Sheaves with connection on complex tori

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Contents

1 Introduction

1.1 Background

Mukai [Muk81, Sec. 2] introduces an analog of the Fourier transform for sheaves of modules on abelian varieties, known as the *Fourier-Mukai transform.* Laumon [Lau96] and Rothstein [Rot96] study independently its lift to sheaves with connection (integrable or not). They both prove the Fourier inversion formula for the lift. Laumon [Lau96, Thm. 6.3.3] applies it to investigate generalized 1-motives. Meanwhile, as an application, Rothstein [Rot96, Thm. 3.2] recovers Matsushima's theorem ([Mat59]): every vector bundle on an abelian variety admitting a connection is translation invariant. Schnell's work [Sch15] about holonomic D-modules on abelian varieties relies upon the lift of the Fourier-Mukai transform.

Let k be an algebraically closed field. Let A, B be abelian varieties over k dual to each other. Set $g = \dim A$. Let p_A (resp. p_B) denote the projection from $A \times B$ to A (resp. B). Let $\mathcal{P}$ be the normalized Poincaré line bundle on $A \times B$. We adopt the following sign convention for the Fourier-Mukai transform:

$$\begin{aligned} R\mathscr{S}_1 &= Rp_{A*}(\mathcal{P} \otimes^L p_B^* \cdot) : D(O_B) \to D(O_A); \\ R\mathscr{S}_2 &= Rp_{B*}(\mathcal{P}^{-1} \otimes^L p_A^* \cdot) : D(O_A) \to D(O_B), \end{aligned} \tag{1}$$

For a triangulated category, let T denote the degree shift automorphism. For an algebraic variety V over k, denote by $D_{\mathrm{qc}}(O_V) \subset D(O_V)$ (resp. $D_c^b(O_V) \subset D^b(O_V)$) the full subcategory of objects whose cohomologies are quasi-coherent (resp. coherent) O_V-modules. Mukai establishes an analog of the Fourier inversion formula for this triangulated subcategory.

Fact 1.1.1 (Mukai, [Muk81, Thm. 2.2], [Rot96, p.569])**.**

1. *There are natural isomorphisms of functors $R\mathscr{S}_1 \circ R\mathscr{S}_2 \cong T^{-g}$ on $D_{\mathrm{qc}}(O_A)$ and $R\mathscr{S}_2 \circ R\mathscr{S}_1 \cong T^{-g}$ on $D_{\mathrm{qc}}(O_B)$. In particular, $R\mathscr{S}_1 : D_{\mathrm{qc}}(O_B) \to D_{\mathrm{qc}}(O_A)$ is an equivalence of triangulated categories, with a quasi-inverse $T^g R\mathscr{S}_2$.*
2. *The functor $R\mathscr{S}_1 : D(O_B) \to D(O_A)$ restricts to an equivalence $D_c^b(O_B) \to D_c^b(O_A)$.*

Let $0 \to H^0(A, \Omega_A^1) \to B^\natural \xrightarrow{p} B \to 0$ be the universal vectorial extension of B (constructed in [Ros58, Prop. 11]). For an algebraic variety V, denote the forgetful functor $D(D_V) \to D(O_V)$ by for_V. Let $D_{\mathrm{qc}}(D_A) \subset D(D_A)$ (resp. $D_c^b(D_A) \subset D^b(D_A)$) be the full subcategory of objects whose cohomologies are quasi-coherent O_A-modules (resp. coherent D_A-modules). Laumon and Rothstein lift the Fourier-transform to D-modules and establish a duality result similar to Fact 1.1.1.

Fact 1.1.2 (Laumon, Rothstein)**.**

1. *There are functors $RS_1 : D(O_{B^\natural}) \to D(D_A)$ and $RS_2 : D(D_A) \to D(O_{B^\natural})$ fitting into commutative squares*

$$\begin{array}{ccc} D_{\mathrm{qc}}(O_{B^\natural}) & \xrightarrow{RS_1} & D_{\mathrm{qc}}(D_A) \\ \downarrow{\scriptstyle Rp_*} & & \downarrow{\scriptstyle \mathrm{for}_A} \\ D_{\mathrm{qc}}(O_B) & \xrightarrow{R\mathscr{S}_1} & D_{\mathrm{qc}}(O_A), \end{array} \qquad \begin{array}{ccc} D_{\mathrm{qc}}(O_{B^\natural}) & \xleftarrow[RS_2]{} & D_{\mathrm{qc}}(D_A) \\ \downarrow{\scriptstyle Rp_*} & & \downarrow{\scriptstyle \mathrm{for}_A} \\ D_{\mathrm{qc}}(O_B) & \xleftarrow[R\mathscr{S}_2]{} & D_{\mathrm{qc}}(O_A). \end{array}$$

2. ([Lau96, Thm. 3.2.1], [Rot96, Thm. 4.5], [Rot97], [Vig21, Thm. 2.2.21]) *There are natural isomorphisms of functors $RS_1RS_2 \cong T^{-g}$ on $D_{\mathrm{qc}}(D_A)$ and $RS_2RS_1 \cong T^{-g}$ on $D_{\mathrm{qc}}(O_{B^\natural})$, hence an equivalence $RS_1 : D_{\mathrm{qc}}(O_{B^\natural}) \to D_{\mathrm{qc}}(D_A)$.*

3. ([Lau96, Cor. 3.1.3], [Rot96, Thm. 6.2]) *The functor $RS_1 : D(O_{B^\natural}) \to D(D_A)$ restricts to an equivalence $RS_1 : D^b_c(O_{B^\natural}) \to D^b_c(D_A)$.*

1.2 Extension to complex tori

Let X, Y be complex tori dual to each other and of dimension g. Define the analytic Fourier-Mukai transform $R\mathscr{S}_1 : D(O_X) \to D(O_Y)$ and $R\mathscr{S}_2 : D(O_Y) \to D(O_X)$ by formulae similar to (1). For a complex manifold Z, let $D_{\mathrm{good}}(O_Z) \subset D(O_Z)$ be the full subcategory of objects whose cohomologies are good O_Z-modules (in the sense of [Kas03, Def. 4.22]). In [BBBP07, Thm. 2.1], a result similar to Fact 1.1.1 is established for complex tori.

Fact 1.2.1 (Mukai, Ben-Bassat, Block, Pantev)**.**

1. ([Liu23a, Thm. 4.1.1]) *There are natural isomorphisms of functors*

$$R\mathscr{S}_1R\mathscr{S}_2 \cong T^{-g} : D_{\mathrm{good}}(O_Y) \to D_{\mathrm{good}}(O_Y),$$
$$R\mathscr{S}_2R\mathscr{S}_1 \cong T^{-g} : D_{\mathrm{good}}(O_X) \to D_{\mathrm{good}}(O_X).$$

   In particular, $R\mathscr{S}_1 : D_{\mathrm{good}}(O_X) \to D_{\mathrm{good}}(O_Y)$ is an equivalence of categories with a quasi-inverse $T^gR\mathscr{S}_2$.

2. ([PPS17, Thm. 13.1]) *The functor $R\mathscr{S}_1 : D(O_X) \to D(O_Y)$ restricts to an equivalence $D^b_c(O_X) \to D^b_c(O_Y)$.*

We lift the analytic Fourier-Mukai transform to D-modules, and give an analog of Fact 1.1.2. Good D-modules are reviewed in Section 6.1. For a complex manifold Z and an O_Z-algebra $\mathcal{R}$, let $D_{O-\mathrm{good}}(\mathcal{R}) \subset D(\mathcal{R})$ (resp. $D^b_{\mathrm{good}}(\mathcal{R}) \subset D^b(\mathcal{R})$) be the full subcategory of objects whose cohomologies are good over O_Z (resp. $\mathcal{R}$).

Theorem 1.2.2.

- (Prop. 5.1.2) *There is a canonical commutative O_X-algebra $\mathcal{A}_X$, such that the functors $R\mathscr{S}_1$ and $R\mathscr{S}_2$ lift naturally to triangulated functors $RS_1 : D(\mathcal{A}_X) \to D(D_Y)$ and $RS_2 : D(D_Y) \to D(\mathcal{A}_X)$ respectively.*

- (Thm. 5.1.3) *The functors RS_i restrict to equivalences $RS_1 : D_{O-\text{good}}(\mathcal{A}_X) \to D_{O-\text{good}}(D_Y)$ and $RS_2 : D_{O-\text{good}}(D_Y) \to D_{O-\text{good}}(\mathcal{A}_X)$.*

- (Thm. 6.3.1) *The functors RS_i restricts to equivalences $RS_1 : D^b_{\text{good}}(\mathcal{A}_X) \to D^b_{\text{good}}(D_Y)$ and $RS_2 : D^b_{\text{good}}(D_Y) \to D^b_{\text{good}}(\mathcal{A}_X)$.*

Notation and convention

For a sheaf F on a topological space, let Supp$F$ be its support. For a (not necessarily commutative) ringed space $(X, \mathcal{R})$, let $\text{Mod}(\mathcal{R})$ be the category of left $\mathcal{R}$-modules. Let $\text{Coh}(\mathcal{R}) \subset \text{Mod}(\mathcal{R})$ be the full subcategory of coherent $\mathcal{R}$-modules. Given a symbol $* \in \{\emptyset, +, -, b\}$, the notation $D^*(\mathcal{R})$ refers to the unbounded/bounded below/bounded above/bounded derived category of the abelian category $\text{Mod}(\mathcal{R})$ in order. Let $D^*_c(\mathcal{R}) \subset D^*(\mathcal{R})$ be the full subcategory of objects whose cohomologies are coherent $\mathcal{R}$-modules (in the sense of [Sta23, Tag 01BV]).

Let k be an algebraically closed field. An algebraic variety refers to an integral scheme of finite type and separated over k. For a complex manifold Z and $z \in Z$, let $i_z : (z, \mathbb{C}) \to (Z, O_Z)$ be the closed embedding of complex manifolds. Set $\mathbb{C}_z := (i_z)_*\mathbb{C}$, which is a coherent O_Z-module. Let X, Y be complex tori dual to each other and of dimension g.

2 Preliminaries

For the convenience of the reader, we recall the notation of [Rot97, Sec. 2.1].

2.1 Categories of splittings

For a complex manifold Z and a (holomorphic) vector bundle $M \to Z$, by [Har77, III, Prop. 6.3 (c)], one has $H^1(Z, M) = \text{Ext}^1(O_Z, M)$. Thus, every $\alpha \in H^1(Z, M)$ determines a short exact sequence in $\text{Mod}(O_Z)$

$$0 \to M \to \mathcal{E}_\alpha \overset{\mu_\alpha}{\to} O_Z \to 0. \tag{2}$$

Since O_Z is a flat O_Z-module, by [Sta23, Tag 05NJ], for every $F \in \text{Mod}(O_Z)$, the sequence (2) remains exact after tensored with F:

$$0 \to M \otimes_{O_Z} F \to \mathcal{E}_\alpha \otimes_{O_Z} F \overset{\mu_\alpha \otimes \text{Id}_F}{\to} F \to 0. \tag{3}$$

Definition 2.1.1. Define a category $\text{Mod}(O_Z)_{\alpha-\text{sp}}$ as follows: the objects are pairs (F, ψ), where $F \in \text{Mod}(O_Z)$ and $\psi : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ is an α-splitting on F, *i.e.*, an O_Z-linear splitting of $\mu_\alpha \otimes \text{Id}_F$. The morphisms in $\text{Mod}(O_Z)_{\alpha-\text{sp}}$ are required to be compatible with the splittings.

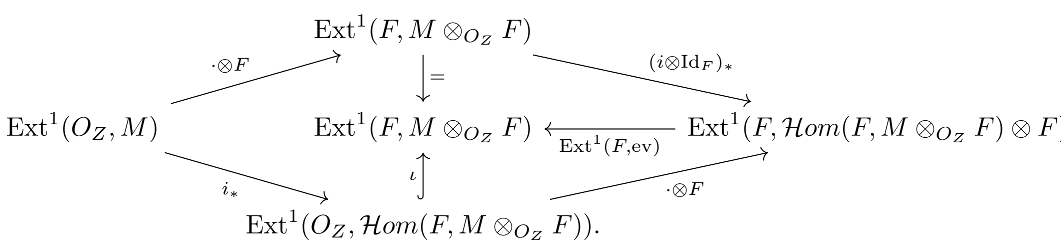
Example 2.1.2. When $\alpha = 0$, the sequence (2) identifies $\mathcal{E}_0$ with $M \oplus O_Z$. There is a natural functor $\mathrm{Mod}(O_Z) \to \mathrm{Mod}(O_Z)_{0-\mathrm{sp}}$ defined by $F \mapsto (F, \psi)$, where $\psi : F \to \mathcal{E}_0 \otimes F = (M \otimes_{O_Z} F) \oplus F$ is the canonical injection to the second factor. If further $M = \Omega^1_Z$, then an α-splitting ϕ on a vector bundle $E \to Z$ is exactly a holomorphic 1-form on Z with values in $\mathcal{E}nd(E)$. The pair (E, ϕ) is a Higgs bundle (in the sense of [Sim92, p.6]) if and only if $[\phi, \phi] = 0$.

Lemma 2.1.3. *For an O_Z-module F, there is an α-splitting on F if and only if the map $i_* : H^1(Z, M) \to H^1(Z, M \otimes_{O_Z} \mathcal{E}nd(F))$ (induced by the natural morphism $O_Z \to \mathcal{E}nd(F)$) sends α to 0. In that case, the set of α-splittings on F has a natural simple transitive action of the abelian group $\mathrm{Hom}_{O_Z}(F, M \otimes_{O_Z} F)$.*

Proof. The natural morphism $O_Z \to \mathcal{E}nd(F)$ induces a morphism $i : M \to \mathcal{H}om_{O_Z}(F, M \otimes_{O_Z} F), \quad i(m)(f) = m \otimes f$. There is a canonical evaluation morphism $\mathrm{ev} : \mathcal{H}om_{O_Z}(F, M \otimes_{O_Z} F) \otimes F \to M \otimes_{O_Z} F, \quad \mathrm{ev}(\phi \otimes f) = \phi(f)$. The five-term exact sequence of the spectral sequence

$$E_2^{i,j} = \mathrm{Ext}^i(O_Z, \mathcal{E}xt^j(F, M \otimes_{O_Z} F)) \Rightarrow \mathrm{Ext}^{i+j}(F, M \otimes_{O_Z} F)$$

gives an injection $\iota : \mathrm{Ext}^1(O_Z, \mathcal{H}om(F, M \otimes_{O_Z} F)) \to \mathrm{Ext}^1(F, M \otimes_{O_Z} F)$, which is $\mathrm{Ext}^1(F, \mathrm{ev}) \circ (\cdot \otimes F)$:



One has

$$\mathrm{ev} \circ (i \otimes \mathrm{Id}_F)(m \otimes f) = \mathrm{ev}(i(m) \otimes f) = i(m)(f) = m \otimes f,$$

so $\mathrm{ev} \circ (i \otimes \mathrm{Id}_F) = \mathrm{Id}_{M \otimes_{O_Z} F}$ as morphisms $M \otimes_{O_Z} F \to M \otimes_{O_Z} F$. Therefore, the diagram is commutative. Then F admits an α-splitting if and only if $\alpha \otimes F = 0$ if and only if $i_*(\alpha) = 0$. Any two α-splittings on F differ by a unique element of $\mathrm{Hom}(F, M \otimes_{O_Z} F)$. ☐

To each object $(F, \psi) \in \mathrm{Mod}(O_Z)_{\alpha-\mathrm{sp}}$, we assign an element

$$[\psi, \psi] \in \Gamma(Z, (\wedge^2 M) \otimes_{O_Z} \mathcal{E}nd(F)) \tag{4}$$

as follows. The sequence (2) induces a short exact sequence

$$0 \to \wedge^2 M \to \wedge^2 \mathcal{E}_\alpha \xrightarrow{\omega_\alpha} M \to 0,$$

where

$$\omega_\alpha(\rho_1 \wedge \rho_2) = \mu_\alpha(\rho_1)\rho_2 - \mu_\alpha(\rho_2)\rho_1.$$

The flatness of M ensures the exactness when tensoring with F:

$$0 \to (\wedge^2 M) \otimes F \to (\wedge^2 \mathcal{E}_\alpha) \otimes F \stackrel{\omega_\alpha \otimes \mathrm{Id}_F}{\to} M \otimes_{O_Z} F \to 0. \tag{5}$$

Let $a : \mathcal{E}_\alpha \otimes \mathcal{E}_\alpha \to \wedge^2 \mathcal{E}_\alpha$ be the morphism defined by $e \otimes e' \mapsto e \wedge e'$. Let ψ^1 be the composition

$$\mathcal{E}_\alpha \otimes F \stackrel{\mathrm{Id}_{\mathcal{E}_\alpha} \otimes \psi}{\to} \mathcal{E}_\alpha \otimes (\mathcal{E}_\alpha \otimes F) \xrightarrow{\sim} (\mathcal{E}_\alpha \otimes \mathcal{E}_\alpha) \otimes F \stackrel{a \otimes \mathrm{Id}_F}{\to} (\wedge^2 \mathcal{E}_\alpha) \otimes F,$$

where the isomorphism in the middle is from the associativity of tensor product.

Lemma 2.1.4. *One has* $(\omega_\alpha \otimes \mathrm{Id}_F)\psi^1\psi = 0$.

Proof. Locally, the vector bundle $\mathcal{E}_\alpha$ has a (holomorphic) frame $\{e_1, \ldots, e_r\}$. For a local section $f \in F$, write $\psi(f) = \sum_{i=1}^r e_i \otimes f_i$, where f_i are local sections of F. For every $1 \leq i \leq r$, write $\psi(f_i) = \sum_{j=1}^r e_j \otimes f_j^{(i)}$, where $f_j^{(i)}$ are local sections of F. As ψ is a section to $\mu_\alpha \otimes \mathrm{Id}_F$, one has

$$f = (\mu_\alpha \otimes \mathrm{Id}_F)\psi(f) = \sum_{i=1}^r \mu_\alpha(e_i) f_i; \tag{6}$$

$$f_i = (\mu_\alpha \otimes \mathrm{Id}_F)\psi(f_i) = \sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)}. \tag{7}$$

Thus,

$$\psi(f) \stackrel{(6)}{=} \sum_{i=1}^r \mu_\alpha(e_i)\psi(f_i). \tag{8}$$

By construction, $\psi^1\psi(f) = \sum_{i,j=1}^r (e_i \wedge e_j) \otimes f_j^{(i)}$. Then

$$\begin{aligned}
(\omega_\alpha \otimes \mathrm{Id}_F)\psi^1\psi(f) &= \sum_{i,j=1}^r [\mu_\alpha(e_i)e_j - \mu_\alpha(e_j)e_i] \otimes f_j^{(i)} \\
&= \sum_{i=1}^r \mu_\alpha(e_i) \sum_{j=1}^r e_j \otimes f_j^{(i)} - \sum_{i=1}^r e_i \otimes [\sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)}] \\
&\stackrel{(7)}{=} \sum_{i=1}^r \mu_\alpha(e_i)\psi(f_i) - \sum_{i=1}^r e_i \otimes f_i \\
&\stackrel{(8)}{=} \psi(f) - \psi(f) = 0.
\end{aligned}$$

□

From Lemma 2.1.4 and (5), one has $\psi^1\psi(F) \subset (\wedge^2 M) \otimes F$. The morphism $\psi^1\psi : F \to (\wedge^2 M) \otimes F$ gives an element $[\psi, \psi] \in \Gamma(Z, (\wedge^2 M) \otimes_{O_Z} \mathcal{E}nd(F))$.

Example 2.1.5. For the complex torus X, set $\mathfrak{g} = H^1(X, O_X)$. Then

$$H^1(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) = \mathfrak{g}^* \otimes_{\mathbb{C}} \mathfrak{g} = \mathrm{End}(\mathfrak{g}).$$

Hence a category $\mathrm{Mod}(O_X)_{T-\mathrm{sp}}$ for each $T \in \mathrm{End}(\mathfrak{g})$. The identity element $1 \in \mathrm{End}(\mathfrak{g})$ corresponds to the tautological exact sequence [Rot96, (1.3)]:

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} O_X \to \mathcal{E} \to O_X \to 0. \tag{9}$$

We also write $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ for $\mathrm{Mod}(O_X)_{1-\mathrm{sp}}$. For $(F, \psi) \in \mathrm{Mod}(O_X)_{\mathrm{sp}}$, the element $[\psi, \psi]$ lies in

$$\Gamma(X, \wedge^2 \mathfrak{g}^* \otimes_{\mathbb{C}} O_X \otimes_{O_X} \mathcal{E}nd(F)) = \wedge^2 \mathfrak{g}^* \otimes_{\mathbb{C}} \mathrm{End}(F),$$

and we recover [Rot96, (4.8)]. Similarly, $H^1(X \times X, \mathfrak{g}^* \otimes O_{X \times X}) = \mathrm{End}(g) \oplus \mathrm{End}(g)$, so for every pair $T_1, T_2 \in \mathrm{End}(g)$, the category $\mathrm{Mod}(O_{X \times X})_{(T_1, T_2)-\mathrm{sp}}$ is defined.

2.2 Categories of twisted connection

We continue to review the twisted (relative) connection introduced in [Rot97, p.206]. Consider a smooth morphism of complex manifolds $f : Z \to S$, with relative cotangent sheaf Ω^1_f. As f is smooth, Ω^1_f is a vector bundle on Z. Let $d_f : O_Z \to \Omega^1_f$ denote the differential relative to f. An element $\alpha \in H^1(Z, \Omega^1_f)$ determines an extension

$$0 \to \Omega^1_f \to \mathcal{E}_\alpha \overset{\mu_\alpha}{\to} O_Z \to 0. \tag{10}$$

Definition 2.2.1. On an O_Z-module F, an α-connection is an $f^{-1}(O_S)$-linear splitting $\nabla : F \to \mathcal{E}_\alpha \otimes_{O_Z} F$ to $\mu_\alpha \otimes \mathrm{Id}_F$, satisfying the Leibniz rule

$$\nabla(h\phi) = h\nabla(\phi) + d_f(h) \otimes \phi, \tag{11}$$

where h and ϕ are local sections of O_Z and F respectively. Let $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ be the category of pairs (F, ∇), where $F \in \mathrm{Mod}(O_Z)$ and ∇ is an α-connection on F.

Example 2.2.2. If $\alpha = 0$, then α-connection are exactly f-relative connection. Define a sheaf $\tilde{D}_{Z/S}$ of noncommutative O_Z-algebras by gluing the following local data. If $\{\xi_1, \ldots, \xi_n\}$ is a local frame of $(\Omega^1_f)^\vee$ (the vector bundle dual to Ω^1_f) on an open subset $U \subset Z$, then a multiplication law on $O_U\{\xi_1, \ldots, \xi_n\}$ is introduced by imposing the commutation relation $[\xi_i, h] = \xi_i(h)$ for local sections h of O_Z. Let it be $\tilde{D}_{Z/S}|_U$. Then $\mathrm{Mod}(Z)_{f,0-\mathrm{cxn}} = \mathrm{Mod}(\tilde{D}_{Z/S})$. The category $\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}$ is denoted by $\mathrm{Mod}(O_Z)_{\mathrm{cxn}}$ when f is the structure morphism $Z \to \mathrm{Specan}(\mathbb{C})$.

Remark 2.2.3. In fact, a twisted connection is a particular splitting. Define another extension

$$0 \to \Omega^1_f \to \mathcal{E}_{\alpha'} \to O_Z \to 0 \tag{12}$$

in $\mathrm{Mod}(O_Z)$ as follows. As an extension of abelian sheaves, (12) is same as (10). Let h (resp. s') be a local section of O_Z (resp. $\mathcal{E}_{\alpha'}$) and s denote the local section of $\mathcal{E}_\alpha$ induced by s'. The O_Z-module structure on $\mathcal{E}_{\alpha'}$ is defined such that the local section $hs + \mu_\alpha(s)d_f h$ of $\mathcal{E}_\alpha$ induces the local section hs' of $\mathcal{E}_{\alpha'}$.

We claim this indeed defines an O_Z-module structure on $\mathcal{E}_{\alpha'}$. For local sections h_1, h_2 of O_Z, let t be the local section of $\mathcal{E}_\alpha$ induced by $h_2 s'$. Then $t = h_2 s + \mu_\alpha(s) d_f h_2$, so $\mu_\alpha(t) = h_2\mu_\alpha(s)$. Thus, the local section of $\mathcal{E}_\alpha$ corresponding to $h_1(h_2 s')$ is

$$h_1 t + \mu_\alpha(t) d_f h_1 = h_1 h_2 s + h_1\mu_\alpha(s) d_f h_2 + h_2\mu_\alpha(s) d_f h_1 = (h_1h_2)s + \mu_\alpha(s) d_f(h_1h_2).$$

Therefore, $h_1(h_2s') = (h_1h_2)s'$. The claim is proved.

By construction, the morphisms in (12) are O_Z-linear. Then (12) is indeed an extension in $\mathrm{Mod}(O_Z)$, hence a new extension class $\alpha' \in \mathrm{Ext}(O_Z, \Omega^1_f)$. An α-connection on $F \in \mathrm{Mod}(O_Z)$ is equivalent to an α'-splitting on F. Hence an equivalence of categories

$$\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}} \to \mathrm{Mod}(O_Z)_{\alpha'-\mathrm{sp}}.$$

There is a notion of integrable α-connection ([Rot97, Remark, p.206]). Let $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn},\mathrm{fl}}$ be the full subcategory of $\mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$ comprised of objects whose connection are integrable. Then $\mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn},\mathrm{fl}}$ coincides with $\mathrm{MIC}(f)$ defined in [ABC20, 4.3.7], which is further equivalent to $\mathrm{Mod}(D_{Z/S})$. Here $D_{Z/S}$ is the sheaf of ring of relative differential operators on Z/S defined in [SS94, p.9].

Example 2.2.4. For the dual complex tori X, Y, consider the projection $p_X : X \times Y \to X$. Since $\Omega^1_{p_X} = p_X^*(\mathfrak{g}^* \otimes_{\mathbb{C}} O_X)$, there is a natural morphism

$$p_X^* : \mathrm{End}(\mathfrak{g}) = H^1(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) \to H^1(X \times Y, \Omega^1_{p_X}).$$

For every $T \in \mathrm{End}(g)$, the category $\mathrm{Mod}(O_{X\times Y})_{p_X, p_X^*T-\mathrm{cxn}}$ (resp. $\mathrm{Mod}(O_{X\times Y})_{p_X, p_X^*T-\mathrm{cxn},\mathrm{fl}}$) is also written as $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}$ (resp. $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn},\mathrm{fl}}$).

Fact 2.2.5 is taken from the two remarks in [Rot97, pp.206–207].

Fact 2.2.5. *The Poincaré bundle $\mathcal{P}$ is naturally an object of* $\mathrm{Mod}(O_{X\times Y})_{-1-\mathrm{cxn},\mathrm{fl}}$.

In local coordinates, the $p_X^*(-1)$-connection on $\mathcal{P}$ is explained in [Rot96, (1.10) and p.575*ff.*] (except that we use a Stein open cover of X instead of Rothestein's affine open cover).

2.3 Functors between them

Recall that the Fourier-Mukai transform (1) is the composition of the pullback, the tensor product with $\mathcal{P}$ as well as the derived direct image. Rothstein's lift to modules with connection keeps an extra track of the splittings and connection.

Remark 2.3.1. Combining [Rot97, (2.21)] with the fact that twisted relative connection are kinds of splittings (Remark 2.2.3), the categories under consideration ($\mathrm{Mod}(O_X)_{\mathrm{sp}}$, $\mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}$, *etc.*) are equivalent to categories of modules over sheaves of certain noncommutative *flat* O-algebras. In particular, each of them is a Grothendieck abelian category. Each has enough K-injectives ([Sta23, Tag 079P]) and enough objects flat over O ([HT07, Lem. 1.5.2 (ii)]), *cf.* [Rot97, Cor. 2.3]. Thus, all the (left exact) direct image functors involved below admit right derived functors on the unbounded derived categories (see [Sta23, Tag 070K] and [Sta23, Tag 079P]).

From splittings to connection

Given $T \in \mathrm{End}(\mathfrak{g})$ and $(F, \psi) \in \mathrm{Mod}(O_X)_{T-\mathrm{sp}}$, the induced morphism

$$p_X^{-1}\psi : p_X^{-1}F \to p_X^{-1}\mathcal{E} \otimes_{p_X^{-1}O_X} p_X^{-1}F$$

is $p_X^{-1}O_X$-linear. By Example 2.2.4, the sequence (9) induces a short exact sequence in $\mathrm{Mod}(O_{X\times Y})$

$$0 \to \Omega^1_{p_X} \to p_X^*\mathcal{E} \to O_{X\times Y} \to 0.$$

Its extension class is $p_X^*T \in H^1(X \times Y, \Omega^1_{p_X})$. Define another $p_X^{-1}O_X$-linear morphism

$$\nabla_\psi : p_X^*F = (O_{X\times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}F) \to p_X^*\mathcal{E} \otimes_{O_{X\times Y}} p_X^*F(=$$
$$p_X^*\mathcal{E} \otimes_{p_X^{-1}O_X} p_X^{-1}F = O_{X\times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}\mathcal{E} \otimes_{p_X^{-1}O_X} p_X^{-1}F)$$

by

$$\nabla_\psi(h \otimes s) = d_{p_X}(h) \otimes s + h \otimes [(p_X^{-1}\psi)(s)],$$

where h (resp. s) is a local section of $O_{X\times Y}$ (resp. $p_X^{-1}F$). By construction, ∇_ψ satisfies the Leibniz rule (11). So it is a p_X^*T-connection on p_X^*F. Thus, we get the *exact* functor in [Rot97, (2.5)]:

$$p_X^* : \mathrm{Mod}(O_X)_{T-\mathrm{sp}} \to \mathrm{Mod}(O_{X\times Y})_{T-\mathrm{cxn}}. \tag{13}$$

Tensoring with Poincaré bundle

By Fact 2.2.5 and [Rot97, (2.10)], the functor

$$\cdot \otimes_{O_{X\times Y}} \mathcal{P} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \tag{14}$$

restricts to a functor $\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn,fl}}(\cong \mathrm{Mod}(D_{X\times Y/X}))$. The functor (14) is an equivalence of abelian categories, with a quasi-inverse $\cdot \otimes_{O_{X\times Y}} \mathcal{P}^{-1}$.

From connection to splittings

For every $(F,\nabla)\in \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}}$, the morphism

$$\nabla : F \to p_X^*\mathcal{E}\otimes_{O_{X\times Y}} F (= p_X^{-1}\mathcal{E}\otimes_{p_X^{-1}O_X} F)$$

is a $p_X^{-1}O_X$-splitting to $(p_X^{-1}\mu_1)\otimes \mathrm{Id}_F$. By projection formula (see *e.g*, [KS13, Prop. 2.6.6]), the induced morphism

$$p_{X*}\nabla : p_{X*}F \to \mathcal{E}\otimes_{O_X} p_{X*}F$$

is an O_X-linear splitting to $\mu_1\otimes_{O_X}\mathrm{Id}_{p_{X*}F}$. Hence $(p_{X*}F, p_{X*}\nabla)\in \mathrm{Mod}(O_X)_{\mathrm{sp}}$. Thus, we get a left exact functor (a special case of [Rot97, (2.13)]):

$$p_{X*} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn}} \to \mathrm{Mod}(O_X)_{\mathrm{sp}}. \tag{15}$$

If (F,∇) is integrable, then $[p_{X*}\nabla, p_{X*}\nabla]$ defined in (4) is zero.

Between connection

We define the inverse image and the direct image of relative connection on changing bases. Consider a cartesian square of complex manifolds

$$\begin{array}{ccc} W & \xrightarrow{g'} & Z \\ \downarrow{\scriptstyle f'} & \square & \downarrow{\scriptstyle f} \\ T & \xrightarrow{g} & S, \end{array} \tag{16}$$

where f is smooth. For every $(F,\nabla)\in \mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}}$, by [ABC20, Sec. 4.2], the relative connection ∇ is equivalent to an O_Z-linear splitting to the natural projection $P^1_f\otimes_{O_Z} F \to F$, where $P^1_\bullet$ denotes the sheaf of first order jets (defined in [ABC20, Sec. 4.1.2]). Applying g'^* to the induced splitting, we get an O_W-linear splitting to the natural projection $P^1_{f'}\otimes_{O_W} g'^*F \to g'^*F$. This is equivalent to an f'-connection on g'^*F. Hence an inverse image functor

$$g'^* : \mathrm{Mod}(O_Z)_{f,0-\mathrm{cxn}} \to \mathrm{Mod}(O_W)_{f',0-\mathrm{cxn}}. \tag{17}$$

It is right exact. By [ABC20, Sec. 5.1], the connection induced by ∇ is integrable if ∇ is so.

Now for direct image. Fix $\alpha\in F^1(Z,\Omega^1_f)$. For every

$$(F,\nabla)\in \mathrm{Mod}(O_W)_{f',g'^*\alpha-\mathrm{cxn}},$$

by projection formula (see *e.g*, [Har77, II, Ex. 5.1 (d)]), one has

$$g'_*(F\otimes_{O_W} g'^*\mathcal{E}_\alpha) = (g'_*F)\otimes_{O_Z}\mathcal{E}_\alpha.$$

Then the induced morphism

$$g'_*\nabla : g'_*F \to (g'_*F)\otimes_{O_Z}\mathcal{E}_\alpha$$

is $f^{-1}(O_S)$-linear. Since $d_{f'} : O_W \to \Omega^1_{f'}$ and $d_f : O_Z \to \Omega^1_f$ are related by $g'^* d_f = d_{f'}$, the induced map $g'_*\nabla$ satisfies the Leibniz rule (11). Hence, the pair $(g'_*F, g'_*\nabla) \in \mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}$. In this manner, we get a left exact functor

$$g'_* : \mathrm{Mod}(O_W)_{f', g'^*\alpha-\mathrm{cxn}} \to \mathrm{Mod}(O_Z)_{f,\alpha-\mathrm{cxn}}. \tag{18}$$

When $\alpha = 0$, the functor (18) sends $\mathrm{MIC}(f')$ to $\mathrm{MIC}(f)$.

Example 2.3.2. Take (16) to be

$$\begin{array}{ccc} X \times Y & \xrightarrow{p_Y} & Y \\ \Big\downarrow{\scriptstyle p_X} & \square & \Big\downarrow \\ X & \longrightarrow & \mathrm{Specan}(\mathbb{C}), \end{array}$$

then $p_Y^* : \mathrm{Mod}(O_Y)_{\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}$ sits on the left of the diagram [Rot97, (2.15)] and

$$p_{Y*} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(Y)_{\mathrm{cxn}} \tag{19}$$

is [Rot97, (2.12)]. They restrict respectively to functors

$$p_{Y*} : \mathrm{MIC}(p_X) \to \mathrm{Mod}(D_Y); \tag{20}$$

$$p_Y^* : \mathrm{Mod}(D_Y) \to \mathrm{MIC}(p_X). \tag{21}$$

Remark 2.3.3. Take $\alpha = 0 \in H^1(Z, \Omega^1_f)$. From another point of view, the morphism $O_Z \to g'_*O_W$ between sheaves of rings extends to a morphism $\tilde{D}_{Z/S} \to g'_*\tilde{D}_{W/T}$. Then (17) and (18) are respectively the pullback and the pushout along the induced morphism $(W, \tilde{D}_{W/T}) \to (Z, \tilde{D}_{Z/S})$ of ringed spaces. By [Sta23, Tag 0096], the functor (17) is the left adjoint to (18). Then from [Sta23, Tag 09T5], the derived functor

$$Lg'^* : D(\mathrm{Mod}(Z)_{f,0-\mathrm{cxn}}) \to D(\mathrm{Mod}(W)_{f',0-\mathrm{cxn}})$$

is the left adjoint to

$$Rg'_* : D(\mathrm{Mod}(W)_{f',0-\mathrm{cxn}}) \to D(\mathrm{Mod}(Z)_{f,0-\mathrm{cxn}}).$$

3 Rothstein transform on modules with connection

3.1 Construction

Definition 3.1.1. Define functors $R\mathfrak{S}_1 : D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ and $R\mathfrak{S}_2 : D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \to D(\mathrm{Mod}(O_X)_{\mathrm{sp}})$ by

$$R\mathfrak{S}_1 = Rp_{Y*}(\mathcal{P} \otimes_{O_{X\times Y}} p_X^* \cdot),$$
$$R\mathfrak{S}_2 = Rp_{X*}(\mathcal{P}^{-1} \otimes_{O_{X\times Y}} p_Y^* \cdot).$$

Here Rp_{Y*} (resp. Rp_{X*}) is the right derived functor of (19) (resp. (15)). The pair $(R\mathfrak{S}_1, R\mathfrak{S}_2)$ is called the *Rothstein transform*.

Let $D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \subset D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ (resp. $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \subset D(\mathrm{Mod}(O_X)_{\mathrm{sp}})$) be the full subcategory of objects whose cohomologies are good O-modules (in the sense of [Kas03, Def. 4.22]). In view of Proposition 3.1.2, Rothstein transform is compatible with Fourier-Mukai transform.

Proposition 3.1.2. *There are commutative squares*

$$\begin{array}{ccc} D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) & \xrightarrow{R\mathfrak{S}_1} & D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \\ \downarrow & & \downarrow \\ D(O_X) & \xrightarrow{R\mathscr{S}_1} & D(O_Y), \end{array} \qquad \begin{array}{ccc} D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) & \xrightarrow{R\mathfrak{S}_2} & D(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \\ \downarrow & & \downarrow \\ D(O_Y) & \xrightarrow{R\mathscr{S}_2} & D(O_X), \end{array}$$

where the vertical functors are forgetful. In particular, $R\mathfrak{S}_1$ and $R\mathfrak{S}_2$ restrict to functors $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}}) \to D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ and $D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \to D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}})$.

Proof. All the functors $p_X^* : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X\times Y})$, (13), (14) and

$$\mathcal{P} \otimes_{O_{X\times Y}} \cdot : \mathrm{Mod}(O_{X\times Y}) \to \mathrm{Mod}(O_{X\times Y})$$

are exact. To prove the commutativity of the first square, it remains to do so for the square

$$\begin{array}{ccc} D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) & \xrightarrow{Rp_{Y*}} & D(\mathrm{Mod}(O_Y)_{\mathrm{cxn}}) \\ \downarrow{\scriptstyle \mathrm{for}_{X\times Y}} & & \downarrow{\scriptstyle \mathrm{for}_Y} \\ D(O_{X\times Y}) & \xrightarrow{Rp_{Y*}} & D(O_Y). \end{array} \tag{22}$$

Since the forgetful functor $\mathrm{for}_Y : \mathrm{Mod}(O_Y)_{\mathrm{cxn}} \to \mathrm{Mod}(O_Y)$ is exact, the composition $\mathrm{for}_Y Rp_{Y*} : D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) \to D(O_Y)$ is the right derived functor of

$$\mathrm{for}_Y \circ p_{Y*} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_Y).$$

From Remark 2.3.1, [Sta23, Tag 0096] and [Sta23, Tag 08BJ], the functor $\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})$ preserves K-injective complexes. By Lemma A.0.9, the composition $Rp_{Y*}\mathrm{for}_{X\times Y} : D(\mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}) \to D(O_Y)$ is the right derived functor of

$$p_{Y*}\mathrm{for}_{X\times Y} : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_Y).$$

Since $\mathrm{for}_Y \circ p_{Y*} = p_{Y*} \circ \mathrm{for}_{X\times Y}$, the first square is indeed commutative.

By the commutativity of the first square and [Liu23a, Cor. 3.1.14], the transform $R\mathfrak{S}_1$ preserves O-goodness. The other half about $R\mathfrak{S}_2$ is similar. □

3.2 Rothstein's theorem

Theorem 3.2.1 (Rothstein)**.** *There are natural isomorphisms $R\mathfrak{S}_1 R\mathfrak{S}_2 \cong T^{-g}$ on $D_{O-\mathrm{good}}(\mathrm{Mod}(O_Y)_{\mathrm{cxn}})$ and $R\mathfrak{S}_2 R\mathfrak{S}_1 \cong T^{-g}$ on $D_{O-\mathrm{good}}(\mathrm{Mod}(O_X)_{\mathrm{sp}})$.*

We begin the proof of Theorem 3.2.1 with Lemma 3.2.2, a direct adaption of [Rot97, Prop. 2.4] for complex tori.

Lemma 3.2.2. *Let $\Delta \subset X \times X$ be the diagonal. Define a morphism of complex tori $\epsilon_X : X \times X \to X, \quad (x_1, x_2) \mapsto x_2 - x_1$. Then*

$$Rp_{12*}(\epsilon_X \times 1_Y)^* \mathcal{P} \cong O_\Delta[-g]$$

in $D^b(\mathrm{Mod}(O_{X \times X})_{(1,-1)-\mathrm{sp}})$, where $p_{12} : X \times X \times Y \to X \times X$ is the projection.

Proof. The identification $Rp_{X*}\mathcal{P} \cong \mathbb{C}_0[-g]$ in $D^b(O_X)$ from [Kem91, Thm. 3.15] can be lifted to an isomorphism in $D^b(\mathrm{Mod}(O_X)_{-1-\mathrm{sp}})$. As stated in the last sentence of the proof of [Vig21, Prop. 2.1.21], a morphism of modules with splittings (or connection) is an isomorphism whenever the underlying morphism of O-modules is so. Then apply [Liu23a, Thm. 3.2.3] to the cartesian square

$$\begin{array}{ccc} X \times X \times Y & \xrightarrow{\epsilon_X \times 1_Y} & X \times Y \\ \downarrow{\scriptstyle p_{12}} & & \downarrow{\scriptstyle p_X} \\ X \times X & \xrightarrow{\epsilon_X} & X. \end{array}$$

□

Arguing as in Lemma 3.2.2, we can prove the analytic version of [Rot97, Prop. 2.5; Prop. 3.1]. These three results are used in the proof of Theorem 3.2.1 below.

Proof of Theorem 3.2.1. Repeat the proof of [Rot97, Thm. 3.2], which requires the projection formula and smooth base change theorem for modules with *connection*. For this, we first construct the corresponding comparison morphism that is compatible with the underlying O-module comparison morphism. The construction reduces to the adjunction between derived inverse image and derived direct image of relative connection in Remark 2.3.3.

The compatibility with O-module comparison morphism can be proved in a way similar to Proposition 3.1.2. On the level of O-modules, the comparison morphism is an isomorphism by [Liu23a, Fact 3.2.13; Thm. 3.2.3]. (This type of arguments can also be found in the proof of [Vig21, Prop. 2.1.21; Thm. 2.1.33].) □

3.3 Matsushima's theorem

A holomorphic vector bundle $H \to Y$ is called homogeneous if $T_y^* H$ is isomorphic to H for all $y \in Y$, where $T_y : Y \to Y$ is the translation by y. The first half of Theorem 3.3.1 is a special case of [Mat59, Thm. 1].

Theorem 3.3.1 (Matsushima)**.** *Let E be a coherent O_Y-module with a connection ∇. Then E is a homogeneous vector bundle and the pair (E, ∇) is translation invariant.*

Proof. By Proposition 3.1.2, for every integer i, the coherent O_X-module $H^iRS_2(E)$ admits a 1-splitting. By Lemma 3.3.2, the support of $H^iRS_2(E)$ is finite. Consequently, in $D^b_c(O_X)$ there is an isomorphism $RS_2(E) \cong \oplus_{i\in\mathbb{Z}} T^{-i}H^iRS_2(E)$. From [Liu23a, Prop. 5.3.2 3] and Fact 1.2.1 2, it induces an isomorphism in $D^b_c(O_Y)$

$$T^{-g}E \to \oplus_{i\in\mathbb{Z}} T^{-i}H^0RS_1(H^iRS_2(E)),$$

and each $H^0RS_1(H^iRS_2(E))$ is a homogeneous vector bundle on Y. Then E is isomorphic to $H^0RS_1(H^gRS_2(E))$, hence a homogeneous vector bundle.

We adopt the argument in [BK09, Footnote (6), p.388]. For every $y \in Y$, $T_y^*\nabla$ is a connection on $T_y^*E \xrightarrow{\sim} E$ and $T_0^*\nabla = \nabla$. The map

$$Y \to H^0(Y, \Omega^1_Y \otimes \mathcal{E}nd(E)), \quad y \mapsto T_y^*\nabla - \nabla$$

is holomorphic. It is constantly 0 since Y is compact and $H^0(Y, \Omega^1_Y \otimes \mathcal{E}nd(E))$ is a finite-dimensional vector space (Cartan-Serre's theorem). Hence $T_y^*(E,\nabla) = (E,\nabla)$ for all $y \in Y$. □

Lemma 3.3.2 ([Rot96, Lem. 3.1])**.** *Let F be a* coherent *module with a 1-splitting on the complex torus X, then F is finitely supported.*

Proof. Suppose to the contrary that $\mathrm{Supp}(F)$ is infinite. By [GR84, p.76], $\mathrm{Supp}(F)$ is an analytic set in X. Then $\dim \mathrm{Supp}(F) \geq 1$. Let C be an irreducible component of $\mathrm{Supp}(F)$ of maximal dimension. Write $i : C \to X$ for the inclusion. Take a morphism $h : Z \to X$ provided by [Liu23a, Lem. 5.3.3]. Then $h(Z) = C$ and $F'' := F'/T(F')$ is a *vector bundle* on Z of positive rank r, where $F' = h^*F$ and $T(*)$ denotes the torsion part of a sheaf of modules. In consequence, the morphism of complex tori $h^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Z)$ is nonzero. However, we claim that its tangent map at origin $h^* : \mathfrak{g} \to H^1(Z, O_Z)$ is zero.

Let $\mathcal{E}' = h^*\mathcal{E}$. Because O_X is flat over itself, pulling back (9) to Y and tensoring with F'', by [Sta23, Tag 05NJ] we get a short exact sequence

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} F'' \to \mathcal{E}' \otimes_{O_Z} F'' \to F'' \to 0. \tag{23}$$

Since $\mathcal{E}'$ is a vector bundle on Z, one has

$$\frac{\mathcal{E}' \otimes F'}{T(\mathcal{E}' \otimes F')} = \mathcal{E}' \otimes F''.$$

Then the splitting on F induces a splitting $F'' \xrightarrow{\psi'} \mathcal{E}' \otimes F''$ of (23). Let β be the natural morphism $\beta : O_Z \to \mathcal{E}nd(F'')$. By Lemma 2.1.3, the composition

$$\mathrm{End}(g) \xrightarrow{\mathrm{Id}_{\mathfrak{g}^*} \otimes h^*} \mathfrak{g}^* \otimes_{\mathbb{C}} H^1(Z, O_Z) \xrightarrow{\mathrm{Id}_{\mathfrak{g}^*} \otimes H^1(Z,\beta)} \mathfrak{g}^* \otimes_{\mathbb{C}} H^1(Z, \mathcal{E}nd(F''))$$

sends $1 \in \mathrm{End}(\mathfrak{g})$ to 0. Therefore, the map $H^1(Z,\beta)h^* : \mathfrak{g} \to H^1(Z, \mathcal{E}nd(F''))$ is zero. Taking trace, we get a morphism $\tau : \mathcal{E}nd(F'') \to O_Z$ with $\tau\beta = r \cdot \mathrm{Id}_{O_Z}$. Then $h^* = \frac{1}{r}\tau_* H^1(Z,\beta)h^* = 0$ as a map $\mathfrak{g} \to H^1(Z, O_Z)$. The claim follows. The claim gives a contradiction. □

Corollary 3.3.3. *Every local system (of finite dimensional $\mathbb{C}$-vector spaces) on a complex torus is translation invariant.*

Proof. Let L be a local system on Y. By Theorem 3.3.1, the pair $(L \otimes_{\mathbb{C}} O_Y, \mathrm{Id}_L \otimes d)$ is translation invariant. The result follows from the Riemann-Hilbert correspondence [Del70, I, Thm. 2.17]. $\square$

4 Laumon-Rothstein sheaf of algebras

4.1 Construction

To lift the Fourier-Mukai transform to D-modules, we recall (in Definition 4.1.1) the sheaf $\mathcal{A}_X$ from [Rot96, p.576]. In the notation of (9), fix a $\mathbb{C}$-basis $\{\omega^1, \ldots, \omega^g\}$ of the $\mathbb{C}$-vector space

$$H^0(Y, \Omega^1_Y) = \mathfrak{g}^* = \Gamma(X, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) \subset \Gamma(X, \mathcal{E}).$$

For each Stein open subset $U \subset X$, by Cartan's Theorem B (see, *e.g.*, [KK11, Sec. 52, Thm. B]) one has $H^1(U, \mathfrak{g}^* \otimes_{\mathbb{C}} O_X) = 0$. Thence (9) induces a short exact sequence

$$0 \to \mathfrak{g}^* \otimes_{\mathbb{C}} O_X(U) \to \mathcal{E}(U) \xrightarrow{\mu} O_X(U) \to 0.$$

Whence, there is $\rho \in \mathcal{E}(U)$ with $\mu(\rho) = 1 \in O_X(U)$. For two such pairs (U, ρ) and $(\tilde{U}, \tilde{\rho})$ with $U \cap \tilde{U} \neq \emptyset$, one has $\mu(\tilde{\rho} - \rho) = 0 \in O_X(U \cap \tilde{U})$, so $\tilde{\rho} - \rho \in \mathfrak{g}^* \otimes_{\mathbb{C}} O_X(U \cap \tilde{U})$. There exists a unique tuple $f_1, \ldots, f_g \in O_X(U \cap \tilde{U})$ such that

$$\tilde{\rho} - \rho = \sum_{i=1}^{g} \omega^i \otimes f_i$$

in $\mathcal{E}(U \cap \tilde{U})$.

Definition 4.1.1. For each chosen pair (U, ρ) as above, introduce independent variables $x^\rho_1, \ldots, x^g_\rho$ and put

$$\mathcal{A}_X|_U = O_U[x^\rho_1, \ldots, x^\rho_g].$$

For another choice $(\tilde{U}, \tilde{\rho})$ with the tuple $(f_1, \ldots, f_g)$ as above, we glue $\mathcal{A}_X|_U$ and $\mathcal{A}_X|_{\tilde{U}}$ by the rule

$$x^\rho_i - x^{\tilde{\rho}}_i|_{U \cap \tilde{U}} = f_i. \tag{24}$$

The resulting sheaf $\mathcal{A}_X$ is a sheaf of commutative O_X-algebra.

Let

$$0 \to \mathfrak{g}^* \to X^\natural \xrightarrow{\pi} X \to 0 \tag{25}$$

be the universal vectorial extension of X constructed in [Liu23b, (22)]. In coordinate-free terms, $\mathcal{A}_X$ is the O_X-subalgebra of $\pi_* O_{X^\natural}$ of sections whose restriction to each fiber of π is a polynomial on $\mathfrak{g}^*$. For every integer $m \geq 0$, let

$O_{X^\natural}(m) \subset O_{X^\natural}$ denote the subsheaf of sections whose restriction to the fibers of π are homogeneous polynomials of degree m. Similar to [Bjö93, Def 1.6.1], there exists a sheaf of graded rings $O_{[X^\natural]} := \oplus_{m\geq 0} O_{X^\natural}(m) (\subset O_{X^\natural})$ on $X^\natural$. Then $\mathcal{A}_X = \pi_* O_{[X^\natural]}$ and $\Gamma(X, \mathcal{A}_X) = \mathbb{C}$.

Remark 4.1.2. Unlike the analytic case, if X is an abelian variety, then the notation $\mathcal{A}_X$ in [Rot96, p.576] is the *algebraic* direct image $\pi_* O_{X^\natural}$. Morally, such difference also lies between algebraic and analytic D-modules. For a complex manifold or a smooth algebraic variety V, let $p : T^*V \to V$ be the natural projection of the cotangent bundle. Denote by GD_V the associated graded ring of the degree filtration on D_V. Then $GD_V = p_* O_{T^*V}$ in the algebraic case ([HT07, p.57]). By contrast, in the analytic case, GD_V is the O_V-submodule of $p_* O_{T^*V}$ of sections whose restriction to each fiber of p is a polynomial.

Remark 4.1.3. The sheaf of rings $\mathcal{A}_X$ is functorial in X in the following sense. Let $\phi : X' \to X$ be a morphism of complex tori. Let $\hat{\phi} : Y \to Y'$ be the morphism dual to ϕ. By [Liu23b, Prop. 5.4.7], it induces a morphism $\phi^\natural : X'^\natural \to X^\natural$ of complex Lie groups fitting into a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^0(Y', \Omega^1_{Y'}) & \longrightarrow & X'^\natural & \xrightarrow{\pi'} & X' & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \hat{\phi}^*} & & \downarrow{\scriptstyle \phi^\natural} & & \downarrow{\scriptstyle \phi} & & \\ 0 & \longrightarrow & H^0(Y, \Omega^1_Y) & \longrightarrow & X^\natural & \xrightarrow{\pi} & X & \longrightarrow & 0. \end{array}$$

For each local section of $O_{[X^\natural]}$, its $\phi^\natural$-pullback (a local section of $O_{X'^\natural}$) restricts to a polynomial on each fiber of π'. Indeed, this restriction is the $\hat{\phi}^*$-pullback of a restriction to a fiber of π. Therefore, the natural morphism $O_{X^\natural} \to \phi^\natural_* O_{X'^\natural}$ restricts to a morphism $O_{[X^\natural]} \to \phi^\natural_* O_{[X'^\natural]}$. The resulting morphism of ringed spaces $(X'^\natural, O_{[X'^\natural]}) \to (X^\natural, O_{[X^\natural]})$ descends to another morphism of ringed spaces

$$\tilde{\phi} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X), \tag{26}$$

which is compatible with ϕ. In particular, the following square

$$\begin{array}{ccc} D(\mathcal{A}_{X'}) & \xrightarrow{R\tilde{\phi}_*} & D(\mathcal{A}_X) \\ \downarrow & & \downarrow \\ D(O_{X'}) & \xrightarrow{R\phi_*} & D(O_X) \end{array} \tag{27}$$

is commutative, where the vertical functors are forgetful. If M is an O_X-module, then

$$L\tilde{\phi}^*(\mathcal{A}_X \otimes_{O_X} M) = \mathcal{A}_{X'} \otimes_{O_{X'}} L\phi^* M. \tag{28}$$

4.2 Basic properties

Notice that $\mathcal{A}_X$ has a natural degree filtration $\{\mathcal{A}_X(m)\}_{m\in\mathbb{Z}}$, where

$$\mathcal{A}_X(m) = \pi_*(\oplus_{j=0}^m O_{X^\natural}(j))$$

is the O_X-submodule of $\mathcal{A}_X$ of polynomials of degree at most m. See also [Rot96, Sec. 5.3] and the end of [Lau96, p.10]. Then $\mathcal{A}_X(0) = O_X$, $\mathcal{A}_X(1) = \mathcal{E}^\vee$ (*cf.* the start of [Lau96, p.10]), and every $\mathcal{A}_X(m)$ is a locally free O_X-module of finite rank. Moreover, for any integers $m, n \geq 0$, one has

$$\mathcal{A}_X(n)\mathcal{A}_X(m) = \mathcal{A}_X(n+m). \tag{29}$$

Thus, $\mathcal{A}_X$ is a sheaf of positively filtered rings (in the sense of [Bjö93, p.459; p.464]) on the complex torus X.

We review some terminology from [Bjö93, A:III]. A coherent sheaf of rings on a locally compact Hausdorff space is called noetherian if every increasing sequence of ideal sheaves is stationary over relatively compact subsets ([Bjö93, 2.24, p.470]). Let R be a commutative filtered ring. If the subring $\oplus_{v\in\mathbb{Z}} R_v T^v$ of $R[T, T^{-1}]$ is a noetherian ring, then R is called a *noetherian filtered ring*.

Definition 4.2.1 ([Bjö93, A.III, 1.7; Def. 1.11; 1.19])**.** A filtration on an R-module M is a family of additive subgroups $\{M_v\}_{v\in\mathbb{Z}}$ such that

$$M_v \subset M_{v+1}; \quad R_k M_v \subset M_{k+v}; \quad \cup_v M_v = M.$$

This filtration is called *separated* if $\cap_{v\in\mathbb{Z}} M_v = 0$, and called *good* if $\oplus_{v\in\mathbb{Z}} M_v T^v$ is a finitely generated $\oplus_{v\in\mathbb{Z}} R_v T^v$-module.

A *zariskian filtered ring* is a noetherian filtered ring such that all the good filtrations on every finitely generated module are separated. A filtered sheaf of rings is called *stalkwise zariskian* if every stalk is a zariskian filtered ring ([Bjö93, Def. 2.6, p.465]).

Lemma 4.2.2. *The sheaf of rings $\mathcal{A}_X$ is coherent and noetherian. The sheaf of filtered rings $\mathcal{A}_X$ is stalkwise zariskian.*

Proof. By (24), the graded ring associated to the degree filtration of $\mathcal{A}_X$ is

$$G\mathcal{A}_X := \oplus_{m\geq 0}\mathcal{A}_X(m)/\mathcal{A}_X(m-1) = \mathrm{Sym}(\mathfrak{g}) \otimes_{\mathbb{C}} O_X = O_X[x_1, \ldots, x_g]. \tag{30}$$

Here for each chosen pair (U, ρ) as above, $x_i|_U \in \Gamma(U, \mathcal{A}_X(1)/\mathcal{A}_X(0)) \subset \Gamma(U, G\mathcal{A}_X)$ is the image of $x_i^\rho \in \Gamma(U, \mathcal{A}_X(1))$. From [Bjö79, Thm. 1.26, p.460], $\mathcal{A}_X$ is stalkwise zariskian. The other part follows from [Bjö79, Prop. 1.27, p.460; Thm. 2.7, p.465]. (See also the proof of [Bjö93, Thm. 1.2.5].) □

In view of the difference mentioned in Remark 4.1.2, the statement of [Rot96, Prop. 4.4] is slightly modified as Fact 4.2.3. For every $\mathcal{A}_X$-module F and every chosen pair (U, ρ) as above, define $\psi_U^\rho : F(U) \to \mathcal{E}(U) \otimes_{O_X(U)} F(U)$ by

$$\psi_U^\rho(s) = \rho \otimes s + \sum_{i=1}^{g} \omega^i|_U \otimes (x_i^\rho s).$$

Then $(\mu_1 \otimes \mathrm{Id}_F)(\psi_U^\rho(s)) = s$. In light of (24), the family $\{\psi_U^\rho\}_{(U,\rho)}$ glue to a 1-splitting ψ on F. By the commutativity of $\mathcal{A}_X$ and [Rot96, (4.9)], one has $[\psi, \psi] = 0$.

Fact 4.2.3. *The resulting functor* $\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X)_{\mathrm{sp}}, \quad F \mapsto (F, \psi)$ *induces an equivalence from* $\mathrm{Mod}(\mathcal{A}_X)$ *to the full subcategory of* $\mathrm{Mod}(O_X)_{\mathrm{sp}}$ *comprised of objects* (F, ψ) *with* $[\psi, \psi] = 0$*.*

From Fact 4.2.3 and the proof of [Rot96, Prop. 4.1], the functor (13) restricts to an exact functor $p_X^* : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}}$. Similarly by [Rot96, Prop. 4.2], the functor (15) restricts to a functor

$$p_{X*} : \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}} \to \mathrm{Mod}(\mathcal{A}_X). \tag{31}$$

5 Laumon-Rothstein transform

5.1 Construction and properties

Definition 5.1.1. Define functors

$$RS_1 = Rp_{Y*}(\mathcal{P} \otimes^L_{O_{X\times Y}} p_X^* \cdot) : D(\mathcal{A}_X) \to D(D_Y); \tag{32}$$

$$RS_2 = Rp_{X*}(\mathcal{P}^{-1} \otimes^L_{O_{X\times Y}} p_Y^* \cdot) : D(D_Y) \to D(\mathcal{A}_X), \tag{33}$$

where $Rp_{Y*} : D(\mathrm{MIC}(p_X)) \to D(D_Y)$ (resp. $Rp_{X*} : D(\mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}}) \to D(\mathcal{A}_X)$) is the right derived functor of (20) (resp. (31)). The pair is called the *Laumon-Rothstein transform*.

The situation is depicted below.

$$\begin{array}{ccc} \mathrm{Mod}(\mathcal{A}_X) & \xrightarrow{H^0 RS_1} & \mathrm{Mod}(D_Y) \\ \downarrow{\scriptstyle p_X^*} & & \uparrow{\scriptstyle p_{Y*}} \\ \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}} & \xrightarrow{\mathcal{P}\otimes\cdot} & \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn,fl}}; \end{array} \qquad \begin{array}{ccc} \mathrm{Mod}(D_Y) & \xrightarrow{H^0 RS_2} & \mathrm{Mod}(\mathcal{A}_X) \\ \downarrow{\scriptstyle p_Y^*} & & \uparrow{\scriptstyle p_{X*}} \\ \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn,fl}} & \xrightarrow{\mathcal{P}^{-1}\otimes\cdot} & \mathrm{Mod}(O_{X\times Y})_{1-\mathrm{cxn,fl}}. \end{array}$$

Proposition 5.1.2. *There are commutative squares*

$$\begin{array}{ccc} D(\mathcal{A}_X) & \xrightarrow{RS_1} & D(D_Y) \\ \downarrow & & \downarrow \\ D(O_X) & \xrightarrow{R\mathscr{S}_1} & D(O_Y); \end{array} \qquad \begin{array}{ccc} D(D_Y) & \xrightarrow{RS_2} & D(\mathcal{A}_X) \\ \downarrow & & \downarrow \\ D(O_Y) & \xrightarrow{R\mathscr{S}_2} & D(O_X), \end{array}$$

where the vertical functors are forgetful. In particular, RS_1 *(resp.* RS_2*) sends* $D_{O-\mathrm{good}}(\mathcal{A}_X)$ *(resp.* $D_{O-\mathrm{good}}(D_Y)$*) to* $D_{O-\mathrm{good}}(D_Y)$ *(resp.* $D_{O-\mathrm{good}}(\mathcal{A}_X)$*).*

Proof. The proof is similar to that of Proposition 3.1.2, as $\mathcal{A}_X$ (resp. D_Y) is flat over O_X (resp. O_Y). □

With Proposition 5.1.2, the proof of Theorem 5.1.3 is similar to that of Theorem 3.2.1.

Theorem 5.1.3 (Laumon, Rothstein)**.** *There are natural isomorphisms of functors* $RS_1 RS_2 \cong T^{-g}$ *on* $D_{O-\mathrm{good}}(D_Y)$ *and* $RS_2 RS_1 \cong T^{-g}$ *on* $D_{O-\mathrm{good}}(\mathcal{A}_X)$*.*

Proposition 5.1.4 follows from Proposition 5.1.2, Theorem 5.1.3 and Fact 1.1.1 1 as in the proof of [Rot96, Thm. 6.1], *cf.* [Lau96, Prop. 3.1.2; Cor. 3.2.4].

Proposition 5.1.4. *There are natural isomorphisms of functors*

$$RS_2(D_Y \otimes^L_{O_Y} \cdot) \cong \mathcal{A}_X \otimes^L_{O_X} R\mathscr{S}_2(\cdot) : D_{\text{good}}(O_Y) \to D_{O-\text{good}}(\mathcal{A}_X);$$
$$RS_1(\mathcal{A}_X \otimes^L_{O_X} \cdot) \cong D_Y \otimes^L_{O_Y} R\mathscr{S}_1(\cdot) : D_{\text{good}}(O_X) \to D_{O-\text{good}}(D_Y).$$

For $x \in X$ (resp. $y \in Y$), let $P_x = \mathcal{P}|_{x \times Y}$ (resp. $P_y = \mathcal{P}|_{X \times y}$) be the pullback line bundle on Y (resp. X). For a closed analytic subset S of a complex manifold Z, [Kas03, (3.30), p.51] defines a D_Z-module $\mathcal{B}_{S|Z}$.

Corollary 5.1.5. *For any $x \in X$ and $y \in Y$, one has*

$$RS_2(D_Y \otimes_{O_Y} \mathbb{C}_y) = \mathcal{A}_X \otimes_{O_X} P_{-y};$$
$$T^g RS_1(\mathcal{A}_X \otimes_{O_X} P_{-y}) = D_Y \otimes_{O_Y} \mathbb{C}_y = i_{y+}\mathbb{C} = \mathcal{B}_{\{y\}|Y};$$
$$RS_1(\mathcal{A}_X \otimes_{O_X} \mathbb{C}_x) = D_Y \otimes_{O_Y} P_x;$$
$$T^g RS_2(D_Y \otimes_{O_Y} P_x) = \mathcal{A}_X \otimes_{O_X} \mathbb{C}_x.$$

Proof. By [HT07, Example 1.6.4], one has $D_Y \otimes_{O_Y} \mathbb{C}_y = \mathcal{B}_{\{y\}|Y}$. The result follows from Theorem 5.1.3, Proposition 5.1.4, Fact 1.2.1 and [Liu23a, Lem. 2.0.8]. □

5.2 Matsushima-Morimoto theorem

Proposition 5.2.1, due to Matsushima [Mat59, Thm. 1] and Morimoto [Mor59, Thm. 2], is a converse to Theorem 3.3.1. For abelian varieties, Nakayashiki [Nak94, Prop. 5.9] gives a proof using the Fourier-Mukai transform.

Proposition 5.2.1. *A homogeneous vector bundle on a complex torus admits an integrable connection.*

Proof. Let $E \to Y$ be a homogeneous vector bundle. Set $\hat{E} = H^g R\mathscr{S}_2(E)$. According to [Liu23a, Prop. 5.3.2] and Fact 1.1.1, one has $E = H^0 R\mathscr{S}_1(\hat{E})$ and $\mathrm{Supp}(\hat{E})$ is finite. By Lemma 5.2.2, $\hat{E}$ has an $\mathcal{A}_X$-module structure. By Proposition 5.1.2, the O_Y-module underlying $H^0 RS_1(\hat{E})$ is E. The D_Y-module $H^0 RS_1(\hat{E})$ carries naturally an integrable connection. □

The proof of Proposition 5.2.1 needs Lemma 5.2.2, a converse to Lemma 3.3.2.

Lemma 5.2.2. *If F is an O_X-module with finite support on the complex torus X, then F admits a 1-splitting ψ with $[\psi, \psi] = 0$.*

Proof. There is a decomposition $F = \oplus_{i=1}^m F_i$, where $\mathrm{Supp}(F_i)$ is a singleton for each i. Thus, one may assume that $\mathrm{Supp}(F)$ is a singleton. Then there exists an open neighborhood $U \subset X$ of $\mathrm{Supp}(F)$ and a morphism of complex manifolds $s : U \to X^\natural$ that is a local section to (25). Let $\iota : U \to X$ be the inclusion. Applying π_* to the morphism of sheaves of rings $O_{X^\natural} \to s_*O_U$, one gets a morphism $\pi_*O_{X^\natural} \to \iota_*O_U$. As $\mathcal{A}_X$ is an O_X-subalgebra of $\pi_*O_{X^\natural}$, this endows ι_*O_U an $\mathcal{A}_X$-module structure.[1] Since the canonical O_X-morphism $\mathrm{Id}_F \otimes \iota^\# : F \to F \otimes_{O_X} \iota_*U$ is an isomorphism, F also obtains an $\mathcal{A}_X$-module structure. This induces such a splitting by Fact 4.2.3. □

Proposition 5.2.1, together with Theorem 3.3.1, yields (a slight generalization of) Morimoto's theorem [Mor59, Thm. 2, p.91].

Corollary 5.2.3 (Morimoto)**.** *A coherent module admitting a connection on a complex torus is a vector bundle admitting an integrable connection.*

6 Good modules

6.1 Definition

We define good $\mathcal{A}_X$-modules. We also review several definitions of good D-modules in the literature, and show that they are equivalent.

Let Z be a complex manifold.

Definition 6.1.1. [Bjö93, 2.5, p.465] Let $\mathcal{R}$ be a positively filtered sheaf of rings on Z such that the associated graded ring $G\mathcal{R}$ is coherent. Let M be a coherent left $\mathcal{R}$-module. A filtration on M is an increasing sequence of subsheaves $\{M_v\}_{v\in\mathbb{Z}}$ satisfying $\cup_{v\in\mathbb{Z}} M_v = M$ and $\mathcal{R}_k M_v \subset M_{k+v}$ for all integers $k \geq 0$ and v. This filtration is called

- *B-good* ([Bjö93, Remark 2.16, p.467]) if for every $x \in Z$, there exists an open neighborhood U, a finite set $\{m_1, \ldots, m_s\} \subset \Gamma(U, M)$ and integers $k_1, \ldots, k_s$ such that $M_v|_U = \sum_{i=1}^s \mathcal{R}_{v-k_i} m_i$ for all integers v.
- *locally good* ([Meb89, Prop. 2.1.12 (i)]) if every M_v is coherent over O_Z, and if for every $x \in Z$, there is an open neighborhood U of x and an integer $k_0 \geq 0$ such that $\mathcal{R}_m M_{k_0} = M_{m+k_0}$ on U for all integers $m \geq 0$.

The proof of Lemma 6.1.2 is similar to that of [HT07, Prop. 2.1.1; Def. 2.1.2].

Lemma 6.1.2. *Let $M. = (M_v)_{v\in\mathbb{Z}}$ be a filtration on a coherent $\mathcal{A}_X$-module M. Then $M.$ is B-good if and only if $M.$ is locally good. (In that case, we call $M.$ a good filtration on M.)*

Proof.
- Assume that $M.$ is B-good. By Lemma 4.2.2 and [Bjö93, Thm. 2.17, p.467], the $G\mathcal{A}_X$-module $\oplus_{v\in\mathbb{Z}} M_v/M_{v-1}$ is coherent. Because of (30) and the proof of [Bjö93, Prop. 1.4.5], for every integer v, the O_X-module M_v/M_{v-1} is coherent. From [Bjö93, Prop. 2.23, p.470], the filtration $M.$ is locally bounded blow. Then by induction on $v \in \mathbb{Z}$, one proves that the O_X-module M_v is coherent.

[1] This example shows that Lemma 3.3.2 fails without coherent condition.

For every $x \in X$, by definition, there is an open neighborhood $U \subset X$ of x, sections $m_1, \dots, m_s \in \Gamma(U, M)$ and integers $k_1, \dots, k_s$ such that $M_v|_U = \sum_{i=1}^s \mathcal{A}_X(v - k_i)m_i$ for all integers v. Put $k_0 = \max_{j=1}^s k_j$. For every integer $k \geq 0$, one has $\mathcal{A}_X(k)M_{k_0} \subset M_{k+k_0}$. Moreover,

$$M_{k+k_0}|_U = \sum_{i=1}^s \mathcal{A}_X(k+k_0-k_i)m_i \overset{\text{(a)}}{\subset} \sum_{i=1}^s \mathcal{A}_X(k)\mathcal{A}_X(k_0-k_i)m_i \subset \mathcal{A}_X(k)M_{k_0},$$

where (a) uses (29). Hence $\mathcal{A}_X(k)M_{k_0} = M_{k+k_0}$ on U.

- Conversely, assume that $M.$ is locally good. For a fixed $x \in X$, take U and k_0 provided by the definition of local goodness. Since M_{k_0} is coherent over O_X, by shrinking U, one may assume that the O_U-module $M_{k_0}|_U$ is generated by sections $s_1, \dots, s_m \in \Gamma(U, M_{k_0})$. Define a morphism of $\mathcal{A}_X$-modules $\phi : \mathcal{A}_X^m|_U \to M|_U, \quad (f_1, \dots, f_m) \mapsto \sum_{j=1}^m f_j s_j$. Since $M.$ is a filtration, for every integer v, one has $\mathcal{A}_X(v - k_0)M_{k_0} \subset M_v$. Hence $\phi(\mathcal{A}_X(v - k_0)^m) \subset M_v$. By construction, one has $\phi(\mathcal{A}_X(0)^m) = M_{k_0}|_U$. For every integer $k \geq k_0$, on U one has

$$M_k = \mathcal{A}_X(k - k_0)M_{k_0} = \mathcal{A}_X(k - k_0)\phi(\mathcal{A}_X(0)^m) \subset \phi(\mathcal{A}_X(k - k_0)^m).$$

  Therefore, the filtration $M.$ is B-good.

□

From [HT07, Thm. 2.1.3 (i)], a coherent D_V-module on a smooth algebraic variety V admits a globally defined good filtration. By contrast, Malgrange [Mal04, p.405] gives a coherent D-module on the complex manifold $\mathbb{C}^* \times \mathbb{CP}^1$ that does not admit any global good filtration.

Definition 6.1.3. An O_Z-module F is called

- *countably quasi-good* ([KS97, p.942]) if every compact subset of Z has an open neighborhood U such that $F|_U$ is the union of an increasing sequence of coherent O_U-submodules.
- *quasi-good* ([KS16, p.12]) if for every relatively compact open subset $U \subset Z$, the restriction $F|_U$ is a sum of coherent O_U-submodules.

A D_Z-module M is called

- *good coherent* if for every relatively compact open subset U of Z, there is a finite filtration $\{M_k\}_{k \in \mathbb{Z}}$ of $M|_U$ such that each quotient M_k/M_{k-1} is a coherent D_U-modules admitting a good filtration. ([Sai89, p.369], [SS94, p.10] and [KS96, p.43].)
- *S-quasi-good* ([KS96, p.43]) if for every relatively compact open subset $U \subset Z$, the restriction $M|_U$ admits a filtration $\{M_v\}_{v \in \mathbb{Z}}$ by coherent D_U-submodule such that each quotient M_v/M_{v-1} admits a good filtration and $M_v = 0$ for $v \ll 0$.

Proposition 6.1.4. *Let M be a coherent D_Z-module. Then the following are equivalent.*

1. *For every relatively compact open subset U of Z, there is a coherent O_U-submodule $F \subset M|_U$ with $D_U \cdot F = M|_U$.*
2. *For every relatively compact open subset U of Z, the D_U-module $M|_U$ admits a good filtration.*
3. *The D_Z-module M is good coherent.*
4. *The D_Z-module M is S-quasi-good.*
5. *The O_Z-module M is countably quasi-good.*
6. *The O_Z-module M is good.*
7. *The O_Z-module M is quasi-good.*

Proof. We follow the circular chain.

1 implies 2 See [Bjö93, 1.4.10].

2 implies 3 For every relatively compact open subset U of Z, define a finite filtration of $M|_U$ by $M_0 = 0$ and $M_1 = M|_U$. Then the graded piece M_1/M_0 admits a good filtration over U.

3 implies 4 For every relatively compact open subset U of Z, consider the filtration $\{M_k\}$ in the definition. By induction on k, one proves that each M_k is D_U-coherent.

4 implies 5 Every quotient M_v/M_{v-1} admits a good filtration, then by [Bjö93, Cor. 1.4.6], it is countably quasi-good. By induction on v and using [KS97, Lem. 2.1.1], one proves that every M_v is countably quasi-good. Therefore, for every integer v, there is an increasing sequence $\{M_v^k\}_{k\geq 1}$ of coherent O_U-submodules of M_v with $M_v = \cup_{k\geq 1} M_v^k$. For every integer $k \geq 1$, let $M^k := \sum_{i\leq k, v\leq k} M_v^i$. By [Sta23, Tag 01BY], M^k is a coherent O_U-submodule of M_k. Then

$$M = \cup_{v\in\mathbb{Z}} M_v = \cup_{v\in\mathbb{Z}} \cup_{i\geq 1} M_v^i = \cup_{k\geq 1} M^k,$$

so M is countably quasi-good.

5 implies 6 An increasing sequence forms a directed family.

6 implies 7 By definition.

7 implies 1 Let U be a relatively compact open subset of Z. Because M is a finite type D_Z-module, for every $x \in \bar{U}$, there is a relatively compact open neighborhood $U(x) \subset Z$ of x, an integer $n(x) \geq 1$ and sections

$$\{s_i^x\}_{1\leq i\leq n(x)} \subset \Gamma(U(x), M)$$

generating the $D_{U(x)}$-module $M|_{U(x)}$. By compactness of $\bar{U}$, the open cover $\{U(x)\}_{x\in\bar{U}}$ of $\bar{U}$ has a finite subcover $\{U(x_j)\}_{1\le j\le r}$. Then $V = \cup_{j=1}^r U(x_j)$ is a relatively compact open subset of Z containing U. By Condition 7, one may write $M|_V = \sum_{\alpha\in I} G_\alpha$, where I is an index set, and each G_α is a coherent O_V-submodule of $M|_V$.

For every $x \in \bar{U}$, there is an open neighborhood $V(x) \subset U(x)$ of x, such that for each $1 \le i \le n(x)$, the restriction $s_i^x|_{V(x)} \in \Gamma(V(x), G_{\alpha(x,i)})$ for some index $\alpha(x,i) \in I$. By compactness of $\bar{U}$ again, the open cover $\{V(x)\}_{x\in\bar{U}}$ has a finite subcover $\{V(x'_k)\}_{1\le k\le m}$. Then

$$F := \sum_{1\le k\le m, 1\le i\le n(x'_k)} G_{\alpha(x'_k,i)}$$

is a finite type O_V-submodule of $M|_V$. By Lemma 6.2.7, it is coherent over O_V. Moreover, $D_U \cdot F|_U = M|_U$.

□

The proof of Proposition 6.1.5 is similar to that of Proposition 6.1.4.

Proposition 6.1.5. *Let M be a coherent $\mathcal{A}_X$-module on the complex torus X. Then the O_X-module M is good if and only if there is a coherent O_X-submodule $F \subset M$ with $\mathcal{A}_X \cdot F = M$.*

Let the sheaf of rings $\mathcal{R}$ be either D_Z or $\mathcal{A}_X$ on the fixed complex torus X.

Definition 6.1.6. [Kas03, Def. 4.24] A coherent $\mathcal{R}$-module is *good* if the underlying O-module is good.

For example, by Lemma 4.2.2 and [Bjö93, Thm. 1.2.5], the left $\mathcal{R}$-module $\mathcal{R}$ is good. Let $\mathrm{Good}(\mathcal{R}) \subset \mathrm{Coh}(\mathcal{R})$ (resp. $D^b_{\mathrm{good}}(\mathcal{R}) \subset D^b_{O-\mathrm{good}}(\mathcal{R})$) be the full subcategory of good $\mathcal{R}$-modules (resp. objects whose cohomologies are good $\mathcal{R}$-modules). By Proposition 6.1.4, the category $D^b_{\mathrm{good}}(D_Z)$ is what Björk denotes by $D^b_{\mathrm{coh}}(D_Z)_f$ in [Bjö93, p.119].

A coherent D_Z-module is called *holonomic* if its characteristic variety is of (minimal) dimension $\dim Z$ ([Bjö93, Def. 3.1.1]). Malgrange ([Mal94, p.35], [Mal96, p.367], see also [Sab11, Thm. 4.3.4 (2)]) claims to have proved that every holonomic D_Z-module is generated by a coherent O_Z-submodule, so it is a good D_Z-module. Let $D^b_h(D_Z) \subset D^b(D_Z)$ be the full subcategory of objects with holonomic cohomologies.

6.2 Basic properties

Let $\mathcal{R}$ be either D_Z on a complex manifold Z or $\mathcal{A}_X$ on the fixed complex torus X.

Lemma 6.2.1 (Induced modules)**.** *The functor $\mathcal{R}\otimes_{O_Z} \cdot : \mathrm{Mod}(O_Z) \to \mathrm{Mod}(\mathcal{R})$ is exact. It restricts to a functor $\mathcal{R}\otimes_{O_Z} \cdot : \mathrm{Coh}(Z) \to \mathrm{Good}(\mathcal{R})$, and induces a t-exact functor $\mathcal{R}\otimes^L_{O_Z} \cdot : D^b_c(O_Z) \to D^b_{\mathrm{good}}(\mathcal{R})$.*

Proof. As $\mathcal{R}$ is flat over O_Z, the functor is exact. Consider the degree filtration $\{\mathcal{R}(m)\}_{m\geq 0}$ of $\mathcal{R}$, where $\mathcal{R}(m) \subset \mathcal{R}$ is the O_Z-submodule of polynomials of degree at most m. Each $\mathcal{R}(m)$ is vector bundle on Z and $\mathcal{R} = \mathrm{colim}_m \mathcal{R}(m)$. Therefore, the O-module $\mathcal{R}$ is good. By [Liu23a, Prop. 3.1.5 2], for every coherent O_Z-module F, the O-module $\mathcal{R} \otimes_{O_Z} F$ is good. Because F is an O_Z-module of finite presentation, $\mathcal{R} \otimes_{O_Z} F$ is an $\mathcal{R}$-module of finite presentation. Then it is $\mathcal{R}$-coherent by [Bjö93, Thm. 1.2.5] and Lemma 4.2.2. The other part follows. □

Lemma 6.2.2. *The category* $\mathrm{Good}(\mathcal{R})$ *is a weak Serre subcategory of* $\mathrm{Mod}(\mathcal{R})$*. In particular,* $D^b_{\mathrm{good}}(\mathcal{R})$ *is a triangulated subcategory of* $D^b(\mathcal{R})$*.*

Proof. The first half is a combination of [Kas03, Prop. 4.23], [Sta23, Tag 01BY] and [Sta23, Tag 0754]. The second half follows from [Yek19, Prop. 7.4.5]. □

For a morphism of complex manifolds $f : M \to N$, the direct image of D-modules $f_+ : D(D_M) \to D(D_N)$ is constructed in [Bjö93, 2.3.12].

Fact 6.2.3 ([Bjö93, Thm. 2.8.1, 2.8.7])**.** *Let* $f : W \to Z$ *be a morphism of complex manifolds. For every* $M \in D^b_{\mathrm{good}}(D_W)$*, if* $f|_{\mathrm{Supp}(M)} : \mathrm{Supp}(M) \to Z$ *is proper, then* $f_+M \in D^b_{\mathrm{good}}(D_Z)$*.*

Lemma 6.2.4. *Let* $f : W \to Z$ *be a proper morphism of complex manifolds. Then the direct image functor* $f_+ : D(D_W) \to D(D_Z)$ *restricts to a functor* $D_{O-\mathrm{good}}(D_W) \to D_{O-\mathrm{good}}(D_Z)$*.*

Proof. Take $M \in D_{O-\mathrm{good}}(D_W)$. By [Sab11, Remark 3.3.4 (4)], the functor f_+ has finite cohomological dimension. So to prove $f_+M \in D_{O-\mathrm{good}}(D_Z)$, by [Har66, I, Prop. 7.3 (iii)], one may assume that $M \in \mathrm{Mod}(D_W)$. Define a morphism $i : W \to W \times Z, \quad w \mapsto (w, f(w))$, which is a closed embedding. Let $q : W \times Z \to Z$ be the projection. By [Sab11, Thm. 3.3.6 (1)], one has $f_+ = q_+i_+$. The restriction $q|_W : W \to Z$ is proper. By [Bjö93, Prop. 2.4.8], one has $f_+M = Rq_*DR_{W\times Z/Z}(i_+M)[\dim Z]$. As each term of the (relative) de Rham complex $DR_{W\times Z/Z}(i_+M)$ is $O_{W\times Z}$-good and supported on W, by [Liu23a, Thm. 3.1.6], $Rq_*[DR_{W\times Z/Z}(i_+M)] \in D_{\mathrm{good}}(O_Z)$. □

For a closed embedding $i : M \to N$ of complex manifolds, the inverse image $i^* : \mathrm{Mod}(D_N) \to \mathrm{Mod}(D_M)$ may not preserve D-coherence ([HT07, Rk. 1.5.10]). For smooth morphisms, Fact 6.2.5 can be proved by applying [Kas03, Thm. 4.7] or repeating the proof of [HT07, Prop. 1.5.13 (ii)].

Fact 6.2.5. *Let* $f : M \to N$ *be a smooth morphism of complex manifolds. Then* $Lf^* : D^b(D_N) \to D^b(D_M)$ *restricts to functors* $D^b_c(D_N) \to D^b_c(D_M)$ *and* $D^b_{\mathrm{good}}(D_N) \to D^b_{\mathrm{good}}(D_M)$*.*

Lemma 6.2.6 concerns the local existence of good filtrations on coherent $\mathcal{A}_X$-modules.

Lemma 6.2.6. *Let M be a coherent $\mathcal{A}_X$-module on the complex torus X. For every $x \in X$, there is an open neighborhood U of x and a positive good filtration on $M|_U$.*

Proof. Let $\mathcal{A}_X^q|_U \xrightarrow{\phi} \mathcal{A}_X^p|_U \xrightarrow{\epsilon} M|_U \to 0$ be a local presentation of M on a relatively compact open neighborhood U of x. For every integer v, set $M_v = \epsilon(\mathcal{A}_X(v)^p)$, which is an O_U-submodule of $M|_U$. Then $M_v = 0$ when $v < 0$. Moreover, $\cup_{v\in\mathbb{Z}} M_v = M|_U$ and for any integers $m, k \geq 0$, one has $\mathcal{A}_X(m)M_k \subset M_{k+m}$. Thus, $\{M_v\}_{v\in\mathbb{Z}}$ is a positive filtration of $M|_U$. For every integer $k \geq 0$, one has $\mathcal{A}_X(k)M_0 = M_k$. It remains to prove that M_k is coherent over O_U.

We claim that $\phi(\mathcal{A}_X(m)^q) \cap \mathcal{A}_X(k)^p$ is coherent over O_U. In fact, for every $y \in U$, there is an integer $s \geq \max(0, k-m)$ such that $\phi(\mathcal{A}_X(m)^q) \subset \mathcal{A}_X(m+s)^p$ near y. In side the coherent O_X-module $\mathcal{A}_X(m+s)^p$, the two O_X-submodules $\phi(\mathcal{A}_X(m)^q)$ and $\mathcal{A}_X(k)^p$ are finite type. By [Sta23, Tag 01BY], their intersection $\phi(\mathcal{A}_X(m)^q) \cap \mathcal{A}_X(k)^p$ is coherent near y. The claim is proved.

Because $\mathcal{A}_X(k)^p$ is a noetherian O_X-module, the increasing sequence of submodules $\{\phi(\mathcal{A}_X(m)^q) \cap \mathcal{A}_X(k)^p\}_{m\geq 0}$ is stationary on U. Therefore, the union $\phi(\mathcal{A}_X^q) \cap \mathcal{A}_X(k)^p = \ker(\epsilon) \cap \mathcal{A}_X(k)^p$ is coherent over O_U. Since the sequence

$$0 \to \ker(\epsilon) \cap \mathcal{A}_X(k)^p \to \mathcal{A}_X(k)^p \to M_k|_U \to 0$$

is exact in $\mathrm{Mod}(O_U)$, the restriction $M_k|_U$ is O_U-coherent. The constructed filtration is therefore good. □

When $\mathcal{R} = D_Z$, Lemma 6.2.7 is [Sab11, Exercise E.2.4 (4)]. On a complex manifold Z, an O_Z-module F is *pseudo-coherent* if for every open subset U of X, every finite type O_U-submodule of $F|_U$ is of finite presentation ([Kas03, Def. A.5]).

Lemma 6.2.7. *If M is a coherent $\mathcal{R}$-module, then M is pseudo-coherent over O_Z.*

Proof. Let $F \subset M$ be a finite type O-submodule. For every point x, by [Meb89, Prop. 2.1.9] (in the case $\mathcal{R} = D_Z$) and Lemma 6.2.6 (in the case $\mathcal{R} = \mathcal{A}_X$), there exists an open neighborhood U of x and a good filtration on $M|_U$. By [Bjö93, Cor. 1.4.6] (in the case $\mathcal{R} = D_Z$) and Lemma 6.1.2 (in the case $\mathcal{R} = \mathcal{A}_X$), $M|_U$ is the sum of an increasing sequence of coherent O_U-submodules. Hence $M|_U$ is good over O_U. By [Liu23a, Lem. A.4.2 1], the O_U-module $M|_U$ is pseudo-coherent. As pseudo-coherence is a local property, M is pseudo-coherent over O_Z. □

Lemma 6.2.8. *Let M be a good $\mathcal{R}$-module. Let N be a finite type $\mathcal{R}$-submodule of M. Then N is good over $\mathcal{R}$.*

Proof. By [Sta23, Tag 01BY (1)], N is coherent over $\mathcal{R}$. For every relatively compact open subset U of X and every $x \in \bar{U}$, there is an open neighborhood $U(x) \subset X$ of x, an integer $n(x) > 0$ and sections $\{s_i(x)\}_{i=1}^{n(x)} \subset \Gamma(U(x), N)$ generating the $\mathcal{R}|_{U(x)}$-module $N|_{U(x)}$. The open cover $\{U(x)\}_{x\in\bar{U}}$ of $\bar{U}$ has a finite subcover $\{U(x_j)\}_{j=1}^m$. Let N_0 be the O_U-submodule of $N|_U$ generated by the finitely many local sections

$$\{s_i(x_j)\}_{1\le j\le m, 1\le i\le n(x_j)}.$$

Then N_0 is a finite type O_U-module. Because $M|_U$ is good over $\mathcal{R}|_U$, by Lemma 6.2.7, the O_U-module N_0 is coherent. By construction, one has $\mathcal{R}|_U \cdot N_0 = N|_U$. Therefore, the $\mathcal{R}$-module N is good by Propositions 6.1.4 (in the case $\mathcal{R} = D_Z$) and 6.1.5 (in the case $\mathcal{R} = \mathcal{A}_X$). □

6.3 Preservation of goodness

Theorem 6.3.1. *The functor $RS_1 : D(\mathcal{A}_X) \to D(D_Y)$ restricts to an equivalence $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$, with a quasi-inverse $T^g RS_2 : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$.*

Proof. 1. For every coherent O_Y-module F, one has $RS_2(D_Y \otimes^L_{O_Y} F) \in D^b_{\mathrm{good}}(\mathcal{A}_X)$.

By Proposition 5.1.4, one has $RS_2(D_Y \otimes^L_{O_Y} F) = \mathcal{A}_X \otimes^L_{O_X} R\mathscr{S}_2(F)$. By Fact 1.2.1 2, one has $R\mathscr{S}_2(F) \in D^b_c(O_X)$. From Lemma 6.2.1, one gets $\mathcal{A}_X \otimes^L_{O_X} R\mathscr{S}_2(F) \in D^b_{\mathrm{good}}(\mathcal{A}_X)$.

2. For every $M \in \mathrm{Good}(D_Y)$ and every integer i, the $\mathcal{A}_X$-module $H^i RS_2(M)$ is good.

Descending induction on $i \in \mathbb{Z}$. The O_X-module underlying $H^i RS_2(M)$ is $H^i R\mathscr{S}_2(M)$. By Lemma 6.3.2, one has $H^i R\mathscr{S}_2(M) = 0$ when $i > 2g$. In particular, $H^i RS_2(M)$ is good over $\mathcal{A}_X$.

Assume the statement for $i + 1$. By Proposition 6.1.4, there is a coherent O_Y-submodule $F \subset M$ with $D_Y \cdot F = M$. Let M' be the kernel of the natural epimorphism $D_Y \otimes_{O_Y} F \to M$. Then

$$0 \to M' \to D_Y \otimes_{O_Y} F \to M \to 0 \tag{34}$$

is a short exact sequence in $\mathrm{Mod}(D_Y)$. By Lemma 6.2.1, the D_Y-module $D_Y \otimes_{O_Y} F$ is good. By Lemma 6.2.2, so is M'. From (34), one gets an exact sequence in $\mathrm{Mod}(\mathcal{A}_X)$

$$H^i RS_2(M') \to H^i RS_2(D_Y \otimes_{O_Y} F) \to H^i RS_2(M) \to H^{i+1} RS_2(M') \to H^{i+1} RS_2(D_Y \otimes_{O_Y} F). \tag{35}$$

By 1, the $\mathcal{A}_X$-module $H^j RS_2(D_Y \otimes_{O_Y} F)$ is good for $j \in \{i, i+1\}$. By the inductive hypothesis, so is $H^{i+1} RS_2(M')$.

Let $G = \ker[H^{i+1} RS_2(M') \to H^{i+1} RS_2(D_Y \otimes_{O_Y} F)]$. By Lemma 6.2.2, the $\mathcal{A}_X$-module G is good (hence of finite type). The sequence (35) yields an exact sequence

$$H^i RS_2(D_Y \otimes_{O_Y} F) \to H^i RS_2(M) \to G \to 0,$$

so $H^i RS_2(M)$ is a finite type $\mathcal{A}_X$-module for every coherent D_Y-module M. In particular, $H^i RS_2(M')$ is a finite type $\mathcal{A}_X$-module.

Let $N = \mathrm{im}(H^i RS_2(M') \to H^i RS_2(D_Y \otimes_{O_Y} F))$. It is a finite type $\mathcal{A}_X$-submodule of the good $\mathcal{A}_X$-module $H^i RS_2(D_Y \otimes_{O_Y} F)$. By Lemma 6.2.8, the $\mathcal{A}_X$-module N is a good. The sequence (35) yields an exact sequence

$$0 \to N \to H^i RS_2(D_Y \otimes_{O_Y} F) \to H^i RS_2(M) \to H^{i+1} RS_2(M') \to H^{i+1} RS_2(D_Y \otimes_{O_Y} F).$$

By Lemma 6.2.2, the $\mathcal{A}_X$-module $H^i RS_2(M)$ is good. The induction is completed.

From 2, Lemma 6.2.2 and [Har66, I, Prop. 7.3 (i)], the functor $R\mathscr{S}_2$ restricts to a functor $D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$. Similarly, using Proposition 6.1.5, one can prove that RS_1 restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y)$. By Theorem 5.1.3, the restrictions are equivalences. □

The proof of Theorem 6.3.1 needs a cohomological dimension estimation.

Lemma 6.3.2. *For an O_X-module F, we have $R\mathscr{S}_1(F) \in D^{[0,2g]}(O_Y)$. Similarly, for an O_Y-module G, we have $R\mathscr{S}_2(G) \in D^{[0,2g]}(O_X)$.*

Proof. By left exactness of the functor $p_{Y*} : \mathrm{Mod}(O_{X\times Y}) \to \mathrm{Mod}(O_Y)$, one has $R^i\mathscr{S}_1(F) = 0$ for every integer $i < 0$. For every $y \in Y$, let M be the restriction (as sheaves) of $\mathcal{P} \otimes_{O_{X\times Y}} p_X^* F$ to $X \times y$. For every integer j, by the proper base change theorem (see *e.g.*, [Mil13, Thm. 17.2]), one has $R^j\mathscr{S}_1(F)_y = H^j(X \times y, M)$. When $j > 2g$, by [KS13, Prop. 3.2.2 (iv)], one has $H^j(X \times y, M) = 0$. Therefore, $R^j\mathscr{S}_1(F) = 0$. The other part is similar. □

7 Relations with other functors

The properties [Muk81, (3.1), (3.4), (3.8)] of the Fourier-Mukai transform have analogs for the Laumon-Rothstein transform.

7.1 Exchange of translation and multiplication

For every $y \in Y$, we view P_y as an object of $\mathrm{Mod}(O_X)_{0-\mathrm{sp}}$ *via* Example 2.1.2. There is a canonical isomorphism $T^*_{(0,y)}\mathcal{P} \cong \mathcal{P} \otimes_{O_{X\times Y}} p_X^* P_y$ in $\mathrm{Mod}(X \times Y)_{-1-\mathrm{cxn}}$, where $p_X^* : \mathrm{Mod}(O_X)_{0-\mathrm{sp}} \to \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}}$ is defined in (13) and the functor

$$\mathcal{P} \otimes_{O_{X\times Y}} (\cdot) : \mathrm{Mod}(O_{X\times Y})_{0-\mathrm{cxn}} \to \mathrm{Mod}(O_{X\times Y})_{-1-\mathrm{cxn}}$$

is from [Rot97, (2.10)]. Arguing as in [Muk81, (3.1)], we get Proposition 7.1.1 from the projection formula.

Proposition 7.1.1.

$$RS_2 \circ T_y^* \cong (\cdot \otimes_{O_X} P_y) \circ RS_2 : D(D_Y) \to D(\mathcal{A}_X);$$
$$RS_2 \circ (\cdot \otimes_{O_Y} P_x) \cong T_{-x}^* \circ RS_2 : D(D_Y) \to D(\mathcal{A}_X);$$
$$RS_1 \circ (\cdot \otimes_{O_X} P_y) \cong T_y^* \circ RS_1 : D(\mathcal{A}_X) \to D(D_Y);$$
$$RS_1 \circ T_x^* \cong (\cdot \otimes_{O_Y} P_{-x}) \circ RS_1 : D(\mathcal{A}_X) \to D(D_Y).$$

Similar results hold for $R\mathfrak{S}_1$ and $R\mathfrak{S}_2$.

7.2 Duality

Let Z be a complex manifold. Denote by Δ^{O_Z} the duality (contravariant) functor $R\mathcal{H}om_{O_Z}(\cdot, \omega_Z^{-1})[\dim Z] : D_c^b(O_Z) \to D_c^b(O_Z)$. The duality functor on D_Z-modules $\Delta^{D_Z} : D(D_Z) \to D(D_Z)$ is defined by $\Delta^{D_Z} F = G[\dim Z]$, where G is the complex of left D_Z-modules associated with the complex $R\mathcal{H}om_{D_Z}(F, D_Z)$ of *right* D_Z-modules. By [Bjö93, Def. 2.11.1], Δ^{D_Z} restricts to a functor $D_c^b(D_Z) \to D_c^b(D_Z)$, and the natural transformation $\mathrm{Id} \to \Delta^{D_Z} \circ \Delta^{D_Z}$ is an isomorphism of functors $D_c^b(D_Z) \to D_c^b(D_Z)$.

Lemma 7.2.1 ([KS16, p.16])**.** *The functor $\Delta^{D_Z} : D(D_Z) \to D(D_Z)$ restricts to a functor $D^b_{\mathrm{good}}(D_Z) \to D^b_{\mathrm{good}}(D_Z)$.*

Proof. Suppose F is a coherent O_Z-module and $N = D_Z \otimes_{O_Z} F$, then by [Bjö93, (ii), p.122], there is $G \in D_c^b(O_Z)$ with $\Delta^{D_Z} N = D_Z \otimes_{O_Z} G$. By Lemma 6.2.1, $\Delta^{D_Z} N \in D^b_{\mathrm{good}}(D_Z)$.

Take $M \in D^b_{\mathrm{good}}(D_Z)$. To prove $\Delta^{D_Z} M \in D^b_{\mathrm{good}}(D_Z)$, by [Har66, I, Prop. 7.3 (i)], one may assume $M \in \mathrm{Good}(D_Z)$. For every relatively compact open subset $U \subset Z$, by [Bjö93, Thm. 1.5.8] and Proposition 6.1.4 , there is a finite length exact sequence in $\mathrm{Mod}(D_U)$:

$$0 \to D_U \otimes_{O_U} F^{-n} \to \cdots \to D_U \otimes_{O_U} F^0 \to M|_U \to 0,$$

where each F^i is a coherent O_U-module. For every i, one has $\Delta^{D_U}(D_U \otimes_{O_U} F^i) \in D^b_{\mathrm{good}}(D_U)$. By Lemma 6.2.2, one has $(\Delta^{D_Z} M)|_U = \Delta^{D_U}(M|_U) \in D^b_{\mathrm{good}}(D_U)$. Hence $\Delta^{D_Z} M \in D^b_{\mathrm{good}}(D_Z)$. □

For algebraic varieties, an analogue of Fact 7.2.2 is stated as [HT07, Cor. 2.6.8 (iii), Prop. 3.2.1]. From [HT07, p.101], all the arguments in [HT07, Sec. 2.6] are valid for analytic D-modules.

Fact 7.2.2.

1. *The contravariant functor $\Delta^{D_Z} : D_h^b(D_Z) \to D_h^b(D_Z)$ an equivalence.*
2. *Let M be a coherent D_Z-module. Then M is holonomic if and only if $H^i(\Delta^{D_Z} M) = 0$ for all integers $i \neq 0$.*

Fact 7.2.3. *Let $f : W \to Z$ be a morphism of complex manifolds. Then:*

1. [Bjö93, Thm. 3.2.13 (1)] *The inverse image $Lf^* : D^b(D_Z) \to D^b(D_W)$ restricts to a functor $D_h^b(D_Z) \to D_h^b(D_W)$.*
2. [Sab11, Thm. 4.4.1] *If $F \in D_h^b(D_W)$ is such that $f|_{\mathrm{Supp}(F)}$ is proper, then $f_+ F \in D_h^b(D_Z)$.*
3. [Bjö93, Thm. 3.2.13 (3)] *The bifunctor $- \otimes^L_{O_W} + : D^b(D_W) \times D^b(D_W) \to D^b(D_W)$ restricts to a bifunctor $D_h^b(D_W) \times D_h^b(D_W) \to D_h^b(D_W)$.*

Restricted to the complex torus Y, [Bjö93, (ii), p.122] becomes [Rot96, (6.12)]:

$$\Delta^{D_Y}(D_Y \otimes^L_{O_Y} \cdot) \cong D_Y \otimes^L_{O_Y} \Delta^{O_Y} \cdot : D^b_c(O_Y) \to D^b_c(D_Y).$$

Define the duality (contravariant) functor $\Delta^{\mathcal{A}_X} : D^b(\mathcal{A}_X) \to D^b(\mathcal{A}_X)$ as

$$\Delta^{\mathcal{A}_X} = T^g R\mathcal{H}om_{\mathcal{A}_X}(\cdot, \mathcal{A}_X).$$

It restricts to a functor $D^b_c(\mathcal{A}_X) \to D^b_c(\mathcal{A}_X)$. Similar to Lemma 7.2.1, it restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(\mathcal{A}_X)$. Theorem 7.2.4 follows from Proposition 7.2.5 and Fact 7.2.2 2, in the same way how Theorem 6.5 follows from Propositions 6.3 and 6.4 in [Rot96].

Theorem 7.2.4 (Rothstein). *Let $F \in D^b_{\mathrm{good}}(\mathcal{A}_X)$ be an object such that $RS_1(F)$ is concentrated in a single degree $i \in \mathbb{Z}$. Then $H^i RS_1(F)$ is holonomic if and only if $RS_1 \Delta^{\mathcal{A}_X} F$ is concentrated in degree $g - i$.*

Proposition 7.2.5 can be deduced from Corollary 7.2.7, Proposition 5.1.4 and [Liu23a, Prop. 5.1.6], in the same way that [Rot96, Prop. 6.3] is proved.

Proposition 7.2.5.

$$RS_2 \Delta^{D_Y} = [-1]^*_X T^{-g} \Delta^{\mathcal{A}_X} RS_2 : D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X); \tag{36}$$

$$\Delta^{D_Y} RS_1 = [-1]^*_Y T^g RS_1 \Delta^{\mathcal{A}_X} : D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y). \tag{37}$$

Lemma 7.2.6 ([Huy06, (3.13)]). *For any objects $K, L \in D(O_Z)$ and $M \in D^-_c(O_Z)$, the natural morphism* (provided by [Sta23, Tag 0BYS])

$$K \otimes^L_{O_Z} R\mathcal{H}om_{O_Z}(M, L) \to R\mathcal{H}om_{O_Z}(M, K \otimes^L_{O_Z} L) \tag{38}$$

is an isomorphism in $D(O_Z)$.

Proof. By [Har66, I, Prop. 7.1 (ii)], one may assume that $M \in \mathrm{Coh}(O_Z)$. By [Sta23, Tag 08DL] and [GH78, p.696], one may shrink Z such that M admits a globally free resolution $F \to M$, where the complex F is

$$0 \to O_Z^{k_n} \to \cdots \to O_Z^{k_1} \to O_Z^{k_0} \to 0$$

with $O_Z^{k_i}$ placed in degree $-i$. The morphism (38) becomes

$$K \otimes^L_{O_Z} \mathcal{H}om_{O_Z}(F, L) \to \mathcal{H}om_{O_Z}(F, K \otimes^L_{O_Z} L),$$

which is an isomorphism. □

Corollary 7.2.7 proves the analytic counterpart of [Rot96, (6.12)].

Corollary 7.2.7. *There is a canonical isomorphism $\Delta^{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} \cdot) \cong \mathcal{A}_X \otimes^L_{O_X} \Delta^{O_X} \cdot$ of functors $D^b_c(O_X) \to D^b_c(\mathcal{A}_X)$.*

Proof. By [Rot96, (6.2)], one has

$$\Delta^{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} \cdot) = T^g R\mathcal{H}om_{\mathcal{A}_X}(\mathcal{A}_X \otimes^L_{O_X} \cdot, \mathcal{A}_X) = T^g R\mathcal{H}om_{O_X}(\cdot, \mathcal{A}_X).$$

By Lemma 7.2.6, it equals $T^g R\mathcal{H}om_{O_X}(\cdot, O_X) \otimes^L_{O_X} \mathcal{A}_X = \mathcal{A}_X \otimes^L_{O_X} \Delta^{O_X}\cdot$. □

Example 7.2.8. Let $F = T^g\mathcal{A}_X \in D^b_{\text{good}}(\mathcal{A}_X)$. By Corollary 5.1.5, one has $RS_1(F) = D_Y \otimes_{O_Y} \mathbb{C}_0$. One has $\Delta^{\mathcal{A}_X} F = \mathcal{A}_X$, and $RS_1\Delta^{\mathcal{A}_X} F$ is concentrated in degree g. Then by Theorem 7.2.4, the D_Y-module $D_Y \otimes_{O_Y} \mathbb{C}_0$ is holonomic.

7.3 Pullback and pushout

Proposition 7.3.1 ([Lau96, Prop. 3.3.2])**.** *Let $f : X' \to X$ be a morphism of complex tori, with $\dim X' = g'$. Let $\hat{f} : Y \to Y'$ be the morphism dual to f. Let $\tilde{f} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X)$ be the induced morphism (26). Then there are canonical isomorphisms of functors*

1.

$$L\hat{f}^* RS_1' \cong RS_1 R\tilde{f}_* : D_{O-\text{good}}(\mathcal{A}_{X'}) \to D_{O-\text{good}}(D_Y); \qquad (39)$$

$$R\tilde{f}_* RS_2' \cong T^{g-g'} RS_2 L\hat{f}^* : D_{O-\text{good}}(D_{Y'}) \to D_{O-\text{good}}(\mathcal{A}_X). \qquad (40)$$

2.

$$RS_2'\hat{f}_+ \cong L\tilde{f}^* RS_2 : D^b_{\text{good}}(D_Y) \to D^b_{\text{good}}(\mathcal{A}_{X'}); \qquad (41)$$

$$\hat{f}_+ RS_1 \cong T^{g'-g} RS_1' L\tilde{f}^* : D^b_{\text{good}}(\mathcal{A}_X) \to D^b_{\text{good}}(D_{Y'}). \qquad (42)$$

Proof. 1. The isomorphism (40) follows from (39) as follows:

$$\begin{aligned} R\tilde{f}_* RS_2' &\overset{(a)}{\cong} T^g RS_2 RS_1 R\tilde{f}_* RS_2' \\ &\overset{(b)}{\cong} T^g RS_2 L\hat{f}^* RS_1' RS_2' \\ &\overset{(c)}{\cong} T^{g-g'} RS_2 L\hat{f}^*, \end{aligned}$$

where (39) (resp. Theorem 5.1.3) is used in (b) (resp. (a) and (c)). Then we prove (39).

By (27) (resp. the proof of [HT07, Prop. 1.5.8]), the derived direct image (resp. inverse image) functor of $\mathcal{A}$-modules (resp. D-modules) regards that of the underlying O-modules. From [Liu23a, Prop. 3.1.2 2], the functor $\mathcal{P}' \otimes^L_{O_{X'\times Y'}} p^*_{X'}\cdot : D(\mathcal{A}_{X'}) \to D(O_{X'\times Y'})$ restricts to a functor $D_{O-\text{good}}(\mathcal{A}_{X'}) \to D_{\text{good}}(O_{X'\times Y'})$. An application of [Liu23a, Lem. 3.2.11] to the cartesian square

$$\begin{array}{ccc} X' \times Y & \xrightarrow{1_{X'} \times \hat{f}} & X' \times Y' \\ {\scriptstyle p_2}\downarrow & \square & \downarrow{\scriptstyle p_{Y'}} \\ Y & \xrightarrow[\hat{f}]{} & Y' \end{array}$$

yields a canonical isomorphism of functors

$$L\hat{f}^* Rp_{Y'} \to Rp_{2*} L(1_{X'} \times \hat{f})^* : D_{\mathrm{good}}(O_{X' \times Y'}) \to D_{\mathrm{good}}(O_Y). \tag{43}$$

Applying [Liu23a, Thm. 3.2.3] to the cartesian square

$$\begin{array}{ccc} X' \times Y & \xrightarrow{p_1} & X' \\ {\scriptstyle f \times 1_Y}\downarrow & \square & \downarrow{\scriptstyle f} \\ X \times Y & \xrightarrow[p_X]{} & X, \end{array}$$

of complex manifolds, one gets a natural isomorphism

$$p_X^* R\tilde{f}_* \to R(f \times 1_Y)_* p_1^* \tag{44}$$

of functors $D_{O-\mathrm{good}}(\mathcal{A}_{X'}) \to D(\mathrm{Mod}(O_{X \times Y})_{1-\mathrm{cxn,fl}})$.

Then

$$\begin{aligned} L\hat{f}^* RS_1' =& L\hat{f}^* Rp_{Y'}(\mathcal{P}' \otimes^L_{O_{X' \times Y'}} p_{X'}^* \cdot) \\ \stackrel{(a)}{\cong}& Rp_{2*} L(1_{X'} \times \hat{f})^* (\mathcal{P}' \otimes^L_{O_{X' \times Y'}} p_{X'}^* \cdot) \\ \cong& Rp_{2*}[L(1_{X'} \times \hat{f})^* \mathcal{P}' \otimes^L_{O_{X' \times Y}} L(1_{X'} \times \hat{f})^* p_{X'}^* \cdot] \\ \cong& Rp_{2*}[(1_{X'} \times \hat{f})^* \mathcal{P}' \otimes^L_{O_{X' \times Y}} p_1^* \cdot] \\ \stackrel{(b)}{\cong}& Rp_{2*}[(f \times 1_Y)^* \mathcal{P} \otimes^L_{O_{X' \times Y}} p_1^* \cdot] \\ \cong& Rp_{Y*} R(f \times 1_Y)_*[(f \times 1_Y)^* \mathcal{P} \otimes^L_{O_{X' \times Y}} p_1^* \cdot] \\ \stackrel{(c)}{\cong}& Rp_{Y*}[\mathcal{P} \otimes^L_{O_{X \times Y}} R(f \times 1_Y)_* p_1^* \cdot] \\ \stackrel{(d)}{\cong}& Rp_{Y*}[\mathcal{P} \otimes^L_{O_{X \times Y}} p_X^* R\tilde{f}_* \cdot] \\ =& RS_1 R\tilde{f}_*, \end{aligned}$$

where (a), (b), (c) and (d)) use (43), [Liu23a, (23)], [Liu23a, Fact 3.2.13] and (44) respectively. This proves (39).

2. The isomorphism (42) follows from (41) as follows:

$$\begin{aligned} \hat{f}_+ RS_1 \stackrel{(a)}{\cong}& T^{g'} RS_1' RS_2' \hat{f}_+ RS_1 \\ \stackrel{(b)}{\cong}& T^{g'} RS_1' L\tilde{f}^* RS_2 RS_1 \\ \stackrel{(c)}{\cong}& T^{g'-g} RS_1' L\tilde{f}^*, \end{aligned}$$

where (a) and (c) use Theorem 6.3.1, and (b) uses (41). Then we prove (41).

Using (28), one can prove that $L\tilde{f}^* : D(\mathcal{A}_X) \to D(\mathcal{A}_{X'})$ restricts to a functor $D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(\mathcal{A}_{X'})$. From Fact 6.2.3, the direct image functor $\hat{f}_+ : D^b(D_Y) \to D^b(D_{Y'})$ restricts to a functor $D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(D_{Y'})$. There are canonical isomorphism of bifunctors $D^b_{\mathrm{good}}(D_Y)^{\mathrm{op}} \times D^b_{\mathrm{good}}(\mathcal{A}_{X'}) \to \mathrm{Ab}$:

$$\begin{aligned}
\mathrm{Hom}_{D^b_{\mathrm{good}}(\mathcal{A}_{X'})}(RS'_2\hat{f}_+-,+) &\overset{(a)}{\cong} \mathrm{Hom}_{D^b_{\mathrm{good}}(D_{Y'})}(\hat{f}_+-, T^{g'}RS'_1+) \\
&\overset{(b)}{\cong} \mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(-, T^g L\hat{f}^* RS'_1+) \\
&\overset{(c)}{\cong} \mathrm{Hom}_{D^b_{\mathrm{good}}(D_Y)}(-, T^g RS_1 R\tilde{f}_*+) \\
&\overset{(d)}{\cong} \mathrm{Hom}_{D^b_{\mathrm{good}}(\mathcal{A}_X)}(RS_2-, R\tilde{f}_*+) \\
&\cong \mathrm{Hom}_{D^b_{\mathrm{good}}(\mathcal{A}_{X'})}(L\tilde{f}^* RS_2-,+),
\end{aligned}$$

where (a) and (d) use Theorem 6.3.1, (a) uses [Bjö93, Thm. 2.11.8], and (c) uses (39). From Yoneda's lemma, there is a canonical isomorphism $RS'_2\hat{f}_+ \cong L\tilde{f}^* RS_2$ of functors $D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_{X'})$.

□

7.4 External tensor product

For two complex manifolds U, V, recall the (exact) external tensor product bifunctor

$$(\cdot) \boxtimes_O (\cdot) : \mathrm{Mod}(D_U) \times \mathrm{Mod}(D_V) \to \mathrm{Mod}(D_{U\times V}) \tag{45}$$

defined in [Bjö93, 2.4.4]. By exactness, it descends to

$$D(D_U) \times D(D_V) \to D(D_{U\times V}). \tag{46}$$

Remark 7.4.1. By [Bjö93, 2.4.13], the bifunctor (45) restricts to bifunctors $\mathrm{Coh}(D_U)\times\mathrm{Coh}(D_V) \to \mathrm{Coh}(D_{U\times V})$ and $\mathrm{Good}(D_U)\times\mathrm{Good}(D_V) \to \mathrm{Good}(D_{U\times V})$. Then by [Har66, I, Prop. 7.3 (i)], the bifunctor (46) restricts to bifunctors $D^b_c(D_U)\times D^b_c(D_V) \to D^b_c(D_{U\times V})$ and $D^b_{\mathrm{good}}(D_U)\times D^b_{\mathrm{good}}(D_V) \to D^b_{\mathrm{good}}(D_{U\times V})$. By [Bjö93, p.139], it also restricts to a bifunctor $D^b_h(D_U)\times D^b_h(D_V) \to D^b_h(D_{U\times V})$.

Using [Liu23a, Lem. 5.1.4] (at the place of [HT07, Lem. 1.5.31]), Lemma 6.2.4 and [Sab11, Thm. 3.3.6 (1)], one can argue as in [HT07, Prop. 1.5.30] to get Fact 7.4.2.

Fact 7.4.2.

1. *Let U, V, Z be complex manifolds. Let $f : U \to V$ be a* proper *morphism. Then the natural transformation*

$$f_+(-)\boxtimes_O(+) \to (f\times\mathrm{Id}_Z)_+(-\boxtimes_O+) : D_{O-\mathrm{good}}(D_U)\times D(D_Z) \to D(D_{V\times Z})$$

is an isomorphism.

2. *Let $f_i : U_i \to V_i$ $(i = 1, 2)$ be two* proper *morphisms of complex manifolds. Then the natural transformation*

$$(f_{1+}-)\boxtimes_O(f_{2+}+) \to (f_1\times f_2)_+(-\boxtimes_O+) : D_{O-\mathrm{good}}(D_{U_1})\times D_{O-\mathrm{good}}(D_{U_2}) \to D_{O-\mathrm{good}}(D_{V_1\times V_2})$$

is an isomorphism.

For a complex torus X, let $\mathrm{for}_X : \mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_X)$ be the forgetful functor. Let X' be another complex torus. Set $X'' = X \times X'$. Write $u : X'' \to X$ and $u' : X'' \to X'$ for the projections. Let Y', Y'' be the dual of X' and X'' respectively. For an $\mathcal{A}_X$-module F and an $\mathcal{A}_{X'}$-module G, denote $\tilde{u}^*F \otimes_{\mathcal{A}_{X''}} \tilde{u'}^*G$ by $F \boxtimes_{\mathcal{A}_X} G$. As

$$F \boxtimes_{\mathcal{A}_X} G = u^{-1}F \otimes_{u^{-1}\mathcal{A}_X} \mathcal{A}_{X''} \otimes_{u'^{-1}\mathcal{A}_{X'}} u'^{-1}G,$$

and $\mathcal{A}_{X''}$ is flat over $u^{-1}\mathcal{A}_X$ and over $u'^{-1}\mathcal{A}_{X'}$, the bifunctor

$$- \boxtimes_{\mathcal{A}_X} + : \mathrm{Mod}(\mathcal{A}_X) \times \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(\mathcal{A}_{X''})$$

is exact in both arguments. Consider the diagonal morphism $\delta : X \to X^2$. There is a canonical isomorphism of bifunctors

$$L\tilde{\delta}^*[- \boxtimes_{\mathcal{A}_X} +] \cong (-) \otimes^L_{\mathcal{A}_X} (+) : D(\mathcal{A}_X) \times D(\mathcal{A}_X) \to D(\mathcal{A}_X). \tag{47}$$

Although the tensor product of two $\mathcal{A}_X$-modules is different from the tensor product of the underlying O_X-module, Lemma 7.4.3 shows that external products do agree. It is used in the proof of Lemma 7.4.4.

Lemma 7.4.3. *There is a natural isomorphism of bifunctors*

$$\mathrm{for}_{X''}(-\boxtimes_{\mathcal{A}}+) \to (\mathrm{for}_X-)\boxtimes_O(\mathrm{for}_{X'}+) : \mathrm{Mod}(\mathcal{A}_X)\times\mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(O_{X''}).$$

Proof. By construction, one has

$$\mathcal{A}_{X''} = \mathcal{A}_X \boxtimes_O \mathcal{A}_{X'} = u^{-1}\mathcal{A}_X \otimes_{u^{-1}O_X} u'^*\mathcal{A}_{X'}. \tag{48}$$

There are natural isomorphisms of functors $\mathrm{Mod}(\mathcal{A}_X) \to \mathrm{Mod}(O_{X''})$:

$$\begin{aligned}
\mathrm{for}_{X''}\tilde{u}^* :=& u^{-1} \cdot \otimes_{u^{-1}\mathcal{A}_X}\mathcal{A}_{X''} \\
\overset{(a)}{=}& u^{-1} \cdot \otimes_{u^{-1}\mathcal{A}_X}(u^{-1}\mathcal{A}_X \otimes_{u^{-1}O_X} u'^*\mathcal{A}_{X'}) \\
\cong& u^{-1} \cdot \otimes_{u^{-1}O_X} u'^*\mathcal{A}_{X'} \\
\cong& (u^{-1} \cdot \otimes_{u^{-1}O_X} O_{X''}) \otimes_{O_{X''}} u'^*\mathcal{A}_{X'} \\
\cong& u^*\mathrm{for}_X \cdot \otimes_{O_{X''}} u'^*\mathcal{A}_{X'},
\end{aligned}$$

where (a) uses (48). Similarly, there is a natural isomorphism of functors $\mathrm{for}_{X''}\tilde{u'}^* \cong u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathrm{for}_{X'}\cdot : \mathrm{Mod}(\mathcal{A}_{X'}) \to \mathrm{Mod}(O_{X''})$. One has natural isomorphisms of bifunctors

$$\begin{aligned}\mathrm{for}_{X''}(- \boxtimes_{\mathcal{A}_X} +) :=& \tilde{u}^* - \otimes_{\mathcal{A}_{X''}} \tilde{u'}^* + \\ \cong& (u^*\mathrm{for}_X - \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}) \otimes_{u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathcal{A}_{X'}} (u^*\mathcal{A}_X \otimes_{O_{X''}} u'^*\mathrm{for}_{X'}+) \\ \cong& (u^*\mathrm{for}_X -) \otimes_{O_{X''}} (u'^*\mathrm{for}_{X'}+) \\ :=& (\mathrm{for}_X -) \boxtimes_O (\mathrm{for}_{X'}+).\end{aligned}$$

□

Lemma 7.4.4. *There are canonical isomorphisms of bifunctors*

$$RS''_2[- \boxtimes_O +] \cong RS_2 - \boxtimes_{\mathcal{A}} RS'_2+ : D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_{Y'}) \to D_{O-\mathrm{good}}(\mathcal{A}_{X''}); \tag{49}$$

$$RS''_1[- \boxtimes_{\mathcal{A}} +] \cong RS_1 - \boxtimes_O RS'_1+ : D_{O-\mathrm{good}}(\mathcal{A}_X) \times D_{O-\mathrm{good}}(\mathcal{A}_{X'}) \to D_{O-\mathrm{good}}(D_{Y''}). \tag{50}$$

Proof. It follows from [Liu23a, Prop. 5.1.3], Lemma 7.4.3 and Proposition 5.1.2.

□

7.5 Convolution and tensor product

For the dual complex tori X and Y, let $m : X^2 \to X$ and $\mu : Y^2 \to Y$ be their respective group law.

Definition 7.5.1 (Convolution, [Lau96, p.22])**.** Define bifunctors

$$\begin{aligned}*_D : D(D_Y) \times D(D_Y) \to D(D_Y), &\quad - *_D + = \mu_+[- \boxtimes_O +], \\ *_{\mathcal{A}} : D(\mathcal{A}_X) \times D(\mathcal{A}_X) \to D(\mathcal{A}_X), &\quad - *_{\mathcal{A}} + = R\tilde{m}_*[- \boxtimes_{\mathcal{A}} +].\end{aligned}$$

As μ is proper, by Fact 6.2.3, Lemma 6.2.4 and Fact 7.2.3 2, the direct image μ_+ restricts to functors $D^b_{\mathrm{good}}(D_{Y^2}) \to D^b_{\mathrm{good}}(D_Y)$, $D_{O-\mathrm{good}}(D_{Y^2}) \to D_{O-\mathrm{good}}(D_Y)$ and $D^b_h(D_{Y^2}) \to D^b_h(D_Y)$. Together with Remark 7.4.1, this implies that the bifunctor $*_D$ restricts to bifunctors $D^b_{\mathrm{good}}(D_Y) \times D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(D_Y)$, $D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(D_Y)$ and $D^b_h(D_Y) \times D^b_h(D_Y) \to D^b_h(D_Y)$.

Lemma 7.5.2. *The pair $(D(D_Y), *_D)$ is a symmetric tensor triangulated category* (in the sense of [Bal10, Def. 3]) *with unit $D_Y \otimes_{O_Y} \mathbb{C}_0$.*

Proof. Let $i : \mathrm{Specan}(\mathbb{C}) \to Y$ be the inclusion of $0 \in Y$. Then $D_Y \otimes_{O_Y} \mathbb{C}_0 = i_+\mathbb{C}$. There are canonical isomorphisms

$$\begin{aligned}(i_+\mathbb{C}) *_D \cdot :=& \mu_+[(i_+\mathbb{C}) \boxtimes_O \cdot] \\ =& \mu_+[(i_+\mathbb{C}) \boxtimes_O (\mathrm{Id}_{Y+}\cdot)] \\ \overset{(a)}{\cong}& \mu_+(i \times \mathrm{Id}_Y)_+(\mathbb{C} \boxtimes_O \cdot) \\ \overset{(b)}{\cong}& \mathrm{Id}_{Y+} = \mathrm{Id}_{D(D_Y)}\end{aligned}$$

of functors $D(D_Y) \to D(D_Y)$, where (a) and (b) use Fact 7.4.2 1 and [Sab11, Thm. 3.3.6 (1)] respectively, Therefore, $D_Y \otimes_{O_Y} \mathbb{C}_0$ is the unit. The other axioms can be verified as in [Wei07, pp. 10-11]. □

Proposition 7.5.3 ([Wei11]). *For every $M \in D^b_{\text{good}}(D_Y)$, the functor $\cdot *_D M : D^b_{\text{good}}(D_Y) \to D^b_{\text{good}}(D_Y)$ admits a right adjoint $([-1]_Y^* \Delta^{D_Y} M) *_D \cdot$.*

Proof. Define an automorphism $f : Y^2 \to Y^2$ of the complex torus Y^2 by $f(a, b) = (a + b, -a)$. Then $p_1 f = \mu$, $p_2 f = [-1]_Y p_1$ and $\mu f = p_2$. One has $Lf^* O_{Y^2} = O_{Y^2}$ in $D^b(D_{Y^2})$.

For any objects $F, G \in D^b_{\text{good}}(D_Y)$, there are canonical bijections

$$
\begin{aligned}
&\operatorname{Hom}_{D^b_{\text{good}}(D_Y)}(F *_D M, G) := \operatorname{Hom}_{D^b_{\text{good}}(D_Y)}(\mu_+(F \boxtimes_O M), G) \\
\overset{\text{(a)}}{=}&\operatorname{Hom}_{D(D_{Y^2})}(F \boxtimes_O M, T^g \mu^* G) \\
\overset{\text{(b)}}{=}&\operatorname{Hom}_{D(D_{Y^2})}(O_{Y^2}, \Delta^{D_{Y^2}}(F \boxtimes_O M) \otimes^L_{O_{Y^2}} T^g \mu^* G) \\
\overset{\text{(c)}}{=}&\operatorname{Hom}_{D(D_{Y^2})}(O_{Y^2}, (\Delta^{D_Y} F) \boxtimes_O (\Delta^{D_Y} M) \otimes^L_{O_{Y^2}} T^g \mu^* G) \\
:=&\operatorname{Hom}_{D(D_{Y^2})}(O_{Y^2}, p_1^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_2^* \Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g \mu^* G) \\
=&\operatorname{Hom}_{D(D_{Y^2})}(f^* O_{Y^2}, f^*[p_1^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_2^* \Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g \mu^* G]) \\
=&\operatorname{Hom}_{D(D_{Y^2})}(O_{Y^2}, \mu^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} p_1^*[-1]_Y^* \Delta^{D_Y} M \otimes^L_{O_{Y^2}} T^g p_2^* G) \\
:=&\operatorname{Hom}_{D(D_{Y^2})}(O_{Y^2}, T^g \mu^* \Delta^{D_Y} F \otimes^L_{O_{Y^2}} ([-1]_Y^* \Delta^{D_Y} M \boxtimes_O G)) \\
\overset{\text{(d)}}{=}&\operatorname{Hom}_{D(D_{Y^2})}(O_{Y^2}, T^g \Delta^{D_Y}(\mu^* F) \otimes^L_{O_{Y^2}} ([-1]_Y^* \Delta^{D_Y} M \boxtimes_O G)) \\
\overset{\text{(e)}}{=}&\operatorname{Hom}_{D(D_{Y^2})}(\mu^* F, T^g([-1]_Y^* \Delta^{D_Y} M \boxtimes_O G)) \\
\overset{\text{(f)}}{=}&\operatorname{Hom}_{D(D_Y)}(F, \mu_+([-1]_Y^* \Delta^{D_Y} M \boxtimes_O G)) \\
\overset{\text{(g)}}{=}&\operatorname{Hom}_{D^b_{\text{good}}(D_Y)}(F, ([-1]_Y^* \Delta^{D_Y} M) * G),
\end{aligned}
$$

where (a), (c), (d), (f) and (g) use [Bjö93, Thm. 2.11.8], Proposition 7.5.4, [Kas03, Thm. 4.12], [Kas03, Thm. 4.40] and Lemma 7.2.1 in order, and both (b), (e) use [Kas03, (3.13)]. As the bijections are functorial in F and G, the adjunction follows. □

The proof of Proposition 7.5.3 needs the commutativity of duality with external tensor product for D-modules.

Proposition 7.5.4. *Let Z_i $(i = 1, 2)$ be two complex manifolds. Then there is a canonical isomorphism*

$$(\Delta^{D_{Z_1}} -) \boxtimes_O (\Delta^{D_{Z_2}} +) \to \Delta^{D_{Z_1 \times Z_2}}(- \boxtimes_O +) : D^b_c(D_{Z_1}) \times D^b_c(D_{Z_2}) \to D^b_c(D_{Z_1 \times Z_2})^{\text{op}}.$$

Proof. For a complex manifold Z, the sheaf $D_Z \otimes_{\mathbb{C}_Z} D_Z^{\mathrm{op}}$ is naturally a $\mathbb{C}_Z$-algebra, and D_Z is naturally a left $D_Z \otimes_{\mathbb{C}_Z} D_Z^{\mathrm{op}}$-module. For $N_i \in D(D_{Z_i^{\mathrm{op}}})$, by [HT07, p.39], there is a natural isomorphism in $D(D^{\mathrm{op}}_{Z_1\times Z_2})$:

$$N_1 \boxtimes_O N_2 = (N_1 \boxtimes_{\mathbb{C}} N_2) \otimes_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1\times Z_2}. \tag{51}$$

First, we construct the natural transformation. Take $M_i \in D^b_c(D_{Z_i})$.

Claim 7.5.5. Then there is a natural morphism in $D^b((D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2})^{\mathrm{op}})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1}}(M_1, D_{Z_1}) \boxtimes_{\mathbb{C}} R\mathcal{H}om_{D_{Z_2}}(M_2, D_{Z_2}) \\ \to &R\mathcal{H}om_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}). \end{aligned} \tag{52}$$

Claim 7.5.6. There is a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1\times Z_2})$:

$$\begin{aligned} &R\mathcal{H}om_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1\times Z_2} \\ \to &R\mathcal{H}om_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1\times Z_2}). \end{aligned} \tag{53}$$

Again, there is a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1\times Z_2})$:

$$R\mathcal{H}om_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, D_{Z_1\times Z_2}) \to R\mathcal{H}om_{D_{Z_1\times Z_2}}(M_1 \boxtimes_O M_2, D_{Z_1\times Z_2}), \tag{54}$$

which can be defined by taking a $D_{Z_1\times Z_2} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1\times Z_2}$-injective resolution of $D_{Z_1\times Z_2}$.

Composing the morphisms (51), (52), (53) and (54) in order, one gets a natural morphism in $D^b(D^{\mathrm{op}}_{Z_1\times Z_2})$:

$$R\mathcal{H}om_{D_{Z_1}}(M_1, D_{Z_1})\boxtimes_O R\mathcal{H}om_{D_{Z_2}}(M_2, D_{Z_2}) \to R\mathcal{H}om_{D_{Z_1\times Z_2}}(M_1\boxtimes_O M_2, D_{Z_1\times Z_2}). \tag{55}$$

We prove that the constructed natural transformation is an isomorphism. To show (55) is an isomorphism, by [Har66, I, Prop. 7.1 (i)], one may assume $M_i \in \mathrm{Coh}(D_{Z_i})$ for $i = 1, 2$. By shrinking Z_i and using [KS13, Prop. 11.2.6], one may find a bounded resolution of M_i by free D_{Z_i}-modules of finite rank. Thus, one may further assume that $M_i = D_{Z_i}$. Since $\omega_{Z_1\times Z_2} = \omega_{Z_1} \boxtimes_O \omega_{Z_2}$ in $\mathrm{Mod}(D^{\mathrm{op}}_{Z_1\times Z_2})$, by [HT07, Eg. 2.6.3], in this case (55) is an isomorphism. □

Proof of Claim 7.5.5. Take a $D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$-injective resolution $D_{Z_i} \to I_i^*$. Then $I_1^* \boxtimes_{\mathbb{C}} I_2^*$ is a complex of modules over

$$(D_{Z_1} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1}) \boxtimes_{\mathbb{C}} (D_{Z_2} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_2}) = (D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{\mathbb{C}} (D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2})^{\mathrm{op}}. \tag{56}$$

By [Sta23, Tag 013K (2)], there exists an injective resolution $I_1^* \boxtimes_{\mathbb{C}} I_2^* \to I^*$ (hence an induced injective resolution $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2} \to I^*$) over (56). The natural morphism $D_{Z_i} \to D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$ is flat, so every injective $D_{Z_i} \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_i}$-module is injective over D_{Z_i}. Similarly, every term of the complex I^* is injective over $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}$. Then (52) is defined to be the composition of the natural morphisms

$$\begin{aligned} &\mathcal{H}om_{D_{Z_1}}(M_1, I_1^*) \boxtimes_{\mathbb{C}} \mathcal{H}om_{D_{Z_2}}(M_2, I_2^*) \to \mathcal{H}om_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, I_1^* \boxtimes_{\mathbb{C}} I_2^*) \\ &\to \mathcal{H}om_{D_{Z_1}\boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, I^*). \end{aligned}$$

□

Proof of Claim 7.5.6. Take an injective resolution $D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2} \to J^*$ over (56). By [Sta23, Tag 013K (2)], over $(D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}) \otimes_{\mathbb{C}} D^{\mathrm{op}}_{Z_1 \times Z_2}$ there exists an injective resolution $J^* \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \to K^*$. Then (53) is defined to be the composition of the natural morphisms

$$\begin{aligned} &\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, J^*) \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2} \\ \to &\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, J^* \otimes_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}} D_{Z_1 \times Z_2}) \\ \to &\mathcal{H}om_{D_{Z_1} \boxtimes_{\mathbb{C}} D_{Z_2}}(M_1 \boxtimes_{\mathbb{C}} M_2, K^*). \end{aligned}$$

□

Corollary 7.5.7 ([Lau96, Cor. 3.3.3])**.** *The equivalence $RS_2 : (D^b_{\mathrm{good}}(D_Y), *_D) \to (D^b_{\mathrm{good}}(\mathcal{A}_X), \otimes^L_{\mathcal{A}_X})$ is a strong monoidal functor. In fact, there are canonical isomorphisms of bifunctors*

$$RS_2(- *_D +) \cong (RS_2 -) \otimes^L_{\mathcal{A}_X} (RS_2 +) : D^b_{\mathrm{good}}(D_Y) \times D^b_{\mathrm{good}}(D_Y) \to D^b_{\mathrm{good}}(\mathcal{A}_X); \tag{57}$$

$$(RS_1 -) *_D (RS_1 +) \cong T^{-g} RS_1(- \otimes^L_{\mathcal{A}_X} +) : D^b_{\mathrm{good}}(\mathcal{A}_X) \times D^b_{\mathrm{good}}(\mathcal{A}_X) \to D^b_{\mathrm{good}}(D_Y); \tag{58}$$

$$RS_1(- *_{\mathcal{A}} +) \cong (RS_1 -) \otimes^L_{O_Y} (RS_1 +) : D_{O-\mathrm{good}}(\mathcal{A}_X) \times D_{O-\mathrm{good}}(\mathcal{A}_X) \to D_{O-\mathrm{good}}(D_Y); \tag{59}$$

$$(RS_2 -) *_{\mathcal{A}} (RS_2 +) \cong T^{-g} RS_2(- \otimes^L_{O_Y} +) : D_{O-\mathrm{good}}(D_Y) \times D_{O-\mathrm{good}}(D_Y) \to D_{O-\mathrm{good}}(\mathcal{A}_X). \tag{60}$$

Proof. Let $\delta_X : X \to X^2 =: X'$ be the diagonal morphism. Its dual morphism is $\mu : Y^2 \to Y$. There are canonical isomorphisms of bifunctors

$$\begin{aligned} RS_2(- *_D +) :=& RS_2 \mu_+(- \boxtimes_O +) \\ \overset{(a)}{\cong}& L\tilde{\delta}^*_X RS_2'(- \boxtimes_O +) \\ \overset{(b)}{\cong}& L\tilde{\delta}^*_X (RS_2 - \boxtimes_{\mathcal{A}} RS_2 +) \\ \overset{(c)}{\cong}& (RS_2 -) \otimes^L_{\mathcal{A}_X} (RS_2 +), \end{aligned}$$

where (a), (b) and (c) use (41), (49) and (47) respectively. This shows (57).

By Corollary 5.1.5, the functor RS_2 preserves units, so it is strong monoidal.

In addition, (58) follows:

$$\begin{aligned}(RS_1-) *_D (RS_1+) &\overset{\text{(a)}}{\cong} T^g RS_1 RS_2 (RS_1 - *_D RS_1+)\\ &\overset{\text{(b)}}{\cong} T^g RS_1 (RS_2 RS_1 - \otimes^L_{\mathcal{A}_X} RS_2 RS_1+)\\ &\overset{\text{(c)}}{\cong} T^g RS_1 (T^{-g} - \otimes^L_{\mathcal{A}_X} T^{-g}+)\\ &= T^{-g} RS_1 (- \otimes^L_{\mathcal{A}_X} +),\end{aligned}$$

where (a) and (c) (resp. (b)) use Theorem 6.3.1, (resp. (57)).

Because the diagonal morphism $\delta_Y : Y \to Y^2$ is dual to $m : X' = X^2 \to X$, there are canonical isomorphisms of bifunctors

$$\begin{aligned}RS_1(- *_{\mathcal{A}} +) &:= RS_1 R\tilde{m}_*(- \boxtimes_{\mathcal{A}} +)\\ &\overset{\text{(a)}}{\cong} L\delta_Y^* RS_1'(- \boxtimes_{\mathcal{A}} +)\\ &\overset{\text{(b)}}{\cong} L\delta_Y^* (RS_1 - \boxtimes_O RS_1+)\\ &\overset{\text{(c)}}{\cong} (RS_1-) \otimes^L_{O_Y} (RS_1+),\end{aligned}$$

where (a), (b) and (c) use (39), (50) and [HT07, p.39] respectively. This demonstrates (59). Then (60) follows:

$$\begin{aligned}(RS_2-) *_{\mathcal{A}} (RS_2+) &\overset{\text{(a)}}{\cong} T^g RS_2 RS_1 (RS_2 - *_{\mathcal{A}} RS_2+)\\ &\overset{\text{(b)}}{\cong} T^g RS_2 (RS_1 RS_2 - \otimes^L_{O_Y} RS_1 RS_2+)\\ &\overset{\text{(c)}}{\cong} T^g RS_2 (T^{-g} - \otimes^L_{O_Y} T^{-g}+)\\ &= T^{-g} RS_2 (- \otimes^L_{O_Y} +),\end{aligned}$$

where (a) and (c) (resp. (b)) use Theorem 5.1.3 (resp. (59)). □

A Unbounded Bernstein's equivalence

In Section A, let X be a smooth algebraic variety over be an algebraically closed field k of characteristic 0. Let $\mathrm{Qch}(O_X) \subset \mathrm{Mod}(O_X)$ and $\mathrm{Mod}_{\mathrm{qc}}(D_X) \subset \mathrm{Mod}(D_X)$ be the full subcategories of objects quasi-coherent over O_X. They are weak Serre subcategories.

Fact A.0.1 (Bernstein, [B$^+$87, VI, Thm. 2.10])**.** *The natural functor*

$$\iota'_X : D^b(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \to D^b_{\mathrm{qc}}(D_X)$$

is an equivalence.

Theorem A.0.2 is an unbounded generalization of Fact A.0.1. It is left "to the reader to state and prove" in [Nee96, p.207]. We follow the strategy pointed out in [gh], and do not claim originality here.

Theorem A.0.2. *The functor*

$$\iota'_X : D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \to D_{\mathrm{qc}}(D_X) \tag{61}$$

induced by the inclusion $\mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_X)$ *is an equivalence of categories.*

We need a series of lemmas for the proof of Theorem A.0.2.

Lemma A.0.3. *Every object of* $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ *is the inductive limit of its coherent* D_X*-submodules.*

Proof. Let F be such an object. Then the family of coherent D_X-submodules of F is directed. In fact, if G_1, G_2 are coherent D_X-submodules of F, then both have finite type over D_X. Their sum $G_1 + G_2 (\subset F)$ is of finite type over D_X. As $\mathrm{Qch}(O_X)$ is an abelian subcategory of $\mathrm{Mod}(O_X)$, the image $G_1 + G_2$ of the natural morphism $G_1 \oplus G_2 \to F$ is quasi-coherent over O_X. By [HT07, Prop. 1.4.9 (ii)], the D_X-submodule $G_1 + G_2$ of F is coherent.

We prove that F is the union of its coherent D_X-submodules. (It is stated as [HT07, Cor. 1.4.17 (iii)], whose poof is omitted.) Let $U \subset X$ be an affine open, $s \in \Gamma(U, F)$ be a section, and $G \subset F|_U$ be the D_U-submodule generated by s. By [HT07, Prop. 1.4.3, 1.4.4 and 1.4.13], the D_U-module G is coherent. By [Meb89, Prop. 2.5.7], there is a coherent D_X-submodule $G' \subset F$ with $G'|_U = G$. Since X has a basis for the Zariski topology consisting of affine opens, every local section of F is locally contained in a coherent D_X-submodule. □

For an open immersion $j : U \to X$, we have a natural morphism of ringed spaces $j : (U, D_U) \to (X, D_X)$. From [B+87, VI, 5.2] and [HT07, Prop. 1.5.29], the functor $j_+ : D(D_U) \to D(D_X)$ is the right derived functor of the corresponding (left exact) direct image $j_* : \mathrm{Mod}(D_U) \to \mathrm{Mod}(D_X)$. By [Ber83, 2, p.12] and [Sta23, Tag 0096], the inverse image $j^* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_U)$ is left adjoint to j_*. Lemma A.0.4 2 helps to construct a quasi-inverse to (61).

Lemma A.0.4.

1. *The category* $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ *is locally noetherian.*

2. *The inclusion functor* $\iota' : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_X)$ *admits a right adjoint* $Q' = Q'_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$*. The unit natural transform* $\eta' : \mathrm{Id}_{\mathrm{Mod}_{\mathrm{qc}}(D_X)} \to Q'\iota'$ *is an isomorphism.*

Proof. By [Sta23, Tag 01LA (4)], $\mathrm{Qch}(O_X) \subset \mathrm{Mod}(O_X)$ is an abelian subcategory closed under colimits. Then so is $\mathrm{Mod}_{\mathrm{qc}}(D_X) \subset \mathrm{Mod}(D_X)$.

1. When X is affine, by [HT07, Prop. 1.4.4 (ii)], the functor $\Gamma(X, \cdot) : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_X(X))$ is an equivalence of abelian categories. As the ring $D_X(X)$

is left noetherian, the category $\mathrm{Mod}(D_X(X))$ is locally noetherian by the last paragraph of [Gab62, p.402].

For a general X, one may assume that there exists an open covering $X = U \cup V$, such that the statement holds for U and V. Arguing as in [Gab62, Prop. 2, p.441], one can prove that $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ is the gluing of $\mathrm{Mod}_{\mathrm{qc}}(D_U)$ and $\mathrm{Mod}_{\mathrm{qc}}(D_V)$ along $\mathrm{Mod}_{\mathrm{qc}}(D_{U\cap V})$ in the sense of [Gab62, VI. 1]. Let $j : U \to X$ be the inclusion. Then

$$j^* : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_U)$$

is exact and left adjoint to

$$j_* : \mathrm{Mod}_{\mathrm{qc}}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_X).$$

The (counit) natural transformation $\epsilon : j^*j_* \to \mathrm{Id}_{\mathrm{Mod}_{\mathrm{qc}}(D_U)}$ is an isomorphism. From [Gab62, Prop. 5, p.374], the subcategory $\ker(j^*)$ is localizing in $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ (in the sense of [Gab62, p372]) and j^* induces an equivalence

$$\mathrm{Mod}_{\mathrm{qc}}(D_X)/\ker(j^*) \to \mathrm{Mod}_{\mathrm{qc}}(D_U).$$

A similar result holds for V. Then by [Gab62, Lem. 2, p.442], the gluing category $\mathrm{Mod}_{\mathrm{qc}}(D_X)$ is locally noetherian.

2. It follows from 1 and Lemma A.0.5.

□

Lemma A.0.5. *Let $\mathcal{A}$ be a Grothendieck abelian category. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor preserving all colimits.*

1. *Then F admits a right adjoint $G : \mathcal{B} \to \mathcal{A}$.*
2. *If further F is fully faithful, then the unit natural transformation $\eta : \mathrm{Id}_{\mathcal{A}} \to GF$ is an isomorphism.*

Proof. 1. Let Set be the category of sets. For each object $Y \in \mathcal{B}$, consider the functor

$$\mathrm{Hom}_{\mathcal{B}}(F(\cdot), Y) : \mathcal{A}^{\mathrm{op}} \to \mathrm{Set}.$$

It transforms colimits into limits. Then by [Sta23, Tag 07D7], it is representable. From [ML13, Cor. 2, p.85], the functor F admits a right adjoint.

2. If follows from Yoneda's lemma.

□

By [Sta23, Tag 077P (2)], the inclusion $\iota = \iota_X : \mathrm{Qch}(O_X) \to \mathrm{Mod}(O_X)$ admits a right adjoint $Q_X = Q : \mathrm{Mod}(O_X) \to \mathrm{Qch}(O_X)$, called the *coherator* of X. To reduce the problem to the study of O_X-modules, consider the square

$$\begin{array}{ccc} \mathrm{Mod}(D_X) & \xrightarrow{Q'_X} & \mathrm{Mod}_{\mathrm{qc}}(D_X) \\ \downarrow{\scriptstyle \mathrm{for}_X} & & \downarrow{\scriptstyle \mathrm{for}_X} \\ \mathrm{Mod}(O_X) & \xrightarrow{Q_X} & \mathrm{Qch}(O_X), \end{array} \tag{62}$$

where the vertical functors are forgetful.

Lemma A.0.6. *Suppose that X is affine. Write $R = \Gamma(X, D_X)$. Then:*

1. *The functor $\tilde{\ } := D_X \otimes_R \cdot : \mathrm{Mod}(R) \to \mathrm{Mod}(D_X)$ is left adjoint to the global section functor $\Gamma(X, \cdot) : \mathrm{Mod}(D_X) \to \mathrm{Mod}(R)$;*

2. *The square (62) is commutative.*

Proof.

1. Let $(\sigma, \sigma^{\#}) : (X, D_X) \to (\{*\}, R)$ be the morphism of ringed spaces, with $\sigma : X \to \{*\}$ the unique map and $\sigma^{\#}$ given by Id_R. Then $\Gamma(X, \cdot) = \sigma_* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(R)$. By [Sta23, Tag 01BH], the functor $\tilde{\ } = \sigma^*$. The adjunction follows from [Sta23, Tag 0096].

2. From 1 and [HT07, Prop. 1.4.4 (ii)], the functor $Q' : \mathrm{Mod}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$ is the composition of $\Gamma(X, \cdot) : \mathrm{Mod}(D_X) \to \mathrm{Mod}(R)$ with $\tilde{\ } : \mathrm{Mod}(R) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$. The largest rectangle in the following diagram

$$\begin{tikzcd}
\mathrm{Mod}(D_X) \arrow[r, "\Gamma(X,\cdot)"] \arrow[d] \arrow[rr, bend left, "Q'"] & \mathrm{Mod}(R) \arrow[r, "D_X\otimes_R\cdot"] \arrow[d] & \mathrm{Mod}_{\mathrm{qc}}(D_X) \arrow[r, "\Gamma(X,\cdot)"] \arrow[d] & \mathrm{Mod}(R) \arrow[d] \\
\mathrm{Mod}(O_X) \arrow[r, "\Gamma(X,\cdot)"] \arrow[rr, bend right, "Q"'] & \mathrm{Mod}(O_X(X)) \arrow[r, "O_X\otimes_{O_X(X)}\cdot"] & \mathrm{Qch}(O_X) \arrow[r, "\Gamma(X,\cdot)"] & \mathrm{Mod}(O_X(X))
\end{tikzcd}$$

   is same as the small square on the left, hence commutative. Moreover, the two horizontal functors $\Gamma(X, \cdot)$ on the right are equivalences, so Q' is compatible with Q.

□

The abelian categories $\mathrm{Mod}(D_X)$ and $\mathrm{Mod}(O_X)$ are Grothendieck. By [Sta23, Tag 079P] and [Sta23, Tag 070K], the functor $Q' : \mathrm{Mod}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$ and $Q : \mathrm{Mod}(O_X) \to \mathrm{Qch}(O_X)$ admit right derived functors $RQ' : D(D_X) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_X))$ and $RQ : D(O_X) \to D(\mathrm{Qch}(O_X))$.

Lemma A.0.7. 1. *The square (62) is commutative.*

2. *The square*

$$\begin{array}{ccc}
D(D_X) & \xrightarrow{RQ'_X} & D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \\
\big\downarrow{\scriptstyle \mathrm{for}_X} & & \big\downarrow{\scriptstyle \mathrm{for}_X} \\
D(O_X) & \xrightarrow{RQ_X} & D(\mathrm{Qch}(O_X)),
\end{array}$$

is commutative.

Proof.

1. We deduce a formula for Q'_X. Since X is quasi-compact, there is a finite cover $\{U_\alpha\}_{\alpha\in I}$ of X by affine opens. For any $\alpha \neq \beta$ in I, since X is separated over k, the scheme $U_{\alpha\beta} := U_\alpha \cap U_\beta$ is affine. Denote all the various open immersions $U_{\alpha\beta} \to X$ and $U_\alpha \to X$ as j. For every D_X-module F, the sheaf axiom gives an equalizer diagram in $\mathrm{Mod}(D_X)$:

$$0 \to F \to \oplus_\alpha j_*(F|_{U_\alpha}) \rightrightarrows \oplus_{(\alpha,\beta)} j_*(F|_{U_{\alpha\beta}}),$$

where the two right morphisms are induced by the inclusions $U_{\alpha\beta} \to U_\alpha$ and $U_{\alpha\beta} \to U_\beta$. By Lemma A.0.8, it induces another equalizer diagram in $\mathrm{Mod}_{\mathrm{qc}}(D_X)$:

$$0 \to Q'_X F \to \oplus_\alpha j_* Q'_{U_\alpha}(F|_{U_\alpha}) \rightrightarrows \oplus_{(\alpha,\beta)} j_* Q'_{U_{\alpha\beta}}(F|_{U_{\alpha\beta}}). \qquad (63)$$

There is a natural transformation $\iota' Q'_X \to \mathrm{Id}_{\mathrm{Mod}(D_X)} : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_X)$. Applying $\mathrm{for}_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}(O_X)$, one gets a natural transformation $\mathrm{for}_X \circ \iota' \circ Q'_X \to \mathrm{for}_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}(O_X)$. Since $\mathrm{for}_X \circ \iota' = \iota \circ \mathrm{for}_X : \mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(O_X)$ and Q_X is right adjoint to ι, there is a natural transformation

$$\mu_X : \mathrm{for}_X \circ Q'_X \to Q_X \circ \mathrm{for}_X$$

of functors $\mathrm{Mod}(D_X) \to \mathrm{Qch}(O_X)$. By Lemma A.0.6 2, it is an isomorphism when X is affine.

For a general X, by (63) and [TT07, (B.14.2)], there is a commutative diagram of functors $\mathrm{Mod}(D_X) \to \mathrm{Qch}(O_X)$:

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{for}_X Q'_X & \longrightarrow & \oplus_\alpha j_* \mathrm{for}_{U_\alpha} Q'_{U_\alpha}(\cdot|_{U_\alpha}) & \rightrightarrows & \oplus_{(\alpha,\beta)} j_* \mathrm{for}_{U_{\alpha\beta}} Q'_{U_{\alpha\beta}}(\cdot|_{U_{\alpha\beta}}) \\ & & \downarrow{\scriptstyle \mu_X} & & \downarrow & & \downarrow \\ 0 & \longrightarrow & Q_X \mathrm{for}_X & \longrightarrow & \oplus_\alpha j_* Q_{U_\alpha} \mathrm{for}_{U_\alpha}(\cdot|_{U_\alpha}) & \rightrightarrows & \oplus_{(\alpha,\beta)} j_* Q_{U_{\alpha\beta}} \mathrm{for}_{U_{\alpha\beta}}(\cdot|_{U_{\alpha\beta}}), \end{array}$$

where the two vertical arrows on the right are isomorphisms. Therefore, μ_X is an isomorphism.

2. The morphism $(X, D_X) \to (X, O_X)$ of ringed spaces is flat, and the direct image functor is the forgetful functor $\mathrm{for}_X : \mathrm{Mod}(D_X) \to \mathrm{Mod}(O_X)$. By [Sta23, Tag 08BJ], it preserves K-injective complexes. The conclusion follows from Point 1, Lemma A.0.9 and [Sta23, Tag 070K].

□

Lemma A.0.8. *Let $j : U \to X$ be an open immersion. Then the natural transformation $j_* \circ Q'_U \to Q'_X \circ j_* : \mathrm{Mod}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$ is an isomorphism.*

Proof. As $j^* : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_U)$ restricts to a functor $\mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}_{\mathrm{qc}}(D_U)$, one has $\iota'_U j^* = j^* \iota'_X$ as functors $\mathrm{Mod}_{\mathrm{qc}}(D_X) \to \mathrm{Mod}(D_U)$. The functor $j_* : \mathrm{Mod}(D_U) \to \mathrm{Mod}(D_X)$ regards the direct image $j_* : \mathrm{Mod}(O_U) \to \mathrm{Mod}(O_X)$, so it also restricts to a functor $\mathrm{Mod}_{\mathrm{qc}}(D_U) \to \mathrm{Mod}_{\mathrm{qc}}(D_X)$. As Q' is right adjoint to ι' and j_* is right adjoint to j^*, the isomorphism follows. □

Lemma A.0.9. *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors of abelian categories. Assume that $\mathcal{A}$, $\mathcal{B}$ are Grothendieck. If for ever K-injective complex I over $\mathcal{A}$, the natural morphism $GF(I) \to RG(F(I))$ in $D(\mathcal{C})$ is an isomorphism,[2] then the canonical natural transformation (constructed in [Sta23, Tag 05T2 (1)]) $t : R(G \circ F) \to RG \circ RF$ is an isomorphism of functors from $D(\mathcal{A}) \to D(\mathcal{C})$.*

Proof. Let A be a complex over $\mathcal{A}$. As $\mathcal{A}$ is Grothendieck, by [Sta23, Tag 079P], there is a quasi-isomorphism $A \to I$ such that I is a K-injective complex. By [Sta23, Tag 070K], the morphism t_A is the composition of isomorphisms

$$R(G \circ F)(A) = GF(I) \to RG(F(I)) = RG(RF(A)).$$

□

Proof of Theorem A.0.2. By [Sta23, Tag 09T5], $RQ' : D(D_X) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_X))$ is right adjoint to $L\iota' = \iota' : D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \to D(D_X)$. Let $\Psi' : D_{\mathrm{qc}}(D_X) \to D(\mathrm{Mod}_{\mathrm{qc}}(D_X))$ (resp. $\Psi : D_{\mathrm{qc}}(O_X) \to D(\mathrm{Qch}(O_X))$) be the restriction of RQ' (resp. RQ). By Lemma A.0.7 2, there are natural commutative squares

$$\begin{array}{ccc} D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) & \xrightarrow{L\iota'} & D_{\mathrm{qc}}(D_X) \\ \downarrow{\scriptstyle \mathrm{for}} & & \downarrow{\scriptstyle \mathrm{for}} \\ D(\mathrm{Qch}(O_X)) & \xrightarrow{L\iota} & D_{\mathrm{qc}}(O_X), \end{array} \qquad \begin{array}{ccc} D_{\mathrm{qc}}(D_X) & \xrightarrow{\Psi'} & D(\mathrm{Mod}_{\mathrm{qc}}(D_X)) \\ \downarrow{\scriptstyle \mathrm{for}} & & \downarrow{\scriptstyle \mathrm{for}} \\ D_{\mathrm{qc}}(O_X) & \xrightarrow{\Psi} & D(\mathrm{Qch}(O_X)), \end{array}$$

where $L\iota$ is induced by the inclusion $\iota : \mathrm{Qch}(O_X) \to \mathrm{Mod}(O_X)$.

Since Ψ is right adjoint to ι, the counit $\epsilon' : \iota'\Psi' \to \mathrm{Id}_{D_{\mathrm{qc}}(D_X)}$ (resp. unit $\eta' : \mathrm{Id}_{D(\mathrm{Mod}_{\mathrm{qc}}(D_X))} \to \Psi'\iota'$) is compatible with the counit $\epsilon : \iota\Psi \to \mathrm{Id}_{D_{\mathrm{qc}}(O_X)}$ (resp. unit $\eta : \mathrm{Id}_{D(\mathrm{Qch}(O_X))} \to \Psi\iota$). The functor for : $D(D_X) \to D(O_X)$ is conservative. By [Sta23, Tag 09T4], the counit ϵ and the unit η are isomorphisms, so are the counit ϵ' and the unit η'. In particular, the functor (61) is an equivalence with a quasi-inverse Ψ'. □

B When is an induced D-module holonomic?

Proposition B.0.1. *Let X be a complex manifold. Let F be an O_X-module. Then the following conditions are equivalent:*

1. *the induced module $D_X \otimes_{O_X} F$ is holonomic;*

[2] *i.e.*, $F(I)$ computes RG in the sense of [Sta23, Tag 05SX (1)]

2. *F is coherent with* $\operatorname{Supp}(F)$ *discrete.*

Lemma B.0.2 and Lemma B.0.3 are needed for the proof of Proposition B.0.1.

Lemma B.0.2. *Let A be a Gorenstein local ring* (in the sense of [Sta23, Tag 0DW7 (1)]) *of Krull dimension n. Let M be a finite A-module. Then the following conditions are equivalent:*

1. *For all integers $i \neq n$, one has* $\operatorname{Ext}^i(M, A) = 0$*;*

2. *the length of M is finite.*

Proof. Let k be the residue field of A.

- Assume Condition 1. To prove 2, one may assume $M \neq 0$. As A is Gorenstein, $A[0]$ is a dualizing complex of A. By [Mat87, Thm. 18.1, p.141], one has $R\mathcal{H}om_A(k, A[n]) = k[0]$, so $A[n]$ is the normalized dualizing complex of A (in the sense of [Sta23, Tag 0A7M]). Let d be the depth of M. By [Sta23, Tag 0B5A], the module M is Cohen-Macaulay and

$$M = \operatorname{Ext}_A^{n-d}(\operatorname{Ext}_A^{n-d}(M, A), A).$$

  Thus, $\operatorname{Ext}_A^{n-d}(M, A) \neq 0$. By Condition 1, one has $n - d = n$. Hence $\dim \operatorname{Supp}(M) = d = 0$. By [Ati69, Exercise 19 v), p.46], one has $\dim A/\operatorname{Ann}(M) = 0$. Then $A/\operatorname{Ann}(M)$ is an artinian ring. From [Eis13, Cor. 2.17], the length of M is finite.

- Assume Condition 2. Induction on the length $l(M)$ of M. When $l(M) = 0$, one has $M = 0$ and Condition 1 holds. Now assume $l(M) > 0$ and the statement holds for all modules of length less than $l(M)$. There is a submodule N of M such that M/N is a simple module and $l(N) < l(M)$. By [Sta23, Tag 00J2], the module M/N is isomorphic to k. For every integer $i \neq n$, the short exact sequence $0 \to N \to M \to M/N \to 0$ induces an exact sequence $\operatorname{Ext}^i(M/N, A) \to \operatorname{Ext}^i(M, A) \to \operatorname{Ext}^i(N, A)$. By the inductive hypothesis, $\operatorname{Ext}^i(N, A) = 0$. By [Mat87, Thm. 18.1, p.141], one has $\operatorname{Ext}^i(M/N, A) = 0$. Hence $\operatorname{Ext}^i(M, A) = 0$.

□

Lemma B.0.3. *Let X be a complex analytic space. Let F be a coherent O_X-module. Then the length of the $O_{X,x}$-module F_x is finite for all $x \in X$ if and only if the subspace* $\operatorname{Supp}(F) \subset X$ *is discrete.*

Proof. The "if" part follows from [Liu23a, Lem. 5.2.4 1]. We prove the "only if" part. By coherence of F and [GR84, p.76], $\operatorname{Supp}(F)$ is a closed analytic set of X. Assume to the contrary that $\operatorname{Supp}(F)$ is not discrete. Then $\dim \operatorname{Supp}(F) > 0$. Let C be an irreducible component of $\operatorname{Supp}(F)$ of maximal dimension. Endow C with the reduced induced closed subspace structure. Let $i : C \to X$ be the closed embedding of complex analytic spaces.

For every $x \in C$, the morphism $O_{X,x} \to O_{C,x}$ is surjective. Then by [Sta23, Tag 00IX], one has $l_{O_{C,x}}(i^*F)_x = l_{O_{X,x}}(i^*F)_x$. The morphism $F_x \to (i^*F)_x$ of $O_{X,x}$-modules is surjective, so $l_{O_{X,x}}(i^*F)_x \leq l_{O_{X,x}} F_x$. In particular, the length of $(i^*F)_x$ over $O_{C,x}$ is finite. By [**?**, Cor. 5.2.4.1], the support of i^*F is C. Replacing (X, F) by (C, i^*F), one may assume further that X is irreducible with $\dim X > 0$.

By the generic freeness [Ros68, Prop. 3.1], there is $x_0 \in X$ such that F_{x_0} is a free O_{X,x_0}-module. As the support of F is X, from [RS17, p.238], F is not a torsion sheaf. Then by irreducibility of X and [Ros68, p.69], the O_{X,x_0}-module F_{x_0} has positive rank. Thus, O_{X,x_0} has finite length over itself, hence an artinian ring. The dimension formula in [GR84, p.96] and [CD94, (14.14), p.89] yield $\dim X = \dim_{x_0} X = \dim O_{X,x} = 0$, a contradiction. □

Proof of Proposition B.0.1. Let $M = D_X \otimes_{O_X} F$ and $\hat{F} = R\mathcal{H}om_{O_X}(F, O_X)$. By [Sta23, Tag 08DJ], one has

$$\mathcal{H}om_{O_X}(\omega_X, \hat{F}) = R\mathcal{H}om_{O_X}(\omega_X \otimes_{O_X} F, O_X). \tag{64}$$

Provided that F is *coherent*, [Bjö93, (ii) p.122] gives

$$\Delta^{D_X} M = D_X \otimes_{O_X} \mathcal{H}om_{O_X}(\omega_X, \hat{F})[\dim X]. \tag{65}$$

Plugging (64) into (65), one gets

$$\Delta^{D_X} M = D_X \otimes_{O_X} R\mathcal{H}om_{O_X}(\omega_X \otimes_{O_X} F, O_X)[\dim X].$$

For every nonzero integer i, one has

$$H^i(\Delta^{D_X} M) = D_X \otimes_{O_X} \mathcal{E}xt^{i+\dim X}_{O_X}(\omega_X \otimes_{O_X} F, O_X).$$

By [Sta23, Tag 01CB] and [GH78, 1. p.700], its stalk at $x \in X$ is isomorphic to

$$D_{X,x} \otimes_{O_{X,x}} \mathrm{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x})$$

.

- Assume Condition 2. By [Bjö93, 1.5.1], the D_X-module M is coherent. By Lemma B.0.3, the $O_{X,x}$-module F_x has finite length. As $O_{X,x}$ is a noetherian regular local ring of Krull dimension $\dim_x X$, by Lemma B.0.2, one has $\mathrm{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}) = 0$ for all $x \in X$. Hence $H^i(\Delta^{D_X} M) = 0$. From Fact 7.2.2 2, the D_X-module M is holonomic.
- Assume Condition 1. From [SS94, p.55], the O_X-module F is coherent. From Fact 7.2.2 2, for every nonzero integer i, one has $H^i(\Delta^{D_X} M) = 0$. As $D_{X,x}$ is a nonzero free $O_{X,x}$-module, one gets $\mathrm{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}) = 0$. By Lemma B.0.2, the $O_{X,x}$-module F_x has finite length for every $x \in X$. From Lemma B.0.3, the support of F is discrete.

□

The proof of Proposition B.0.4 (an algebraic analog of Proposition B.0.1) is similar.

Proposition B.0.4. *Let X be a smooth algebraic variety over an algebraically closed field of characteristic* 0*. Let F be an O_X-module. Then the following conditions are equivalent:*

1. *the induced module $D_X \otimes_{O_X} F$ is holonomic;*
2. *F is coherent with* $\mathrm{Supp}(F)$ *finite.*

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