Infinitesimal cohomology and singular cohomology

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Abstract

Grothendieck conjectures that for every algebraic variety X over $\mathbb{C}$, the singular cohomology $H^*(X^{\mathrm{an}}, \mathbb{C})$ is canonically isomorphic to the infinitesimal cohomology $H^*_{\mathrm{inf}}(X/\mathbb{C})$. The conjecture is proved by Deligne, so that infinitesimal cohomology provides a purely algebraic way to compute singular cohomology, even for singular varieties. Hartshorne introduces the de Rham cohomology $H^*_{\mathrm{DR}}(X/\mathbb{C})$, and shows that it is also isomorphic to $H^*(X^{\mathrm{an}}, \mathbb{C})$. We prove the folklore comparison isomorphism $H^*_{\mathrm{inf}}(X/\mathbb{C}) \to H^*_{\mathrm{DR}}(X/\mathbb{C})$. Combined with Hartshorne's theorem, it provides an alternative proof of Grothendieck's conjecture. The definition of $H^*_{\mathrm{DR}}(X/\mathbb{C})$ is extrinsic, and the isomorphism gives it an intrinsic interpretation. The comparison result is proved for crystals. . It also incorporates the comparison of the infinitesimal filtration with the Hodge filtration.

We work out several fundamental properties of infinitesimal cohomology over a general base scheme in characteristic 0, parallel to those of crystalline cohomology in characteristic $p > 0$. Some examples are given to illustrate certain properties of crystalline sites that fail for infinitesimal sites.

1 Introduction

For a complex algebraic variety X, let $H^*(X^{\mathrm{an}}, \mathbb{C})$ be its singular cohomology defined in terms of the complex analytification X^{an}. Suppose that X is smooth. Then the de Rham theorem shows that the singular cohomology can be computed using differential forms. Let $\Omega^\bullet_{X^{\mathrm{an}}}$ be the complex of holomorphic differential forms on the complex manifold X^{an}. Let $H^*_{\mathrm{dR}}(X^{\mathrm{an}})$ be the hypercohomology $H^*(X^{\mathrm{an}}, \Omega^\bullet_{X^{\mathrm{an}}})$. Then by Poincaré's lemma, as X is smooth, the morphism $\mathbb{C} \to \Omega^\bullet_{X^{\mathrm{an}}}$ is a resolution of sheaves on X^{an}, so the canonical morphism $H^*(X^{\mathrm{an}}, \mathbb{C}) \to H^*_{\mathrm{dR}}(X^{\mathrm{an}})$ is an isomorphism.

Grothendieck [Gro66, Theorem 1'] offers a purely *algebraic* way to compute the singular cohomology. Let $\Omega^\bullet_{X/\mathbb{C}}$ be the de Rham complex of algebraic differential forms. Its hypercohomology $H^*_{\mathrm{dR}}(X/\mathbb{C}) := H^*(X, \Omega^\bullet_{X/\mathbb{C}})$ is called the *algebraic de Rham cohomology* of X.

Fact 1.1. *Let X be a scheme* smooth *over $\mathbb{C}$. Then the canonical morphism $H^*_{\mathrm{dR}}(X/\mathbb{C}) \to H^*_{\mathrm{dR}}(X^{\mathrm{an}})$ is an isomorphism. In particular, there is a canonical isomorphism $H^*_{\mathrm{dR}}(X/\mathbb{C}) \cong H^*(X^{\mathrm{an}}, \mathbb{C})$.*

Over a field of characteristic 0, the algebraic de Rham cohomology is a Weil cohomology theory (see, e.g., [Sta25, Tag 0FWC]). In characteristic $p > 0$, it no longer has reasonable properties, roughly because of $d(x^p) = 0$. To construct a satisfactory p-adic cohomology, Grothendieck [Gro68] uses a "topology" for which locally means differentiably. In characteristic 0, this topology is know as the infinitesimal site, which provides an alternative to differential forms. Let $H^*_{\mathrm{inf}}(X/\mathbb{C})$ be the infinitesimal cohomology of X associated with the infinitesimal site $\mathrm{Inf}(X/\mathbb{C})$.

Fact 1.2 ([Gro68, Theorem 4.1])**.** *Let X be a scheme smooth over $\mathbb{C}$. Then there is a canonical isomorphism $H^*_{\mathrm{inf}}(X/\mathbb{C}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X/\mathbb{C})$ and hence $H^*_{\mathrm{inf}}(X/\mathbb{C}) \cong H^*(X^{\mathrm{an}}, \mathbb{C})$.*

Grothendieck [Gro68, Conjecture 4.2] conjectures that even if X is a singular variety, $H^*_{\mathrm{inf}}(X/\mathbb{C}) \cong H^*(X^{\mathrm{an}}, \mathbb{C})$ still holds. According to [HL71, p.98], the conjecture is proved by Deligne, whose proof remains unpublished.

Theorem 1.3 (Theorem 14.1)**.** *Let X be a scheme locally of finite type over $\mathbb{C}$. Then there is a canonical isomorphism $H^*_{\mathrm{inf}}(X/\mathbb{C}) \cong H^*(X^{\mathrm{an}}, \mathbb{C})$.*

It is well-known that Hartshorne's work [Har75] can be used to provide an alternative proof of Theorem 1.3. For embeddable X, i.e., admitting a closed immersion $X \hookrightarrow Y$ into a smooth variety Y over $\mathbb{C}$, Hartshorne introduces another de Rham cohomology $H^*_{\mathrm{DR}}(X/\mathbb{C})$. He shows that this group is independent of the choice of the embedding, and establishes a comparison isomorphism

$$H^*_{\mathrm{DR}}(X/\mathbb{C}) \cong H^*(X^{\mathrm{an}}, \mathbb{C}). \tag{1}$$

For a general X which may not be embeddable, the theory is technically more difficult, as sketched in [Har75, Remark, p.28]. As the reduction of Grothendieck's conjecture to Hartshorne's theorem is not documented in the literature, we provide some details. We shall give a proof of the folklore comparison theorem of infinitesimal cohomology and Hartshorne's de Rham cohomology. Combined with (1), Theorem 1.4 implies Theorem 1.3.

Theorem 1.4. *Let S be a locally Noetherian scheme of characteristic 0. Let $i : X \to Y$ be a closed immersion of schemes over S, with Y smooth over S. Let $\hat{\Omega}^\bullet_{Y/S}$ be the formal completion of $\Omega^\bullet_{Y/S}$ along X. Then for every $i \geq 0$, there is a canonical isomorphism $H^i_{\mathrm{inf}}(X/S) \xrightarrow{\sim} H^i(Y, \hat{\Omega}^\bullet_{Y/S})$.*

In Theorem 11.2 (b), we show a comparison result for the infinitesimal cohomology with coefficients in a crystal. The special case with the structure crystal $O_{X/S}$ gives Theorem 1.4. We also compare the natural infinitesimal filtration on infinitesimal cohomology with the Hodge filtration on de Rham cohomology. If X is smooth proper over $\mathbb{C}$, then the infinitesimal filtration $F^q H^i_{\mathrm{inf}}(X/\mathbb{C})$ agrees with the Hodge filtration $F^q H^i(X^{\mathrm{an}}, \mathbb{C})$ (Corollary 11.11). For a singular proper complex variety X, the infinitesimal filtration is finer than the Hodge filtration (Remark 14.2 (b)). The infinitesimal Chern class $c_p : K^0(X) \to H^{2p}_{\mathrm{inf}}(X/\mathbb{C})$ has image in $F^p H^{2p}_{\mathrm{inf}}(X/\mathbb{C})$, so the infinitesimal cohomology imposes more constrains on the Chern classes of singular varieties.

The proof of Theorem 1.4 follows Grothendieck's approach to Fact 1.2, i.e., the case where $i : X \to Y$ is id_X. He introduces stratifications as linear counterparts of non-linear differential operators, and assigns a special sheaf, called crystal, on $\mathrm{Inf}(X/S)$ to every stratified O_Y-module. Thus, he constructs a linearization functor L, turning differential operators of O_Y-modules to linear morphism of crystals. The de Rham complex $\Omega^\bullet_{Y/S}$ is a complex of differential operators of order ≤ 1. By a Poincaré lemma, the linearization $L(\Omega^\bullet_{Y/S})$ of the the de Rham complex is a resolution of the infinitesimal structure sheaf $O_{X/S}$. This gives $H^*_{\mathrm{inf}}(X/S) \xrightarrow{\sim} H^*_{\mathrm{inf}}(X/S, L(\Omega^\bullet_{Y/S}))$. The Čech-Alexander technique computes cohomology in a topos. In this way, Grothendieck computes $H^*_{\mathrm{inf}}(X/S, L(\Omega^\bullet_{Y/S}))$ by unfolding the Čech-Alexander complex $\mathrm{CA}^\bullet_Y(L(\Omega^\bullet_{Y/S}))$ of the linearized complex $L(\Omega^\bullet_{Y/S})$.

In characteristic p, the definition of infinitesimal site using nilpotent thickenings can be modified to crystalline site ([Gro68, p.351]). Berthelot [Ber74] shows that it establishes a p-adic Weil cohomology, known as the crystalline cohomology.

As [Ber74, p.11] mentions, compared with crystalline cohomology, the extra technicalities of infinitesimal cohomology arise from inverse limits related to the nilpotence condition. On the crystalline side, the Čech-Alexander complex $\mathrm{CA}^\bullet_Y$ is the evaluation at a particular object of the crystalline site $\mathrm{Cris}(X/S)$, namely the divided power infinitesimal neighborhood of infinite order $D_X(Y)$ ([Ber74, I, Définition 4.1.7]). On the infinitesimal side, however, the corresponding infinitesimal neighborhood of infinite order $\Delta_X(Y)$ is a formal scheme and no longer a scheme. Although not an object of $\mathrm{Inf}(X/S)$, it is represented by a direct system $(\Delta^i_X(Y))_{i\geq 0}$ in $\mathrm{Inf}(X/S)$ comprised of infinitesimal neighborhood of finite order. For this reason, the definition of $\mathrm{CA}^\bullet_Y$ in infinitesimal cohomology contains an inverse limit. Similarly, the crystalline linearization functor is defined using the structure sheaf of $D_Y(Y \times_S Y)$, while to define the infinitesimal linearization, Grothendieck replaces it with the limits of the inverse system of structure sheaves of $(\Delta^i_Y(Y \times_S Y))_{i\geq 0}$.

Several problems with the inverse limit functor are that it is not right exact, and does not commute with tensor product. An inverse limit of quasi-coherent sheaves may not be quasi-coherent. Inverse limit of sheaves may not commute with pullback.

Remark 1.5. Bhatt and de Jong [BJ11, Remark 3.7] sketches another strategy to proving Theorem 1.4, without stratifications nor linearizations. For crystalline site, they give a vanishing theorem [BJ11, Theorem 3.2] for the crystalline sheaf of differentials $\Omega^i_{X/S,\mathrm{cris}}$ with $i > 0$, defined as $(\Omega^i_{X/S,\mathrm{cris}})_T = \Omega^i_{T/S}$ for all $(U, T, \delta) \in \mathrm{Cris}(X/S)$. This cohomological vanishing result reduces the crystalline version of Theorem 1.4 to affine case, which in turn is proved with a Čech-theoretic approach. An infinitesimal analog of the vanishing theorem for the infinitesimal sheaf of differentials is [Cor03, Proposition 7.9] (see also [CHW09, Theorem 1.9]). The strategy in [BJ11, Remark 3.7] also counts on the Čech-Alexander technique, so carrying it out shall lead to similar technical problems with inverse limits.

Notation

For an abelian category $\mathcal{A}$, let $\mathrm{Ch}^+(\mathcal{A})$ (resp. $\mathrm{Ch}^{\geq 0}(\mathcal{A})$) be the category of bounded below cochain complexes (resp. complexes in non-negative degrees) over $\mathcal{A}$. For a topological space X, let $\mathrm{Ab}(X)$ be the category of abelian sheaves on X. Given a category $\mathcal{C}$, let $\mathrm{PSh}(\mathcal{C})$ be the category of presheaves on $\mathcal{C}$, i.e., functors $\mathcal{C}^{\mathrm{op}} \to \mathrm{Set}$. For a site $\mathcal{S}$, let $\mathrm{Sh}(\mathcal{S})$ be the category of sheaves on $\mathcal{S}$. Let e be the presheaf assigning the singleton $\{*\}$ to all objects of $\mathcal{S}$. Then $e \in \mathrm{Sh}(\mathcal{S})$ and is a final object of $\mathrm{PSh}(\mathcal{S})$. The global section functor $\mathrm{Sh}(\mathcal{S}) \to \mathrm{Set}$ is $\Gamma(\mathrm{Sh}(\mathcal{S}), \cdot) := \mathrm{Hom}_{\mathrm{Sh}(\mathcal{S})}(e, \cdot)$. For a ringed site $(\mathcal{C}, O_{\mathcal{C}})$, let $\mathrm{Mod}(O_{\mathcal{C}}) = \mathrm{Mod}(\mathcal{C}, O_{\mathcal{C}})$ be the category of $O_{\mathcal{C}}$-modules on $\mathcal{C}$.

For an integer $n \geq 0$ and an immersion of schemes $X \hookrightarrow Y$, let $\Delta^n_X(Y)$ be the n-th infinitesimal neighborhood of X in Y. Let $P^n_X(Y)$ be the structure sheaf of the scheme $\Delta^n_X(Y)$. Let $P_X(Y) := \lim_{n\geq 0} P^n_X(Y)$ be the completion of O_Y along X in the sense of [GD71, Définition 10.8.2], where each $P^n_X(Y)$ is a pseudo-discrete sheaf of rings. Then $P_X(Y)$ is an O_Y-algebra supported on X. By [GD71, Proposition 10.6.3], the topologically ringed space $(X, P_X(Y))$ is a formal scheme, denoted by $\Delta_X(Y)$.

Remark 1.6. From [GD71, Proposition 10.8.3], if $X \to Y$ is a closed immersion defined by an ideal sheaf I of finite type, then $\Delta_X(Y)$ is an adic formal scheme, which is called the *formal completion* of Y along X.

2 Infinitesimal site

We recall the infinitesimal site, infinitesimal topos, the structure sheaf, the infinitesimal ideal sheaf. We only emphasize differences compared with crystalline cohomology theory. If a result for crystalline cohomology has an analog for infinitesimal cohomology and can be proved similarly, we use it directly.

Fix two universes $\underline{U}$ and $\underline{V}$ with $\underline{U} \in \underline{V}$. Let $f : X \to S$ be a morphism of schemes. Recall the *infinitesimal site* $\mathrm{Inf}(X/S)$ introduced in [Gro68, p.331].

Definition 2.1. Define a category $\mathrm{Inf}(X/S)$ as follows. An object of it is a finite order thickening $U \to T$ over S, where U is a Zariski open subset of X, and T is an S-scheme belonging to $\underline{U}$. By a covering family $\{(U_i, T_i) \to (U, T)\}_{i\in I}$, we mean that for every $i \in I$, $T_i \to T$ is an open immersion, and $T = \cup_{i\in I} T_i$.

From $\underline{U} \in \underline{V}$, $\mathrm{Inf}(X/S)$ is a small site in $\underline{V}$. Let $(X/S)_{\mathrm{inf}}$ be the corresponding topos, called the *infinitesimal topos* of X over S. For a description of sheaves of sets on $\mathrm{Inf}(X/S)$, see [Ber74, III, 1.1.4]. For a sheaf E on $\mathrm{Inf}(X/S)$ and $(U, T) \in \mathrm{Inf}(X/S)$, let $E_T = E_{(U,T)}$ be the induced Zariski sheaf on T.

Remark 2.2. The infinitesimal topos has enough points in the following sense. Let P denote the punctual topos. For $(U,T) \in \mathrm{Inf}(X/S)$ and $t \in T$, the functor

$$\xi^*_{T,t} : \mathrm{Sh}(\mathrm{Inf}(X/S)) \to \mathrm{Set}, \quad E \mapsto E_{T,t}$$

is the pullback functor of a morphism of topoi $\xi_{T,t} : P \to (X/S)_{\mathrm{inf}}$, called a point of $(X/S)_{\mathrm{inf}}$. From the proof of [Ber74, III, Proposition 2.1.10], the family of points $\xi_{T,t}$ (with $(U,T) \in \mathrm{Inf}(X/S)$ and $t \in T$) is conservative.

Let $O_{X/S}$ be the structure sheaf of rings on $\mathrm{Inf}(X/S)$. Then $((X/S)_{\mathrm{inf}}, O_{X/S})$ is a ringed topos. Let

$$u_{X/S} : (X/S)_{\mathrm{inf}} \to X_{\mathrm{Zar}}$$

be the morphism of topoi from [Ber74, III, Proposition 3.2.3], called the *projection from the infinitesimal topos to the Zariski topos.* For every $(U,T) \in \mathrm{Inf}(X/S)$ and every sheaf $F \in \mathrm{Sh}(X)$, the inverse image $(u^*_{X/S}F)_{(U,T)}$ is $F|_U$ viewed as a sheaf on T.

Remark 2.3. The morphism of topoi $u_{X/S} : (X/S)_{\mathrm{inf}} \to X_{\mathrm{Zar}}$ may not define a morphism of ringed topoi $((X/S)_{\mathrm{inf}}, O_{X/S}) \to (X, O_X)$. We prove that

$$u_{X/S} : ((X/S)_{\mathrm{inf}}, O_{X/S}) \to (X, f^{-1}O_S)$$

is a morphism of ringed topoi, for which we need to construct a morphism $\phi : u^*_{X/S}f^{-1}O_S \to O_{X/S}$ of shaves of rings. For every object $(U,T) \in \mathrm{Inf}(X/S)$, let $g_T : T \to S$ be the structure morphism. As $(u^*_{X/S}f^{-1}O_S)_{(U,T)}$ is $(f^{-1}O_S)|_U$ seen as a sheaf on T, it equals $g_T^{-1}O_S$. The canonical morphism $g_T^{-1}O_S \to O_T$ gives

$$\phi_{(U,T)} : (u^*_{X/S}f^{-1}O_S)_T \to O_T.$$

These morphisms are compatible and define ϕ.

Let

$$i_{X/S} : X_{\mathrm{Zar}} \to (X/S)_{\mathrm{inf}}$$

be the morphism of ringed topoi from [Ber74, III, Proposition 3.3.2], known as the *immersion of the Zariski topos into the infinitesimal topos.* It satisfies $u_{X/S} \circ i_{X/S} = \mathrm{Id}_{X_{\mathrm{Zar}}}$. Let $J_{X/S}$ be the sheaf on $\mathrm{Inf}(X/S)$ defined in [Ber74, III, 1.1.4]. For every $(U,T) \in \mathrm{Inf}(X/S)$, one has $J_{X/S,T} = \ker(O_T \to O_U)$. There is a canonical exact sequence

$$0 \to J_{X/S} \to O_{X/S} \to i_{X/S*}O_X \to 0$$

of $O_{X/S}$-modules.

3 Functoriality of infinitesimal topos

We prove that the formation of infinitesimal topos is functorial, in the sense that for every commutative square

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ S' \arrow[r, "u"] & S \end{tikzcd} \tag{2}$$

of schemes, there is a canonical morphism of ringed topoi $g_{\inf} : (X'/S')_{\inf} \to (X/S)_{\inf}$. Let $R\Gamma_{\inf}(X/S) := R\Gamma((X/S)_{\inf}, O_{X/S})$, and for every $i \geq 0$, let $H^i_{\inf}(X/S) := H^i R\Gamma_{\inf}(X/S)$ be the i-th infinitesimal cohomology group. This functoriality shows that there is a canonical morphism $R\Gamma_{\inf}(X/S) \to R\Gamma_{\inf}(X'/S')$ and hence $H^i_{\inf}(X/S) \to H^i_{\inf}(X'/S')$.

Definition 3.1. For objects $(U, T) \in \mathrm{Inf}(X/S)$ and $(U', T') \in \mathrm{Inf}(X'/S')$, a morphism $h : T' \to T$ over S is called a *g-morphism*, if $g(U') \subset U$, and if the diagram

$$\begin{tikzcd}
U' \arrow[r, "g|_{U'}"] \arrow[d, hook] & U \arrow[d, hook] \\
T' \arrow[r, "h"] & T
\end{tikzcd}$$

is commutative. Let $\mathrm{Hom}_g(T', T)$ be the set of g-morphisms $T' \to T$.

For every $(U, T) \in \mathrm{Inf}(X/S)$, define a presheaf $g^*T = g^*(U, T)$ on $\mathrm{Inf}(X'/S')$ by

$$g^*(U, T)(U', T') = \mathrm{Hom}_g(T', T). \tag{3}$$

By gluing morphisms of schemes, one can prove that g^*T is a sheaf on $\mathrm{Inf}(X'/S')$. Thus, one has a functor

$$g^* : \mathrm{Inf}(X/S) \to \mathrm{Sh}(\mathrm{Inf}(X'/S')).$$

Remark 3.2. In Diagram (2), if $g = \mathrm{Id}_X$ and $u = \mathrm{Id}_S$, then a morphism in $\mathrm{Inf}(X/S)$ is exactly an Id_X-morphism. For every $(U, T) \in \mathrm{Inf}(X/S)$, $h_T := g^*(U, T)$ is the sheaf representable by (U, T). Therefore, the topology on $\mathrm{Inf}(X/S)$ is subcanonical, i.e., every representable functor $\mathrm{Inf}(X/S)^{\mathrm{op}} \to \mathrm{Set}$ is a sheaf.

For every $F \in \mathrm{Sh}(\mathrm{Inf}(X'/S'))$, let $g_{\inf *}F := F \circ g^*$ denote the pullback presheaf on $\mathrm{Inf}(X/S)$, i.e., for every $(U, T) \in \mathrm{Inf}(X/S)$, one has

$$(g_{\inf *}F)(U, T) = \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Inf}(X'/S'))}(g^*(U, T), F). \tag{4}$$

By [Ber74, III, Lem 2.2.2], $g_{\inf *}F$ is a sheaf on $\mathrm{Inf}(X/S)$.

Theorem 3.3. *For the commutative diagram* (2)*, there is a unique morphism of topoi* $g_{\inf} : (X'/S')_{\inf} \to (X/S)_{\inf}$ *such that for every* $(U, T) \in \mathrm{Inf}(X/S)$*, one has* $g^{-1}_{\inf} h_T = g^*T$*. Moreover, it is naturally a morphism of ringed topoi.*

Proof. Assume that $g_{\inf}$ is such a morphism. As $g_{\inf *}$ is right adjoint to $g^{-1}_{\inf}$, it is defined by (4). The uniqueness of $g_{\inf}$ follows. Similar to [BO78, pp. 5.8–5.11], the existence of $g_{\inf}$ follows from Remark 2.2 and Lemma 3.4. Similar to [Ber74, III, Corollaire 2.2.4], one can prove that $g_{\inf}$ is a morphism of ringed topoi. $\square$

Lemma 3.4. *In the notation of* (2)*, fix an object* $(U', T') \in \mathrm{Inf}(X'/S')$ *and a point* $t' \in T'$*. Define a category* $I_{t', T', g}$ *as follows. An object of it is a g-morphism* $h : T'_0 \to T$*, where* $(U, T) \in \mathrm{Inf}(X/S)$*,* T'_0 *is an open neighborhood of*

t' in T', and $U'_0 := U' \times_{T'} T'_0$ so that $(U'_0, T'_0) \in \mathrm{Inf}(X'/S')$. A morphism from $h_1 : T'_1 \to T_1$ to $h_2 : T'_2 \to T_2$ in $I_{t',T',g}$ is a morphism $(U_1, T_1) \to (U_2, T_2)$ in $\mathrm{Inf}(X/S)$ such that the diagram

$$\begin{tikzcd}
T'_1 \arrow[r, hook] \arrow[d, "h_1"] & T'_2 \arrow[d, "h_2"] \\
T_1 \arrow[r] & T_2
\end{tikzcd}$$

is commutative, where $T'_1 \hookrightarrow T'_2$ is the inclusion.

Then the opposite category $(I_{t',T',g})^{\mathrm{op}}$ is a nonempty filtered category in the sense of [Sta25, Tag 002V].

Proof. We prove that $I_{t',T',g}$ is nonempty. Let $u' \in U'$ be the preimage of $t' \in T'$. Let $x = g(u') \in X$. Choose an affine neighborhood U of x in X. Let U'_0 be an affine neighborhood of u' in $U' \cap g^{-1}(U)$. Let T'_0 be the open subscheme of T' with underlying subset U'_0. Then $U'_0 \to T'_0$ is the base change of $U' \to T'$ along $T'_0 \to T'$, so $(U'_0, T'_0) \in \mathrm{Inf}(X'/S')$. By Lemma 3.8, as $g|_{U'_0} : U'_0 \to U$ is affine, there is an object $(U, T) \in \mathrm{Inf}(X/S)$ and a $g|_{U'_0}$-morphism $h : T'_0 \to T$. Then h is an object of $I_{t',T',g}$.

We prove that $I_{t',T',g}$ is a connected category. For objects $h_1 : T'_1 \to T_1$ and $h_2 : T'_2 \to T_2$ of $I_{t',T',g}$, let

$$U_3 := U_1 \cap U_2, \quad U'_3 := U'_1 \cap U'_2, \quad T'_3 := T'_1 \cap T'_2.$$

Then $U'_3 \to T'_3$ is the base change of $U' \to T'$ along $T'_3 \to T'$, so $(U'_3, T'_3) \in \mathrm{Inf}(X'/S')$. One has $g(U'_3) \subset g(U'_1) \cap g(U'_2) \subset U_3$. Then by Lemma 3.7, there is an object $(U_3, T_3) \in \mathrm{Inf}(X/S)$ and a g-morphism $h : T'_3 \to T_3$ fitting into a commutative diagram

$$\begin{tikzcd}
T'_3 \arrow[r, dashed, "h" description] \arrow[d] \arrow[rr, bend left] & T_3 \arrow[r, dashed] \arrow[d, dashed] & T_1 \arrow[d] \\
S' \arrow[rr, bend right, "u"'] & T_2 \arrow[r] & S.
\end{tikzcd}$$

Then h is an object of $I_{t',T',g}$ and $h \to h_1$ and $h \to h_2$ are morphisms in $I_{t',T',g}$.

Consider objects $h_1 : T'_1 \to T_1$ and $h_2 : T'_2 \to T_2$ of $I_{t',T',g}$, and two morphisms $a, b : h_1 \to h_2$, depicted as

$$\begin{tikzcd}
T'_1 \arrow[r, hook] \arrow[d, "h_1"] & T'_2 \arrow[d, "h_2"] & T'_1 \arrow[r, hook] \arrow[d, "h_1"] & T'_2 \arrow[d, "h_2"] \\
T_1 \arrow[r, "a"] & T_2, & T_1 \arrow[r, "b"] & T_2.
\end{tikzcd}$$

By Lemma 3.5, there is an object (U_1, T_0) and a morphism $p : (U_1, T_0) \to (U_1, T_1)$ in $\mathrm{Inf}(X/S)$, which satisfy $a \circ p = b \circ p$ as morphism $T_0 \to T_2$ and the universal property. As morphisms $T_1' \to T_2$, both $a \circ h_1$ and $b \circ h_1$ coincide with the composition $T_1' \hookrightarrow T_2' \overset{h_2}{\to} T_2$, the universal property implies that there is a unique g-morphism $h_3 : T_1' \to T_0$ with $p \circ h_3 = h_1$. Then h_3 is an object of $I_{t',T',g}$, and $p : T_0 \to T_1$ gives rise to a morphism $c : h_3 \to h_1$. By construction, one has $a \circ c = b \circ c$. ☐

Lemma 3.5. *Let Y be a scheme over S. Let $(U, T) \in \mathrm{Inf}(X/S)$. Let $q_1, q_2 : T \to Y$ be two morphisms over S with $q_1|_U = q_2|_U$. Then there is an object (U, T_0) and a morphism $p : (U, T_0) \to (U, T)$ in $\mathrm{Inf}(X/S)$ with $q_1 \circ p = q_2 \circ p$ as morphism $T_0 \to Y$, such that the following universal property holds. For every commutative square (2), every object $(U', T') \in \mathrm{Inf}(X'/S')$ and every g-morphism $h : T' \to T$ with $q_1 \circ h = q_2 \circ h$, i.e., making the diagram*

$$\begin{tikzcd}
T' \arrow[r, dashed, "h_0"'] \arrow[rr, bend left, "h"] & T_0 \arrow[r, "p"'] & T \arrow[r, shift left, "q_1"] \arrow[r, shift right, "q_2"'] & Y
\end{tikzcd}$$

commutative, there is a unique g-morphism $h_0 : T' \to T_0$ with $p \circ h_0 = h$ as morphism $T' \to T$.

Proof. By [Sta25, Tag 01KM], the category of schemes has equalizers. Let $p : T_0 \to T$ be the equalizer of $q_1, q_2 : T \rightrightarrows Y$. One has $q_1 \circ p = q_2 \circ p$, and $p : T_0 \to T$ is an immersion and hence separated. As $q_1|_U = q_2|_U$, the closed immersion $U \to T$ factors through a closed immersion $U \to T_0$. Then the square

$$\begin{tikzcd}
U \arrow[r] \arrow[d, equal] \arrow[dr, phantom, "\square"] & T_0 \arrow[d, "p"] \\
U \arrow[r] & T
\end{tikzcd}$$

is cartesian, so $U \to T_0$ is a thickening of finite order. Thus, (U, T_0) belongs to $\mathrm{Inf}(X/S)$, and $p : (U, T_0) \to (U, T)$ is a morphism in $\mathrm{Inf}(X/S)$.

We verify the universal property. From $q_1 \circ h = q_2 \circ h$, there is a unique morphism $h_0 : T' \to T_0$ of schemes with $p \circ h_0 = h$. As the outer rectangular of the diagram

$$\begin{tikzcd}
U' \arrow[r, "g|_{U'}"] \arrow[d] & U \arrow[r, equal] \arrow[d] & U \arrow[d] \\
T' \arrow[r, dashed, "h_0"] \arrow[rr, bend right, "h"'] & T_0 \arrow[r, hook, "p"] & T
\end{tikzcd}$$

is commutative, and $p : T_0 \to T$ is a monomorphism, the left square is commutative. Therefore, $h_0 : T' \to T_0$ is a g-morphism. ☐

Remark 3.6. The category $\mathrm{Inf}(X/S)$ has fiber products. By Lemma 3.5, it also has equalizers. It may not have a final object. By [Ber74, III, Corollaire 2.1.4 i)], the crystalline site has finite nonempty products. By contrast, we prove that for every irreducible algebraic variety X over $\mathbb{C}$, if the product of two copies of (X, X) exists in $\mathrm{Inf}(X/\mathbb{C})$, then $\dim X = 0$. Let (U, T) be the product. Let $\Delta : X \to X^2$ be the diagonal immersion. The two projections $p_0, p_1 : (U, T) \to (X, X)$ induce a commutative diagram

$$\begin{tikzcd}
U \arrow[rr, hook] \arrow[d, hook] & & T \arrow[d] \arrow[dl, dashed] \\
X \arrow[r, hook] \arrow[rr, bend right, "\Delta"'] & \Delta^n_X(X^2) \arrow[r, hook] & X^2
\end{tikzcd}$$

of schemes over $\mathbb{C}$. As $U \to T$ is a nilpotent thickening, there is $n > 0$ such that $T \to X^2$ factors through $\Delta^n_X(X^2)$. The two projections $(X, \Delta^{n+1}_X(X^2)) \to (X, X)$ induce a morphism $(X, \Delta^{n+1}_X(X^2)) \to (U, T)$ in $\mathrm{Inf}(X/\mathbb{C})$, so the inclusion $\Delta^n_X(X^2) \to \Delta^{n+1}_X(X^2)$ is an isomorphism. As $X^2_{\mathbb{C}}$ is irreducible, $X \to X^2$ is a thickening. Then $\dim X = 2 \dim X$ and hence $\dim X = 0$.

By Remark 3.6, the product of two objects in $\mathrm{Inf}(X/S)$ may not be representable. Lemma 3.7 shows that it is still ind-representable.

Lemma 3.7. *Let S be a scheme, X, Y be schemes over S. For $i = 1, 2$, let (U_i, T_i) be an object of $\mathrm{Inf}(X/S)$. Let $U = U_1 \cap U_2$. Let $q_i : T_i \to Y$ be a morphism over S with $q_1|_U = q_2|_U$ as morphism $U \to Y$. Then there is a direct system of objects $(U, T^m)_{m \geq 0}$ in $\mathrm{Inf}(X/S)$, each of which is equipped with two morphisms $p^m_i : (U, T^m) \to (U_i, T_i)$ in $\mathrm{Inf}(X/S)$, with the following universal property. Given a commutative square (2), let $(U', T') \in \mathrm{Inf}(X'/S')$. Consider a solid commutative diagram*

$$\begin{tikzcd}
T' \arrow[r, dashed, "h^m"'] \arrow[rr, bend left, "h_1"] \arrow[dr, "h_2"'] \arrow[dd] & T^m \arrow[r, dashed, "p^m_1"] \arrow[d, "p^m_2"] & T_1 \arrow[d, "q_1"] \\
 & T_2 \arrow[r, "q_2"] \arrow[dr] & Y \arrow[d] \\
S' \arrow[rr, "u"] & & S,
\end{tikzcd}$$

where $h_i : T' \to T_i$ $(i = 1, 2)$ are g-morphisms. Then there is an integer $m \geq 0$, and a g-morphism $h^m : T' \to T^m$ keeping the diagram commutative. Moreover, such a $h^m : T' \to T^m$ is unique once m is determined.

Proof. Since $q_1|_U = q_2|_U$, there is an induced morphism $U \to U_1 \times_Y U_2$. It is a base change of the diagonal immersion $X \to X \times_Y X$, so also an immersion. Since $U_i \to T_i$ $(i = 1, 2)$ are immersions over Y, $U_1 \times_Y U_2 \to T_1 \times_Y T_2$ is an immersion. Hence the composition $U \to T_1 \times_Y T_2$ is an immersion over Y. For every $m \geq 0$, let T^m be the m-th infinitesimal neighborhood of U in $T_1 \times_Y T_2$. Then $(U, T^m) \in \mathrm{Inf}(X/S)$. Let $p^m_i : T^m \to T_i$ be the composition of the inclusion $T^m \hookrightarrow T_1 \times_Y T_2$ with the projection $T_1 \times_Y T_2 \to T_i$. Then it defines a morphism $p^m_i : (U, T^m) \to (U_i, T_i)$ in $\mathrm{Inf}(X/S)$.

As the diagram

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[d, "g|_{U'}"] & T' \arrow[dd, "h_1"] \arrow[dr, "h_2"] & \\
U \arrow[rr, hook, dashed] \arrow[dr, hook] & & T_2 \arrow[d, "q_2"] \\
& T_1 \arrow[r, "q_1"] & Y
\end{tikzcd}$$

is commutative, it induces a commutative diagram

$$\begin{tikzcd}
U' \arrow[r, hook] \arrow[d, "g|_{U'}"] & T' \arrow[d] \\
U \arrow[r] & T_1 \times_Y T_2.
\end{tikzcd}$$

As $U' \to T'$ is a finite order thickening, there is an integer $m > 0$, such that $T' \to T_1 \times_Y T_2$ factors through T^m. The induced morphism $h^m : T' \to T^m$ is a g-morphism. □

Lemma 3.8. *In the notation of* (2)*, assume that $g : X' \to X$ is an* affine *morphism. Let (X', T') be an object of* $\mathrm{Inf}(X'/S')$*. Then there is an object (X, T) of* $\mathrm{Inf}(X/S)$ *and a g-morphism $\bar{g} : T' \to T$ with the following universal property. For every commutative square*

$$\begin{tikzcd}
X \arrow[r, "g''"] \arrow[d] & X'' \arrow[d] \\
S \arrow[r, "u''"] & S''
\end{tikzcd}$$

of schemes, every object $(X'', T'') \in \mathrm{Inf}(X''/S'')$ and every $g'' \circ g$-morphism $h'' : T' \to T''$, there is a unique g''-morphism $h : T \to T''$ fitting into a commutative diagram

$$\begin{tikzcd}
X' \arrow[r, hook] \arrow[d, "g"] & T' \arrow[d, "\bar{g}"'] \arrow[ddr, "h''"] & \\
X \arrow[r, hook] \arrow[dr, "g''"'] & T \arrow[dr, dashed, "h"] & \\
& X'' \arrow[r, hook] & T''.
\end{tikzcd}$$

Proof. By the proof of [Sta25, Tag 07RT], as $g : X' \to X$ is affine and $X' \to T'$ is a thickening of finite order, there is a pushout

$$\begin{tikzcd}
X' \arrow[r, hook] \arrow[d, "g"] & T' \arrow[d, "\bar{g}"] \\
X \arrow[r, hook] & T
\end{tikzcd}$$

in the category of schemes, where $X \to T$ is also a thickening of finite order. From the solid commutative diagram

$$\begin{tikzcd}
X' \arrow[r, hook] \arrow[d, "g"] & T' \arrow[r] \arrow[d, "\bar{g}"] & S' \arrow[dd, "u"] \\
X \arrow[r, hook] \arrow[rrd, "f"'] & T \arrow[rd, dashed] & \\
 & & S,
\end{tikzcd}$$

there is a unique morphism $T \to S$ keeping the diagram commutative. Thus, one defines an object $(X, T) \in \mathrm{Inf}(X/S)$, and $\bar{g} : T' \to T$ is a g-morphism. From the universal property of pushout, the existence and uniqueness of $h : T \to T''$ follow. Moreover, h is a morphism over S'' and hence a g''-morphism. $\square$

4 Stratifying topos and Čech-Alexander complex

Let S be a scheme, $g : X \to Y$ be a morphism of schemes over S. We review the stratifying topos, and show that the corresponding cohomology can be computed as the cohomology of a cosimplicial Zariski sheaf.

Definition 4.1. Let $Y\,\mathrm{Strat}(X/S)$ be the full subcategory of $\mathrm{Inf}(X/S)$ of objects $U \to T$ such that there is a morphism $h : T \to Y$ over S with $h|_U = g|_U$ as morphism $U \to Y$. This subcategory inherits the induced topology of $\mathrm{Inf}(X/S)$, making $Y\,\mathrm{Strat}(X/S)$ a site, called the stratifying site. Let $(X/S)_{Y\,\mathrm{Strat}}$ be the associated topos, known as the *stratifying topos.*

By [Ber74, III, Proposition 1.1.5], the restriction functor $\mathrm{Sh}(\mathrm{Inf}(X/S)) \to \mathrm{Sh}(Y\,\mathrm{Strat}(X/S))$ commutes with $\underline{V}$-limits and $\underline{V}$-colimits. Then by [SGA4I, IV, Corollaire 1.7], it is the inverse image j^{-1} for a morphism of topoi

$$j : (X/S)_{Y\,\mathrm{Strat}} \to (X/S)_{\mathrm{inf}}. \tag{5}$$

We still write $O_{X/S}$ for $j^* O_{X/S}$. Let $u'_{X/S}$ be the composition

$$(X/S)_{Y\,\mathrm{Strat}} \overset{j}{\to} (X/S)_{\mathrm{inf}} \overset{u_{X/S}}{\to} X_{\mathrm{Zar}}$$

of morphisms of topoi. Then for every $E \in \mathrm{Sh}(Y\,\mathrm{Strat}(X/S))$ and every open subset U of X, one has

$$\Gamma(U, u'_{X/S*}E) = \Gamma((U/S)_{Y\,\mathrm{Strat}}, E). \tag{6}$$

Let $\iota : X \to Y$ be an *immersion* of schemes over a scheme S. Given an $O_{X/S}$-module F, we recall the construction of the Čech-Alexander complex $\mathrm{CA}^\bullet_Y(F)$. It is a complex of sheaves on X, whose hypercohomology computes the infinitesimal cohomology of F, as Lemma 4.6 shows.

For every integer $v \geq 0$, let $Y^{v+1}_S := Y \times_S \times \cdots \times_S Y$ ($v+1$ times). Let $\Delta^{v+1} : Y \to Y^{v+1}_S$ be the diagonal immersion. For every integer $i \geq 0$, let $j^i(v+1) : \Delta^i_X(Y^{v+1}) \to Y^{v+1}_S$ be the i-th infinitesimal neighborhood of X for the composed immersion

$$X \overset{\iota}{\to} Y \overset{\Delta^{v+1}}{\to} Y^{v+1}_S$$

in the sense of [EGA IV 4, Définition 16.1.2]. Then the inclusion $X \hookrightarrow \Delta^i_X(Y^{v+1})$ is a thickening of finite order. For each of the $v+1$ projections $p : Y^{v+1}_S \to Y$, $p \circ j^i(v+1) : \Delta^i_X(Y^{v+1}) \to Y$ is a morphism over S whose restriction to X is $\iota : X \to Y$. Therefore, $X \hookrightarrow \Delta^i_X(Y^{v+1})$ is an object of $Y\operatorname{Strat}(X/S)$.

For a sheaf of sets E on $Y\operatorname{Strat}(X/S)$, we define a cosimplicial sheaf of sets on X, denoted by $\mathrm{CA}^\bullet_Y(E)$ and called the *Čech-Alexander complex* of E relative to $\iota : X \to Y$. For every integer $v \geq 0$, set

$$\mathrm{CA}^v_Y(E) := \lim_i E_{(X,\Delta^i_X(Y^{v+1}))} \tag{7}$$

in $\mathrm{Sh}(X)$, where every sheaf $E_{(X,\Delta^i_X(Y^{v+1}))}$ on $\Delta^i_X(Y^{v+1})$ is considered as a sheaf on X via the homeomorphism $X \hookrightarrow \Delta^i_X(Y^{v+1})$. By [Ber74, III, Proposition 1.1.5], the resulting functor

$$\mathrm{CA}^v_Y : \mathrm{Sh}(Y\operatorname{Strat}(X/S)) \to \mathrm{Sh}(X)$$

is left exact.

Example 4.2. For every $v \geq 0$, one has

$$\mathrm{CA}^v_Y(O_{X/S}) = P_X(Y^{v+1}),$$
$$\mathrm{CA}^v_Y(J_{X/S}) = \lim_i \ker(P^i_X(Y^{v+1}) \to O_X) = \ker(P_X(Y^{v+1}) \to O_X).$$

Remark 4.3. By [GD71, Remarque 10.6.6], for an $O_{X/S}$-module F on $Y\operatorname{Strat}(X/S)$, $\mathrm{CA}^v_Y(F)$ is naturally an $P_X(Y^{v+1})$-module. From [GD71, Théorème 10.11.3], if Y is a locally Noetherian scheme, and if F is a locally coherent (i.e., for every $(U,T) \in Y\operatorname{Strat}(X/S)$, the O_T-module F_T is coherent) crystal in $O_{X/S}$-modules in the sense of Definition 6.1, then $\mathrm{CA}^v_Y(F)$ is a coherent sheaf on the formal scheme $\Delta_X(Y^{v+1})$.

We define the linking morphisms of the cosimplicial object $\mathrm{CA}^\bullet_Y(E)$. For integers $v > 0$ and $0 \leq j \leq v$, let $p^v_j : Y^{v+1}_S \to Y^v_S$ be the projection skipping the j-th factor of Y^{v+1}_S. It defines a projective system of morphisms $\{(X, \Delta^i_X(Y^{v+1})) \to (X, \Delta^i_X(Y^v))\}_i$ in the category $Y\operatorname{Strat}(X/S)$. This system induces a morphism

$$\delta^v_j : \mathrm{CA}^{v-1}_Y(E) \to \mathrm{CA}^v_Y(E).$$

Consider the immersion

$$\iota^v_j : Y^{v+1}_S \to Y^{v+2}_S, \quad (y_0, \dots, y_v) \mapsto (y_0, \dots, y_j, y_j, y_{j+1}, \dots, y_v).$$

It induces a projective of morphisms $\{(X, \Delta^i_X(Y^{v+1})) \to (X, \Delta^i_X(Y^{v+2}))\}_i$ in the category $Y\operatorname{Strat}(X/S)$, hence a morphism

$$\sigma^v_j : \mathrm{CA}^{v+1}_Y(E) \to \mathrm{CA}^v_Y(E).$$

Let Δ be the category of finite ordered sets. Thus, one defines a functor

$$\mathrm{CA}^\bullet_Y(E) : \Delta \to \mathrm{Sh}(X),$$

i.e., a cosimplicial sheaf on X. The construction is functorial in E, so one gets a functor

$$\mathrm{CA}_Y^\bullet : \mathrm{Sh}(Y\,\mathrm{Strat}(X/S)) \to \mathrm{CoSimp}(\mathrm{Sh}(X)).$$

Remark 4.4. The Čech-Alexander complex is a sheaf version of the Čech complex, as we explain. Let $\tilde{Y}$ be the presheaf on $Y\,\mathrm{Strat}(X/S)$ sending each object (U, T) to the set of morphisms $T \to Y$ of schemes over S extending $\iota|_U : U \hookrightarrow Y$. It turns out to be a sheaf. From [SGA4I, II, Proposition 4.8 a)], every topos has finite limits. For every $v > 0$, let $\tilde{Y}^v$ be the v-fold product of Y in $\mathrm{Sh}(Y\,\mathrm{Strat}(X/S))$, which is a colimit of representable sheaves $\mathrm{colim}_i\, h_{\Delta^i_X(Y^v)}$. From (7), for an open subset U of X, one has

$$\Gamma(U, \mathrm{CA}_Y^v(E)) = \lim_{i>0} E(U, \Delta_U^i(Y^{v+1})). \tag{8}$$

In particular, one has

$$\Gamma(X, \mathrm{CA}_Y^v(E)) = \mathrm{Hom}_{\mathrm{Sh}(Y\,\mathrm{Strat}(X/S))}(\tilde{Y}^{v+1}, E). \tag{9}$$

Let $\mathcal{U} = (\tilde{Y} \to e)$, which is a representable covering morphism by [Ber74, V, Lemme 1.2.1]. Let $C^\bullet(\mathcal{U}, E)$ be the *Čech complex* of the sheaf E relative to the covering $\mathcal{U}$, i.e., the cosimplicial object defined by [SGA 4II, V, (2.3.3.1)]. The isomorphisms (9) for variable $v \geq 0$ are compatible with linking morphisms, so $\Gamma(X, \mathrm{CA}_Y^\bullet(E)) = C^\bullet(\mathcal{U}, E)$.

By (6) and (8), given a sheaf of rings A on $Y\,\mathrm{Strat}(X/S)$ and an A-module E, each $\mathrm{CA}_Y^v(E)$ is a $u'_{X/S*}A$-module. Thus, one can similarly define a functor

$$\mathrm{CA}_Y^\bullet : \mathrm{Mod}(Y\,\mathrm{Strat}(X/S), A) \to \mathrm{CoSimp}(\mathrm{Mod}(u'_{X/S*}A)).$$

Remark 4.5. Given a cochain complex of abelian sheaves $F^\bullet$ on $Y\,\mathrm{Strat}(X/S)$, one defines a functor $\mathrm{CA}_Y^\bullet(F^\bullet) : \Delta \to \mathrm{Ch}(\mathrm{Ab}(X))$, a cosimplicial complex of sheaves on X_{Zar}. We also write $\mathrm{CA}_Y^\bullet(F^\bullet) \in \mathrm{Ch}(\mathrm{Ab}(X))$ for the associated cochain complex.

The assumption of vanishing higher $R^q \lim$ in Lemma 4.6 is a variant of [Gro68, Conditions (1) and (2), p.336]. As the linearization $L(O_Y)$ (Definition 8.1) may not be locally quasi-coherent, we have to slightly relax Grothendieck's conditions.

Lemma 4.6. *Let $\iota : X \hookrightarrow Y$ be an immersion of schemes over a scheme S. Let A be a sheaf of rings on $Y\,\mathrm{Strat}(X/S)$. Let $F^\bullet \in \mathrm{Ch}^+(\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), A))$. Assume that for any integers $v \geq 0$ and k, the inverse system $(F^k_{(X, \Delta^i_X(Y^{v+1}))})_{i \geq 0}$ in $\mathrm{Ab}(X)$ has vanishing $R^q \lim_i$ for all $q > 0$. Then there exists a canonical isomorphism*

$$Ru'_{X/S*}F^\bullet \cong \mathrm{CA}_Y^\bullet(F^\bullet) \tag{10}$$

in $D^+(X, \mathrm{Mod}(u'_{X/S*}A))$, which is functorial in $F^\bullet$. It induces a canonical isomorphism

$$R\Gamma((X/S)_{Y\,\mathrm{Strat}}, F^\bullet) \cong R\Gamma(X, \mathrm{CA}_Y^\bullet(F^\bullet))$$

in $D^+(\mathrm{Mod}(\Gamma((X/S)_{Y\,\mathrm{Strat}}, A)))$.

Proof. By [Sta25, Tag 07A5], $\mathrm{Mod}(A)$ is a Grothendieck abelian category. So it has enough injectives. Then by [Sta25, Tag 013K], as $F^\bullet$ is bounded below, there is a complex $I^\bullet$ such that each term is an injective object of $\mathrm{Mod}(A)$, and a morphism $F^\bullet \to I^\bullet$ in $\mathrm{Ch}^+(\mathrm{Mod}(A))$ that is a quasi-isomorphism. The morphism

$$Ru'_{X/S*}F^\bullet \to u'_{X/S*}I^\bullet \tag{11}$$

is an isomorphism in $D^+(\mathrm{Mod}(u'_{X/S*}A))$. By [Ber74, V, Lemme 1.2.4] (whose infinitesimal analog can be proved using Remark 4.4), for every integer k, as I^k is an injective in $\mathrm{Mod}(A)$, there exists a natural resolution $u'_{X/S*}I^k \to \mathrm{CA}^\bullet_Y(I^k)$. Therefore, there is a natural morphism

$$u'_{X/S*}I^\bullet \to \mathrm{CA}^\bullet_Y(I^\bullet) \tag{12}$$

in $\mathrm{Ch}^+(\mathrm{Mod}(u'_{X/S*}A))$ that is a quasi-isomorphism.

We prove that the morphism

$$\mathrm{CA}^\bullet_Y(F^\bullet) \to \mathrm{CA}^\bullet_Y(I^\bullet) \tag{13}$$

in $\mathrm{Ch}^+(\mathrm{Mod}(u'_{X/S*}A))$ is a quasi-isomorphism.

By [Ber74, III, Proposition 1.1.5], for every $v \geq 0$, the morphism of complexes of projective systems

$$(F^\bullet_{(X,\Delta^i_X(Y^{v+1}))})_{i\geq 0} \to (I^\bullet_{(X,\Delta^i_X(Y^{v+1}))})_{i\geq 0}$$

in $\mathrm{Ch}^+(\mathrm{Mod}(u'_{X/S*}A)^{\mathbb{N}})$ is a quasi-isomorphism. By assumption, the termwise limit complex $\mathrm{CA}^v_Y(F^\bullet)$ represents $R\lim_i F^\bullet_{(X,\Delta^i_X(Y^{v+1}))} \in D(u'_{X/S*}A)$. From Lemma 4.10, for every integer k, the inverse system $(I^k_{(X,\Delta^i_X(Y^{v+1}))})_{i\geq 0}$ satisfies condition (*). Then by Fact 4.8, $\mathrm{CA}^v_Y(I^\bullet)$ represents $R\lim_i I^\bullet_{(X,\Delta^i_X(Y^{v+1}))} \in D(u'_{X/S*}A)$. Therefore, the morphism $\mathrm{CA}^v_Y(F^\bullet) \to \mathrm{CA}^v_Y(I^\bullet)$ in $\mathrm{Ch}^+(\mathrm{Mod}(u'_{X/S*}A))$ is a quasi-isomorphism. The quasi-isomorphisms for variable $v \geq 0$ are compatible, so (13) is indeed a quasi-isomorphism.

Composing the isomorphisms (11), (12) and (13), one gets an isomorphism $Ru'_{X/S*}F^\bullet \to \mathrm{CA}^\bullet_Y(F^\bullet)$ in $D^+(X, u'_{X/S*}A)$. Using that two injective resolutions are homotopic equivalent, one can show that this isomorphism is independent of the choice of the injective resolution $F^\bullet \to I^\bullet$. Moreover, the isomorphism (10) is functorial in $F^\bullet$. □

In a Grothendieck abelian category $\mathcal{A}$, an inverse system $(A_i)_{i\geq 0}$ satisfying the Mittag-Leffler condition (see, e.g., [Sta25, Tag 0595]) may have non-vanishing $R^1\lim_i A_i$. We recall a condition for sheaves ensuring the vanishing of higher derived limits, due to Grothendieck [EGA III 1, Proposition 13.3.1].

Definition 4.7. Let X be a ringed space. We say that an object $(F_i)_{i\geq 0}$ of $\mathrm{Mod}(O_X)^{\mathbb{N}}$ satisfies condition (*) if there is a base $\mathcal{B}$ of X, such that for every $U \in \mathcal{B}$,

(a) for any integers $p > 0$ and $i \geq 0$, one has $H^p(U, F_i) = 0$;

(b) one has $R^1 \lim_i F_i(U) = 0$.

Fact 4.8 ([Sta25, Tag 0BKS]). *Let X be a ringed space. Let $(F_i)_{i\geq 0}$ be an inverse system satisfying condition (*). Then $\lim_i F_i \to R\lim_i F_i$ is an isomorphism in $D(\mathrm{Mod}(O_X))$.*

Example 4.9. (a) Let X be a ringed space. Let $(F_i)_{i\geq 0}$ be a projective system in $\mathrm{Mod}(O_X)$. If for every $i \geq 0$, $F_i \in \mathrm{Mod}(O_X)$ is flasque, then Condition (a) holds. If for every $U \in \mathcal{B}$, the inverse system of $O_X(U)$-modules $\{F_i(U)\}_{i\geq 0}$ satisfies the Mittag-Leffler condition, then from [Wei95, Proposition 3.5.7], Condition (b) holds.

(b) Let X be a scheme. Let $(F_i)_{i\geq 0}$ be an inverse system in the category $\mathrm{Qch}(X)$ of quasi-coherent O_X-modules. Suppose that (F_i) satisfies the Mittag-Leffler condition. Then for every affine open subset U of X, the inverse system $(F_i(U))_{i\geq 0}$ satisfies the Mittag-Leffler condition. By the Serre vanishing theorem, the system $(F_i)_{i\geq 0}$ satisfies condition (*).

Lemma 4.10. *Notation as in Lemma 4.6. Let $\mathcal{B}$ be a base for the underlying topological space X. Let I be an injective object of $\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), A)$. Then for every integer $v > 0$, the inverse system $\{I_{(X,\Delta^i_X(Y^v))}\}_{i\geq 0}$ satisfies condition (*) relative to $\mathcal{B}$.*

Proof. By [Ber74, VI, Proposition 1.1.5], for every integer $i > 0$, the $A_{(X,\Delta^i_X(Y^v))}$-module $I_{(X,\Delta^i_X(Y^v))}$ on $\Delta^i_X(Y^v)$ is injective. Therefore, the underlying abelian sheaf on X is flasque. For every open subset U of X and every integer $i \geq 0$, the inclusion

$$(U, \Delta^i_U(Y^v)) \to (U, \Delta^{i+1}_U(Y^v))$$

is a monomorphism in $Y\,\mathrm{Strat}(X/S)$. By the Yoneda lemma, the canonical functor $Y\,\mathrm{Strat}(X/S) \to (X/S)_{Y\,\mathrm{Strat}}$ is left exact. By [SGA 4II, V, 4.6], the injective A-module I is flasque. Then by [SGA 4II, V, Proposition 4.7], the map

$$I(U, \Delta^{i+1}_U(Y^v)) \to I(U, \Delta^i_U(Y^v))$$

is surjective. In particular, the inverse system $\{I(U, \Delta^i_U(Y^v))\}_i$ satisfies the Mittag-Leffler condition. Then by Example 4.9 (a) , $\{I_{(X,\Delta^i_X(Y^v))}\}_{i\geq 0}$ satisfies condition (*). ☐

5 Stratification and formalization functor

We recollect the notion of stratification on sheaves of modules, which is closely related to connection. Stratification can be viewed as linearizing differential operators, as Grothendieck's construction of formalizing functor shows.

Definition 5.1. Let $\mathcal{C}$ be a category. Let $\mathcal{C}^{\mathbb{N}}$ be the category of projective systems over $\mathcal{C}$ indexed over $\mathbb{N}$. For an inverse system $X = (X_n, d_n)_{n\geq 0} \in \mathcal{C}^{\mathbb{N}}$ and an integer $k \geq 0$, let $X[k]$ be the inverse system $(X_{k+n}, d_{k+n})_{n\geq 0}$. Define a category $\mathrm{AR}(\mathcal{C})$ as follows. Its objects are the same as $\mathcal{C}^{\mathbb{N}}$. A morphism $(X_n) \to (Y_n)$ in $\mathrm{AR}(\mathcal{C})$ refers to an element of $\mathrm{colim}_{k\geq 0} \mathrm{Hom}_{\mathcal{C}^{\mathbb{N}}}(X[k], Y)$. An object of $\mathrm{AR}(\mathcal{C})$ is known as an *Artin-Rees pro-object* of $\mathcal{C}$.

Viewing objects of $\mathcal{C}$ as constant projective systems, one gets an inclusion functor $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{AR}(\mathcal{C})$ which is fully faithful. Every functor $F : \mathcal{C} \to \mathcal{D}$ induces a functor $\mathrm{AR}(F) : \mathcal{C} \to \mathcal{D}$.

Remark 5.2. Let $\mathcal{C}$ be a category such that every projective system indexed over $\mathbb{N}$ in $\mathcal{C}$ has a limit. Then the limit functor $\lim : \mathcal{C}^{\mathbb{N}} \to \mathcal{C}$ induces a functor

$$\lim : \mathrm{AR}(\mathcal{C}) \to \mathcal{C},$$

which is right adjoint to the inclusion $\iota_{\mathcal{C}} : \mathcal{C} \to \mathrm{AR}(\mathcal{C})$. The natural morphism $\mathrm{Id} \to \lim \circ \iota_{\mathcal{C}}$ of functors $\mathcal{C} \to \mathcal{C}$ is an isomorphism.

Remark 5.3. Let $\mathcal{C}$ be an abelian category. An object $X \in \mathcal{C}^{\mathbb{N}}$ is called *AR-zero* if there is an integer $d \geq 0$ such that the canonical morphism $X[d] \to X$ in $\mathcal{C}^{\mathbb{N}}$ is zero. By [SGA 5, V, Proposition 2.4.4], the natural functor $\mathcal{C}^{\mathbb{N}} \to \mathrm{AR}(\mathcal{C})$ is the quotient by the Serre subcategory of $\mathcal{C}^{\mathbb{N}}$ consisting of AR-zero objects. In particular, $\mathrm{AR}(\mathcal{C})$ admits a natural structure of abelian category. The inclusion $\mathcal{C} \to \mathrm{AR}(\mathcal{C})$ is exact and exhibits C as a weak Serre subcategory of $\mathrm{AR}(C)$.

Assume that C has enough injectives. Then by [Jan88, Proposition 1.1], $C^{\mathbb{N}}$ has enough injectives, and the inclusion $C \to C^{\mathbb{N}}$ preserve injective objects. By [Kri23, Proposition 4.1.5], $\mathrm{AR}(C)$ also has enough injectives, and the quotient functor $C^{\mathbb{N}} \to \mathrm{AR}(C)$ preserves injective objects.

For every left exact functor $F : A \to B$ of abelian categories, the induced functor $F^{\mathbb{N}} : A^{\mathbb{N}} \to B^{\mathbb{N}}$ is left exact. Then by [Gab62, III, Corollaire 1], it descends to a left exact functor $\mathrm{AR}(F) : \mathrm{AR}(A) \to \mathrm{AR}(B)$, called the AR-extension of F. From [Sta25, Tag 015M], the right derived functors exist, and are compatible in the sense that they fit into a commutative diagram

$$\begin{tikzcd}
D^+(A) \arrow[r] \arrow[d, "RF"] & D^+(A^{\mathbb{N}}) \arrow[r] \arrow[d, "R(F^{\mathbb{N}})"] & D^+(\mathrm{AR}(A)) \arrow[d, "R(\mathrm{AR}(F))"] \\
D^+(B) \arrow[r] & D^+(B^{\mathbb{N}}) \arrow[r] & D^+(\mathrm{AR}(B)).
\end{tikzcd}$$

By [Jan88, Proposition 1.2], for every integer $i \geq 0$, one has $\mathrm{AR}(R^iF) = R^i(\mathrm{AR}(F))$ as functor $\mathrm{AR}(A) \to \mathrm{AR}(B)$.

Let $Y \to S$ be a morphism of schemes. For every integer $n \geq 0$, let $P^n_{Y/S} := P^n_Y(Y^2)$ be the *sheaf of principal parts* of order n on Y defined in [EGA IV 4, Définition 16.3.1]. In particular, one has $P^0_{Y/S} = O_Y$. By [EGA IV 4, Corollaire 16.1.7], there is a canonical projective system $(P^n_{Y/S})_{n\geq 0}$ in the category $\mathrm{Ring}(Y)$ of sheaves of rings on Y. By [EGA IV 4, 16.3.2], for every $n > 0$, the two projections $p_0, p_1 : Y_S^2 \to Y$ induces two right inverses $O_Y \to P^n_{Y/S}$ to the augmentation morphism $\pi^n : P^n_{Y/S} \to O_Y$, giving two structures of quasi-coherent O_Y-algebra on $P^n_{Y/S}$, called the *left structure* and the *right structure* in order. For integers $n \geq m \geq 0$, let $\pi^{n,m} : P^n_{Y/S} \to P^m_{Y/S}$ be the projection. For integers $i, k \geq 0$, let

$$\delta^{i,k} : P^{i+k}_{Y/S} \to P^i_{Y/S} \otimes_{O_Y} P^k_{Y/S}$$

be the O_Y-linear (for both left and right O_Y-module structures) ring morphism constructed in [EGA IV 4, Lemme 16.8.9.1]. Let $q_0^{i,k}$ be the composition

$$P^{i+k}_{Y/S} \overset{\pi^{i+k,i}}{\to} P^i_{Y/S} \to P^i_{Y/S} \otimes_{O_Y} P^k_{Y/S}.$$

For an integer $n \geq 0$, let $\sigma^n : P^n_{Y/S} \to P^n_{Y/S}$ be the canonical symmetry defined in [EGA IV 4, Proposition 16.3.4]. It is an involutive automorphism and exchanges the two structures of O_Y-algebra on $P^n_{Y/S}$. Let $P^\infty_{Y/S} := \lim_n P^n_{Y/S}$.

Definition 5.4. Let $Y \to S$ be a morphism of schemes. Let M be an object of $\mathrm{AR}(\mathrm{Mod}(O_Y))$. For an integer $n \geq 0$, an *n-connection* on M relative to S is an isomorphism

$$\epsilon_n : P^n_{Y/S} \otimes_{O_Y} M \xrightarrow{\sim} M \otimes_{O_Y} P^n_{Y/S}$$

in $\mathrm{AR}(\mathrm{Mod}(P^n_{Y/S}))$, which induces Id_M when base changed along the augmentation $\pi^n : P^n_{Y/S} \to O_Y$. An 1-connection is called a *connection.* A *stratification* on M relative to S is the datum of an n-connection ϵ_n relative to S for every $n \geq 0$, such that for any integers $0 \leq m \leq n$,

- the diagram

$$\begin{tikzcd} P^n_{Y/S} \otimes_{O_Y} M \arrow[r, "\epsilon_n"] \arrow[d, "\pi^{n,m}\otimes \mathrm{Id}"] & M \otimes_{O_Y} P^n_{Y/S} \arrow[d, "\mathrm{Id}\otimes\pi^{n,m}"] \\ P^m_{Y/S} \otimes_{O_Y} M \arrow[r, "\epsilon_m"] & M \otimes_{O_Y} P^m_{Y/S} \end{tikzcd} \tag{14}$$

  in $\mathrm{AR}(\mathrm{Mod}(P^n_{Y/S}))$ is commutative,

- and

$$\delta^{m,n-m*}(\epsilon_n) = q_0^{m,n-m*}(\epsilon_n) \circ q_1^{m,n-m*}(\epsilon_n). \tag{15}$$

Let $f : M \to M'$ be a morphism in $\mathrm{AR}(\mathrm{Mod}(O_Y))$, where M and M' are equipped with stratifications ϵ and ϵ'. If f is compatible with stratifications, then it is called *horizontal.*

Remark 5.5. By [Ber74, II, Exemple 3.1.4 i) and Lemme 3.2.1], a connection on an O_Y-module $M \in \mathrm{Mod}(O_Y)$ is equivalent to an O_S-linear morphism $\nabla : M \to M \otimes_{O_Y} \Omega^1_{Y/S}$ satisfying the Leibniz rule.

Remark 5.6. For an integer $n \geq 0$, let $p_i^n : \Delta^n_Y(Y^2) \to Y$ $(i = 0, 1)$ be the two projections. Let $M \in \mathrm{AR}(\mathrm{Mod}(O_Y))$. Then an n-connection on M is equivalent to an isomorphism $(p_1^n)^*M \to (p_0^n)^*M$ in the category $\mathrm{AR}(\mathrm{Mod}(O_{\Delta^n_Y(Y^2)}))$, whose restriction along the diagonal inclusion $Y \hookrightarrow \Delta^n_Y(Y^2)$ is the identity.

Remark 5.7. Let M be an O_Y-module. By [Ber74, II, Proposition 3.2.5], every integrable connection $\nabla : M \to M \otimes_{O_Y} \Omega^1_{Y/S}$ relative to S naturally fits into a complex

$$0 \to M \xrightarrow{\nabla} M \otimes_{O_Y} \Omega^1_{Y/S} \to M \otimes_{O_Y} \Omega^2_{Y/S} \to \dots,$$

called the *de Rham complex* with coefficients in M. For every integer $i \geq 0$, the morphism $M \otimes_{O_Y} \Omega^i_{Y/S} \to M \otimes_{O_Y} \Omega^{i+1}_{Y/S}$ is a differential operator of order ≤ 1. Similar to the proof of [Sta25, Tag 07J6], one can prove that the connection underlying a stratification on M relative to S is integrable.

Fact 5.8 ([Gro68, Appendix], [Ber74, p.81])**.** *Let $f : Y \to S$ be a smooth morphism of schemes, with Y of characteristic 0. Let $D_{Y/S}$ be the sheaf of differential operators relative to S. Then for every O_Y-module M, a stratification on M relative to S is equivalent to an integrable connection $M \to M \otimes_{O_Y} \Omega^1_{Y/S}$, i.e., a structure of left $D_{Y/S}$-module structure on M.*

Definition 5.9. For an O_Y-module E and an integer $n \geq 0$, set $Q^0(E)^n_Y := P^n_{Y/S} \otimes_{O_Y} E$, where the tensor product is taken with respect to the *right* structure of O_Y-module on $P^n_{Y/S}$. Via the *left* structure of O_Y-module on $P^n_{Y/S}$, we regard $Q^0(E)^n_Y$ as an O_Y-module. As functors

$$Q^0(\cdot)^n = p^n_{0*} \circ (p^n_1)^* : \quad \mathrm{Mod}(O_Y) \to \mathrm{Mod}(O_Y).$$

Set

$$Q^0(E)^\bullet_Y := P^\bullet_{Y/S} \otimes_{O_Y} E := (Q^0(E)^n_Y)_{n \geq 0} \in \mathrm{Mod}(O_Y)^{\mathbb{N}}.$$

Let $\mathrm{Diff}(Y/S)$ be the category of O_Y-modules, with differential operators of finite order relative to S in the sense of [EGA IV 4, Définition 16.8.1] as morphisms. Let $\mathrm{Strat}-\mathrm{AR}(O_Y)$ be the category of Artin-Rees pro-modules over O_Y equipped with a stratification relative to S, with horizontal O_Y-linear morphisms. By Lemmas 5.11, 5.12 and 5.13,

$$Q^0 : \mathrm{Diff}(Y/S) \to \mathrm{Strat}-\mathrm{AR}(O_Y)$$

is a well-defined functor, called the *formalization functor*.

Remark 5.10. Let $\mathrm{Strat}(O_Y)$ be a category, where objects are O_Y-modules equipped with a stratification, and morphisms are horizontal. It is a full subcategory of $\mathrm{Strat}-\mathrm{AR}(O_Y)$. The category $\mathrm{Strat}-\mathrm{AR}(O_Y)$ is different from $\mathrm{AR}(\mathrm{Strat}(O_Y))$. Both $\mathrm{Diff}(Y/S)$ and $\mathrm{Strat}-\mathrm{AR}(O_Y)$ are additive categories, and Q^0 is an additive functor. Every O_Y-linear morphism is a differential operator, so there is a natural faithful, essentially surjective additive functor $\mathrm{Mod}(O_Y) \to \mathrm{Diff}(Y/S)$. However, the forgetful functor $\mathrm{Diff}(Y/S) \to \mathrm{Ab}(Y)$ may not be faithful, i.e., a nonzero differential operator $E \to F$ may have zero morphism of underlying abelian sheaves. By a slight variant of [Ber74, II, Proposition 1.5.2], if Y is *smooth* over S, then $\mathrm{Strat}-\mathrm{AR}(O_Y)$ is naturally an abelian category, and the forgetful functor $\mathrm{Strat}-\mathrm{AR}(O_Y) \to \mathrm{AR}\,\mathrm{Mod}(O_Y)$ is exact.

Lemma 5.11. *For every O_Y-module E, the Artin-Rees pro-module $Q^0(E)^\bullet_Y \in \mathrm{AR}\,\mathrm{Mod}(O_Y)$ has a canonical stratification relative to S.*

Proof. For an integer $m \geq 0$, define a morphism of abelian sheaves $\theta_{n,m} : P^{n+m}_{Y/S} \otimes E \to (P^m_{Y/S} \otimes E) \otimes P^n_{Y/S}$ as the composition

$$P^{n+m}_{Y/S} \otimes E \overset{\delta^{n,m} \otimes \mathrm{Id}_E}{\to} P^n_{Y/S} \otimes P^m_{Y/S} \otimes E \overset{\sigma^n \otimes \mathrm{Id} \otimes \mathrm{Id}}{\to} P^n_{Y/S} \otimes' P^m_{Y/S} \otimes E \overset{\cong}{\to} (P^m_{Y/S} \otimes E) \otimes P^n_{Y/S},$$

where tensor products are over O_Y, $\otimes'$ signifies that $P^n_{Y/S}$ uses the left structure of O_Y-algebra, and the last morphism is the symmetry of tensor products. Then $\theta_{n,m}$ induces a morphism of $P^n_{Y/S}$-modules

$$\epsilon_{n,m} : P^n_{Y/S} \otimes (P^{n+m}_{Y/S} \otimes E) \to (P^m_{Y/S} \otimes E) \otimes P^n_{Y/S}.$$

For variable $m \geq 0$, they are compatible, so give rise to a morphism

$$\epsilon_n : P^n_{Y/S} \otimes (P^\bullet_{Y/S} \otimes E) \to (P^\bullet_{Y/S} \otimes E) \otimes P^n_{Y/S}$$

in $\mathrm{AR}\,\mathrm{Mod}(P^n_{Y/S})$. By [Ber74, II, Corollaire 1.4.4], the $(\epsilon_n)_{n \geq 0}$ define a stratification. □

Let E, F be O_Y-modules. Let $D : E \to F$ be a morphism in $\mathrm{Diff}(Y/S)$. Choose an integer $k \geq 0$ such that D is a differential operator of order $\leq k$. Then D factors uniquely as

$$E \overset{d^k_{Y/S,E}}{\to} P^k_{Y/S} \otimes_{O_Y} E \overset{u}{\to} F,$$

where $u : P^k_{Y/S} \otimes_{O_Y} E \to F$ is morphism in $\mathrm{Mod}(O_Y)$. Define a morphism

$$Q^0(E)^{i+k}_Y \to Q^0(F)^i_Y \tag{16}$$

in $\mathrm{Mod}(O_Y)$ as the composition

$$P^{i+k}_{Y/S} \otimes_{O_Y} E \overset{\delta^{i,k}}{\to} P^i_{Y/S} \otimes_{O_Y} P^k_{Y/S} \otimes_{O_Y} E \overset{\mathrm{Id}_{P^i} \otimes u}{\to} P^i_{Y/S} \otimes_{O_Y} F.$$

For variable i, they fit to a morphism $Q^0(D)_Y : Q^0(E)_Y[k] \to Q^0(F)_Y$ in $\mathrm{Mod}(O_Y)^{\mathbb{N}}$. It induces a morphism

$$Q^0(D) : Q^0(E)^\bullet_Y \to Q^0(F)^\bullet_Y \tag{17}$$

in $\mathrm{AR}(\mathrm{Mod}(O_Y))$.

Lemma 5.12. *The morphism* (17) *is independent of the choice of* $k \geq 0$.

Proof. Let $k' \geq 0$ be another choice with corresponding factorization $u' : P^{k'}_{Y/S} \otimes E \to F$. By symmetry, one may assume $k' \geq k$. For $i \geq 0$, write P^i for $P^i_{Y/S}$. Consider a diagram

$$\begin{tikzcd}
P^{i+k'} \arrow[r, "\delta^{i,k'}"] \arrow[d, "\delta^{i+k,k'-k}"] \arrow[dd, bend right=60, "\pi^{i+k',i+k}"'] & P^i \otimes P^{k'} \arrow[d, dashed, "\mathrm{Id} \otimes \delta^{k,k'-k}"] \arrow[dd, bend left=60, "\mathrm{Id} \otimes \pi^{k',k}"] \\
P^{i+k} \otimes P^{k'-k} \arrow[r, "\delta^{i,k} \otimes \mathrm{Id}"] \arrow[d, dashed, "\mathrm{Id} \otimes \pi^{k'-k}"] & P^i \otimes P^k \otimes P^{k'-k} \arrow[d, dashed, "\mathrm{Id}_{P^i \otimes P^k} \otimes \pi^{k'-k}"] \\
P^{i+k} \arrow[r, "\delta^{i,k}"] & P^i \otimes P^k,
\end{tikzcd}$$

where the lower square is commutative. From [Ber74, II, Exemple 1.1.5 a)], $\hat{P}(X/S)$ is a formal groupoid, so by [Ber74, II, (1.1.10)], the triangles on both sides are commutative. From [Ber74, II, (1.1.11)], the upper square is commutative.

By uniqueness of factorization, one has $u' = u \circ (\pi^{k',k} \otimes \mathrm{Id}_E)$. Then the diagram

$$\begin{tikzcd}
P^{i+k'} \otimes E \arrow[r, "\delta^{i,k'} \otimes \mathrm{Id}"] \arrow[d, dashed, "\pi^{i+k',i+k} \otimes \mathrm{Id}"'] & P^i \otimes P^{k'} \otimes E \arrow[d, dashed, "\mathrm{Id} \otimes \pi^{k',k} \otimes \mathrm{Id}"'] \arrow[dr, "\mathrm{Id} \otimes u'"] & \\
P^{i+k} \arrow[r, "\delta^{i,k} \otimes \mathrm{Id}"] & P^i \otimes P^k \otimes E \arrow[r, "\mathrm{Id} \otimes u"] & P^i \otimes F
\end{tikzcd}$$

is commutative. Therefore, the morphism $Q^0(E)_Y^{\bullet+k} \to Q^0(F)_Y^{\bullet}$ in $\mathrm{Mod}(O_Y)^{\mathbb{N}}$ induces $Q^0(E)_Y^{\bullet+k'} \to Q^0(F)_Y^{\bullet}$. $\square$

The proof of Lemma 5.13 is similar to that of [Ber74, IV, Lem 3.1.2 ii), iii)], so it is omitted.

Lemma 5.13. *Let $D : E \to F$ be a morphism in* $\mathrm{Diff}(Y/S)$.

(a) *The morphism $Q^0(D) : Q^0(E) \to Q^0(F)$ in* $\mathrm{AR}\,\mathrm{Mod}(O_Y)$ *is horizontal for the canonical stratifications.*

(b) *For another morphism $D' : F \to G$ in* $\mathrm{Diff}(Y/S)$, *one has $Q^0(D' \circ D) = Q^0(D') \circ Q^0(D)$.*

6 Crystal

Let $X \to Y$ be a morphism of schemes over a scheme S. Grothendieck introduces a sort of "special"sheaves on the infinitesimal site, known as crystals. He also gives an interpretation of stratified modules on Y as crystals on $Y\,\mathrm{Strat}(X/S)$.

Definition 6.1. An object $E \in \mathrm{AR}(\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S}))$ is called a crystal or an *Artin-Rees pro-crystal*, if for every morphism $g : (U', T') \to (U, T)$ in $Y\,\mathrm{Strat}(X/S)$, the morphism $g_E^* : g^*E_{(U,T)} \to E_{(U',T')}$ is an isomorphism in $\mathrm{AR}(\mathrm{Mod}(O_{T'}))$. Let $\mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}}$ be the full subcategory of $\mathrm{AR}(\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S}))$ consisting of crystals. Let $C_{X/S, Y\,\mathrm{Strat}}$ be the full subcategory consisting of crystals in $\mathrm{Mod}(O_{X/S})$ on $Y\,\mathrm{Strat}(X/S)$. An object of $C_{X/S, Y\,\mathrm{Strat}}$ is called a crystal in $O_{X/S}$-modules. Similar definition extends to the site $\mathrm{Inf}(X/S)$ instead of $Y\,\mathrm{Strat}(X/S)$. Let $C_{X/S}$ be the category of crystals in $O_{X/S}$-modules on $\mathrm{Inf}(X/S)$.

Remark 6.2. An $O_{X/S}$-module M on $\mathrm{Inf}(X/S)$ is called locally quasi-coherent, if for every $(U,T) \in \mathrm{Inf}(X/S)$, the O_T-module M_T is quasi-coherent. By [Ber74, IV, Proposition 1.1.3], an $O_{X/S}$-module M is quasi-coherent if and only if it is a locally quasi-coherent crystal.

Remark 6.3. For a commutative ring R, the tensor product $\otimes : \mathrm{Mod}(R) \times \mathrm{Mod}(R) \to \mathrm{Mod}(R)$ induces a bifunctor

$$\mathrm{AR}\,\mathrm{Mod}(R) \times \mathrm{AR}\,\mathrm{Mod}(R) \to \mathrm{AR}\,\mathrm{Mod}(R).$$

From this, one may define a tensor product

$$\mathrm{AR}\,\mathrm{Mod}(O_Y) \times \mathrm{AR}\,\mathrm{Mod}(O_Y) \to \mathrm{AR}\,\mathrm{Mod}(O_Y).$$

By [Ber74, II, 1.5.3], it upgrades to a tensor product of stratified pro-modules

$$\mathrm{Strat}-\mathrm{AR}(O_Y) \times \mathrm{Strat}-\mathrm{AR}(O_Y) \to \mathrm{Strat}-\mathrm{AR}(O_Y). \tag{18}$$

Similarly, one has a tensor product

$$\mathrm{AR}\,\mathrm{Mod}(O_{X/S}) \times \mathrm{AR}\,\mathrm{Mod}(O_{X/S}) \to \mathrm{AR}\,\mathrm{Mod}(O_{X/S}),$$

which restricts to a bifunctor

$$\mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}} \times \mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}} \to \mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}}. \tag{19}$$

Lemma 6.4. *Let F be a crystal in $O_{X/S}$-modules on $\mathrm{Inf}(X/S)$ (resp. $Y\,\mathrm{Strat}(X/S)$). Then for every object (U,T) of $\mathrm{Inf}(X/S)$ (resp. $Y\,\mathrm{Strat}(X/S)$), the O_T-module $F_{(U,T)}$ admits a natural stratification and an integrable connection relative to S.*

Proof. We prove the case without parentheses. For every integer $n \geq 0$, as $T \to \Delta^n_T(T^2)$ is a nilpotent thickening, $(U, \Delta^n_T(T^2))$ is an object of $\mathrm{Inf}(X/S)$. As F is a crystal, for the two projections $p_i : \Delta^n_T(T^2) \to T$ $(i = 0,1)$, the morphisms

$$p_i^* F_{(U,T)} \to F_{(U,\Delta^n_T(T^2))}$$

are isomorphisms of $P^n_{T/S} = P^n_T(T^2)$-modules. Thus, one has an isomorphism

$$\epsilon_{n,T} : P^n_{T/S} \otimes_{O_T} F_{(U,T)} \cong F_{(U,T)} \otimes_{O_T} P^n_{T/S}$$

of $P^n_{T/S}$-modules. Its base change along $P^n_{T/S} \to O_T$ is the identity, so $\epsilon_{n,T}$ is an n-connection on $F_{(U,T)}$. From [Ber74, II, Proposition 1.3.3 i)], using the crystal property for the three projections $\Delta^n_T(T^3) \to \Delta^n_T(T^2)$, one shows that $(\epsilon_{n,T})_{n\geq 0}$ is a stratification on $F_{(U,T)}$ relative to S. Let $\nabla_T : F_T \to F_T \otimes_{O_T} \Omega^1_{T/S}$ be the O_S-linear morphism corresponding to the connection $\epsilon_{1,T}$. Similar to [Sta25, Tag 07J6], one shows that ∇_T is an integrable connection. $\square$

Lemma 6.5. *Let $X \to Y$ be a morphism over S. There is a natural functor*

$$\mathrm{Strat} - \mathrm{AR}(O_Y) \to \mathrm{Pro} - \mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}}, \tag{20}$$

which regards tensor products (18) *and* (19)*. It restricts to a functor*

$$\mathrm{Strat}(O_Y) \to C_{X/S, Y\,\mathrm{Strat}}. \tag{21}$$

When Y is smooth over S, one can replace the site $Y\,\mathrm{Strat}(X/S)$ by $\mathrm{Inf}(X/S)$ to get a functor $\mathrm{Strat}(O_Y) \to C_{X/S}$.

Proof. Let $M \in \mathrm{AR}(\mathrm{Mod}(O_Y))$ be equipped with a stratification ϵ. For every object $(U,T) \in Y\,\mathrm{Strat}(X/S)$, choose a morphism $h : T \to Y$ over S with $h|_U = g|_U$. We prove that up to canonical isomorphism, $h^*M \in \mathrm{AR}(\mathrm{Mod}(O_T))$ is independent of the choice of h, and we shall define an object $\mathcal{M} \in \mathrm{AR}\,\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S})$ with $\mathcal{M}_T := h^*M$.

In fact, let $h_i : T \to Y$ $(i = 0, 1)$ be two such choices. As $U \hookrightarrow T$ is a thickening of finite order, one may choose an integer $n \geq 0$ such that the $(n+1)$-th power of the defining ideal sheaf vanishes. By [Ber74, II, Proposition 1.2.4 i)] (which is stated for modules but holds for pro-modules), ϵ_n induces an isomorphism $\epsilon_{h_0,h_1} : h_1^*M \to h_0^*M$ in $\mathrm{AR}(\mathrm{Mod}(O_T))$. By commutativity of (14), ϵ_{h_0,h_1} is independent of the choice of n. From [Ber74, II, Proposition 1.3.7 i)] (which holds for pro-modules), the cocycle condition (15) implies that for another such morphism $h_2 : T \to Y$, one has $\epsilon_{h_0,h_2} = \epsilon_{h_0,h_1} \circ \epsilon_{h_1,h_2}$ as morphism $h_2^*M \to h_0^*M$.

For every morphism $u : (U', T') \to (U, T)$ in $Y\,\mathrm{Strat}(X/S)$, let the natural isomorphism

$$u_M^* : u^*(h^*M) \to (hu)^*M$$

be the transition morphism $u^*\mathcal{M}_T \to \mathcal{M}_{T'}$ in $\mathrm{AR}\,\mathrm{Mod}(O_{T'})$. For another morphism $u' : (U'', T'') \to (U', T')$ in $Y\,\mathrm{Strat}(X/S)$, the transitivity condition

$$(u \circ u')_M^* = u_M'^* \circ u'^{-1}(u_M^*).$$

holds. Thus, one defines a crystal $\mathcal{M}$ in Artin-Rees pro-modules over $O_{X/S}$ on $Y\,\mathrm{Strat}(X/S)$. Similarly, every horizontal morphism $(M', \epsilon') \to (M, \epsilon)$ induces a morphism of crystals. Thus, one defines the stated functor (20). ☐

Remark 6.6. Take $X \to Y$ to be Id_X. Then [Gro68, Section 4.2] shows that the functor (21) is an equivalence of categories, with a quasi-inverse

$$C_{X/S, X\,\mathrm{Strat}} \to \mathrm{Strat}(O_X), \quad \mathcal{M} \mapsto \mathcal{M}_{(X,X)}.$$

Similarly, (20) is also an equivalence. Suppose further that X is smooth over S and of characteristic 0. Then by Fact 5.8, (21) is identified with an equivalence $\mathrm{Mod}(D_{X/S}) \to C_{X/S}$, which sends the $D_{X/S}$-module O_X to the crystal $O_{X/S}$.

Let X' be an other scheme smooth over S and of characteristic 0. Let $f : X \to X'$ be a morphism over S. Then by [Ber74, IV, Corollaire 1.2.4], the pullback of a crystal is a crystal, and there is a canonical commutative diagram

$$\begin{tikzcd}
\mathrm{Mod}(D_{X'/S}) \arrow[r, "\cong"] \arrow[d, "f^*"] & C_{X'/S} \arrow[d, "f^*_{\mathrm{inf}}"] \\
\mathrm{Mod}(D_{X/S}) \arrow[r, "\cong"] & C_{X/S}.
\end{tikzcd}$$

7 Direct image of a crystal along a closed immersion

Let $i : X \to Y$ be a closed immersion of schemes over a scheme S. The direct image of a structural crystal $O_{X/S}$ along the closed immersion may no longer be a crystal, as Example 7.6 shows. We show that it is nevertheless an Artin-Rees pro-crystal, which gives a "pro-connection" on the system of infinitesimal neighborhoods $\{\Delta^i_X(Y)\}_{i\geq 0}$. This "pro-connection" is involved in a complex of differential operators which computes the infinitesimal cohomology.

For every object $(U, T) \in \mathrm{Inf}(Y/S)$, let $V := U \times_Y X$. As $i : X \to Y$ is a closed immersion, so is the composition $V \to U \to T$.

Lemma 7.1. *For every object* $(U, T) \in \mathrm{Inf}(Y/S)$*, the sheaf* $i^*T \in \mathrm{Sh}(\mathrm{Inf}(X/S))$ *defined by* (3) *is pro-representable. It is representable by* (V, T) *if* $i : X \to Y$ *is a nilpotent thickening.*

Proof. By definition (3), for every object $(V', T') \in \mathrm{Inf}(X/S)$, one has $i^*T(V', T') = \mathrm{Hom}_i(T', T)$. For every integer $n \geq 0$, $V \to \Delta^n_V(T)$ is a finite order thickening, so $(V, \Delta^n_V(T)) \in \mathrm{Inf}(X/S)$. For every morphism $(V', T') \to (V, \Delta^n_V(T))$ in $\mathrm{Inf}(X/S)$, the composition $T' \to \Delta^n_V(T) \hookrightarrow T$ is an i-morphism. Thus, there is a natural map

$$\phi : \mathrm{colim}_{n\geq 0} \mathrm{Hom}_{\mathrm{Inf}(X/S)}((V', T'), (V, \Delta^n_V(T))) \to \mathrm{Hom}_i(T', T).$$

From [EGA IV 4, Proposition 16.1.5 (ii)], every $\Delta^n_V(T) \to T$ is a closed immersion, so ϕ is injective. Conversely, as $V' \to T'$ is a finite order thickening, every i-morphism $T' \to T$ induces a factorization $T' \to \Delta^n_V(T)$ for some $n > 0$. Thus, ϕ is surjective. Therefore, i^*T is pro-representable by the direct system of objects $(V, \Delta^n_V(T))_{n\geq 0}$ in $\mathrm{Inf}(X/S)$. ☐

Lemma 7.2. *Let* $(U, T) \in \mathrm{Inf}(Y/S)$*.*

(a) *For every* $n \geq 0$ *and every sheaf* $E \in \mathrm{Sh}(\mathrm{Inf}(X/S))$*, the presheaf*

$$\lambda_n E : \mathrm{Inf}(Y/S)^{\mathrm{op}} \to \mathrm{Set}, \quad (U, T) \mapsto E(V, \Delta^n_V(T))$$

is a sheaf. The resulting functor $\lambda_n : \mathrm{Sh}(\mathrm{Inf}(X/S)) \to \mathrm{Sh}(\mathrm{Inf}(Y/S))$ *is exact.*

(b) *Define a functor*

$$\lambda : \mathrm{Sh}(\mathrm{Inf}(X/S)) \to \mathrm{Sh}(\mathrm{Inf}(Y/S))^{\mathbb{N}}, \quad E \mapsto (\lambda_n E)_{n\geq 0}.$$

Then there is a canonical isomorphism of functors

$$i_{\mathrm{inf}\,*} \xrightarrow{\sim} \lim \circ \lambda : \quad \mathrm{Sh}(\mathrm{Inf}(X/S)) \to \mathrm{Sh}(\mathrm{Inf}(Y/S)). \tag{22}$$

In particular, there is a canonical isomorphism

$$(i_{\inf *}E)_{(U,T)} \to \lim_{n\geq 0} E_{(V,\Delta_V^n(T))}$$

in $\mathrm{Sh}(T)$*, where every* $E_{(V,\Delta_V^n(T))} \in \mathrm{Sh}(\Delta_V^n(T))$ *is seen as a sheaf on* T *via extension by zero for the closed immersion* $\Delta_V^n(T) \to T$*.*

(c) *There is a canonical isomorphism* $(i_{\inf *}O_{X/S})_{(U,T)} \xrightarrow{\sim} P_V(T)$ *of* O_T*-algebras. In particular, if the ideal sheaf of* $i : X \hookrightarrow Y$ *is locally nilpotent, then the canonical morphism* $O_{Y/S} \to i_{\inf *}O_{X/S}$ *is an isomorphism.*

(d) *There is a canonical isomorphism of functors*

$$Ri_{\inf *} \to R\lim_n \circ \lambda : \quad D^+(O_{X/S}, \mathrm{Inf}(X/S)) \to D^+(O_{Y/S}, \mathrm{Inf}(Y/S)).$$

Proof. (a) For every covering $\{(U_i, T_i) \to (U, T)\}_{i\in I}$ in the site $\mathrm{Inf}(Y/S)$, $\{(V_i, \Delta_{V_i}^n(T_i)) \to (V, \Delta_V^n(T))\}$ is a covering in the site $\mathrm{Inf}(X/S)$. As E is a sheaf, the diagram

$$E(V, \Delta_V^n(T)) \to \prod_{i\in I} E(V_i, \Delta_{V_i}^n(T_i)) \rightrightarrows \prod_{i,j\in I} E(V_{ij}, \Delta_{V_{ij}}^n(T_{ij}))$$

is exact, which is identified with

$$(\lambda_n E)(U, T) \to \prod_i (\lambda_n E)(U_i, T_i) \rightrightarrows \prod_{ij} (\lambda_n E)(U_{ij}, T_{ij}).$$

Therefore, $\lambda_n(E)$ is a sheaf on $\mathrm{Inf}(Y/S)$. By construction, one has

$$(\lambda_n E)_{(U,T)} = E_{(V,\Delta_V^n(T))}. \tag{23}$$

From [Ber74, III, Proposition 1.1.5], λ_n is an exact functor.

(b) One has

$$\begin{aligned}(i_{\inf *}E)(U,T) &\overset{\text{(a)}}{=} \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Inf}(Y/S))}(i^*T, E)\\ &\overset{\text{(b)}}{=} \lim_{n\geq 0} \mathrm{Hom}_{\mathrm{Sh}(\mathrm{Inf}(Y/S))}(h_{(V,\Delta_V^n(T))}, E)\\ &\overset{\text{(c)}}{=} \lim_{n\geq 0} E(V, \Delta_V^n(T)) =: (\lim_n \lambda_n E)(U,T),\end{aligned} \tag{24}$$

where (a) is from (4), (b) and (c) use Lemma 7.1 and the Yoneda lemma respectively. For every open subset T_0 of T, let $U_0 = T_0 \times_T U$ and $V_0 = T_0 \times_T V$. Then $\Delta_{V_0}^n(T_0) = \Delta_V^n(T) \times_T T_0$ for all $n \geq 0$. One has a natural identification

$$\begin{aligned}\Gamma(T_0, (i_{\inf *}E)_{(U,T)}) &= (i_{\inf *}E)(U_0, T_0)\\ &\overset{(a)}{=} \lim_{n\geq 0} E(V_0, \Delta^n_{V_0}(T_0))\\ &= \lim_{n\geq 0} \Gamma(\Delta^n_{V_0}(T_0), E_{(V,\Delta^n_V(T))})\\ &= \lim_{n\geq 0} \Gamma(T_0, E_{(V,\Delta^n_V(T))})\\ &= \Gamma(T_0, \lim_{n\geq 0} E_{(V,\Delta^n_V(T))}),\end{aligned}$$

where (a) uses (24). Thus, there is an isomorphism of sheaves $(i_{\inf *}E)_{(U,T)} \to \lim_{n\geq 0} E_{(V,\Delta^n_V(T))}$.

(c) The first statement is from $(\lambda_n O_{X/S})_T = P^n_V(T)$ and (22). Now assume that $X \to Y$ is a locally nilpotent thickening. Then so is $V \to T$. Hence $P_V(T) = O_T$ and the second statement follows.

(d) By Part (a) and [Sta25, Tag 015M], it remains to prove that for every injective object I of $\mathrm{Mod}(O_{X/S})$ on $\mathrm{Inf}(X/S)$, $\lambda(I)$ is right acyclic for $\lim : \mathrm{Mod}(O_{Y/S})^{\mathbb{N}} \to \mathrm{Mod}(O_{Y/S})$. By [Ber74, VI, Proposition 1.1.5], for every $(U,T) \in \mathrm{Inf}(Y/S)$, the sheaf $I_{(V,\Delta^n_V(T))}$ on $\Delta^n_V(T)$ is flasque. By (23), so is the sheaf $\lambda_n(I)_{(U,T)}$ on T. For every integer $q > 0$, one has

$$H^q((U,T), \lambda_n(I)) = H^q(T, \lambda_n(I)_{(U,T)}) = 0.$$

As the inclusion $(V, \Delta^n_V(T)) \to (V, \Delta^{n+1}_V(T))$ is a monomorphism in $\mathrm{Inf}(X/S)$, and I is injective, the map

$$I(V, \Delta^{n+1}_V(T)) \to I(V, \Delta^n_V(T))$$

is surjective. Equivalently, the map $(\lambda_{n+1}I)(U,T) \to (\lambda_n I)(U,T)$ is surjective. Therefore, the system $\{(\lambda_n I)(U,T)\}_{n\geq 0}$ has vanishing $R^1 \lim$. By [Sta25, Tag 0BKY], the result follows.

□

Remark 7.3. We do not know whether the functor $i_{\inf *} : \mathrm{Sh}(\mathrm{Inf}(X/S)) \to \mathrm{Sh}(\mathrm{Inf}(Y/S))$ is exact. The analogous exactness holds for crystalline topoi ([Ber74, IV, Corollaire 1.3.2]).

Lemma 7.4. *For every crystal in $O_{X/S}$-modules E on* $\mathrm{Inf}(X/S)$*, $\lambda(E)$ is an Artin-Rees pro-crystal.*

Proof. Let $u : (U', T') \to (U, T)$ be a morphism in $\mathrm{Inf}(Y/S)$. We prove that the canonical morphism

$$u^{-1}\lambda(E)_T \otimes_{u^{-1}O_T} O_{T'} \to \lambda(E)_{T'} \tag{25}$$

in $\mathrm{Mod}(O_{T'})^{\mathbb{N}}$ induces an isomorphism in $\mathrm{AR}\,\mathrm{Mod}(O_{T'})$. Fix an integer $d \geq 0$ such that $U' \to T'$ is a thickening of order $\leq d$. We shall construct an inverse $\{(\lambda_{n+d}E)_{T'}\}_n \to \{(\lambda_n E)_T \otimes_{O_T} O_{T'}\}_n$ as follows.

First consider the special case $E = O_{X/S}$. From the commutative diagram

$$\begin{tikzcd}
U' \arrow[drr, hook] \arrow[dr, dashed] \arrow[ddr] & & \\
& U \times_T T' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & T' \arrow[d] \\
& U \arrow[r, hook] & T,
\end{tikzcd}$$

the factorization $U' \to U \times_T T'$ of $U' \to T'$ is also a thickening of order $\leq d$. Then so is its base change $V' \to V \times_T T'$ along $i : X \to Y$. For every $n \geq 0$, since $V \to \Delta^n_V(T)$ is a thickening of order $\leq n$, so is its base change $V \times_T T' \to \Delta^n_V(T) \times_T T'$. Thus, the composition $V \to \Delta^n_V(T) \times_T T'$ is a thickening of order $\leq n + d$. Therefore, the solid commutative diagram

$$\begin{tikzcd}
V' \arrow[r, hook] \arrow[d, equal] & V \times_T T' \arrow[r, hook] & \Delta^n_V(T) \times_T T' \arrow[d] \arrow[dl, dashed, "h"] \\
V' \arrow[r, hook] & \Delta^{n+d}_{V'}(T') \arrow[r, hook] & T'
\end{tikzcd}$$

induces a factorization $h : \Delta^n_V(T) \times_T T' \to \Delta^{n+d}_{V'}(T')$. Both the compositions

$$\Delta^n_V(T) \times_T T' \xrightarrow{h} \Delta^{n+d}_{V'}(T') \to \Delta^{n+d}_V(T) \hookrightarrow T,$$
$$\Delta^n_V(T) \times_T T' \to \Delta^n_V(T) \hookrightarrow \Delta^{n+d}_V(T) \hookrightarrow T$$

coincide with the canonical morphism $\Delta^n_V(T) \times_T T' \to T' \xrightarrow{u} T$. As the inclusion $\Delta^{n+d}_V(T) \hookrightarrow T$ is a monomorphism, the left square of the diagram

$$\begin{tikzcd}
\Delta^n_V(T) \times_T T' \arrow[r, "h"] \arrow[d] & \Delta^{n+d}_{V'}(T') \arrow[r, hook] \arrow[d] & T' \arrow[d, "u"] \\
\Delta^n_V(T) \arrow[r, hook] & \Delta^{n+d}_V(T) \arrow[r, hook] & T
\end{tikzcd} \tag{26}$$

is commutative. The morphism $P^{n+d}_V(T) \to P^n_V(T)$ makes the $P^n_V(T)$-module $P^n_V(T) \otimes_{O_T} O_{T'}$ a $P^{n+d}_V(T)$-module. The morphism h induces a morphism

$$h^{\#} : P^{n+d}_{V'}(T') \to P^n_V(T) \otimes_{O_T} O_{T'}$$

of $O_{T'}$-algebras. By (26), it fits into a commutative diagram

$$\begin{tikzcd}
P^{n+d}_V(T) \arrow[r] \arrow[d] & P^n_V(T) \arrow[d] \\
P^{n+d}_{V'}(T') \arrow[r, dashed, "h^{\#}"] & P^n_V(T) \otimes_{O_T} O_{T'}.
\end{tikzcd} \tag{27}$$

Now we consider general E. By (27), the morphism

$$E_{\Delta_V^{n+d}(T)} \times P_{V'}^{n+d}(T') \to E_{\Delta_V^n(T)} \otimes_{P_V^n(T)} (P_V^n(T) \otimes_{O_T} O_{T'}), \quad (s, y) \mapsto s \otimes h^{\#}(y)$$

is $P_V^{n+d}(T)$-bilinear. It induces a morphism

$$E_{\Delta_V^{n+d}(T)} \otimes_{P_V^{n+d}(T)} P_{V'}^{n+d}(T') \to E_{\Delta_V^n(T)} \otimes_{P_V^n(T)} (P_V^n(T) \otimes_{O_T} O_{T'}) \xrightarrow{\sim} E_{\Delta_V^n(T)} \otimes_{O_T} O_{T'} \tag{28}$$

of $P_V^{n+d}(T)$-modules and of $O_{T'}$-modules.

Because E is a crystal, the natural morphism

$$E_{\Delta_V^{n+d}(T)} \otimes_{P_V^{n+d}(T)} P_{V'}^{n+d}(T') \to E_{\Delta_{V'}^{n+d}(T')}$$

is an isomorphism. Thus, one gets a morphism

$$\begin{aligned}
(\lambda_{n+d}E)_{T'} :=& E_{\Delta_{V'}^{n+d}(T')} \\
\xleftarrow{\sim}& E_{\Delta_V^{n+d}(T)} \otimes_{P_V^{n+d}(T)} P_{V'}^{n+d}(T') \\
\overset{(28)}{\to}& E_{\Delta_V^n(T)} \otimes_{O_T} O_{T'} \\
=&(\lambda_n E)_T \otimes_{O_T} O_{T'},
\end{aligned}$$

For variable n, they are compatible, so give rise to a morphism $\lambda(E)_{T'} \to u^{-1}\lambda(E)_T \otimes_{u^{-1}O_T} O_{T'}$ in $\mathrm{AR}\,\mathrm{Mod}(O_{T'})$. By universal property, it is inverse to (25). ☐

Remark 7.5. By Lemma 7.4 and a slight variant of Lemma 6.4, the object $\{\lambda(O_{X/S})\}_{(Y,Y)} = \{P_X^i(Y)\}_{i\geq 0}$ of $\mathrm{AR}\,\mathrm{Mod}(O_Y)$ admits a canonical stratification relative to S. The underlying connection

$$\nabla : \{P_X^i(Y)\} \to \{P_X^i(Y)\} \otimes_{O_Y} \Omega^1_{Y/S}$$

is represented by the system of morphisms $P_X^{n+1}(Y) \to P_X^n(Y) \otimes_{O_Y} \Omega^1_{Y/S}$, such that for every local section σ of the ideal sheaf of $X \to Y$ and every $j \geq 0$,

$$\nabla(\sigma^j) = j\sigma^{j-1} \otimes d\sigma, \tag{29}$$

and that for every local section y of O_Y, one has $\nabla(y) = 1 \otimes dy$.

Then by a variant of [Ber74, II, Proposition 2.2.1], every differential operator $u : M \to N$ of O_Y-modules of order $\leq n$ induces a morphism

$$\{P^n_{Y/S} \otimes_{O_Y} P_X^i(Y) \otimes_{O_Y} M\}_{i\geq 0} \to \{P_X^i(Y) \otimes_{O_Y} N\}_{i\geq 0}$$

in $\mathrm{AR}\,\mathrm{Mod}(O_Y)$. Taking limits, one gets a morphism

$$\lim_i (P^n_{Y/S} \otimes_{O_Y} P_X^i(Y) \otimes_{O_Y} M) \to \lim_i (P_X^i(Y) \otimes_{O_Y} N) \tag{30}$$

in $\mathrm{Mod}(O_Y)$. In addition, the natural morphisms

$$d^n_{Y/S, P_X^i(Y)\otimes M} : P_X^i(Y) \otimes M \to P^n_{Y/S} \otimes_{O_Y} P_X^i(Y) \otimes_{O_Y} M$$

induce a morphism

$$\lim_i \left(P^i_X(Y) \otimes M\right) \to \lim_i (P^n_{Y/S} \otimes_{O_Y} P^i_X(Y) \otimes_{O_Y} M) \tag{31}$$

in $\mathrm{Mod}(Y, O_S)$. Taking composition of (30) and (31), one obtains a morphism

$$\lim_i (P^i_X(Y) \otimes_{O_Y} M) \to \lim_i (P^i_X(Y) \otimes_{O_Y} N) \tag{32}$$

in $\mathrm{Mod}(Y, O_S)$.

By [Ber74, IV, Théorème 1.3.4], on crystalline site, the direct image of a crystal along a closed immersion is a crystal, which is "an extremely nontrivial and useful example" of crystal ([BO78, p.6.1]). The infinitesimal analog fails, as Example 7.6 shows.

Example 7.6. Let k be a field. Let $X = S = \operatorname{Spec} k$, $Y = \operatorname{Spec} k[y] = \mathbb{A}^1_k$, $i : X \to Y$ be the inclusion of the origin. We prove that $E := i_{\inf *} O_{X/S}$ is not a crystal in $O_{Y/S}$-modules.

By Lemma 7.2 (c), for every finite order thickening $Y \to T$ over S, $E_{(Y,T)} = P_X(T)$ as an O_T-module. It is supported at a single point. Then the O_Y-module $E_{(Y,Y)}$ is not quasi-coherent, and its stalk at the origin is $k[[y]]$. Take

$$R = k[y, x_1, x_2, \dots]/(x_i x_j : i, j > 0), \quad T = \operatorname{Spec} R.$$

The morphism of k-algebras

$$\phi : R \to k[y], \quad y \mapsto y,\ x_i \mapsto 0$$

is surjective, with kernel $I := (x_1, x_2, \dots)$. One has $I^2 = 0$ in R. Then ϕ induces a first order thickening $\iota : Y \to T$ over S.

Call the composition $X \to Y \to T$ the origin $0 \in T$. The corresponding ring map $R \to k$ has kernel $J := (y, x_1, x_2, \dots)$. The stalk of $E_{(Y,T)}$ at the origin is $\lim_{n \geq 0} R/J^n$. It has an element, formally written as $\sum_{i>0} x_i y^i$, whose image in R/J^n is $\sum_{i=1}^{n-1} x_i y^i$.

The inclusion $k[y] \to R$ makes R a free $k[y]$-module, with a basis $\{1, x_1, x_2, \dots\}$. It induces a morphism $h : T \to Y$. Thus, the induced morphism $O_{Y,0} \to O_{T,0}$ makes $O_{T,0}$ a free $O_{Y,0}$-module with a basis $\{1, x_1, x_2, \dots\}$. One has $h \circ \iota = \mathrm{Id}_Y$. Thus, $h : (Y, T) \to (Y, Y)$ is a morphism in $\mathrm{Inf}(Y/S)$.

Consider the morphism $\psi : h^* E_{(Y,Y)} \to E_{(Y,T)}$ of O_T-modules. By [Sta25, Tag 0098], the stalk of $h^* E_{(Y,Y)}$ at the origin is $k[[y]] \otimes_{O_{Y,0}} O_{T,0}$, which is a free $k[[y]]$-module with a basis $\{1, x_1, x_2, \dots\}$. The element $\sum_{i>0} x_i y^i$ is not in the image of the morphism of stalks

$$\psi_0 : (h^* E_{(Y,Y)})_0 \to (E_{(Y,T)})_0,$$

so ψ is not surjective. Therefore, E is not a crystal.

Remark 7.7. In Example 7.6, take $k = \mathbb{C}$. Then it shows that the functor $i_{\inf *}$ is not compatible with the direct image of D-modules $\int_i^0 : \mathrm{Mod}(D_X) \to \mathrm{Mod}(D_Y)$ in [HT07, Proposition 1.5.24].

8 Linearization

Let $X \to Y$ be a morphism of schemes over a scheme S. We recall Grothendieck's linearization functor. We shall see that as Poincaré's lemma, in characteristic 0, the linearization of the de Rham complex $\Omega^\bullet_{Y/S}$ is a resolution of $O_{X/S}$ when Y is smooth.

By [Sta25, Tag 01AH (1)], the category $\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S})$ has limits. Let

$$\lim : \mathrm{AR}\,\mathrm{Mod}(O_{X/S}) \to \mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S}) \tag{33}$$

be the limit functor from Remark 5.2.

Definition 8.1. Let

$$\mathrm{Gro} : \mathrm{Diff}(Y/S) \overset{Q^0}{\to} \mathrm{Strat}-\mathrm{AR}(O_Y) \overset{(20)}{\to} \mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}},$$

$$L : \mathrm{Diff}(Y/S) \overset{\mathrm{Gro}}{\to} \mathrm{AR}\,\mathrm{Mod}(O_{X/S}) \overset{(33)}{\to} \mathrm{Mod}(O_{X/S})$$

be the compositions. The functor L is called the *linearization functor.*

Remark 8.2. From [Ber74, III, Proposition 1.2.3], when $Y \to S$ is quasi-smooth, the morphism of topoi (5) is an equivalence of categories. In this case, the functor Gro takes values in crystals in $\mathrm{AR}\,\mathrm{Mod}(\mathrm{Inf}(X/S), O_{X/S})$, and L takes value in $\mathrm{Mod}(\mathrm{Inf}(X/S), O_{X/S})$.

By the proof of Lemma 6.5, for every O_Y-module E, every $(U,T) \in Y\,\mathrm{Strat}(X/S)$ and every morphism $h : T \to Y$ with $h|_U = g|_U$, there is a canonical isomorphism

$$\mathrm{Gro}(E)_{(U,T)} \xrightarrow{\sim} h^*(P^\bullet_{Y/S} \otimes_{O_Y} E) \tag{34}$$

in $\mathrm{AR}(\mathrm{Mod}(O_T))$. By [Ber74, III, Proposition 1.1.5], there is an isomorphism

$$L(E)_{(U,T)} = \lim \mathrm{Gro}(E)_{(U,T)} = \lim_{n\geq 0} h^*\left(Q^0(E)^n\right) = \lim_{n\geq 0} h^*(P^n_{Y/S} \otimes_{O_Y} E) \tag{35}$$

in $\mathrm{Mod}(O_T)$. Then $L(O_Y)$ is a sheaf of $O_{X/S}$-algebras, and $L(E)$ is naturally an $L(O_Y)$-module. By Lemma 8.3, the morphism $L(O_Y) \to \mathrm{Gro}(O_Y)$ is surjective.

Lemma 8.3. *For a quasi-coherent O_Y-module E, the canonical morphism $L(E) \to \mathrm{Gro}(E)$ in $\mathrm{AR}\,\mathrm{Mod}(O_{X/S})$ is surjective.*

Proof. We need to prove that for every object $(U,T) \in Y\,\mathrm{Strat}(X/S)$, the morphism $L(E)_{(U,T)} \to \mathrm{Gro}(E)_{(U,T)}$ in $\mathrm{AR}\,\mathrm{Mod}(O_T)$ is surjective. Choose a morphism $h : T \to Y$ over S with $h|_U = g|_U$. We show that the morphism $\lim_{n\geq 0} h^*Q^0(E)^n \to h^*Q^0(E)$ in $\mathrm{Mod}(O_T)^{\mathbb{N}}$ is surjective, i.e., for every integer $n_0 \geq 0$, the morphism

$$\lim_n h^*(P^n_{Y/S} \otimes_{O_Y} E) \to h^*(P^{n_0}_{Y/S} \otimes_{O_Y} E)$$

in $\mathrm{Mod}(O_T)$ is surjective. By [EGA IV 4, 16.7.4], as E is quasi-coherent, the left and the right structure of O_Y-modules on $P^{n_0}_{Y/S} \otimes_{O_Y} E$ are quasi-coherent. The result then follows from Lemma 8.4. $\square$

Lemma 8.4. *Let S be a scheme. Let $(F_n)_{n\geq 0} \in \mathrm{Qch}(S)^{\mathbb{N}}$ be an inverse system of quasi-coherent sheaves on S. Assume that for every integer $n \geq 0$, the transition morphism $F_{n+1} \to F_n$ is surjective. Then for every integer $n_0 \geq 0$, the projection $\lim_n F_n \to F_{n_0}$ is surjective.*

Proof. For every integer $n \geq n_0$, let K_n be the kernel of $F_n \to F_{n_0}$. By the Serre vanishing theorem (see, e.g., [Sta25, Tag 01XB]), as the F_n are quasi-coherent, for every affine open subset U of S, the sequence

$$0 \to K_n(U) \to F_n(U) \to F_{n_0}(U) \to 0$$

is exact. By the four lemma, as the transition morphisms $F_{n+1}(U) \to F_n(U)$ are surjective, so is $K_{n+1}(U) \to K_n(U)$. Then by [Ati69, Proposition 10.2], the morphism $\lim_n F_n(U) \to F_{n_0}(U)$ is surjective. As affine opens form a base of the topology of S, $\lim_n F_n \to F_{n_0}$ is surjective. $\square$

Remark 8.5. Let E, F and G be O_Y-modules. Equip G with a stratification relative to S. Then by [Ber74, II, Proposition 2.2.1], the stratification induces a canonical map

$$\mathrm{Hom}_{\mathrm{Diff}(Y/S)}(E, F) \to \mathrm{Hom}_{\mathrm{Diff}(Y/S)}(G \otimes_{O_Y} E, G \otimes_{O_Y} F). \tag{36}$$

Let $\mathcal{G}$ be the crystal in $O_{X/S}$-modules on $Y\,\mathrm{Strat}(X/S)$ induced by G via (21). Let $u : E \to F$ be a differential operator. Let $v : G \otimes_{O_Y} E \to G \otimes_{O_Y} F$ be the differential operator induced by u via (36). As in [Ber74, IV, Proposition 3.1.4], one can prove that there is a canonical isomorphism

$$\mathrm{Gro}(G \otimes_{O_Y} E) \to \mathcal{G} \otimes_{O_{X/S}} \mathrm{Gro}(E)$$

of Artin-Rees pro-crystals, and a similar one for F which fits into a commutative diagram

$$\begin{tikzcd}
\mathrm{Gro}(G \otimes_{O_Y} E) \arrow[r, "\mathrm{Gro}(v)"] \arrow[d] & \mathrm{Gro}(G \otimes_{O_Y} F) \arrow[d] \\
\mathcal{G} \otimes_{O_{X/S}} \mathrm{Gro}(E) \arrow[r, "\mathrm{id} \otimes \mathrm{Gro}(u)"] & \mathcal{G} \otimes_{O_{X/S}} \mathrm{Gro}(F)
\end{tikzcd}$$

in $\mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}}$.

Lemma 8.6 computes the local expression of the formalization of the de Rham complex. It can be proved as in [Ber74, IV, Lemme 3.2.5].

Lemma 8.6. *Let $q_1, \ldots, q_n > 0$ and $m, k \geq 0$ be integers. Let $a, x_1, \ldots, x_n$ be local sections of O_Y. Let ξ_i be the local section of $P^m_{Y/S}$ which is the image of the local section $1 \otimes x_i - x_i \otimes 1$ of $O_{Y^2_S}$. Let ω be a local section of $\Omega^k_{Y/S}$. Let*

$$Q^0(d) : P^{m+1}_{Y/S} \otimes_{O_Y} \Omega^k_{Y/S} \to P^m_{Y/S} \otimes \Omega^{k+1}_{Y/S}$$

be the O_Y-linear morphism (16)*. One has*

$$Q^0(d)(a\xi_1^{q_1}\dots\xi_n^{q_n}\otimes\omega) = a\xi_1^{q_1}\dots\xi_n^{q_n}\otimes d(\omega)+\sum_{i=1}^{n} a\xi_1^{q_1}\dots(q_i\xi_i^{q_i-1})\dots\xi_n^{q_n}\otimes(d(x_i)\wedge\omega)$$

as local section of $P^m_{Y/S}\otimes_{O_Y}\Omega^{k+1}_{Y/S}$.

Remark 8.7. Notation as in Lemma 8.6. By [EGA IV 4, 16.11.1], locally the O_Y-module $P^m_{Y/S}$ is generated by the local sections $\xi^t := \xi_1^{t_1}\dots\xi_n^{t_n}$ with $|t|\le m$.

The image of O_Y with its natural stratification under the functor (21) is the crystal $O_{X/S}$ on $Y\,\mathrm{Strat}(X/S)$. The morphisms $(\pi^n : P^n_{Y/S}\to O_Y)_{n\ge 0}$ glue to a morphism $\pi : Q^0(O_Y)\to O_Y$ in $\mathrm{AR}\,\mathrm{Mod}(O_Y)$ that is horizontal. It induces a surjective augmentation morphism $\pi : \mathrm{Gro}(O_Y)\to O_{X/S}$ in $\mathrm{AR}\,\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S})$. Taking limit, one gets a morphism

$$L(O_Y)\to O_{X/S} \tag{37}$$

of $O_{X/S}$-algebras on $Y\,\mathrm{Strat}(X/S)$. The morphisms $(d_0^n : O_Y \to P^n_{Y/S})_{n\ge 0}$ induced by the first projection $Y_S^2\to Y$ glue to a morphism $d_0 : O_Y\to Q^0(O_Y)$ in $\mathrm{AR}(\mathrm{Mod}(O_Y))$ that is horizontal. By Lemma (20), it gives rise to a natural co-augmentation morphism

$$O_{X/S}\to \mathrm{Gro}(O_Y) \tag{38}$$

in the category $\mathrm{Pro}-\mathrm{Cris}(X/S)_{Y\,\mathrm{Strat}}$ of Artin-Rees pro-crystals. Taking limits, one has a morphism

$$O_{X/S}\to L(O_Y) \tag{39}$$

of sheaves of rings on $Y\,\mathrm{Strat}(X/S)$, which is a right inverse of (37). Thus, $O_{X/S}$ is a direct factor of $L(O_Y)$. By Lemma 8.6, for every $n\ge 0$, the composition

$$O_Y \overset{d_0^{n+1}}{\to} P^{n+1}_{Y/S} \overset{Q^0(d)}{\to} P^n_{Y/S}\otimes\Omega^1_{Y/S}$$

is zero, so the composition of horizontal morphisms $O_Y\to Q^0(O_Y)\to Q^0(\Omega^1_{Y/S})$ in $\mathrm{AR}(\mathrm{Mod}(O_Y))$ is zero. Therefore, the corresponding composition of morphisms of crystals

$$O_{X/S}\overset{(38)}{\to}\mathrm{Gro}(O_Y)\to\mathrm{Gro}(\Omega^1_{Y/S})$$

vanishes. Let M be an O_Y-module with an integrable connection. By [Ber74, p.165], $M\otimes_{O_Y}\Omega^\bullet_{Y/S}$ is a differential complex of order ≤ 1. Then from [Ber74, IV, Proposition 3.2.7], $\mathrm{Gro}(M\otimes_{O_Y}\Omega^\bullet_{Y/S})$ and hence $L(M\otimes_{O_Y}\Omega^\bullet_{Y/S})$ are complexes. Thus, there is a canonical morphism

$$O_{X/S}\to\mathrm{Gro}(\Omega^\bullet_{Y/S}) \tag{40}$$

in $\mathrm{Ch}^{\ge 0}(\mathrm{AR}(\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S})))$. Passing to limit, it induces a canonical morphism

$$O_{X/S}\to L(\Omega^\bullet_{Y/S})$$

in $\mathrm{Ch}^{\ge 0}(\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S}))$.

Lemma 8.8. *Let $g : X \to Y$ be a morphism over S. Let $\mathrm{Alg}(Y\,\mathrm{Strat}(X/S), O_{X/S})$ be the category of $O_{X/S}$-algebras on $Y\,\mathrm{Strat}(X/S)$. Then there is a canonical surjective morphism*

$$\phi : \mathrm{Gro}(O_Y) \to j^* i_{X/S*} O_X \tag{41}$$

in $\mathrm{AR}\,\mathrm{Alg}(Y\,\mathrm{Strat}(X/S), O_{X/S})$. It induces a surjective morphism

$$L(O_Y) \to j^* i_{X/S*} O_X \tag{42}$$

of $O_{X/S}$-algebras on $Y\,\mathrm{Strat}(X/S)$.

Proof. Pulling back the surjective morphism $(P^n_{Y/S})_{n \geq 0} \to O_Y$ in $\mathrm{Mod}(O_Y)^{\mathbb{N}}$ along $g : X \to Y$, one has a surjective morphism

$$(g^* P^n_{Y/S})_{n \geq 0} \to O_X \tag{43}$$

in $\mathrm{Mod}(O_X)^{\mathbb{N}}$. By (34), one has $\mathrm{Gro}(O_Y)_{(X,X)} = (g^* P^n_{Y/S})_{n \geq 0}$ in $\mathrm{AR}\,\mathrm{Mod}(O_X)$. From [Sta25, Tag 077I], as the inclusion functor $Y\,\mathrm{Strat}(X/S) \to \mathrm{Inf}(X/S)$ is fully faithful, the canonical morphism

$$\mathrm{Gro}(O_Y) \to j^* j_! \,\mathrm{Gro}(O_Y)$$

in $\mathrm{AR}\,\mathrm{Mod}(Y\,\mathrm{Strat}(X/S), O_{X/S})$ is an isomorphism. Therefore, the canonical morphism

$$\mathrm{Gro}(O_Y)_{(X,X)} \to i^*_{X/S} j_! \,\mathrm{Gro}(O_Y)$$

in $\mathrm{AR}\,\mathrm{Mod}(O_X)$ is an isomorphism. Combine it with (43), one has a morphism $i^*_{X/S} j_! \,\mathrm{Gro}(O_Y) \to O_X$ in $\mathrm{AR}\,\mathrm{Mod}(O_X)$. By adjunction, it induces the morphism (41).

For every object $(U,T) \in Y\,\mathrm{Strat}(X/S)$, one has an induced morphism

$$\phi_{(U,T)} : \mathrm{Gro}(O_Y)_{(U,T)} \to (j^* i_{X/S*} O_X)_{(U,T)}$$

in $\mathrm{AR}\,\mathrm{Alg}(O_T)$. Choose a morphism $h : T \to Y$ over S with $h|_U = g|_U$. By construction, $\phi_{(U,T)}$ is induced by the composition $(h^* P^n_{Y/S})_{n \geq 0} \to h^* O_Y = O_T \to O_U$ of surjective morphisms in $\mathrm{Alg}(O_T)^{\mathbb{N}}$. Therefore, $\phi_{(U,T)}$ is a surjective morphism in $\mathrm{AR}\,\mathrm{Alg}(O_T)$. Hence, ϕ is a surjective morphism in $\mathrm{AR}\,\mathrm{Alg}(Y\,\mathrm{Strat}(X/S), O_{X/S})$. The second statement follows from Lemma 8.3. □

Let $\mathcal{K}$ be the kernel of (41) in the $\mathrm{AR}\,\mathrm{Mod}(O_{X/S})$, which is an abelian category by Remark 5.3. Let K be the kernel of (42) in the abelian category $\mathrm{Mod}(O_{X/S})$. Then K is an ideal of $L(O_Y)$ and $K = \lim \mathcal{K}$. From the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & K \arrow[r] \arrow[d] & L(O_Y) \arrow[r] \arrow[d, shift right, "(37)"'] & j^* i_{X/S*} O_X \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & j^* J_{X/S} \arrow[r] & O_{X/S} \arrow[r] \arrow[u, bend right, "(39)"'] & j^* i_{X/S*} O_X \arrow[r] & 0
\end{tikzcd}$$

with exact rows, $j^* J_{X/S}$ is naturally a direct summand of the $O_{X/S}$-module K.

We give a concrete description of $\mathcal{K}$. For every integer $n \geq 0$, let $H_n := \ker(P^n_{Y/S} \to O_Y)$ be the kernel of the augmentation morphism, i.e., the ideal sheaf of the nilpotent thickening $Y \to \Delta^n_Y(Y^2)$. For an object $(U, T) \in Y \operatorname{Strat}(X/S)$ and a *chosen* morphism $h : T \to Y$ with $h|_U = g|_U$, let $K_{n,(U,T)} = K_{n,T}$ be the kernel of the morphism $h^* Q^0(O_Y)^n \to O_U$, which is an ideal of $h^* P^n_{Y/S}$. For both left and right structures of O_T-module, $K_{n,(U,T)}$ is quasi-coherent. Then $(K_{n,(U,T)})_{n \geq 0} \in \operatorname{Mod}(O_T)^{\mathbb{N}}$ represents $\mathcal{K}_{(U,T)} \in \operatorname{AR}\operatorname{Mod}(O_T)$. By [Ber74, III, Proposition 1.1.5], for left structures one has

$$K_{(U,T)} = \lim_n K_{n,(U,T)}$$

in $\operatorname{Mod}(O_T)$. The short exact sequence $0 \to H_n \to P^n_{Y/S} \to O_Y \to 0$ in $\operatorname{Mod}(P^n_{Y/S})$ admits two natural splittings, so the middle row of the commutative diagram

$$\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & h^* H_n \arrow[r] \arrow[d, equal] & K_{n,(U,T)} \arrow[r] \arrow[d] & J_{X/S,(U,T)} \arrow[d] \arrow[ddll, dashed] & \\
0 \arrow[r] & h^* H_n \arrow[r] \arrow[d] & h^* P^n_{Y/S} \arrow[r] \arrow[d] & O_T \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] \arrow[d] & O_U \arrow[r, equal] \arrow[d] & O_U \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

is exact. By the snake lemma, the diagram induces a short exact sequence

$$0 \to h^* H_n \to K_{n,(U,T)} \to J_{X/S,(U,T)} \to 0 \tag{44}$$

in $\operatorname{Mod}(O_T)$ with two natural splittings. As $P^{n+1}_{Y/S} \to P^n_{Y/S}$ is surjective, so are $H_{n+1} \to H_n$ and $K_{n+1,(U,T)} \to K_{n,(U,T)}$.

Remark 8.9. For variable $n \geq 0$, the left splittings for (44) are compatible, and same for the right splittings. Thus, taking limits one gets an exact sequence

$$0 \to \lim_n h^* H_n \to K_{(U,T)} \to J_{X/S,(U,T)} \to 0$$

with two natural splittings.

9 Poincaré lemma

Remark 9.1. Let $X \to Y$ be a morphism over S. For an $O_{X/S}$-module M on $Y \operatorname{Strat}(X/S)$, we define a filtration of the complex $M \otimes_{O_{X/S}} \operatorname{Gro}(\Omega^\bullet_{Y/S})$ as follows. By (44) and Lemma 8.6, for every $k \geq 0$, the morphisms

$$Q^0(d) : P^{n+1}_{Y/S} \otimes_{O_Y} \Omega^k_{Y/S} \to P^n_{Y/S} \otimes \Omega^{k+1}_{Y/S}$$

with variable $n \geq 0$ induce a morphism

$$\operatorname{Gro}(d) : \mathcal{K}^{q+1} \cdot \operatorname{Gro}(\Omega^k_{Y/S}) \to \mathcal{K}^q \cdot \operatorname{Gro}(\Omega^{k+1}_{Y/S})$$

for every $q \geq 0$. Thus, one gets a subcomplex $F^q \operatorname{Gro}(\Omega^\bullet_{Y/S})$ of $\operatorname{Gro}(\Omega^\bullet_{Y/S})$, whose k-th term is $\mathcal{K}^{q-k} \cdot \operatorname{Gro}(\Omega^k_{Y/S})$. (By convention, $\mathcal{K}^i = \operatorname{Gro}(O_Y)$ when $i \leq 0$.)

By (44), the morphism (38) restricts to a morphism $J^q_{X/S} \to \mathcal{K}^q$. Thus, the morphism of complexes (40) restricts to a morphism

$$J^q_{X/S} \to F^q \operatorname{Gro}(\Omega^\bullet_{Y/S}). \tag{45}$$

Let

$$F^q(M \otimes_{O_{X/S}} \operatorname{Gro}(\Omega^\bullet_{Y/S})) \tag{46}$$

be the image of the morphism (Id_M tensor product with the inclusion)

$$M \otimes_{O_{X/S}} F^q \operatorname{Gro}(\Omega^\bullet_{Y/S}) \to M \otimes_{O_{X/S}} \operatorname{Gro}(\Omega^\bullet_{Y/S})$$

in $\mathrm{Ch}^{\geq 0}(\mathrm{AR}\operatorname{Mod}(O_{X/S}))$. Its k-th term is the image of

$$M \otimes_{O_{X/S}} \mathcal{K}^{q-k} \cdot \operatorname{Gro}(\Omega^k_{Y/S}) \to M \otimes_{O_{X/S}} \operatorname{Gro}(\Omega^k_{Y/S})$$

in $\mathrm{AR}\operatorname{Mod}(O_{X/S})$. Then (45) induces a canonical morphism

$$J^q_{X/S} \cdot M \to F^q \left(M \otimes_{O_{X/S}} \operatorname{Gro}(\Omega^\bullet_{Y/S}) \right). \tag{47}$$

The filtered Poincaré lemma, Theorem 9.2, is well-known. For instance, when $S = \operatorname{Spec} \mathbb{C}$, X is a scheme separated *smooth* of finite type over $\mathbb{C}$, $X \to Y$ is the identity, $M = O_{X/S}$, $q = 0$ and $(U, T) = (X, X)$, then [Fio11, Lemma 1] proves that (48) is locally homotopic to zero. We need the case that X is singular while Y is smooth to prove Theorem 14.1.

Theorem 9.2. *Let $X \to Y$ be a morphism over S. Assume that $Y \to S$ is smooth, and that X is of characteristic 0. Then for every $O_{X/S}$-module M and every $q \geq 0$, (47) is a resolution.*

Proof. For every $(U, T) \in \operatorname{Inf}(X/S)$, we prove that the complex

$$(J^q_{X/S} \cdot M)_T \to \left(F^q(M \otimes_{O_{X/S}} \operatorname{Gro}(\Omega^\bullet_{Y/S})) \right)_T \tag{48}$$

belonging to $\mathrm{Ch}^{\geq 0}(\mathrm{AR}\operatorname{Mod}(O_T))$ is locally homotopic to zero. This property is local in T. We shall shrink T to prove that (48) is represented by an object of $\mathrm{Ch}^{\geq 0}(\operatorname{Mod}(O_T)^{\mathbb{N}})$, each level of which is homotopic to zero.

Recall Remark 8.2 that because $Y \to S$ is quasi-smooth, shrinking T one may assume that there is morphism $h : T \to T$ over S with $h|_U = g|_U$. As $Y \to S$ is differentially smooth, shrinking Y one may find $y_1, \dots, y_t \in \Gamma(Y, O_Y)$ such that $\Omega^1_{Y/S} = \oplus_{i=1}^t O_Y dy_i$. For every $1 \le i \le t$, let $\eta_i \in \Gamma(Y, P^\infty_{Y/S})$ be the image of the global section $1 \otimes y_i - y_i \otimes 1$ under the morphism $O_Y \otimes_{O_S} O_Y \to P^\infty_{Y/S}$.

Let $\mathfrak{m} = (\eta_1, \dots, \eta_t)$, which is an ideal sheaf in $O_Y[\eta_1, \dots, \eta_t] = O_{\mathbf{A}^t_Y}$. Then by a slight variant of [Ber74, I, Corollaire 4.5.3 i)], for every $n \ge 0$, $P^n_{Y/S} = O_Y[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n+1}$ as a sheaf of rings on Y, and these identifications are compatible with the projections $P^{n+1}_{Y/S} \to P^n_{Y/S}$.

Consider the isomorphism

$$P^n_{Y/S} \otimes_{O_Y} \Omega^1_{Y/S} \to O_Y[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n+1} \otimes_{O_{\mathbf{A}^t_Y}} \Omega^1_{\mathbf{A}^t_Y/Y}, \quad 1 \otimes dy_i \mapsto 1 \otimes d\eta_i.$$

For every integer $k \ge 0$, it induces an isomorphism

$$P^n_{Y/S} \otimes_{O_Y} \Omega^k_{Y/S} \to O_Y[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n+1} \otimes_{O_{\mathbf{A}^t_Y}} \Omega^k_{\mathbf{A}^t_Y/Y}. \tag{49}$$

By definition, $\mathrm{Gro}(\Omega^k_{Y/S})_T$ is represented by the inverse system

$$\{h^*(P^n_{Y/S} \otimes_{O_Y} \Omega^k_{Y/S})\}_{n \ge 0}.$$

From (49), this system is identified with the system

$$\{O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n+1} \otimes_{O_{\mathbf{A}^t_T}} \Omega^k_{\mathbf{A}^t_T/T}\}_{n \ge 0}. \tag{50}$$

Consider the object

$$O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{\bullet+1} \otimes_{O_{\mathbf{A}^t_T}} \Omega^\bullet_{\mathbf{A}^t_T/T} \tag{51}$$

of $\mathrm{Ch}^{\ge 0}(\mathrm{Mod}(T, O_S)^{\mathbb{N}})$. Its n-th level is the complex

$$O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n+1} \to O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n} \otimes_{O_{\mathbf{A}^t_T}} \Omega^1_{\mathbf{A}^t_T/T} \to \cdots \to O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n-t+1} \otimes_{O_{\mathbf{A}^t_T}} \Omega^t_{\mathbf{A}^t_T/T} \to 0,$$

where each differential

$$O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n-k+1} \otimes_{O_{\mathbf{A}^t_T}} \Omega^k_{\mathbf{A}^t_T/T} \to O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n-k} \otimes_{O_{\mathbf{A}^t_T}} \Omega^{k+1}_{\mathbf{A}^t_T/T}$$

is induced by the "pro-connection"

$$\nabla : O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n-k+1} \to O_T[\eta_1, \dots, \eta_t]/\mathfrak{m}^{n-k} \otimes_{O_{\mathbf{A}^t_T}} \Omega^1_{\mathbf{A}^t_T/T}, \quad \eta_i^r \mapsto r\eta_i^{r-1} \otimes d\eta_i.$$

From Lemma 8.6, the identifications given by (50) are compatible with differentials, so $\mathrm{Gro}(\Omega^\bullet_{Y/S})_T$ is represented by (51). As X is of characteristic 0, so is U. As $U \hookrightarrow T$ is surjective, T is also of characteristic 0, the result follows from Lemma 9.3. □

Lemma 9.3. *Let A be a commutative ring of characteristic 0. Let J be an ideal of A. Let $B = A[\eta_1, \dots, \eta_t]$ be a polynomial algebra over A. Let $\mathfrak{m} = (\eta_1, \dots, \eta_t) \subset B$. For every $n \geq 0$, let $C_n = B/\mathfrak{m}^{n+1}$. Let K_n be the ideal of C_n generated by J and $\mathfrak{m}$. For every $q \geq 0$, let $F^q(C_\bullet \otimes_B \Omega^\bullet_{B/A})$ be a subcomplex of $C_\bullet \otimes_B \Omega^\bullet_{B/A} \in \mathrm{Ch}(\mathrm{Mod}(A)^{\mathbb{N}})$, where the k-th term of the n-th level complex is $K^{q-k}_{n-k} \cdot (C_{n-k} \otimes_B \Omega^k_{B/A})$. Let M be an A-module. Let $F^q(M \otimes_A C_\bullet \otimes_B \Omega^\bullet_{B/A})$ be the image of $M \otimes_A F^q(C_\bullet \otimes_B \Omega^\bullet_{B/A}) \to M \otimes_A C_\bullet \otimes_B \Omega^\bullet_{B/A}$. Then the A-linear complex*

$$0 \to J^q \cdot M \to F^q(M \otimes_A C_\bullet \otimes_B \Omega^\bullet_{B/A}) \tag{52}$$

is homotopic to zero.

Proof. The n-th level of (52) is

$$0 \to J^q \cdot M \to K^q_n \cdot (M \otimes_A C_n) \to K^{q-1}_{n-1} \cdot (M \otimes_A C_{n-1} \otimes_B \Omega^1_{B/A}) \to \cdots \to K^{q-t}_{n-t} \cdot (M \otimes_A C_{n-t} \otimes_B \Omega^t_{B/A}) \to 0.$$

First, we construct a homotopy for (52) with $M = A$ and $q = 0$, i.e., the complex belonging to $\mathrm{Ch}(\mathrm{Mod}(A)^{\mathbb{N}})$

$$0 \to A \to C_\bullet \otimes_B \Omega^\bullet_{B/A}. \tag{53}$$

For an *ordered* subset $I = \{i_1, \dots, i_p\}$ of $\{1, \dots, t\}$, let $d\eta_I = d\eta_{i_1} \wedge \cdots \wedge d\eta_{i_p}$. For $p > 0$, define an A-linear map

$$h_p = h_p(n) : C_{n-p} \otimes_B \Omega^p_{B/A} \to C_{n-p+1} \otimes_B \Omega^{p-1}_{B/A},$$
$$\eta^\alpha \otimes d\eta_I \mapsto \frac{1}{p + |\alpha|} \sum_{m=1}^{p} (-1)^{m+1} \eta^\alpha \eta_{i_m} \otimes d\eta_{i_1} \wedge \dots \widehat{d\eta_{i_m}} \cdots \wedge d\eta_{i_p},$$

where $\alpha = (\alpha_1, \dots, \alpha_t) \in \mathbb{N}^t$ is a multi-index, and $\eta^\alpha = \eta_1^{\alpha_1} \dots \eta_t^{\alpha_t}$. It is well-defined as $p + |\alpha|$ is invertible in A. Define a morphism of A-algebras

$$h_0 = h_0(n) : C_n \to A, \quad \eta^\alpha \mapsto 0.$$

By computation, we prove that the $h_\bullet(n)$ form a homotopy for the complex

$$0 \to A \to C_n \to C_{n-1} \otimes_B \Omega^1_{B/A} \to \cdots \to C_{n-t} \otimes_B \Omega^t_{B/A} \to 0, \tag{54}$$

which is the n-th level of the inverse system (53). Let $e_j = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{N}^t$, where 1 is at the j-th place. As an endomorphism $C_{n-p} \otimes_B \Omega^p_{B/A} \to$

$C_{n-p} \otimes_B \Omega^p_{B/A}$, we have

$$\begin{aligned}
&d^{p-1} \circ h_p(\eta^\alpha \otimes d\eta_I)\\
=&\frac{1}{p+|\alpha|}\sum_{m=1}^{p}(-1)^{m+1}d^{p-1}(\eta^\alpha\eta_{i_m}\otimes d\eta_{i_1}\wedge\ldots\widehat{d\eta}_{i_m}\cdots\wedge d\eta_{i_p})\\
=&\frac{1}{p+|\alpha|}\sum_{m=1}^{p}(-1)^{m+1}((\alpha_{i_m}+1)\eta^\alpha\otimes d\eta_{i_m}\wedge d\eta_{i_1}\wedge\ldots\widehat{d\eta}_{i_m}\cdots\wedge d\eta_{i_p}\\
+&\sum_{j\notin I}\alpha_j\eta^{\alpha-e_j}\eta_{i_m}\otimes d\eta_j\wedge d\eta_{i_1}\wedge\ldots\widehat{d\eta}_{i_m}\cdots\wedge d\eta_{i_p})\\
=&\frac{1}{p+|\alpha|}\sum_{m=1}^{p}((\alpha_{i_m}+1)\eta^\alpha\otimes d\eta_I\\
+&(-1)^{m+1}\sum_{j\notin I}\alpha_j\eta^{\alpha-e_j}\eta_{i_m}\otimes d\eta_j\wedge d\eta_{i_1}\wedge\ldots\widehat{d\eta}_{i_m}\cdots\wedge d\eta_{i_p}).
\end{aligned}$$

We have

$$\begin{aligned}
&h_{p+1}\circ d^p(\eta^\alpha\otimes d\eta_I)\\
=&h_{p+1}\left(\sum_{1\le j\le t, j\notin I}(\alpha_j\eta^{\alpha-e_j}\otimes d\eta_j\wedge d\eta_I)\right)\\
=&\sum_{1\le j\le t, j\notin I}\alpha_j\frac{1}{p+1+|\alpha-e_j|}(\eta^\alpha\otimes d\eta_I\\
+&\sum_{1\le m\le p}(-1)^m\eta^{\alpha-e_j}\eta_{i_m}\otimes d\eta_j\wedge d\eta_{i_1}\wedge\ldots\widehat{d\eta}_{i_m}\cdots\wedge d\eta_{i_p})
\end{aligned}$$

Therefore, we have

$$(d^{p-1}\circ h_p+h_{p+1}\circ d^p)(\eta^\alpha\otimes d\eta_I)=\frac{1}{p+|\alpha|}\left(\sum_{m=1}^{p}(\alpha_{i_m}+1)+\sum_{j\notin I}\alpha_j\right)\eta^\alpha\otimes d\eta_I=\eta^\alpha\otimes d\eta_I.$$

Thus, the family $\{h_p(n)\}$ is a homotopy for the complex (54).

By construction, for fixed p, the $h_p(n)$ are compatible, so define an A-linear morphism $h_p : C_{\bullet-p}\otimes_B\Omega^p_{B/A}\to C_{\bullet-p+1}\otimes_B\Omega^{p-1}_{B/A}$ of inverse systems. Tensoring with id_M, one gets a homotopy for the complex

$$0\to M\to M\otimes_A C_\bullet\otimes_B\Omega^\bullet_{B/A}, \tag{55}$$

which is (52) with $q=0$. For general q, since K^q_n is generated by elements of the form $a\eta^\alpha$ ($|\alpha|\le q$ and $a\in J^{q-|\alpha|}$), $h_p(n)$ induces a morphism

$$K^{q-p}_{n-p}\cdot(M\otimes_A C_{n-p}\otimes_B\Omega^p_{B/A})\to K^{q-p+1}_{n-p+1}\cdot(M\otimes_A C_{n-p+1}\otimes_B\Omega^{p-1}_{B/A}).$$

Therefore, the constructed homotopy for (55) restricts to a homotopy of the subcomplex (52). □

10 Čech-Alexander complex of linearization

We compute the composition of the Čech-Alexander functor with the linearization functor. Combined with Lemma 4.6, the computation for an immersion $X \hookrightarrow Y$ over S implies that the linearization $L(E)$ of a quasi-coherent sheaf E on Y is acyclic for the functor $u'_{X/S*}$.

Remark 10.1. Let $X \to Y$ be a morphism over S. Let E be an O_Y-module with a stratification relative to S. Let $\mathcal{E}$ be the induced crystal on $Y\,\mathrm{Strat}(X/S)$ via (21). Let $E \otimes_{O_Y} \Omega^\bullet_{Y/S}$ be the de Rham complex with coefficients in E as in Remark 5.7. By Remark 8.5, there is a canonical isomorphism

$$\mathrm{Gro}(E \otimes_{O_Y} \Omega^\bullet_{Y/S}) \xrightarrow{\sim} \mathcal{E} \otimes_{O_{X/S}} \mathrm{Gro}(\Omega^\bullet_{Y/S})$$

in $\mathrm{Ch}^{\geq 0}(\mathrm{AR}\,\mathrm{Mod}(O_{X/S}))$. For every $q \geq 0$, it identifies the subcomplex $F^q(\mathcal{E} \otimes_{O_{X/S}} \mathrm{Gro}(\Omega^\bullet_{Y/S}))$ defined in (46) with a subcomplex

$$F^q\,\mathrm{Gro}(E \otimes_{O_Y} \Omega^\bullet_{Y/S}) \subset \mathrm{Gro}(E \otimes_{O_Y} \Omega^\bullet_{Y/S}),$$

whose k-th term is $\mathcal{K}^{q-k} \cdot \mathrm{Gro}(E \otimes_{O_Y} \Omega^k_{Y/S})$. From (47), one gets a morphism in $\mathrm{Ch}(\mathrm{AR}\,\mathrm{Mod}(O_{X/S}))$ on $Y\,\mathrm{Strat}(X/S)$

$$J^q_{X/S} \cdot \mathcal{E} \to F^q\,\mathrm{Gro}(E \otimes_{O_Y} \Omega^\bullet_{Y/S}). \tag{56}$$

Remark 10.2. Let $X \to Y$ be a morphism over S. For every $q > 0$ and every O_Y-module M, define $K^{[q]}L(M) := \lim \mathcal{K}^q \cdot \mathrm{Gro}(M)$, which is an $O_{X/S}$-submodule of $L(M)$ containing $K^q \cdot L(M)$. For every $(U,T) \in Y\,\mathrm{Strat}(X/S)$, choose a morphism $h : T \to Y$ over S with $h|_U = g|_U$. As $K_{n,(U,T)}$ is the ideal sheaf of the nilpotent thickening $U \to T \to T \times_Y \Delta^n_Y(Y^2)$ of schemes, and $E_n := h^*(P^n_{Y/S} \otimes_{O_Y} M)$ is a sheaf of module on $T \times_Y \Delta^n_Y(Y^2)$, one has

$$\begin{aligned} K^{[q]}L(M)_{(U,T)} &= \lim_{n \geq 0} K^q_{n,(U,T)} \cdot E_n \\ &\overset{\text{(a)}}{=} \lim_n \ker\left(E_n \to P^{q-1}_U(T \times_Y \Delta^n_Y(Y^2)) \otimes_{O_{T \times_Y \Delta^n_Y(Y^2)}} E_n\right) \\ &= \lim_n \ker\left(h^*(P^n_{Y/S} \otimes_{O_Y} M) \to P^{q-1}_U(T \times_Y \Delta^n_Y(Y^2)) \otimes_{O_Y} M\right), \end{aligned} \tag{57}$$

where (a) uses Remark 10.3.

Remark 10.3. For a closed immersion $i : X \to Y$ of schemes, let I be the ideal sheaf. Let $q > 0$ be an integer. Let $i : \Delta^{q-1}_X(Y) \to Y$ be the inclusion. For every O_Y-module M, $I^q \cdot M$ is the kernel of the natural morphism $M \to M \otimes_{O_Y} P^{q-1}_X(Y) = i_* i^* M$.

Let $i : X \to Y$ be an immersion of schemes over S. Let J_i be the ideal sheaf of the closed immersion $X \to \Delta^i_X(Y)$. Let J be the kernel of the morphism $P_X(Y) \to O_X$ induced by the inclusion $X \to \Delta_X(Y)$, which is an ideal of $P_X(Y)$, and an abelian sheaf on X.

Lemma 10.4. (a) *Let E be an O_Y-module. For every integer $q \geq 0$, let*

$$J^{[q]}E := \lim_i (J_i^q \cdot (P_X^i(Y) \otimes_{O_Y} E)),$$

which is a $P_X(Y)$-submodule of $\lim_i (P_X^i(Y) \otimes_{O_Y} E)$. Then there is a natural complex of abelian sheaves on X

$$J^{[q]}E \to \mathrm{CA}_Y^\bullet(K^{[q]}L(E))$$

that is homotopic to zero. In particular, for $q = 0$, there is a natural resolution

$$\lim_i (P_X^i(Y) \otimes_{O_Y} E) \to \mathrm{CA}_Y^\bullet(L(E)). \tag{58}$$

(b) *Let $u : E \to F$ be a differential operator of O_Y-modules. Let*

$$v : \lim_i (P_X^i(Y) \otimes E) \to \lim_i (P_X^i(Y) \otimes F)$$

be the morphism induced by u as in (32)*. Then under the resolution* (58)*, v is compatible with $\mathrm{CA}_Y^\bullet(L(u))$.*

Proof. (a) First, we prove the case with $q = 0$. Define a cosimplicial object of $\mathrm{Ab}(X)$ as follows. For every $v \geq 0$, let

$$\mathcal{E}^v = \lim_{i \geq 0} P_X^i(Y^{v+1}) \otimes_{O_Y} E.$$

For $v > 0$ and $0 \leq j \leq v$, the projection $Y^{v+1} \to Y^v$ over S skipping the j-th factor induces a system of morphisms $\{P_X^i(Y^v) \to P_X^i(Y^{v+1})\}_{i \geq 0}$. Tensoring product with id_E and passing to limits, the system induces a morphism $\delta_j^v : \mathcal{E}^{v-1} \to \mathcal{E}^v$. The closed immersion

$$Y^{v+1} \to Y^{v+2}, \quad (y_0, \ldots, y_v) \mapsto (y_0, \ldots, y_{j-1}, y_j, y_j, y_{j+1}, \ldots, y_v)$$

induces a system of morphisms $\{P_X^i(Y^{v+2}) \to P_X^i(Y^{v+1})\}_{i \geq 2}$. They induce a morphism $\sigma_j^v : \mathcal{E}^{v+1} \to \mathcal{E}^v$. From [Sta25, Tag 016K], one obtains $\mathcal{E} \in \mathrm{CoSimp}(\mathrm{Ab}(X))$.

For every $i \geq 0$, let $h : \Delta_X^i(Y^{v+1}) \to Y$ be the projection to the last factor, which fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, equal] & \Delta_X^i(Y^{v+1}) \arrow[d, "h"] \\ X \arrow[r] & Y. \end{tikzcd}$$

By (35), one has

$$L(E)_{(X, \Delta_X^i(Y^{v+1}))} = \lim_{n \geq 0} h^*(P_{Y/S}^n \otimes_{O_Y} E) = \lim_{n \geq 0} P_X^i(Y^{v+1}) \otimes_{O_Y} P_{Y/S}^n \otimes_{O_Y} E.$$

Then one has

$$\begin{aligned}\mathrm{CA}^v_Y(L(E)) &:= \lim_{i\geq 0}\lim_{n\geq 0} P^i_X(Y^{v+1})\otimes P^n_{Y/S}\otimes_{O_Y} E\\ &\overset{\text{(a)}}{=} \lim_{i\geq 0} P^i_X(Y^{v+2})\otimes_{O_Y} E = \mathcal{E}^{v+1},\end{aligned}\tag{59}$$

where (a) uses (62).

For every integer v, let $\mathcal{F}^v$ be $\mathcal{E}^{v+1}$ when $v \geq -1$, and be 0 when $v < -1$. Define the differential morphism $d^v : \mathcal{F}^v \to \mathcal{F}^{v+1}$ as the alternating sum of degeneracy maps $\sum_{k=0}^{v+1}(-1)^k\delta^{v+2}_k$. Then $(\mathcal{F}^\bullet, d^\bullet)$ is a complex. By construction, the truncation $(\mathrm{CA}^v_Y(L(E)), d^v)_{v\geq 0}$ of $\mathcal{F}^\bullet$ is exactly the cochain complex associated with the cosimplicial object $\mathrm{CA}^\bullet_Y(L(E)) \in \mathrm{CoSimp}(\mathrm{Ab}(X))$. By [Ber74, V, Lemme 2.2.1], the complex $\mathcal{F}^\bullet$ is homotopy equivalent to zero, and a homotopy is given by

$$h^v_E = (-1)^{v+1}\sigma^{v+1}_{v+1} : \mathcal{F}^{v+1} \to \mathcal{F}^v.$$

Therefore, $J^{[0]}E \to \mathrm{CA}^\bullet_Y(L(E))$ is a resolution.

Now assume $q > 0$. We compute an expression of $K^{[q]}L(E)$. For any $i, v \geq 0$, the pair $(X, \Delta^i_X(Y^{v+1}))$ is an object of $Y\,\mathrm{Strat}(X/S)$. Therefore, by (57) one has

$$\begin{aligned}&\Big(K^{[q]}L(E)\Big)_{\Delta^i_X(Y^{v+1})}\\ =&\lim_n \ker\Big(P^i_X(Y^{v+1})\otimes_{O_Y} P^n_{Y/S}\otimes_{O_Y} E \to P^{q-1}_X(\Delta^i_X(Y^{v+1})\times_Y \Delta^n_Y(Y^2))\otimes_{O_Y} E\Big).\end{aligned}$$

Then one has

$$\begin{aligned}&\mathrm{CA}^v_Y(K^{[q]}L(E)) = \lim_i \Big(K^{[q]}L(E)\Big)_{\Delta^i_X(Y^{v+1})}\\ =&\lim_{i,n}\ker\Big(P^i_X(Y^{v+1})\otimes_{O_Y} P^n_{Y/S}\otimes_{O_Y} E \to P^{q-1}_X(\Delta^i_X(Y^{v+1})\times_Y \Delta^n_Y(Y^2))\otimes_{O_Y} E\Big)\\ =&\ker\Big(\lim_{i,n} P^i_X(Y^{v+1})\otimes_{O_Y} P^n_{Y/S}\otimes_{O_Y} E \to \lim_{i,n} P^{q-1}_X(\Delta^i_X(Y^{v+1})\times_Y \Delta^n_Y(Y^2))\otimes_{O_Y} E\Big)\\ \overset{\text{(a)}}{=}&\ker\Big(\lim_i P^i_X(Y^{v+2})\otimes_{O_Y} E \to \lim_i P^{q-1}_X(\Delta^i_X(Y^{v+2}))\otimes_{O_Y} E\Big)\\ =&\lim_i\ker\Big(P^i_X(Y^{v+2})\otimes_{O_Y} E \to P^{q-1}_X(\Delta^i_X(Y^{v+2}))\otimes_{O_Y} E\Big).\end{aligned}\tag{60}$$

where (a) uses (63).

Similarly, one has

$$J^{[q]}E := \lim_i (J^q_i\cdot(P^i_X(Y)\otimes_{O_Y} E)) = \lim_i \ker\Big(P^i_X(Y)\otimes_{O_Y} E \to P^{q-1}_X(\Delta^i_X(Y))\otimes_{O_Y} E\Big).\tag{61}$$

Therefore, $d^{-1} : \mathcal{E}^0 \to \mathrm{CA}^0_Y(L(E))$ restricts to a morphism $J^{[q]}E \to \mathrm{CA}^0_Y(K^{[q]}L(E))$. Hence, $J^{[q]}E \to \mathrm{CA}^\bullet_Y(K^{[q]}L(E))$ is a subcomplex of $\mathcal{F}^\bullet$. We check that the homotopy $(h^v)_{v\in\mathbb{Z}}$ of $\mathcal{F}^\bullet$ restricts to a homotopy of this subcomplex. The closed immersion $Y^{v+2} \to Y^{v+3}$ defining σ^{v+1}_{v+1} induces a commutative diagram

$$\begin{tikzcd}
\Delta^{q-1}_X(\Delta^i_X(Y^{v+2})) \arrow[r, hook] \arrow[d] & \Delta^i_X(Y^{v+2}) \arrow[d] \\
\Delta^{q-1}_X(\Delta^i_X(Y^{v+3})) \arrow[r, hook] & \Delta^i_X(Y^{v+3})
\end{tikzcd}$$

of schemes over Y, and hence a commutative diagram

$$\begin{tikzcd}
P^i_X(Y^{v+3}) \arrow[r] \arrow[d, "h^v_{O_Y}"] & P^{q-1}_X(\Delta^i_X(Y^{v+3})) \arrow[d] \\
P^i_X(Y^{v+2}) \arrow[r] & P^{q-1}_X(\Delta^i_X(Y^{v+2}))
\end{tikzcd}$$

of O_Y-algebras. Therefore,

$$P^i_X(Y^{v+3}) \otimes_{O_Y} E \to P^i_X(Y^{v+2}) \otimes_{O_Y} E$$

restricts to a morphism

$$\begin{aligned}
&\ker(P^i_X(Y^{v+3}) \otimes_{O_Y} E \to P^{q-1}_X(\Delta^i_X(Y^{v+3})) \otimes_{O_Y} E) \\
\to &\ker(P^i_X(Y^{v+2}) \otimes_{O_Y} E \to P^{q-1}_X(\Delta^i_X(Y^{v+2})) \otimes_{O_Y} E).
\end{aligned}$$

From (60) and (61), passing to limits, $h^v_E : \mathcal{F}^{v+1} \to \mathcal{F}^v$ restricts to a morphism $\mathrm{CA}^{v+1}_Y(K^{[q]}L(E)) \to \mathrm{CA}^v_Y(K^{[q]}L(E))$ (when $v \geq 0$) and $\mathrm{CA}^0_Y(K^{[q]}L(E)) \to J^{[q]}E$ (when $v = -1$).

(b) The proof is similar to that of [Ber74, V, Proposition 2.2.2 ii)].

□

Lemma 10.5. *Let $X \to Y$ be an immersion of schemes over S. Let E be an O_Y-module. Then for any integers $k, k' \geq 0$, there is a canonical isomorphism*

$$\lim_{i,j\geq 0} \left(P^i_X(Y^{k+1}) \otimes_{O_Y} P^j_Y(Y^{k'+1}) \otimes_{O_Y} E\right) \to \lim_n (P^n_X(Y^{k+k'+1}) \otimes_{O_Y} E) \quad (62)$$

of O_Y-modules. For every integer $q \geq 0$, there is a canonical isomorphism

$$\lim_{i,j\geq 0} P^q_X(\Delta^i_X(Y^{k+1}) \times_Y \Delta^j_Y(Y^{k'+1})) \otimes_{O_Y} E \to \lim_n P^q_X(\Delta^n_X(Y^{k+k'+1})) \otimes_{O_Y} E \quad (63)$$

of O_Y-modules.

Proof. Taking cofinal inverse subsystems does not change limits. For every integer $n \geq 0$, the projection $Y_S^{k+k'+1} \to Y_S^{k+1}$ to the first $k+1$ factors induces a morphism $\Delta_X^n(Y^{k+k'+1}) \to \Delta_X^n(Y^{k+1})$. Similarly, the projection $Y_S^{k+k'+1} \to Y_S^{k'+1}$ to the last $k'+1$ factors induces a morphism $\Delta_X^n(Y^{k+k'+1}) \to \Delta_Y^n(Y^{k'+1})$. Thus, one gets a morphism

$$\Delta_X^n(Y^{k+k'+1}) \to \Delta_X^n(Y^{k+1}) \times_Y \Delta_Y^n(Y^{k'+1})$$

of schemes over Y. It restricts to a morphism

$$\Delta_X^q(\Delta_X^n(Y^{k+k'+1})) \to \Delta_X^q\left(\Delta_X^n(Y^{k+1}) \times_Y \Delta_Y^n(Y^{k'+1})\right).$$

They induce morphisms

$$\begin{gathered} P_X^n(Y^{k+1}) \otimes_{O_Y} P_Y^n(Y^{k'+1}) \to P_X^n(Y^{k+k'+1}), \\ P_X^q\left(\Delta_X^n(Y^{k+1}) \times_Y \Delta_Y^n(Y^{k'+1})\right) \to P_X^q(\Delta_X^n(Y^{k+k'+1})) \end{gathered}$$

of O_Y-algebras. Thus, one has morphisms

$$\lim_{n\geq 0}\left(P_X^n(Y^{k+1}) \times_{O_Y} P_Y^n(Y^{k'+1}) \otimes_{O_Y} E\right) \to \lim_{n \geq 0}\left(P_X^n(Y^{k+k'+1}) \otimes_{O_Y} E\right), \tag{64}$$

$$\lim_n P_X^q\left(\Delta_X^n(Y^{k+1}) \times_Y \Delta_Y^n(Y^{k'+1})\right) \otimes_{O_Y} E \to \lim_n\left(P_X^q(\Delta_X^n(Y^{k+k'+1})) \otimes_{O_Y} E\right) \tag{65}$$

of O_Y-modules.

Conversely, for any integers $i, j \geq 0$, as $Y \to \Delta_Y^j(Y^{k'+1})$ is a thickening of order $\leq j$, so is its base change $\Delta_X^i(Y^{k+1}) \to \Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1})$. As $X \to \Delta_X^i(Y^{k+1})$ is a thickening of order $\leq i$, the composition

$$X \to \Delta_X^i(Y^{k+1}) \to \Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1})$$

is a thickening of order $\leq i+j$. The inclusions $\Delta_X^i(Y^{k+1}) \to Y_S^{k+1}$ and $\Delta_Y^j(Y^{k'+1}) \to Y_S^{k'+1}$ induce a morphism

$$\Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1}) \to Y_S^{k+k'+1}$$

fitting into a solid commutative diagram

$$\begin{tikzcd}
X \arrow[rr, hook] \arrow[d, equal] & & \Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1}) \arrow[d] \arrow[dl, dashed] \\
X \arrow[r, hook] & \Delta_X^{i+j}(Y^{k+k'+1}) \arrow[r, hook] & Y_S^{k+k'+1}.
\end{tikzcd}$$

The diagram induces a morphism

$$\Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1}) \to \Delta_X^{i+j}(Y^{k+k'+1})$$

of schemes. It restricts to a morphism

$$\Delta_X^q\left(\Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1})\right) \to \Delta_X^q(\Delta_X^{i+j}(Y^{k+k'+1})).$$

They correspond to morphisms

$$\begin{gathered} P_X^{i+j}(Y^{k+k'+1}) \to P_X^i(Y^{k+1}) \otimes_{O_Y} P_Y^j(Y^{k'+1}), \\ P_X^q(\Delta_X^{i+j}(Y^{k+k'+1})) \to P_X^q\left(\Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1})\right). \end{gathered}$$

of O_Y-algebras. Thus, one has morphisms

$$\lim_{i,j\geq 0}\left(P_X^{i+j}(Y^{k+k'+1}) \otimes_{O_Y} E\right) \to \lim_{i,j\geq 0}\left(P_X^i(Y^{k+1}) \otimes_{O_Y} P_Y^j(Y^{k'+1}) \otimes_{O_Y} E\right), \tag{66}$$

$$\lim_{i,j}\left(P_X^q(\Delta_X^{i+j}(Y^{k+k'+1})) \otimes_{O_Y} E\right) \to \lim_{i,j}\left(P_X^q\left(\Delta_X^i(Y^{k+1}) \times_Y \Delta_Y^j(Y^{k'+1})\right) \otimes_{O_Y} E\right). \tag{67}$$

By universal properties, (64) and (66) are inverse to each other, and so are (65) and (67). ☐

11 Comparison of infinitesimal cohomology and de Rham cohomology

Let $X \to Y$ be an immersion of schemes over a scheme S. Let E be an O_Y-module with a stratification relative to S. We prove a comparison isomorphism between the de Rham cohomology with coefficients in E and the infinitesimal cohomology of the corresponding crystal $\mathcal{E}$. In particular, for $E = O_Y$, the result reduces to a a comparison isomorphism of $H^i_{\mathrm{inf}}(X/S)$ and Hartshorne's de Rham cohomology $H^i_{\mathrm{DR}}(X/S)$.

The stratification of E together with the stratification of the Artin-Rees system $\{P_X^i(Y)\}_{i\geq 0}$ (Remark 7.5) induces a complex

$$P_X^n(Y)\otimes_{O_Y} E \to P_X^{n-1}(Y)\otimes_{O_Y} E\otimes_{O_Y}\Omega^1_{Y/S} \to P_X^{n-2}(Y)\otimes_{O_Y} E\otimes_{O_Y}\Omega^2_{Y/S} \to \dots \tag{68}$$

for every $n \geq 0$. They fit to an object $(P_X^\bullet(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^\bullet_{Y/S})_{n\geq 0}$ of $\mathrm{Ch}^{\geq 0}(\mathrm{Mod}(X, O_S)^{\mathbb{N}})$. Let

$$E\hat{\otimes}\Omega_D^\bullet \in \mathrm{Ch}^{\geq 0}(\mathrm{Mod}(X, O_S))$$

be the termwise limit complex, whose k-th term is

$$E\hat{\otimes}\Omega_D^k := \lim_n\left(P_X^n(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^k_{Y/S}\right).$$

Remark 11.1. In Lemma 10.4, assume that Y is locally Noetherian, $X \to Y$ is a closed immersion, and E is a coherent O_Y-module. Then by [GD71, Proposition 10.8.8 (ii)], the canonical morphism $P_X(Y) \otimes_{O_Y} E \to J^{[0]}E$ is an isomorphism. Suppose further that $Y \to S$ is locally of finite type. Then from [EGA IV 4, Proposition 16.3.9], $\Omega^1_{Y/S}$ is a coherent O_Y-module. Therefore, the morphism $P_X(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^k_{Y/S} \to E\hat{\otimes}\Omega^k_D$ is also an isomorphism.

Let $f : X \to S$ be a finite type morphism of Noetherian $\mathbb{Q}$-schemes. Let $X \hookrightarrow Y$ be a closed immersion with Y smooth over S. Let E be O_Y with the natural stratification. Then $E\hat{\otimes}\Omega^\bullet_D$ coincides with $\Omega^H_{X/S} \in D(X, O_S)$ introduced in [Bha12, Construction 4.25].

From (29), for any integers $q > k \geq 0$, the differential

$$P^{n-k}_X(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^k_{Y/S} \to P^{n-k-1}_X(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^{k+1}_{Y/S}$$

in the complex (68) restricts to a morphism

$$J^{q-k}_{n-k} \cdot (P^{n-k}_X(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^k_{Y/S}) \to J^{q-k-1}_{n-k-1} \cdot (P^{n-k-1}_X(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^{k+1}_{Y/S}).$$

Thus, one gets a subcomplex

$$F^q(E\hat{\otimes}\Omega^\bullet_D) \subset E\hat{\otimes}\Omega^\bullet_D,$$

whose k-th term is

$$J^{[q-k]}(E \otimes_{O_Y} \Omega^k_{Y/S}) := \lim_i J^{q-k}_i \cdot (P^i_X(Y) \otimes_{O_Y} E \otimes_{O_Y} \Omega^k_{Y/S}). \tag{69}$$

Theorem 1.4 follows from Remark 11.1 and Theorem 11.2 (b) (where we take $q = 0$ and E to be O_Y with its natural stratification).

Theorem 11.2. *Let $i : X \to Y$ be an immersion over S. Let E be a* quasi-coherent *O_Y-module with a stratification relative to S. Let $\mathcal{E}$ be the crystal in $O_{X/S}$-modules induced by E on $Y\,\mathrm{Strat}(X/S)$ via* (21)*.*

(a) *Then for every $q \geq 0$, there is a canonical morphism*

$$Ru'_{X/S*}(J^q_{X/S} \cdot \mathcal{E}) \to F^q(E\hat{\otimes}\Omega^\bullet_D) \tag{70}$$

in $D^+(X, O_S)$. It induces a morphism

$$R\Gamma((X/S)_{Y\,\mathrm{Strat}}, J^q_{X/S} \cdot \mathcal{E}) \to R\Gamma(Y, F^q(E\hat{\otimes}\Omega^\bullet_D))$$

in $D^+(O_S(S))$.

(b) *Assume further that X is of characteristic* 0*, and $Y \to S$ is smooth. Then the morphisms in Part* (a) *are isomorphisms, and one can write $Ru_{X/S}$ for $Ru'_{X/S*}$, and $R\Gamma((X/S)_{\mathrm{inf}}, -)$ for $R\Gamma((X/S)_{Y\,\mathrm{Strat}}, -)$.*

Proof. (a) From [Ber74, IV, Proposition 1.1.3], as E is quasi-coherent over O_Y, $\mathcal{E}$ is a quasi-coherent $O_{X/S}$-module. Then by Lemma 11.3, for every $v \geq 0$, the inverse system $((J^q_{X/S} \cdot \mathcal{E})_{(X,\Delta^i_X(Y^{v+1}))})_{i\geq 0}$ satisfies Condition (*). By Fact 4.8, it has vanishing $R^q \lim_i$ for all $q > 0$. Hence by Lemma 4.6 and Remark 2.3, there is a canonical isomorphism

$$Ru'_{X/S*}(J^q_{X/S} \cdot \mathcal{E}) \cong \mathrm{CA}^\bullet_Y(J^q_{X/S} \cdot \mathcal{E}) \tag{71}$$

in $D^+(X, u'_{X/S*}O_{X/S})$. For every $v \geq 0$, the canonical morphism

$$\mathrm{CA}^v_Y(J^q_{X/S} \cdot \mathcal{E}) \xrightarrow{\sim} R\lim_i (J^q_{X/S} \cdot \mathcal{E})_{\Delta^i_X(Y^{v+1})} \tag{72}$$

is an isomorphism.

For every $i \geq 0$, let $h_i : \Delta^i_X(Y^{v+1}) \to Y$ be the last projection. Let $\mathscr{K}^\bullet_{v,i} \in \mathrm{Ch}^{\geq 0}(P^i_X(Y^{v+1}))$ be the complex

$$K^q_{i,\Delta^i_X(Y^{v+1})} \cdot h_i^*(P^i_{Y/S} \otimes_{O_Y} E) \to K^{q-1}_{i-1,\Delta^i_X(Y^{v+1})} \cdot h_i^*(P^{i-1}_{Y/S} \otimes_{O_Y} E \otimes_{O_Y} \Omega^1_{Y/S}) \to \dots$$

The differential morphisms are constructed as follows. By (36), the stratification on E turns the differential operator $d : \Omega^k_{Y/S} \to \Omega^{k+1}_{Y/S}$ to another $E \otimes_{O_Y} \Omega^k_{Y/S} \to E \otimes_{O_Y} \Omega^{k+1}_{Y/S}$. From (16), one gets a morphism

$$P^{i-k}_{Y/S} \otimes_{O_Y} E \otimes_{O_Y} \Omega^k_{Y/S} \to P^{i-k-1}_{Y/S} \otimes_{O_Y} E \otimes_{O_Y} \Omega^{k+1}_{Y/S}.$$

From (56), there is a canonical morphism

$$(J^q_{X/S} \cdot \mathcal{E})_{\Delta^i_X(Y^{v+1})} \to \mathscr{K}^\bullet_{v,i} \tag{73}$$

in $\mathrm{Ch}^{\geq 0}(P^i_X(Y^{v+1}))$. For variable $i \geq 0$, the morphisms fit to a morphism of inverse systems in $\mathrm{Ch}^{\geq 0}(\mathrm{Ab}(X))$. Therefore, there is a canonical morphism

$$R\lim_i (J^q_{X/S} \cdot \mathcal{E})_{\Delta^i_X(Y^{v+1})} \to R\lim_i \mathscr{K}^\bullet_{v,i} \tag{74}$$

in $D^+(X, P_X(Y^{v+1}))$. By Lemma 11.4, for every $k \geq 0$, the inverse system $(\mathscr{K}^k_{v,i})_{i\geq 0}$ satisfies Condition (*), so it is right acyclic for the functor lim. Then by Leray's acyclicity lemma (see, e.g., [Sta25, Tag 015E]), the termwise limit complex $\lim_i \mathscr{K}^\bullet_{v,i}$ represents $R\lim_i \mathscr{K}^\bullet_{v,i}$.

For every $k \geq 0$, one has

$$\begin{aligned}
\lim_i \mathscr{K}^k_{v,i} &= \lim_i K^{q-k}_{i-k,\Delta^i_X(Y^{v+1})} \cdot h_i^*(P^{i-k}_{Y/S} \otimes_{O_Y} E \otimes \Omega^k_{Y/S}) \\
&= \lim_{i,n} K^{q-k}_{n-k,\Delta^i_X(Y^{v+1})} \cdot h_i^*(P^{n-k}_{Y/S} \otimes_{O_Y} E \otimes \Omega^k_{Y/S}) \\
&=: \mathrm{CA}^v_Y(K^{[q-k]}L(E \otimes_{O_Y} \Omega^k_{Y/S})).
\end{aligned} \tag{75}$$

Let $\lim_i \mathscr{K}^\bullet_{*,i}$ be the cosimplicial object

$$\Delta \to \mathrm{Ch}(\mathrm{Ab}(X)), \quad v \mapsto \lim_i \mathscr{K}^\bullet_{v,i}.$$

Together with Lemma 10.4, (75) implies that there is a quasi-isomorphism

$$F^q(E \hat{\otimes} \Omega^\bullet_D) \xrightarrow{\sim} \lim_i \mathscr{K}^\bullet_{*,i} \tag{76}$$

of complexes. As the diagram

$$\begin{tikzcd}
Ru'_{X/S*}(J^q_{X/S} \cdot \mathcal{E}) \arrow[r, dash, "\cong"] \arrow[d, dashed] & \mathrm{CA}^\bullet_Y(J^q_{X/S} \cdot \mathcal{E}) \arrow[r, "\cong"] & R\lim_i (J^q_{X/S} \cdot \mathcal{E})_{\Delta^i_X(Y^{v+1})} \arrow[d, "(74)"] \\
F^q(E \hat{\otimes} \Omega^\bullet_D) \arrow[r, "\cong"] & \lim_i \mathscr{K}^\bullet_{v,i} \arrow[r, "\cong"] & R\lim_i \mathscr{K}^\bullet_{v,i}
\end{tikzcd}$$

shows, combining (71), (72), (74) and (76), one gets a morphism (70).

(b) By Remark 8.2, as Y is quasi-smooth over S, $\mathcal{E}$ is defined on $\mathrm{Inf}(X/S)$. By the filtered Poincaré lemma, as Y is smooth over S and X is of characteristic 0, for the object $(X, \Delta^i_X(Y^{v+1})) \in \mathrm{Inf}(X/S)$ the complex (48) is locally homotopic to zero. Therefore, (73) is a resolution. Then (74) is an isomorphism.

□

Lemma 11.3. *Let $X \to Y$ be a morphism over S. Let $\mathcal{E}$ be a quasi-coherent $O_{X/S}$-module on $Y\,\mathrm{Strat}(X/S)$. Then*

(a) *For any $i > 0$, $q \geq 0$, every $(U, T) \in Y\,\mathrm{Strat}(X/S)$ and every affine open subset V of T, one has $H^i(V, (J^q_{X/S} \cdot \mathcal{E})_T) = 0$.*

(b) *Let $u : (U, T) \to (U, T')$ be a morphism in $Y\,\mathrm{Strat}(X/S)$ such that T' is affine and $u : T \to T'$ is a closed immersion. Then for every $q \geq 0$, $(J^q_{X/S} \cdot \mathcal{E})(T') \to (J^q_{X/S} \cdot \mathcal{E})(T)$ is surjective.*

Proof. (a) As $\mathcal{E}$ is quasi-coherent, $\mathcal{E}_T$ is a quasi-coherent O_T-module. By [Ber74, III, Proposition 1.1.5], one has $(J^q_{X/S} \cdot \mathcal{E})_T = (J_{X/S,T})^q \cdot \mathcal{E}_T$, so it is a quasi-coherent O_T-module. The result follows from [Sta25, Tag 01XB].

(b) From [Ber74, IV, Proposition 1.1.3], as $\mathcal{E}$ is quasi-coherent, it is a crystal. Whence, the morphism $u^*\mathcal{E}_{T'} \to \mathcal{E}_T$ of O_T-modules is an isomorphism. As T' is affine, the map $\mathcal{E}(T') \otimes_{O(T')} O(T) \to \mathcal{E}(T)$ is an isomorphism. Since $u : T \to T'$ is a closed immersion, $O(T') \to O(T)$ is surjective. Thus, $\mathcal{E}(T') \to \mathcal{E}(T)$ is surjective. By the four lemma, the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & J_{X/S}(T') \arrow[r] \arrow[d] & O(T') \arrow[r] \arrow[d, two heads] & O(U) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & J_{X/S}(T) \arrow[r] & O(T) \arrow[r] & O(U) \arrow[r] & 0
\end{tikzcd}$$

with exact rows shows that $J_{X/S}(T') \to J_{X/S}(T)$ is surjective. As T' is affine, one has

$$(J^q_{X/S} \cdot \mathcal{E})(T') = \big(J_{X/S}(T')\big)^q \cdot \mathcal{E}(T'), \quad (J^q_{X/S} \cdot \mathcal{E})(T) = \big(J_{X/S}(T)\big)^q \cdot \mathcal{E}(T).$$

The surjectivity follows.

□

Lemma 11.4. *Let $X \to Y$ be an immersion over S. Let E be a quasi-coherent O_Y-module. Fix integers $k, v \geq 0$. For every $i \geq k$, let $K_i := K_{i-k,(X,\Delta^i_X(Y^{v+1}))}$ be the ideal sheaf of the closed immersion $X \hookrightarrow \Delta^i_X(Y^{v+1}) \times_Y \Delta^{i-k}_Y(Y^2)$. Then for every $q \geq 0$, the inverse system*

$$\{K^q_i \cdot (P^i_X(Y^{v+1}) \otimes_{O_Y} P^{i-k}_{Y/S} \otimes_{O_Y} E)\}_{i \geq k}$$

in $\mathrm{Ab}(X)$ *satisfies Condition (*).*

Proof. Write $(F_i)_{i \geq k}$ for this system. As E is quasi-coherent, every F_i is a quasi-coherent sheaf on $\Delta^i_X(Y^{v+1}) \times_Y \Delta^{i-k}_Y(Y^2)$. For every affine open subset U of X, the open subscheme of $\Delta^i_X(Y^{v+1}) \times_Y \Delta^{i-k}_Y(Y^2)$ with underlying set U is a thickening of the scheme U. Then by [Sta25, Tag 06AD], it is also affine. Therefore, by [Sta25, Tag 01XB], for every $j > 0$, one has $H^j(U, F_i) = 0$.

Assume further that there is an affine open subset V of Y containing the image of U. We prove that $\Gamma(U, F_{i+1}) \to \Gamma(U, F_i)$ is surjective. As U and V are affine, one has

$$\Gamma(U, K_i) = \ker\Big(\Gamma(U, P^i_X(Y^{v+1})) \otimes_{O_Y(V)} \Gamma(V, P^{i-k}_{Y/S}) \to O_X(U)\Big),$$

$$\Gamma(U, F_i) = \Gamma(U, K_i)^q \cdot \Big(\Gamma(U, P^i_X(Y^{v+1})) \otimes_{O_Y(V)} \Gamma(V, P^{i-k}_{Y/S}) \otimes_{O_Y(V)} \Gamma(V, E)\Big).$$

Since both

$$\Gamma(U, P^{i+1}_X(Y^{v+1})) \to \Gamma(U, P^i_X(Y^{v+1})), \quad \Gamma(V, P^{i-k+1}_{Y/S}) \to \Gamma(V, P^{i-k}_{Y/S})$$

are surjective, so is $\Gamma(U, K_{i+1}) \to \Gamma(U, K_i)$. The surjectivity of $\Gamma(U, F_{i+1}) \to \Gamma(U, F_i)$ follows. □

Remark 11.5. We discuss the comparison between the infinitesimal cohomology and Hartshorne's algebraic de Rham cohomology, from which the finiteness and Künneth formula for infinitesimal cohomology follows. Let k be a field of characteristic 0. Let X be a scheme of finite type over k, which is embeddable, i.e., there is a closed immersion $X \to Y$ over k with Y smooth over k. Hartshorne

[Har75, p.24] defines $H^q_{\mathrm{DR}}(X) := H^q(X, \Omega^\bullet_D)$. From Theorem 11.2 (b), there is a canonical isomorphism $Ru_{X/k*}O_{X/k} \cong \Omega^\bullet_D$. So up to isomorphism in $D^+(X, k)$, $\Omega^\bullet_D := \hat{\Omega}^\bullet_{Y/k}$ depends only on X and is independent of the choice of the embedding $X \hookrightarrow Y$. This gives a canonical isomorphism

$$H^q_{\mathrm{inf}}(X/k) \cong H^q_{\mathrm{DR}}(X).$$

Thus, one recovers [Har75, II, Theorem 1.4].

Hartshorne [Har75, Remark, p.28] sketches how to define $H^q_{\mathrm{DR}}(Z)$ for a (possibly non-embeddable) scheme Z of finite type over k, which is still canonically isomorphic to $H^q_{\mathrm{inf}}(Z/k)$. One can extend the finiteness result [Har75, II, Theorem 6.1] to $H^q_{\mathrm{DR}}(Z)$, so that $H^q_{\mathrm{inf}}(Z/k)$ is a finite dimensional k-vector space.

Based on Geisser's result, Huber and Jörder [HJ14, Remark 7.5] interpret $H^q_{\mathrm{DR}}(Z)$ in the h-topology when Z is moreover separated. From [HJ14, Proposition 7.29], for two schemes Z and Z' separated of finite type over k, the Künneth formula

$$H^n_{\mathrm{inf}}(Z \times_k Z'/k) = \oplus_{a+b=n} H^q_{\mathrm{inf}}(Z/k) \otimes_k H^b_{\mathrm{inf}}(Z'/k)$$

holds.

Remark 11.6. In Theorem 11.2 (b), assume further that $f : X \to S$ is a finite type morphism of Noetherian schemes, and that $i : X \to Y$ is a closed immersion. Let E be O_Y with the natural stratification. Then $\Omega^\bullet_D$ is the formal completion $\hat{\Omega}^\bullet_{Y/S}$ of the de Rham complex $\Omega^\bullet_{Y/S}$ along X. For $q \geq 0$, Hartshorne [Har75, p.74] defines the q-th *relative algebraic de Rham cohomology* of X over S as $R^q_{\mathrm{DR}} f_*(X) := R^q f_* \Omega^\bullet_D$, which is an O_S-module. In particular, there is a canonical isomorphism

$$R^q f_{X/S*} O_{X/S} \xrightarrow{\sim} R^q_{\mathrm{DR}} f_*(X).$$

They are the sheafification of the presheaf $U \mapsto H^q_{\mathrm{inf}}(X \times_S U/U)$ on S.

Remark 11.7. Let k be a field of characteristic 0. Let $f : X \to S$ be a morphism of finite type of reduced schemes over k. Then by Remark 11.6 and [Har75, III, Theorem 5.1], there is an open dense subset U of S, such that for every integer $i \geq 0$, $R^i f_{X/S*} O_{X/S}$ is finite locally free on U. For a flat base change theorem, see [Har75, III, Proposition 5.2]. Combined with Remark 11.6, it implies the following. If further S is irreducible with generic point η, then for every $q \geq 0$, the natural morphism

$$(R^q f_{X/S*} O_{X/S}) \otimes_{O_S} k(S) \to H^q_{\mathrm{inf}}(X_\eta/k(S))$$

is an isomorphism. In particular, the natural morphism $\mathrm{colim}_V\, H^q_{\mathrm{inf}}(X \times_S V/V) \to H^q_{\mathrm{inf}}(X_\eta/k(S))$ is an isomorphism, where V runs through nonempty open subsets of S.

Remark 11.8. Hartshorne [Har75, p.75] extends the Gauss-Manin connection to a singular morphism, which is assumed to be the factorization of a closed immersion followed by a smooth morphism. Using Remark 11.6, we rewrite his construction in terms of infinitesimal cohomology to allow the non-embeddable case. Let S be a Noetherian scheme of characteristic 0. Let X, Y be schemes of finite type over S. Let $f : X \to Y$ be a morphism over S. Then there is a spectral sequence

$$E_1^{pq} = R^q f_*(f^*\Omega^p_{Y/S} \otimes_{O_Y} Ru_{X/Y*}O_{X/Y}) \Rightarrow R^{p+q} f_* Ru_{X/S*}O_{X/S}.$$

Suppose further that the structure morphism $g : Y \to S$ is smooth. Then the canonical morphism $\Omega^p_{Y/S} \otimes_{O_Y} R^q f_{X/Y*}O_{X/Y} \xrightarrow{\sim} E_1^{pq}$ is an isomorphism for any $p, q \geq 0$. The differential

$$d_1^{0q} : R^q f_{X/Y*}O_{X/Y} \to R^q f_{X/Y*}O_{X/Y} \otimes_{O_Y} \Omega^1_{Y/S}$$

is an integrable connection on the O_Y-module $R^q f_{X/Y*}O_{X/Y}$ relative to S.

Take $S = \operatorname{Spec}\mathbb{C}$ and assume that Y is smooth. Assume that $f : X \to Y$ is either smooth or proper. Then by [Del70, II, Théorème 6.13] (in smooth case) [Har75, IV, Corollary 4.3] (in proper case), up to shrinking Y to a dense open subset, the following holds. For every $q \geq 0$, $R^q f_*^{\mathrm{an}}\mathbb{C}$ is a local system on Y^{an}, and $R^q f_{X/Y*}O_{X/Y}$ is finite locally free over O_Y. The vector bundle $R^q f_{X/Y*}O_{X/Y}$ equipped with the Gauss-Manin connection corresponds the local system $R^q f_*^{\mathrm{an}}\mathbb{C}$ via the Riemann-Hilbert correspondence.

Corollary 11.9. *Let $X \to S$ be a morphism of schemes. Let $S_0 \to S$ be a nilpotent thickening. Let $X_0 = X \times_S S_0$. Then there is a canonical morphism*

$$Ru_{X_0/S*}O_{X_0/S*} \to \Omega^\bullet_{X/S} \tag{77}$$

in $D^+(X, O_S)$. Assume further that X is of characteristic 0 and $X \to S$ is smooth. Then (77) is an isomorphism, so up to quasi-isomorphism the de Rham complex $\Omega^\bullet_{X/S}$ depends only on X_0.

Proof. As $X_0 \to X$ is a nilpotent thickening, $P^i_{X_0}(X) = O_X$ when i is large enough. Then the result follows from Theorem 11.2, and the canonical morphism $Ru_{X_0/S*}O_{X_0/S} \to Ru'_{X_0/S*}O_{X_0/S}$. ☐

Definition 11.10. Let $X \to S$ be a morphism of schemes. Let M be an $O_{X/S}$-module on $\operatorname{Inf}(X/S)$. For any $q, i \geq 0$, let $F^q H^i_{\mathrm{inf}}(X/S, M)$ be the image of the map $H^i_{\mathrm{inf}}(X/S, J^q_{X/S}M) \to H^i_{\mathrm{inf}}(X/S, M)$. The decreasing filtration $F^\bullet H^i_{\mathrm{inf}}(X/S, M)$ on $H^i_{\mathrm{inf}}(X/S, M)$ is called the *infinitesimal filtration.*

We give an infinitesimal interpretation of the Hodge filtration.

Corollary 11.11. *Let $X \to S$ be a morphism of schemes. Then for every $q \geq 0$, there is a canonical morphism*

$$Ru_{X/S*}J^q_{X/S} \to \sigma_{\geq q}\Omega^\bullet_{X/S} \tag{78}$$

in $D^+(X, O_S)$*, which induces a morphism* $F^qH^*_{\mathrm{inf}}(X/S) \to F^qH^*_{\mathrm{dR}}(X/S)$*. When* $q = 0$*, (78) becomes*

$$Ru_{X/S*}O_{X/S} \to \Omega^\bullet_{X/S}$$

and induces a morphism

$$R\Gamma_{\mathrm{inf}}(X/S) \to R\Gamma_{\mathrm{dR}}(X/S). \tag{79}$$

If X *is smooth over* S *and of characteristic* 0*, then they are all isomorphisms.*

Proof. Let $X\,\mathrm{Strat}(X/S)$ be the stratifying topos for $\mathrm{id} : X \to X$ over S. Let $J'_{X/S} := j^{-1}J_{X/S}$, which is a sheaf on $X\,\mathrm{Strat}(X/S)$. Since (5) is a morphism of ringed topoi, there is a canonical morphism

$$Ru_{X/S*}J^q_{X/S} \to Ru'_{X/S*}J'^q_{X/S}.$$

From (69), one has $F^q(\Omega^\bullet_D) = \sigma_{\geq q}\Omega^\bullet_{X/S}$. Then by Theorem 11.2, there is a canonical morphism

$$Ru'_{X/S*}J'^q_{X/S} \to \sigma_{\geq q}\Omega^\bullet_{X/S}.$$

By composition, one gets (78), which is an isomorphism when X is smooth over S and of characteristic 0. □

Remark 11.12. Let Ybe a scheme of characteristic 0. Let $Y \to S$ be a smooth morphism. Let $E \in \mathrm{AR}\,\mathrm{Qch}(Y)$ be an object with a stratification relative to S. Let $\mathcal{E} \in \mathrm{Pro} - \mathrm{Cris}(X/S)$ be the Artin-Rees pro-crystal on $\mathrm{Inf}(Y/S)$ induced by E via (20). Write $\Gamma : \mathrm{AR}\,\mathrm{Mod}(O_{X/S}) \to \mathrm{AR}\,\mathrm{Mod}(O_S(S))$ for the AR-extension of the left exact functor $\Gamma : \mathrm{Mod}(O_{X/S}) \to \mathrm{Mod}(O_S(S))$. Then similar to Theorem 11.2, one can show that there is a canonical isomorphism

$$R\Gamma_{\mathrm{inf}}(Y/S, \mathcal{E}) \xrightarrow{\sim} R\Gamma(Y, E \otimes_{O_Y} \Omega^\bullet_{Y/S})$$

in $D^+(\mathrm{AR}\,\mathrm{Mod}(O_S(S)))$.

Theorem 11.2 concerns the infinitesimal cohomology of crystals induced by stratified modules. The case with a general crystal follows.

Corollary 11.13. *Let* $Y \to S$ *be a smooth morphism of schemes. Let* $i : X \to Y$ *be a closed immersion over* S*, with* X *of characteristic* 0*. Let* $\mathcal{F}$ *be a quasi-coherent* $O_{X/S}$*-module on* $\mathrm{Inf}(X/S)$*. Then there is a canonical isomorphism*

$$R\Gamma_{\mathrm{inf}}(X/S, \mathcal{F}) \xrightarrow{\sim} R\Gamma(Y, \lim_n \mathcal{F}_{\Delta^n_X(Y)} \otimes_{O_Y} \Omega^\bullet_{Y/S}).$$

Proof. As each $\Omega^k_{Y/S}$ is a finite locally free O_Y-module, $(\lim_n \mathcal{F}_{\Delta^n_X(Y)}) \otimes_{O_Y} \Omega^\bullet_{Y/S} \to \lim_n(\mathcal{F}_{\Delta^n_X(Y)} \otimes_{O_Y} \Omega^\bullet_{Y/S})$ is an isomorphism. By Lemma 7.4, $\mathcal{E} := \lambda(\mathcal{F})$ is an Artin-Rees pro-crystal on $\mathrm{Inf}(Y/S)$. By Remark 6.6, it is induced by a stratified Artin-Rees pro-module $E := \mathcal{E}_{(Y,Y)}$. From (23), $E = (\mathcal{F}_{\Delta^n_X(Y)})_{n\geq 0}$ is in $\mathrm{AR}\,\mathrm{Qch}(Y)$. By Lemma 11.3, for every $k \geq 0$, the inverse system $(\mathcal{F}_{\Delta^n_X(Y)} \otimes_{O_Y}$

$\Omega^k_{Y/S})_{n\geq 0}$ satisfies condition (*). Then by Fact 4.8, the termwise limit complex $\lim_n(E\otimes_{O_Y}\Omega^\bullet_{Y/S})$ represents

$$R\lim(E\otimes_{O_Y}\Omega^\bullet_{Y/S}).$$

One has

$$\begin{aligned} R\Gamma_{\mathrm{inf}}(X/S,\mathcal{F}) &= R\Gamma_{\mathrm{inf}}(Y/S, Ri_{\mathrm{inf}\,*}\mathcal{F}) \\ &\overset{\text{(a)}}{=} R\Gamma_{\mathrm{inf}}(Y/S, R\lim\mathcal{E}) \\ &= R\lim R\Gamma_{\mathrm{inf}}(Y/S,\mathcal{E}) \\ &\overset{\text{(b)}}{\xrightarrow{\sim}} R\lim R\Gamma(Y, E\otimes_{O_Y}\Omega^\bullet_{Y/S}) \\ &\overset{\text{(c)}}{=} R\Gamma\left(Y, R\lim(E\otimes_{O_Y}\Omega^\bullet_{Y/S})\right) \\ &= R\Gamma\left(Y, \lim_n(\mathcal{F}_{\Delta^n_X(Y)}\otimes_{O_Y}\Omega^\bullet_{Y/S})\right), \end{aligned}$$

where (a) uses Lemma 7.2 (d), (b) relies on Remark 11.12, (c) uses [Sta25, Tag 0BKP]. ☐

Remark 11.14. From Theorem 11.2 and Corollary 11.13, one can derive cohomological finiteness for quasi-coherent crystals, the smooth base change theorem [Ber74, V, 3.2.4, 3.2.7, 3.5.1, 3.5.2], as well as perfectness [BO78, 7.16, 7.24].

12 Algebraic infinitesimal topos

Let X be a scheme locally of finite type over a field k. The infinitesimal site $\mathrm{Inf}(X/k)$ contains objects (U,T) with T non-Noetherian. We introduce a subsite of the infinitesimal site, consisting of objects with finiteness condition.

Let $\mathrm{Inf}(X/k)^{\mathrm{alg}}$ be the full subcategory of $\mathrm{Inf}(X/k)$ of objects (U,T) with T separated of finite type over k. Let $\mathrm{Inf}(X/k)^{\mathrm{eft}}$ be the full subcategory of $\mathrm{Inf}(X/k)$ of objects (U,T) admitting a morphism

$$(U,T)\to(V,Z)$$

in $\mathrm{Inf}(X/k)$ with $(V,Z)\in\mathrm{Inf}(X/k)^{\mathrm{alg}}$. Endow $\mathrm{Inf}(X/k)^{\mathrm{alg}}$ and $\mathrm{Inf}(X/k)^{\mathrm{eft}}$ with the topology induced by $\mathrm{Inf}(X/k)$ in the sense of [SGA4I, III, 3.1], so that both are sites. Let $(X/k)^{\mathrm{alg}}_{\mathrm{inf}}$ be the topos associated with the site $\mathrm{Inf}(X/k)^{\mathrm{alg}}$.

Lemma 12.1. *The restriction functor* $\mathrm{Sh}(\mathrm{Inf}(X/k))\to\mathrm{Sh}(\mathrm{Inf}(X/k)^{\mathrm{eft}})$ *is an equivalence of categories.*

Proof. By [SGA4I, III, Théorème 4.1], it remains to prove that every $(U,T)\in\mathrm{Inf}(X/k)$ can be covered by objects in $\mathrm{Inf}(X/k)^{\mathrm{eft}}$. Thus, one may assume that T is affine. As X is locally of finite type over k, one may choose a closed immersion $U\to\mathbf{A}^n_k$ over k for some integer $n>0$. Since $\mathbf{A}^n_k$ is formally smooth over k and $U\to T$ is a nilpotent thickening of affine schemes, there is a morphism $T\to\mathbf{A}^n_k$ fitting into a commutative diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, equal] & T \arrow[d, dashed] \arrow[dr] & \\
U \arrow[r, hook] & \mathbf{A}^n_k \arrow[r] & \operatorname{Spec} k.
\end{tikzcd}$$

Then there is an integer m with $T \to \mathbf{A}^n_k$ factoring through the affine variety $\Delta^m_U(\mathbf{A}^n_k)$. Then $(U, \Delta^m_U(\mathbf{A}^n_k)) \in \operatorname{Inf}(X/k)^{\mathrm{alg}}$, and $(U,T) \to (U, \Delta^m_U(\mathbf{A}^n_k))$ is a morphism in $\operatorname{Inf}(X/k)$. Hence (U,T) is in $\operatorname{Inf}(X/k)^{\mathrm{eft}}$. $\square$

The inclusion functor $u : \operatorname{Inf}(X/k)^{\mathrm{alg}} \to \operatorname{Inf}(X/k)$ is continuous and cocontinuous. By [Sta25, Tag 00XO], it defines a morphism of topoi

$$\iota_{X/k} : (X/k)^{\mathrm{alg}}_{\mathrm{inf}} \to (X/k)_{\mathrm{inf}}$$

with $\iota^{-1}_{X/k} = u^s$. Let $O^{\mathrm{alg}}_{X/k} := \iota^{-1}_{X/k} O_{X/k}$. Then for every $(U,T) \in \operatorname{Inf}(X/k)^{\mathrm{alg}}$, $(O^{\mathrm{alg}}_{X/k})_T = O_T$. Moreover, $((X/k)^{\mathrm{alg}}_{\mathrm{inf}}, O^{\mathrm{alg}}_{X/k})$ is a ringed topos, and $\iota_{X/k}$ is a morphism of ringed topoi.

Lemma 12.2. *The inclusion* $u : \operatorname{Inf}(X/k)^{\mathrm{alg}} \to \operatorname{Inf}(X/k)$ *defines a morphism of ringed topoi* $p_{X/k} : (X/k)_{\mathrm{inf}} \to (X/k)^{\mathrm{alg}}_{\mathrm{inf}}$. *Furthermore,*

$$p_{X/k*} = u^s = \iota^{-1}_{X/k} : \operatorname{Sh}(\operatorname{Inf}(X/k)) \to \operatorname{Sh}(\operatorname{Inf}(X/k)^{\mathrm{alg}})$$

is an exact functor, and $p_{X/k} \circ \iota_{X/k} = \mathrm{id}_{(X/k)^{\mathrm{alg}}_{\mathrm{inf}}}$ *up to a canonical isomorphism.*

Proof. The functor u factors through a continuous functor $u : \operatorname{Inf}(X/k)^{\mathrm{alg}} \to \operatorname{Inf}(X/k)^{\mathrm{eft}}$. We prove that it induces a morphism of sites $\operatorname{Inf}(X/k)^{\mathrm{eft}} \to \operatorname{Inf}(X/k)^{\mathrm{alg}}$. For every object $(U,T) \in \operatorname{Inf}(X/k)^{\mathrm{eft}}$, let I_T be the category of morphisms $h : (U,T) \to (V,Z)$ in $\operatorname{Inf}(X/k)$ with $(V,Z) \in \operatorname{Inf}(X/k)^{\mathrm{alg}}$. By [Sta25, Tag 00X5], it remains to prove that I_T^{op} is a filtered category. By definition of $\operatorname{Inf}(X/k)^{\mathrm{eft}}$, I_T is nonempty. By Lemma 3.7, it is connected. Then from Lemma 3.5, I_T^{op} is filtered.

The morphism of sites $\operatorname{Inf}(X/k)^{\mathrm{eft}} \to \operatorname{Inf}(X/k)^{\mathrm{alg}}$ induces a morphism of topoi

$$\operatorname{Sh}(\operatorname{Inf}(X/k)^{\mathrm{eft}}) \to (X/k)^{\mathrm{alg}}_{\mathrm{inf}}.$$

Combined with Lemma 12.1, it defines a morphism of topoi $p_{X/k} : (X/k)_{\mathrm{inf}} \to (X/k)^{\mathrm{alg}}_{\mathrm{inf}}$.

Fix a sheaf $F \in \operatorname{Sh}(\operatorname{Inf}(X/k)^{\mathrm{alg}})$. Define a functor

$$F_T : I_T^{\mathrm{op}} \to \mathrm{Set}, \quad (h : (U,T) \to (V,Z)) \mapsto F(V,Z).$$

Then by construction, $p^{-1}_{X/k} F$ is the sheafification of the presheaf

$$\operatorname{Inf}(X/k)^{\mathrm{op}} \to \mathrm{Set}, \quad (U,T) \mapsto \operatorname{colim}_{I_T^{\mathrm{op}}} F_T.$$

As every object $\phi : (U,T) \to (W,S)$ of I_T induces a map $O_S(S) \to O_T(T)$, there is a natural morphism $p^{-1}_{X/k} O^{\mathrm{alg}}_{X/k} \to O_{X/k}$. Thus, $p_{X/k} : (X/k)_{\mathrm{inf}} \to (X/k)^{\mathrm{alg}}_{\mathrm{inf}}$ is a morphism of ringed topoi.

Let ${}_TI$ be the category of morphisms $\psi : (V,Z) \to (U,T)$ in $\mathrm{Inf}(X/k)$ with $(V,Z) \in \mathrm{Inf}(X/k)^{\mathrm{alg}}$. Consider the functor

$$ {}_TF : {}_TI^{\mathrm{op}} \to \mathrm{Set}, \quad (\psi : (V,Z) \to (U,T)) \mapsto F(V,Z). $$

By construction, one has $(\iota_{X/k*}F)(U,T) = \lim_{{}_TI^{\mathrm{op}}} {}_TF$. When $(U,T) \in \mathrm{Inf}(X/k)^{\mathrm{alg}}$, $\mathrm{Id}_{(U,T)}$ is a final object of ${}_TI$, so $(\iota_{X/k*}F)(U,T) = F(U,T)$. This shows $p_{X/k*}\iota_{X/k*} = \iota^{-1}_{X/k}\iota_{X/k*} = \mathrm{Id}_{\mathrm{Sh}(\mathrm{Inf}(X/k)^{\mathrm{alg}})}$. ☐

Restricting from the small infinitesimal topos to the algebraic infinitesimal topos preserves the cohomology.

Corollary 12.3. *Let A be a sheaf of rings on $\mathrm{Inf}(X/k)$. Let F be a bounded below complex of A-modules on $\mathrm{Inf}(X/k)$. Then there is a canonical isomorphism*

$$ R\Gamma((X/k)_{\mathrm{inf}}, F) \xrightarrow{\sim} R\Gamma((X/k)^{\mathrm{alg}}_{\mathrm{inf}}, \iota^{-1}_{X/k}F) $$

in $D^+(\Gamma((X/k)_{\mathrm{inf}}, A))$.

Proof. It follows from Lemma 12.2 and the canonical isomorphism of functors

$$ R\Gamma((X/k)_{\mathrm{inf}}, \cdot) \xrightarrow{\sim} R\Gamma((X/k)^{\mathrm{alg}}_{\mathrm{inf}}, \cdot) \circ Rp_{X/k*} : \quad D^+(A) \to D^+(\Gamma((X/k)_{\mathrm{inf}}, A)). $$

☐

13 Analytic infinitesimal topos

Given an algebraic variety X, we do not know how to construct a comparison morphism between the infinitesimal cohomology $H^*_{\mathrm{inf}}(X/\mathbb{C})$ and the Betti cohomology $H^*(X^{\mathrm{an}}, \mathbb{C})$ directly. Instead, we introduce an auxiliary site $\mathrm{Inf}(X^{\mathrm{an}})$, which is a complex analytic analog of the infinitesimal site, and compare both cohomology groups with the cohomology group $H^*_{\mathrm{inf}}(X^{\mathrm{an}})$ defined by the site $\mathrm{Inf}(X^{\mathrm{an}})$.

Let M be a complex analytic space in the sense of [Gro60, Définition 2.1]. Let M_{cl} be the topos associated with the topological space underlying M. Let $\mathrm{Inf}(M)$ be the category of pairs (U,T), where U is an open subset of M, and $U \hookrightarrow T$ is a nilpotent thickening of complex analytic spaces. Morphisms and covering families are defined similar to the infinitesimal topos $\mathrm{Inf}(X/S)$ for schemes. Similar to $O_{X/S}$, there is a structure sheaf $O_{M,\mathrm{inf}}$ on $\mathrm{Inf}(M)$. Let M_{inf} be the topos corresponding to $\mathrm{Inf}(M)$. Parallel to Remark 2.2, every $(U,T) \in \mathrm{Inf}(M)$ and every $t \in T$ define a point of the topos $\xi_{T,t} : P \to M_{\mathrm{inf}}$, and the family of points $\{\xi_{T,t}\}$ is conservative. Similar to the construction of $u_{X/S} : (X/S)_{\mathrm{inf}} \to X_{\mathrm{Zar}}$, there is a canonical morphism of topoi $u_M : M_{\mathrm{inf}} \to M_{\mathrm{cl}}$.

Let X be a scheme locally of finite type over $\mathbb{C}$. Let $\phi_X : X^{\mathrm{an}}_{\mathrm{cl}} \to X_{\mathrm{Zar}}$ be the morphism of topoi induced by the continuous map $X^{\mathrm{an}} \to X$. We also write ϕ for ϕ_X. Let $\mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft}}$ be the full subcategory of $\mathrm{Inf}(X^{\mathrm{an}})$ of objects (U,T) admitting a morphism $(U,T) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})$ for certain $(V,Z) \in \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}$. Endow it with the topology induced by $\mathrm{Inf}(X^{\mathrm{an}})$.

Lemma 13.1. *The restriction functor* $\mathrm{Sh}(\mathrm{Inf}(X^{\mathrm{an}})) \to \mathrm{Sh}(\mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft}})$ *is an equivalence of categories.*

Proof. The proof is similar to that of Lemma 12.1. We prove that every $(U,T) \in \mathrm{Inf}(X^{\mathrm{an}})$ can be covered by objects in $\mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft}}$. Shrinking X, one may assume that X is affine. Since X is of finite type over $\mathbb{C}$, one can choose a closed immersion $X \to \mathbf{A}^n_{\mathbb{C}}$ for some integer $n > 0$. As $U \to T$ is a nilpotent thickening, locally on T, there is a morphism $h : T \to \mathbb{C}^n$ fitting into a commutative diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, hook] & T \arrow[d, dashed, "h"] \\
X^{\mathrm{an}} \arrow[r, hook] & \mathbb{C}^n.
\end{tikzcd}$$

There is an integer $m > 0$ such that $h : T \to \mathbb{C}^n$ factors through $\Delta^m_{X^{\mathrm{an}}}(\mathbb{C}^n)$, which is the analytification of $\Delta^m_X(\mathbf{A}^n)$. Then $(X, \Delta^m_X(\mathbf{A}^n)) \in \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}$, and there is a morphism $(U,T) \to (X^{\mathrm{an}}, (\Delta^m_X(\mathbf{A}^n))^{\mathrm{an}})$. ☐

Lemma 13.2. *The analytification functor*

$$u : \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}} \to \mathrm{Inf}(X^{\mathrm{an}}), \quad (U,T) \mapsto (U^{\mathrm{an}}, T^{\mathrm{an}})$$

induces a morphism of ringed topoi $\epsilon : X^{\mathrm{an}}_{\mathrm{inf}} \to (X/\mathbb{C})^{\mathrm{alg}}_{\mathrm{inf}}$.

Proof. By [SGA 1, XII, 1.2], the analytification functor Φ from the category of schemes locally of finite type over $\mathbb{C}$ to the category of complex analytic spaces commutes with finite projective limits. So $u : \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}} \to \mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft}}$ is a continuous functor. For every $(U,T) \in \mathrm{Inf}(X^{\mathrm{an}})$, let I_T be the category of morphisms $\phi : (U,T) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})$ in $\mathrm{Inf}(X^{\mathrm{an}})$ with $(V,Z) \in \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}$. Assume $(U,T) \in \mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft}}$, so that I_T is nonempty. By Lemma 3.7, as Φ preserves finite products, I_T is connected. From Lemma 3.5, as Φ preserves equalizers, I_T^{op} is a filtered category. Then by [Sta25, Tag 00X5], the continuous functor u defines a morphism of sites

$$\mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft}} \to \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}.$$

Together with Lemma 13.1, it induces a morphism of topoi $\epsilon : X^{\mathrm{an}}_{\mathrm{inf}} \to (X/\mathbb{C})^{\mathrm{alg}}_{\mathrm{inf}}$, such that for every sheaf $G \in \mathrm{Sh}(\mathrm{Inf}(X^{\mathrm{an}}))$, $\epsilon_* G$ is the sheaf

$$\mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg,op}} \to \mathrm{Set}, \quad (V,Z) \mapsto G(V^{\mathrm{an}}, Z^{\mathrm{an}}).$$

For every $(U,T) \in \mathrm{Inf}(X^{\mathrm{an}})$ and every $F \in \mathrm{Sh}(\mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}})$, define a functor

$$F_T : I_T^{\mathrm{op}} \to \mathrm{Set}, \quad (\phi : (U,T) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})) \mapsto F(V,Z).$$

Then $\epsilon^{-1} F$ is the sheafification of the presheaf

$$\mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{op}} \to \mathrm{Set}, \quad (U,T) \mapsto \mathrm{colim}_{I_T^{\mathrm{op}}} F_T. \tag{80}$$

For every object $\phi : (U,T) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})$ of I_T, there is a canonical ring map $O_Z(Z) \to O_T(T)$. Thus, for $F = O_{X/\mathbb{C}}$ one gets a morphism $\mathrm{colim}_{I_T^{\mathrm{op}}} F_T \to O_T(T)$. Whence, there is a canonical morphism $\epsilon^{-1} O^{\mathrm{alg}}_{X/\mathbb{C}} \to O_{X^{\mathrm{an}},\mathrm{inf}}$ of sheaves of rings on $\mathrm{Inf}(X)^{\mathrm{an}}$. Thus, ϵ is a morphism of ringed topoi. ☐

Lemma 13.3. *The diagram*

$$\begin{tikzcd} X^{\mathrm{an}}_{\mathrm{inf}} \arrow[r, "\epsilon"] \arrow[d, "u_{X^{\mathrm{an}}}"] & X^{\mathrm{alg}}_{\mathrm{inf}} \arrow[r, "\iota_{X/\mathbb{C}}"] & (X/\mathbb{C})_{\mathrm{inf}} \arrow[d, "u_{X/\mathbb{C}}"] \\ X^{\mathrm{an}}_{\mathrm{cl}} \arrow[rr, "\phi"] & & X_{\mathrm{Zar}} \end{tikzcd} \tag{81}$$

of topoi is commutative.

Proof. We construct a canonical morphism of functors

$$\epsilon^{-1}\iota^{-1}_{X/\mathbb{C}}u^{-1}_{X/\mathbb{C}} \to u^{-1}_{X^{\mathrm{an}}}\phi^{-1} : \quad \mathrm{Sh}(X) \to \mathrm{Sh}(\mathrm{Inf}(X^{\mathrm{an}})) \tag{82}$$

as follows. For every $F \in \mathrm{Sh}(X)$ and every $(V, Z) \in \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}$, one has

$$(\iota^{-1}_{X/\mathbb{C}}u^{-1}_{X/\mathbb{C}}F)(V, Z) = (u^{-1}_{X/\mathbb{C}}F)(V, Z) = F(V),$$
$$(\epsilon_* u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F)(V, Z) = (u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F)(V^{\mathrm{an}}, Z^{\mathrm{an}}) = (\phi^{-1}F)(V^{\mathrm{an}}).$$

The natural maps $F(V) \to (\phi^{-1}F)(V^{\mathrm{an}})$ induce a morphism in $\mathrm{Sh}(\mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}})$

$$\iota^{-1}_{X/\mathbb{C}}u^{-1}_{X/\mathbb{C}}F \to \epsilon_* u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F.$$

By adjunction, it induces a morphism

$$\epsilon^{-1}\iota^{-1}_{X/\mathbb{C}}u^{-1}_{X/\mathbb{C}}F \to u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F$$

in $\mathrm{Sh}(\mathrm{Inf}(X^{\mathrm{an}}))$ which is functorial in F.

We prove that (82) is an isomorphism. As the family of points $\{\xi_{T,t}\}$ is conservative, it suffices to prove that for every $(U, T) \in \mathrm{Inf}(X^{\mathrm{an}})$ and every $t \in T$,

$$\xi_{T,t}\epsilon^{-1}\iota^{-1}_{X/\mathbb{C}}u^{-1}_{X/\mathbb{C}}F \to \xi_{T,t}u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F$$

is an isomorphism. Let $x \in U \subset X^{\mathrm{an}}$ be the preimage of t. Define a category I_t, whose objects are open neighborhoods of x in X, and whose morphisms are inclusions. Define a functor

$$F_t : I^{\mathrm{op}}_t \to \mathrm{Set}, \quad V \mapsto F(V).$$

Then

$$\xi_{T,t}u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F = \big((u^{-1}_{X^{\mathrm{an}}}\phi^{-1}F)_T\big)_t = (\phi^{-1}F)_x = F_x = \mathrm{colim}_{I^{\mathrm{op}}_t} F_t.$$

Similarly, let I_T be the category of morphisms $(U, T) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})$ in $\mathrm{Inf}(X^{\mathrm{an}})$ with $(V, Z) \in \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}$. Define a functor

$$F_T : I^{\mathrm{op}}_T \to \mathrm{Set}, \quad ((U, T) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})) \mapsto F(V).$$

From (80), $\epsilon^{-1}\iota^{-1}_{X/\mathbb{C}}u^{-1}_{X/\mathbb{C}}F$ is the sheafification of the presheaf

$$\mathrm{Inf}(X^{\mathrm{an}})^{\mathrm{eft,op}} \to \mathrm{Set}, \quad (U, T) \mapsto \mathrm{colim}_{I^{\mathrm{op}}_T} F_T.$$

Define a category $I_{T,t}$ as follows. An object of it is a morphism $(U_0, T_0) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})$ in $\mathrm{Inf}(X^{\mathrm{an}})$, where T_0 is an open neighborhood of t in T, $U_0 := U \times_T T_0$ and $(V, Z) \in \mathrm{Inf}(X/\mathbb{C})^{\mathrm{alg}}$. There is a forgetful functor

$$I_{T,t} \to I_t, \quad ((U_0, T_0) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})) \mapsto V.$$

Define a functor

$$F_{T,t} : I_{T,t}^{\mathrm{op}} \to \mathrm{Set}, \quad ((U_0, T_0) \to (V^{\mathrm{an}}, Z^{\mathrm{an}})) \mapsto F(V),$$

which is the composition $I_{T,t}^{\mathrm{op}} \to I_t^{\mathrm{op}} \xrightarrow{F_t} \mathrm{Set}$. Then

$$\xi_{T,t}\epsilon^{-1}\iota_{X/\mathbb{C}}^{-1}u_{X/\mathbb{C}}^{-1}F = \mathrm{colim}_{T_0}\, \mathrm{colim}_{I_{T_0}^{\mathrm{op}}} F_{T_0} = \mathrm{colim}_{I_{T,t}^{\mathrm{op}}} F_{T,t}.$$

By [Sta25, Tag 04E7], it suffices to prove that for every $(U, T) \in \mathrm{Inf}(X^{\mathrm{an}})$ and every $t \in T$, the forgetful functor $I_{T,t} \to I_t$ is initial.

(a) For every $V \in I_t$, we need to find an object $(U_0, T_0) \to (W^{\mathrm{an}}, S^{\mathrm{an}})$ of $I_{T,t}$ with $W \subset V$. Shrinking V to an affine neighborhood of x in X, one may assume that V is affine. The remaining proof is similar to that of Lemma 13.1. Choose a closed immersion $V \to \mathbf{A}^n_{\mathbb{C}}$ over $\mathbb{C}$. Take $U_0 = V^{\mathrm{an}}$. Let T_0 be the image of $U_0 \hookrightarrow U \hookrightarrow T$, which is an open subset of T. Then $U_0 \to T_0$ is a nilpotent thickening of complex analytic spaces. As $\mathbb{C}^n$ is smooth, shrinking T_0 one may assume that there is a morphism $h : T_0 \to \mathbb{C}^n$ extending $U_0 = V^{\mathrm{an}} \hookrightarrow \mathbb{C}^n$. Then h factors through some $\Delta^m_V(\mathbf{A}^n)^{\mathrm{an}}$, and $(U_0, T_0) \to (V^{\mathrm{an}}, \Delta^m_V(\mathbf{A}^n)^{\mathrm{an}})$ is an object of $I_{T,t}$.

(b) For every $W \in I_t$ and any two objects $\psi : (U_0, T_0) \to (V, Z)^{\mathrm{an}}$ and $\psi' : (U_0', T_0') \to (V', Z')^{\mathrm{an}}$ of $I_{T,t}$ with $V \subset W$ and $V' \subset W$, we shall find an object $\psi'' : (U_0'', T_0'') \to (V'', Z'')^{\mathrm{an}}$ of $I_{T,t}$ with two morphisms $\psi'' \to \psi$ and $\psi'' \to \psi'$. This follows from Lemma 3.7.

□

The proof of Theorem 13.4 is similar to that of Theorem 11.2. We use Cartan's Theorem B instead of Serre's vanishing theorem.

Theorem 13.4. *Let $X \to Y$ be a closed immersion of complex analytic spaces, with Y smooth. Let E be a finite locally free O_Y-module with an integrable connection. Let $\mathcal{E}$ be the crystal in $O_{X,\mathrm{inf}}$-modules on $\mathrm{Inf}(X)$ defined by E. Let $\Omega_Y^\bullet$ be the complex of sheaves of holomorphic differential forms on Y. Let $\hat{\Omega}_Y^\bullet$ be its formal completion along X. Then there is a canonical isomorphism $Ru_{X*}\mathcal{E} \to E \otimes_{O_Y} \hat{\Omega}_Y^\bullet$ in $D^+(X, \mathbb{C})$.*

14 Comparison of infinitesimal cohomology and singular cohomology

We finish the proof of [Gro68, Conjecture 4.2].

Theorem 14.1. *Let X be a scheme locally of finite type over $\mathbb{C}$. Then the canonical morphisms*

$$R\Gamma_{\mathrm{inf}}(X/\mathbb{C}) \to R\Gamma_{\mathrm{inf}}(X^{\mathrm{an}}) \leftarrow R\Gamma(X^{\mathrm{an}}, \mathbb{C}) \tag{83}$$

are isomorphisms. In particular, for every $i \geq 0$, there exist a canonical commutative diagram

$$\begin{tikzcd} H^i(X^{\mathrm{an}}, \mathbb{C}) \arrow[r, dash, "\cong"] \arrow[d] & H^i_{\mathrm{inf}}(X/\mathbb{C}) \arrow[d] & \\ H^i_{\mathrm{dR}}(X^{\mathrm{an}}) & H^i_{\mathrm{dR}}(X/\mathbb{C}) \arrow[l] \arrow[r] & H^i(X, O_X). \end{tikzcd} \tag{84}$$

Proof. We use the notation in Diagram (81). From Lemma 13.2, there is a natural morphism in $D^+(\mathrm{Inf}(X/\mathbb{C}), O_{X/\mathbb{C}})$

$$O_{X/\mathbb{C}} \to R(\iota_{X/\mathbb{C}}\epsilon)_* O_{X^{\mathrm{an}},\mathrm{inf}}.$$

By Lemma 13.3, it induces a morphism

$$\alpha : Ru_{X/\mathbb{C}*} O_{X/\mathbb{C}} \to R\phi_* Ru_{X^{\mathrm{an}}*} O_{X^{\mathrm{an}},\mathrm{inf}}$$

in $D^+(X, \mathbb{C})$. By adjunction, the morphism $u_{X^{\mathrm{an}}}^{-1}\mathbb{C} \to O_{X^{\mathrm{an}},\mathrm{inf}}$ in $\mathrm{Sh}(\mathrm{Inf}(X^{\mathrm{an}}))$ induces a morphism $\mathbb{C} \to Ru_{X^{\mathrm{an}}*} O_{X^{\mathrm{an}},\mathrm{inf}}$ in $D^+(X^{\mathrm{an}}, \mathbb{C})$, and hence a morphism

$$\beta : R\phi_* \mathbb{C} \to R\phi_* Ru_{X^{\mathrm{an}}*} O_{X^{\mathrm{an}},\mathrm{inf}}$$

in $D^+(X, \mathbb{C})$.

We prove that α and β are isomorphisms. It remains to prove that for every integer q, both morphisms of sheaves on X

$$\mathcal{H}^q Ru_{X/\mathbb{C}*} O_{X/\mathbb{C}} \overset{\mathcal{H}^q(\alpha)}{\to} \mathcal{H}^q R\phi_* Ru_{X^{\mathrm{an}}*} O_{X^{\mathrm{an}},\mathrm{inf}} \overset{\mathcal{H}^q(\beta)}{\leftarrow} \mathcal{H}^q R\phi_* \mathbb{C}$$

are isomorphisms. By [Sta25, Tag 0BKJ], it suffices to prove that for every affine open subset U of X, the maps

$$H^q(U, Ru_{X/\mathbb{C}*} O_{X/\mathbb{C}}) \overset{\alpha^q}{\to} H^q(U, R\phi_* Ru_{X^{\mathrm{an}}*} O_{X^{\mathrm{an}},\mathrm{inf}}) \overset{\beta^q}{\leftarrow} H^q(U, R\phi_* \mathbb{C})$$

are isomorphisms.

The formation of $u_{X/\mathbb{C}} : (X/\mathbb{C})_{\mathrm{inf}} \to X_{\mathrm{Zar}}$ and $\phi : X_{\mathrm{cl}} \to X_{\mathrm{Zar}}$ commutes with restriction to open subsets of X, so we may assume that $X = U$ is *affine*, and we need to prove that the maps

$$H^q_{\mathrm{inf}}(X/\mathbb{C}) \overset{\alpha^q}{\to} H^q R\Gamma(X^{\mathrm{an}}_{\mathrm{inf}}, O_{X^{\mathrm{an}},\mathrm{inf}}) \overset{\beta^q}{\leftarrow} H^q(X^{\mathrm{an}}, \mathbb{C})$$

are isomorphisms.

As X is affine, one may choose a smooth algebraic variety Y and a closed immersion $X \to Y$ over $\mathbb{C}$. By Theorems 11.2 (b) and 13.4, α^q and β^q are identified with the functorial maps

$$H^q(X, \hat{\Omega}^\bullet_{Y/\mathbb{C}}) \xrightarrow{\alpha^q} H^q(X^{\mathrm{an}}, \hat{\Omega}_{Y^{\mathrm{an}}}) \xleftarrow{\beta^q} H^q(X^{\mathrm{an}}, \mathbb{C}).$$

By [Har75, IV, Theorem 1.1], both are isomorphisms.

For a general X, applying $R\Gamma(X, \cdot) : D^+(X, \mathbb{C}) \to D^+(\mathbb{C})$ to the isomorphisms α and β, one gets an isomorphism (83). The second statement follows from (79). □

Remark 14.2. Let X be a scheme separate of finite type over $\mathbb{C}$. Deligne [Del74, Proposition 8.2.2] shows that the singular cohomology $H^*(X^{\mathrm{an}}, \mathbb{Z})$ carries a natural mixed Hodge structure. Using the isomorphism $H^*(X^{\mathrm{an}}, \mathbb{C}) \cong H^*_{\mathrm{inf}}(X/\mathbb{C})$ from Theorem 14.1, we compare the Hodge filtration on $H^*(X^{\mathrm{an}}, \mathbb{C})$ and the infinitesimal filtration on $H^*_{\mathrm{inf}}(X/\mathbb{C})$.

(a) Assume that X is smooth over $\mathbb{C}$. We show that the infinitesimal filtration is coarser than the Hodge filtration. From Corollary 11.11, the isomorphism $H^*_{\mathrm{inf}}(X/\mathbb{C}) \xrightarrow{\sim} H^*_{\mathrm{dR}}(X/\mathbb{C})$ identifies the infinitesimal filtration with the naive Hodge filtration.

   By Fact 1.1, as X is smooth, for every $i \geq 0$ there is a natural isomorphism

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \cong H^i_{\mathrm{dR}}(X/\mathbb{C}). \tag{85}$$

   The logarithmic de Rham complex in [Del71, p.31] is the analytification of a $\mathbb{C}$-linear complex of *algebraic* coherent sheaves on X, which by [Del71, Théorème 3.2.5 (i)] defines the Hodge filtration on $H^*(X^{\mathrm{an}}, \mathbb{C})$. Therefore, (85) restricts to an injection $F^q H^i(X^{\mathrm{an}}, \mathbb{C}) \subset F^q H^i_{\mathrm{dR}}(X/\mathbb{C})$. As [EZT14, Example 3.4.9] shows, there is a smooth affine curve X such that the inclusion is strict.

(b) Assume that X is proper over $\mathbb{C}$. By Remark 11.1 and [Bha12, Proposition 5.2], the infinitesimal filtration is finer than the Hodge filtration. More precisely, for any $q, i \geq 0$, the isomorphism $H^i_{\mathrm{inf}}(X/\mathbb{C}) \cong H^i(X^{\mathrm{an}}, \mathbb{C})$ restricts to an injection

$$F^q H^i_{\mathrm{inf}}(X/\mathbb{C}) \hookrightarrow F^q H^i(X^{\mathrm{an}}, \mathbb{C}). \tag{86}$$

   By properness and GAGA, the canonical map $F^q H^i_{\mathrm{dR}}(X/\mathbb{C}) \to F^q H^i_{\mathrm{dR}}(X^{\mathrm{an}})$ is an isomorphism. Then by [AK11, Example 4.5], there is a projective curve X over $\mathbb{C}$ such that the map $H^1(X^{\mathrm{an}}, \mathbb{C}) \to H^1_{\mathrm{dR}}(X/\mathbb{C})$ in (84) does not send $F^1 H^1(X^{\mathrm{an}}, \mathbb{C})$ inside $F^1 H^1_{\mathrm{dR}}(X/\mathbb{C})$. For this curve, (86) (with $q = i = 1$) is a strict inclusion. Such an example with X a *normal* projective surface is in [BVS94, p.39].

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