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An element of infinite order in TQFT-representations of mapping class groups

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Quantum invariants of 3-manifolds and the associated 'Topological Quantum Field Theories' (TQFT) give rise to interesting finite-dimensional representations of mapping class groups of surfaces. Recently, there has been some interest in the question of whether these representations have finite image. For the $SU(2)$-theory, Gilmer [G] has shown that this is the case in genus 1, while Funar [F] has shown that the image of the mapping class group in genus $g > 1$ is an infinite group, except for the representations corresponding to a few small levels.

To do so, Funar observed that it is enough to prove this for the mapping class group of a 4-punctured sphere. In this case, his proof that the image cannot be a finite group uses the classification of finite subgroups of $SO(3)$, following an argument used by V. Jones to show the generic infiniteness of braid group representations into Hecke algebras. Although this is a nice argument, it does not actually show that there is a mapping class whose image in the TQFT-representations has infinite order.

In this note we will complete Funar's result, and at the same time simplify his proof, by exhibiting such a mapping class explicitly. This class has infinite order in the TQFT-representations for $SU(2)$ at level k except if $k = 1, 2, 4,$ or 8 (in other words, except if the order $r = k + 2$ of the root of unity q is $3, 4, 6,$ or 10). Although the basic idea is the same as Funar's, the proof here is by a simple Kauffman bracket computation which is perhaps easier to check than the considerations in [F].

Let Σ be a surface of genus $g > 1$, and let α and β be Dehn twists along simple closed curves a and b meeting transversely in exactly two points but with opposite signs. Note that a regular neighborhood of $a \cup b$ in Σ is a four-holed sphere, S, say. Let us assume that none of the four boundary circles of S is null-homologous in $\Sigma - S$. Such a pair of simple closed curves always exists if $g > 1$.

Proposition. *The image of the mapping class $\alpha^{-1}\beta$ in the TQFT-representation corresponding to $SU(2)$ at level k has infinite order except if $k = 1, 2, 4,$ or 8.*

Note. As the TQFT-representations of the mapping class group are only projective-linear, the above statement means more precisely that no power of the mapping class $\alpha^{-1}\beta$ acts as a scalar matrix. Also note that every Dehn twist has

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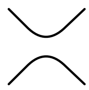
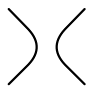
finite order in the TQFT-representations, so that a product of at least two twists is needed to get a matrix of infinite order.

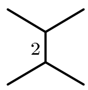
The proof is very simple and goes as follows. We will take the point of view of the TQFT-functors V_p constructed in [BHMV] from the Kauffman bracket, with Kauffman's skein variable A being a primitive root of unity of order $2p$. The $SU(2)$-theory at level k corresponds to the case $p = 2k+4$.

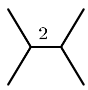
Consider the four-holed sphere S which is a regular neighborhood of $a \cup b$. We think of the curves a and b as the equator and a meridian of that sphere, respectively. Since (by assumption) none of its boundary circles is homologically trivial in $\Sigma - S$, it follows from the 'fusion rules' (as shown in theorem 1.14 of [BHMV]) that the vector space $V_p(\Sigma)$ contains a direct summand $V \otimes W$ where $V = V_p(S^2, (1,1,1,1))$ is the vector space associated to a four-punctured sphere with all punctures (corresponding to the boundary circles of our 4-holed sphere S) colored by 1 (corresponding to the fundamental representation of $SU(2)$). We will compute a matrix for the mapping class $\alpha^{-1}\beta$ acting on V, and then show it has infinite order, proving the proposition.

The vector space V is 2-dimensional, and is isomorphic to the skein module of banded $(2,2)$-tangles in a 3-ball, modulo the usual Kauffman skein relations. Since the mapping classes α and β extend to the 3-ball, their action on V is just the natural action on tangles (see [BHMV, MR]). A basis of V is given by the tangles

$v_0 =$  $h_0 =$ 

The mapping classes α and β are given by a 'horizontal' resp. a 'vertical' full twist, so that α leaves fixed v_0 and β leaves fixed h_0. The skein elements v_2 (resp. h_2) determined by the following graphs

$v_2 =$ 

$h_2 =$ 

are also eigenvectors for α (resp. β), with eigenvalue A^8 in both cases (with the notations of [BHMV], this is μ_2 where $\mu_i = (-1)^i A^{i^2+2i}$). Here the 2 beneath an edge means cabling by a Jones-Wenzl idempotent f_2. Thus one has $v_2 = h_0 - \delta^{-1} v_0$ and $h_2 = v_0 - \delta^{-1} h_0$, where $\delta = -A^2 - A^{-2}$. (Indeed, f_2 is given by $f_2 = h_0 - \delta^{-1} v_0$ where h_0 is viewed as the identity $(2,2)$-tangle.) From this, it is easy to work out the matrix for the base change from (v_0, v_2) to (h_0, h_2). One finds that $\alpha^{-1}\beta$ in the basis (v_0, v_2) is given by the product of matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & A^{-8} \end{pmatrix} \begin{pmatrix} \delta^{-1} & 1-\delta^{-2} \\ 1 & -\delta^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & A^{8} \end{pmatrix} \begin{pmatrix} \delta^{-1} & 1-\delta^{-2} \\ 1 & -\delta^{-1} \end{pmatrix}$$
$$= \begin{pmatrix} A^8 + \delta^{-2}(1-A^8) & \star \\ \star & A^{-8} + \delta^{-2}(1-A^{-8}) \end{pmatrix}$$

This matrix has determinant 1 and its trace, t, say, is easily computed to be

$$t = 2 - q - q^{-1} + q^2 + q^{-2},$$

where we have put $q = A^4$. If a power of this matrix is a scalar multiple of the identity matrix, then the scalar must be ± 1, and the eigenvalues λ, λ^{-1} of the

matrix must be roots of unity. Hence for every embedding $\mathbb{Q}(q) \to \mathbb{C}$, we must have
$$|t| = |\lambda + \lambda^{-1}| \leq 2.$$
But if the order $r = k + 2$ of q is ≥ 5 and different from 6 or 10, then there is a primitive r-th root of unity $q \in \mathbb{C}$ such that $Re(q^2) > Re(q)$ and hence $|t| > 2$. (Just take $q = e^{2i\pi l/r}$, where $l = 1 + r/2$ if $r \equiv 0$ (4) and $r \geq 8$, $l = (1 + r)/2$ if $r \equiv 1$ (2) and $r \geq 5$, and $l = 2 + r/2$ if $r \equiv 2$ (4) and $r \geq 14$.) Thus, $\alpha^{-1}\beta$ has infinite order in the TQFT-representation except if the level is $k = 1, 2, 4$, or 8, as asserted.

REMARKS. (i) If $k = 1, 2, 4$, or 8, then the matrix has order $1, 4, 4, 6$, respectively.

(ii) The same argument shows that $\alpha^{-1}\beta$ has infinite order in the TQFT-representation in the $SO(3)$-case (corresponding to V_p with p odd) under the same conditions on the order of $q = A^4$. (See [BHMV], thm. 4.14.)

(iii) Funar [F] conjectured that a class similar to $\alpha^{-1}\beta$ but made out of half-twists in place of full twists has infinite order. He also has an argument (which I haven't checked) in the case of level 8 (i.e., $r = 10$) but he needs the genus to be at least 3.

(iv) The projective-linear representation of the mapping class group may also be thought of as a linear representation of a central extension of the mapping class group by $\mathbb{Z}$. Although this extension is non-trivial, there are 'preferred' lifts of Dehn twists to this extension (see [BHMV, MR]). The matrices for α and β described above correspond to the 'geometric' lifts considered in [MR].

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