Higher order Hochschild cohomology

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Abstract

Following ideas of Pirashvili, we define higher order Hochschild cohomology over spheres S^d defined for any commutative algebra A and module M. When $M = A$, we prove that this cohomology is equipped with graded commutative algebra and degree d Lie algebra structures as well as with Adams operations. All operations are compatible in a suitable sense. These structures are related to Brane topology.

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Résumé

Cohomologie de Hochschild supérieure. A la manière de Pirashvili, on peut associer une cohomologie de Hochschild supérieure associée aux sphères S^d définie pour toute algèbre commutative A et module M. Lorsque $M = A$, cette cohomologie est munie d'un produit gradué commutatif, d'un crochet de Lie de degré d et d'opérations d'Adams. Ces structures sont compatibles entre elles et sont reliées à la topologie des Branes.

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La topologie des cordes [3] est l'étude des structures algébriques de $H_*(\mathrm{Map}(S^1, M))$ (où M est une variété) induites par des opérations sur le cercle telles que la multiplication $S^1 \times S^1 \to S^1$ ou la composition de lacets. La topologie des cordes est intimement reliée à la cohomologie de Hochschild via l'isomorphisme $H_{*+\dim(M)}(\mathrm{Map}(S^1, M)) \cong HH^*(C^*(M), C^*(M))$ pour M 1-connexe. De fait, la plupart des structures algébriques apparaissant en topologie des cordes ont un analogue pour la cohomologie de Hochschild $HH^*(A, A)$ d'une algèbre A ce qui permet, entre autres, d'étendre la topologie des cordes au cas des espaces à dualité de Poincaré. La topologie des Branes est une généralisation de la topologie des cordes où le cercle est remplacé par une sphère de dimension d. Sullivan et Voronov ont montré que

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$H_{*+\dim(M)}(\mathrm{Map}(S^d, M))$ est une $d+1$-algèbre (c'est à dire une algèbre sur l'homologie de l'opérades des petits cubes de dimension $d+1$). On peut consulter [4] pour plus de détails sur tout ceci. Une interprétation de la topologie des Branes en cohomologie "de Hochschild" des d-algèbres a été donnée par Hu [6] en utilisant une version topologique de la généralisation par Kontsevich de la conjecture de Deligne [7].

Une très intéréssante et différente généralisation de la (co)homologie de Hochschild pour les algèbres commutatives est due à Pirashvili [11] : à tout ensemble simplicial $X_\bullet$ est associé *fonctoriellement* une homologie de Hochschild $HH^{X_\bullet}(-,-)$ qui coincide avec la définition usuelle lorsque $X_\bullet$ est le modèle simplicial standard $S^1_\bullet$ du cercle. Un point clé est que cette théorie homologique ne dépend en fait que du type d'homologie simpliciale de $X_\bullet$. En particulier, on obtient facilement de nombreux complexes calculant $HH^X(A, M)$ (pour toute algèbre commutative A et A-module M). Le dual de la construction de Pirashvili permet de définir une cohomologie de Hochschild $HH^*_X(A, A)$. On s'intéresse au cas $X = S^d$, $d > 1$. Par fonctorialité en $X_\bullet$ de $HH^*_X(A, A)$, on obtient des opérations d'Adams ψ^k comme la composition

$$HH^*_{S^d}(A, M) \xrightarrow{\mathrm{dg}^*} HH^*_{S^d\vee\cdots\vee S^d}(A, M) \xrightarrow{p^*} HH^*_{S^d}(A, M)$$

et donc une décomposition de Hodge en caractéristique zéro. Les applications $p : S^d \to S^d \vee \cdots \vee S^d$ et $\mathrm{dg} : S^d \vee \cdots \vee S^d \to S^d$ sont respectivement des itérations du pincement et de la codiagonale. Une idée similaire permet de définir des $\cup_i$-produits $(i = 0 \dots d)$ sur le complexe singulier. On en déduit

Théorème 0.1 *Soit A une algèbre commutative. Il existe une structure de $d+1$-algèbre munie d'opérations d'Adams sur $HH^*_{S^d}(A, A)$. De plus les opérations d'Adams sont des morphismes de $d+1$-algèbres.*

En caractéristique zéro, l'isomorphisme de Hochschild Kostant Rosenberg a un analogue pour $d > 1$:

Théorème 0.2 *Soit (A, d_A) une algèbre différentielle graduée commutative libre. Il existe un isomorphisme naturel de $d+1$-algèbres préservant la décomposition de Hodge*

$$HKR : H^*(\mathrm{Hom}_A(S^*(\Omega_A[d]), A), d_A) \to HH^*_{S^d}(A, A).$$

De plus, tout quasi-isomorphisme $(A, d_A) \to (B, d_B)$ induit un isomorphisme $HH^*_{S^d}(A, A) \cong HH^*_{S^d}(B, B)$ de $d+1$-algèbres préservant les opérations d'Adams.

On en déduit alors que si X est un espace d-connexe à dualité de Poincaré et (A, d_A) un modèle rationnel pour X, il y a un isomorphisme rationel $H_{*+\dim(X)}(\mathrm{Map}(S^d, X)) \cong HH^*_{S^d}(A, A)$. En particulier, l'homologie de $\mathrm{Map}(S^d, X)$ est munie d'une structure de Hodge et de $d+1$-algèbre compatibles.

1. Introduction

String topology [3] and its relation to Hochschild cohomology have recently drawn considerable attention. String topology deals with the rich algebraic structure of $H_*(\mathrm{Map}(S^1, M))$ where M is a manifold. Most of these structures have a counterpart in Hochschild cohomology of an algebra with value in itself. Note that if M is 1-connected, then $H_{*+\dim(M)}(\mathrm{Map}(S^1, M)) \cong HH^*(C^*(M), C^*(M))$. The latter result extends the string topology structure to Poincaré duality spaces X. Brane topology is a higher dimensional version of string topology where S^1 is replaced by d-dimensional spheres S^d. It was proved by Sullivan Voronov that $H_{*+\dim(M)}(\mathrm{Map}(S^d, M))$ is a $d+1$-algebra (that is an algebra over the little $d+1$-cube operad). See [4] for details on this and above. On one hand, a nice interpretation of Brane topology in terms of "Hochschild" cohomology of d-algebras was given by Hu [6] using a topological analog of Kontsevich generalization of Deligne conjecture [7].

On the other hand, Pirashvili [11] has shown how to define a Hochschild homology theory for commutative algebras associated *functorially* to any simplicial set $X_\bullet$, such that the classical Hochschild homology is given by the standard simplicial model of S^1. Since this homology depends only on the homology type of the simplicial set, one gets for free many cochain complexes computing it and a lot of flexibility to build operations in homology. A very important point in Pirashvili's construction is that the resulting homology depends only of the homology type of the simplicial set. In particular, one gets freely many quasi-isomorphic cochain complex computing the same cohomology which, thanks to the functoriality on $X_\bullet$, gives a lot of flexibility to describe operations in Hochschild homology. It is trivial to dualize Pirashvili's construction in order to define Hochschild cohomology $HH^*_X(A,A)$. In this paper we study $HH_{S^d}(A,A)$ and prove that it is a $d+1$-algebra equipped with compatible Adams operations see Theorems 5.3 and 6.1. Moreover, in characteristic 0 if A is a model for a d-connected Poincaré duality space X, then $HH^*_{S^d}(A,A) \cong H_{*+\dim(X)}(\mathrm{Map}(S^d,X))$. In particular it adds a Hodge decomposition into the framework of Brane topology and provide a new higher order Hochschild cohomology analog of it. We also make explicit these algebraic structures when A is free commutative, thus providing an efficient tool for computations.

Notations : Let $\mathbf{k}$ be a field. The category of $\mathbf{k}$-vector spaces will be denoted Vect. The standard n-dimensional simplex will be written Δ^n. We simply write Δ for the simplicial category and $I=[0,1]$ for the interval. If X is a finite set we write $\#X$ for its cardinal.

2. Γ-modules and Hochschild cochain complexes over spheres

Let Γ be the category of finite pointed sets. We write k_+ for the set $\{0,1,\dots,k\}$ with 0 as base point. A right Γ-module is a functor $\Gamma^{\mathrm{op}} \to \mathrm{Vect}$. The category $\mathrm{Mod}-\Gamma$ of right Γ-modules is abelian with enough projectives and injectives. Details can be found in [11]. The significance of Γ-modules in Hochschild (co)homology was first understood by Loday [9] who initiated the following constructions. Let A be a commutative unital algebra and M a symmetric A-bimodule. The right Γ-module $\mathcal{H}(A,M)$ is defined on objects k_+ by $\mathcal{H}(A,M)(k_+) = \mathrm{Hom}_{\mathbf{k}}(A^{\otimes k},M)$. For a map $n_+ \xrightarrow{\phi} m_+$ and $f \in \mathrm{Hom}_{\mathbf{k}}(A^{\otimes m},M)$, the linear map $\mathcal{H}(A,M)(\phi)(f) \in \mathrm{Hom}_{\mathbf{k}}(A^{\otimes n},M)$ is given, for any $a_1,\dots,a_n \in A$, by

$$\mathcal{H}(A,M)(\phi)(f)(a_1\otimes\cdots\otimes a_n) = b_0.f(b_1\otimes\cdots\otimes b_m)$$

where $b_i = \prod_{0\neq j\in\phi^{-1}(i)} a_j$ (the empty product is set to be the unit 1 of A). Given a cocommutative coalgebra C and a C-comodule N, Pirashvili [11] defined a right Γ-module ${}^{\mathrm{co}}\mathcal{L}(C,N)$ given on objects by ${}^{\mathrm{co}}\mathcal{L}(C,N)(k_+) = N\otimes C^{\otimes k}$. The action on arrows is as for $\mathcal{H}(A,M)$ replacing multiplications by comultiplications. Both constructions make sense with differential graded algebras and coalgebras. For example, if $L_\bullet$ is a simplicial set, then its homology is a cocommutative coalgebra and ${}^{\mathrm{co}}\mathcal{L}(H_*(L),H_*(L))$ is a graded right Γ-module. In particular its degree q part yields the right Γ-module ${}^{\mathrm{co}}\mathcal{L}_q(H_*(L),H_*(L))$.

A right Γ-module R can be extended to a functor $\mathrm{Fin}^{\mathrm{op}} \to \mathrm{Vect}$, where Fin is the category of pointed sets, by taking limits : $\mathrm{Fin} \ni Y \mapsto R(Y) := \lim_{\Gamma\ni X\to Y} R(X)$. Thus, given any pointed simplicial set $Y_\bullet$ and right Γ-module R one gets a cosimplicial vector space $R(Y_\bullet)$. The dual of Theorem 2.4 in [11] is

Lemma 2.1 *Let $R \in \mathrm{Mod}-\Gamma$ and $L_\bullet$ be a pointed simplicial set. There exists a spectral sequence*

$$E_1^{p,q} = \mathrm{Ext}^p_{\mathrm{Mod}-\Gamma}({}^{\mathrm{co}}\mathcal{L}_q(H_*(L),H_*(L))\,,\,R) \Longrightarrow H^{p+q}R(L_\bullet).$$

In particular if $\alpha : X_\bullet \to Y_\bullet$ is a map of pointed simplicial sets, by functoriality it induces a map of cosimplicial vector spaces $R(Y_\bullet) \to R(X_\bullet)$ which is an isomorphism in cohomology when $\alpha_* : H_*(X_\bullet) \to H_*(Y_\bullet)$ is an isomorphism. This motivates the following definition.

Definition 2.2 *Let X be a topological space, $X_\bullet$ a simplicial set whose realisation is homeomorphic to X, A a commutative unital algebra and M a A-module. The Hochschild cohomology over X of A with value in M, denoted $HH^*_X(A,M)$, is the cohomology $H^*(\mathcal{H}(A,M)(X_\bullet))$.*

By Lemma 2.1 it is independent of the choice of $X_\bullet$. Furthermore any simplicial set $Y_\bullet$ connected to $X_\bullet$ by a zigzag of quasi-isomorphisms gives a cochain complex computing $HH^*_X(A,M)$. This complex, denoted $C^*_{Y_\bullet}(A,M)$, is the one underlying the cosimplicial vector space $\mathcal{H}(A,M)(Y_\bullet)$.

3. Hochschild cochain complexes over spheres

Taking $X = S^d$, we get three canonical complexes computing $HH^*_{S^d}(A,M)$:

- The **standard complex** $C^*_{S^d}(A,M)$ is the cochain complex associated to $\mathcal{H}(A,M)(S^d_\bullet)$ where $S^d_\bullet := S^1_\bullet \wedge \cdots \wedge S^1_\bullet$ (d-factors). Here $S^1_\bullet$ is the standard simplicial set representing the circle which has a nondegenerate simplex in dimension 0 and 1 so that $S^1_n = n_+$. In particular $C^*_{S^1}(A,M)$ is the usual Hochschild cochain complex of A with value in M.
- The **small complex** $C^*_{S^d_{sm}}(A,M)$ is the cochain complex of $\mathcal{H}(A,M)((S^d_{sm})_\bullet)$ where $(S^d_{sm})_\bullet$ is a simplicial set with one nondegenerate simplex in degree 0 and d. Thus $(S^d_{sm})_n \cong \binom{n}{d}_+$.
- The **singular complex** $C^*_{\Delta_\bullet(S^d)}(A,M)$ is the cochain complex associated to $\mathcal{H}(A,M)(\Delta_\bullet(S^d))$ where $\Delta_\bullet(S^d)$ is the fibrant simplicial set which in dimension n is the set of maps $\Delta^n \to S^d$. By functoriality, there is a chain complex map $C^*_{\Delta_\bullet(S^d)}(A,M) \to C^*_{X_\bullet}(A,M)$ for any simplicial set $X_\bullet$ whose realisation is S^d.

All cochain complexes above came from cosimplicial vector spaces structure. Thus they are quasi-isomorphic to their normalized complexes, that is the subcomplexes obtained by taking the kernel of degeneracies. Henceforth, we tacitly assume that our cochain complexes are normalized ones.

Now assume that B is a commutative A-algebra (for example $B = A$). Let $X_\bullet, Y_\bullet$ be finite pointed simplicial sets. Given pointed finite simplicial sets $X_\bullet, Y_\bullet$ we can form the cosimplicial vector spaces $\mathcal{H}(A,B)(X_\bullet) \otimes \mathcal{H}(A,B)(Y_\bullet)$ (with the diagonal cosimplicial structure) and $\mathcal{H}(A,B)(X_\bullet \vee Y_\bullet)$. There is a cosimplicial map $\mu : \mathcal{H}(A,B)(X_\bullet) \otimes \mathcal{H}(A,B)(Y_\bullet) \to \mathcal{H}(A,B)(X_\bullet \vee Y_\bullet)$ given for any $f \in \mathrm{Hom}(A^{\otimes \# X_n}, B)$, $g \in \mathrm{Hom}(A^{\otimes \# Y_n}, B))$ by

$$\mu(f,g)(x_1,\dots x_{\#X_n}, y_1,\dots,y_{\#Y_n}) = f(x_1,\dots x_{\#X_n}).g(y_1,\dots,y_{\#Y_n}).$$

By limit arguments it extends to (nonnecessarily finite) pointed simplicial sets $X_\bullet, Y_\bullet$.

Lemma 3.1 *Composing μ with the Eilenberg-Zilber quasi-isomorphisms gives "associative" cochain maps*

i) $m_{st} : C^*_{S^d}(A,B) \otimes C^*_{S^d}(A,B) \to C^*_{S^d_\bullet \vee S^d_\bullet}(A,B)$;

ii) $m_{sm} : C^*_{S^d_{sm}}(A,B) \otimes C^*_{S^d_{sm}}(A,B) \to C^*_{(S^d_{sm})_\bullet \vee (S^d_{sm})_\bullet}(A,B)$;

iii) $m_{sg} : C^*_{\Delta_\bullet(S^d)}(A,B) \otimes C^*_{\Delta_\bullet(S^d)}(A,B) \to C^*_{\Delta_\bullet(S^d)\vee\Delta_\bullet(S^d)}(A,B) \xrightarrow{j^*} C^*_{\Delta_\bullet(S^d \vee S^d)}(A,B)$ *where* $j : \mathbf{k}[\Delta_\bullet(S^d \vee S^d)] \to \mathbf{k}[\Delta_\bullet(S^d) \vee \Delta_\bullet(S^d)]$ *is a quasi-inverse of the inclusion map* $\Delta_\bullet(S^d) \vee \Delta_\bullet(S^d) \hookrightarrow \Delta_\bullet(S^d \vee S^d)$.

Explicitly, for $\sigma : \Delta^{n\geq 1} \to S^d \vee S^d$, one defines $j(\sigma) = \sigma_1 \vee \mathrm{cst} + \mathrm{cst} \vee \sigma_2$ where σ_i are the respective projections on each factor and cst is the constant map to the basepoint of S^d.

4. Adams operations, Hodge decomposition and $d+1$-algebra structure

The edgewise subdivison functor [2] $\mathrm{sd}_k : \Delta \to \Delta$ (where $k \geq 1$) is defined on objects by $\mathrm{sd}_k(n-1)_+ = (kn-1)_+$ and if $f : (n-1)_+ \to (m-1)_+$ is non-decreasing, $\mathrm{sd}_k(f)(in+j) = im + f(j)$. It is well-known [10] that for any $R \in \mathrm{Mod} - \Gamma$ and pointed simplicial set $X_\bullet$, one has $|R(X_\bullet)| \cong |R(\mathrm{sd}_k(X)_\bullet)|$. There is an explicit quasi-isomorphism $\mathcal{D}_k : R(\mathrm{sd}_k(X)_\bullet) \to R(X_\bullet)$ due to McCarthy [10] representing this equivalence. Let $\widetilde{\varphi}^k_n : (kn-1)_+ \to (n-1)_+$ be the maps defined by $\widetilde{\varphi}^k_n(in+j) = j$. By functoriality these maps yield simplicial maps $\varphi^k = R(\widetilde{\varphi}^k) : R(X_\bullet) \to R(\mathrm{sd}_k(X_\bullet))$. We denote $\psi^k = \mathcal{D}^k \circ \varphi^k$. Note that $\psi^1 = \mathrm{id}$.

Proposition 4.1 *The maps ψ^k defined on the standard complex and the singular complex agree in cohomology and satisfy the identity $\psi^p \circ \psi^q = \psi^{pq}$ for any $p, q \geq 1$. Moreover*

i) *if $\mathbf{k}$ is of characteristic 0, then there is a splitting $HH^*_{S^d}(A,M) = \prod_{j\geq 0} HH^{*,(j)}_{S^d}(A,M)$ where the vector spaces $HH^{*,(j)}_{S^d}(A,M)$ are isomorphic to $\ker(\psi^k - k^j.\mathrm{id})$.*

ii) *The map ψ^k is the composition*

$$HH^*_{S^d}(A,M) \xrightarrow{\mathrm{dg}^*} HH^*_{S^d\vee\cdots\vee S^d}(A,M) \xrightarrow{p^*} HH^*_{S^d}(A,M)$$

where $p : S^d \to S^d \vee \cdots \vee S^d$ (k-factors) is the iterated pinch map and $\mathrm{dg} : S^d \vee \cdots \vee S^d \to S^d$ is the identity on each factor of the wedges.

In particular **ii)** identifies $\psi^k : C^*_{\Delta_\bullet(S^d)}(A,M) \to C^*_{\Delta_\bullet(S^d)}(A,M)$ with the map $(F^k)^*$ where F^k is the the canonical map $F^k : \Delta(S^d) \to \Delta(S^d)$ of degree k (that is $\pi_d(F^k)(1) = k$).

Remark 1 *The maps $\psi^k : C^n_{S^d}(A,M) \to C^n_{S^d}(A,M)$ are explicitly given by $\sum_{i=0}^{k-1} \sum_{\sigma\in\Sigma_{n,k-i}} \mathrm{sgn}(\sigma)\binom{n+k}{n}\sigma^*$ where $\Sigma_{n,j}$ is the subset of permutations of Σ_n with $j-1$ descents and, for $f \in C^n_{S^d}(A,M) = \mathrm{Hom}(A^{\otimes n^d}, M))$, one has $\sigma^*(f)(\cdots \otimes a_{i_1,\dots,i_d} \otimes \cdots) = f(\cdots \otimes a_{\sigma(i_1),\dots,\sigma(i_d)} \otimes \cdots)$.*

5. *d*-algebra structure

For $d \geq 1$, a structure of $d+1$-*algebra* on a graded vector space B is the data of a graded commutative product and a degree d Lie bracket satisfying the Leibniz rule

$$[a, bc] = [a,b]c + (-1)^{(|a|-d)|b|} b[a,c].$$

In other words, a $d+1$-algebra is an algebra over the operad $H_*(\mathcal{C}_{d+1})$ where $\mathcal{C}_n = (\mathcal{C}_n(1), \mathcal{C}_n(2), \dots)$ is the little n-cubes operad. Recall that an element $c \in \mathcal{C}_n(k)$ is a configuration of k n-dimensional cubes in I^n. Such an element c defines a map $p_c : S^n \to \bigvee_k S^n$ by collapsing to the base point the complementary of the interiors of the k cubes. Composing with the map m_{sg} of Lemma 3.1.iii) we get a cochain map

$$\mu_c : C^*_{\Delta_\bullet(S^d)}(A,B)^{\otimes k} \xrightarrow{m_{sg}} C^*_{\Delta_\bullet(\bigvee_k S^d)}(A,B) \xrightarrow{p_c^*} C^*_{\Delta_\bullet(S^d)}(A,B). \tag{1}$$

Let $c_0 \in \mathcal{C}_d(2)$ be given by the configuration of the two cubes $[0,1/2]^d$ and $[1/2,1]^d$ in I^d.

Proposition 5.1 *The map (1) induces a structure of $C_*(\mathcal{C}_d)$-algebra on the singular Hochschild complex $C^*_{\Delta_\bullet(S^d)}(A,B)$ and thus of $H_*(\mathcal{C}_{d+1})$-algebra on $HH^*_{S^d}(A,B)$.*

Note that for $d > 1$, it implies that $HH^*_{S^d}(A,B)$ is a graded commutative algebra. Furthemore the product is given by the product $\cup_0 := \mu_{c_0}$ on the singular complex and is associative on $C^*_{\Delta_\bullet(S^d)}(A,B)$. The commutativity is induced by a $\cup_1$-product which preserves the base point. Using this fact and the description of the Adams operation given in Proposition 4.1.ii) we get

Proposition 5.2 *For $d > 1$, the Adams operations ψ^k acting on $HH^*_{S^d}(A,B)$ commutes with the cup-product. That is one has $\psi^k(f) \cup_0 \psi^k(g) = \psi^k(f \cup_0 g)$ for all $f, g \in HH^*_{S^d}(A,B)$.*

Recall [1] that this result is false for $d = 1$.

Remark 2 It is easy to describe the product $\cup_0$ (as well as $\cup_1$ indeed) on the standard chain complex. For $f \in C^p_{S^d}(A,B)$, $g \in C^q_{S^d}(A,B)$, the product $f \cup_0 g \in C^{p+q}_{S^d}(A,B)$ is defined by

$$f \cup_0 g\big((a_{i_1,\dots,i_d})_{1\leq i_1,\dots,i_d\leq p+q}\big) = f\big((a_{i_1,\dots,i_d})_{1\leq i_1,\dots,i_d\leq p}\big) g\big((a_{i_1,\dots,i_d})_{p+1\leq i_1,\dots,i_d\leq p+q}\big) \prod a_{j_1,\dots,j_d}$$

where the last product is over all indices which are not in the argument of f or g.

When $B = A$, Proposition 5.1 yields a Lie bracket of degree $d-1$ in cohomology, induced by the antisymmetrization of the $\cup_{d-1}$-product, where we expect a degree d Lie bracket. In fact, as in the case $d=1$, one can use the fact that $B = A$ to get a (non pointed) $\cup_d$-product. Using the notations of the end of Section 3, let $\eta : Z_\bullet \to X_\bullet \vee Y_\bullet$ be a (non based) map of simplicial sets. Let $f \in \mathrm{Hom}(A^{\otimes \# X_n}, A)$, $g \in \mathrm{Hom}(A^{\otimes \# Y_n}, A)$ and assume $\eta(0) = i+1 \in X_n$. We define $\widetilde{\eta}(f,g) \in \mathrm{Hom}(A^{\otimes \# Z_n}, A)$ by the formula

$$\widetilde{\eta}(f,g)(z_1,\dots,z_{\# Z_n}) = x_0 f(x_1,\dots,x_i, g(y_1,\dots,y_{\# Y_n})\overline{x_{i+1}}, x_{i+2},\dots,x_{\# X_n})$$

where $x_k = \prod_{l/\eta(l)=k\in X_n} z_l$, $y_k = \prod_{l/\eta(l)=k\in Y_n} z_l$, $\overline{x_{i+1}} = \prod_{0\neq l/\eta(l)=i+1\in X_n} z_l$. Note that if η is base point preserving, then $\widetilde{\eta} = \eta^* \circ \mu$. As in Section 3 we extend the previous construction to $C^*_{\Delta_\bullet(S^d)}(A,A)$ and apply it to the map $I^d \times S^d \to S^d \vee S^d$ obtained from c_0 by moving the base point along the canonical map $I^d \to I^d/\partial I^d \cong [0,1/2]^d$. This yields a $\cup_d$-product $\cup_d : S^p_{S^d}(A,A) \otimes S^q_{S^d}(A,A) \to S^{p+q-d}_{S^d}(A,A)$ giving an homotopy for the commutativity of $\cup_{d-1}$. Let $[f,g]_d := f \cup_d g - (-1)^{(|f|-d)(|g|-d)} g \cup_d f$.
Theorem 5.3 *The $\cup_0$-product and bracket $[\,,\,]_d$ give a structure of $d+1$-algebra to $HH^*_{S^d}(A,A)$.*

6. Free commutative algebras and Brane topology in characteristic zero

By definition of the small complex, one has $C^{n<d}_{S^d_{sm}}(A,M) = M$ and $C^d_{S^d_{sm}}(A,M) = \mathrm{Hom}(A,M)$. Furthermore one checks that $f \in \mathrm{Hom}(A,M) = C^d_{S^d_{sm}}(A,M)$ is a cocycle if and only if $f \in \mathrm{Der}(A,M)$. Thanks to the commutative cup-product there is a canonical map

$$HKR : \mathrm{Hom}_A(S^*(\Omega_A[d]), M) \to HH^*_{S^d}(A,M)) \qquad (2)$$

where Ω_A is the space of Kähler differentials (recall that $\mathrm{Hom}_A(\Omega A, M) \cong \mathrm{Der}(A,M)$) and S^* is the graded symmetric algebra functor. Note that $\mathrm{Hom}_A(S^*(\Omega_A[d]), A)$ is a $d+1$-algebra with product induced by the symmetric power and bracket given by the identification $\mathrm{Hom}_A(\Omega A, A) \cong \mathrm{Der}(A,A)$ and extended to the whole space by the Leibniz rule. Moreover there are Adams operations ψ^k defined on $\mathrm{Hom}_A(S^j(\Omega_A[d]), A)$ by the multiplication by k^j. As in the classical case, if A is free, the map HKR is an isomorphism preserving all the algebraic structures. Furthermore, all of the above makes sense for differential graded commutative algebras as well. When $\mathbf{k}$ is of characteristic zero, any (dg) commutative algebra (A,d_A) is quasi-isomorphic to a dg free one $(F,d_F) \xrightarrow{\sim} (A,d_A)$. Proposition 5.2 implies
Theorem 6.1 *Let $d>1$ and $\mathrm{char}(\mathbf{k}) = 0$. The map*

$$HKR : H^*(\mathrm{Hom}_F(S^*(\Omega_F[d]), F), d_F) \to HH^*_{S^d}(A,A)$$

is an isomorphism of $d+1$-algebras commuting with the Adams operations. Moreover a quasi-isomorphism $(A,d_A) \to (B,d_B)$ of dg-commutative algebras induces an isomorphism $HH^*_{S^d}(A,A) \cong HH^*_{S^d}(B,B)$ of $d+1$-algebras and Hodge structures.
This theorem gives an efficient way to compute the structure of higher order Hochschild homology.
Remark 3 In particular for d odd, the groups appearing in the Hodge decomposition are those in the Hodge decomposition for $d=1$ but they are dispatched in different degrees. The same is true for d even with the groups appearing in the decomposition for $d=2$. Note that for $d=1$, the Hodge decomposition coincides with the classical one [5], [9].

Let X be a d-connected Poincaré duality space of dimension n. By [8], there exists a free dg-commutative algebra $(\mathcal{A}^*_X, d_X)$ quasi-isomorphic to the minimal model of X together with a quasi-isomorphism $\mathcal{A}^*_X \to (\mathcal{A}^*_X)'[n]$ of $\mathcal{A}^*_X$-modules inducing the Poincaré duality in cohomology. Moreover by Theorem 6.1 and direct inspection on a minimal model of X, there is an isomorphism $HH^i_{S^d}(\mathcal{A}^*_X, (\mathcal{A}^*_X)') \cong H_i(\mathrm{Map}(S^d, X))$.

$$HH^i_{S^d}(\mathcal{A}^*_X, \mathcal{A}^*_X) \cong HH^{i+d}_{S^d}(\mathcal{A}^*_X, (\mathcal{A}^*_X)') \cong H_{i+d}(\mathrm{Map}(S^d, X)).$$

The last isomorphism come from Theorem 6.1 applied to a minimal model of X.

Corollary 6.2 *For any commutative model $\mathcal{M}_X$ for X, one has $HH^i_{S^d}(\mathcal{M}_X, \mathcal{M}_X) \cong H_{i+d}(\mathrm{Map}(S^d, X))$.*

In particular, the shifted homology of the mapping space $\mathrm{Map}(S^d, X)$ inherits a structure of d-algebra which is graded with respect to the Hodge decomposition. Corollary 6.2 adds the Hodge decomposition to the Brane topology story studied in [4] and [6].

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