A NEW MATRIX-TREE THEOREM

GREGOR MASBAUM AND ARKADY VAINTROB

Abstract. The classical Matrix-Tree Theorem allows one to list the spanning trees of a graph by monomials in the expansion of the determinant of a certain matrix. We prove that in the case of three-graphs (that is, hypergraphs whose edges have exactly *three* vertices) the spanning trees are generated by the Pfaffian of a suitably defined matrix. This result can be interpreted topologically as an expression for the lowest order term of the Alexander-Conway polynomial of an algebraically split link. We also prove some algebraic properties of our Pfaffian-tree polynomial.

Contents

1. Introduction

The classical Matrix-Tree Theorem of Kirchhoff provides the following way of listing all the spanning trees in a graph. Consider a finite graph G with vertex set V and set of edges E. Multiple edges between two vertices are allowed, and we denote by $v(e) \subset V$ the set of endpoints of the edge e. If we label each edge $e \in E$ by a variable x_e, then a subgraph of G given as a collection of edges $S \subset E$ corresponds to the monomial

$$x_S = \prod_{e \in S} x_e.$$

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Form a symmetric matrix $\Lambda = (\ell_{ij})$, whose rows and columns are indexed by the vertices of the graph and entries given by

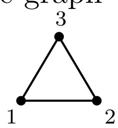
$$\ell_{ij} = -\sum_{\substack{e\in E,\\ v(e)=\{i,j\}}} x_e, \quad \text{if } i \neq j, \quad \text{and} \quad \ell_{ii} = \sum_{\substack{e\in E,\\ i\in v(e)}} x_e.$$

(This matrix arises in the theory of electrical networks and is called sometimes the *Kirchhoff matrix* of the graph.) Since the entries in each row of Λ add up to zero, the determinant of this matrix vanishes and the determinant of the submatrix $\Lambda^{(p)}$ obtained by deleting the pth row and column of Λ is independent of p.[1] This gives a polynomial $\mathcal{D}_G$ in variables x_e which is called the Kirchhoff polynomial of G. The *Matrix-Tree Theorem* [Tut, Theorem VI.29] states that non-vanishing monomials appear in the polynomial $\mathcal{D}_G$ with coefficient 1 and correspond to the spanning trees of G (*i.e.* connected acyclic subgraphs of G with vertex set V). In other words,

$$\mathcal{D}_G := \det \Lambda^{(p)} = \sum_T x_T \ , \tag{1}$$

where the sum is taken over all the spanning trees in G.

For example, if G is the complete graph

$K_3 =$ 

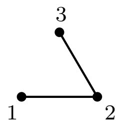
with vertices $1, 2, 3$ and edges $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$, we have

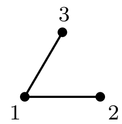
$$\Lambda = \begin{pmatrix} x_{13}+x_{12} & -x_{12} & -x_{13} \\ -x_{12} & x_{12}+x_{23} & -x_{23} \\ -x_{13} & -x_{23} & x_{13}+x_{23} \end{pmatrix},$$

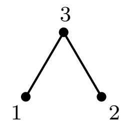
and

$$\mathcal{D}_G = \det \Lambda^{(1)} = \det \Lambda^{(2)} = \det \Lambda^{(3)} = x_{12}x_{23} + x_{12}x_{13} + x_{13}x_{23},$$

which corresponds to the three spanning trees of K_3 :



, 

, and 

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[1] In fact, all entries of $\mathrm{adj}(\Lambda)$, the matrix of cofactors of Λ, are equal. This can be seen as follows. By hypothesis, $\Lambda v = 0$, where v is the column vector with all entries equal to 1. If Λ has rank $\leq m-2$, where m is the size of Λ, the matrix $\mathrm{adj}(\Lambda)$ is identically zero. Otherwise, Λ has rank $m-1$, and its kernel is generated by v. Now the formula $\Lambda\,\mathrm{adj}(\Lambda) = 0$ shows that all columns of $\mathrm{adj}(\Lambda)$ are multiples of v. Since Λ is symmetric, so is $\mathrm{adj}(\Lambda)$, proving that all entries of $\mathrm{adj}(\Lambda)$ are equal.

In this paper we present an analog of this theorem for *three-graphs* (or 3-graphs). Edges of a 3-graph have three vertices and can be visualized as triangles or Y-shaped objects  with the three vertices at their endpoints. We prove that the spanning trees of a 3-graph can be generated by the terms in the expansion of the Pfaffian of a suitably defined *skew-symmetric* matrix. (A sub-3-graph of a 3-graph G is called a spanning tree if its vertex set coincides with that of G and the ordinary graph obtained by gluing together Y-shaped objects  corresponding to the edges of T is a tree. See Figure 1 for an example.)

Let us describe our result for the complete 3-graph Γ_m with the vertex set $V(\Gamma_m) = \{1, 2, \ldots, m\}$. The edges of Γ_m are the three-element subsets $\{i, j, k\}$ of $V(\Gamma_m)$.

Introduce variables y_{ijk}, with $i, j, k \in V(\Gamma_m)$, anti-symmetric in i, j, k, *i.e.*

$$y_{ijk} = -y_{jik} = y_{jki} \quad \text{and} \quad y_{iij} = 0\ . \tag{2}$$

Consider the $m \times m$ matrix

$$\Lambda = (\lambda_{ij}),\ 1 \le i, j \le m, \quad \text{with} \quad \lambda_{ij} = \sum_k y_{ijk}. \tag{3}$$

This matrix is skew-symmetric and its entries in each row add up to zero. This implies that the determinant of $\Lambda^{(p)}$ is independent of p. (Here, as before, $\Lambda^{(p)}$ denotes the result of removing the pth row and column from Λ.)

For example, if $m = 3$, we have

$$\Lambda = \begin{pmatrix} 0 & y_{123} & y_{132} \\ y_{213} & 0 & y_{231} \\ y_{312} & y_{321} & 0 \end{pmatrix} = \begin{pmatrix} 0 & y_{123} & -y_{123} \\ -y_{123} & 0 & y_{123} \\ y_{123} & -y_{123} & 0 \end{pmatrix}$$

and $\det \Lambda^{(3)} = -y_{123}y_{213} = y_{123}^2$.

Since the submatrix $\Lambda^{(p)}$ is still skew-symmetric, it has a *Pfaffian*, $\mathrm{Pf}(\Lambda^{(p)})$, whose square is equal to $\det \Lambda^{(p)}$. (See Section 4 for a review of Pfaffians and their properties.)

It turns out that $(-1)^{p-1}\,\mathrm{Pf}(\Lambda^{(p)})$ does not depend on p (see Lemma 4.1) which allows us to define the polynomial

$$\mathcal{P}_m := (-1)^{p-1}\,\mathrm{Pf}(\Lambda^{(p)}) \tag{4}$$

in variables y_{ijk}. We will call $\mathcal{P}_m$ the *Pfaffian-tree polynomial* because of its connections with spanning trees in Γ_m.

In the example with $m = 3$ above, one has

$$\mathcal{P}_3 = \mathrm{Pf}(\Lambda^{(3)}) = y_{123}.$$

Note that since the determinant of a skew-symmetric matrix of odd size is always zero, $\mathcal{P}_m = 0$ if m is even.

As in the case of ordinary graphs, the correspondence

$$\text{variable} \;\; y_{ijk} \quad \mapsto \quad \text{edge } \{i, j, k\} \text{ of } \;\Gamma_m$$

assigns to each monomial in y_{ijk} a sub-3-graph of Γ_m. A remarkable property of the polynomial $\mathcal{P}_m$ is that the sub-3-graphs of Γ_m corresponding to its monomials are precisely the spanning trees of Γ_m. In particular, if m is odd, the 3-graph Γ_m has no spanning trees (this, of course, can be easily proved directly by a simple combinatorial argument).

Note, however, that because of condition (2) the correspondence between monomials and sub-3-graphs is not one-to-one. A sub-3-graph determines a monomial only up to a sign.

In order to express the signs of the monomials of $\mathcal{P}_m$ in terms of spanning trees, we introduce a notion of orientation of 3-graphs (see Section 3.3). The 3-graph Γ_m has a canonical orientation $\mathfrak{o}_{can}$ given by the natural ordering of the vertices. If T is a spanning tree in Γ_m, this orientation allows us to define unambiguously a monomial $y(T, \mathfrak{o}_{can})$ (see Section 3.4 for details) which is, up to sign, just the product of the variables y_{ijk} over the edges of T. The sum of these monomials is the generating function for spanning trees in Γ_m, denoted by $P(\Gamma_m, \mathfrak{o}_{can})$. Our Pfaffian Matrix-Tree Theorem in the case of the complete 3-graph states, then, that this generating function is given by the Pfaffian-tree polynomial $\mathcal{P}_m$:

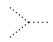
$$P(\Gamma_m, \mathfrak{o}_{can}) = \mathcal{P}_m \tag{5}$$

For example, if $m = 5$, we have

$$\mathcal{P}_5 = P(\Gamma_5, \mathfrak{o}_{can}) = y_{123}\, y_{145} \;\pm\; \ldots\;, \tag{6}$$

where the right-hand side is a sum of 15 terms corresponding to the 15 spanning trees of Γ_5. In Section 3.5 we will explain how to write the terms of $P(\Gamma_m, \mathfrak{o}_{can})$ explicitly including signs. In the case $m = 5$, all spanning trees are obtained from each other by permutations and the right-hand side of (6) can be written with signs as

$$\frac{1}{8} \sum_{\sigma \in \mathfrak{S}_5} (-1)^\sigma y_{\sigma(1)\sigma(2)\sigma(3)}\, y_{\sigma(1)\sigma(4)\sigma(5)} \;\;.$$

If we visualize the edges of Γ_m as Y-shaped objects  , then the spanning tree corresponding to the first term of (6) will look like on Figure 1.

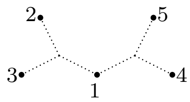


Figure 1. A spanning tree in the complete 3-graph Γ_5. It has two edges, $\{1,2,3\}$ and $\{1,4,5\}$, and contributes the term $y_{123}\,y_{145}$ to $\mathcal{P}_5 = P(\Gamma_5, \mathfrak{o}_{can})$.

This paper grew out of our work [MV] on connections between the Alexander-Conway polynomial of links in $\mathbb{R}^3$ and the Milnor higher linking numbers. In Section 2 we give a brief overview of the relations between the matrix-tree theorems and invariants of links. In Section 3, we formally introduce 3-graphs, their spanning trees, and orientations. There we also deal with the issue of signs and define the generating function $P(G, \mathfrak{o})$ for spanning trees in an oriented 3-graph G. In Section 4, we review Pfaffians and prove several properties we need. Our Matrix-Tree Theorem expressing $P(G, \mathfrak{o})$ as the Pfaffian-tree polynomial $\mathcal{P}_G$ of G is proved in Section 5. Finally in Section 6, we establish some interesting algebraic properties of the Pfaffian-tree polynomial. In particular, we show that $\mathcal{P}_G$ satisfies a contraction-deletion relation. We also prove a 3-term relation and a 4-term relation for the Pfaffian-tree polynomial $\mathcal{P}_m$ of the complete 3-graph Γ_m, and give a recursion formula for $\mathcal{P}_m^2$ which can be used to identify this polynomial in other contexts.

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2. Alexander-Conway polynomial, linking numbers, and Milnor invariants

In this section we briefly discuss the topological motivation of our work. It is not necessary for understanding the rest of the paper and may be safely skipped by a reader interested only in the combinatorial aspect of our results.

Let L be an oriented link in $\mathbb{R}^3$. The best known classical isotopy invariants of L are the linking numbers $\ell_{ij}(L)$ between the ith and jth components of L and its Alexander-Conway polynomial (see *e.g.* [Ka])

$$\nabla_L = \sum_{i\geq 0} c_i(L) z^i \in \mathbb{Z}[z].$$

After the work of Hosokawa [Hs], Hartley [Ha, (4.7)], and Hoste [Ht], it is known that the coefficients $c_i(L)$ of ∇_L for an m-component link L vanish when $i \leq m-2$ and the coefficient $c_{m-1}(L)$ depends only on the linking numbers $\ell_{ij}(L)$ via the determinantal formula

$$c_{m-1}(L) = \det \Lambda^{(p)}, \tag{7}$$

where $\Lambda = (\lambda_{ij})$ is the matrix formed by linking numbers

$$\lambda_{ij} = \begin{cases} -\ell_{ij}(L), & \text{if } \ i \neq j \\ \sum_{k\neq i} \ell_{ik}(L), & \text{if } \ i = j \ . \end{cases}$$

Hartley and Hoste also gave a second expression for $c_{m-1}(L)$ as a sum over trees:

$$c_{m-1}(L) = \sum_T \prod_{\{i,j\}\in edges(T)} \ell_{ij}(L) \ , \tag{8}$$

where T runs through the maximal trees in the complete graph K_m with vertices $\{1, 2, \ldots, m\}$.

For example, if $m = 2$ then $c_1(L) = \ell_{12}(L)$, and if $m = 3$, then

$$c_2(L) = \ell_{12}(L)\ell_{23}(L) + \ell_{12}(L)\ell_{13}(L) + \ell_{13}(L)\ell_{23}(L) \ ,$$

corresponding to the three spanning trees of K_3.

The equality of the expressions (7) and (8) for $c_{m-1}(L)$ follows from the classical Matrix-Tree Theorem (1) applied to the complete graph with m vertices.

If the link L is *algebraically split*, *i.e.* all linking numbers ℓ_{ij} vanish, then not only $c_{m-1}(L) = 0$, but, as was proved by Traldi [Tr1] and Levine [Lev], the next $m-2$ coefficients of ∇_L also vanish

$$c_{m-1}(L) = c_m(L) = \ldots = c_{2m-3}(L) = 0.$$

For an algebraically split oriented link L with three components, there exists an integer-valued isotopy invariant $\mu_{123}(L)$ called the *Milnor triple linking number* (see [Mi]). (This invariant is equal to 1 for the standard Borromean rings.) For an algebraically split link L with

m components, we thus have $\binom{m}{3}$ triple linking numbers $\mu_{ijk}(L)$ corresponding to the different 3-component sublinks of L. Unlike the ordinary linking numbers, the triple linking numbers are anti-symmetric in their indices

$$\mu_{ijk}(L) = -\mu_{jik}(L) = \mu_{jki}(L) \ .$$

Levine [Lev] (see also Traldi [Tr2, Theorem 8.2]) found an expression for the first non-vanishing coefficient $c_{2m-2}(L)$ of ∇_L for an algebraically split link in terms of Milnor triple linking numbers

$$c_{2m-2}(L) = \det \Lambda^{(p)} \ , \tag{9}$$

where $\Lambda = (\lambda_{ij})$ is the $m \times m$ skew-symmetric matrix with entries given by

$$\lambda_{ij} = \sum_k \mu_{ijk}(L)$$

(cf. (3)).

This formula is analogous to the determinantal expression (7). One of the starting points of the present paper was an attempt to find an analogue of the sum over trees formula (8) for algebraically split links. As a corollary of the determinantal formula (9) and our Matrix-Tree Theorem for complete 3-graphs we obtain a combinatorial expression for c_{2m-2} as (the square of) a sum over trees.

Theorem 2.1. *If L is an algebraically split link, then*

$$c_{2m-2}(L) = P_m(L)^2 \ , \tag{10}$$

where $P_m(L)$ is the spanning tree generating function $P(\Gamma_m, \mathfrak{o}_{can})$ evaluated at $y_{ijk} = \mu_{ijk}(L)$.

Proof. This follows from (9) and (5), since $\det \Lambda^{(p)} = \mathrm{Pf}(\Lambda^{(p)})^2$. $\square$

Using the theory of finite type invariants, we give in [MV] another, direct, proof of formula (10) for the coefficient $c_{2m-2}(L)$ which does not use the determinantal formula (9). In fact, the proof in [MV] together with our Matrix-Tree Theorem for 3-graphs provides an alternative proof of formula (9).

The following table summarizes the analogy which was our guiding principle in this work:

Linking numbers	Milnor's triple linking numbers
Edges of ordinary graphs	(Oriented) Edges of 3-graphs
Formulas (7) and (8)	Formulas (9) and (10)
The classical Matrix-Tree Theorem	The Pfaffian Matrix-Tree Theorem

3. Three-graphs

3.1. **Basic definitions.**

Three-graphs (or, more officially, 3-*uniform hypergraphs*, see [Ber]) are analogues of graphs. The only difference is that the edges of a 3-graph are 'triangular', *i.e.* they have three vertices, while edges of ordinary graphs have only two vertices. For our purposes, it will not be necessary to consider degenerate edges, *i.e.* edges with less than 3 vertices (they are analogous to loop edges for ordinary graphs), and so we will use the following definition.

Definition 3.1. A *three-graph* (or 3-graph) is a triple $G = (V, E, v)$ where $V = V(G)$ and $E = E(G)$ are finite sets, whose elements will be called, respectively, *vertices* and *edges* of G, and v is a map from E to the set of three-element subsets of V. For an edge $e \in E$ the elements of $v(e)$ are called the *vertices* or the *endpoints* of e.

A 3-graph $G' = (V', E', v')$ is called a *sub*-3-*graph* of $G = (V, E, v)$ if $V' \subset V$, $E' \subset E$ and $v' = v_{|E'}$. If $V' = v(E')$, we say that G' is the sub-3-graph of G generated by the subset of edges E'.

Note that by replacing "three-element" with "two-element" in the above definition, we recover exactly the definition of a (finite) graph without loops, but, possibly, with multiple edges.

Visually, an edge of a 3-graph can be thought of as a triangle or as a Y-shaped object  with the three vertices at its endpoints.

In this paper we adopt the latter point of view and, accordingly, we define the *topological realization* of a 3-graph by gluing together all the Y's corresponding to its edges.

More formally, the topological realization $|G|$ of a 3-graph G is a one-dimensional cell complex obtained by taking one 0-cell for every element of $V(G) \cup E(G)$ and then gluing a 1-cell for every pair $(v, e) \in V(G) \times E(G)$ such that $v \in v(e)$.

Note that $|G|$ is the same as the topological realization of the *bipartite graph* naturally associated with the 3-graph G.

3.2. **Trees.**

Definition 3.2. A 3-graph G is called a *tree* if its topological realization $|G|$ is connected and simply connected. (*I.e.* the bipartite graph formed by the Y-shaped objects  corresponding to the edges of G is a tree.)

The following proposition states that this definition is equivalent to the standard definition of trees for hypergraphs (see [Ber]) and that a 3-graph with an even number of vertices cannot be a tree.

A *path* in a 3-graph G is a sequence $v_0, e_1, v_1, \dots, v_{n-1}, e_n, v_n$ of vertices v_i and edges e_i of G such that $v_i \in v(e_{i+1})$ for $i = 0, \dots, n-1$, and $v_i \in v(e_i)$ for $i = 1, \dots, n$. A path is called a *cycle* if $v_0 = v_n$ and $e_i \neq e_j$ for $i \neq j$. A 3-graph G is called *connected* if for any two vertices v and u of G there exists a path in G which begins in v and ends in u. (In particular, a 3-graph with only one vertex and no edges is connected.)

Proposition 3.3.

(i) A 3-graph is a tree if and only if it is connected and has no cycles.

(ii) If a 3-graph with m vertices is a tree, then m is odd and the number of edges is equal to $(m-1)/2$.

Both statements can be easily proved by induction on the number of edges.

A sub-3-graph T of a 3-graph G is called a *spanning tree* if T is a tree and $V(T) = V(G)$. By the above proposition, only 3-graphs with odd number of vertices may have spanning trees.

In the next section we will need the following characterization of trees.

Proposition 3.4. *Let T be a 3-graph with n edges and $2n+1$ vertices. Fix an ordering of the vertex set $E(T)$, and for each edge $e \in E(T)$, choose a (non-trivial) cyclic permutation $\sigma(e)$ of the three-element set $v(e)$. Then T is a tree if and only if the product*

$$\sigma(T) = \prod_{e \in E(T)} \sigma(e) \tag{11}$$

is a cyclic permutation of the vertex set $V(T)$. In particular, if the product in some order is a cyclic permutation of $V(T)$, then the same is true for any other order as well.

Proof. First, assume that T is a tree. We will prove that $\sigma(T)$ is a cycle by induction on the number of edges in T. In the case $n = 1$, the statement is a tautology. If $n \geq 2$, number the edges $e_1, e_2, \dots, e_n \in E(T)$ according to the order they appear in the product (11), so that

$$\sigma(T) = \sigma(e_1)\sigma(e_2)\dots\sigma(e_n) \ .$$

Since T is a tree, it has an edge e_k which has only one common vertex with the sub-3-graph T' generated by the remaining edges (in which case, T' is itself a tree). Since a conjugate of any cyclic permutation is again a cyclic permutation, the equation

$$\sigma(e_1)\sigma(e_2)\ldots\sigma(e_n) = \pi^{-1}(\sigma(e_k)\ldots\sigma(e_n)\sigma(e_1)\ldots\sigma(e_{k-1}))\pi \ , \tag{12}$$

where $\pi = \sigma(e_k)\ldots\sigma(e_n)$, shows that we can assume that $k = 1$, *i.e.* that the sub-3-graph T' of T generated by the edges $e_2, e_3, \ldots, e_n$ is also a tree. Therefore, by induction, the permutation

$$\sigma(T') = \sigma(e_2)\sigma(e_3)\ldots\sigma(e_n)$$

is a cyclic permutation of the set $V(T')$. This implies that the permutation $\sigma(T) = \sigma(e_1)\sigma(T')$ is a cycle, since it is a product of two cyclic permutations having only one common element.

Conversely, assume that T is not a tree. Then it is disconnected, because a connected 3-graph with n edges can have $2n+1$ vertices only if it is a tree. Therefore, the permutation $\sigma(T)$ cannot be a $2n+1$-cycle, since it is equal to the product of several commuting permutations corresponding to the different components of T. □

Remark 3.5. A similar description of trees exists for ordinary graphs. Namely, every edge of a graph G determines a transposition (a two-cycle) of the vertex set $V(G)$; and a graph with m vertices and $m-1$ edges is a tree if and only if the product (taken in any order) of the corresponding $m-1$ transpositions is an m-cycle.

3.3. **Orientations.**

In order to keep track of signs, we introduce a notion of *orientation* as follows.

Definition 3.6. An *orientation* of a finite set $\mathcal{X}$ is an ordering of $\mathcal{X}$ up to even permutations. An *orientation* of a 3-graph G is an orientation of its vertex set $V(G)$. An *orientation* of an edge $e \in E$ is an orientation of its vertex set $v(e)$.

Note that an orientation of an edge $e \in E(G)$ is the same as an orientation of the sub-3-graph of G generated by that edge.

Remark 3.7. The term 'orientation' here is justified because an orientation of a set $\mathcal{X}$ induces an orientation of the vector space $\mathbb{R}^{\mathcal{X}}$.

If a 3-graph G has an odd number, m, of vertices, then an orientation of G can also be specified by a *cyclic ordering* of $V(G)$, *i.e.* an ordering up to cyclic permutation. (This is because an m-cycle is an even permutation if m is odd.) We will usually write such an ordering as a cyclic permutation. In particular, an orientation of an edge of a

3-graph with vertex set $\{i, j, k\}$ will be indicated by choosing one of the two three-cycles (ijk) or (jik).

Note that the orientations given by two m-cycles σ and σ' are the same if and only if $\sigma' = s\sigma s^{-1}$ where s is an even permutation of $V(G)$.

The following is a key construction needed for the definition of the generating function of spanning trees in a 3-graph. Let $\tilde{T}$ denote the data consisting of a tree T together with a choice of orientation for every edge of T. For each edge $e \in E(T)$, denote by $\sigma(e)$ the cyclic permutation of the three-element set $v(e)$ induced by the orientation of this edge in $\tilde{T}$. If we choose an ordering of the set $E(T)$, then by Proposition 3.4 the product

$$\sigma(\tilde{T}) = \prod_{e \in E(T)} \sigma(e) \tag{13}$$

is a cyclic permutation of the set $V(T)$ and, therefore, gives a cyclic ordering of $V(T)$. Since T has an odd number of vertices, this cyclic ordering induces an orientation of T. As we will show now, this orientation does not depend on the choice of ordering of the edges and we denote it by $\mathfrak{o}(\tilde{T})$.

Proposition 3.8. *The orientation $\mathfrak{o}(\tilde{T})$ is well-defined. Moreover, $\mathfrak{o}(\tilde{T})$ changes sign whenever the orientation of an edge in $\tilde{T}$ is reversed.*

Proof. The proof that the orientation $\mathfrak{o}(\tilde{T})$ is well-defined, *i.e.* it does not depend on the order of factors in (13), is similar to the proof of Proposition 3.4. Indeed, if we change the order of the factors cyclically then $\sigma(\tilde{T})$ is replaced by $\pi^{-1}\sigma(\tilde{T})\pi$ (see (12)), where π is an even permutation (a product of several three-cycles $\sigma(e_i)$), and thus, the orientation will not change. This implies that, as in the proof of Proposition 3.4, we may assume that the first factor $\sigma(e_1)$ in (13) comes from an edge that has only one common vertex with the sub-3-graph T' generated by the remaining edges. But in this case, T' is a tree and by induction we see that the orientation $\mathfrak{o}(\tilde{T}')$, and therefore $\mathfrak{o}(\tilde{T})$ too, is well-defined.

Let us now prove that changing the orientation of one edge e in $\tilde{T}$ reverses the orientation $\mathfrak{o}(\tilde{T})$.

Denote the vertices of the tree T by $V(T) = \{1, 2, \ldots, m\}$ so that $v(e) = \{1, 2, 3\}$. Let T_i be the maximal subtree of T which contains the vertex i but not the edge e.[2] The data $\tilde{T}$ induces orientations of

[2]It may happen that T_i consists of just the vertex i and no edge. Note that with our definition, such a 3-graph is a tree.

each of the subtrees T_1, T_2 and T_3, which we represent, respectively, by cyclic permutations $(1A)$, $(2B)$ and $(3C)$, where A, B and C are ordered disjoint sets (some of which may be empty) with (unordered) union $\{4, 5, \ldots, m\}$.

If the orientation of the edge e in $\tilde{T}$ is (123), then the orientation $\mathfrak{o}(\tilde{T})$ is represented by the product of cycles[3]

$$\sigma = (1A)(123)(2B)(3C) = (A12B3C) \ .$$

If we change the orientation of e to (132), then the new orientation of T is represented by

$$\sigma' = (1A)(132)(2B)(3C) = (A13C2B)$$

(see Figure 3 for an illustration). Therefore,

$$\sigma' = s\sigma s^{-1} \ ,$$

where s is an odd permutation, since both B and C have an even number of elements. Thus, σ and σ' define opposite orientations of the tree T.

This completes the proof of the proposition. □

In concrete situations it is often convenient to specify orientations of the edges of trees by embedding them into the plane. This leads to the following pictorial way of computing the orientation $\mathfrak{o}(\tilde{T})$.

If we embed the tree T (or, more precisely, its topological realization $|T|$) into the plane, then every edge of T acquires an orientation induced from the standard (counterclockwise) orientation of the plane. We call such an embedding *admissible* with respect to the data $\tilde{T}$ if for every edge its induced orientation coincides with the one specified by $\tilde{T}$. By induction on the number of edges, it is easy to see that for every $\tilde{T}$ there exists an admissible embedding.

Given an admissible embedding of T, we can obtain an orientation of $V(T)$ as follows. Go around the embedded tree in the counterclockwise order and write down the *cyclic* sequence of vertices in the order they were visited. Some vertices will be visited more than once, but when we remove extra vertices from this sequence (in an arbitrary way) we obtain a cyclic ordering of the set $V(T)$ (see an example on Figure 2). It is easy to see that the orientation of T given by this ordering is independent of the choices involved and coincides with the orientation $\mathfrak{o}(\tilde{T})$.

As an example of this approach, Figure 3 gives a pictorial illustration of the above proof of the sign change property of the orientation $\mathfrak{o}(\tilde{T})$.

[3]Our convention is that $\sigma_1\sigma_2$ means first apply σ_2 then σ_1.

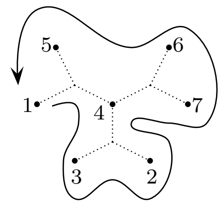

Figure 2. An admissible embedding of $\tilde{T}$ having three oriented edges (145), (243), and (476). Orderings representing $\mathfrak{o}(\tilde{T})$ include, for example, 1432765 and 1327645. They can be read off going counterclockwise around the diagram, as indicated by the solid arrow.

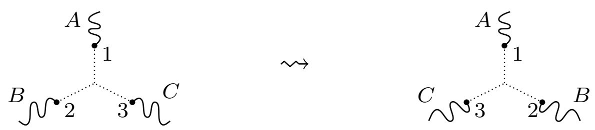

Figure 3. An admissible embedding of $\tilde{T}$ with the orientation of the edge $\{1, 2, 3\}$ reversed can be obtained by exchanging the subtrees at the vertices 2 and 3.

3.4. **The tree generating function.**

We now introduce a function which generates spanning trees in a 3-graph.

Given a 3-graph $G = (V, E, v)$, we denote by $\tilde{E} = \tilde{E}(G)$ the set of oriented edges of G. Thus, an element $\tilde{e}$ of $\tilde{E}(G)$ is a pair consisting of an edge $e \in E(G)$ and an orientation of e. We denote by $-\tilde{e}$ the edge with the opposite orientation. The assignment $\tilde{e} \mapsto -\tilde{e}$ is a fixed-point free involution on $\tilde{E}(G)$, with quotient $E(G)$.

To every oriented edge $\tilde{e} \in \tilde{E}(G)$, we associate an indeterminate $y_{\tilde{e}}$ with the relation

$$y_{-\tilde{e}} = -y_{\tilde{e}}.$$

Definition 3.9. Let T be a 3-graph which is a tree, and let $\mathfrak{o}$ be an orientation of T. We define the monomial $y(T, \mathfrak{o})$ by the formula

$$y(T, \mathfrak{o}) = \frac{\mathfrak{o}(\tilde{T})}{\mathfrak{o}} \prod_{e \in E(T)} y_{\tilde{e}} \ ,$$

where $\tilde{T}$ is any choice of orientations for the edges (giving a lift $\tilde{e} \in \tilde{E}(G)$ of every edge $e \in E(G)$), $\mathfrak{o}(\tilde{T})$ is the orientation defined in Proposition 3.8, and $\mathfrak{o}(\tilde{T})/\mathfrak{o}$ is the sign relating the two orientations.

If one changes the orientation of an edge e in $\tilde{T}$, then by Proposition 3.8 the orientation $\mathfrak{o}(\tilde{T})$ picks up a minus sign which is cancelled by the change of sign for $y_{\tilde{e}}$. This shows that $y(T, \mathfrak{o})$ is well defined.

Definition 3.10. Let G be a 3-graph, and let $\mathfrak{o}$ be an orientation of G. The *generating function for spanning trees* of G is

$$P(G, \mathfrak{o}) = \sum_T y(T, \mathfrak{o}) \ , \tag{14}$$

where T runs through the spanning trees of G.

Note that $P(G, -\mathfrak{o}) = -P(G, \mathfrak{o})$. Thus, only the sign of the generating function depends on the orientation.

3.5. **The tree generating function of the complete 3-graph.**
Let us now give a more explicit combinatorial expression for the tree generating function (14) of a complete 3-graph.

By definition, the *complete* 3*-graph* Γ_m has vertex set

$$V(\Gamma_m) = \{1, 2, \ldots, m\},$$

and has exactly one edge e for every unordered triple $\{i, j, k\}$ of vertices. Every cyclic permutation (ijk) determines an *oriented* edge $\tilde{e}$. Therefore we can identify the indeterminates $y_{\tilde{e}}$ with indeterminates y_{ijk} which are totally antisymmetric in their indices (as in (2)).

We denote by $\mathfrak{o}_{can}$ the orientation of Γ_m given by the natural ordering of $V(\Gamma_m)$. If m is even, then Γ_m has no spanning trees. Therefore let us assume m is odd. Put $d = (m-1)/2$.

If $\sigma_1, \ldots, \sigma_d$ are 3-cycles in $\mathfrak{S}_m$, we set

$$\sigma = \prod_i \sigma_i \in \mathfrak{S}_m$$

and define

$$\varepsilon(\sigma_1, \ldots, \sigma_d) = \begin{cases} 0 & \text{if } \sigma \text{ is not an } m\text{-cycle} \\ (-1)^s & \text{if } \sigma = (s(1) \ldots s(m)) \text{ is an } m\text{-cycle,} \end{cases}$$

where $(-1)^s$ is the sign of the permutation $s \in \mathfrak{S}_m$. Notice that in the case when σ is an m-cycle, s is defined by the condition

$$\sigma = (s(1)\, s(2) \ldots s(m)) = s(1\, 2 \ldots m)s^{-1}$$

only up to powers of the standard cycle $(1\, 2 \ldots m)$. However, since m is odd, the sign $(-1)^s$ is well-defined.

If T is a spanning tree on Γ_m, the associated monomial is

$$y(T, \mathfrak{o}_{can}) = \varepsilon\big((i_1 j_1 k_1), \ldots, (i_d j_d k_d)\big) \prod_{\alpha=1}^{d} y_{i_\alpha j_\alpha k_\alpha}$$

where T is given by a collection of (unoriented) edges $e_\alpha = \{i_\alpha, j_\alpha, k_\alpha\}$, $1 \leq \alpha \leq d$.

Since by Proposition 3.4 the sign $\varepsilon\big((i_1 j_1 k_1), \ldots, (i_d j_d k_d)\big)$ is zero if T is not a spanning tree, we can write

$$P(\Gamma_m, \mathfrak{o}_{can}) = \sum \varepsilon\big((i_1 j_1 k_1), \ldots, (i_d j_d k_d)\big) \prod_{\alpha=1}^{d} y_{i_\alpha j_\alpha k_\alpha}$$

where the sum is taken over all monomials $\prod_\alpha y_{i_\alpha j_\alpha k_\alpha}$ of degree d, with the convention that monomials which differ only by changing the order of indices in some of the variables are taken only once. Alternatively, since $\varepsilon \neq 0$ implies that the triples $(i_\alpha, j_\alpha, k_\alpha), \alpha = 1, \ldots, d$ are distinct, we can write

$$P(\Gamma_m, \mathfrak{o}_{can}) = \frac{1}{6^d d!} \sum \varepsilon\big((i_1 j_1 k_1), \ldots, (i_d j_d k_d)\big) \prod_{\alpha=1}^{d} y_{i_\alpha j_\alpha k_\alpha}$$

where the sum is now over all $3d$-tuples of indices $i_1, j_1, \ldots k_d$.

Some interesting algebraic properties of the tree generating function $P(\Gamma_m, \mathfrak{o}_{can})$ will be given in Section 6.

4. Pfaffians

In this section, we review the definition and the main properties of Pfaffians (see *e.g.* [La, Chapter XIV.§10] and [BR, Chapter 9.5] for more details). Let

$$A = (a_{ij}),\ 1 \leq i, j \leq 2n, \qquad a_{ij} = -a_{ij}$$

be a skew-symmetric matrix. One way to define its Pfaffian, $\mathrm{Pf}(A)$, is as follows. Associate to A the 2-form

$$\alpha = \sum_{i<j} a_{ij}\ e_i \wedge e_j = \frac{1}{2} \sum_{i,j} a_{ij}\ e_i \wedge e_j$$

where $e_1, \ldots, e_{2n}$ is a basis of one-forms on a vector space of dimension $2n$. Then

$$\frac{\alpha^n}{n!} = \mathrm{Pf}(A)\ e_1 \wedge e_2 \wedge \ldots \wedge e_{2n} \tag{15}$$

For example, if $n = 1$, then $\mathrm{Pf}(A) = a_{12}$, and if $n = 2$, then $\mathrm{Pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

Equation (15) implies the following explicit formula for $\mathrm{Pf}(A)$.

$$\begin{aligned} \mathrm{Pf}(A) &= \frac{1}{2^n n!} \sum_{\sigma\in\mathfrak{S}_{2n}} (-1)^\sigma\, a_{\sigma(1)\sigma(2)}\, a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2n-1)\sigma(2n)} \\ &= \sum_{\substack{\sigma\in\mathfrak{S}_{2n} \\ \sigma(1)<\sigma(3)<\ldots<\sigma(2n-1) \\ \sigma(2i-1)<\sigma(2i) \text{ for all } i}} (-1)^\sigma\, a_{\sigma(1)\sigma(2)}\, a_{\sigma(3)\sigma(4)} \cdots a_{\sigma(2n-1)\sigma(2n)} \qquad (16) \end{aligned}$$

We will need the following *row development formula*. Let $A^{(i,j)}$ denote the $(2n-2)\times(2n-2)$-matrix obtained by removing the ith and jth row and column from A. Then

$$\mathrm{Pf}(A) = a_{12}\ \mathrm{Pf}(A^{(1,2)}) - a_{13}\ \mathrm{Pf}(A^{(1,3)}) + \ldots + a_{1,2n}\ \mathrm{Pf}(A^{(1,2n)}) \ . \tag{17}$$

This formula can be deduced from (16) by noticing that $\sigma(1)=1$ for every permutation σ appearing in the sum in (16).

Here are two more standard properties of the Pfaffian:

$$\mathrm{Pf}(A)^2 = \det A \ , \tag{18}$$

$$\mathrm{Pf}(S^TAS) = \det(S)\ \mathrm{Pf}(A) \ . \tag{19}$$

We omit the proofs, but notice that if $\det S \neq 0$, then S^TAS is just the matrix of the 2-form α in a new basis, which allows one to deduce (19) from (15).

Finally, we need the following lemma.

Lemma 4.1. *Let $A=(a_{ij})_{1\le i,j\le 2n+1}$ be a skew-symmetric $(2n+1)\times(2n+1)$ matrix such that*

$$\sum_i a_{ij} = 0$$

for all j. Then

$$(-1)^{p-1}\ \mathrm{Pf}(A^{(p)})$$

is independent of p.

Proof. Define a bilinear form α on a $2n$-dimensional vector space with basis $b_1, b_2, \ldots, b_{2n}$ by $\alpha(b_i,b_j)=a_{ij}$. Thus, α is the two-form associated to the matrix $A^{(2n+1)}$. Set

$$b' = -\sum_{i=1}^{2n} b_i$$

and observe that

$$\alpha(b', b_j) = -\sum_{i=1}^{2n} a_{ij} = a_{2n+1,j} \ .$$

Thus, the matrix of the form α in the basis $b_1, b_2, \ldots, b_{2n-1}, b'$ is precisely $A^{(2n)}$. Since the corresponding base-change transformation has determinant -1, it follows from (19) that

$$\mathrm{Pf}(A^{(2n)}) = -\,\mathrm{Pf}(A^{(2n+1)}) \ .$$

More generally, $A^{(p)}$ is the matrix of α in the basis

$$b_1, b_2, \ldots, b_{p-1}, b_{p+1}, \ldots, b_{2n-1}, b' \ ,$$

which is related to the standard basis by a base change of determinant $(-1)^{p-1}$, which proves the lemma. □

5. The Pfaffian Matrix-Tree Theorem

We now state and prove a formula expressing the generating function for spanning trees in terms of a Pfaffian. Let G be a 3-graph with vertices numbered from 1 to m. Define the matrix

$$\Lambda(G) = (\lambda_{ij})_{1\le i,j\le m}$$

as follows. The diagonal terms λ_{ii} are zero. If $i \neq j$, then

$$\lambda_{ij} = \sum_{\tilde{e}} y_{\tilde{e}} \ ,$$

where $\tilde{e}$ runs through the oriented edges $\tilde{e} \in \tilde{E}(G)$ such that i and j are vertices of $\tilde{e}$, and the orientation of $\tilde{e}$ is represented by the cyclic ordering (ijk), where k denotes the third vertex of $\tilde{e}$. Since $y_{-\tilde{e}} = -y_{\tilde{e}}$, the matrix $\Lambda(G)$ is skew-symmetric.

Definition 5.1. Let G be a 3-graph with vertices numbered from 1 to m. The *Pfaffian-tree polynomial* of G is

$$\mathcal{P}_G = (-1)^{p-1}\,\mathrm{Pf}(\Lambda(G)^{(p)}) \ , \tag{20}$$

where $\Lambda(G)^{(p)}$ for $p = 1, 2, \ldots, m$ is the matrix obtained by removing the pth row and column from $\Lambda(G)$.

Note that the the right-hand side of (20) is independent of p by Lemma 4.1.

Example. In the case of the complete 3-graph Γ_m, we can write the variables $y_{\tilde{e}}$ of $\mathcal{P}_{\Gamma_m}$ as y_{ijk}, where the y_{ijk}'s are totally antisymmetric with respect to their indices. Then the matrix $\Lambda(\Gamma_m)$ is precisely the matrix Λ in (3), so that the Pfaffian-tree polynomial $\mathcal{P}_{\Gamma_m}$ is equal to the polynomial $\mathcal{P}_m$ defined in (4).

The name 'Pfaffian-tree' is justified by the following theorem, which is the main result of this paper.

Theorem 5.2. *Let G be a 3-graph with vertices numbered from 1 to m. Then the generating function of spanning trees in G is equal to the Pfaffian-tree polynomial of G*

$$P(G, \mathfrak{o}_{can}) = \mathcal{P}_G \ . \tag{21}$$

Here, $\mathfrak{o}_{can}$ is the canonical orientation of G determined by the ordering of the vertices.

Proof. We will assume that $\mathcal{P}_G$ is given by formula (20) with $p = 1$. We may assume m is odd, since otherwise both sides of (21) are zero.

First, we will show that it is enough to prove the theorem in the case of the complete 3-graph Γ_m. Indeed, let us write the indeterminates for Γ_m as y_{ijk}, where the y_{ijk} are totally antisymmetric in their indices. Then the generating function for G is obtained from the one for Γ_m by the substitution

$$y_{ijk} = \sum_{\tilde{e}} y_{\tilde{e}} \ ,$$

where $\tilde{e}$ runs through the oriented edges $\tilde{e} = (e, \mathfrak{o})$ of G such that i, j and k are the vertices of e, and the orientation $\mathfrak{o}$ of $\tilde{e}$ is represented by the cyclic permutation (ijk). The same substitution applied to the matrix $\Lambda = \Lambda(\Gamma_m)$ yields the matrix $\Lambda(G)$. Thus, if the theorem holds for Γ_m, then it holds for any G.

Let us now prove the result for $G = \Gamma_m$. We denote the spanning tree generating function $P(\Gamma_m, \mathfrak{o}_{can})$ by P_m. Note that both P_m and the Pfaffian-tree polynomial $\mathcal{P}_m$ are polynomials in the indeterminates y_{ijk}.

We will prove that

$$P_m = \mathcal{P}_m \tag{22}$$

by induction on m.

The $m = 3$ case was checked in the introduction.

It will be convenient to use the following notation. Consider an m-dimensional vector space $\mathcal{V}$ with a basis $v_1, \ldots, v_m$. Then we can identify the variables y_{ijk} in P_m and $\mathcal{P}_m$ with the standard basis of the space of three-vectors

$$y_{ijk} = v_i \wedge v_j \wedge v_k \in \Lambda^3 \mathcal{V}$$

and consider both polynomials P_m and $\mathcal{P}_m$ as elements of the space

$$S^{(m-1)/2}(\Lambda^3\mathcal{V}) \ .$$

There is a natural embedding of this space into the tensor algebra of $\mathcal{V}$, *i.e.,* the free associative algebra generated by $v_1, \dots, v_m$. Therefore we may view both the left- and the right-hand side of (22) as polynomials in the *non-commuting* variables v_i

$$\begin{aligned} P(\Gamma_m, \mathfrak{o}_{can}) &= P_m(y_{ijk}) = P_m(v_1, v_2, \dots, v_m) \ , \\ \mathrm{Pf}(\Lambda(\Gamma_m)^{(1)}) &= \mathcal{P}_m(y_{ijk}) = \mathcal{P}_m(v_1, v_2, \dots, v_m) \ . \end{aligned}$$

The proof that $P_m = \mathcal{P}_m$ proceeds by showing that both P_m and $\mathcal{P}_m$ satisfy the same recursion formula.

First, the generating function P_m. We have

$$P_m = P(\Gamma_m, \mathfrak{o}_{can}) = P'_m + P''_m \ , \tag{23}$$

where P'_m generates the spanning trees of Γ_m which contain the edge $\{1,2,3\}$, and P''_m generates those which do not. Given a spanning tree of the first kind, we can collapse the edge $\{1,2,3\}$ to a new vertex, 0, say, and obtain a spanning tree in the complete 3-graph Γ' with $m-2$ vertices $0, 4, 5, \dots, m$. Conversely, every spanning tree in Γ' can be lifted (in a non-unique way) to Γ_m, and every such lift together with the edge $\{1,2,3\}$ constitutes a spanning tree of Γ_m. This correspondence gives a relation between the generating functions which can be conveniently expressed in terms of the variables v_i:

$$P'_m = y_{123}\, P_{m-2}(v_1 + v_2 + v_3, v_4, \dots, v_m) \ . \tag{24}$$

On the other hand, notice that P''_m is just the polynomial obtained from $P_m(y_{ijk})$ by setting $y_{123} = 0$.

Now for the Pfaffian polynomial $\mathcal{P}_m$. Recall that $\Lambda(\Gamma_m)$ is the $m \times m$ matrix

$$\Lambda = (\lambda_{ij}), \quad \text{with} \quad \lambda_{ij} = \sum_k y_{ijk}, \ 1 \le i, j \le m \ .$$

Write

$$\mathcal{P}_m = \mathrm{Pf}(\Lambda^{(1)}) = \mathcal{P}'_m + \mathcal{P}''_m \ , \tag{25}$$

where $\mathcal{P}''_m$ is, *by definition,* the polynomial obtained from $\mathcal{P}_m$ by setting $y_{123} = 0$. We claim that

$$\mathcal{P}'_m = y_{123}\, \mathcal{P}_{m-2}(v_1 + v_2 + v_3, v_4, \dots, v_m) \ . \tag{26}$$

To show this, we apply the row development formula (17) to $\Lambda^{(1)}$ to obtain

$$\begin{aligned}\mathcal{P}_m &= \mathrm{Pf}(\Lambda^{(1)}) \\ &= \lambda_{23}\ \mathrm{Pf}(\Lambda^{(1,2,3)}) - \lambda_{24}\ \mathrm{Pf}(\Lambda^{(1,2,4)}) + \ldots + \lambda_{2,m}\ \mathrm{Pf}(\Lambda^{(1,2,m)}) \ .\end{aligned}$$

The only entry in $\Lambda^{(1)}$ where y_{123} appears is $\lambda_{23} = -\lambda_{32}$, so that setting $y_{123} = 0$ affects only the very first term of this expansion. It follows that

$$\mathcal{P}'_m = \mathcal{P}_m - \mathcal{P}''_m = y_{123}\ \mathrm{Pf}(\Lambda^{(1,2,3)}) \ .$$

It remains to show that $\mathrm{Pf}(\Lambda^{(1,2,3)})$ is equal to $\mathcal{P}_{m-2}(v_1+v_2+v_3, v_4, \ldots, v_m)$. To see this, consider again the complete 3-graph Γ' with $m-2$ vertices $0, 4, 5, \ldots, m$. The (i,j) entry of the associated matrix is

$$y_{ij0} + \sum_{k=4}^{m} y_{ijk} \ .$$

Thus, if we delete the 0-th row and column and substitute $v_0 = v_1 + v_2 + v_3$, we get exactly the matrix $\Lambda^{(1,2,3)}$ (whose (i,j) entry is $y_{ij1} + y_{ij2} + y_{ij3} + \sum_{k=4}^{m} y_{ijk}$). This proves (26).

Thus we have shown that both P_m and $\mathcal{P}_m$ satisfy the same recursion relation

$$P_m = y_{123} P_{m-2}(v_1 + v_2 + v_3, v_4, \ldots, v_m) + [P_m]_{y_{123}=0} \tag{27}$$

(and similarly for $\mathcal{P}_m$.)

This implies that $P_m = \mathcal{P}_m$ as follows. Since $P_{m-2} = \mathcal{P}_{m-2}$ by the induction hypothesis, the recursion (27) shows that $P_m - \mathcal{P}_m$ is divisible by y_{123}. Since a similar recursion obviously holds for every edge $\{i,j,k\}$, the difference $P_m - \mathcal{P}_m$ must be divisible by every y_{ijk}. Therefore, $P_m - \mathcal{P}_m$ must be zero by degree count. This completes the proof. □

6. Properties of the Pfaffian-tree polynomial $\mathcal{P}_m$

In this section, we establish some algebraic properties of the Pfaffian-tree polynomial of the complete 3-graph Γ_m.

6.1. **Antisymmetry.**

By definition, $\mathcal{P}_m$ is a homogeneous polynomial of degree $(m-1)/2$ in the indeterminates y_{ijk}. Thinking of the y_{ijk} as elements of $\Lambda^3\mathcal{V}$, as in the proof of Theorem 5.2, we may consider $\mathcal{P}_m$ as an element of the space $S^{(m-1)/2}\Lambda^3\mathcal{V}$. The following result shows that $\mathcal{P}_m$ belongs to the subspace

$$(S^{(m-1)/2}\Lambda^3\mathcal{V})^- \ ,$$

where the superscript $^-$ indicates the subspace which is totally antisymmetric with respect to the action of the symmetric group $\mathfrak{S}_m$. Recall that $v_1 \dots, v_m$ denotes a basis of $\mathcal{V}$ such that $y_{ijk} = v_i \wedge v_j \wedge v_k$.

Proposition 6.1. *For every permutation $\sigma \in \mathfrak{S}_m$, one has*

$$\mathcal{P}_m(v_{\sigma(1)}, \dots, v_{\sigma(m)}) = (-1)^\sigma \mathcal{P}_m(v_1, \dots, v_m). \tag{28}$$

Proof. It is enough to prove this when σ is a transposition. Thus, we must show that $\mathcal{P}_m$ changes sign if two entries v_i and v_j are permuted. In terms of the definition of $\mathcal{P}_m$ as a Pfaffian (see (20)), this follows from the fact that the Pfaffian of a skew-symmetric matrix changes sign if one simultaneously permutes the ith and jth row and the ith and jth column (see (19)). Alternatively, another proof can be given if $\mathcal{P}_m$ is viewed as the spanning tree generating function $P(\Gamma_m, \mathfrak{o}_{can})$. There the proof comes down to the fact that permuting two vertices reverses the orientation of Γ_m. We leave the details of this alternative argument to the reader. $\square$

6.2. **Contraction-deletion relation.**

Recall the recursion formula (27) shown in the proof of Theorem 5.2:

$$\mathcal{P}_m = y_{123}\mathcal{P}_{m-2}(v_1 + v_2 + v_3, v_4, \dots, v_m) + [\mathcal{P}_m]_{y_{123}=0}$$

This formula can be viewed as a *contraction-deletion* relation, as we shall now explain. It can be written as

$$P(\Gamma_m, \mathfrak{o}_{can}) = y_{\tilde{e}} P(\Gamma_m/e, \mathfrak{o}_{can}/\mathfrak{o}_{\tilde{e}}) + P(\Gamma_m - e, \mathfrak{o}_{can}) \ ,$$

where the notation is as follows. We have denoted by e the edge $\{1, 2, 3\} \in E(\Gamma_m)$, and by $\mathfrak{o}_{\tilde{e}}$ the orientation of the oriented edge $\tilde{e} = (123)$. The notation $\Gamma_m - e$ stands for the 3-graph Γ_m with the edge e deleted, and Γ_m/e is the 3-graph obtained from Γ_m by contracting the subgraph induced by the edge e (that is, by replacing the three vertices of the edge e in $V(\Gamma_m)$ by a new vertex, say, 0, and discarding all edges from $E(\Gamma_m)$ that become degenerate after this identification). Notice that the quotient 3-graph Γ_m/e has multiple edges and, therefore, is not isomorphic to Γ_{m-2}. Finally, $\mathfrak{o}_{can}/\mathfrak{o}_{\tilde{e}}$ is the induced orientation of Γ_m/e. In our example, it is represented by the cyclic permutation $(045 \dots m)$ of the vertex set $V(\Gamma_m/e)$.

This *contraction-deletion* relation can be formulated in general:

Proposition 6.2 (**Contraction-deletion relation**).

$$P(G,\mathfrak{o}) = y_{\tilde e} P(G/e, \mathfrak{o}/\mathfrak{o}_{\tilde e}) + P(G-e,\mathfrak{o}) \ , \tag{29}$$

where G is a 3-graph with an orientation $\mathfrak{o}$ and an oriented edge $\tilde e = (e, \mathfrak{o}_{\tilde e})$ and $\mathfrak{o}/\mathfrak{o}_{\tilde e}$ is the induced orientation of the quotient 3-graph G/e.

We omit the easy proof, and merely spell out the rule to compute the induced orientation $\mathfrak{o}/\mathfrak{o}_{\tilde e}$. Assume $V(G) = \{1,2,\dots,m\}$, and, as above, let 0 be the new vertex of G/e obtained by contracting the vertices of e. Then $\mathfrak{o}/\mathfrak{o}_{\tilde e}$ is represented by a cyclic permutation σ of $V(G/e)$ such that if one inserts a cyclic permutation representing $\mathfrak{o}_{\tilde e}$ in place of 0 into σ, one obtains a cyclic permutation representing the original orientation $\mathfrak{o}$ of G. For example,

$$(12345\dots m)/(124) = -(12435\dots m)/(124) = -(035\dots m)$$

which shows that the analogue of (27) for the edge $\{1,2,4\}$ is

$$\mathcal{P}_m = -y_{124}\mathcal{P}_{m-2}(v_1+v_2+v_4, v_3, v_5, \dots, v_m) + [\mathcal{P}_m]_{y_{124}=0} \tag{30}$$

Remark 6.3. A contraction-deletion relation analogous to (29) exists for ordinary graphs and relates the Kirchhoff polynomial $\mathcal{D}_G$ of a graph G to those of G/e and $G-e$, the graphs obtained from G by, respectively, contracting and deleting an edge $e \in E(G)$. In fact, one of the standard proofs of the classical Matrix-Tree Theorem is based on this relation (see *e.g.,* the proof given in [Bol, Theorem II.12]). Note that our proof of Theorem 5.2 is similar in spirit. The Pfaffian-tree polynomial $\mathcal{P}_G$ satisfies a contraction-deletion relation analogous to (29), and a possible variant of our proof would be to prove this relation for $\mathcal{P}_G$ independently and then deduce Theorem 5.2 from it. However, we have found it more convenient to derive Theorem 5.2 from its particular case of complete 3-graphs: in our proof, the contraction-deletion relation for complete 3-graphs takes the form of equations (23) and (24) for the tree generating function, and of equations (25) and (26) for the Pfaffian, respectively.

6.3. **Three-term relation.**

Proposition 6.4. *The polynomial $\mathcal{P}_m$ satisfies the relation*

$$\mathcal{P}_m(v_2+v_3, v_4, \dots) + \mathcal{P}_m(v_3+v_4, v_2, \dots) + \mathcal{P}_m(v_2+v_4, v_3, \dots) = 0 \tag{31}$$

where the dots stand for $v_5, v_6, \dots, v_{m+2}$.

Proof. Here we think of $\mathcal{P}_m$ as the tree-generating function $P(\Gamma_m, \mathfrak{o}_{can})$. The first summand in (31) is obtained from $\mathcal{P}_m(v_2, v_4, \dots)$ by replacing every y_{2ij} occurring in it by $y_{2ij} + y_{3ij}$ and expanding by multilinearity. In particular, a monomial in $\mathcal{P}_m(v_2, v_4, \dots)$ corresponding to a tree T

gets replaced by 2^n terms, where n is the valency of the vertex 2 in T. If the other two summands of (31) are expanded similarly, then each of these 2^n terms coming from the tree T also occurs in exactly one of the two other summands, but with opposite sign (by the antisymmetry of $\mathcal{P}_m$, see (28)). Thus, all terms in (31) cancel, and the result follows. □

The three-term relation implies the following properties of $\mathcal{P}_m^2$ which play an important role in our study of the lowest order term of the Alexander-Conway polynomial in [MV].

Corollary 6.5. *The polynomial $\mathcal{P}_m^2$ satisfies the following relations:*

$$(i)\quad \left[\frac{\partial^2 \mathcal{P}_{m+2}^2}{\partial y_{123}^2}\right]_{v_1=0} = 2\,\mathcal{P}_m^2(v_2+v_3,\ldots)$$

$$(ii)\quad \left[\frac{\partial^2 \mathcal{P}_{m+2}^2}{\partial y_{123}\,\partial y_{124}}\right]_{v_1=0} = \mathcal{P}_m^2(v_2+v_3,v_4,\ldots)+\mathcal{P}_m^2(v_2+v_4,v_3,\ldots) \\ -\mathcal{P}_m^2(v_3+v_4,v_2,\ldots)$$

$$(iii)\quad \left[\frac{\partial^2 \mathcal{P}_{m+2}^2}{\partial y_{123}\,\partial y_{145}}\right]_{v_1=0} = \mathcal{P}_m^2(v_3+v_4,v_2,v_5,\ldots)+\mathcal{P}_m^2(v_2+v_5,v_3,v_4,\ldots) \\ -\mathcal{P}_m^2(v_2+v_4,v_3,v_5,\ldots)-\mathcal{P}_m^2(v_3+v_5,v_2,v_4,\ldots)$$

where the dots stand for the v_i with indices not involved on the left-hand side (for example, in the first equation, the dots stand for $v_4, v_5, \ldots, v_{m+2}$.) It also satisfies all equations obtained from the above ones by some permutation of the indices $1, 2, \ldots, m+2$.

Proof. The contraction-deletion formula (27) shows that

$$\frac{\partial\,\mathcal{P}_{m+2}}{\partial y_{123}} = \mathcal{P}_m(v_1+v_2+v_3, v_4,\ldots).$$

Since $[\mathcal{P}_{m+2}]_{v_1=0} = 0$, it follows that

$$\left[\frac{\partial^2 \mathcal{P}_{m+2}^2}{\partial y_{123}^2}\right]_{v_1=0} = 2\left(\left[\frac{\partial\,\mathcal{P}_{m+2}}{\partial y_{123}}\right]_{v_1=0}\right)^2 = 2\,\mathcal{P}_m^2(v_2+v_3,v_4,\ldots)\ ,$$

proving relation (i).

For relation (ii), we have

$$\left[\frac{\partial^2 \mathcal{P}_{m+2}^2}{\partial y_{123}\,\partial y_{124}}\right]_{v_1=0} = 2\left[\frac{\partial\,\mathcal{P}_{m+2}}{\partial y_{123}}\right]_{v_1=0}\left[\frac{\partial\,\mathcal{P}_{m+2}}{\partial y_{124}}\right]_{v_1=0} \\ = -2\,\mathcal{P}_m(v_2+v_3,v_4,\ldots)\mathcal{P}_m(v_2+v_4,v_3,\ldots)\ ,$$

where the minus sign comes from the minus sign in (30). Since

$$\mathcal{P}_m(v_2+v_3,v_4,\ldots)+\mathcal{P}_m(v_2+v_4,v_3,\ldots) = -\mathcal{P}_m(v_3+v_4,v_2,\ldots)$$

by the three-term relation (31), relation (ii) now follows from the identity

$$-2AB = A^2 + B^2 - (A+B)^2. \tag{32}$$

The proof of relation (iii) is similar in spirit but more complicated. Let us abbreviate $\mathcal{P}_m(v_2+v_3, v_4, v_5, \ldots)$ by $\mathcal{P}(2+3,4,5)$, and similarly for the other terms. After computing derivatives as above, the left-hand side of relation (iii) is

$$2\,\mathcal{P}(2+3,4,5)\mathcal{P}(4+5,2,3)\ . \tag{33}$$

Applying the three-term relation to the second factor, and using the antisymmetry of the polynomial $\mathcal{P}_m$ (see (28)), we see that (33) is equal to

$$\begin{aligned} &-2\,\mathcal{P}(2+3,4,5)\big(\mathcal{P}(5+2,4,3)+\mathcal{P}(2+4,5,3)\big)\\ &\qquad = -2\,\mathcal{P}(2+3,5,4)\mathcal{P}(2+5,3,4) + 2\mathcal{P}(2+3,4,5)\mathcal{P}(2+4,3,5)\ . \end{aligned}$$

Applying the three-term relation again to the two products above, and using (32) as in the proof of (ii), we find that (33) is equal to

$$\begin{aligned} &\big(\mathcal{P}(2+3,5,4)^2 + \mathcal{P}(2+5,3,4)^2 - \mathcal{P}(3+5,2,4)^2\big)\\ &\qquad - \big(\mathcal{P}(2+3,4,5)^2 + \mathcal{P}(2+4,3,5)^2 - \mathcal{P}(3+4,2,5)^2\big)\\ &= \mathcal{P}(2+5,3,4)^2 + \mathcal{P}(3+4,2,5)^2 - \mathcal{P}(2+4,3,5)^2 - \mathcal{P}(3+5,2,4)^2 \end{aligned}$$

which is the right-hand side of relation (iii). This completes the proof. □

Relations (i)-(iii) can be used to compute $\mathcal{P}_m^2$ recursively. This relies on the following simple observation.

Lemma 6.6. *If a monomial $\prod_\alpha y_{i_\alpha j_\alpha k_\alpha}$ occurs with non-zero coefficient in $\mathcal{P}_m^2$, then there exists $p \in \{1,2,\ldots,m\}$ such that p occurs exactly twice in the list of indices $i_1, j_1, k_1, i_2, \ldots j_{m-1}, k_{m-1}$.*

Proof. Recall that $\mathcal{P}_m$ is the tree generating function $\sum_T y(T, \mathfrak{o}_{can})$ of the complete 3-graph Γ_m. A monomial $M = \prod_\alpha y_{i_\alpha j_\alpha k_\alpha}$ determines a 3-graph G_M whose edges are given by the y_{ijk} occurring in M. The coefficient of M in $\mathcal{P}_m^2$ is the number (with signs) of ways the 3-graph G_M can be written as the union of two spanning trees T and T'. Since every vertex is incident with at least one edge of T and one edge of T', every $p \in \{1,2,\ldots,m\}$ occurs at least twice in the list of indices $i_1, j_1, k_1, i_2, \ldots j_{m-1}, k_{m-1}$ of M. Since the total number of indices in this list is $< 3m$, there must be an index which occurs exactly twice. □

Corollary 6.7. *Relations (i)-(iii) of Corollary 6.5 together with Lemma 6.6 allow to compute $\mathcal{P}_m^2$ recursively with initial condition $\mathcal{P}_1 = 1$.*

Proof. For every monomial occurring with non-zero coefficient in $\mathcal{P}_m^2$, we can find by Lemma 6.6 an index p such that the monomial contains $y_{pij}y_{pkl}$ for some i, j, k, l, but no other $y_{\alpha\beta\gamma}$ with $p \in \{\alpha, \beta, \gamma\}$. The coefficient of such a monomial in $\mathcal{P}_m^2$ is equal to its coefficient in

$$y_{pij}\, y_{pkl} \left[\frac{\partial^2\, \mathcal{P}_m^2}{\partial y_{pij}\, \partial y_{pkl}}\right]_{v_p=0}$$

if $\{i, j\} \neq \{k, l\}$, or to one half of this coefficient if $\{i, j\} = \{k, l\}$. But this coefficient can be computed recursively by relations (i)-(iii) of Corollary 6.5. □

Remark 6.8. Monomials in $\mathcal{P}_m$ correspond to spanning trees and always occur with coefficient ± 1. Therefore the coefficient of a monomial M in $\mathcal{P}_m^2$ is equal to the number (with signs) of ordered tree decompositions of the associated 3-graph G_M, divided by the symmetry factor $|Aut(G_M)|$. Here, by an ordered tree decomposition of G_M we mean a sub-3-graph T which is a tree and whose complement is also a tree, and by $Aut(G_M)$ the group of automorphisms of G_M inducing the identity map on the set of vertices $V(G_M)$. The cardinality $|Aut(G_M)|$ is equal to 2^d, where d is the number of (unordered) triples of vertices in G_M with 2 edges attached to them. (This is because there can be at most two edges attached to a triple of vertices of G_M if M has non-zero coefficient in $\mathcal{P}_m^2$.)

Here are three examples to illustrate this. The monomial y_{123}^2 has two ordered tree decompositions and $|Aut(G_M)| = 2$; therefore it occurs with coefficient 1 in $\mathcal{P}_3^2$. The monomial $y_{123}^2 y_{245} y_{345}$ has four ordered tree decompositions (each occurring with a plus sign) and $|Aut(G_M)| = 2$; therefore it occurs with coefficient 2 in $\mathcal{P}_3^2$. Finally, the monomial

$$M = y_{145}\, y_{146}\, y_{256}\, y_{257}\, y_{347}\, y_{367}$$

has six ordered tree decompositions (again each contributing $+1$) and $|Aut(G_M)| = 1$; therefore it occurs with coefficient 6 in $\mathcal{P}_7^2$.

6.4. **Four-term relation.**

Proposition 6.9. *The polynomial $\mathcal{P}_m$ satisfies the relation*

$$\frac{\partial \mathcal{P}_m}{\partial y_{ijk}} - \frac{\partial \mathcal{P}_m}{\partial y_{ijl}} = \frac{\partial \mathcal{P}_m}{\partial y_{jkl}} - \frac{\partial \mathcal{P}_m}{\partial y_{ikl}} \tag{34}$$

for every set $\{i, j, k, l\}$ of four distinct vertices.

See Figure 4 for a pictorial illustration of (34).

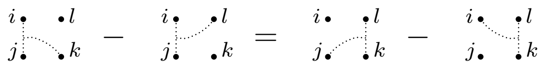

Figure 4. An illustration of Equation (34).

Proof. The four-term relation (34) can be deduced from the three-term relation, as follows.

Without loss of generality, we may assume that $(i, j, k, l) = (1, 2, 3, 4)$. By the contraction-deletion relation (29), the left-hand side of (34) is

$$\mathcal{P}_{m-2}(v_1 + v_2 + v_3, v_4, \ldots) + \mathcal{P}_{m-2}(v_1 + v_2 + v_4, v_3, \ldots)$$

and the right-hand side is equal to

$$-\mathcal{P}_{m-2}(v_2 + v_3 + v_4, v_1, \ldots) - \mathcal{P}_{m-2}(v_1 + v_3 + v_4, v_2, \ldots)\ .$$

(The signs are obtained as in (30); see the discussion following Proposition 6.2.) Using the three-term relation (31) and antisymmetry, we see that both sides are equal to

$$\mathcal{P}_{m-2}(v_1 + v_2, v_3 + v_4, \ldots)$$

which proves (34). $\square$

Here is an equivalent formulation of Proposition 6.9, which we use in [MV].

Let us consider the polynomial $\mathcal{P}_m \in S^{(m-1)/2}\Lambda^3\mathcal{V}$ as a *polynomial function* of degree $(m-1)/2$ on the vector space $W = \Lambda^3 V$, where V is the dual of $\mathcal{V}$. Note that $\{y_{ijk}\}$ is a basis of the space of linear forms on W. Let $\mathrm{Y}_{ijk} \in W$ be the dual basis, *i.e.* the evaluation

$$\langle y_{ijk}, \mathrm{Y}_{\alpha\beta\gamma}\rangle$$

is the sign of the permutation $\binom{i\,j\,k}{\alpha\beta\gamma}$ if $\{i, j, k\} = \{\alpha, \beta, \gamma\}$, and is zero otherwise.

Proposition 6.10. *The polynomial* $\mathcal{P}_m$ *descends to a well-defined polynomial function on* W/W_0*, where* W_0 *is the subspace of* W *generated by vectors of the form*

$$(\mathrm{Y}_{ijk} - \mathrm{Y}_{ijl}) - (\mathrm{Y}_{jkl} - \mathrm{Y}_{ikl})$$

for every set $\{i, j, k, l\}$ *of four distinct vertices.*

Proof. Equation (34) in 6.9 shows that the derivative of $\mathcal{P}_m$ in the direction of any vector in W_0 is identically zero. This implies Proposition 6.10 by Taylor's formula. $\square$

Remark 6.11. The four-term relation for the tree-generating function $\mathcal{P}_m = P(\Gamma_m, \mathfrak{o}_{can})$ has a simple combinatorial meaning, as follows.

The partial derivative $\partial\mathcal{P}_m/\partial y_{123}$ is equal to $y_{123}^{-1}\mathcal{P}_m'$, where $\mathcal{P}_m'$ is the generating function for those trees which contain the edge $\{1,2,3\}$. Given such a tree T, let T' denote T with the edge $\{1,2,3\}$ removed. Note that T' is the disjoint union of 3 subtrees

$$T' = T_1 \cup T_2 \cup T_3 \ ,$$

where T_i denotes the subtree containing the vertex i. The key point is to observe that gluing the edge $\{1,2,4\}$ (resp. $\{1,3,4\}$, $\{2,3,4\}$) to T' yields a tree if and only if the vertex 4 is contained in the component T_3 (resp. T_2, T_1).

It follows that for every tree contributing to $\partial\mathcal{P}_m/\partial y_{123}$, there is a unique way of replacing the edge $\{1,2,3\}$ by one of the three other edges, $\{1,2,4\}$ $\{1,3,4\}$, or $\{2,3,4\}$, so that the result is again a tree. Note that the tree thus obtained contributes to one and only one of $\partial\mathcal{P}_m/\partial y_{124}$, $\partial\mathcal{P}_m/\partial y_{134}$, and $\partial\mathcal{P}_m/\partial y_{234}$.

This observation already implies (34) *modulo* 2. Indeed, the trees contributing to each of the four partial derivatives $\partial\mathcal{P}_m/\partial y_{ijk}$ are partitioned into three disjoint subsets; the set of these altogether 12 subsets is divided into six pairs of two; the two subsets in every pair are in bijective correspondence with each other (so, in particular, they have the same cardinality), but contribute to different $\partial\mathcal{P}_m/\partial y_{ijk}$. This is the combinatorial meaning of the four-term relation up to sign.

The signs can also be checked combinatorially, as follows. Let us do for example the case corresponding to the pair $\{(123),(124)\}$. In this case, we must look at a tree T which is the union of the edge $\{1,2,3\}$ and a remainder, T', so that T' plus the edge $\{1,2,4\}$ gives again a tree, $\widehat{T}$, say. (This means that the vertex 4 must lie in the component T_3 of T'.) Thus, we have

$$\frac{\partial y(T,\mathfrak{o}_{can})}{\partial y_{123}} \ = \ \pm\,\frac{\partial y(\widehat{T},\mathfrak{o}_{can})}{\partial y_{124}} \tag{35}$$

and we must show that the sign is $+1$ in this case. This can be seen by comparing the orientations induced by planar embeddings of T and $\widehat{T}$ which coincide on T'.

Figure 5. Replacing (123) with (124) in a planar embedding of T gives a planar embedding of $\widehat{T}$.

We proceed as in the proof of the sign change property in Proposition 3.8. From the embedding of T we may read off an ordering $\mathfrak{o}$ of the form $1A2B3C$ (see the left diagram on Figure 3). But now we need to take into account where the vertex 4 is placed. Therefore we decompose C as $C''4C'$ (see Figure 5) and write $\mathfrak{o} = 1A2B3C''4C'$. The corresponding embedding of $\widehat{T}$ gives an ordering $\hat{\mathfrak{o}}$ of the form $1A2B4C'3C''$. Observing that $C' \cup C''$ has an odd number of elements, it is easy to check that $\mathfrak{o}$ and $\hat{\mathfrak{o}}$ induce the same orientation, showing that the sign in (35) is indeed $+1$, as asserted.

The recursion formula for $\mathcal{P}^2_m$ of Corollary 6.5 can be understood combinatorially in a similar way.

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Institut de Mathématiques de Jussieu (UMR 7586 CNRS), Equipe 'Topologie et géométrie algébriques', Case 7012, Université Paris VII, 75251 Paris Cedex 05, France

E-mail address: masbaum@math.jussieu.fr

Department of Mathematics, University of Oregon, Eugene, OR 97405, USA

E-mail address: vaintrob@math.uoregon.edu