MILNOR NUMBERS, SPANNING TREES, AND THE ALEXANDER-CONWAY POLYNOMIAL

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ABSTRACT. We study relations between the Alexander-Conway polynomial ∇_L and Milnor higher linking numbers of links from the point of view of finite-type (Vassiliev) invariants. We give a formula for the first non-vanishing coefficient of ∇_L of an m-component link L all of whose Milnor numbers $\mu_{i_1\dots i_p}$ vanish for $p \le n$. We express this coefficient as a polynomial in Milnor numbers of L. Depending on whether the parity of n is odd or even, the terms in this polynomial correspond either to spanning trees in certain graphs or to decompositions of certain 3-graphs into pairs of spanning trees. Our results complement determinantal formulas of Traldi and Levine obtained by geometric methods.

CONTENTS

1. INTRODUCTION

The Alexander-Conway polynomial

$$\nabla_L(z) = \sum_{i\ge 0} c_i(L) z^i \in \mathbb{Z}[z]$$

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of a link L in $\mathbb{R}^3$ is one of the most thoroughly studied classical isotopy invariants of links. In this paper we study relations between ∇_L and and the Milnor higher linking numbers of L from the point of view of the theory of finite-type (Vassiliev) invariants.

Hosokawa [Hs], Hartley [Ha, (4.7)], and Hoste [Ht] showed that the coefficients $c_i(L)$ of ∇_L for an m-component link L vanish when $i \leq m-2$ and that the coefficient $c_{m-1}(L)$ depends only on the linking numbers $\ell_{ij}(L)$ between the ith and jth components of L. Namely,

$$c_{m-1}(L) = \det \Lambda^{(p)}, \tag{1}$$

where $\Lambda = (\lambda_{ij})$ is the matrix formed by linking numbers

$$\lambda_{ij} = \begin{cases} -\ell_{ij}(L), & \text{if } \ i \neq j \\ \sum_{k\neq i} \ell_{ik}(L), & \text{if } \ i = j \end{cases}$$

and $\Lambda^{(p)}$ denotes the matrix obtained by removing from Λ the pth row and column (it is easy to see that $\det \Lambda^{(p)}$ does not depend on p).

If the link L is *algebraically split*, *i.e.* all linking numbers ℓ_{ij} vanish, then not only $c_{m-1}(L) = 0$, but, as was proved by Traldi [Tr1, Tr2] and Levine [Le1], the next $m-2$ coefficients of ∇_L also vanish

$$c_{m-1}(L) = c_m(L) = \ldots = c_{2m-3}(L) = 0.$$

For algebraically split oriented links, there exist well-defined integer-valued isotopy invariants $\mu_{ijk}(L)$ called the *Milnor triple linking numbers.* These invariants generalize ordinary linking numbers (see Section 5), but unlike ℓ_{ij}, the triple linking numbers are antisymmetric with respect to their indices, $\mu_{ijk}(L) = -\mu_{jik}(L) = \mu_{jki}(L)$. Thus, for an algebraically split link L with m components, we have $\binom{m}{3}$ triple linking numbers $\mu_{ijk}(L)$ corresponding to the different 3-component sublinks of L.

Levine [Le1] (see also Traldi [Tr2, Theorem 8.2]) found an expression for the coefficient $c_{2m-2}(L)$ of ∇_L for an algebraically split m-component link in terms of triple Milnor numbers

$$c_{2m-2}(L) = \det \Lambda^{(p)}, \tag{2}$$

where $\Lambda = (\lambda_{ij})$ is an $m \times m$ skew-symmetric matrix with entries

$$\lambda_{ij} = \sum_k \mu_{ijk}(L),$$

and $\Lambda^{(p)}$, as before, is the result of removing the pth row and column. (In particular, since the determinant of an antisymmetric matrix of odd size is always zero, this implies that $c_{2m-2}(L) = 0$ if m is even.[1])

It is well known that the coefficient c_n of $\nabla(z)$ is a finite-type (Vassiliev) invariant of order n whose weight system can be computed by a simple combinatorial rule (see Section 3). This leads to a simple proof of vanishing of $c_k(L)$ for $k \leq m-2$ and also to an explicit expression for the coefficient c_{m-1} as a sum of products of linking numbers corresponding to maximal trees in the complete graph K_m with vertices $\{1, \ldots, m\}$. Thus, there is a second formula for c_{m-1} (see [Ha] and [Ht])

$$c_{m-1}(L) = \mathcal{D}_m(\ell_{ij}(L)) \ . \tag{3}$$

Here $\mathcal{D}_m(x_{ij})$ is the *Kirchhoff polynomial* whose monomials correspond to spanning trees in the complete graph K_m (see Section 2).

The equality of the expressions (1) and (3) for $c_{m-1}(L)$ follows from the classical *Matrix-Tree Theorem* (see *e.g.* [Bol, Tut]) applied to the graph K_m.

Formula (2) is similar to the first determinantal expression (1). One of our initial goals in this work was to find an analog of the tree sum formula (3) for algebraically split links, and one of our results is a formula expressing c_{2m-2} as the square of a sum over trees

$$c_{2m-2} = (\mathcal{P}_m(\mu_{ijk}))^2 \ , \tag{4}$$

where $\mathcal{P}_m$ is the Pfaffian-tree polynomial introduced in [MV]. Similarly to the Kirchhoff polynomial $\mathcal{D}_m$, the polynomial $\mathcal{P}_m$ is the generating function of spanning trees in the complete 3-graph Γ_m with m vertices (see Section 2).

For $m = 3$, both formula (2) and our theorem give the same known result first proved by Cochran[2] in [Co, Theorem 5.1]

$$c_4(L) = (\mu_{123}(L))^2 \ .$$

In the $m \geq 5$ case, our formula is new. For example, when $m = 5$, we obtain that the first non-vanishing coefficient of $\nabla_L(z)$ for algebraically split links with 5 components is equal to

$$\begin{aligned} c_8(L) &= \mathcal{P}_5(\mu_{ijk}(L))^2 \\ &= \big(\mu_{123}(L)\mu_{145}(L) - \mu_{124}(L)\mu_{135}(L) + \mu_{125}(L)\mu_{134}(L) \ \pm \ \ldots\big)^2, \end{aligned}$$

[1]This follows, of course, already from the well-known fact that the Alexander-Conway polynomial of an m-component link has non-zero terms only in degrees congruent to $m-1$ *mod* 2.

[2]Before that, Kidwell and Morton showed that the coefficient c_{m-2} for algebraically split links is a perfect square.

where $\mathcal{P}_5(\mu_{ijk}(L))$ consists of 15 terms corresponding to the spanning trees in the complete 3-graph with 5 vertices.

It is worth pointing out that our formula (4) has some computational advantages over (2). For example, the straightforward expansion of $\det \Lambda^{(p)}$ in (2) for $m = 5$ would consist (before cancellations) of 729 terms, each of which is a product of four μ_{ijk}'s. In contrast, the computation based on (4) only requires finding the sum of 15 products of two μ_{ijk}'s each, and taking the square.

If all triple Milnor linking numbers of a link L vanish, then its Milnor numbers of length four are well-defined, *etc.*, and so it is natural to ask whether the lowest term of the Alexander-Conway polynomial $\nabla_L(z)$ can be expressed via the first non-vanishing Milnor numbers of L. If L is an m-component link all of whose Milnor higher linking numbers $\mu_{i_1\ldots i_p}$ vanish for $p \leq n$, then, as it was proved by Traldi [Tr2] and Levine [Le2], the polynomial $\nabla_L(z)$ begins with the term of degree $n(m-1)$. The corresponding coefficient can be computed, as before, by the determinantal formula

$$c_{n(m-1)}(L) = \det \Lambda^{(p)}, \tag{5}$$

where the (i,j)th entry of the matrix Λ is expressed via Milnor invariants of length $n+1$

$$\lambda_{ij} = \sum_{r_1,\ldots,r_{n-1}} \mu_{r_1,\ldots,r_{n-1},j,i}(L)\ .$$

Our approach based on the theory of finite-type invariants leads to a new proof of vanishing of the coefficients $c_i(L)$ for $i < n(m-1)$ of the Alexander-Conway polynomial of such a link and to a tree-sum counterpart of formula (5). Namely, we show here that $c_{n(m-1)}(L)$ is equal either to the Kirchhoff polynomial or to the square of the Pfaffian-tree polynomial whose variables are linear combinations of the first non-vanishing Milnor numbers of L. Our results also lead to a similar formula for the next coefficient $c_{n(m-1)+1}(L)$ in the case when both n and m are even (and, therefore, $c_{n(m-1)}(L)$ is always zero); this formula is described in Section 8.

Whereas the original proofs of the determinantal formulas (1), (2) and (5) use classical topological tools (Seifert surfaces, geometric interpretation of the Milnor linking numbers, *etc.*), our approach based on finite-type invariants is, in some sense, more of a combinatorial nature. The connection between the Alexander-Conway polynomial and the Milnor numbers is established by studying their weight systems and then using the Kontsevich integral. In the dual language of the space of chord diagrams, the Milnor numbers correspond to the tree diagrams (see [HM]) and the Alexander-Conway polynomial can be described in terms of certain trees and wheel diagrams (see [KSA] and [Vai]). From this point of view, the trees in our formulas for the first non-vanishing term of ∇_L appear very naturally. Note that one may recover the corresponding (less symmetric) determinantal expressions from our tree-sum formulas by applying either the classical matrix-tree theorem [Bol, Tut] or our Pfaffian Matrix-Tree Theorem for 3-graphs [MV] (see Section 2). (In fact, it was an attempt to find a diagrammatic interpretation of formula (2) that led us to the discovery of the Pfaffian Matrix-Tree Theorem.)

The work presented in this paper is a first step in the study of the Alexander-Conway polynomial of links by means of finite type invariants. We believe that our approach will lead to a diagrammatic (and perhaps more canonical) interpretation of a factorization of this polynomial given by Levine in [Le2]. The factors are the Alexander-Conway polynomial of a knot obtained by banding together the components of the link, and a power series involving the Milnor invariants of a certain string link representative of the link. They should correspond respectively to the wheel and tree parts of the weight system of ∇_L.

The paper is organized as follows. In Section 2 we define the spanning-tree polynomials $\mathcal{D}_m$ and $\mathcal{P}_m$ and formulate our results. In Section 3 we recall some facts about the Alexander-Conway polynomial $\nabla_L(z)$ and its weight system and show, as a warm-up, how the philosophy of finite-type invariants leads naturally to the tree-sum formula (3) for $c_{m-1}(L)$. In Section 4 we study the weight systems corresponding to the coefficients of $\nabla_L(z)$ and prove that they vanish on a certain class of diagrams. The proof of our Vanishing Lemma 4.1 uses properties of the Alexander-Conway weight system from [FKV] which are based on the connection between ∇ and the Lie superalgebra $gl(1|1)$. Milnor linking numbers and their connection with finite-type invariants and the Kontsevich integral found in [HM] are described in Section 5. In Section 6 using the Vanishing Lemma of Section 4 and the methods of [HM], we show that for a link L whose Milnor invariants of degree $\leq n-1$ vanish, the first non-vanishing coefficient of $\nabla_L(z)$ can be expressed as a polynomial $F_m^{(n)}$ in the degree-n Milnor numbers of L. In Section 7 we express this polynomial in terms of the spanning tree polynomials $\mathcal{D}_m$ and $\mathcal{P}_m$ introduced in Section 2 and complete the proofs of Theorems 2.3 and 2.4. This identification is based on some algebraic properties of the Pfaffian-tree polynomial $\mathcal{P}_m$ which have been established in [MV]. In particular, one has $F_m^{(2)} = \mathcal{P}_m^2$, and we explain how this allows one to compute the coefficients of the polynomial $F_m^{(2)}$ by counting tree decompositions of certain associated 3-graphs. Finally in Section 8 we give a formula for the coefficient $c_{n(m-1)+1}$ in the case where $c_{n(m-1)}$ is identically zero for parity reasons.

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2. Spanning tree polynomials and statement of results

Recall the following result of Hartley [Ha] and Hoste [Ht].

Theorem 2.1. *Let L be an oriented link in S^3 with m (numbered) components. Let $\ell_{ij}(L)$ be the linking number between the ith and jth components. Then the first $m-1$ coefficients of the Alexander-Conway polynomial vanish,*

$$c_i(L) = 0 \textit{ for } i \leq m-2,$$

and the coefficient $c_{m-1}(L)$ is equal to the Kirchhoff polynomial (7) *evaluated at the linking numbers of L:*

$$c_{m-1}(L) = \mathcal{D}_m(\ell_{ij}(L)) \ . \tag{6}$$

Here the *Kirchhoff polynomial* (or the *spanning trees generating function*) of the complete graph K_m with m vertices is the polynomial

$$\mathcal{D}_m(x_{ij}) = \sum_T x_T \tag{7}$$

whose $\binom{m}{2}$ variables

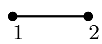
$$x_{ij},\ 1 \leq i,j \leq m,\ i \neq j,\ x_{ij} = x_{ji},$$

correspond to the edges of K_m and whose monomials

$$x_T = \prod_{e \in T} x_e$$

correspond to the maximal (spanning) trees T in K_m.

For example, if $m = 2$, then $\mathcal{D}_2 = x_{12}$, corresponding to the only spanning tree in

$K_2 =$ 

and so $c_1(L) = \ell_{12}(L)$. If $m = 3$, then $\mathcal{D}_3 = x_{12}x_{23} + x_{23}x_{31} + x_{31}x_{12}$ (see Figure 1) and so

$$c_2(L) = \ell_{12}(L)\ell_{23}(L) + \ell_{23}(L)\ell_{31}(L) + \ell_{31}(L)\ell_{12}(L) \ .$$

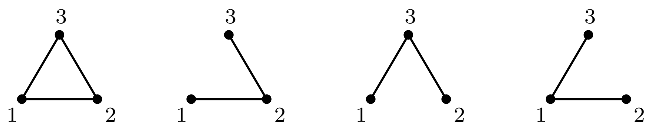


Figure 1. The complete graph K_3 and its three spanning trees.

To state an analog of Theorem 2.1 for algebraically split links we need to introduce another tree-generating polynomial analogous to the Kirchhoff polynomial.

Namely, instead of usual graphs whose edges can be thought of as segments joining pairs of points, we consider 3-*graphs* whose edges have three (distinct) vertices and can be visualized as triangles or Y-shaped objects  with the three vertices at their endpoints.

Similarly to variables x_{ij} of $\mathcal{D}_m$, for each triple of distinct numbers $i, j, k \in \{1, 2, \ldots, m\}$ we introduce variables y_{ijk} antisymmetric in i, j, k

$$y_{ijk} = -y_{jik} = y_{jki}, \text{ and } y_{iij} = 0 \ .$$

These variables correspond to edges $\{i, j, k\}$ of the *complete* 3-*graph* Γ_m with vertices $\{1, \ldots, m\}$. As in the case of ordinary graphs, the correspondence

$$\text{variable } \ y_{ijk} \quad \mapsto \quad \text{edge } \{i, j, k\} \text{ of } \ \Gamma_m$$

assigns to each monomial in y_{ijk} a sub-3-graph of Γ_m. However, because of the antisymmetry, the correspondence between monomials and sub-3-graphs is not one-to-one. A sub-3-graph determines a monomial only up to sign, and to define an analog of the tree-generating function for the complete 3-graph we need to fix signs.

The generating function of spanning trees in the complete 3-graph with m vertices, or the *Pfaffian-tree polynomial* $\mathcal{P}_m$ is defined as follows [MV]. If m is even, then we set

$$\mathcal{P}_m = 0$$

(there are no spanning trees in 3-graphs with even number of vertices). If m is odd, then

$$\mathcal{P}_m = \sum_T \varepsilon(T) y_T \ , \tag{8}$$

where the sum is taken over all monomials

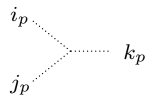
$$y_T = \prod_{p=1}^{d} y_{i_p j_p k_p}$$

of degree $d = (m-1)/2$ in y_{ijk} (with the convention that monomials which differ only by changing the order of indices in some of the variables are taken only once) and the coefficient $\varepsilon(T) \in \{0, \pm 1\}$ for the collection of triples

$$T = ((i_1 j_1 k_1), \ldots, (i_d j_d k_d))$$

is defined by the following rule.

Glue together the d Y-shaped objects



corresponding to the variables in the monomial y_T (see *e.g.* Figure 2). If the resulting 1-complex is a tree (*i.e.* it is connected and simply connected, or, equivalently, the corresponding sub-3-graph in Γ_m is a tree), then the product

$$\sigma_T = \sigma_1 \ldots \sigma_d$$

obtained by multiplying (in an arbitrary order) the 3-cycles $\sigma_p = (i_p j_p k_p)$ in the symmetric group $\mathfrak{S}_m$ is an m-cycle

$$\sigma = (s(1)s(2)\ldots s(m)),$$

for some $s \in \mathfrak{S}_m$ and we define

$$\varepsilon(T) = (-1)^s \ . \tag{9}$$

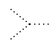
If the monomial T does not produce a tree, we set $\varepsilon(T) = 0$.

Remark 2.2. In fact, σ_T is an m-cycle if and only if the monomial y_T corresponds to a tree (see [MV]). The terms of the Kirchhoff polynomial $\mathcal{D}_m$ also admit a similar description: a monomial $x_{i_1 j_1} \ldots x_{i_{m-1} j_{m-1}}$ enters $\mathcal{D}_m$ (always with the plus sign) if and only if the product of the $m-1$ transpositions $\tau_k = (i_k j_k)$ (taken in any order) in the symmetric group $\mathfrak{S}_m$ is an m-cycle.

For example, if $m = 5$, we have

$$\mathcal{P}_5 = y_{123}\, y_{145} - y_{124}\, y_{135} + y_{125}\, y_{134} \ \pm \ \ldots \ , \tag{10}$$

where the right-hand side is a sum of 15 similar terms corresponding to the 15 spanning trees of Γ_5.

If we visualize the edges of Γ_m as Y-shaped objects  , then the spanning tree corresponding to the first term of (10) will look like on Figure 2.

In this paper we prove the following two theorems analogous to Theorem 2.1 for algebraically split links.

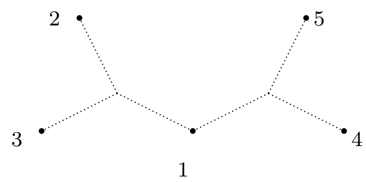


Figure 2. A spanning tree in the complete 3-graph Γ_5. It has two edges, $\{1,2,3\}$ and $\{1,4,5\}$, and contributes the term $y_{123}\,y_{145}$ to $\mathcal{P}_5$.

Theorem 2.3. *Let L be an algebraically split oriented link with m components. Then*

$$c_{2m-2}(L) = \big(\mathcal{P}_m(\mu_{ijk}(L))\big)^2 \ . \tag{11}$$

In fact, it follows from our *Pfaffian Matrix-Tree Theorem* [MV] (which is an analog for 3-graphs of the classical Matrix-Tree Theorem [Bol, Tut]) that the polynomial $\mathcal{P}_m(\mu_{ijk}(L))$ is equal to the Pfaffian of the skew-symmetric matrix $\Lambda^{(p)}$. This gives a combinatorial explanation of the equality of the expressions (2) and (11). However, we will not use the Pfaffian Matrix-Tree Theorem in the present paper and will give a direct proof of Theorem 2.3 based on the theory of finite type invariants.

Theorem 2.4. *Let L be an oriented m-component link with vanishing Milnor numbers of length $p \leq n$ and let*

$$\nabla_L(z) = \sum_{i\geq 0} c_i(L) z^i$$

be its Alexander-Conway polynomial. Then $c_i = 0$ for $i < n(m-1)$ and

$$c_{n(m-1)}(L) = \begin{cases} \mathcal{D}_m(\ell_{ij}^{(n)}) \ , & \textit{if } n \textit{ is odd} \\ (\mathcal{P}_m(\mu_{ijk}^{(n)}))^2 \, , & \textit{if } n \textit{ is even,} \end{cases} \tag{12}$$

where $\ell_{ij}^{(n)}$ and $\mu_{ijk}^{(n)}$ are certain linear combinations of the Milnor numbers of L of length $n+1$.

In the case when both n and m are even the coefficient $c_{n(m-1)}(L)$ is always zero for degree reasons and so $c_{n(m-1)+1}(L)$ becomes the first non-vanishing coefficient of ∇_L. In Section 8 we give a formula for this coefficient analogous to (12).

3. Alexander-Conway polynomial and spanning trees

In this section we recall the basic properties of the Alexander-Conway polynomial ∇_L and its weight system and explain the spanning-trees formula (3) for the first non-vanishing term of ∇_L from the point of view of finite-type invariants.

Let

$$\nabla_L(z) = \sum_{i\geq 0} c_i(L) z^i$$

be the Alexander-Conway polynomial of an oriented link L in S^3. It satisfies the *skein relation*

$$\nabla_{L_+} - \nabla_{L_-} = z\nabla_{L_0} \ , \tag{13}$$

where (L_+, L_-, L_0) is any skein triple (see Figure 3).

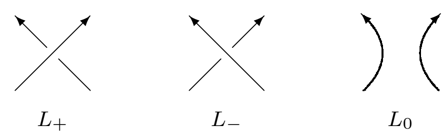


Figure 3. A skein triple.

The Alexander-Conway polynomial is uniquely determined by relation (13) and the initial conditions

$$\nabla_{U_m} = \begin{cases} 1 & \text{if } m = 1 \\ 0 & \text{if } m \geq 2, \end{cases} \tag{14}$$

where U_m is the trivial link with m components.

The skein relations (13) and (14) immediately give the well-known fact that for an m-component link

$$c_i(L) = 0 \ \text{ if } \ i \equiv m \ mod\ 2. \tag{15}$$

They also easily imply that for a 2-component link L, one has

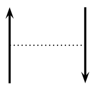
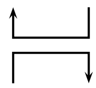
$$\nabla_L = \ell_{12}(L) z + \dots \ ,$$

where $\ell_{12}(L)$ is the linking number between the components of L.

Theorem 2.1 generalizes this fact to links with arbitrary number of components. One of the starting points for the present paper was the observation that the appearance of spanning trees in this theorem is very natural from the point of view of the theory of finite type invariants. As a warm-up, and also to motivate the techniques that we use in the subsequent sections, let us briefly discuss how one can understand this using the properties of the Alexander-Conway weight system.

It is well known that the coefficient c_n of the Alexander-Conway polynomial is a finite type invariant of degree n. Therefore it has a *weight system* W_n which is a linear form on the vector space of chord diagrams of degree n on m circles, where m is the number of components of the link. Here, by a chord diagram of degree n we understand a collection of n pairs of points on the union of these m circles. It is represented pictorially by n *dashed* lines, each line connecting the two points in a pair.

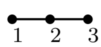
The skein relation (13) implies the following formula

(16) $W_n($  $) = W_{n-1}($  $)$

which allows to compute these weight systems recursively. For a chord diagram D on m circles with n chords, let D' be the result of smoothing of all chords by means of (16). Note that D' is a disjoint union of several, say m' circles (no chords are left). Thus,

$$W_n(D) = W_0(D') = \begin{cases} 1 & \text{if } m' = 1 \\ 0 & \text{if } m' \geq 2, \end{cases} \tag{17}$$

where the second equality follows from (14).

To see how this relates to formula (3), let us compute $W_n(D)$ for a chord diagram D of degree n on m circles. From (17) we see that $W_n(D)$ can only be non-zero if the diagram D' obtained by applying (16) to all the chords of D consists of just one circle. Since a smoothing of a chord cannot reduce the number of circles by more than one, this means that we need at least $m-1$ chords. Moreover, the diagrams D with exactly $m-1$ chords satisfying $W_{m-1}(D) \neq 0$ must have the property that if each circle of D is shrinked to a point, the resulting graph formed by the chords[3] is a tree. See Figure 4 for an example of a chord diagram D whose associated graph is the tree .

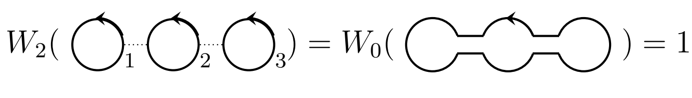


Figure 4. A degree 2 chord diagram D with $W_2(D) = 1$.

Thus, we come to the following result.

[3]In general, this will actually be a multi-graph, that is, it may have multiple edges, and also loop edges.

Lemma 3.1. *For chord diagrams on m circles, the Alexander-Conway weight system satisfies $W_i = 0$ for $i \leq m-2$. Moreover, W_{m-1} takes the value 1 on precisely those chord diagrams whose associated graph is a spanning tree on the complete graph K_m, and W_{m-1} is zero on all other chord diagrams.*

On the other hand, the linking number ℓ_{ij} is a finite type invariant of order 1 whose weight system is the linear form dual to the chord diagram having just one chord connecting the ith and jth circle. It follows that the Kirchhoff polynomial $\mathcal{D}_m$ (see (7)) in the linking numbers ℓ_{ij} is a finite type invariant of order $m-1$ and that its weight system, by our lemma, is equal to W_{m-1}. Thus, we have proved Theorem 2.1 on the level of weight systems.

Remark 3.2. This actually implies the theorem, since the Alexander-Conway polynomial is (almost) a canonical invariant, *i.e.* it can be recovered from its weight system by the Kontsevich integral. We will discuss this in a more general situation in the proof of Proposition 6.4.

This proof of Theorem 2.1 can be given without mentioning the theory of finite type invariants. Indeed, it can be reformulated as a proof by induction on the number of components using the skein relation (13) (see the original arguments in [Ha] and [Ht]). The point of our formulation is that it can be generalized from ordinary linking numbers to higher order Milnor numbers, where finite type invariants technology will play a crucial role.

4. A Vanishing Lemma

In this section, we generalize Lemma 3.1 and show that the Alexander-Conway weight system vanishes also on some classes of diagrams of degree higher than $m-2$.

We now allow not just chord diagrams, but also diagrams with internal (trivalent) vertices. Let us briefly review the relevant concepts and fix our notation. (For more on diagrams and finite-type invariants see *e.g.* [BN1].)

A *diagram* D on an oriented 1-manifold X is a finite uni-trivalent graph Γ whose univalent vertices are attached to (the interior of) X, and so that each trivalent vertex is equipped with a cyclic ordering of the three half-edges meeting at that vertex. It is assumed that every component of Γ has at least one univalent vertex.

The *degree* of a diagram D is one half of the total number of vertices of Γ (both univalent and trivalent vertices count). We denote by $\mathcal{A}_d(X)$ the $\mathbb{Q}$-vector space spanned by degree d diagrams on X, modulo IHX, AS, and STU relations (see Figure 5).

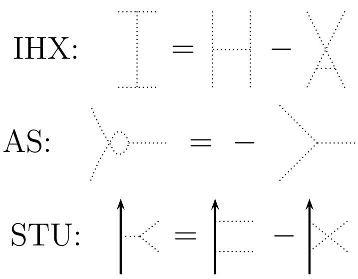


Figure 5. IHX, AS, and STU relations. Orientations at 3-valent vertices are those induced from the orientation of the plane. Four-valent vertices are an artifact of the planar pictorial representation and should be ignored.

Traditionally, the components of X are referred to as *solid*, while the components of Γ are referred to as *dashed*, since the edges of Γ are usually represented pictorially by dashed lines. Thus, for example, elements of the space $\mathcal{A}_d(\amalg_m S^1)$ are referred to as (linear combinations of) diagrams of degree d on m solid circles.

Let us agree that by the *components* of a diagram D, we always mean the components of its *dashed* part Γ. A component is called a *tree component* if its underlying graph is a tree. Note that the degree of a tree component is one less than the number of its univalent vertices. For example, a chord component has degree one, and a Y-shaped component  has degree two.

Recall that the weight system of the coefficient c_n of the Alexander-Conway polynomial is denoted by W_n. This is a linear form on the $\mathbb{Q}$-vector space $\mathcal{A}_n(\amalg_m S^1)$. We will simply write W for W_n, if the degree is clear from the context.

Note that (15) gives

$$W_n = 0 \text{ if } n \equiv m \; mod \, 2 \tag{18}$$

(this also follows immediately from (16) and (17)).

Proposition 4.1 (**Vanishing Lemma**)**.** *Let D be a degree-d diagram on $m \geq 2$ solid circles, such that D has no tree components of degree $\leq n-1$. If $d \leq n(m-1)+1$, then $W(D) = 0$ unless D has exactly $m-1$ components, each of which is a tree of degree $\geq n$.*

Corollary 4.2. *If a diagram D satisfies the hypotheses of the Vanishing Lemma, and $W(D) \neq 0$, then either $d = n(m-1)$ or $d = n(m-1)+1$. Moreover, the only two possibilities are the following:*

- *If at least one of the numbers n and m is odd, then $d = n(m-1)$ and each of the $m-1$ components of D is a tree of degree n.*
- *If both n and m are even, then $d = n(m-1)+1$ and D has $m-1$ components of which $m-2$ are trees of degree n and one is a tree of degree $n+1$.*

Proof of Corollary 4.2. This follows from Proposition 4.1 and (18), since D must be of degree $d \geq n(m-1)$.

□

In the proof of the Vanishing Lemma 4.1, it will be convenient to use the following terminology. If the Alexander-Conway weight system takes equal values

$$W(D) = W(D')$$

on two elements

$$D \in \mathcal{A}_d(X) \text{ and } D' \in \mathcal{A}_{d'}(X')$$

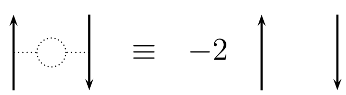
(possibly of different degrees, or maybe even with $X \neq X'$, in which case the correspondence between X and X' should be clear from the context), then we call D and D' *equivalent* and write this as

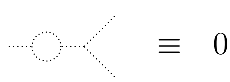
$$D \equiv D' .$$

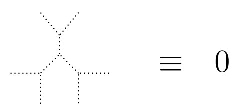
We begin with general formulas that allow to evaluate the Alexander-Conway weight system on $\mathcal{A}_d(\amalg_m S^1)$ for all d and m.

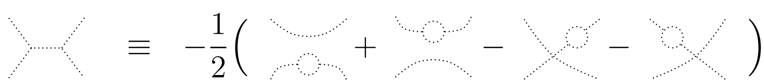
Lemma 4.3 ([FKV])**.** *The Alexander-Conway weight system satisfies the following reduction relations:*

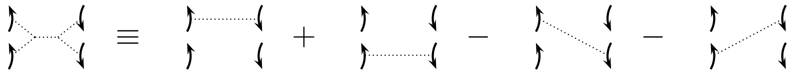
(19) 

(20) 

(21) 

(22) 

(23) 

(24) 

Proof. Relations (20)—(23) have been proved in [FKV]. They follow from the connection between the Alexander-Conway weight system W and the Lie superalgebra $gl(1|1)$. Equation (19) is equivalent to the fact that the invariant metric on $gl(1|1)$ needed to produce W is given by the trace in the defining representation. Finally, equation (24) is a corollary of (23) and (20).

□

Remark 4.4. In this paper, we will mainly need to apply these relations to diagrams where all univalent vertices lie on solid circles. For such diagrams, these relations can also be deduced from the smoothing relations (16) and (17), together with the STU relation.

Lemma 4.5. *Let D be a diagram on $m \geq 2$ solid circles with $a+b$ components that consist of a chords* *and b Y-shaped components*  *. If $a+b < m-1$, then $W(D) = 0$.*

(Note that if $b = 0$, this has already been proved in Lemma 3.1.)

Proof. The proof is by induction on m. If $m = 2$, then $a = b = 0$ and so $W(D) = 0$ by (14). Now assume $m > 2$. Since $a + b < m - 1$, the number of univalent vertices of D, which is $2a + 3b$, is strictly less than $3m$. Therefore at least one of the solid circles has at most 2 univalent vertices on it. If this solid circle has no univalent vertices, then $W(D) = 0$ by (14). If it has one or two univalent vertices, we apply the reduction relations of Lemma 4.3 to get $D \equiv D'$, where D' is a diagram on $m-1$ circles with a' chords and b' Y's, and $a'+b' = a+b-1$ (see Figure 6). Thus $W(D) = W(D') = 0$, where the second equality is by the induction hypothesis. This proves the lemma.

□

Lemma 4.6. *A tree component of odd (resp., even) degree is equivalent to a linear combination of chords (resp., Y's).*

Proof. Equation (24) in Lemma 4.3 shows that a tree of degree 3 is equivalent to a linear combination of chord diagrams. Figure 7 shows that a tree of degree 4 is equivalent to a linear combination of Y's. Trees of higher degree are either equivalent to zero by (22) or can be reduced recursively to trees of degree 3 or 4 by the move shown in Figure 8.

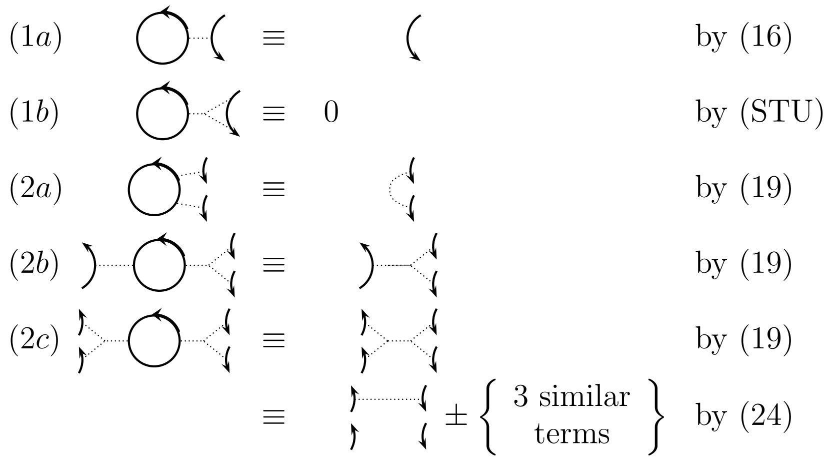


Figure 6. Reducing D to D'. There are five cases to consider. Note that in the case (2c), a increases by 1, but b decreases by 2. In all cases, both m and $a+b$ decrease by 1.

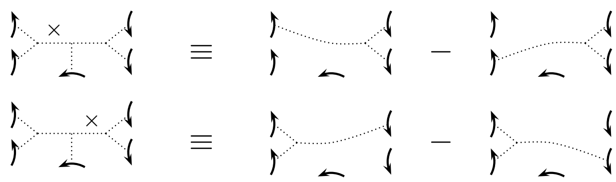


Figure 7. Two ways to reduce a tree of degree 4. We apply (23) at the marked edge, and then apply (20) and (21).

Figure 8. How to reduce trees of degree ≥ 5. Because of (21), only one term survives after applying (23) at the marked edge.

□

Proof of Proposition 4.1. Let $d \leq n(m-1)+1$ be minimal such that there exists a diagram $D \in \mathcal{A}_d(\amalg_m S^1)$ without tree components of degree $\leq n-1$ and $W(D) \neq 0$.

First, we will show that every component of D is a tree. Indeed, assume that a component of D is a wheel (see Figure 9(i)) with $k \geq 2$

legs. (A wheel with only one leg is zero by the STU relation.) If $k = 2$, then we can remove the wheel using (20). The diagram D' thus obtained has degree $d - 2$, and still has no trees of degree $\leq n - 1$. But $W(D') = -2W(D) \neq 0$, which contradicts the minimality of d. Similarly, if the wheel component has $k \geq 3$ legs, then we can apply (23) and (20) as in Figure 8 and get a wheel with $k - 2$ legs. Again, the diagram D' thus obtained has smaller degree and hence contradicts the minimality of d. Therefore no component of D can be a wheel. Furthermore, if a component of D is neither a wheel nor a tree (see *e.g.* Figure 9(ii)), then it has a trivalent vertex all of whose neighboring vertices are also trivalent vertices, hence $W(D) = 0$ by (22).

Hence every component of D is a tree as claimed.

Figure 9. (i) A wheel with 6 legs, and (ii) a component which is neither a wheel nor a tree.

Now let a (resp. b) be the number of trees of odd (resp. even) degree in D. By hypothesis, there are no trees of degree $\leq n-1$, hence $d \geq n(a+b)$. Since $d \leq n(m-1)+1$, it follows that $a+b \leq m-1+\frac{1}{n}$. Since the case $n = 1$ has already been dealt with in Lemma 3.1, we may assume $n \geq 2$. This gives $a + b \leq m - 1$.

Reducing the tree components to chords or Y's by Lemma 4.6, we see that our diagram D is equivalent to a linear combination of diagrams each of which has exactly a chords and b Y's. If $a + b < m - 1$, then $W(D) = 0$ by Lemma 4.5. Thus we must have $a + b = m - 1$, and D has exactly $m - 1$ components each of which is a tree of degree $\geq n$.

Thus we have proved the result for the diagram D.

We have assumed that D had minimal degree, but since this degree is already $\geq n(m-1)$, and $W_d \neq 0$ implies $W_{d+1} = 0$ by (18), diagrams of higher degree will either be killed by W or do not satisfy the hypotheses any more.

This completes the proof.

□

5. The first non-vanishing Milnor invariants

In [M1, M2], Milnor introduced integer-valued invariants $\mu_{i_1 i_2 \ldots i_k}(L)$ of oriented links, known as Milnor higher linking numbers (the number k of indices is called the *length* of a Milnor invariant).

In fact, these invariants are not universally defined, that is, unless the lower order (*i.e.* of lower length) invariants vanish, they are either not defined or have indeterminacies.

The first Milnor invariants are the ordinary linking numbers $\mu_{ij}(L) = \ell_{ij}(L)$ which are always defined. If they all vanish, then the triple linking numbers $\mu_{ijk}(L)$ are well-defined. In general, if all invariants of length $\leq n-1$ are defined and are zero, then all invariants of length n are defined.

Habegger and Lin [HL] introduced the philosophy that Milnor's invariants are actually invariants of string links which are always defined. The first non-vanishing Milnor invariants of a link L are well-defined and equal to the lowest degree non-vanishing invariants of any string link representative of L, and the indeterminacies of invariants of higher degree come precisely from the indeterminacy of representing a link as the closure of a string link. From this point of view, Bar-Natan [BN2] and Lin [L] showed that Milnor's invariants of string links are of finite type.

Here, the *degree* of an invariant is one less than its length (the number of its indices). For example, linking numbers have degree one, and triple linking numbers have degree two.

The universal finite type invariant of a string link σ is its Kontsevich integral $Z(\sigma)$. It was shown in [HM] how to compute the Milnor invariants of σ from the tree part of $Z(\sigma)$. This leads to the following description of the first non-vanishing invariants.

By a *labelled diagram*[4] of degree n we mean a uni-trivalent graph with $2n$ vertices with a cyclic ordering of the three half-edges meeting at each trivalent vertex and whose univalent vertices are labelled by elements of the set $\{1,\ldots,m\}$. We denote by $C_n(m)$ the $\mathbb{Q}$-vector space of labelled diagrams of degree n modulo AS and IHX relations (see Figure 5 in Section 4).

We will be interested in the subspace

$$C^t_n(m) \subset C_n(m)$$

generated by those labelled diagrams whose underlying graphs are trees. In other words, $C^t_n(m)$ is just the space of tree diagrams of degree n with labels from $\{1,\ldots,m\}$ on their univalent vertices, modulo AS and IHX relations.

Denote by $\mathcal{L}_n(m)$ the set of oriented (framed) links L with m (numbered) components, such that all Milnor invariants of L of degree less

[4] These diagrams are sometimes also called Chinese characters, Feynman diagrams, web diagrams, Jacobi diagrams, *etc.*

than n vanish. This defines a decreasing filtration

$$\mathcal{L}_1(m) \supset \mathcal{L}_2(m) \supset \ldots \mathcal{L}_n(m) \supset \ldots$$

on the set of links. (Note that every link lies in $\mathcal{L}_1(m)$.) For links in $\mathcal{L}_n(m)$, Milnor invariants of degree n are well-defined.

Remark 5.1. We consider unframed links as elements of $\mathcal{L}_1(m)$ by equipping them with the zero framing. However, for our purposes the choice of framing does not really affect anything, since the only framing-dependent Milnor invariants are the self-linking numbers $\ell_{ii}(L)$ and the Alexander-Conway polynomial ignores framing.

As explained in [HM], the set of all degree-n Milnor invariants of a link $L \in \mathcal{L}_n(m)$ can be encoded by a universal invariant

$$\xi_n : \mathcal{L}_n(m) \to C_n^t(m) \tag{25}$$

with values in the space of tree diagrams of degree n.

Definition 5.2. We call $\xi_n(L) \in C_n^t(m)$ the *universal degree-n Milnor invariant* of a link $L \in \mathcal{L}_n(m)$.

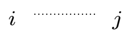
The cases of $n = 1$ and $n = 2$ can be described very explicitly, as follows. Note that

$$C_1^t(m) \cong S^2V$$

and

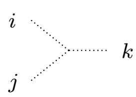
$$C_2^t(m) \cong \Lambda^3 V,$$

where V is a vector space of dimension m with a fixed basis corresponding to the labels $1, 2, \ldots, m$. Thus $C_1^t(m)$ has a basis represented by 'struts'



The coefficient of this strut in $\xi_1(L)$ is equal to the linking number $\ell_{ij}(L)$ between the ith and jth component of L. (We assume $i \neq j$ here.) We can therefore view ℓ_{ij} as the linear form on $C_1^t(m) \cong S^2V$ which sends $\xi_1(L)$ to $\ell_{ij}(L)$.

If $L \in \mathcal{L}_2(m)$ (that is, if all linking numbers of L vanish), then $\xi_1(L) = 0$ and $\xi_2(L)$ is well-defined. It lies in $C_2^t(m) \cong \Lambda^3 V$ which has a basis represented by (antisymmetric) Y-shaped labelled diagrams



The coefficient of this diagram in $\xi_2(L)$ is the Milnor triple linking number $\mu_{ijk}(L)$. In other words, μ_{ijk} is the linear form on $C_2^t(m) \cong \Lambda^3 V$ which sends $\xi_2(L)$ to $\mu_{ijk}(L)$.

Similarly, for links $L \in \mathcal{L}_n(m)$, we have $\xi_i(L) = 0$ for $i < n$, and $\xi_n(L)$ is well-defined. Milnor invariants of degree n can be considered as linear forms on $C^t_n(m)$ sending $\xi_n(L)$ to the numerical invariants, *e.g.* to $\mu_{i_1 i_2 \ldots i_{n+1}}(L)$. In particular, the dimension of $C^t_n(m)$ is equal to the number of linearly independent Milnor invariants of degree n for links in $\mathcal{L}_n(m)$.[5] We refer the reader to [HM, Sections 5 and 6] for an explicit description of the linear forms giving numerical Milnor invariants. (We will not need it in this paper.) The general case is slightly more complicated than the cases of degrees one and two, because the space $C^t_n(m)$ for $n \geq 3$ does not have a canonical basis.

One motivation for encoding the first non-vanishing Milnor invariants of a link $L \in \mathcal{L}_n(m)$ into the universal invariant ξ_n is that $\xi_n(L)$ can be identified with the degree-n term $Z^t_n(\sigma)$ in the tree part of the Kontsevich integral of a string link σ whose closure is L. Since this identification will be used in the next section, we will now review it here (see [HM] for details).

The Kontsevich integral $Z(L)$ of an m-component (framed) link L lies in the space $\mathcal{A}(\amalg_m S^1)$ of diagrams on m solid circles. Thus it is a power series

$$Z(L) = \sum_{n \geq 0} Z_n(L),$$

whose nth term $Z_n(L) \in \mathcal{A}_n(\amalg_m S^1)$ lies in the space of diagrams of degree n.

Suppose now that L is the closure of some string link σ. Its Kontsevich integral is a power series

$$Z(\sigma) = \sum Z_n(\sigma),$$

whose nth term $Z_n(\sigma)$ lies in the space

$$\mathcal{A}_n(m) := \mathcal{A}_n(\amalg_m \mathbf{I})$$

of degree n diagrams on m solid intervals.

Denote by $\mathcal{A}^t(m)$ the quotient space of $\mathcal{A}(m)$ modulo diagrams containing non-simply connected (dashed) components. (The superscript t stands for tree diagrams, which generate $\mathcal{A}^t(m)$ multiplicatively.) By definition, the *tree part* of the Kontsevich integral

$$Z^t(\sigma) = \sum Z^t_n(\sigma)$$

is the image of $Z(\sigma)$ in $\mathcal{A}^t(m)$.

[5] This number, which was first computed by Orr [O, Theorem 15], is equal to $mN_n(m) - N_{n+1}(m)$, where $\sum_i N_i(m) t^i$ is the Hilbert series of the free Lie algebra on m letters. See also [HM, Section 8].

For every string link σ we have

$$Z_0^t(\sigma) = 1,$$

where 1 is the diagram consisting of the trivial string link without any dashed part. It is shown in [HM] that if $L \in \mathcal{L}_n(m) - \mathcal{L}_{n+1}(m)$ (*i.e.*, the first non-vanishing Milnor invariants of L occur precisely in degree n), then the first non-vanishing term of $Z^t(\sigma) - 1$ also occurs in degree n. (In other words, the Z^t*-filtration degree* of σ is equal to the *Milnor filtration degree* of L.) Moreover, since $Z^t(\sigma)$ is group-like, $Z_n^t(\sigma)$ lies in the primitive part of $\mathcal{A}_n^t(m)$, which is naturally identified with $C_n^t(m)$. Under this identification, $Z_n^t(\sigma)$ becomes $\xi_n(L)$, the universal degree-n Milnor invariant of L. In particular, all first non-vanishing Milnor invariants of L can be computed from $Z_n^t(\sigma)$ by an explicit formula given in [HM, Theorem 6.1].

The identification of the primitive part of $\mathcal{A}_n^t(m)$ with $C_n^t(m)$ is very straightforward. As in [HM, Section 4], let us denote the primitive part by $\mathcal{P}_n^t(m) \subset \mathcal{A}_n^t(m)$. It is the subspace generated by diagrams with connected dashed parts. The dashed part in such a diagram is a tree, hence after removing the m solid intervals and retaining only their labels in $\{1, \dots, m\}$ on the univalent vertices, we get an element of $C_n^t(m)$. The relations in $\mathcal{A}^t(m)$ together with the STU relation imply that this gives a well-defined map

$$\mathcal{P}_n^t(m) \xrightarrow{\sim} C_n^t(m),$$

and moreover, this map is an isomorphism.

6. A formula for the coefficient $c_{n(m-1)}$

In this section, we show that if the first non-vanishing Milnor invariants of an m-component link L have degree n (*i.e.* if $L \in \mathcal{L}_n(m)$), then the coefficients of its Alexander-Conway polynomial $\nabla_L(z)$ vanish in degrees smaller than $n(m-1)$ and

$$c_{n(m-1)}(L) = F_m^{(n)}(\xi_n(L)),$$

where $\xi_n(L) \in C_n^t(m)$ is the universal degree-n Milnor invariant (25) of L, and $F_m^{(n)}$ is a homogeneous polynomial of degree $m-1$ on $C_n^t(m)$. Using results and techniques of [HM] this will follow from the Vanishing Lemma 4.1.

The homogeneous polynomial $F_m^{(n)}$ will be given in terms of the corresponding multilinear form $\widetilde{F}_m^{(n)}$ on $C_n^t(m)$ defined as follows.

We call a *lift* of a labelled diagram $\gamma \in C_d(m)$ any diagram

$$D \in \mathcal{A}_d(\amalg_m S^1)$$

on m solid circles whose dashed components are the same as the components of γ and for each i, the univalent vertices of γ with label i are attached in D to the ith solid circle.

Given tree diagrams

$$\xi^{(1)}, \dots, \xi^{(m-1)} \in C_n^t(m),$$

consider their disjoint union γ as an element of $C^t_{n(m-1)}(m)$, and let D be a lift of γ to $\mathcal{A}_{n(m-1)}(\amalg_m S^1)$. Then we set

$$\widetilde{F}_m^{(n)}(\xi^{(1)}, \dots, \xi^{(m-1)}) = W_{n(m-1)}(D), \tag{26}$$

where $W_{n(m-1)}$ is the Alexander-Conway weight system.

Lemma 6.1. *The multilinear form $\widetilde{F}_m^{(n)}$ is well-defined.*

Proof. If D' is another lift of γ, then D' is obtained from D by permuting the cyclic order of the univalent vertices on every component of $\amalg_m S^1$. All such permutations can be obtained by composing simple transpositions of two adjacent vertices on a single circle. Therefore, by applying the STU relation, we see that D and D' differ by diagrams containing non-simply connected components or tree components of degree $\geq n+1$. Since $W_{n(m-1)}$ is zero on such diagrams by the Vanishing Lemma 4.1, the result follows.

□

Now, for a tree diagram $\xi \in C_n^t(m)$ we define

$$F_m^{(n)}(\xi) := \frac{1}{(m-1)!} \widetilde{F}_m^{(n)}(\xi, \dots, \xi), \tag{27}$$

so that $\widetilde{F}_m^{(n)}$ is the polarization of $F_m^{(n)}$.

Remark 6.2. Note that $F_m^{(n)} = 0$ if both m and n are even. This follows immediately from (18).

Proposition 6.3. *Let $L \in \mathcal{L}_n(m)$,* i.e. *all Milnor invariants of L of degree $\leq n-1$ vanish.*

(i) *The Kontsevich integral $Z(L)$ can be written as a linear combination of diagrams none of which contains a tree component of degree $\leq n-1$.*

(ii) *Modulo diagrams containing non-simply connected components or tree components of degree $\geq n+1$, we have*

$$Z_{n(m-1)}(L)) = \frac{1}{(m-1)!}\, \xi_n(L)^{m-1},$$

where $\xi(L) \in C_n^t(m)$ is the universal degree-n Milnor invariant (25) of L.

Here, by abuse of notation, $\xi_n(L)^{m-1}$ is considered as an element of $\mathcal{A}_{n(m-1)}(\amalg_m S^1)$. This is acceptable, since any two lifts of $\xi_n(L)^{m-1}$ to the solid circles differ by diagrams containing non-simply connected components or tree components of degree $\geq n+1$ (see the proof of Lemma 6.1).

Proof. Part (i) is Corollary 9.3 of [HM]. The proof of (ii) is based on a similar argument. Here, we only sketch the idea and refer to [HM] for more details. One has

$$\log Z^t(\sigma) = Z^t_n(\sigma) + O(n+1),$$

where $O(n+1)$ stands for terms of degree $\geq n+1$. It follows that modulo trees of degree $\geq n+1$ we have

$$Z^t_{n(m-1)}(\sigma) = \frac{1}{(m-1)!} \left(Z^t_n(\sigma)\right)^{m-1} .$$

Now $Z^t(L)$ is obtained from $Z^t(\sigma)$ by a procedure involving a special element ν in the algebra of diagrams (it is related to the Kontsevich integral of the unknot). Since ν does not contain trees, this procedure does not introduce any new trees. Since $Z^t_n(\sigma)$ is identified with $\xi_n(L)$, as explained in the previous section, this gives (ii).

□

Proposition 6.4. *Let $L \in \mathcal{L}_n(m)$, and let $\xi_n(L) \in C^t_n(m)$ be its universal degree-n Milnor invariant. Then for the coefficients $c_i(L)$ of the Alexander-Conway polynomial $\nabla_L(z) = \sum_{i\geq 0} c_i(L) z^i$ we have*

(i) $c_i(L) = 0$ *for* $i < n(m-1)$,

(ii) $c_{n(m-1)}(L) = F^{(n)}_m(\xi_n(L))$.

Proof. Consider the following renormalized version of the Alexander-Conway polynomial $\nabla_L(z)$:

$$\widetilde{\nabla}_L(z) = \frac{z}{e^{z/2} - e^{-z/2}} \nabla_L(e^{z/2} - e^{-z/2}) = \sum_{n\geq 0} \tilde{c}_n(L) z^n \ . \tag{28}$$

Note that the coefficient $\tilde{c}_n$ is a finite type invariant of order n. Its weight system is equal to W_n (the weight system of c_n).

It was shown in [BNG] that $\tilde{c}_n$ is a *canonical invariant*, *i.e.* it can be recovered from its weight system by the Kontsevich integral:

$$\tilde{c}_n(L) = W_n(Z_n(L)) \ . \tag{29}$$

Now assume that $L \in \mathcal{L}_n(m)$, *i.e.* the first non-vanishing Milnor invariants of L have degree n. By the Vanishing Lemma 4.1 and Proposition 6.3, non-vanishing coefficients of $\widetilde{\nabla}_L(z)$ can only occur in degrees

$\geq n(m-1)$, and

$$\begin{aligned}\tilde{c}_{n(m-1)}(L) =& W_{n(m-1)}(Z_{n(m-1)}(L)) = \frac{1}{(m-1)!} W_{n(m-1)}(\xi_n(L)^{m-1}) \\ =& \frac{1}{(m-1)!} \widetilde{F}^{(n)}_m(\xi_n(L), \ldots, \xi_n(L)) = F^{(n)}_m(\xi_n(L)) \ .\end{aligned}$$

Since the first non-vanishing coefficient of $\nabla_L(z)$ is equal to the one of $\widetilde{\nabla}_L(z)$, this gives the desired result.

□

7. Expressing $c_{n(m-1)}$ via spanning-tree polynomials

In this section we will rewrite the polynomials $F^{(n)}_m$ introduced in the previous section in terms of the spanning-tree polynomials $\mathcal{D}_m$ and $\mathcal{P}_m$ (see equations (7) and (8)).

This will furnish the proofs of the tree-sum formulas of Theorems 2.3 and 2.4 for the coefficient $c_{n(m-1)}$ of the Alexander-Conway polynomial.

7.1. **The case $n = 1$.**

The case $n = 1$ is given by Theorem 2.1. With the identification $C^t_1(m) = S^2V$, the polynomial $F^{(1)}_m \in S^{m-1}(C^t_1(m))^*$ is just the Kirchhoff polynomial $\mathcal{D}_m$ in the linking numbers ℓ_{ij} (see (7)). This is also clear from the discussion in Section 3 (see Lemma 3.1).

7.2. **The case $n = 2$.**

As already observed in Remark 6.2, we have $F^{(2)}_m = 0$ if m is even.

Let us now assume that m is odd. Since $F^{(2)}_m$ is a homogeneous polynomial of degree $m-1$ on $C^t_2(m) \cong \Lambda^3 V$, it can be written as a sum of monomials of degree $m-1$ in the variables μ_{ijk} (which represent the standard basis of the dual space $C^t_2(m)^*$). The coefficient in $F^{(2)}_m$ of each monomial in the μ_{ijk}'s can be computed by using the Alexander-Conway weight system.

This gives the following characterization of $F^{(2)}_m$. We will write

$$\mu_{ijk} = v_i \wedge v_j \wedge v_k \ ,$$

where $v_1, \ldots, v_m$ is the standard basis of V^*. This allows us to consider $F^{(2)}_m$ as an expression in the variables v_i:

$$F^{(2)}_m = F^{(2)}_m(v_1, v_2, \ldots, v_m) \ .$$

Note that we know already that this expression is invariant under any permutation of the v_i's. (This follows from Proposition 6.4, since the coefficient $c_{n(m-1)}(L)$ does not depend on an ordering of the components of L.)

Proposition 7.1. *Assume that $m \geq 3$ is odd. Then the polynomial $F_m^{(2)}$ is characterized by the following properties. (For simplicity, we write F_m instead of $F_m^{(2)}$.)*

(i) $F_3 = \mu_{123}^2$

(ii) *If a monomial $\prod_\alpha \mu_{i_\alpha j_\alpha k_\alpha}$ occurs with non-zero coefficient in F_m, then there exists $p \in \{1, 2, \ldots, m\}$ such that p occurs exactly twice in the list of indices $i_1, j_1, k_1, i_2, \ldots j_{m-1}, k_{m-1}$.*

(iii) *F_m satisfies the following relations (and all the relations obtained from them by permuting the indices $1, 2, \ldots, m$):*

$$\left[\frac{\partial^2 F_m}{\partial \mu_{123}^2}\right]_{v_1=0} = 2F_{m-2}(v_2 + v_3, \ldots),$$

$$\left[\frac{\partial^2 F_m}{\partial \mu_{123}\, \partial \mu_{124}}\right]_{v_1=0} = F_{m-2}(v_2 + v_3, v_4, \ldots) + F_{m-2}(v_2 + v_4, v_3, \ldots) - F_{m-2}(v_3 + v_4, v_2, \ldots),$$

$$\left[\frac{\partial^2 F_m}{\partial \mu_{123}\, \partial \mu_{145}}\right]_{v_1=0} = F_{m-2}(v_3 + v_4, v_2, v_5, \ldots) + F_{m-2}(v_2 + v_5, v_3, v_4, \ldots) - F_{m-2}(v_2 + v_4, v_3, v_5, \ldots) - F_{m-2}(v_3 + v_5, v_2, v_4, \ldots)\ .$$

(Here the dots stand for the v_i with indices which do not appear in the left hand side; for example, in the first equation, the dots mean $v_4, v_5, \ldots, v_m$).

Proof. The key point is to observe that monomials in the μ_{ijk}'s are dual to diagrams consisting of Y's glued to the m solid circles. With this in mind, statement (ii) is a reformulation of the following fact, already used in Section 4 (see the proof of Lemma 4.5): if a diagram D on m solid circles consists of $m-1$ Y's and has $W(D) \neq 0$, then at least one of the m solid circles has exactly two univalent vertices on it.

To check (i), see the computation in Figure 10.

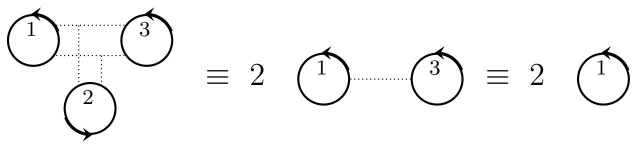


Figure 10. Applying relation (2c) of Figure 6, we see that W_4 takes the value 2 on this diagram. Thus $\partial^2 F_3/\partial\mu_{123}^2 = 2$. By (ii), no other diagram can contribute to F_3. This shows that $F_3 = \mu_{123}^2$.

Finally, formulas (iii) relate the coefficients of monomials occurring in F_m and the coefficients of monomials in F_{m-2}. Note that these coefficients can be computed by applying the Alexander-Conway weight system to the diagram corresponding to a monomial. The identities in (iii) follow from applying to this diagram relation (2c) of Figure 6 (and smoothing the resulting chords in the four remaining terms) in the case when the solid circle with the label 1 has exactly two univalent vertices on it. Let us elaborate. The operator

$$F_m \mapsto \mu_{1ij}\mu_{1kl}\left[\frac{\partial^2 F_m}{\partial\mu_{1ij}\,\partial\mu_{1kl}}\right]_{v_1=0}$$

projects F_m onto the space of monomials of degree $m-1$ which contain $\mu_{1ij}\mu_{1kl}$, but no other $\mu_{\alpha\beta\gamma}$ with $1 \in \{\alpha, \beta, \gamma\}$. Such monomials correspond exactly to those diagrams where we *can apply* relation (2c) with the distinguished solid circle in (2c) labelled by 1 and the other four solid circles labelled by i, j, k, l. The operator

$$F_m \mapsto \left[\frac{\partial^2 F_m}{\partial\mu_{1ij}\,\partial\mu_{1kl}}\right]_{v_1=0}$$

corresponds to *actually applying* (2c) to each of these diagrams, and smoothing the chords in the resulting diagrams. If in one of these new diagrams the two endpoints of the chord lie on the same solid circle, then smoothing the chord increases the number of solid circles and the corresponding diagram does not contribute (the number of remaining Y's is too small). If, however, the endpoints of the chord lie on the ith and jth solid circle, with $i \neq j$, then the two circles are replaced after the smoothing by their connected sum, and such a term contributes a summand of the form

$$F_{m-2}(\ldots, v_i + v_j, \ldots)\ .$$

There are three cases to consider corresponding to whether the set $\{i, j, k, l\}$ has two, three, or four distinct elements. This leads to the three identities in (iii).

Finally, conditions (i)—(iii) characterize F_m uniquely, since they allow to compute F_m recursively. Indeed, (ii) shows that for every nonzero monomial in F_m there is an index p where one can apply (iii) and thus reduce the computation to F_{m-2}.

This completes the proof.

□

Corollary 7.2. *The polynomial $F_m = F_m^{(2)}$ is equal to $\mathcal{P}_m^2$, the square of the Pfaffian-tree polynomial (8).*

Proof. This follows from Corollaries 6.5—6.7 of [MV], where it is shown that the polynomial $\mathcal{P}_m^2$ satisfies (and is determined by) the recursion relations in Proposition 7.1.

□

Remark 7.3. The proof that $\mathcal{P}_m^2$ satisfies the recursion relations is independent of the Pfaffian Matrix-Tree Theorem for 3-graphs of [MV]. In fact, it follows from a three-term deletion-contraction relation for $\mathcal{P}_m$ which is proved using the interpretation of $\mathcal{P}_m$ as the spanning tree generating function of the complete 3-graph with m vertices.

Thus we arrive at one of our main results (stated as Theorem 2.3 in Section 2).

Corollary 7.4. *Let L be an m-component algebraically split link, with triple Milnor linking numbers $\mu_{ijk}(L)$. Let $\nabla_L(z) = \sum_{i\geq 0} c_i(L)z^i$. Then $c_i(L) = 0$ for $i \leq 2m-3$, and*

$$c_{2m-2}(L) = \mathcal{P}_m(\mu_{ijk}(L))^2.$$

Proof. Recall that the numbers $\mu_{ijk}(L)$ are the coefficients of $\xi_2(L)$. Thus, the result follows from Proposition 6.4, since $F_m^{(2)} = \mathcal{P}_m^2$.

□

7.3. **Coefficients of $F_m^{(2)}$ and tree decompositions.**

Let us fix an odd $m \geq 3$. The coefficients of the monomials in the polynomial $F_m^{(2)} = \mathcal{P}_m^2$ can be computed by counting decompositions of related 3-graphs into pairs of spanning trees. Recall from Section 2 that edges of a 3-*graph* have three (distinct) vertices and can be visualized as Y-shapes with the three vertices at their endpoints (see [MV] for details). A diagram D whose components are Y's on m ordered solid circles defines a 3-graph G_D with vertex set $\{1, 2, \ldots, m\}$ and edges given by these components. Since the edges of 3-graphs are not oriented, the assignment $D \mapsto G_D$ is not one-to-one. Note, however, that the 3-graph G_D determines the diagram D up to sign. Similarly, each monomial M in the variables μ_{ijk} gives a 3-graph G_M, and G_M determines M up to sign.

Definition 7.5. Let G be a 3-graph. An *ordered tree decomposition* of G is a sub-3-graph T which is a tree and whose complement T' is also a tree.

In the case when G has m vertices and $m-1$ edges, both trees in a tree decomposition must be spanning trees. In particular, for a diagram D with $m-1$ Y-shaped components or for a monomial M of degree

$m-1$ in the μ_{ijk}'s each tree decomposition of the corresponding 3-graph $G = G_D$ or $G = G_M$ gives two monomials y_T and $y_{T'}$ of degree $(m-1)/2$ in the Pfaffian-tree polynomial $\mathcal{P}_m$. (Note that the monomials y_T and $y_{T'}$ depend on D or M and cannot be determined by the 3-graph G alone.) Let $\varepsilon(T)$ and $\varepsilon(T')$ be the signs of these monomials in $\mathcal{P}_m$. By counting each tree decomposition with the sign

$$\varepsilon = \varepsilon(T)\varepsilon(T'),$$

we define the *algebraic number of ordered tree decompositions* of the diagram D and of the monomial M.

With these definitions in hand, the equality $F_m^{(2)} = \mathcal{P}_m^2$ (Corollary 7.2) and the definition of $\mathcal{P}_m$ as the spanning tree generating function of the complete 3-graph Γ_m (see Section 2) immediately give the following combinatorial rules for computing $W(D)$ and $F_m^{(2)}$.

Proposition 7.6.

(i) Let D be a diagram consisting of $m-1$ Y-shaped components on m solid circles. Then $W(D)$ is equal to the algebraic number of ordered tree decompositions of D.

(ii) Let M be a monomial of degree $m-1$ in the variables μ_{ijk}. Then the coefficient of M in the polynomial $F_m^{(2)}$ is equal to the algebraic number of ordered tree decompositions of M divided by the symmetry factor $|Aut(G_M)|$, where $Aut(G_M)$ is the group of automorphisms of the 3-graph G_M inducing the identity map on the set of vertices of G_M.

□

In our situation, the cardinality $|Aut(G_M)|$ is equal to 2^d, where d is the number of μ_{ijk}'s which occur twice in M (up to sign). In other words, d is the number of (unordered) triples of vertices in G_M with 2 edges attached to them. (Note that if M has non-zero coefficient in $F_m^{(2)} = \mathcal{P}_m^2$, they G_M can have at most two edges with the same vertex set.)

To see the need for the symmetry factor $|Aut(G_M)|$ in (ii), consider for example the monomial y_{123}^2. It occurs in $\mathcal{P}_3^2$ with coefficient 1, but the weight system W takes the value 2 on the diagram in Figure 10 which is dual to y_{123}^2. This corresponds to the fact that the associated 3-graph G_M has two ordered tree decompositions and $|Aut(G_M)| = 2$.

Here are two more examples to illustrate this. The monomial

$$M = y_{123}^2 y_{245} y_{345}$$

has four ordered tree decompositions, each contributing $+1$, and $|Aut(G_M)| = 2$. Therefore, M appears in $\mathcal{P}_3^2$ with coefficient 2. Finally, the monomial

$$M = y_{145}\, y_{146}\, y_{256}\, y_{257}\, y_{347}\, y_{367}$$

has six ordered tree decompositions (again each contributing $+1$) and $|Aut(G_M)| = 1$; therefore it has coefficient 6 in $\mathcal{P}_7^2$.

Let us finish this subsection by illustrating how to compute the coefficients of monomials in $\mathcal{P}_m^2$ using the recursion relations of Proposition 7.1. For example, let us compute the monomials in F_5 which contain $\mu_{123}\mu_{145}$, but no other μ_{ijk} with $1 \in \{i, j, k\}$. The coefficient of such a monomial in F_5 is equal to its coefficient in

$$\mu_{123}\, \mu_{145} \left[\frac{\partial^2 F_5}{\partial \mu_{123}\, \partial \mu_{145}} \right]_{v_1=0} . \tag{30}$$

Applying 7.1(iii), we see that (30) is equal to

$$\begin{aligned}
&\mu_{123}\, \mu_{145} \big(F_3(v_3 + v_4, v_2, v_5) + F_3(v_2 + v_5, v_3, v_4) \\
&\qquad\qquad - F_3(v_2 + v_4, v_3, v_5) - F_3(v_3 + v_5, v_2, v_4) \big) \\
&= \mu_{123}\, \mu_{145} \big((\mu_{325} + \mu_{425})^2 + (\mu_{234} + \mu_{534})^2 \\
&\qquad\qquad - (\mu_{235} + \mu_{435})^2 - (\mu_{324} + \mu_{524})^2 \big) \\
&= 2\, \mu_{123}\, \mu_{145}\, (\mu_{234} + \mu_{235})(\mu_{245} + \mu_{345}) .
\end{aligned}$$

For example, we see that the coefficient of $\mu_{123}\, \mu_{145}\, \mu_{235}\, \mu_{345}$ in F_5 is equal to 2 (see Figure 11).

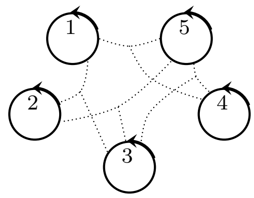


Figure 11. This diagram contributes the term $2\, \mu_{123}\, \mu_{145}\, \mu_{235}\, \mu_{345}$ to $F_5 = \mathcal{P}_5^2$. The associated 3-graph can be decomposed into spanning trees corresponding to $\mu_{123}\, \mu_{345}$ and $\mu_{145}\, \mu_{235}$.

7.4. **The case $n \geq 3$.**

Denote by $\overline{C}_2^t(m)$ the quotient $C_2^t(m)/W_0$ where W_0 is the subspace of $C_2^t(m) = \Lambda^3 V$ defined as follows. Let Y_{ijk} be the basis of $\Lambda^3 V$ dual to $\mu_{ijk} \in \Lambda^3 V^*$. Then W_0 is the subspace generated by elements of the form

$$(\mathrm{Y}_{ijk} - \mathrm{Y}_{ijl}) - (\mathrm{Y}_{jkl} - \mathrm{Y}_{ikl})$$

for every set $\{i, j, k, l\}$ of four distinct vertices (see Figure 12).

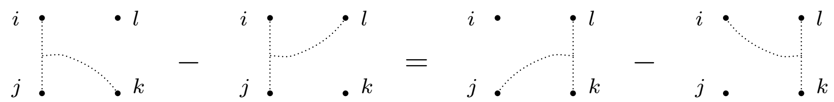


Figure 12. The relations in $\overline{C}^t_2(m)$.

The Pfaffian-tree polynomial $\mathcal{P}_m$ can be viewed as a polynomial function on $C^t_2(m)$. It is shown in [MV, Proposition 6.10] that $\mathcal{P}_m$ descends to a well-defined polynomial on $\overline{C}^t_2(m)$. Hence, the same is true for $F^{(2)}_m = \mathcal{P}^2_m$. Alternatively, the fact that $F^{(2)}_m$ is well-defined on $\overline{C}^t_2(m)$ can be proved from the definition of $F^{(2)}_m$ via the Alexander-Conway weight system and the relations in Figure 7.

Proposition 7.7. *The Alexander-Conway relations define a linear map*

$$\phi_n : C^t_n(m) \to \begin{cases} C_1(m) & \textit{if } n \textit{ is odd} \\ \overline{C}^t_2(m) & \textit{if } n \textit{ is even} \end{cases} \tag{31}$$

such that

$$F^{(n)}_m = \begin{cases} F^{(1)}_m \circ \phi_n = \mathcal{D}_m \circ \phi_n \ , & \textit{if } n \textit{ is odd} \\ F^{(2)}_m \circ \phi_n = \mathcal{P}^2_m \circ \phi_n \ , & \textit{if } n \textit{ is even} \end{cases} \tag{32}$$

Proof. Let D be a labelled tree diagram of degree n. We define $\phi_n(D)$ to be the unique element of $C_1(m)$ or $\overline{C}^t_2(m)$, depending on whether n is odd or even, such that $D \equiv \phi_n(D)$. The existence of $\phi_n(D)$ was proved in the tree reduction Lemma 4.6.

To prove uniqueness of $\phi_n(D)$, we need to show that applying the tree reduction relations (see Equation (24) and Figures 7 and 8) in any order always gives the same result. This verification is an easy application of the diamond lemma. Note that the apparent indeterminacy coming from the relation in Figure 7 is compensated by taking the quotient $\overline{C}^t_2(m) = C^t_2(m)/W_0$ in the case when n is even. Finally, Equation (32) follows immediately from the definition of ϕ_n. This completes the proof.

□

Thus, we arrive at the general formula of Theorem 2.4.

Corollary 7.8. *Let L be an m-component link with vanishing Milnor invariants of degree $\leq n-1$ (*i.e. $L \in \mathcal{L}_n(m)$*). Then $c_i(L) = 0$ for $i < n(m-1)$, and*

$$c_{n(m-1)}(L) = \begin{cases} (\mathcal{D}_m \circ \phi_n)(\xi_n(L)) & = \mathcal{D}_m(\ell_{ij}^{(n)}(L)) \ , \qquad \text{if } n \text{ is odd} \\ (\mathcal{P}_m \circ \phi_n)(\xi_n(L))^2 & = (\mathcal{P}_m(\mu_{ijk}^{(n)}(L)))^2 \ , \quad \text{if } n \text{ is even} \end{cases}$$

where ϕ_n is the reduction map (31) and $\xi_n(L)$ is the universal degree-n Milnor invariant (25) of L.

Here we used the linear forms on $C_n^t(m)$ given by

$$\ell_{ij}^{(n)} = \phi_n \circ \ell_{ij}$$

(when n is odd) and

$$\mu_{ijk}^{(n)} = \mu_{ijk} \circ \tilde{\phi}_n,$$

where

$$\tilde{\phi}_n : C_n^t(m) \to C_2^t(m)$$

is an arbitrary lift of the map ϕ_n to $C_2^t(m) = \Lambda^3(V)$ (when n is even).[6]

Proof. This follows from Propositions 6.4 and 7.7.

□

8. The case when both n and m are even

If both n and m are even, then by (15) the coefficient $c_{n(m-1)}(L)$ of the Alexander-Conway polynomial ∇_L is zero for every m-component link L. Therefore, by Corollary 7.8, if $L \in \mathcal{L}_n(m)$, then the first non-vanishing coefficient of ∇_L can occur only in degree $n(m-1)+1$. The Vanishing Lemma 4.1 and the methods of Section 6 give an expression of this coefficients as a polynomial in Milnor numbers, analogous to the formula $c_{n(m-1)}(L) = F_m^{(n)}(\xi_n(L))$ of Proposition 6.4.

Proposition 8.1. *There exists a polynomial on $C_n^t(m) \oplus C_{n+1}^t(m)$*

$$G_m^{(n)} \in (S^{m-2}C_n^t(m)^*) \otimes C_{n+1}^t(m)^* \subset S^{m-1}(C_n^t(m)^* \oplus C_{n+1}^t(m)^*), \tag{33}$$

such that

$$c_{n(m-1)+1}(L) = G_m^{(n)}(\xi_n(L), \xi_{n+1}(\sigma)). \tag{34}$$

Here $\xi_n(L)$ is the universal degree-n Milnor invariant of L and $\xi_{n+1}(\sigma) \in C_{n+1}^t(m)$ corresponds to $Z_{n+1}^t(\sigma) \in \mathcal{A}_{n+1}^t(m)$, where σ is any string link representative of L.

[6]The linear form $\mu_{ijk}^{(n)}$ depends on the lift $\tilde{\phi}_n$, but $\mathcal{P}_m(\mu_{ijk}^{(n)})$ is independent of this choice.

Proof. The homogeneous polynomial $G_m^{(n)}$ is defined similarly to the polynomial $F_m^{(n)}$ by lifting a disjoint union of $m-2$ labelled diagrams from $C_n^t(m)$ and one diagram from $C_{n+1}^t(m)$ to $\mathcal{A}_{n(m-1)+1}(\amalg_m S^1)$ and applying W. Independence of the result on the choice of lift follows from the STU relations and the Vanishing Lemma similarly to the proof of Lemma 6.1. □

Although $\xi_{n+1}(\sigma) \in C_{n+1}^t(m)$ may depend on the string link representative σ and not just on the link L, formula (34) shows that $G_m^{(n)}(\xi_n(L), \xi_{n+1}(\sigma))$ depends only on L, since it is equal to the coefficient $c_{n(m-1)+1}(L)$ of ∇_L.

For example, if L is an algebraically split m-component link where m is even, then the first non-vanishing coefficient $c_{2m-1}(L)$ is a polynomial of degree $m-2$ in the triple linking numbers $\mu_{ijk}(L)$ (which are the coefficients of $\xi_2(L)$) and of degree one in the quadruple linking numbers $\mu_{ijkl}(\sigma)$ (which are encoded by the coefficients of $\xi_3(\sigma)$).

If $m=2$, there is only one possible diagram contributing to $G_2^{(2)}$ (see Figure 13). The coefficient of this diagram in $Z_3^t(L)$ is $\pm\frac{1}{2}\mu_{1122}(L)$ (see [HM, section 8]). Since W takes the value -2 on this diagram, we recover the well-known fact [Co] that for 2-component algebraically split links

$$c_3(L) = \pm\mu_{1122}(L)$$

(the sign depends on conventions which we don't want to specify here).

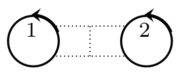


Figure 13. The only diagram contributing to $G_2^{(2)}$.

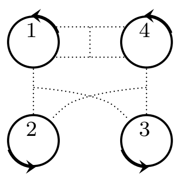


Figure 14. A diagram contributing to $G_4^{(2)}$.

The existence of some expression for $c_{n(m-1)+1}(L)$ as a polynomial in Milnor numbers also follows from Levine's general determinantal formula in [Le2] (see also Traldi [Tr2]). In fact, it is even possible to infer from Levine's formula that the polynomial $G_m^{(n)}$ must be of the form given in (33), *i.e.* it is homogeneous of degree $m-2$ in Milnor numbers of length $n+1$ and of degree one in Milnor numbers of length $n+2$. But while Levine's formula describes the polynomial $G_m^{(n)}$ as a determinant, our methods again lead to an expression for it in terms of tree decompositions.

Indeed, the computation of the coefficients of $G_m^{(n)}$ can be reduced to those of $F_m^{(n)}$. Consider, for example, the case $n = 2$. Then we only need to look at diagrams like the one in Figure 14. Applying relation (19) to the vertical dashed edge of the H-shaped component of this diagram, we may insert an additional solid circle into that edge without changing the value of the Alexander-Conway weight system. Thus, we see that the diagram in Figure 14 is equivalent to a diagram consisting of four Y's on five solid circles. Similarly, the diagram in Figure 13 is equivalent to a diagram consisting of two Y's on three solid circles (which is equal to minus the diagram in Figure 10).

The same argument shows that in general, a diagram consisting of $m-2$ Y's and one H-shaped component on m solid circles is equivalent to a diagram consisting of m Y's on $m+1$ solid circles, and thus, we arrive at the following result.

Proposition 8.2. *Let D be a labelled diagram (with labels from the set $1, 2, \ldots, m$) which has $m-2$ Y-shaped components and one H-shaped component ${}_i^{\ell}H_j^k$ (with obvious notation). Then*

$$W(D) = W(D'),$$

where D' is the diagram (with an additional label 0) obtained from D by replacing the component ${}_i^{l}H_j^k$ with two Y-shaped components Y_{0jk} and $Y_{0\ell i}$.

$\square$

It follows that $W(D)$ is equal to the algebraic number of tree decompositions of the diagram D' (see Proposition 7.6(i)). We leave it to the reader to translate this fact into a formula for the polynomial $G_m^{(2)}$ in terms of monomials involving $m-2$ $\mu_{\alpha\beta\gamma}$'s and one ${}_i^{\ell}\eta_j^k$, where ${}_i^{\ell}\eta_j^k$ is the dual of ${}_i^{\ell}H_j^k$. For example, one has

$$G_2^{(2)} = -2\ {}_2^1\eta_2^1\ .$$

To do this for $m \geq 4$, one has to take into account the symmetry factors as in Proposition 7.6(ii). A certain technical point arises here because of the fact that the dual of a diagram does not make sense in $C_n^t(m)^*$ for $n \geq 3$. (This point did not arise in the description of the polynomial $F_m^{(2)} = \mathcal{P}_m^2$, since the IHX relation is not needed to describe Y-shaped labelled diagrams.) The standard solution is to consider the space $\tilde{C}_n^t(m)$ of labelled tree diagrams modulo AS relations (but not modulo IHX relations). The dual ${}_i^\ell\eta_j^k$ of ${}_i^\ell H_j^k$ makes sense in $\tilde{C}_3^t(m)^*$, and one obtains from Proposition 8.2 a tree decomposition-type formula for the coefficients of monomials of the polynomial $G_m^{(2)}$. Each of these monomials is a function on $\tilde{C}_2^t(m) \oplus \tilde{C}_3^t(m)$ and in general does not descend to $C_2^t(m) \oplus C_3^t(m)$. However, their sum is equal to the polynomial $G_m^{(2)}$ which satisfies the IHX relations and therefore is a well-defined function on the quotient.

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