Higher order Hochschild cohomology

Grégory Ginot

Université Paris VI, Équipe Analyse Algébrique, Case 82, 4 place Jussieu 75252 Paris, France

Received ****; accepted after revision +++++

Presented by £££££

Abstract

Following ideas of Pirashvili, we define higher order Hochschild cohomology over spheres $S^d$ defined for any commutative algebra $A$ and module $M$. When $M = A$, we prove that this cohomology is equipped with graded commutative algebra and degree $d$ Lie algebra structures as well as with Adams operations. All operations are compatible in a suitable sense. These structures are related to Brane topology.

To cite this article: A. Name1, A. Name2, C. R. Acad. Sci. Paris, Ser. I 340 (2005).

Résumé

Cohomologie de Hochschild supérieure. A la manière de Pirashvili, on peut associer une cohomologie de Hochschild supérieure associée aux sphères $S^d$ définie pour toute algèbre commutative $A$ et module $M$. Lorsque $M = A$, cette cohomologie est munie d’un produit gradué commutatif, d’un crochet de Lie de degré $d$ et d’opérations d’Adams. Ces structures sont compatibles entre elles et sont reliées à la topologie des Branes.


Version française abrégée

La topologie des cordes [3] est l’étude des structures algébriques de $H_*(\text{Map}(S^1, M))$ (où $M$ est une variété) induites par des opérations sur le cercle telles que la multiplication $S^1 \times S^1 \to S^1$ ou la composition de lacets. La topologie des cordes est intimement reliée à la cohomologie de Hochschild via l’isomorphisme $H_{*+\dim(M)}(\text{Map}(S^3, M)) \cong HH^*(C^*(M), C^*(M))$ pour $M$ 1-connexe. De fait, la plupart des structures algébriques apparaissant en topologie des cordes ont un analogue pour la cohomologie de Hochschild $HH^*(A, A)$ d’une algèbre $A$ ce qui permet, entre autres, d’étendre la topologie des cordes au cas des espaces à dualité de Poincaré. La topologie des Branes est une généralisation de la topologie des cordes où le cercle est remplacé par une sphère de dimension $d$. Sullivan et Voronov ont montré que
Théorème 0.1

Similaire permet de définir des $\cup$ de Adams sur $dg$ :
fonctorialité en $HH$ permet de définir une cohomologie de Hochschild $X$.

De plus, tout quasi-isomorphisme $X> 1$ :

Théorème 0.2

En caractéristique zéro, l’isomorphisme de Hochschild Kostant Rosenberg a un analogue pour $d>$ et donc une décomposition de Hodge en caractéristique zéro. Les applications $p : S^d \rightarrow S^d \cup \cdots \cup S^d$ et $dg : S^d \cup \cdots \cup S^d \rightarrow S^d$ sont respectivement des itérations du pincement et de la codiagonale. Une idée similaire permet de définir des $\cup_i$-produits ($i = 0 \ldots d$) sur le complexe singulier. On en déduit

Théorème 0.1

Soit $A$ une algèbre commutative. Il existe une structure de $d+1$-algèbre munie d’opérations $A,d$ sur $HH^*_S(A,A)$. De plus les opérations d’Adams sont des morphismes de $d+1$-algèbres. En caractéristique zéro, l’isomorphisme de Hochschild Kostant Rosenberg a un analogue pour $d > 1$ :

Théorème 0.2

Soit $(A,d_A)$ une algèbre différentielle graduée commutative libre. Il existe un isomorphisme naturel de $d+1$-algèbres préservant la décomposition de Hodge

$HH^*_S(A,M) \xrightarrow{dg} HH^*_S(A,M) \xrightarrow{p^*} HH^*_S(A,M)$

de $S^d \cup \cdots \cup S^d$. La construction de Pirashvili est munie d’une structure de Hodge et de $d+1$-algèbres compatibles.

1. Introduction

String topology [3] and its relation to Hochschild cohomology have recently drawn considerable attention. String topology deals with the rich algebraic structure of $H_*(Map(S^1, M))$ where $M$ is a manifold. Most of these structures have a counterpart in Hochschild cohomology of an algebra with value in itself. Note that if $M$ is 1-connected, then $H_*^{dim(M)}(Map(S^1, M)) \cong HH^*(C^*(M), C^*(M))$. The latter result extends the string topology structure to Poincaré duality spaces $X$. Brane topology is a higher dimensional version of string topology where $S^1$ is replaced by $d$-dimensional spheres $S^d$. It was proved by Sullivan Voronov that $H_*^{dim(X)}(Map(S^d, M))$ is a $d+1$-algebra (that is an algebra over the little $d+1$-cube operad). See [4] for details on this and above. On one hand, a nice interpretation of Brane topology in terms of "Hochschild" cohomology of $d$-algebras was given by Hu [6] using a topological analog of Kontsevich generalization of Deligne conjecture [7].

On the other hand, Pirashvili [11] has shown how to define a Hochschild homology theory for commutative algebras associated functorially to any simplicial set $X$, such that the classical Hochschild homology is given by the standard simplicial model of $S^1$. Since this homology depends only on the homology type of the simplicial set, one gets for free many cochain complexes computing it and a lot of flexibility to
build operations in homology. A very important point in Pirashvili’s construction is that the resulting homology depends only of the homology type of the simplicial set. In particular, one gets freely many quasi-isomorphic cochain complex computing the same cohomology which, thanks to the functoriality on \(X_\bullet\), gives a lot of flexibility to describe operations in Hochschild homology. It is trivial to dualize Pirashvili's construction in order to define Hochschild cohomology \(HH_X(A,M)\). In this paper we study \(HH_{S^d}(A,A)\) and prove that it is a \(d+1\)-algebra equipped with compatible Adams operations see Theorems 5.3 and 6.1. Moreover, in characteristic 0 if \(A\) is a model for a \(d\)-connected Poincaré duality space \(X\), then \(HH_{S^d}(A,A) \cong H_{d+\dim(X)}(\text{Map}(S^d,X))\). In particular it adds a Hodge decomposition into the framework of Brane topology and provide a new higher order Hochschild cohomology analog of it. We also make explicit these algebraic structures when \(A\) is free commutative, thus providing an efficient tool for computations.

**Notations** : Let \(k\) be a field. The category of \(k\)-vector spaces will be denoted \(\text{Vect}\). The standard \(n\)-dimensional simplex will be written \(\Delta^n\). We simply write \(\Delta\) for the simplicial category and \(I = [0,1]\) for the interval. If \(X\) is a finite set we write \(#X\) for its cardinal.

### 2. \(\Gamma\)-modules and Hochschild cochain complexes over spheres

Let \(\Gamma\) be the category of finite pointed sets. We write \(k_+\) for the set \(\{0,1,\ldots,k\}\) with 0 as base point. A right \(\Gamma\)-module is a functor \(\Gamma^{op} \to \text{Vect}\). The category \(\text{Mod} - \Gamma\) of right \(\Gamma\)-modules is abelian with enough projectives and injectives. Details can be found in [11]. The significance of \(\Gamma\)-modules in Hochschild (co)homology was first understood by Loday [9] who initiated the following constructions. Let \(A\) be a commutative unital algebra and \(M\) a symmetric \(A\)-bimodule. The right \(\Gamma\)-module \(\mathcal{H}(A,M)\) is defined on objects \(k_+\) by \(\mathcal{H}(A,M)(k_+) = \text{Hom}_k(A^{\otimes k}, M)\). For a map \(n_+ \otimes m_+\) and \(f \in \text{Hom}_k(A^{\otimes m}, M)\), the linear map \(\mathcal{H}(A,M)(f)(f) \in \text{Hom}_k(A^{\otimes n}, M)\) is given, for any \(a_1, \ldots, a_n \in A\), by

\[
\mathcal{H}(A,M)(f)(f)(a_1 \otimes \cdots \otimes a_n) = b_0 f(b_1 \otimes \cdots \otimes b_m)
\]

where \(b_i = \prod_{\theta 
eq \phi \in \phi^{1-(i)}} a_j\) (the empty product is set to be the unit 1 of \(A\)). Given a cocommutative coalgebra \(C\) and a \(C\)-comodule \(N\), Pirashvili [11] defined a right \(\Gamma\)-module \(\mathcal{L}(C,N)\) given on objects by \(\mathcal{L}(C,N)(k_+) = N \otimes C^{\otimes k}\). The action on arrows is as for \(\mathcal{H}(A,M)\) replacing multiplications by coassociations. Both constructions make sense with differential graded algebras and coalgebras. For example, if \(L_\bullet\) is a simplicial set, then its homology is a cocommutative coalgebra and \(\mathcal{L}(H_\bullet(L), H_\bullet(L))\) is a graded right \(\Gamma\)-module. In particular its degree \(q\) part yields the right \(\Gamma\)-module \(\mathcal{L}(H_\bullet(L), H_\bullet(L))\).

A right \(\Gamma\)-module \(R\) can be extended to a functor \(\text{Fin}^{op} \to \text{Vect}\), where \(\text{Fin}\) is the category of pointed sets, by taking limits : \(\text{Fin} \ni Y \mapsto R(Y) := \lim_{U \ni X \to Y} R(X)\). Thus, given any pointed simplicial set \(Y_\bullet\) and right \(\Gamma\)-module \(R\) one gets a cosimplicial vector space \(R(Y_\bullet)\). The dual of Theorem 2.4 in [11] is

**Lemma 2.1** Let \(R \in \text{Mod} - \Gamma\) and \(L_\bullet\) be a pointed simplicial set. There exists a spectral sequence

\[
E_1^{p,q} = \text{Ext}_{\text{Mod}-\Gamma}^p(\mathcal{L}(H_\bullet(L), H_\bullet(L)), R) \Rightarrow H^{p+q}(R(L_\bullet)).
\]

In particular if \(\alpha : X_\bullet \to Y_\bullet\) is a map of pointed simplicial sets, by functoriality it induces a map of cosimplicial vector spaces \(R(Y_\bullet) \to R(X_\bullet)\) which is an isomorphism in cohomology when \(\alpha_\ast : H_\ast(X_\bullet) \to H_\ast(Y_\bullet)\) is an isomorphism. This motivates the following definition.

**Definition 2.2** Let \(X\) be a topological space, \(X_\bullet\) a simplicial set whose realisation is homeomorphic to \(X\), \(A\) a commutative unital algebra and \(M\) an \(A\)-module. The Hochschild cohomology over \(X\) of \(A\) with value in \(M\), denoted \(HH_X(A,M)\), is the cohomology \(H^\ast(\mathcal{H}(A,M)(X_\bullet))\).
By Lemma 2.1 it is independent of the choice of $X_\bullet$. Furthermore any simplicial set $Y_\bullet$ connected to $X_\bullet$ by a zigzag of quasi-isomorphisms gives a cochain complex computing $HH^\bullet_{\Delta}(A, M)$. This complex, denoted $C_{\bullet}^*(A, M)$, is the one underlying the cosimplicial vector space $\mathcal{H}(A, M)(Y_\bullet)$.

3. Hochschild cochain complexes over spheres

Taking $X = S^d$, we get three canonical complexes computing $HH^\bullet_{\Delta}(A, M)$:

- The **standard complex** $C^*_{\Delta d}(A, M)$ is the cochain complex associated to $\mathcal{H}(A, M)((S^d_{sm})_\bullet)$ where $S^d_{sm} := S^d_0 \wedge \cdots \wedge S^d_d$ (d-factors). Here $S^d_0$ is the standard simplicial set representing the circle which has a nondegenerate simplex in dimension 0 and 1 so that $S^1_0 = n_+$. In particular $C^*_0(A, M)$ is the usual Hochschild cochain complex of $A$ with value in $M$.

- The **singular complex** $C^*_{\Delta S_m}(A, M)$ is the cochain complex associated to $\mathcal{H}(A, M)((S^d_{sm})_\bullet)$ and isomorphic to their normalized complexes, that is the subcomplexes obtained by taking the kernel of $\mu$.

- The **cosimplicial map** $\Delta^*_{\Delta(S^d)}(A, M)$ is the cochain complex associated to $\mathcal{H}(A, M)(\Delta\bullet(S^d))$ where $\Delta^n(S^d)$ is the fibrant simplicial set which in dimension $n$ is the set of maps $\Delta^n \to S^d$. By functoriality, there is a chain complex map $C^*_{\Delta(S^d)}(A, M) \to C^*_{\Delta}(A, M)$ for any simplicial set $X_\bullet$ whose realisation is $S^d$.

All cochain complexes above came from cosimplicial vector spaces structure. Thus they are quasi-isomorphic to their normalized complexes, that is the subcomplexes obtained by taking the kernel of degeneracies. Henceforth, we tacitly assume that our cochain complexes are normalized ones.

Now assume that $B$ is a commutative $A$-algebra (for example $B = A$). Let $X_\bullet, Y_\bullet$ be finite pointed simplicial sets. Given pointed finite simplicial sets $X_\bullet, Y_\bullet$ we can form the cosimplicial vector spaces $\mathcal{H}(A, B)(X_\bullet) \otimes \mathcal{H}(A, B)(Y_\bullet)$ (with the diagonal cosimplicial structure) and $\mathcal{H}(A, B)(X_\bullet \vee Y_\bullet)$. There is a cosimplicial map $\mu : \mathcal{H}(A, B)(X_\bullet) \otimes \mathcal{H}(A, B)(Y_\bullet) \to \mathcal{H}(A, B)(X_\bullet \vee Y_\bullet)$ given for any $f \in \text{Hom}(A \otimes X_\bullet, B)$, $g \in \text{Hom}(A \otimes Y_\bullet, B)$ by

$$\mu(f, g)(x_1, \ldots, x_\#X_\bullet, y_1, \ldots, y_\#Y_\bullet) = f(x_1, \ldots, x_\#X_\bullet).g(y_1, \ldots, y_\#Y_\bullet).$$

By limit arguments it extends to (nonnecessarily finite) pointed simplicial sets $X_\bullet, Y_\bullet$.

**Lemma 3.1** Composing $\mu$ with the Eilenberg-Zilber quasi-isomorphisms gives “associative” cochain maps

i) $m_{st} : C^*_{\Delta d}(A, B) \otimes C^*_{\Delta d}(A, B) \to C^*_{\Delta d \vee \Delta d}(A, B)$;

ii) $m_{sm} : C^*_{\Delta S_m}(A, B) \otimes C^*_{\Delta S_m}(A, B) \to C^*_{\Delta(S^d_{sm})\vee(S^d_{sm})}(A, B)$;

iii) $m_{sg} : C^*_{\Delta(S^d_{sm})}(A, B) \otimes C^*_{\Delta(S^d_{sm})}(A, B) \to C^*_{\Delta(S^d_{sm})\vee(S^d_{sm})}(A, B)$ where $j : k[\Delta(S^d \vee S^d)] \to k[\Delta(S^d) \vee \Delta(S^d)]$ is a quasi-inverse of the inclusion map $\Delta(S^d) \vee \Delta(S^d) \to \Delta(S^d \vee S^d)$. Explicitly, for $\sigma : \Delta^{n+1} \to S^d \vee S^d$, one defines $j(\sigma) = \sigma_1 \vee \sigma_2 + \sigma_3 \vee \sigma_2$ where $\sigma_i$ are the respective projections on each factor.

4. Adams operations, Hodge decomposition and $d + 1$-algebra structure

The edgewise subdivision functor $[2] sd_k : \Delta \to \Delta$ (where $k \geq 1$) is defined on objects by $sd_k(n-1)_+ = (kn-1)_+$ and if $f : (n-1)_+ \to (m-1)_+$ is non-decreasing, $sd_k(f)(in + j) = im + f(j)$. It is well-known [10] that for any $R \in \text{Mod} - \Gamma$ and pointed simplicial set $X_\bullet$, one has $|R(X_\bullet)| \cong |R(sd_k(X_\bullet))|$. There is an explicit quasi-isomorphism $D_k : R(sd_k(X_\bullet)) \to R(X_\bullet)$ due to McCarthy [10] representing this equivalence. Let $\varphi^k_n : (kn-1)_+ \to (n-1)_+$ be the maps defined by $\varphi^k_n(in + j) = j$. By functoriality
these maps yield simplicial maps \( \varphi^k = R(\tilde{\varphi}^k) : R(X_\bullet) \rightarrow R(sd_k(X_\bullet)) \). We denote \( \psi^k = D^k \circ \varphi^k \). Note that \( \psi^1 = \text{id} \).

**Proposition 4.1** The maps \( \psi^k \) defined on the standard complex and the singular complex agree in cohomology and satisfy the identity \( \psi^p \circ \psi^q = \psi^{pq} \) for any \( p, q \geq 1 \). Moreover

i) if \( k \) is of characteristic 0, then there is a splitting \( HH^*_{sd}(A, M) = \prod_{j \geq 0} HH^*_{sd}(j)(A, M) \) where the vector spaces \( HH^*_{sd}(j)(A, M) \) are isomorphic to \( \ker(\psi^k - k^j \text{id}) \).

ii) The map \( \psi^k \) is the composition

\[
HH^*_{sd}(A, M) \xrightarrow{\psi^*} HH^*_{sd}(A) \xrightarrow{\text{deg}} HH^*_{sd}(A, M)
\]

where \( p : S^d \rightarrow S^d \vee \cdots \vee S^d \) (k-factors) is the iterated pinch map and \( dg : S^d \vee \cdots \vee S^d \rightarrow S^d \) is the identity on each factor of the wedges.

In particular ii) identifies \( \psi^k : C^*_{\Delta^*}(S^d)(A, M) \rightarrow C^*_{\Delta^*}(S^d)(A, M) \) with the map \( (F^k)^* \) where \( F^k \) is the the canonical map \( F^k : \Delta(S^d) \rightarrow \Delta(S^d) \) of degree \( k \) (that is \( \pi_d(F^k)(1) = k \)).

**Remark 1** The maps \( \psi^k : C^*_{\Delta^*}(S^d)(A, M) \rightarrow C^*_{\Delta^*}(S^d)(A, M) \) are explicitly given by \( \sum_{i=0}^{k-1} \sum_{\sigma \in \Sigma_{n-k, i}} \text{sgn}(\sigma)(\sigma n^k)\sigma^* \) where \( \Sigma_{n,k} \) is the subset of permutations of \( \sigma \) with \( j-1 \) descents and, for \( f \in C^*_{sd}(A, M) = \text{Hom}(A^{\otimes n^d}, M) \), one has \( \sigma^*(f)( \cdots \otimes a_{i_1}, \ldots, a_{i_d} \otimes \cdots ) = f(\cdots \otimes a_{\sigma(i_1)}, \ldots, a_{\sigma(i_d)} \otimes \cdots ) \).

5. \( d \)-algebra structure

For \( d \geq 1 \), a structure of \( d+1 \)-algebra on a graded vector space \( B \) is the data of a graded commutative product and a degree \( d \) Lie bracket satisfying the Leibniz rule

\[
[a, bc] = [a, b]c + (-1)^{(|a|-d)|b|}b[a, c].
\]

In other words, a \( d+1 \)-algebra is an algebra over the operad \( H_*(C_{d+1}) \) where \( C_d = (C_n(1), C_n(2), \ldots) \) is the little n-cubes operad. Recall that an element \( c \in C_d(k) \) is a configuration of \( k \) \( n \)-dimensional cubes in \( I^n \). Such an element \( c \) defines a map \( p_c : S^n \rightarrow \bigvee_k S^n \) by collapsing to the base point the complementary of the interiors of the \( k \) cubes. Composing with the map \( m_{n,q} \) of Lemma 3.1(iii) we get a cochain map

\[
\mu_c : C^*_{\Delta^*}(S^d)(A, B) \otimes_{\Sigma}^m C^*_{\Delta^*}(\bigvee_k S^d)(A, B) \xrightarrow{p_c^*} C^*_{\Delta^*}(S^d)(A, B).
\]

Let \( c_0 \in C_d(2) \) be given by the configuration of the two cubes \([0,1/2]^d \) and \([1/2,1]^d \) in \( I^d \).

**Proposition 5.1** The map \( (1) \) induces a sturcture of \( C_*(C_d) \)-algebra on the singular Hochschild complex \( C^*_{\Delta^*}(S^d)(A, B) \) and thus of \( H_*(C_{d+1}) \)-algebra on \( HH^*_{sd}(A, B) \).

Note that for \( d > 1 \), it implies that \( HH^*_{sd}(A, B) \) is a graded commutative algebra. Furthermore the product is given by the product \( \mu_c \) on the singular complex and is associative on \( C^*_{\Delta^*}(S^d)(A, B) \). The commutativity is induced by a \( \cup_1 \)-product which preserves the base point. Using this fact and the description of the Adams operation given in Proposition 4.1(ii) we get

**Proposition 5.2** For \( d > 1 \), the Adams operations \( \psi^k \) acting on \( HH^*_{sd}(A, B) \) commutes with the cup-product. That is one has \( \psi^k(f) \cup_0 \psi^k(g) = \psi^k(f \cup_0 g) \) for all \( f, g \in HH^*_{sd}(A, B) \).

Recall [1] that this result is false for \( d = 1 \).

**Remark 2** It is easy to describe the product \( \cup_0 \) (as well as \( \cup_1 \) indeed) on the standard chain complex. For \( f \in C^p_{sd}(A, B), g \in C^q_{sd}(A, B) \), the product \( f \cup_0 g \in C^{p+q}_{sd}(A, B) \) is defined by

\[
f \cup_0 g((a_{i_1}, \ldots, a_{i_d})_{1 \leq i_1, \ldots, i_d \leq p}) = f((a_{i_1}, \ldots, a_{i_d})_{1 \leq i_1, \ldots, i_d \leq p}) g((a_{i_{p+1}}, \ldots, a_{i_{p+q}})_{1 \leq i_{p+1}, \ldots, i_{p+q} \leq p+q}) \prod a_{j_1}, \ldots, a_{j_d}
\]

where the last product is over all indices which are not in the argument of \( f \) or \( g \).
When $B = A$, Proposition 5.1 yields a Lie bracket of degree $d - 1$ in cohomology, induced by the antisymmetrization of the $\cup_d$-product, where we expect a degree $d$ Lie bracket. In fact, as in the case $d = 1$, one can use the fact that $B = A$ to get a (non pointed) $\cup_d$-product. Using the notations of the end of Section 3, let $\eta : \mathbb{Z}_\bullet \to X_\bullet \vee Y_\bullet$ be a (non based) map of simplicial sets. Let $f \in \text{Hom}(A^{\#\#X_\bullet}, A)$, $g \in \text{Hom}(A^{\#\#Y_\bullet}, A)$ and assume $\eta(0) = i + 1 \in X_\bullet$. We define $\tilde{\eta}(f, g) \in \text{Hom}(A^{\#\#X_\bullet}, A)$ by the formula
\[
\tilde{\eta}(f, g)(z_1, \ldots, z_{n+1}) = x_0 f(x_1, \ldots, x_i, g(y_1, \ldots, y_{n+i}) x_{i+1}, \ldots, x_{n+i})
\]
where $x_k = \prod_{l=0}^{k} f(\eta(l) = k \in X_\bullet z_1, x_k) = \prod_{l=0}^{k} f(\eta(l) = k \in Y_\bullet z_1, x_{k+1}) = \prod_{l=0}^{k} f(\eta(l) = i + 1 \in X_\bullet z_1)$. Note that if $\eta$ is base point preserving, then $\tilde{\eta} = \eta^* \circ \mu$. As in Section 3 we extend the previous construction to $C^m_{\Delta}(S\Omega(A), A)$ and apply it to the map $I^d \times S^d \to S^d \vee S^d$ obtained from $e_0$ by moving the base point along the canonical map $I^d \to I^d/\partial I^d \cong [0, 1/2]^d$. This yields a $\cup_d$-product $\cup_d : S_{d+1}^d(A, A) \otimes S_{d+1}^d(A, A) \to S_{d+1}^{d+1-d}(A, A)$ giving an homotopy for the commutativity of $\cup_d$. Let $[f, g]_d := f \cup_d g - (-1)^{|f|+d(|g|+d)} g \cup_d f$.

**Theorem 5.3** The $\cup_d$-product and bracket $[\cdot, \cdot]_d$ give a structure of $d + 1$-algebra to $HH_{d+1}^*(A, A)$.

6. Free commutative algebras and Brane topology in characteristic zero

By definition of the small complex, one has $C^n_{Sd}(A, M) = M$ and $C^n_{Sd+1}(A, M) = \text{Hom}(A, M)$. Furthermore one checks that $f \in \text{Hom}(A, M) = C^n_{Sd+1}(A, M)$ is a cocycle if and only if $f \in \text{Der}(A, M)$. Thanks to the commutative cup-product there is a canonical map
\[
HKR : \text{Hom}_A(S^*(\Omega d), M) \to HH_{d+1}^*(A, M)
\]
where $\Omega d$ is the space of Kähler differentials (recall that $\text{Hom}_A(\Omega d, M) \cong \text{Der}(A, M)$) and $S^*$ is the graded symmetric algebra functor. Note that $\text{Hom}_A(S^*(\Omega d), A)$ is a $d + 1$-algebra with product induced by the symmetric power and bracket given by the identification $\text{Hom}_A(\Omega d, A) \cong \text{Der}(A, A)$ and extended to the whole space by the Leibniz rule. Moreover there are Adams operations $\psi^k$ defined on $\text{Hom}_A(S^d(\Omega d), A)$ by the multiplication by $k^d$. As in the classical case, if $A$ is free, the map $HKR$ is an isomorphism preserving all the algebraic structures. Furthermore, all of the above makes sense for differential graded commutative algebras as well. When $A$ is of characteristic zero, any (dg) commutative algebra $(A, d_A)$ is quasi-isomorphic to a dg free one $(F, d_F) \simeq (A, d_A)$. Proposition 5.2 implies

**Theorem 6.1** Let $d > 1$ and $\text{char}(k) = 0$. The map
\[
HKR : \text{Hom}_F(S^*(\Omega F d), F) \to HH_{d+1}^*(A, A)
\]
is an isomorphism of $d + 1$-algebras commuting with the Adams operations. Moreover a quasi-isomorphism $(A, d_A) \to (B, d_B)$ of dg-commutative algebras induces an isomorphism $HH_{d+1}^*(A, A) \cong HH_{d+1}^*(B, B)$ of $d + 1$-algebras and Hodge structures.

This theorem gives an efficient way to compute the structure of higher order Hochschild homology.

**Remark 3** In particular for $d$ odd, the groups appearing in the Hodge decomposition are those in the Hodge decomposition for $d = 1$ but they are dispatched in different degrees. The same is true for $d$ even with the groups appearing in the decomposition for $d = 2$. Note that for $d = 1$, the Hodge decomposition coincides with the classical one [5], [9].

Let $X$ be a $d$-connected Poincaré duality space of dimension $n$. By [8], there exists a free dg-commutative algebra $(A_X, d_X)$ quasi-isomorphic to the minimal model of $X$ together with a quasi-isomorphism $A_X \to (A_X)''[n]$ of $A_X$-modules inducing the Poincaré duality in cohomology. Moreover by Theorem 6.1 and direct inspection on a minimal model of $X$, there is an isomorphism $HH_{d+1}^*(A_X, (A_X)'') \cong H_1(\text{Map}(S^d, X))$.

\[
HH_{d+1}^*(A_X, A_X) \cong HH_{d+1}^*(A_X, (A_X)'') \cong H_{d+1}(\text{Map}(S^d, X)).
\]
The last isomorphism come from Theorem 6.1 applied to a minimal model of $X$. 

6
Corollary 6.2 For any commutative model $\mathcal{M}_X$ for $X$, one has $HH^i(Sd)(\mathcal{M}_X, \mathcal{M}_X) \cong H_i+d(\text{Map}(S^d, X))$.

In particular, the shifted homology of the mapping space $\text{Map}(S^d, X)$ inherits a structure of $d$-algebra which is graded with respect to the Hodge decomposition. Corollary 6.2 adds the Hodge decomposition to the Brane topology story studied in [4] and [6].

References