Signatures in TQFT : Asymptotics and Modularity

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(joint work with Julien Marché)

This talk was a report on joint work [5] with Julien Marché. I began with the following concrete problem. Let $0 < q < p$ be odd coprime integers and define for any integer n the sign $\varepsilon_n = (-1)^{\lfloor nq/p \rfloor}$. Let

$$T_p = \Theta_{p-2} \cap (\mathbb{Z}^3)^{ev} \ ,$$

where $\Theta_n \subset \mathbb{R}^3$ is the tetrahedron with vertices $(0,0,0)$, $(n,n,0)$, $(n,0,n)$, and $(0,n,n)$, and $(\mathbb{Z}^3)^{ev} \subset \mathbb{Z}^3$ is the lattice of triples of integers with even sum. Our goal is to study the following integer-valued sum :

$$\sigma_2(q/p) = \sum_{(j,k,\ell) \in T_p} \varepsilon_{j+1}\varepsilon_{k+1}\varepsilon_{\ell+1} \ . \tag{1}$$

More precisely, we wish to study the asymptotics of $\sigma_2(q/p)$ as q/p goes to some irrational number $\theta \in [0,1]$. We will use this to find a transformation law for $\sigma_2(q/p)$ showing that the sum (1) of signs over lattice points in the tetrahedron has modular properties.

One possible motivation from number theory for studying this sum could be that the similar, but simpler, sum

$$\sum_{n=1}^{p-1} \varepsilon_n \tag{2}$$

is closely related (see [5, Section 6.1]) to the Dedekind sum $s(q,p)$ which is Example 0 in Zagier's Quantum Modular Forms [8]. In particular $s(q,p)$ has modular properties, as it satisfies a reciprocity formula relating $s(q,p)$ with $s(p,q)$. So one might hope that $\sigma_2(q/p)$ also has interesting modular properties.

Our original motivation was, however, the connection with Topological Quantum Field Theory. Indeed, $\sigma_2(q/p)$ is the signature of the natural Hermitian form on the SU(2)-TQFT vector space $\mathcal{V}_p(S_2)$ associated to the closed genus 2 surface S_2, where $\mathcal{V}_p$ is SU(2)-TQFT "at the p-th root of unity" (equivalently, at level $p-2$). The vector space $\mathcal{V}_p(S_2)$ and its Hermitian form can be defined over the cyclotomic field $\mathbb{Q}(\zeta)$, where $\zeta = e^{i\pi q/p}$. (Note that $e^{i\pi q/p}$ has order precisely $2p$, as the coprime integers q and p are both asssumed to be odd.) The connection with the set T_p of lattice points in the tetrahedron Θ_{p-2} is the following: The vector space $\mathcal{V}_p(S_2)$ has a basis b_σ indexed by $\sigma \in T_p$. In particular, its dimension is

$$\dim \mathcal{V}_p(S_2) = |T_p| = \binom{p+1}{3} \ . \tag{3}$$

Moreover, this basis is orthogonal for the Hermitian form, and when $\sigma = (j,k,\ell)$, the sign of the Hermitian form on the basis vector b_σ is $\varepsilon_{j+1}\varepsilon_{k+1}\varepsilon_{\ell+1}$. (See [1, Remark 4.12] but notice that our $\mathcal{V}_p$ corresponds to the V_{2p} of [1].) Thus Formula (1) gives indeed the signature of this Hermitian form.

It is worth observing that when $q = 1$ the Hermitian form is positive definite for all p, as $\varepsilon_n = 1$ for $1 \le n < p$ when $q = 1$. Thus $\sigma_2(1/p) = \dim \mathcal{V}_p(S_2)$.

Our first result is the following

Theorem 1. *(Asymptotics)* [5, Theorem 4.1] *For almost all irrational $\theta \in [0,1]$, if we denote by q_k/p_k the sequence of convergents of the continued fraction expansion of θ, then one has*

$$\lim_{k\to\infty} \frac{\sigma_2(q_k/p_k)}{p_k^2} = \Lambda(\theta)$$

where

$$\Lambda(\theta) = \frac{16}{\pi^3} \sum_{n\geq 1,\,\mathrm{odd}} \frac{1}{n^3 \sin(n\pi\theta)}.$$

Here, we take the limit only over those k such that both q_k and p_k are odd, as we were assuming this to define $\sigma_2(q_k/p_k)$. For almost all irrational θ, the set of those k is infinite (see [5, Remark 4.4]) so that the limit makes sense.

Notice in particular that the signature $\sigma_2(q_k/p_k)$ grows like p_k^2 when $q_k/p_k \to \theta$, whereas the dimension of $\mathcal{V}_p(S_2)$ grows like p^3 (see Formula (3).) In higher genus $g > 2$, numerical experiments seem to indicate that the signature grows like p^{2g-2} whereas the growth rate of the dimension of the TQFT vector spaces is well-known to be p^{3g-3} (see [7, Section 3].)

The key ingredient in the proof of Theorem 1 is Brion's formula [2] for enumerating the lattice points in a lattice polytope as a sum of rational functions indexed by the vertices of that polytope. After applying a finite Fourier transform to the expression (1), we use Brion's formula to get an explicit trigonometric expression for $\sigma_2(q/p)$ (see [5, Section 3]) from which we can then extract a limit when q/p goes to an irrational number θ as stated above.

Let us now discuss a modularity property of the signature which, as far as we know, has not been observed before. We found it after first observing the following modularity properties of the limit function $\Lambda(\theta)$.

Theorem 2. *(Modularity)* [5, Corollary 6.11] *The function $\Lambda(\theta)$ satisfies the transformation laws $\Lambda(\theta+2) = \Lambda(\theta)$ and*

$$\Lambda\Big(\frac{\theta}{2\theta+1}\Big)(2\theta+1)^2 - \Lambda(\theta) = 2\theta^2 + 2\theta + 1 \ . \tag{4}$$

The first transformation law is, of course, obvious from the definition of $\Lambda(\theta)$. For the second one, that is, Formula (4), the proof starts with observing that $\Lambda(\theta)$ is the boundary value of an Eichler integral of a certain modular form of weight 4 for the level 2 congruence subgroup $\Gamma(2)$ (see [5, Proposition 6.9].) Recall that $\Gamma(2)$ is generated by the two transformations $\phi_1(\tau) = \tau + 2$ and $\phi_2(\tau) = \frac{\tau}{2\tau+1}$. The transformation law (4) for $\Lambda(\theta)$ is then obtained by showing that the period polynomial of our Eichler integral for the transformation ϕ_2 is given (up to an appropriate scalar) by $2\tau^2 + 2\tau + 1$.

The above modularity properties of the limit function $\Lambda(\theta)$ suggest that a similar integral version holds for the signatures themselves. This is clear for the transformation $\phi_1(\tau) = \tau+2$ as the signature only depends on the root of unity $e^{i\pi q/p}$, and hence $\sigma_2((q+2p)/p) = \sigma_2(q/p)$. Concerning the transformation $\phi_2(\tau) = \frac{\tau}{2\tau+1}$, we found by numerical experiments (guided by the transformation law (4)) that the signature transforms as follows :

$$\sigma_2\Big(\frac{q}{2q+p}\Big) - \sigma_2\Big(\frac{q}{p}\Big) = 2q^2 + 2qp + p^2 - 1 \ . \tag{5}$$

Taking limits as in Theorem 1 it is easy to check that Formula (5) implies the transformation law (4) for $\Lambda(\theta)$. But the converse is not true.

In the first version of our paper [5] (of December 15, 2025), we stated Formula (5) as a conjecture which we had checked by computer for all $0 < q < p < 100$, but which we were unable to prove in general at that time. Less than two months later, in early February 2026, Y. Murakami [6] obtained a proof of Formula (5) starting from our trigonometric formula for $\sigma_2(q/p)$. A nice feature of his proof is that he makes $\sigma_2(q/p)$ itself (not just its limit $\Lambda(\theta)$) appear as the boundary value of a (linear combination of) Eichler integrals.

A similar but simpler story exists for the signature of the TQFT vector space $\mathcal{V}_p$ of a four-punctured sphere with insertions $(p-1)/2$ at the punctures, as this signature is twice the sum (2) which, as already mentioned, is closely related to Dedekind sums. In particular, this signature also has modular properties.

A natural question is, then, what happens in higher genus ? Note that one can write down a formula for the signature of the TQFT vector space associated to any surface as a sum of signs indexed by lattice points inside some polytope depending on the surface (see [1, Remark 4.12].) But in general the signs are much more complicated than in Formula (1) and we don't know yet if (or how) our results generalize to higher genus. Presumably one needs to employ the Frobenius algebra techniques for computing signatures developed by Deroin and Marché in [3], see also [4] and Marché's talk at this workshop.

References

[1] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket.* Topology **34**, (1995), 883–927.

[2] M. Brion, *Points entiers dans les polyèdres convexes,* Ann. Sci. Ecole Norm. Sup. (4e série) **21** (1988), no. 4, 653–663.

[3] B. Deroin and J. Marché, *Toledo invariants of Topological Quantum Field Theories,* `arXiv:2207.09952`

[4] J. Marché, *Signatures of TQFT and trace fields of two-bridge knots,* `arXiv:2309.03656`

[5] J. Marché and G. Masbaum, *Signatures in TQFT : Asymptotics and Modularity*, `arXiv:2512.13450`

[6] Y. Murakami, *Quantum Modularity of Signatures in TQFT and Generalized Dedekind Sums*, `arXiv:2602.16159`

[7] E. Witten, *On Quantum Gauge Theories in Two Dimensions*, Commun. Math. Phys. **141**, 153–209 (1991).

[8] D. Zagier, *Quantum Modular Forms*, In: Quanta of Maths: Conference in honor of Alain Connes, Clay Mathematics Proceedings **11**, AMS and Clay Mathematics Institute 2010, 659-675