standard points $p_1, \dots, p_l$, in $\mathbb{C} \times 0$ and in $\mathbb{C} \times 1$ (see section 1), into the complement of (the image of) σ in $\mathbb{C} \times \mathrm{I}$. It follows from Stallings' theorem [St] that i_0 and i_1 induce isomorphisms on the lower central series quotients of fundamental groups. Identifying the fundamental group of the complement of $p_1, \dots, p_l$ in $\mathbb{C}$ in a standard way with the free group $F(l)$, the induced map $(i_1)_*^{-1}(i_0)_*$ is an automorphism $\mathcal{A}_n(\sigma)$ of $F(l)/F(l)_{n+1}$. It is easy to see that $\mathcal{A}_n(\sigma)$ lies in $Aut_0(F(l)/F(l)_{n+1})$. We have that

$$\mathcal{A}_n(\sigma)(x_i) = \lambda_i x_i \lambda_i^{-1},$$

where $\lambda_i = \lambda_i^{(n)} \in F(l)/F(l)_{n+1}$ is the *longitude* of σ defined as follows. The framing of σ defines a parallel curve which determines an element in the fundamental group of the complement. Under $(i_1)_*^{-1}$, this element corresponds to an element λ_i of $F(l)/F(l)_{n+1}$, which is called the longitude of σ. Changing the framing modifies λ_i only by multiplying by a power of x_i on the right. Thus the Artin representation of σ, $\mathcal{A}_n(\sigma)$, is independent of the framing.

Similar considerations applied to the complement of a concordance, show that $\mathcal{A}_n(\sigma)$ is a concordance invariant of σ. Thus, $\mathcal{A}_n \colon SL(l) \to Aut_0(F(l)/F(l)_{n+1})$ factors through $CSL(l)$.

Definition. We say that σ has *Milnor filtration* n, if all longitudes λ_i of σ are trivial in $F(l)/F(l)_n$.

Notice that every string link has Milnor filtration 1, and σ has Milnor filtration 2 if and only if all linking numbers and all framings are zero.

Remark 5.1. A string link σ, with all framings zero, has Milnor filtration n if and only if $\mathcal{A}_n(\sigma) = 1$.[14]

Milnor's invariants. Let $\mathcal{P}(l)$ be the ring of power series in non-commuting variables $X_1, \dots, X_l$. The Magnus expansion is the group homomorphism

$$\mu \colon F(l) \to \mathcal{P}(l)$$

[14] This can be proven inductively using lemma 16.3.

defined on the generators of the free group $F(l)$ by $\mu(x_j) = 1 + X_j$. The Milnor μ (mu) invariants of a string link σ are the coefficients (of the monomials in the X_i) in the Magnus expansion, $\mu(\lambda_i)$, of the longitudes of σ.[15]

In this paper, we are particularly interested in the first non-vanishing Milnor invariants. They can be described as follows.

There is a canonical graded isomorphism of $\oplus_{n\geq 1}(F(l)_n/F(l)_{n+1})\otimes\mathbb{Q}$ with the free $\mathbb{Q}$ Lie algebra $\mathrm{Lie}(l) = \oplus_{n\geq 1}\mathrm{Lie}_n(l)$ on l generators $X_1,\dots,X_l$ (in degree 1). (In fact, the graded completion of $\mathrm{Lie}(l)$ is the space of primitives of $\mathcal{P}(l)$, if the latter is regarded as a graded completed Hopf algebra by requiring the X_i to be primitive.) If σ has Milnor filtration n, we can consider the longitudes λ_i as elements of $F(l)_n/F(l)_{n+1}$, and we denote by $\mu_i^{(n)}(\sigma)$ the corresponding element of $\mathrm{Lie}_n(l)$.

Definition. The $\mu_i^{(n)}(\sigma)$ are called *Milnor invariants of degree n.*

Note. This description of Milnor invariants of degree n is valid only for string links of Milnor filtration n. (In the general case, one has to use the Magnus expansion.) Note also that degree n invariants in our definition correspond to what is usually called Milnor's μ invariants of *length* $n+1$ (with possibly repeating indices). For example, the linking number is a Milnor invariant of degree one and of length two. (We consider the self-linking number as a Milnor invariant of degree one.)

6. The reduced Kontsevich integral Z^t and Milnor's invariants.

Definition. We say that a string link σ has *Z^t-filtration n*, if $Z^t(\sigma) = 1+O(n)$, where $O(n)$ denotes terms of degree greater than or equal to n.

Note. We will see that Z^t is a concordance invariant (see section 14), hence a nul-concordant string link has arbitrarily high Z^t-filtration.

If σ has Z^t-filtration n, then by 3.2.5 the first non-vanishing term of $Z^t(\sigma)-1$ lies in $P_n^t(l)$, the primitives of $A_n^t(l)$. In the remainder of this paper, we shall identify $P_n^t(l)$ with $C_n^t(l)$ (the space of degree n labelled Chinese characters which are trees).

Note. The identification of $P_n^t(l)$ with $C_n^t(l)$ is particularly elementary. Since a tree diagram lying on the strings is already symmetric (because of the relations in A^t, permuting two vertices on the same string does not change its value), one simply sends the tree diagram (on the strands) to its underlying Chinese character (the tree with its univalent vertices labelled).

Let $C_n^t(l+1)$ be the space of degree n Chinese characters which are trees, and with labels in $\{0,\dots,l\}$. One can identify $\mathrm{Lie}_n(l)$ with $C_n^t(l,0)$, where $C_n^t(l,0)\subset C_n^t(l+1)$ is the subspace generated by tree diagrams in which the label 0 occurs exactly once. The map $C_n^t(l,0)$ to $\mathrm{Lie}_n(l)$ is described in [B1, Thm. 2] as follows.

Label the edges of a tree in $C_n^t(l,0)$ inductively, by first labelling the edges having a univalent vertex with label in $\{1,\dots,l\}$ by the generator with the same label, and by using the rule that whenever an unlabelled edge meets two other labelled edges in a trivalent

[15]Strictly speaking, here one considers the i-th longitude as an element $\lambda_i = \varprojlim \lambda_i^{(n)}$ of the nilpotent completion $\overline{F(l)} = \varprojlim F(l)/F(l)_{n+1}$. Note that the Magnus expansion extends to $\overline{F(l)}$.

vertex, then it is labelled by the commutator of the two labels (in the order determined by the cyclic orientation of the vertex). (See figure 5.) This procedure terminates with all edges labelled. One associates to this tree, the label on the edge which has the 0-labelled vertex as one of its vertices. This describes a well defined map, since the AS and IHX relations go over into the antisymmetry and Jacobi relations of the Lie bracket. The map is injective, since in a free Lie algebra there are no other relations. Moreover, since any commutator in the generators can be obtained from such a tree, this map is also surjective.

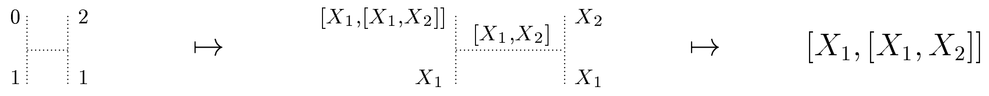


Figure 5. An example for the isomorphism $C^t_n(l,0) \xrightarrow{\approx} \mathrm{Lie}_n(l)$.

We define

$$j_i\colon C^t_n(l) \to C^t_n(l,0) = \mathrm{Lie}_n(l), \quad i = 1, \dots, l,$$

to be the map which takes a labelled tree to the sum of the labelled trees obtained by replacing the label i, on one of the vertices, with the label 0. See figure 8 for an example.

Theorem 6.1. *(i) The Z^t-filtration on string links coincides with the Milnor filtration.*

(ii) Furthermore, the first non-vanishing Milnor invariants of a string link σ are determined by the first non-vanishing term of $Z^t(\sigma) - 1$ (and vice versa) through the formula

$$\mu_i^{(n)}(\sigma) = j_i(\xi),$$

where $Z^t(\sigma) = 1 + \xi + O(n+1)$, with $\xi \in C^t_n(l)$.

The proof of theorem 6.1 will be given in section 11. See 7.2 and section 8 for examples.

Remark 6.2. The *vice versa* part of Theorem 6.1 follows from the fact that the map $j = (j_1, \dots, j_l)$ from $C^t_n(l)$ to $C^t_n(l,0)^l$, the product of l copies of $C^t_n(l,0)$, is *injective*. This fact is easy to see. Indeed, let $r = \Sigma_{i=1}^{i=l} r_i \circ \pi_i$, where π_i is projection onto the i-th factor and $r_i\colon C^t_n(l,0) \to C^t_n(l)$ is the map which associates to a labelled tree, the same labelled tree, but with the vertex labelled 0 replaced with the label i. Then the composition $r \circ j$ is multiplication by $n+1$, proving that the map j is injective.

Corollary 6.3. *Two string links σ_1 and σ_2 have the same Milnor invariants in all degrees (resp. in all degrees $\leq n$) if and only if $Z^t(\sigma_1) = Z^t(\sigma_2)$ (resp. $Z^t_{\leq n}(\sigma_1) = Z^t_{\leq n}(\sigma_2)$).*

Proof. The first non-vanishing Milnor invariants as well as the first non-vanishing term of $Z^t - 1$ are additive and change sign under $\sigma \mapsto \overline{\sigma}$. Thus, it suffices to apply theorem 6.1 to $\sigma_1\overline{\sigma}_2$ and to proceed by induction on the filtration degree.

The following corollary answers a conjecture of X. S. Lin.

Corollary 6.4. *The algebra of rational finite type concordance invariants of string links is generated by Milnor's invariants.*

The proof is deferred to section 15, where we show that Z^t is the universal finite type concordance invariant.

More applications of theorem 6.1 will be discussed in sections 8, 9, and 16.

7. A special case.

Let $PB(l)$ denote the pure braid group on l ordered strands labelled $1, \dots, l$. There is a semidirect product decomposition $PB(l) = F(l-1) \rtimes PB(l-1)$ (see *e.g.* [Bi, cor. 1.8.1]). Here, $PB(l-1)$ is embedded into $PB(l)$ via $\beta \mapsto 1_1 \times \beta$, and $F(l-1)$ is the subgroup generated by x_i, $i = 1, \dots, l-1$, where $x_i = \sigma_{1,i+1}$ wraps the first strand around the $i+1$-th strand. See figure 6. (Note that in this decomposition and in the statement and proof below, we have distinguished the first component. With another choice, we would have obtained a similar result, since these situations are conjugate.)

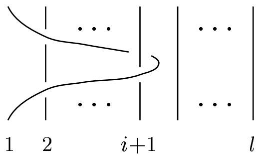


Figure 6. The generator $x_i = \sigma_{1,i+1}$ of $F(l-1) \subset PB(l)$.

Let $x \in F(l-1)_n \subset F(l-1) \subset PB(l)$ be an iterated commutator in the generators x_i such that each x_i appears at most once. The induced element in $(F(l-1)_n / F(l-1)_{n+1}) \otimes \mathbb{Q} = \mathrm{Lie}_n(l-1) = C^t_n(l-1, 0)$ is a labelled tree diagram ξ, say. Let $\xi' \in C^t_n(l)$ be obtained from ξ by increasing all labels by 1. Since each label occurs at most once, there is a unique way of putting the diagram ξ' on the l strings, so that ξ' determines an element of $P_n(l) \subset A_n(l)$ which we denote again by ξ'.

Proposition 7.1. *One has*
$$Z(x) = 1 + \xi' + O(n+1).$$

Remark 7.2. Let us use 7.1 to illustrate theorem 6.1 in this special case. Formula 6.1(ii) says that the first non-vanishing Milnor invariant $\mu^{(n)}_1(x)$ is equal to $j_1(Z^t_n(x))$, which by 7.1 is the diagram obtained from ξ' (considered now as a Chinese character in $C^t_n(l)$) by replacing the label 1 by 0, or, equivalently, the diagram obtained from ξ by increasing all *non-zero* labels by 1. In other words, the first longitude of x is, modulo commutators in $F(l)_{n+1}$, the element $x' \in F(l)_n$ obtained by replacing each occurence of a generator x_i in x by x_{i+1}. (Of course, this can be checked directly in this special case.) See figure 7 for an example.

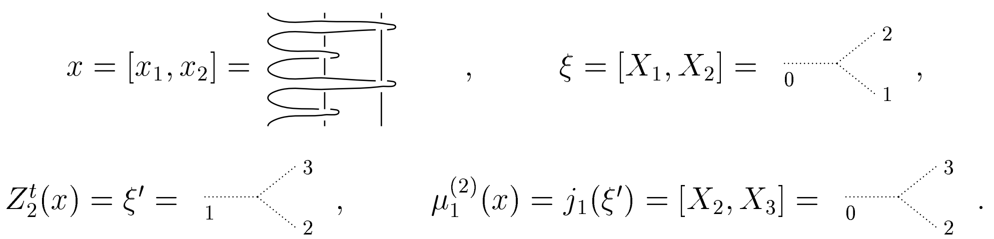


Figure 7. An example for formula 6.1(ii) and 7.1.

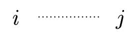
Proof of proposition 7.1. Note that the result is certainly true if $x = x_1 = \sigma_{1,2}$, since in this case, the induced element of $C^t_1(l-1, 0)$ is $\xi_{0,1}$, where $\xi_{i,j}$ is the dashed chord (of degree

1) with vertices labelled i and j, and one has, by definition, $Z(\sigma_{1,2}) = \exp(\xi_{1,2})$. (Here, we are using the canonical q-structure.) For $i \geq 2$, one has that $x_i = \sigma_{1,i+1} = \sigma\sigma_{1,2}\sigma^{-1}$, where σ is a certain braid inducing the permutation $s = (i+1, i, \ldots, 2)$. It follows that $Z(\sigma_{1,i+1}) = s(1+\xi_{1,2})s^{-1} + O(2) = 1 + \xi_{1,i+1} + O(2)$.

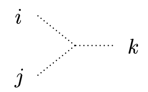
Now suppose inductively for all $m < n$ the result holds and consider an iterated commutator $x \in F(l-1)_n$ so that none of the generators x_i appears twice in x. There are $x^{(i)}$, with degree $x^{(i)}$ strictly less than n, such that $x = [x^{(1)}, x^{(2)}]$, as commutators in $F(l-1) \subset PB(l)$. It follows that $Z(x) = [Z(x^{(1)}), Z(x^{(2)})]$. We have that $Z(x) = \exp(\xi'')$ and $Z(x^{(i)}) = \exp(\xi_i'')$, where ξ'' and ξ_i'' are primitive. Let ξ_i be the diagram corresponding to $x^{(i)}$, and let ξ_i' be the result of increasing all labels by 1. By the induction hypothesis, the lowest degree term of ξ_i'' is ξ_i'. It follows from the Campbell-Baker-Hausdorff formula [MKS], that the lowest degree (*i.e.*, degree n) term of ξ'' is $[\xi_1', \xi_2']$, as elements of $A_n(l)$, where in the latter formula, one uses the Lie bracket $[a,b] = ab - ba$. Using the STU relation at the unique vertices of ξ_1' and ξ_2' on the first string, and the fact that ξ_1' and ξ_2' cannot both have a vertex on the same string other than the first string (since, by hypothesis, none of the generators x_i appears twice in x), one checks that $[\xi_1', \xi_2']$ is equal to the diagram ξ', as asserted.

8. The first non-vanishing Milnor invariants and the lattice $K_n(l)$.

Our formula in theorem 6.1(ii) says that the first non-vanishing Milnor invariants are naturally the *dual* of $C_n^t(l)$. For example, the linking (and self-linking) numbers correspond to the coefficient of



and Milnor's triple linking numbers correspond to the coefficient of



It is quite pleasing to observe how this description nicely encodes the relations among Milnor's invariants.

A less obvious example is the diagram ξ defined in figure 8 which generates $C_3^t(2)$.

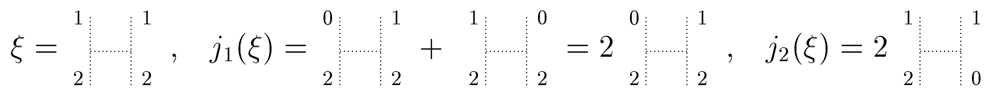


Figure 8. Another example for formula 6.1(ii).

One may interpret this by saying that on 2-component string links σ of Milnor filtration 3, the coefficient of this diagram ξ corresponds to *half* the Milnor invariant μ_{1122} in the classical index notation, since $\mu_{1122}(\sigma)$ is the coefficient of $[X_1, [X_1, X_2]]$ in $\mu_2^{(3)}(\sigma) = j_2(Z_3^t(\sigma))$.[16] Note that for a string link version of the Whitehead link, this Milnor invariant is known to be ± 1, and hence the first non-vanishing term of $Z^t - 1$ must be $\pm\xi/2$ for such a string link.

Generalizing this example, let us compute the image over the integers of Z_n^t on string links of Milnor filtration n. Observe that both the vector spaces $C_n^t(l)$ and $C_n^t(l,0)$ contain a canonical lattice generated by tree diagrams; let us denote these lattices by $D_n^t(l)$ and $D_n^t(l,0)$. (Here, the notation D stands for 'diagram'.) The image of Z_n^t on string links of

[16] Classically, one defines $\mu_{i_1,\ldots,i_n,i}$ as the coefficient of $X_{i_1}\cdots X_{i_n}$ in $\mu(\lambda_i)$.

Milnor filtration n is also a lattice in $C_n^t(l)$; let us denote this lattice by $K_n(l)$. For example, $D_3^t(2)$ is generated by the diagram ξ defined in figure 8, and $K_3(2)$ is generated by $\xi/2$. The lattice $K_n(l)$ can be determined as follows.

Corollary 8.1. *$K_n(l)$ is the sublattice, of $D_n^t(l)\otimes\mathbb{Q}=C_n^t(l)$, consisting of those ξ such that $j(\xi)$ is integral, that is, lies in $D_n^t(l,0)^l\subset C_n^t(l,0)^l$. (It follows that $D_n^t(l)\subset K_n(l)$.)*

Proof. Consider a string link σ of Milnor filtration n. Let $Z^t(\sigma)=1+\xi+O(n+1)$, with $\xi\in C_n^t(l)$. The longitudes λ_i of σ lie in $F(l)_n/F(l)_{n+1}$, and hence can be considered as lying in $D_n^t(l,0)$, as $F(l)_n/F(l)_{n+1}$ is the degree n part of the free $\mathbb{Z}$ Lie algebra on l generators $X_1,\dots,X_l$, and is canonically identified with $D_n^t(l,0)$. Since $j_i(\xi)=\mu_i^{(n)}(\sigma)$, by theorem 6.1, and since the Milnor invariants are computed from the longitudes, one sees by following through the identifications, that $j(K_n(l))$ is contained in $D_n^t(l,0)^l\subset C_n^t(l,0)^l$. In fact, $j(K_n(l))$ is precisely the sublattice of $D_n^t(l,0)^l$ consisting of the l-tuples $(\mu_1,\dots,\mu_l)$ which can be realized as μ-invariants of string links. Hence the rank of $K_n(l)$, which is the same as the rank of $j(K_n(l))$, is the number, $k_n(l)$, say, of linearly independent Milnor invariants of degree n, for l-component links. This number was first computed by Orr [O], who showed that $k_n(l)=lN_n(l)-N_{n+1}(l)$, where $N_n(l)$ is the rank of $D_n^t(l,0)$.

Define $p\colon D_n^t(l,0)^l\to D_{n+1}^t(l,0)$ by $p=\Sigma_{i=1}^{i=l}p_i\circ\pi_i$, where π_i is projection onto the i-th factor, and $p_i\colon D_n^t(l,0)\to D_{n+1}^t(l,0)$ is the map, which associates to a labelled tree the tree obtained from the given one, by replacing the vertex labelled 0 with a trivalent vertex with two additional edges labelled by i and 0. Here, the cyclic ordering of the edges of the additional trivalent vertex is chosen so that under the canonical identification of $D_n^t(l,0)$ with the degree n part of the free $\mathbb{Z}$ Lie algebra on generators $X_1,\dots,X_l$, the map p_i corresponds to the map $Y\mapsto[Y,X_i]$.

It is easy to see that the composite $p\circ j$ is zero and hence $j(K_n(l))\subset ker(p)$, the kernel of p. Clearly, $ker(p)$ has rank $lN_n(l)-N_{n+1}(l)$ (since p is surjective), hence at least $j(K_n(l))$ has finite index in $ker(p)$, by Orr's computation of $k_n(l)$. In fact, Orr's construction can be adapted to string links, showing that any element of the kernel of p is realized by a string link (see [HL2]). Hence $j(K_n(l))$ is equal to $ker(p)$ and the sequence

$$0\to K_n(l)\xrightarrow{j}D_n^t(l,0)^l\xrightarrow{p}D_{n+1}^t(l,0)\to 0$$

is exact, proving the result.

Remark 8.2. *$K_n(l)$ is the lattice in $C_n^t(l)$ generated by all labelled tree diagrams Γ, and by $\frac{1}{2}\Gamma$, if Γ admits a nontrivial symmetry of order 2. (It follows that $2\,K_n(l)\subset D_n^t(l)$.)*

This can be seen as follows. Given a labelled tree Γ with $n+1$ univalent vertices, resp. one with order 2 symmetry, T. Cochran [Co1, Co2] constructs a link $L(\Gamma)$, resp. $L(\frac{1}{2}\Gamma)$, by taking band connected sums of certain Bing doubles of the Hopf, resp. Whitehead, link. Let us denote by $\sigma(\Gamma)$, resp. $\sigma(\frac{1}{2}\Gamma)$, a string link version of these links. By Theorem 3.3 of [Co2], these links realize all 'realizable' Milnor invariants, and hence they generate $K_n(l)$. Thus it suffices to show that $Z^t(\sigma(\Gamma))=1+\xi+O(n+1)$ and $Z^t(\sigma(\frac{1}{2}\Gamma))=1+\xi/2+O(n+1)$, where $\xi\in C_n^t(l)$ is represented by Γ. It is not hard to prove this using theorem 6.1; indeed, it is an exercise to compute the longitudes of these links and to check that the first non-vanishing Milnor invariants are given by $j(\xi)$, resp. $j(\xi/2)$. This verification is left to the reader.

9. Vanishing results.

Note that the first non-vanishing Milnor invariants of a string link σ (*i.e.*, the $\mu_i^{(n)}(\sigma)$ defined in section 5), depend only on $\widehat{\sigma}$, the link which is the 'closure' of σ. (This follows easily from their definition in [M1, M2].) The Milnor invariants of a closed link are called the $\overline{\mu}$-invariants (the bar indicating that higher order invariants other than the first non-vanishing ones have indeterminacies) and depend only on its concordance class (Stallings [St]).

Corollary 9.1. *The first non-vanishing term of $Z^t(\sigma) - 1$ is determined by the first non-vanishing Milnor $\overline{\mu}$ invariants of $\widehat{\sigma}$, the link obtained as the 'closure' of σ. Consequently, if $Z^t(\sigma) = 1 + O(n)$, and if $\widehat{\sigma}'$ is concordant to $\widehat{\sigma}$, then $Z^t(\sigma') - Z^t(\sigma) = O(n+1)$.*

Proof. This follows immediately from theorem 6.1.

The following vanishing results for the Kontsevich expansion, Z, can at times be useful. (Recall that we consider self-linking numbers to be Milnor invariants of degree 1.)

Proposition 9.2. *Let n denote a positive integer or ∞. Let σ be a string link such that all Milnor μ invariants of σ of degree $\leq n$ vanish. Then $Z(\sigma)$ can be written as a linear combination of diagrams, none of which contain a dashed component which is a tree of degree $\leq n$.*

Proof. Write $Z(\sigma) = \exp(\xi)$ where ξ is primitive. The quotient map from $A(l)$ to $A^t(l)$ has a canonical section, coming from the isomorphism of $A(l)$ with $B(l)$ and the fact that $B^t(l)$ is naturally a direct summand of $B(l)$ (since the Euler characteristic of a Chinese character is determined by its degree and the number of its univalent vertices). Thus there is a canonical decomposition $\xi = \xi_1 + \xi_2$, where ξ_1 corresponds to connected Chinese characters in $B^t(l)$, and ξ_2 corresponds to connected Chinese characters in the complementary direct summand of $B(l)$. Then since $Z^t(\sigma) = \exp(\xi_1) = 1 + O(n+1)$ by theorem 6.1 and corollary 9.1, all terms of ξ_1 are of degree $> n$. The result follows.

Corollary 9.3. *Let n denote a positive integer or ∞. Let L be a link such that all Milnor $\overline{\mu}$ invariants of degree $\leq n$ vanish. Then the Kontsevich expansion, $Z(L)$, can be written as a linear combination of diagrams, none of which contain a dashed component which is a tree of degree $\leq n$.*

Proof. Let σ be a string link such that its closure $\widehat{\sigma}$ is isotopic to L. Then $Z(L)$ can be calculated from $Z(\sigma)$, which satisfies the conclusion of proposition 9.2. We must see that this introduces no diagrams which contain a tree of degree $\leq n$. A precise formula can be found in [LM2]: $Z(L)$ is obtained from $Z(\sigma)$ by multiplying by a certain element $\nu_l \in A(l)$ and then projecting to $A(\coprod_{j=1}^{j=l} S^1)$. In fact ν_l is obtained from $\nu = \nu_1$, by the operator $D^{(l)}\colon A(1) \to A(l)$ (which is the sum over all lifts of vertices on an interval to l intervals). But $\nu = 1$ in $A^t(1)$ (see remark 4.1) so that ν contains no diagrams with trees. The result then follows from 9.2.

Example 9.4. If the self-linking and linking numbers vanish, there are no dashed chord components. If L is a boundary link, then all Milnor invariants vanish. Hence $Z(L)$ is a linear combination of diagrams, none of which contain a dashed component which is a tree.

10. The case of homotopy Milnor invariants.

One of the goals of this section is to give the analog of theorem 6.1 for homotopy Milnor invariants (see 10.1 and 10.6). In the next section, this will be used in the proof of theorem 6.1.

We recall the construction of the homotopy Milnor invariants. The *reduced free group* $RF(l)$ is the quotient of $F(l)$ obtained by adding relations which say that each generator commutes with all of its conjugates. Consider a string link $\sigma \in SL(l)$ and its i-th longitude $\lambda_i(\sigma) \in F(l)/F(l)_n$. Let us denote by $R\lambda_i(\sigma)$ the image of $\lambda_i(\sigma)$ in $RF(l-1)/RF(l-1)_n$ under the quotient map which sets the i-th generator x_i equal to 1. Then $R\lambda_i(\sigma)$ depends only on the homotopy class of σ, *i.e.*, the value of σ in $HSL(l)$.

We say that σ has *homotopy Milnor filtration* n, if all longitudes $R\lambda_i(\sigma)$ are trivial in $RF(l-1)/RF(l-1)_n$.

Proposition 10.1. *For $\sigma \in HSL(l)$, the following are equivalent.*
(i) σ *has homotopy Milnor filtration* n,
(ii) $\sigma \in HSL(l)_n$, *the n-th term of the lower central series of* $HSL(l)$,
(iii) $\sigma - 1 \in I^n$, *where I denotes the augmentation ideal of* $\mathbb{Q}HSL(l)$,
(iv) $Z^h(\sigma) = 1 + O(n)$.

Proof. The equivalence of (i) and (ii) is proven in [HL1] using the semidirect product decomposition $HSL(k) = RF(k-1) \rtimes HSL(k-1)$ (analogous to the semidirect product decomposition $PB(k) = F(k-1) \rtimes PB(k-1)$).

This also shows that the lower central series quotients $HSL(l)_m/HSL(l)_{m+1}$ are torsion free (see [HL1]).

Next, recall that for a group G, the augmentation ideal $I \subset \mathbb{Q}G$ is generated by the set of elements $g-1$ where $g \in G$. The filtration of $\mathbb{Q}G$ by powers of I is called the I-adic filtration. It is a general fact (see *e.g.* [Q1, cor. 4.2]) that if G_m/G_{m+1} is torsion free for all m, then for all n, G_n is equal to $\{g \in G \ : \ g-1 \in I^n\}$, the so-called n-th dimension subgroup. Applying this to $G = HSL(l)$ shows that (ii) and (iii) are equivalent.

The equivalence of (iii) and (iv) follows from the following proposition and the fact (see section 4) that Z^h induces the Vassiliev filtration on $\mathbb{Q}HSL(l)$.

Proposition 10.2. *The Vassiliev filtration on $\mathbb{Q}HSL(l)$ coincides with the I-adic filtration.*

Proof. Let $\mathbb{Q}HSL(l)_n$ denote the Vassiliev filtration on homotopy string links. (N.b., this should not be confused with $\mathbb{Q}(HSL(l)_n)$, where $HSL(l)_n$ is the n-th term of the lower central series.) Let I denote the augmentation ideal of the group ring $\mathbb{Q}HSL(l)$. It is easy to see that $I \subset \mathbb{Q}HSL(l)_1$ and hence $I^n \subset \mathbb{Q}HSL(l)_n$. We must show that $I^n = \mathbb{Q}HSL(l)_n$. This follows by induction from the following lemma and the fact that Z^h induces the Vassiliev filtration.

Lemma 10.3. *The map $Z^h_n \colon I^n/I^{n+1} \to A^h_n(l)$ is an isomorphism.*

Proof. Recall that if G is a group, then $\mathbb{Q}G$ is a Hopf algebra with cocommutative comultiplication $\Delta(g) = g \otimes g$ (*i.e.*, group elements are group-like). Since $Z(\sigma)$ is group-like, $Z^h \colon \mathbb{Q}HSL(l) \to A^h(l)$ is a (filtration preserving) Hopf algebra map, and hence induces a map of (graded complete cocommutative) Hopf algebras

$$\widehat{\oplus}_n Z^h_n : \widehat{\oplus}_{n\geq 0}\,(I^n/I^{n+1}) \longrightarrow A^h(l) = \widehat{\oplus}_{n\geq 0}\, A^h_n(l).$$

By [Q2, 2.21], this map is an isomorphism if and only if it is an isomorphism on primitives. By [Q2, 3.11], the set of primitives of $\widehat{\oplus}_{n\geq 0}(I^n/I^{n+1})$ is

$$\widehat{\oplus}_{n\geq 1}\,(HSL(l)_n/HSL(l)_{n+1})\otimes\mathbb{Q}.$$

The space of primitives of $A^h_n(l)$ is $P^h_n(l)$, which is identified with $C^h_n(l)$.

Proposition 7.1 shows that generators of $C^h_n(l)$ are realized by pure braids (in $PB(l)_n$). Hence the map $(HSL(l)_n/HSL(l)_{n+1})\otimes\mathbb{Q}\to C^h_n(l)$ is surjective. On the other hand, in [B1], the *dimension* of $C^h_n(l)$ was computed to be the same as the rank of $HSL(l)_n/HSL(l)_{n+1}$ (computed in [HL1]). Thus the above map is also injective, and the result follows.

The proof of proposition 10.1 is complete. We have also proven the following.

Proposition 10.4. *The map $Z^h\colon \mathbb{Q}HSL(l)\to A^h(l)$ is injective and induces an isomorphism of complete Hopf algebras (after completing $\mathbb{Q}HSL(l)$ with respect to the I-adic filtration).*

Remark 10.5. By [HL1], the group $HSL(l)$ is nilpotent of class $l-1$. Hence the equivalence of 10.1(ii) and 10.1(iv) shows (or use 10.4) that two string links σ, σ' are homotopic if and only if $Z^h(\sigma)=Z^h(\sigma')$. Since Z^h is the universal finite type homotopy invariant, this shows that finite type invariants separate homotopy string links (Bar-Natan [B1], Lin [L]).

There is a canonical isomorphism of $\oplus_{n\geq 1}(RF(l)_n/RF(l)_{n+1})\otimes\mathbb{Q}$ with the reduced free $\mathbb{Q}$ Lie algebra $R\mathrm{Lie}(l)=\oplus_{n\geq 1}R\mathrm{Lie}_n(l)$. Here $R\mathrm{Lie}(l)$ is the quotient of the free Lie algebra on l generators, $X_1,\ldots,X_l$, in degree 1, obtained by setting equal to zero all basic commutators in which one of the variables X_i appears twice. If σ has homotopy Milnor filtration n, we can consider the longitudes $R\lambda_i(\sigma)$ as elements of $RF(l-1)_n/RF(l-1)_{n+1}$, and we denote by $R\mu^{(n)}_i(\sigma)$ the corresponding element of $R\mathrm{Lie}_n(l-1)$. The $R\mu^{(n)}_i(\sigma)$ are called *homotopy Milnor invariants of degree n.* (These correspond to what are usually called Milnor's μ invariants of *length* $n+1$, without repeating indices.)

Consider the map $j_i\colon C^h_n(l)\to C^h_n(l,0)$, induced from $j_i\colon C^t_n(l)\to C^t_n(l,0)$. Recall that $C^h_n(l)$ is generated by trees with distinct labels on its univalent vertices. (See section 4.) Observe that this j_i is simply the map which replaces the label i by the label 0 if it occurs, and sends the labelled tree to zero, otherwise.

The identification of $\mathrm{Lie}_n(l)$ with $C^t_n(l,0)$ induces an identification of $R\mathrm{Lie}_n(l)$ with $C^h_n(l,0)$.

Proposition 10.6. *The first non-vanishing homotopy Milnor invariants of a string link σ are determined by the first non-vanishing term of $Z^h(\sigma)-1$ (and vice versa) through the formula*

$$R\mu^{(n)}_i(\sigma)=j_i(\xi)$$

where $Z^h(\sigma)=1+\xi+O(n+1)$, with $\xi\in C^h_n(l)$.

Proof. Assume $Z^h(\sigma)=1+\xi+O(n+1)$, with $\xi\in C^h_n(l)$. Fix i with $1\leq i\leq l$. Suppose first that σ is a tree diagram (not a linear combination of diagrams). There is an l-component pure braid σ_ξ lying in a subgroup $F(l-1)_n\subset F(l-1)\subset PB(l)$ as in section 7 (but with using the i-th string as preferred string in place of the first string) such that $Z^h(\sigma_\xi)=1+\xi+O(n+1)$ (see proposition 7.1, but notice that our ξ corresponds to the ξ' of 7.1). Thus, all strings except the i-th string of σ_ξ are trivial, and the i-th string (which determines a word in $F(l-1)_n$) also determines the i-th longitude of σ_ξ, and hence the i-th homotopy μ-invariant. Following through the identifications, this shows $R\mu^{(n)}_i(\sigma_\xi)=j_i(\xi)$.

Next, let ξ still be a tree diagram, but let σ be any string link such that $Z^h(\sigma) = 1 + \xi + O(n+1)$. Then $\overline{\sigma}$ (as defined in section 1) is an inverse up to homotopy of σ, and hence $Z^h(\overline{\sigma}) = 1 - \xi + O(n+1)$. Thus $Z^h(\overline{\sigma}\sigma_\xi) = 1 + O(n+1)$, hence $\overline{\sigma}\sigma_\xi$ has Milnor filtration $n+1$. By additivity of homotopy μ-invariants, this shows $R\mu_i^{(n)}(\sigma) = R\mu_i^{(n)}(\sigma_\xi) = j_i(\xi)$.

This proves the result for any $\sigma \in HSL(l)_n$, with ξ a tree diagram, and all i. Since tree diagrams generate $C_n^h(l)$ and are realized by string links in $HSL(l)_n$, the proof is complete upon observing that both $R\mu_i^{(n)}$ and $j_i \circ Z_n^h$ are homomorphisms on $HSL(l)_n/HSL(l)_{n+1}$, and Z_n^h is an isomorphism from $(HSL(l)_n/HSL(l)_{n+1}) \otimes \mathbb{Q}$ to $C_n^h(l)$ (see the proof of 10.3).

11. Proof of Theorem 6.1.

The proof of theorem 6.1 will be based on proposition 10.6 and the following 2 lemmas. Let $D^{(m)}(\sigma)$ be the string link obtained from σ by replacing each component of σ with m parallel copies.[17]

Lemma 11.1. *Suppose that $m \geq \max(2, n-2)$. Then $Z^t(\sigma) = 1 + O(n)$ if and only if $Z^h(D^{(m)}(\sigma))) = 1 + O(n)$.*

Proof. If $Z^t(\sigma) = 1 + \xi + O(n+1)$, where ξ is of degree n, then $Z^h(D^{(m)}(\sigma)) = 1 + \pi(D^{(m)}(\xi)) + O(n+1)$, where $D^{(m)}$ of a diagram is the sum over all lifts of vertices on a component to the components lying over the given component, and where π denotes the projection of A^t onto A^h. (This follows from iterated application of 3.2.6.)

It follows that $Z^t(\sigma) = 1 + O(n)$ implies $Z^h(D^{(m)}(\sigma)) = 1 + O(n)$. To prove the converse implication, note that $D^{(m)}$ commutes with the comultiplication, so that $D^{(m)}$ takes primitives to primitives. Let π also denote the projection from $C_n^t(ml)$ to $C_n^h(ml)$. It is enough to show that $\pi \circ D^{(m)}\colon C_\nu^t(l) \to C_\nu^h(ml)$ is injective in degrees ν, where $1 \leq \nu \leq n-1$. This can be seen as follows. Let p denote the map, from $C_n^h(ml)$ to $C_n^t(l)$, which identifies the m labels over a given label. Then the composition $p \circ \pi \circ D^{(m)}$ is injective, since on trees with the same labels and multiplicities (which generate a direct summand of $C_n^t(l)$), the map is multiplication by a positive integer. Here we need that m be at least as large as the largest possible multiplicity, which is 2 for $\nu = 1$, and is $\nu - 1$ in degrees $\nu \geq 2$.[18] Hence $\pi \circ D^{(m)}$ is injective provided $m \geq \max(2, n-2)$, proving the lemma.

Lemma 11.2. *Suppose that $m \geq \max(2, n-2)$. Then σ has Milnor filtration n if and only if $D^{(m)}(\sigma)$ has homotopy Milnor filtration n.*

Proof. If the i-th longitude $\lambda_i(\sigma)$ is represented as a word in the generators $x_1, \dots, x_l$ of the free group $F(l)$, then the (i,ν)-th (homotopy) longitude $R\lambda_{i,\nu}(D^{(m)}(\sigma))$ is obtained as follows. First replace, for each k, each occurence of an x_k in this word by the product $x_{k,1}x_{k,2}\cdots x_{k,m}$. Second, set $x_{i,\nu}$ equal to 1. The result, viewed in $RF(ml-1)$, represents $R\lambda_{i,\nu}(D^{(m)}(\sigma))$.

By induction, we may assume that the lemma holds up to $n-1$, and hence we may suppose that $\lambda_i(\sigma) \in F(l)_n/F(l)_{n+1}$ and $R\lambda_{i,\nu}(D^{(m)}(\sigma)) \in RF(ml-1)_n/RF(ml-1)_{n+1}$.

[17] We don't need to specify the q-structure on $D^{(m)}(\sigma)$ here because we'll only be concerned with first non-vanishing terms in this section. (See 3.1(iv).) Notice, however, that if the q-structure on $D^{(m)}(\sigma)$ is chosen to be a 'refinement' of the q-structure on σ, then (although the refinement is not unique if $m \geq 3$), the value of $Z(D^{(m)}(\sigma))$ (in all degrees) is independent of the choice of refinement.

[18] Note that a degree ν Chinese character which is a tree has $\nu + 1$ univalent vertices. If $\nu \geq 2$, it follows from the AS relation that the diagram is zero if one label appears with multiplicity $> \nu - 1$.

Thus we may consider $\lambda_i(\sigma)$ as an element of $C^t_n(l,0) = \mathrm{Lie}_n(l)$ and $R\lambda_{i,\nu}(D^{(m)}(\sigma))$ as an element of $C^h_n(ml-1,0) = R\mathrm{Lie}_n(ml-1)$. It is easily seen that $R\lambda_{i,\nu}(D^{(m)}(\sigma)) = \pi_{i,\nu} \circ D^{(m)}(\lambda_i(\sigma))$, where $D^{(m)}\colon C^t_n(l,0) \to C^t_n(ml,0)$ is defined as before and $\pi_{i,\nu}$ is the projection from $C^t_n(ml,0)$ to $C^h_n(ml-1,0)$ (*i.e.*, the map $\pi_{i,\nu}\colon \mathrm{Lie}_n(ml) \to R\mathrm{Lie}_n(ml-1)$ setting the (i,ν)-th variable to zero).

Reasoning as in the proof of 11.1, one implication of 11.2 is obvious, while the converse implication follows from the fact that $\pi_{i,\nu} \circ D^{(m)}$ is injective in degrees $\leq n-1$, provided $m \geq \max(2, n-2)$. The lemma follows.

Proof of theorem 6.1. Fix $m \geq \max(2, n-1)$ and let $D^{(m)}(\sigma)$ be the string link obtained from σ by replacing all components of σ with m parallel copies. By prop. 10.1, $D^{(m)}(\sigma)$ has homotopy Milnor filtration n if and only if $Z^h(D^{(m)}(\sigma)) = 1 + O(n)$. Applying 11.1 and 11.2, this shows that σ has Z^t-filtration n if and only if it has Milnor filtration n. This proves 6.1(i).

Let $SL(l)_n$ be the subset of $SL(l)$ consisting of string links σ of filtration n. In the proof of 11.2, we have established the commutative diagram

$$\begin{array}{ccc} SL(l)_n & \xrightarrow{\mu_i^{(n)}} & C^t_n(l,0) \\ \downarrow & & \downarrow{\scriptstyle \pi_{i,\nu}\circ D^{(m)}} \\ HSL(ml)_n & \xrightarrow{R\mu_{i_\nu}^{(n)}} & C^h_n(ml-1,0) \end{array}$$

Similarly, one has the commutative diagram

$$\begin{array}{ccccc} SL(l)_n & \xrightarrow{Z^t_n} & C^t_n(l) & \xrightarrow{j_i} & C^t_n(l,0) \\ \downarrow & & \downarrow{\scriptstyle \pi\circ D^{(m)}} & & \downarrow{\scriptstyle \pi_{i,\nu}\circ D^{(m)}} \\ HSL(ml)_n & \xrightarrow{Z^h_n} & C^h_n(ml) & \xrightarrow{j_{i,\nu}} & C^h_n(ml-1,0) \end{array}$$

where the commutativity of the left square was already used in the proof of 11.1, and the commutativity of the right square is easily checked.

In order to show 6.1(ii), we must show that the composite $j_i \circ Z^t_n$ and $\mu_i^{(n)}$ agree as maps from $SL(l)_n$ to $C^t_n(l,0)$. (Actually, $SL(l)_n$ is a sub-monoid of $SL(l)$, and both maps are morphisms of monoids, but we don't need this here.) It is sufficient to show that $j_i \circ Z^t_n$ and $\mu_i^{(n)}$ agree after composing both maps with $\pi_{i,\nu} \circ D^{(m)}$, since this map is injective for $m \geq \max(2, n-1)$ (see the proof of 11.2). In view of the commutative diagrams above, this follows from the equality $j_{i,\nu} \circ Z^h_n = R\mu^{(n)}_{i,\nu}$, shown in lemma 10.6.

Remark 11.3. There are several proofs [B1, L] that Milnor's invariants are of finite type. Here, we briefly sketch how to see this using the above methods. It suffices to show that the map which takes a string link to its longitude is a filtration preserving map, *i.e.*, it sends the Vassiliev filtration to the I-adic filtration of the group ring (of the nilpotent completion) of the free group (since the Magnus expansion is multiplicative and hence preserves I-adic filtrations).

This can be seen as follows. For homotopy Milnor invariants, the corresponding assertion is a consequence of our identification of the Vassiliev filtration with the I-adic filtration. Indeed for a semidirect product for which the quotient acts trivially on homology, it is an exercise to see that projecting onto the kernel, although failing to be multiplicative, induces a map of group algebras which preserves the I-adic filtrations. Since taking the (homotopy) longitudes is by definition such a projection, the result follows in the homotopy case.

In the general case, note that it suffices to compose with the Magnus expansion, since this is a filtration preserving isomorphism. Applying the operation D, which doubles each component N times say, and then reducing to homotopy, we see that this gives injections of the associated power series rings, up to order $N-1$. Consequently, the general case follows from the result for homotopy.

12. A global formula for Milnor's invariants.

The combinatorial formula given in Theorem 6.1, which describes the first non-vanishing Milnor invariants, is quite suggestive, since it is obtained from the doubling operation via a forgetful map. One can think of the map j_i (*i.e.*, take the sum over all i-labelled vertices of the result of relabelling the vertex with the label 0) in the following way. Recall that the i-th longitude is the parallel (or double) of the i-th string. The map j_i consists of computing it by means of the doubling formula 3.2.6, and then throwing away diagrams which don't have exactly one vertex on the 0-th component (which represents the longitude).

We now explain how to generalize this to a formula which expresses the full Magnus expansion of the longitudes in terms of the Kontsevich integral. We are indebted to D. Bar-Natan, whose suggestions convinced us that the above interpretation can be made rigorous, and generalizes to higher order Milnor invariants.

Recall that the pure braid group on $l+1$ strands, denoted $PB(l+1)$, is a semi-direct product, $PB(l+1) = F(l) \rtimes PB(l)$, where the action of $PB(l)$ on $F(l)$ is given by the Artin representation $PB(l) \to Aut_0(F(l))$. We consider the strings of elements of $PB(l+1)$ to be labelled from 0 to l, and of elements of $PB(l)$ to be labelled from 1 to l. If $\sigma \in PB(l)$ is given, we denote by $1_0 \times \sigma$ the corresponding element of $PB(l+1)$ obtained by adding a trivial 0-th string on the left.

The key observation is that the i-th longitude $\lambda_i \in F(l)$ of an element $\sigma \in PB(l)$ is easily calculable from σ. Let β_i be the braid which crosses the 0-th string over the strings labelled 1 to $i-1$ (*i.e.*, β_i is the product of the first $i-1$ standard braid generators). Let D_i be the map $PB(l) \to PB(l+1)$ obtained by doubling the i-th component. The i-th longitude λ_i is obtained from the pure braid $\beta_i D_i(\sigma)\beta_i^{-1}$ by 'combing' this braid, as follows.

Proposition 12.1. *Let λ_i be the i-th longitude of $\sigma \in PB(l)$. In $PB(l+1)$, one has the formula*

(2) $$\beta_i D_i(\sigma)\beta_i^{-1} = (1_0 \times \sigma)\lambda_i,$$

and consequently $\lambda_i = (1_0 \times \sigma)^{-1}\beta_i D_i(\sigma)\beta_i^{-1}$.

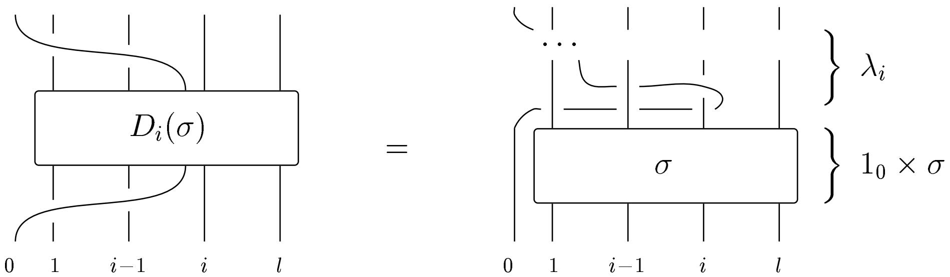


Figure 9. An illustration of formula (2).

Applying the Kontsevich expansion, Z, one deduces immediately the following.

Corollary 12.2. *In $A(l+1)$, one has the formula*

$$Z(\lambda_i) = Z(1_0 \times \sigma)^{-1} D_i(Z(\sigma))^{b_i},$$

where D_i is the doubling operation on $A(l)$ (for the i-th string), $b_i = Z(\beta_i)$, and a^b denotes bab^{-1}. (See 3.2.6 *for the definition of the doubling operation.)*

Remark 12.3. In the above formula, the pure braid $1_0 \times \sigma \in PB(l+1)$ is endowed with the canonical q-structure, whereas the braid β_i has the canonical q-structure on the bottom, and the 'i-th double' of the canonical structure at the top.

In proposition 12.4, we will extend this formula for the Kontsevich expansion of the longitudes of pure braids to the case of string links, where the longitudes λ_i must now be considered to lie in the nilpotent completion of $F(l)$. In order to do this, we consider another quotient of $A(l+1)$, which we now describe.

Suppose the 1-manifold X is a disjoint union of compact 1-manifolds X_0 and X_1. Let $A^{h,t}(X)$ be the quotient of $A^t(X)$ obtained by introducing h relations (see section 4) for diagrams with vertices on X_0. Thus, in addition to the relations in $A^t(X)$, diagrams $X \cup \Gamma$ containing a (dashed) component $\Gamma_0 \subset \Gamma$ with two vertices on X_0 are set equal to zero. We denote the projection onto $A^{h,t}(X)$ by $\pi^{h,t}$, and put $Z^{h,t} = \pi^{h,t} \circ Z$. Then $Z^{h,t}$ is an isotopy (and even concordance, see section 14) invariant, and, in addition, a homotopy invariant of the components in X_0 (in the complement of those in X_1). In accordance with the notation $A(l+1)$, we will use the notation $A^{h,t}(l+1)$ for the case where $X = \coprod_{j=0}^{j=l} \mathrm{I}_j$, and where X_0 consists of the 0-th component.

Proposition 12.4. *Let σ be a string link and λ_i its i-th longitude. In $A^{h,t}(l+1)$, one has the formula*

$$Z^{h,t}(\lambda_i) = \pi^{h,t}(Z(1_0 \times \sigma)^{-1} D_i(Z(\sigma))^{b_i}).$$

The proof makes use of the following lemma.

Lemma 12.5. *Let X be the disjoint union of X_0 and X_1 and suppose T_i, $i = 1, 2$, are two tangles which agree on X_1. Assume that on each component of X_0 the maps differ by an element in the $n+1$-st term of the lower central series of the fundamental group of the complement of X_1. Then the images of $Z(T_i)$ in $A^{h,t}_{\leq n}(X)$ agree.*

Proof. The fundamental group of the complement of X_1 is generated by a collection of meridians (*e.g.*, as in the Wirtinger presentation). More precisely, there is a ball meeting X_1 in a trivial string link, on say m components, such that the map of fundamental groups, induced by the inclusion of complements, is surjective. It follows that the map T_2 can be obtained by a homotopy of the components of X_0 after first modifying the map T_1 on X_0 by pure braids in $F(m)_{n+1} \subset F(m) \subset PB(m+1)$. Since modifying by pure braids in $F(m)_{n+1}$ does not change terms of degree $\leq n$ [S] (see also the proof of 13.1 below), the result follows.

Proof of proposition 12.4. Let $\lambda_i^{(n)} \in F(l)$ represent the longitude λ_i in $F(l)/F(l)_{n+1}$. Then the tangles $(1_0 \times \sigma)\lambda_i^{(n)}$ and $\beta_i D_i(\sigma)\beta_i^{-1}$ satisfy (by Stallings' theorem [St]) the hypotheses of lemma 12.5. Applying the functor Z, solving for $Z(\lambda_i^{(n)})$, projecting onto $A^{h,t}(l)$, and letting n tend to infinity proves the result.

Note that $A^{h,t}(l+1)$ is a graded (completed) cocommutative Hopf algebra, whose space of primitives is isomorphic to $C^{h,t}(l+1)$, *i.e.*, Chinese characters which are trees with labels $\in \{0,1,\dots,l\}$ on their univalent vertices, and such that the label 0 occurs at most once. Let $X_i \in A^{h,t}(l+1)$ denote the element defined by the chord diagram $\xi_{0,i}$, the dashed chord (of degree 1) with vertices labelled 0 and i. Then $X_1,\dots,X_l$ generate a free non-commutative power series ring

$$\mathcal{P}(l) = \mathcal{P}(X_1,\dots,X_l) \subset A^{h,t}(l+1),$$

since the primitives of $A^{h,t}(l+1)$ are naturally decomposed as

$$C^{h,t}(l+1) = C^t(l,0) \oplus C^t(l) = \mathrm{Lie}(l) \oplus C^t(l),$$

and the first summand corresponds to $\mathcal{P}(l)$.

Lemma 12.6. *The image of $F(l)$ in $A^{h,t}(l+1)$ via the inclusion in $PB(l+1)$ followed by the map $Z^{h,t}$, lies in the non-commutative power series ring $\mathcal{P}(l)$.*

Proof. Let $x_i = \sigma_{0,i}$, $i=1,\dots,l$, be the generators of $F(l) \subset PB(l+1)$ (see section 7). Then $Z^{h,t}(x_i) = \exp(Y_i)$, where $Y_i = X_i + O(2)$, since $Z^{h,t}(x_i)$ is a conjugate of $\exp(X_i)$. Note that by 3.2.4, each diagram occuring in Y_i must have a vertex on the 0-th string, since x_i becomes trivial upon forgetting the 0-th string. It follows that Y_i lies in the subspace $C^t(l,0) = \mathrm{Lie}(l)$ of the primitives of $A^{h,t}(l+1)$, proving the lemma.

We can now describe our global formula for Milnor's invariants, *i.e.*, the Magnus expansion of the longitudes. Recall that the Magnus expansion $\mu\colon F(l) \to \mathcal{P}(l)$ is defined by $\mu(x_i) = 1+X_i$. Since $Z^{h,t}(x_i) = 1+X_i+O(2)$, it follows from [L] that one has that $\mu = \Psi \circ Z^{h,t}$, where Ψ is a *unipotent* automorphism of $\mathcal{P}(l)$, *i.e.*, if $a \in \mathcal{P}(l)$ has degree n, then $\Psi(a) = a + O(n+1)$. Hence we have shown the following

Theorem 12.7. *For $\sigma \in SL(l)$ and λ_i its i-th longitude, one has the formula*

$$\mu(\lambda_i) = (\Psi \circ \pi^{h,t})(Z(1_0 \times \sigma)^{-1} D_i(Z(\sigma))^{b_i}),$$

where both sides are considered as elements of $\mathcal{P}(l) \subset A^{h,t}(l+1)$, D_i is the doubling operation on $A(l)$ (for the i-th string), $b_i = Z(\beta_i)$, $\pi^{h,t}$ is the projection onto $A^{h,t}(l+1)$, and Ψ is a unipotent automorphism.

Remark 12.8. The formula in Theorem 12.7 allows one to calculate all the Milnor invariants from the Kontsevich integral, giving yet another proof that Milnor's invariants are of finite type. We leave it to the reader as an exercise to show that it contains the combinatorial formula of Theorem 6.1 (which does not depend on the unipotent automorphism Ψ) as a special case. It would be interesting to have an explicit formula for Ψ. (This would, of course, require fixing a choice of an associator.) On the other hand, the effect of applying Ψ is merely to replace the Magnus expansion by some other expansion; it is therefore not very important as far as the topological information carried by Milnor's invariants is concerned.

13. Realizing primitive diagrams.

T. Stanford [S] introduced the procedure of modifying a knot or link by a pure braid. In this section we will realize certain diagrams by modifying the trivial string link by pure braids. This technique will be needed in section 15.

It is a natural question to ask what are the possible values of the Kontsevich integral. We have the following partial answer.

Proposition 13.1. *Let $\xi \in A_n(l)$ be represented by a linear combination with integer coefficients of diagrams with connected dashed graphs. Then there is a string link σ such that $Z(\sigma) = 1 + \xi + O(n+1)$.*

Proof. Let $\xi = X \cup \Gamma$ be a tree diagram on $X = \coprod_{j=1}^{j=l} \mathrm{I}_j$. Note that Γ has $n+1$ univalent vertices (which lie on X). Embed X as the trivial l-component string link 1_l. Consider first the special case where Γ has exactly one vertex on each string. Then the result follows already fromproposition 7.1 (we take for σ a pure braid which is an iterated commutator as in 7.1). In the general case, we first perform an isotopy, after which we can write 1_l as the composite

$$T_1(T_2 \times 1_{n+1} \times T_3)T_4$$

for some tangles $T_1, \ldots, T_4$, such that each component of 1_{n+1} contains exactly one of the univalent vertices of Γ. (To see this, choose a disk in $\mathbb{C} \times \mathrm{I}$ which meets X transversally in the $n+1$ univalent vertices of Γ lying on X, and isotope this disk into a standard position.) Inserting now the appropriate iterated commutator $x \in F(n)_n \subset PB(n+1)$ in place of 1_{n+1}, gives a string link σ such that $Z(\sigma) = 1 + \xi + O(n+1)$. This follows from proposition 7.1, the fact that $Z(T_i) = 1 + O(1)$, and the observation that $Z(T_1)(Z(T_2) \times 1 \times Z(T_3))Z(T_4) = 1$, by the isotopy invariance of Z.

Thus we can realize any tree diagram. Since the first non-zero term in positive degrees of the Kontsevich integral of a composition of string links is additive, we may realize any diagram which is a linear combination with integer coefficients of tree diagrams. (To realize the negative of a tree diagram, just use the inverse braid in the construction above.) The theorem follows, since any connected diagram which is not a tree can be expressed (using the STU relation) as a linear combination with integer coefficients of tree diagrams.

Since diagrams with connected dashed graphs generate the space of primitive elements of $A(l)$, one has the following immediate consequence of 13.1. (By 3.2.5, the hypothesis below that $\xi \in A_n(l)$ be primitive is certainly necessary.)

Corollary 13.2. *Let ξ be an element of $P_n(l)$, the primitives of $A_n(l)$. Then there is a positive integer N and a string link σ such that $Z(\sigma) = 1 + N\xi + O(n+1)$.*

We conclude this section with a few corollaries pointed out by P. Vogel.

Corollary 13.3. *Let ξ be an element of $P(l)$, the primitives of $A(l)$. Fix any n. Then there is a positive integer N and a string link σ such that $Z(\sigma) = \exp(N\xi) + O(n+1)$.*

Proof. Suppose inductively one has σ_1 such that $Z(\sigma_1) = \exp(N_1\xi) + O(n)$. One has $Z(\sigma_1) = \exp(\xi_1)$ and hence $\exp(N_1\xi)\exp(-\xi_1) = \exp(\xi_2)$, with $\xi_2 = O(n)$. By 13.2, there are N_2 and σ_2, with $Z(\sigma_2) = 1 + N_2\xi_2 + O(n+1) = \exp(N_2\xi_2) + O(n+1)$. Using the Campbell-Hausdorff formula [MKS] and noting that the commutator $[\xi, \xi_2] = O(n+1)$, one checks that 13.3 is satisfied with $\sigma = \sigma_1^{N_2}\sigma_2$ and $N = N_1N_2$.

Corollary 13.4. *Let σ be a string link. Then for any positive integer n, there is a string link σ_1, such that the truncated Kontsevich expansion satisfies $Z_{\leq n}(\sigma\sigma_1) = 1$.*

Proof. Let $Z(\sigma) = \exp(\xi)$. By 13.3, there exist σ_2 and N, such that $Z(\sigma_2) = \exp(-N\xi) + O(n+1)$. Thus $Z(\sigma\sigma_1) = 1 + O(n+1)$, where $\sigma_1 = \sigma^{N-1}\sigma_2$.

As an application, consider the equivalence relation on the monoid $SL(l)$ given by $\sigma_1 \sim_n \sigma_2$ if and only if $Z_{\leq n}(\sigma_1) = Z_{\leq n}(\sigma_2)$. (Compare [Gu, NS].) We denote by $E_nSL(l)$ the set of equivalence classes. (N.b., the equivalence relation is independent of the choice of q-structure.)

Proposition 13.5. *$E_nSL(l)$ is a group. Furthermore it is nilpotent of class n.*

Proof. The truncated Kontsevich expansion $Z_{\leq n}$ induces an injective map of the monoid $E_nSL(l)$ into the group $1 + \oplus_{1\leq j\leq n} A_j(l)$, which is nilpotent of class n. Thus, the result follows from 13.4.

Remark 13.6. P. Vogel has remarked that the values in a fixed degree of the Kontsevich integral of any tangle diagram can be taken to be a linear sum of chord diagrams with rational coefficients, each having a fixed denominator (depending only on the degree). It follows that the group $E_nSL(l)$ is a central extension of $E_{n-1}SL(l)$ by a lattice in $P_n(l)$ (in particular, by a free abelian group of rank the dimension of $P_n(l)$). It follows (inductively) that the groups $E_nSL(l)$ are finitely generated and torsion free.

14. The concordance invariance of Z^t for tangles.

Theorem 14.1. *For all framed q-tangles T, $Z^t(T)$ depends only on the concordance class of T.*

Proof of Theorem 14.1. Consider two tangles T and T'' which are concordant. There is a tangle T' (see [H]) such that T' is obtained from T (and up to isotopy from T'') by a ribbon concordance (*i.e.*, adding cancelling 0- and 1-handles). Specifically, we pass from T to the tangle T' by connecting the boundaries of some trivially embedded disks Δ_i in the complement of T, via band connected sums along bands b_i, lying in a neighborhood of paths γ_i, from a point on the boundary of the disk Δ_i to a point on the tangle T. (These paths may pass through the disks.) It is enough to see that $Z^t(T) = Z^t(T')$, if T' is obtained from T as above.

(It might be instructive for the reader to follow the rest of the proof in the special case that there is just one disk and the path passes through the disk just once, as in figure 10.)

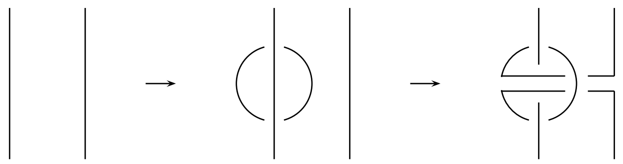


Figure 10. A concordance with one band-disk pair. Note that the string link on the right is isotopic to $\overline{\sigma}\sigma$, where σ is as in figure 1, showing that $\overline{\sigma}\sigma$ is nul-concordant.

If $\varepsilon = (\varepsilon_i)$, with $\varepsilon_i \in \{0,1\}$ for all i, denote by T_ε the tangle obtained from T', by sliding the bands meeting Δ_i off Δ_i, if $\varepsilon_i = 1$, and doing nothing otherwise. Note that $T_{(1,\dots,1)}$ is isotopic to T, and $T_{(0,\dots,0)}$ is T'. Note also that if $\varepsilon_i = 1$, one may remove the i-th band-disk pair, up to isotopy, so that T_ε is obtained from T by a ribbon concordance with fewer disks.

Set $\delta(T) = \Sigma(-1)^{|\varepsilon|}T_\varepsilon$, where $|\varepsilon| = \Sigma\varepsilon_i$.

Lemma 14.2. $Z^t(\delta(T)) = 0$.

Assuming lemma 14.2, we complete the proof of Theorem 14.1 by induction on m, the number of disks. If $m = 1$, the result holds by the lemma, since $\delta(T) = T' - T$ (up to isotopy) in this case.

Assume $Z^t(T) = Z^t(T')$, if T' is obtained from T as above, but with fewer than m disks. It follows that $Z^t(T) = Z^t(T_\varepsilon)$, if $\varepsilon \neq (0, \dots, 0)$ and hence that $Z^t(T') - Z^t(T) = Z^t(\delta(T))$. The result now follows from lemma 14.2.

Proof of Lemma 14.2. Let $\sigma_{2,3}$ denote the 3-component pure braid (endowed with the canonical q-structure) which wraps the second strand around the third. Let τ be obtained from $\sigma_{2,3}$ by capping off the first two components, and let τ_0 denote the cap-off of the trivial 3-component string link (both with the canonical q-structure on the bottom). See figure 11.

Figure 11. The tangles τ and τ_0.

We claim that $Z^t(\tau) - Z^t(\tau_0)$ is a linear combination of graphs $X \cup \Gamma$ (with $X = \mathrm{I}_1 \amalg \mathrm{I}_2$) such that *each connected component* of Γ has at least one univalent vertex on each of the two components of X. To see this, note that $Z^t(\sigma_{2,3}) = \exp(Y)$, where each diagram occuring in Y is connected and (by 3.2.4) must have a vertex on both the second and the third component, since $\sigma_{2,3}$ becomes trivial upon forgetting either the second or the third strand.[19] Since $Z^t(\cap) = 1$ (see remark 4.1), the claim follows.

Thus $Z^t(\tau) - Z^t(\tau_0)$ is a linear combination of graphs $X \cup \Gamma$ such that each univalent vertex of Γ lying on I_1 is the origin of a path in Γ which ends in a univalent vertex on I_2.

Now the local changes near Δ_i in the definition of $\delta(T)$ are obtained from $\tau - \tau_0$ by replacing I_2 by parallel copies, one for each strand of the γ_j piercing Δ_i, and then replacing each of these copies with 2 parallel copies (corresponding to the 2 sides of each band).

Hence, in view of the doubling formula 3.2.6, $Z^t(\delta(T))$ has a description, given as follows:

Recall that T' is the tangle obtained from T by taking band connected sums along bands b_i with disks Δ_i. Identifying the underlying 1-manifold, X, of the tangle T' with its image, we define submanifolds X_i^Δ and X_i^b of X by setting $X_i^\Delta = X \cap \partial(\Delta_i)$ and $X_i^b = X \cap b_i$. Let $\overline{X}$ be obtained from X by identifying the two parallel strands in each X_i^b with each other. (Although $\overline{X}$ is not a manifold, one still has the notion of diagrams $\overline{X} \cup \Gamma$.) See figure 12.

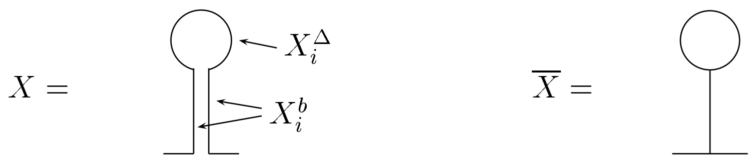


Figure 12. An example of X_i^Δ, X_i^b, and $\overline{X}$.

[19] As we have chosen the canonical q-structure, $Z^t(\sigma_{2,3}) = \exp(Y)$ is not equal to $\exp(\xi_{2,3})$, where $\xi_{2,3}$ is the chord connecting the second and third components, but obtained from it by conjugating by the associator.

Then $Z^t(\delta(T))$ is a linear combination of terms, each of which is the pullback (*i.e.*, the sum over all diagrams obtained by lifting vertices on $\overline{X}$ to X under the projection $X \to \overline{X}$) from a diagram $\overline{X} \cup \Gamma$ (since the two strands in b_i are parallel), with the property that any univalent vertex lying in X_i^{Δ} is the origin of a path in Γ ending in some univalent vertex lying in $\cup_j \overline{X}_j^b$. We choose such a path γ_v for every univalent vertex v lying in $\cup_i X_i^{\Delta}$.

Now consider any univalent vertex of Γ lying on $\overline{X}_j^b$. Assume this is the one lying closest to X_j^{Δ}. This lifts to two terms, one for each component of X_j^b. These two terms are equivalent to a sum of terms (by the STU relation), with the vertex attached to a branch of Γ arriving at X_j^{Δ}. See figure 13 for an example.

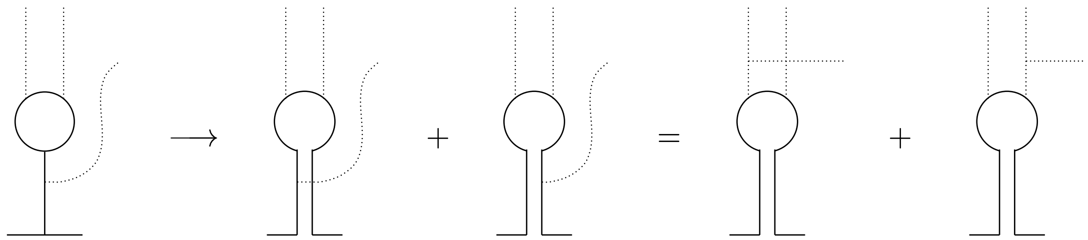


Figure 13. Applying the STU relation.

Repeating this for all vertices lying on $\overline{X}_j^b$, for all j, we thus arrive at a sum of terms, such that in each term, the paths γ_v give rise to paths γ'_v with the following property. Here γ'_v has the same origin as γ_v, but ends in a trivalent vertex in the interior of another path γ'_w (w may be equal to v). Since all paths have a preferred direction (*e.g.*, starting at v), it follows that the union of these paths (which is compact) has a loop, and hence such a term is trivial in $A^t(X)$.

This completes the proof of theorem 14.1.

Remark 14.3. An alternative proof could be given based on theorem 6.1 and the methods of [HL2], along the following lines. First, the concordance invariance of $Z^t(\sigma)$, for string links σ, follows, by induction on the filtration degree, from 6.1, the concordance invariance of Milnor's invariants, and the fact that $Z^t(\overline{\sigma}) = Z^t(\sigma)^{-1}$ (since if σ is concordant to σ', then $\overline{\sigma}\sigma'$ is nul-concordant, hence it has vanishing Milnor invariants, and thus (by 6.1) $1 = Z^t(\overline{\sigma}\sigma') = Z^t(\sigma)^{-1}Z^t(\sigma')$). The concordance invariance of $Z^t(T)$ for an arbitrary tangle (*i.e.*, theorem 14.1) can then be deduced from the (generalized version of the) structure theorem of [HL2].

15. The universal finite type concordance invariant.

Let X be a q-tangle type. Let $\mathbb{Q}\mathcal{CT}(X)$ denote the $\mathbb{Q}$-vector space with basis the set of framed embeddings of X in $\mathbb{C} \times \mathrm{I}$, up to concordance rel. boundary. Let $\mathbb{Q}\mathcal{CT}(X)_n$ be the Vassiliev filtration on $\mathbb{Q}\mathcal{CT}(X)$ (induced from $\mathbb{Q}\mathcal{T}(X)_n$). By 14.1, Z^t is well-defined on $\mathbb{Q}\mathcal{CT}(X)$. Let $Z^t_{\le n}$ be the composite with the projection onto $A^t_{\le n}(X)$.

Theorem 15.1. *The sequence,*

$$0 \to \mathbb{Q}\mathcal{CT}(X)_{n+1} \to \mathbb{Q}\mathcal{CT}(X) \xrightarrow{Z^t_{\le n}} A^t_{\le n}(X) \to 0$$

is exact. In other words, Z^t is the universal finite type concordance invariant.

Note. This theorem answers a question of Bar-Natan [B1], who asked for an interpretation of the algebra $A^t(l)$.

For the proof, we first need to investigate the relationship between loop diagrams and nul-concordant string links.

Let σ be a string link with l components. In section 1, we defined $\overline{\sigma}$, which is a concordance inverse of σ.

Lemma 15.2. *Assume $Z(\sigma) = 1 + \xi + O(n+1)$ with $\xi \in P_n(l) \subset A_n(l)$ represented by a diagram $X \cup \Gamma$ with connected dashed graph Γ with Euler characteristic $e(\Gamma) = 0$ such that Γ has at most one vertex on each component of $\coprod_{j=1}^{j=l} \mathrm{I}_j$. Then $Z(\sigma\overline{\sigma}) = 1 + 2\xi + O(n+1)$.*

Proof. Write $Z(\sigma) = \exp(\xi) + O(n+1)$. Then $Z(\sigma\overline{\sigma}) = \exp(\xi + R(\xi)) + O(n+1)$, where $R = (-1)^n r_*$ (see 3.2.7). Since Γ has at most one vertex on each component, and r is orientation-reversing on each component, one has $r_*(\xi) = (-1)^{v_1}\xi$ as elements of $P_n(l)$, where v_i denotes the number of i-valent vertices of Γ. Note that $n = (v_3 + v_1)/2$ and $2a = 3v_3 + v_1$, where a is the number of edges of Γ. Thus $n = a - v_3$ and hence $e(\Gamma) = v_1 + v_3 - a = v_1 - n$. Since $e(\Gamma) = 0$ by hypothesis, it follows that $R(\xi) = \xi$, proving the lemma.

Remark 15.3. The computation of $e(\Gamma)$ in the proof above shows that on labelled Chinese characters the map R is simply multiplication by $(-1)^e$, where e is the Euler characteristic. (Indeed, the element of $A(l)$ associated to a Chinese character Γ is the average of the ways of putting Γ on the strings; this average, $a(\Gamma)$, say, again satisfies $r_*(a(\Gamma)) = (-1)^{v_1}a(\Gamma)$ and hence $R(a(\Gamma)) = (-1)^{e(\Gamma)}a(\Gamma)$.)

Remark 15.4. One may use the observation in 15.3 to give a direct proof of the fact that $Z^t(\overline{\sigma}) = Z^t(\sigma)^{-1}$ (without knowing that Z^t is a concordance invariant), as follows. Write $Z^t(\sigma) = \exp(\xi_0 + \xi_1)$, with ξ_i primitive and $R(\xi_i) = (-1)^i\xi_i$. Then $Z^t(\overline{\sigma}) = R(Z^t(\sigma)) = \exp(\xi_0 - \xi_1)$. Now consider the (comultiplicative, but not multiplicative) isomorphism of $A(l)$ with the algebra $B(l)$ of labelled Chinese characters. The R-symmetrical Chinese characters have even Euler characteristic and so the primitives ones (*i.e.*, the connected ones) are non-simply connected, and hence trivial in $B^t(l)$. This shows that ξ_0 is zero in $A^t(l)$, and hence $Z^t(\overline{\sigma}) = \exp(-\xi_1) = Z^t(\sigma)^{-1}$, as asserted.

Proof of theorem 15.1. By definition and by 14.1, the short exact sequence

$$0 \to \mathbb{Q}\mathcal{T}(X)_{n+1} \to \mathbb{Q}\mathcal{T}(X)_n \xrightarrow{Z_n} A_n(X) \to 0,$$

maps surjectively to the sequence

$$0 \to \mathbb{Q}\mathcal{C}\mathcal{T}(X)_{n+1} \to \mathbb{Q}\mathcal{C}\mathcal{T}(X)_n \xrightarrow{Z_n^t} A_n^t(X) \to 0$$

from which the theorem follows by induction, provided we can show this latter sequence is exact.

Obviously we have injectivity and surjectivity where required. To see exactness in the middle we proceed as follows. By definition, $\mathbb{Q}\mathcal{C}\mathcal{T}(X)_n$ is the image of $\mathbb{Q}\mathcal{T}(X)_n$. Thus it suffices to see that if $x \in \mathbb{Q}\mathcal{T}(X)_n$ is in the kernel of the map to $A_n^t(X)$, then we may find an element $y \in \mathbb{Q}\mathcal{T}(X)_n$ in the kernel of the map to $\mathbb{Q}\mathcal{C}\mathcal{T}(X)_n$, such that $x - y$ lies in the kernel of the map to $A_n(X)$.

Hence it suffices to see that for all elements $\xi \in A_n(X)$ which lie in the kernel of the map to $A^t_n(X)$ (or by linearity, for a set of generators of this kernel), there is an element $y \in \mathbb{Q}\mathcal{T}(X)_n$, which is in the kernel of the map to $\mathbb{Q}\mathcal{CT}(X)_n$, such that $Z_n(y) = \xi$.

The kernel of the map $A_n(X) \to A^t_n(X)$ is (by STU) generated by diagrams $X \cup \Gamma$ containing a component with Euler characteristic 0. Let Γ_i denote the connected components of Γ. We may assume that Γ_1 is the component with Euler characteristic 0.

Let X_i be a regular neighborhood of the vertices of Γ_i in X. Fix a framed embedding T of X for which the X_i each form trivial string links e_i in disjoint balls B_i. Now $\amalg(X_i \cup \Gamma_i)$ represents an element $\otimes\xi_i \in \otimes A_{n_i}(X_i) \subset A_n(\amalg X_i)$, where n_i is the degree of ξ_i, which maps to $\xi \in A_n(X)$ (under the inclusion $\amalg X_i \subset X$). By proposition 13.1, there is some string link σ_i such that $Z(\sigma_i) = 1 + \xi_i + O(n_i + 1)$. Since Γ_1 has even Euler characteristic, and has precisely 1 vertex on each component of X_1, it follows from lemma 15.2 that $Z(\sigma_1\overline{\sigma}_1) = 1+2\xi_1+O(n_1+1)$. Note that $\sigma_1\overline{\sigma}_1$ is nul-concordant. It follows that the linear combination, y, say, of framed embeddings of X, obtained from T by inserting $\sigma_1\overline{\sigma}_1 - e_1$ in B_1 and $\sigma_i - e_i$ in B_i, maps trivially to $\mathbb{Q}\mathcal{CT}(X)_n$. On the other hand, $Z(y) = 2\xi + O(n+1)$, since the value of Z on the restriction of T to the complement (in X) of $\amalg X_i$ is $1 + O(1)$. Thus $Z_n(y) = 2\xi$ and the theorem follows.

Remark 15.5. Alternatively to our construction above of $2\xi_1$ by a nul-concordant string link, one could also use a construction of K. Y. Ng, to realize a generating set of loop diagrams by ribbon (and hence nul-concordant) string links (see [Ng]).

Remark 15.6. A similar argument as in the proof above shows that if $\overline{A}$ is a monoidal category quotient of A (see section 4) such that for all string links σ, the induced invariant $\overline{Z}$ satisfies $\overline{Z}(\overline{\sigma}) = \overline{Z}(\sigma)^{-1}$ (*e.g.*, if $\overline{Z}(\sigma)$ is a concordance invariant of σ), then $\overline{A}$ is a quotient of A^t.

We conclude this section with the proof of Corollary 6.4, which states that the algebra of rational finite type concordance invariants of string links is generated by Milnor's invariants.

Proof of Corollary 6.4. Rational finite type concordance invariants of string links form a $\mathbb{Q}$-algebra, $\mathcal{V}$, say, equipped with a compatible ascending filtration $\mathbb{Q} = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}$. By 15.1, one has $gr_*\mathcal{V} = A^t(l)^*$, where $A^t(l)^* = \oplus A^t_n(l)^*$ and $A^t_n(l)^*$ denotes the dual vector space of $A^t_n(l)$. Elements of $A^t_n(l)^*$ are usually referred to as *weight systems*. It is known [B1, L] that Milnor's invariants of string links are of finite type. (See also remarks 11.3 and 12.8.) In order to show that they generate $\mathcal{V}$, it suffices to show that $A^t(l)^*$ is generated by weight systems of Milnor invariants.

By theorem 6.1, for every $\varphi \in A^t_n(l)^*$, there is an invariant $M\colon SL(l) \to \mathbb{Q}$ such that M is a linear combination of Milnor invariants and $\varphi \circ Z^t_n$ coincides with M when restricted to $SL(l)_n$, the set of string links of (Milnor or Z^t-) filtration n. Let φ_M be the weight system of M. Then $\varphi - \varphi_M$ is zero on the primitives of $A^t_n(l)$, since every primitive element of $A^t_n(l)$ is realized rationally as a string link (in $SL(l)_n$). But a (homogeneous) weight system $\psi \in A^t_n(l)^*$ is zero on primitives if and only if it is *decomposable*, *i.e.,* a linear combination of products of weight systems of strictly lower degree.[20] Thus, $\varphi - \varphi_M$ is decomposable, and the result follows.

[20] This follows from the fact that $A^t(l)^*$ is the symmetric algebra over $P^t(l)^*$, where $P^t(l)^*$ is the dual of the space of primitives, $P^t(l)$, of $A^t(l)$. (This can be seen using the algebra isomorphism of $A^t(l)^*$ with $B^t(l)^*$.)

16. Appendix on the Vassiliev filtration of pure braids.

We define the Vassiliev filtration on $\mathbb{Q}PB(l)$ to be the filtration induced by the Vassiliev filtration on string links, *i.e.*, we put $\mathbb{Q}PB(l)_n = \mathbb{Q}PB(l) \cap \mathbb{Q}SL(l)_n$. Let I denote the augmentation ideal of $\mathbb{Q}PB(l)$. Let $A'(l)$ be the graded completed subalgebra of $A(l)$ generated by chord diagrams in degree 1 between distinct components. $A'(l)$ is a Hopf subalgebra.

Theorem 16.1. *(i) The Vassiliev filtration on $\mathbb{Q}PB(l)$ coincides with the I-adic filtration,* i.e., *one has $I^n = \mathbb{Q}PB(l) \cap \mathbb{Q}SL(l)_n$.*

(ii) Moreover, Z induces an isomorphism (of complete Hopf algebras) from the I-adic completion of $\mathbb{Q}PB(l)$, to $A'(l)$.

Note. Since $A'(l)$ injects into $A^t(l)$, one may replace Z by Z^t in 16.1(ii).

Remark 16.2. It is easy to see that the I-adic filtration on pure braids corresponds to the filtration 'à la Vassiliev' where one allows only 'horizontal' crossings, *i.e.*, coming from 'singular' braids (all strands remaining vertical).[21] It has been known for some time (see [K1, K2]), that this filtration corresponds to the Hopf algebra *à la* Vassiliev of 'horizontal' chord diagrams and 'horizontal' 4T relations. Theorem 16.1 says that this algebra *injects* into the Hopf algebra $A^t(l)$, and *a fortiori* into $A(l)$. (This latter has recently been proven in a different way by Bar-Natan [B4].)

Lemma 16.3. *If $\lambda \in F(l)$ satisfies $[\lambda, x_i] \in F(l)_{n+1}$ for a generator x_i, then for some $m \in \mathbb{Z}$, $\lambda x_i^m \in F(l)_n$.*

This well-known fact can be proven using the Magnus expansion [MKS].

Lemma 16.4. *Let σ be a pure braid (with zero framing). Then $Z^t(\sigma) = 1 + O(n)$ if and only if $\sigma \in PB(l)_n$.*

Proof. This follows from theorem 6.1 and the fact that a pure braid has Milnor filtration n if and only if it lies in $PB(l)_n$. Here is a proof of this fact. We use the characterization of the Milnor filtration by the Artin representation (remark 5.1). Thus, we will show that $\sigma \in PB(l)_n$ if and only if $\mathcal{A}_n(\sigma) = 1 \in Aut(F(l)/F(l)_{n+1})$.

Note that the image of the Artin representation of a string link lies in the subgroup $Aut_0(F(l)/F(l)_{n+1})$ of $Aut(F(l)/F(l)_{n+1})$, consisting of those automorphisms which conjugate the generators and which preserve the product of the generators. Thus these automorphisms act trivially on homology. In particular, one has that $Aut_0(F(l)/F(l)_2)$ is trivial. Using lemma 16.3, it is easy to see that the projection $Aut_0(F(l)/F(l)_{n+1}) \to Aut_0(F(l)/F(l)_n)$ is a central extension. It follows that $Aut_0(F(l)/F(l)_{n+1})$ is a nilpotent group of class $n-1$. This shows that $PB(l)_n$ is in the kernel of $\mathcal{A}_n$.

To see the converse, we proceed by induction on l and use the semidirect product decomposition $PB(l) = F(l-1) \rtimes PB(l-1)$. Suppose the image of $x \in PB(l)$ is trivial in $Aut_0(F(l)/F(l)_{n+1})$. Let $\overline{x} \in PB(l-1)$ be its image. Then the image of $\overline{x}$ is trivial in $Aut_0(F(l-1)/F(l-1)_{n+1})$, so inductively we have $\overline{x} \in PB(l-1)_n$.

Hence it is enough to show that $y = x\overline{x}^{-1} \in F(l-1)$ lies in $F(l-1)_n$. Recall that $\mathcal{A}_n(\sigma)(x_i) = \lambda_i x_i \lambda_i^{-1}$, where the x_i are the generators of the free groups $F(l)$, and the λ_i are the longitudes. Note that under the forgetful map, $\lambda_1 \in F(l)$ maps to $y \in F(l-1) \subset$

[21] This suggests calling the I-adic filtration the 'horizontal Vassiliev filtration', so that 16.1 may be expressed by saying that this coincides with the actual Vassiliev filtration.

$PB(l)$. Thus if the image of y in $Aut_0(F(l)/F(l)_{n+1})$ is trivial, this says in particular that $[\lambda_1, x_1] \in F(l)_{n+1}$. By lemma 16.3, this implies $\lambda_1 x_1^m \in F(l)_n$, for some $m \in \mathbb{Z}$, and hence $y \in F(l-1)_n \subset PB(l)_n$, completing the proof.

Proof of theorem 16.1. Note that since the associator and braid crossings both have image in the Hopf subalgebra $A'(l)$ generated by chord diagrams in degree 1 between distinct components, one has that Z^t maps $\mathbb{Q}PB(l)$ into this subalgebra. Since I/I^2 maps surjectively to A'_1, it follows that the map

$$\widehat{\oplus}_{n\geq 0}(I^n/I^{n+1}) \longrightarrow A'(l) \tag{3}$$

is surjective (see [Q2, 2.17]). To prove injectivity, as in the proof of 10.3, it is enough to see (by [Q2, 2.21]), that the map is injective on primitives. By [Q2, 3.11], the set of primitives of $\widehat{\oplus}_{n\geq 0}(I^n/I^{n+1})$ is

$$\widehat{\oplus}_{n\geq 1}(PB(l)_n/PB(l)_{n+1}) \otimes \mathbb{Q}.$$

By theorem 6.1 and lemma 16.4, if $x \in PB(l)_n$ and $Z^t_n(x) = 0$, then $x \in PB(l)_{n+1}$. It follows that

$$(PB(l)_n/PB(l)_{n+1}) \otimes \mathbb{Q} \longrightarrow A^t_n(l)$$

is injective, and hence (3) is an isomorphism, proving 16.1(ii).

In particular, this shows that I^n/I^{n+1} injects into $A'_n(l)$, and *a fortiori* into $A_n(l)$. Thus

$$\frac{I^n}{I^{n+1}} \longrightarrow \frac{\mathbb{Q}PB(l) \cap \mathbb{Q}SL(l)_n}{\mathbb{Q}PB(l) \cap \mathbb{Q}SL(l)_{n+1}}$$

is injective, and so, inductively, it follows that $I^n = \mathbb{Q}PB(l) \cap \mathbb{Q}SL(l)_n$. In other words, the Vassiliev filtration on $\mathbb{Q}PB(l)$ coincides with the I-adic filtration, as asserted in 16.1(i). This completes the proof.

Corollary 16.5. *The reduced Kontsevich integral Z^t (and a fortiori the Kontsevich integral Z) separates pure braids.*

Proof. In view of lemma 16.4, it suffices to show that the intersection of the lower central series of $PB(l)$ is trivial. This is well-known [FR], but for completeness, here is a proof. The pure braid group is a semidirect product $PB(l) = F(l-1) \rtimes PB(l-1)$. Moreover, the action of $PB(l-1)$ on the free group $F(l-1)$ is via the Artin representation. Then, since this representation acts trivially on homology, it is an exercise to see that the lower central series of $PB(l)$ is the semidirect product of the lower central series of $F(l-1)$ and that of $PB(l-1)$. Since the intersection of the lower central series of a free group is trivial, by induction, the same holds for the pure braid group.

Remark 16.6. Since the lower central series quotients of the free group are torsion free, the argument above also shows that the lower central series quotients $PB(l)_n/PB(l)_{n+1}$ are torsion free, and hence the pure braid group $PB(l)$ is residually torsion free nilpotent. By Chen [C, 2.2.1], this implies that the I-adic filtration of the group ring $\mathbb{Q}PB(l)$ is complete, *i.e.*, satisfies $\cap I^n = 0$. By 16.1, the same is true for the Vassiliev filtration on pure braids.

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