Lawrence-Venkatesh's p-adic approach to Mordell's conjecture

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1 Introduction

Mordell's conjecture is first proved by Faltings via Arakelov methods (see [Fal83] or its English translation [Fal86]). He proved the following conjectures in order:

- (Tate conjecture) An abelian variety over a number field K is determined up to K-isogeny by its Tate module with Γ_K-action.
- (Shafarevich conjecture) There are only finitely many abelian varieties (resp. smooth projective curves) of fixed dimension $g \geq 0$ (resp. fixed genus $g > 1$) defined over a fixed number field K with good reduction[1] outside a fixed finite set of places of K.
- (Mordell conjecture) Theorem 5.0.1.

The observation that Mordell's conjecture follows from that of Shafarevich was due to Parshin [Par68]. It relies on constructing a non-isotrivial relative curve over the given curve of genus at least 2.

[1] Definition 3.4.5

By now different proofs are presented: Vojta's Diophantine approximation way [Voj91] and Lawrence-Venkatesh's p-adic period method in [LV20]. The method of Lawrence-Venkatesh is a combination of ideas from Faltings' proof and Kim's non-abelian Chabauty theory [Kim05]. Although certain additional assumption is needed to deduce finiteness by the method of Chabauty-Kim, this one can explicitly determine the set of rational points in some examples where the additional information is known.

Mordell's conjecture can be formulated more generally for integral points on smooth hyperbolic curves in order to include genus 0 curves with at least three punctures (cf.[LV20, Theorem 4.1]) and genus 1 curves with at least one puncture (see Theorem 8.0.1).

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2 Preparatory results

This section should be referred to only as necessary when reading the main text.

$Frob_v \in G_{K_v}$ arithmetic Frobenius, see [BC09, p.4]

We gather some notation. A general field is denoted by k and its absolute Galois group is denoted by $G_k = G(k^s/k)$, where k^s denotes a separable closure of k. Denote an algebraic closure of k by $\bar{k}$. A field with a discrete valuation is denoted by E, the ring of integer by O_E and its residue field by $\mathbb{F}$. For a finite *unramified* extension $E/\mathbb{Q}_p$, we call *a* preimage of *the* (arithmetic, that is the field automorphism $\mathbb{F}_p \to \mathbb{F}_p$ defined by $x \mapsto x^p$) Frobenius (element of $G_{\mathbb{F}_p}$) under the isomorphism $G(E/\mathbb{Q}_p) \to G(\mathbb{F}/\mathbb{F}_p)$ *an* (arithmetic) *Frobenius* of $E/\mathbb{Q}_p$ and denote it by $\sigma_{E/\mathbb{Q}_p}$, which is of order $[E : \mathbb{Q}_p]$. Throughout K denotes a number field unless otherwise specified. Repeatedly $S(\supset S_\infty)$ stands for a finite set of places of K, including the set

S_∞ of all archimedean ones. Then $\mathcal{O}_S(\subset K)$ is understood to be the ring of S-integers.

For a finite place w of K, K_w denotes the completion of K at w and $\mathbb{F}_w$ denotes its residue field. Let $q_w = \#\mathbb{F}_w$ be its cardinal. We let $\mathbb{C}_{K_w}$ denote the completion of an algebraic closure of K_w, which is isomorphic to $\mathbb{C}$ and K_w^{ur} be the maximal unramified extension of K_w inside K_w^s. Then the inertia group at w is $G_{K_w^{ur}}$. Recall the natural isomorphism $G_{K_w}/G_{K_w^{ur}} \to G_{\mathbb{F}_w}$. Choosing a place u of K^a above w allows us to identify G_{K_w} with the decomposition subgorup D_u of G_K. Different choices lead to G_K-conjucate closed subgroups. Choose $\mathrm{Frob_w \in G_{K_w}}$ an element maps to the Frobenius in $G_{\mathbb{F}_w}$ and use the same symbol for its restriction to K^s: $\mathrm{Frob}_w \in G_K$. For a scheme X of finite type over K, denote by $X^h(= X_{\mathbb{C}}^{an})$ (resp.X_w^{an}) the analytification of $X_{\mathbb{C}}$ (resp.X_{K_w}). For $\mathcal{X}$ a scheme over a local ring (R, m), its special fiber $\mathcal{X} \otimes_R R/m$ is written as $\bar{\mathcal{X}}$. For a scheme S, we denote $\dim S$ its Krull dimension.

Recall that a subset of a topology space is called rare/thin/nowhere dense if its closure has no interior.

Proposition 2.0.1. *[GR12, Theorem, p.168] Let X be an analytic space, then X is irreducible if and only if every proper analytic subset is rare. Every connected complex manifold is an irreducible analytic space.*

Theorem 2.0.2. *[GR12, p.111] Let X be an analytic space, $\{A_\alpha\}$ a family of analytic subsets of X, then $\cap_\alpha A_\alpha$ is an analytic subset of X.*

From Theorem 2.0.2, for every subset $A \subset X$, there is a smallest analytic subset of X containing A, called the analytic Zariski closure of A in X.

2.1 Riemann-Hilbert correspondence

Let M be a connected, locally path connected, locally simply connected topological space with a base point p.

Proposition 2.1.1. *[Del70, Corollaire 1.4, p.4][ZS09, Corollary 1.9] Denote the category of local systems (i.e., locally constant sheaf) of finite-dimensional $\mathbb{C}$-vector spaces by* $\mathrm{Loc}(M)$*. Then taking stalk $F_p : Loc(M) \to Vec_{\mathbb{C}}$ is a fiber functor making* $\mathrm{Loc}(M)$ *a neutralized Tannakian category. The topological monodromy representation induces an equivalence of neutralized Tannakian categories*

$$Loc(M) \to Rep_{\mathbb{C}}(\pi_1(M, p)). \tag{1}$$

The Tannakian group is the algebraic hull of $\pi_1(M,p)$ over field $\mathbb{C}$. For all $L \in Loc(M)$, its Tannakian monodromy group is the Zariski closure of the image of the associated monodromy representation $\pi_1(M,p) \to GL(L_p)$ and the image of the natural map $\Gamma(M,L) \to L_p$ is an isomorphism onto the set of monodromy invariants $L_p^{\pi_1(M,p)}$. In particular, when M is furthermore simply connected, then any local system on M is a constant sheaf.

Assume further that M is a connected smooth manifold. Let C_M^∞ be the sheaf of smooth complex functions on M. Let $E \to M$ be a smooth vector bundle with a linear (smooth) connection D. Let $\mathcal{E}$ be the sheaf of smooth sections of E. The connection D is called integrable/flat if its curvature R^D vanishes. A smooth local section s of E is called horizontal/flat/parallel if $D(s) = 0$.

Assume that $\{s_1, \dots, s_r\}$ is a smooth local frame of E over an open $U \subset M$, then any section $s \in \Gamma(U,E)$ writes as $s = \sum_{i=1}^r f_i s_i$ and $D(s) = \sum_{i=1}^r s_i \otimes df_i$. That means we get a local trivialization

$$\begin{tikzcd} E|_U \arrow[r] \arrow[dr] & U \times \mathbb{C}^r \arrow[d] \\ & U \end{tikzcd}$$

that carries D to the standard derivative on $U \times \mathbb{C}^r$. Then $R^D|_U = 0$. In fact, a converse holds.

Theorem 2.1.2 (Smooth Frobenius)**.** *[Huy05, Exercise 4.3.10, p.192][CMSP17, Lemma C.4.1; Corollary C.4.2] The connection D is integrable if and only if every $p \in M$ admits an open neighborhood U and a smooth local frame over U consisting of horizontal sections. In that case,* $\ker(D)$ *is a local system on M and the parallel transport along a smooth curve depends only on the homotopy (with fixed ends) class of the curve, hence a representation $\pi_1(M,p) \to GL(E_p)$. The natural map* $\ker(D)_p \to E_p = \mathcal{E}_p \otimes_{C_p^\infty} \mathbb{C}$ *is a $\pi_1(M,p)$-equivariant isomorphism.*

Given another smooth vector bundle E' with a connection D', then a morphism $F : (E,D) \to (E',D')$ is equivalent to a global horizontal section of the Hom-vector bundle $\mathcal{H}om(E,E')$ with the induced connection. When D, D' are integrable, then so is the induced connection. In particular, the stalks $F_p \in Hom_{\mathbb{C}}(E_p, E'_p)$ are of the same rank in view of Theorem 2.1.2.

Here a morphism of smooth vector bundles is *not* required to have a smooth vector bundles as its kernel a priori, but this follows from its compatibility with the flat connections.

Theorem 2.1.3 (Smooth Riemann-Hilbert correspondence)**.** *[CMSP17, Theorem C.4.3] Let* $\mathrm{Fl}(M)$ *be the category of smooth complex vector bundles with a flat connection. Then $Fl(M)$ is an abelian category and the functor $\psi : Fl(M) \to Loc(M)$ by $(E, D) \mapsto \ker(D)$ is an equivalence of abelian categories. A quasi-inverse is given by $\mathcal{L} \mapsto (\mathcal{L} \otimes_{\mathbb{C}} C^\infty_M, Id \otimes d)$.*

Proof. For a morphism $(E, D) \to (E', D')$ in $\mathrm{Fl}(M)$, the kernel is a smooth vector bundle. Given a local system $\mathcal{L}$, then $\mathcal{L} \otimes_{\mathbb{C}} C^\infty_M$ is a smooth vector bundle. Then one can argue as in [Con, Theorem 2.6]. □

Now assume even further that M is a connected complex manifold and retain a base point $p \in M$. Let A^1_M (resp.$A^{1,0}_M$, resp. $A^{0,1}_M$) the sheaf of smooth complex 1 (resp. $(1, 0)$ resp. $(0, 1)$) forms on M, $\Omega^1_M \subset A^{1,0}_M$ the sheaf of holomorphic 1-forms on M, (note that $A^1_M = A^{1,0}_M \oplus A^{0,1}_M = \Omega^1_M \otimes_{O_M} C^\infty_M$). Recall the definition of holomorphic connection, which is more restrictive than a connection compatible with the holomorphic structure.

Definition 2.1.4 (holomorphic connection)**.** [Huy05, Definition 4.2.17, p.179]Let $H \to M$ be a holomorphic vector bundle. A holomorphic connection is a $\mathbb{C}$-linear map $\nabla : H \to H \otimes_{\mathbb{C}} \Omega^1_M$ with

$$\nabla(fs) = s \otimes (\partial f) + f\nabla(s)$$

for any local holomorphic function f on M and any local holomorphic section s of H.

An analogue for Theorem 2.1.2 in the analytic setting.

Theorem 2.1.5 (holomorphic Frobenius)**.** *Let $E \to M$ be a holomorphic vector bundle with a holomorphic connection, then the connection is integrable if and only if every $p \in M$ admits an open neighborhood U and a holomorphic local frame comprised of horizontal sections.*

Proof. Use [Voi02, Theorem 2.26, p.51]. □

Given a smooth vector bundle on M with a smooth connection D, ie a locally free sheaf $\mathcal{E}$ of C^∞_M-module of rank r with $D : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty_M} A^1_M$. We can decompose $D = D^{1,0} + D^{0,1}$, where $D^{1,0} : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty_M} A^{1,0}_M$ (not a smooth connection!) idem for $D^{0,1}$. Then the curvature $R^D = (D^{1,0})^2 + (D^{0,1})^2 + D^{1,0}D^{0,1} + D^{0,1}D^{1,0}$. Therefore, D is integrable if and only if

$$(D^{1,0})^2 = 0, \quad (D^{0,1})^2, D^{1,0}D^{0,1} + D^{0,1}D^{1,0} = 0.$$

If $(D^{0,1})^2 = 0$, then by Koszul-Malgrange theorem [KM58, Théorème 2, p.106] $\mathcal{H} := \ker(D^{0,1}) \subset \mathcal{E}$ is a holomorphic vector bundle (i.e a locally free sheaf of O_M-module) of same rank r, the natural map $\mathcal{H} \otimes_{O_M} C^\infty_M \to \mathcal{E}$ is an isomorphism and $D^{0,1} = \bar{\partial}^{\mathcal{H}}$ is the Dolbeault operator. If D is furthermore integrable, then by [Bis98, p.2829] $D^{1,0}$ restricts to an integrable holomorphic connection $\nabla : \mathcal{H} \to \mathcal{H} \otimes_{O_M} \Omega^1_M$.

Conversely, given a holomorphic vector bundle $\mathcal{H}$, we obtain a smooth vector bundle $\mathcal{E} := \mathcal{H} \otimes_{O_M} C^\infty_M$. If ∇ is a holomorphic connection on $\mathcal{H}$, then $D : \mathcal{E} \to \mathcal{E} \otimes_{C^\infty_M} A^1_M$ by $D(s \otimes f) := (\nabla s) \otimes f + s \otimes df$ (for all holomorphic local section s of $\mathcal{H}$ and smooth function f defined on the same open subset) defines a smooth connection. We find

$$D^{1,0}(s \otimes f) = (\nabla s) \otimes f + s \otimes (\partial f);$$
$$D^{0,1}(s \otimes f) = s \otimes \bar{\partial} f.$$

Therefore $D^{0,1} = \bar{\partial}^{\mathcal{H}}$, $\mathcal{H} = \ker(D^{0,1})$ and $D^{1,0}|_{\mathcal{H}} = \nabla$. If ∇ is integrable, then D is integrable.

Denote the category of holomorphic vector bundles with an integrable connection on M by $DE(M)$. The following is a summary of the preceding discussion.

Theorem 2.1.6. *$DE(M)$ is an abelian category. The functor $Fl(M) \to DE(M)$ by $(\mathcal{E}, D) \mapsto (\ker(D^{0,1}), D^{1,0}|_{\ker(D^{0,1})})$ is an equivalence of categories, with quasi-inverse $DE(M) \to Fl(M)$ by $(\mathcal{H}, \nabla) \mapsto (\mathcal{H} \otimes_{O_M} C^\infty_M, \nabla + d)$.*

Note carefully, a morphism of two holomorphic vector bundles is *not* required to have a holomorphic vector bundle as its kernel a priori. Then and taking fiber $\omega_p : DE(M) \to Vec_{\mathbb{C}}$ by $(\mathcal{E}, \nabla) \mapsto \mathcal{E}_p \otimes_{O_{M,p}} \mathbb{C}$ is a fiber functor making $DE(M)$ a neutralized Tannakian category over $\mathbb{C}$. Under Theorem 2.1.3, Theorem 2.1.6 is equivalent to the following.

Theorem 2.1.7 (Analytic Riemann-Hilbert correspondence, [Del70, Théorème 2.17, p.12], [Kat82, Proposition 5.1], [Mal87, Theorem 1.1], [Con, Theorem 2.6])**.** *The functor $\eta : DE(M) \to Loc(M)$ defined by $(\mathcal{E}, \nabla) \mapsto \ker(\nabla)$, $\ker(\nabla) \subset \mathcal{E}$*

being the subsheaf given by horizontal sections, is an equivalence of neutralized Tannakian categories (a tensor equivalence commuting with chosen fiber functors). A quasi-inverse $Loc(M) \to DE(M)$ is given by

$$\mathcal{L} \mapsto (\mathcal{L} \otimes_{\mathbb{C}} O_M, Id \otimes \partial_M).$$

In particular, for all $(\mathcal{E}, \nabla) \in DE(M)$ the natural inclusion $(\ker \nabla)_p \to \mathcal{E}_p \otimes_{O_{M,p}} \mathbb{C}$ is a $\pi_1(M,p)$-equivariant linear isomorphism.

Example 2.1.8. Let $M = \mathbb{C}^*$ with base point $p = 1 \in M$. The trivial holomorphic line bundle $E = M \times \mathbb{C}$ (ie $\mathcal{E} = O_M$). For every holomorphic function $f \in O_M(M)$, we define a holomorphic connection $D : \mathcal{E} \to \mathcal{E} \otimes_{\mathbb{C}} \Omega^1_M$ by $D(s) = (s' - fs) \otimes dz$. This connection is flat since $\dim M = 1$. Then $L = \ker(D)$ is a local system on M of rank 1. The fundamental group $\pi_1(M,p)$ has a generator $\gamma : [0,1] \to M$ by $\gamma(t) = e^{2\pi i t}$. Its monodromy representation $\pi_1(M,p) \to GL(L_p)$ is identified with $\mathbb{Z} \to \mathbb{C}^*$. By parallel transport, $s(p) \in L_p$ is transported to $s(p)e^{\int_p^q f(z)dz}$ along the chose path. Therefor the image of $1 \in \mathbb{Z}$ is $e^{\int_\gamma f(z)dz} = e^{2\pi i Res(f,0)}$

Corollary 2.1.9. *Given $(\mathcal{E}, \nabla) \in DE(M)$, then $\ker(\nabla)$ is quasi-isomorphic to the de Rham complex $(\Omega^*_M \otimes \mathcal{E})$.*

Since $\ker \nabla$ is a local system, there exists a connected open neighborhood Ω of $p \in M$ such that $(\ker \nabla)|_\Omega = \underline{\mathcal{E}(p)}_\Omega$. Further, $\underline{\mathcal{E}(p)}_\Omega \otimes_{\mathbb{C}} O_\Omega = (\ker \nabla)|_\Omega \otimes_{\mathbb{C}} O_\Omega \xrightarrow{\sim} \mathcal{E}|_\Omega$. For any $y \in \Omega$, taking fibers at y induces a parallel transport isomorphism:

$$P^y_p : \mathcal{E}(y_0) \to \mathcal{E}(y).$$

In short, an integrable connection on a vector bundle provides a way to identify nearby fibers.

Now we present a relative version of Theorem 2.1.7.

Definition 2.1.10 (relative local system)**.** [BE13, Definition 1.17 (ii)] A sheaf $\mathcal{L}$ of $\phi^{-1}(O_X)$-module is called a relative local system if for every $y \in Y$ there exist an open neighborhoods U of y and an open subset V of X with $\phi(U) \subset V$ and a coherent O_V-module M such that $\mathcal{L}|_U$ is isomorphic to $\phi^{-1}(M)|_U$. The full subcategory of the category of $\phi^{-1}(O_X)$-modules comprised of relative local systems is denoted by $LS(Y/X)$.

When X is a point, then we recover local system of finite dimensional $\mathbb{C}$-vector spaces on Y. Given a morphism $X' \to X$, let $\phi' : Y' \to X'$ be the base change, tensor then $LS(Y/X) \to LS(Y'/X')$ functor. In particular, given a relative local system $\mathcal{L}$ on Y/X, for any $x \in X$, $\mathcal{L}|_{Y_x}$ is a local system. If $Y \to X \to X'$, a relative local system on Y/X' is also a relative local system on Y/X.

let $DE(Y/X)$ be the category of coherent O_Y-modules with a flat relative connection.

Proposition 2.1.11. *[Kat70, Proposition 8.8][And01, Corollaire 2.5.2.2] When X is a point, $(\mathcal{E}, \nabla) \in DE(Y/X)$, then $\mathcal{E}$ is a holomorphic vector bundle on the complex manifold Y and hence $DE(Y) = DE(Y/X)$. (I don't think it is true in general.)*

Theorem 2.1.12 (relative Riemann-Hilbert)**.** *[Del70, Théorème 2.23, p.14] then the functor $DE(Y/X) \to LS(Y/X)$ by $(\mathcal{E}, \nabla) \mapsto \ker(\nabla)$ is an equivalence of categories. A quasi-inverse is given by $\mathcal{L} \mapsto \mathcal{L} \otimes_{\phi^{-1}O_X} O_Y$ with relative connection $Id_{\mathcal{L}} \otimes d_{Y/X}$.*

By Proposition 2.1.11, Theorem 2.1.12 contains Theorem 2.1.7 as a special case.

2.2 Symplectic groups

Let $(V, \langle , \rangle)$ be a finite dimensional symplectic space over a field k of characteristic 0. The associated symplectic group is denoted by $Sp(V)$. If $\dim_k V = 2n$, we also write $Sp_{2n}(k)$ for $Sp(V)$.

Definition 2.2.1 (transvection)**.** If $T \in Sp(V)$ is unipotent and $\mathrm{rk(T-I)} = 1$, then we call T a transvection.

Example 2.2.2. For $0 \neq v \in V$, define $T_v(x) = x + \langle v, x\rangle v$. Then T_v is a transvection.

Theorem 2.2.3. *[Hua48, p. 740,Theorem][Dul74, p. 26,V.1.Theorem][Sol77, Theorem 2.5][Kli63, Theorem 3] Let k be a field with characteristic different from 2 and $g \in \mathbb{N}^+$. If $\Lambda \in Aut(Sp_{2g}(k))$, then there exists $B \in GSp_{2g}(k)$ such that $\Lambda(A) = BAB^{-1}$ for any $A \in Sp_{2g}(k)$. Moreover, the natural map $Aut(Sp_{2g}(k)) \to Aut(PSp_{2g}(k))$ is surjective. The only nontrivial normal subgroup of $Sp_{2g}(k)$ is its center $\{\pm I_{2g}\}$.*

We record two Goursat type lemmas for subgroups of $\prod Sp_{2g}(\mathbb{Q})$.

Lemma 2.2.4. *Suppose that $G \leq Sp_{2g}(\mathbb{Q})^N$ is an abstract* normal *subgroup whose projection to each factor is surjective, then $G = Sp_{2g}(\mathbb{Q})^N$.*

The proof of [Rib76, Sublemma, p. 794] works.

The second is a mild generalization of [LV20, Lemma 2.12].

Lemma 2.2.5. *Let G be an* algebraic *subgroup of $Sp_{2g}(\mathbb{Q})^N$ satisfying the following conditions.*

- *For $1 \leq i \leq N$, the projection $\pi_i : G \to Sp_{2g}(\mathbb{Q})$ onto the i-th factor is surjective.*
- *For $1 \leq i \neq j \leq N$, there exists $g \in G$ such that $\pi_i(g)$ is not conjugated to any one of $\pm\pi_j(g)$ in $GSp_{2g}(\mathbb{Q})$.*

Then $G = Sp_{2g}(\mathbb{Q})^N$.

Note that $G = \{(A, \pm A) : A \in SL_2(\mathbb{Q})\}(\leq SL_2(\mathbb{Q})^2)$ satisfies the first condition but not the second.

Proof. Induction on N. The case $N = 1$ is trivial. Now assume that the case $N = k$ is proved and we proceed to $N = k + 1$. Let $P_k : G \to Sp^k$ be the projection onto the first k factors. By induction hypothesis, P_k is surjective. Let $H_2 = \ker(P_k)$ and $H_1 = \ker(\pi_{k+1})$. By Goursat's lemma, H_1 (resp. H_2) can be viewed as a normal algebraic subgroup of the product of first k factors Sp^k (resp. of the last factor Sp) and $\mathrm{im(G \to Sp^k/H_1 \times Sp/H_2)}$ is the graph of an isomorphism

$$f : Sp^k/H_1 \to Sp/H_2, \quad [\pi_k(g)] \mapsto [\pi_{k+1}(g)] \tag{2}$$

between algebraic groups, hence an isomorphism between Lie algebras

$$d_e f : \mathrm{Lie(Sp^k/H_1)} \to \mathrm{Lie(Sp/H_2)}.$$

By Theorem 2.2.3, H_2 is $\{I_{2g}\}, \{\pm I_{2g}\}$ or the full of Sp. We just need to exclude the first two possibilities.

If $H_2 = \{I\}$, in virtue of [Ser09, p. 45,Corollary 3], $\mathrm{Lie(H_1)} \subset \mathfrak{sp}^{\mathrm{k}}$ is direct sum of some (say the first) $(k-1)$ factors. By [Bor12, 7.1, p.105], Sp^{k-1} is the identity component of H_1. Then $\pi_k(H_1)$ is a finite normal subgroup of the k-th factor Sp and π_k induces an isomorphism $Sp^k/H_1 \to Sp/\pi_k(H_1)$. By

(2) the latter group has nontrivial center, so $\pi_k(H_1) = \{I\}$ and $H_1 = Sp^{k-1}$. Now (2) becomes

$$f : Sp \to Sp, \quad \pi_k(g) \mapsto \pi_{k+1}(g)$$

an isomorphism between the last two factors. According to the second condition, there exists $g_0 \in G$ such that $\pi_k(g_0), \pi_{k+1}(g_0)$ are not conjugated, which contradicts Theorem 2.2.3.

If $H_2 = \{\pm I\}$, by the same reasoning as above, we may assume that H_1 is the product of the first $(k-1)$ factors and one piece of $\{\pm I\}$ at the k-th factor which leads to similar contradiction.

Therefore, $H_2 = Sp$ and $G = Sp^{k+1}$. The induction is completed. □

Lemma 2.2.6. *A finite (abstract) subgroup G of $SL_2(\mathbb{R})$ is cyclic.*

Proof. Define a new scalar product of $\mathbb{R}^2$ from the standard one $(-,-)$ by $\langle x, y\rangle := \frac{1}{|G|}\sum_{g\in G}(g\cdot x, g\cdot y)$. Fix an orthonormal basis of $(\mathbb{R}^2, \langle -,-\rangle)$, with regard to which we obtain an injection $\psi : G \to SO_2(\mathbb{R})$. As any finite subgroup of $SO_2(\mathbb{R})$ is cyclic, so is G. □

Remark 2.2.7. We can determine $n = |G|$ if $G \leq SL_2(\mathbb{Q})$. In fact, there exists $g \in G$ such that $\psi(g) = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$, where $\theta = \frac{2\pi}{n}$. Since $\psi(g)$ and g are conjugate, $\mathrm{tr(g)} = \mathrm{tr}(\psi(\mathrm{g})) = 2\cos(\theta) \in \mathbb{Q}$. By Niven's theorem (cf.[KN16]), $n = 1, 2, 3, 4$ or 6.

2.3 Faltings' finiteness theorem

All representations are assumed to be finite dimensional.

Definition 2.3.1. Let v be a finite place of K and ρ be a Galois representation of G_K. Denote the inertia group by $I_v (\leq G_K)$. If $\rho(I_v)$ is trivial, then ρ is called unramified at v.

Fact 2.3.2 ([Dal06, Theorem, p.2])**.** *Let $p \neq l$ be primes. Let $E/\mathbb{Q}_p$ be a finite extension. Let X/E be a smooth proper variety with good reduction. Then for every integer $k \geq 0$, the representation $\Gamma_K \to \mathrm{GL}(H^k_{ét}(X_{K^s}; \mathbb{Q}_l))$ is unramified.*

Definition 2.3.3 (Weil number)**.** [SZ15, Sec.1][Kli, Definition 2.5.1] Fix $m \in \mathbb{N}^+$. Let $\bar{\mathbb{Z}}$ be the set of algebraic integers (i.e., the integral closure of $\mathbb{Z}$ inside $\mathbb{C}$). An m-Weil number is an element $\alpha \in \bar{\mathbb{Z}}$ such that $|\sigma(\alpha)|^2 = m$ for *every* $\sigma \in G_{\mathbb{Q}}$.

Definition 2.3.4 (Purity, weight)**.** [FON12, p.19][FO22, Def. 2.19] Let K be a number field. A p-adic Galois representation $\rho : G_K \to \mathrm{GL}_d(\mathbb{Q}_p)$ is called *pure of weight* $w \in \mathbb{Z}$, if for all but finitely many finite places v of K, ρ is unramified at v and any complex eigenvalue of the *geometric* Frobenius action $\rho(\mathrm{Frob_v})$ is a q_v^m-Weil number.

Theorem 2.3.5. *[Del74, Theorem 1.6][Del80, Corollaire 3.3.9, p.207]Let $X/\mathbb{F}_q$ be a smooth proper variety. The Frobenius morphism $F : X \to X$ (defined by $x \mapsto x^q$) induces $F^* : H^i_{ét}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l) \to H^i_{ét}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l)$ for each $i \geq 0$ and rational prime $l(\neq \mathrm{char}(\mathbb{F}_q))$. Then the characteristic polynomial $\det(tI - F^*, H^i_{ét}(X_{\overline{\mathbb{F}_q}}, \mathbb{Q}_l))$ is of integral coefficients and independent of l. Each complex root of this polynomial is a q-Weil number of weight i.*

Lemma 2.3.6 (Faltings' finiteness, [Del85, Cor. 1], [Del83, Thm. 3.1], [Lan91, Ch. IV, Theorem 4.3], [LV20, Lem. 2.3])**.** *Fix $d \geq 0$, $w \in \mathbb{Z}$, K a number field with a finite set $S(\supset S_\infty)$ of places including all the archimedean ones. Then up to conjugation, there are only finitely many* semisimple *p-adic Galois representations $\rho : G_K \to GL_d(\mathbb{Q}_p)$ such that outside S*

1. *ρ is unramified and*

2. *pure of weight w.*

3. *The characteristic polynomials of a (hence every)* arithmetic *Frobenius (element of Γ_K) have integer coefficients.*

This is a consequence of Theorem 2.3.7 below. It is worth noting that the semisimplicity hypothesis is essential in Lemma 2.3.6.

Theorem 2.3.7 (Hermite-Minkowski)**.** *[SBW89, p. 49]Let K be a number field, $S \supset S_\infty$ a finite set of places of K, and $n \geq 1$ an integer. Then there are (up to isomorphism) only finitely many extensions of K unramified outside S of degree n.*

3 Cohomology theory

A quick guide is [Rom14].

3.1 Algebraic de Rham cohomology

Definition 3.1.1 (hypercohomology)**.** Suppose that $\mathcal{A}$ is an abelian category with enough injectives and $F : \mathcal{A} \to \mathcal{B}$ a left exact functor to another abelian category $\mathcal{B}$. If $C \in Kom^+(\mathcal{A})$ is a cochain complex of objects of $\mathcal{A}$ bounded on the left, $i \in \mathbb{Z}$, then the i-th hypercohomology $\mathcal{H}^i(C) := \mathcal{R}^i F(C) \in \mathcal{B}$ of C is calculated as follows: Take a quasi-isomorphism $\Phi : C \to I$, here I is a complex of injective elements of A. Then $\mathcal{H}^i(C)$ is the i-th cohomology of the complex $F(I)$.

The hypercohomology of C is independent of the choice of the quasi-isomorphism, up to a unique isomorphism. In the language of derived categories, it is the composition of derived functor $\mathcal{R}F : D^+(\mathcal{A}) \to D^+(\mathcal{B})$ and the cohomology functor $H^i : D^+(\mathcal{B}) \to \mathcal{B}$.

Lemma 3.1.2. *[Wei95, Lemma 5.7.5]Settings as in Definition 3.1.1. If* $0 \to C_1 \to C_2 \to C_3 \to 0$ *is a short exact sequence of complexes bounded on the left, then there is a long exact sequence*

$$\cdots \to \mathcal{R}^j F(C_3) \xrightarrow{\delta} \mathcal{R}^{j+1} F(C_1) \to \mathcal{R}^{j+1} F(C_2) \to \mathcal{R}^{j+1} F(C_3) \xrightarrow{\delta} \dots$$

Given a scheme X, let $Mod(X)$ be the category of sheaves of O_X-modules. It is a Grothendieck abelian category, hence with enough injectives. Let $\pi : X \to Y$ be a morphism of schemes. Then the functor $\pi_* : Mod(X) \to Mod(Y)$ is left exact. Its right derived functor $R\pi_* : D^+(Mod(X)) \to D^+(Mod(Y))$ exists.

Definition 3.1.3. [Sta23, Tag 0FL6]For every $n \geq 0$, the n-th *algebraic de Rham cohomology* of X over Y is defined by

$$\mathcal{H}^n_{dR}(X/Y) := \mathcal{R}^n \pi_*(\Omega^*_{X/Y}, d) \in Mod(Y).$$

Put $\Gamma(Y, O_Y)$-module

$$H^n_{dR}(X/Y) := \Gamma(Y, \mathcal{H}^n_{dR}(X/Y)).$$

By [Sta23, Tag 0FLX], if f is quasi-compact and quasi-separated, then each $\mathcal{H}^n_{dR}(X/Y)$ is a quasi-coherent O_Y-module.

Lemma 3.1.4. *[Sta23, Tag 0FLY] If Y is a locally Noether scheme and $f : X \to Y$ is a proper morphism, then $\mathcal{H}^n_{dR}(X/Y)$ is a coherent O_Y-module for each $n \geq 0$.*

Lemma 3.1.5 (finiteness). *[Sta23, Tag 0FLZ] Let A be a Noether ring and $Y = \mathrm{Spec}(A)$. If $f : X \to Y$ is a proper morphism, then $H^i_{\mathrm{dR}}(X/Y)$ is finite A-module for all i.*

Example 3.1.6. Let k be a field of characteristic $p > 0$, then $H^0_{\mathrm{dR}}(A^1_k/k) = k[T^p]$ as $O(A^1_k) = k[T]$-module.

This examples shows that de Rham cohomology is not a reasonable tool in positive characteristic.

Lemma 3.1.7. *Let k be a field and X/k be a proper connected geometrically connected scheme, then $H^0_{\mathrm{dR}}(X/k) = k$.*

Proof. By [Sta23, Tag 0BUG], $\Omega^0_{X/k} = O(X) = k$ and the differential map $d^0 : \Omega^0_{X/k} \to \Omega^1_{X/k}$ is zero. □

Fact 3.1.8. *[Lau96, p.18][Vig21, p.62] Let k be a field and A/k an abelian variety. Then the cup product induces a canonical isomorphism $\bigwedge^q H^1_{\mathrm{dR}}(A/k) \to H^q_{\mathrm{dR}}(A/k)$ for each $q \geq 0$. There is a first Chern class map $c_1 : \mathrm{Pic}(A) \to H^2_{\mathrm{dR}}(A/k)$. If $L \in \mathrm{Pic}(A)$ induces a polarization on A, then the pairing $H^1_{\mathrm{dR}}(A/k) \times H^1_{\mathrm{dR}}(A/k) \to k$ induced by $c_1(L)$ is symplectic.*

Remark 3.1.9. Let $\pi : X \to Y$ be a morphism of complex analytic spaces, then we can define (analytic) relative de Rham cohomology in a similar manner.

Lemma 3.1.10. *[Sta23, Tag 0FM0]Let $\pi : X \to Y$ be a proper smooth morphism of schemes. Then the formation of $\mathcal{H}^*_{dR}(X/Y)$ commutes with arbitrary base change.*

Without properness, it remains true in derived categories, cf [Gro68, p.309].

The stupid truncation $(\Omega^{\geq p}_{X/Y})_{p\geq 0}$ on the complex $\Omega^*_{X/Y}$ gives rise to a spectral sequence (cf.[Sta23, Tag 012K]), called the **Hodge to de Rham spectral sequence** (also known as the Frölicher spectral sequence),

$$E_1^{p,q} = R^q\pi_*\Omega^p_{X/Y} \Rightarrow \mathcal{H}^{p+q}_{dR}(X/Y). \tag{3}$$

The induced decreasing filtration by holomorphic vector subbundles on $\mathcal{H}^n_{dR}(X/Y)$ is called the Hodge filtration, whose terms are

$$Fil^p\mathcal{H}^n_{dR}(X/Y) := \mathrm{Im}(\mathbb{H}^n(X, \Omega^{\geq p}_{X/Y}) \to \mathcal{H}^n_{dR}(X/Y)).$$

$$F^p\mathcal{H}^n = \mathrm{im}[R^n f_*\Omega^{\geq p}_{X/Y} \to \mathcal{H}^n_{dR}(X/Y)].$$

Similar formula holds for $Fil^p H^n_{dR}(X/Y)$. Taking global sections, we get a filtration of $H^n_{\mathrm{dR}}(X/Y)$.

Theorem 3.1.11. [Del68, Thm. 5.5] [Kat70, Proposition 8.8] *Let Y be a scheme of characteristic* 0 *and let $f : X \to Y$ be a proper smooth morphism. Then*

1. *The O_Y-modules $\mathcal{H}^j_{dR}(X/Y)$ and $\mathcal{H}^{p,q}(X/Y) := R^q f_*\Omega^p_{X/Y}$ are locally free of finite type whose formation commutes with arbitrary base change.*
2. *The Hodge to de Rham spectral sequence (3) degenerates at E_1. In particular, for each $n \in \mathbb{N}$, the p-th graded piece of Hodge filtration on $\mathcal{H}^n_{dR}(X/Y)$ is $Gr^p F\mathcal{H}^n = F^p/F^{p-1} = R^{n-p}f_*\Omega^p_{X/Y}$.*
3. *At any point of Y, the O_Y-modules $R^q f_*\Omega^p_{X/Y}$ and $R^p f_*\Omega^q_{X/Y}$ have the same rank (Hodge symmetry).*

For analytic counterparts of Theorem 3.1.11, see Proposition 3.1.16 and Theorem 3.6.6.

Proposition 3.1.12. *Let $f : X \to Y$ be a morphism of locally finite type $\mathbb{C}$-schemes. Then there are unique isomorphisms $\Psi^p : (\Omega^p_{X/Y})^h \to \Omega^p_{X^h/Y^h}$ of O_{X^h}-modules for $p \geq 0$ such that Ψ^0 is the natural isomorphism and the following diagram commutes*

$$\begin{array}{ccc}
(\Omega^p_{X/Y})^h & \xrightarrow{\Psi^p} & \Omega^p_{X^h/y^h} \\
\downarrow{\scriptstyle (d^p_{X/Y})^h} & & \downarrow{\scriptstyle d^p_{X^h/Y^h}} \\
(\Omega^{p+1}_{X/Y})^h & \xrightarrow{\Psi^{p+1}} & \Omega^{p+1}_{X^h/Y^h}
\end{array}$$

Theorem 3.1.13. *Let $f : X \to Y$ be a proper morphism of locally finite type $\mathbb{C}$-schemes. Then the natural maps are isomorphisms:*

1. $(R^q f_*\Omega^p_{X/Y})^h \to R^q f^h_*\Omega^p_{X^h/Y^h}$
2. $\mathcal{H}^j_{dR}(X/Y)^h \to \mathcal{H}^j_{dR}(X^h/Y^h)$

Proposition 3.1.14. *Let $f : X \to Y$ be a smooth proper morphism of complex analytic spaces. Then for every $j \in \mathbb{N}$:*

1. *For every local system L on X, the sheaf $R^j f_* L$ is a local system on Y.*
2. *The natural map $\underline{\mathbb{C}}_X \to O_X$ induces an isomorphism of O_Y-modules*

$$R^j f_* \underline{\mathbb{C}} \otimes_{\underline{\mathbb{C}}_Y} O_Y \to \mathcal{H}^j_{dR}(X/Y).$$

3. *The module $\mathcal{H}^j_{dR}(X/Y)$ is a holomorphic vector bundle on Y.*

Proof. 1. It can be proved as in [Vir21, Prop. 2].

2. By projection formula [KS90, Prop. 2.6.6], $R^j f_* \underline{\mathbb{C}} = R^j f_* f^{-1} O_Y$. By Lemma 3.5.2, the canonical morphism $R^j f_* f^{-1} O_Y \to \mathcal{H}^j_{dR}(X/Y)$ is an isomorphism.

3. It follows from 1 and 2.

□

An analytic analogue for Theorem 3.1.11.

Fact 3.1.15. *[Del70, Cor. 1.4, p.4] Let S be a path-connected, locally path-connected and locally simply connected topological space with a base point s. Then the category of local systems (of finite-dimensional $\mathbb{C}$-vector spaces) on S is equivalent to the category $\mathrm{Rep}_{\mathbb{C}}(\pi_1(S, s))$, by sending a local system L to its monodromy representation on the stalk L_s.*

Proposition 3.1.16. *Let $f : X \to Y$ be a smooth proper morphism of complex analytic manifolds. Then*

1. *$R^j f_* \underline{\mathbb{C}}$ is a local system of finite-dimensional $\mathbb{C}$-vector spaces on Y, so $\mathcal{H}^j_{dR}(X/Y)$ is a holomorphic vector bundle on Y. (by Ehresmann and Riemann-Hilbert correspondence)*
2. *[Dem96, p.58] For each pair $p, q \in \mathbb{N}$, the O_Y-module $R^q f_* \Omega^p_{X/Y}$ is a vector bundle.*

3.2 Cohomology on sites

We follow Tag 00VG to present the notion of site.

Definition 3.2.1 (site)**.** A site is given by a category $\mathcal{C}$ with a set $Cov(\mathcal{C})$ of families of morphisms with fixed target, called coverings of $\mathcal{C}$, satisfying the following axioms.

1. If $f : V \to U$ is an isomorphism in $\mathcal{C}$, then $\{f\} \in Cov(\mathcal{C})$.
2. If $\{V_i \to U : i \in I\} \in Cov(\mathcal{C})$ and for each i $\{V_{ij} \to U_i : j \in I_j\} \in Cov(\mathcal{C})$, then $\{V_{ij} \to U : i \in I, j \in J_i\} \in Cov(\mathcal{C})$.
3. If $\{U_i \to U : i \in I\} \in Cov(\mathcal{C})$ and $V \to U$ is a morphism of $\mathcal{C}$ then $U_i \times_U V$ exists for all i and $\{U_i \times_U V \to V : i \in I\} \in Cov(\mathcal{C})$.

Example 3.2.2. Given a topological space X, let $\mathcal{C}$ be the category whose objects consist of all the opens in X and whose morphisms are inclusion maps. Define $\{U_i \to U : i \in I\} \in Cov(\mathcal{C})$ if and only if $\cup_i U_i = U$. Thus we get a site $(\mathcal{C}, Cov(\mathcal{C}))$.

Fix a site $\mathcal{C}$ and let $Psh(\mathcal{C})$ be the category of presheaves of sets on $\mathcal{C}$.

Definition 3.2.3 (sheaf)**.** [Sta23, Tag 00VM]Let F be a presheaf of sets on a site $\mathcal{C}$. We say F is a sheaf if for every covering $\{U_i \to U : i \in I\} \in Cov(\mathcal{C})$ the diagram

$$F(U) \longrightarrow \prod_{i\in I} F(U_i) \underset{pr_1^*}{\overset{pr_0^*}{\rightrightarrows}} \prod_{(i_0,i_1)\in I^2} F(U_{i_0} \times_U U_{i_1})$$

represents the first arrow as the equalizer of pr_0^* and pr_1^*.

Let $\mathcal{A}$ be a category and let F be a presheaf on $\mathcal{C}$ with values in $\mathcal{A}$. For each $X \in Ob(\mathcal{A})$, define a presheaf of sets F_X by

$$F_X(U) = \mathrm{Hom_A(X, F(U))}.$$

We say that F is a sheaf if for all $X \in Ob(\mathcal{A})$, F_X is a sheaf. Let $Ab(\mathcal{C})$ be the category of abelian sheaves on $\mathcal{C}$.

Theorem 3.2.4. *[Sta23, Tag 03NU]The category of abelian sheaves on a site is an abelian category with enough injectives.*

If $U \in Ob(\mathcal{C})$, then the section functor $\Gamma(U, -) : Ab(\mathcal{C}) \to Ab$ is left exact. Define $H^p(U, -) = R^p\Gamma(U, -)$.

Definition 3.2.5 (global section functor)**.** [Sta23, Tag 071D]Define $\Gamma(\mathcal{C}, -) : Psh(\mathcal{C}) \to Set$ by $\Gamma(\mathcal{C}, F) = Hom_{Psh(\mathcal{C})}(e, F)$, where e is a final object of $Psh(\mathcal{C})$.

Example 3.2.6. Suppose that $\mathcal{C}$ has a final object X, then $\Gamma(\mathcal{C}, F) = F(X)$.

The functor $\Gamma(\mathcal{C}, -) : Ab(\mathcal{C}) \to Ab$ is left exact. Define $H^i(\mathcal{C}, -) = R^i\Gamma(\mathcal{C}, -)$.

3.3 Étale cohomology

Fix a scheme X and $F \in Ab(X_{\text{ét}})$.

Definition 3.3.1 (small étale site)**.** [Sta23, Tag 03XB]Let X be a scheme. The category $X_{\text{ét}}$ is the full subcategory of Sch/X whose objects are étale X-schemes. A covering is $\{f_i : U_i \to U : i \in I\}$ with $U \in Ob(X_{\text{ét}})$, $\cup_{i \in I} f_i(U_i) = U$ and f_i is étale for each $i \in I$.

Clearly X is a final object of $X_{\text{ét}}$. The étale cohomology group $H^p_{\text{ét}}(X, F)$ is defined to be $H^p(X, F)$ on the site $X_{\text{ét}}$. However, this definition is reasonable only when F is a torsion sheaf. We define

$$H^j_{\text{ét}}(X, \mathbb{Z}_p) := \varprojlim_n H^j_{\text{ét}}(X, \mathbb{Z}/p^n);$$
$$H^j_{\text{ét}}(X, \mathbb{Q}_p) := \mathbb{Q}_p \otimes_{\mathbb{Z}_p} H^j_{\text{ét}}(X, \mathbb{Z}_p).$$

If X is a k-scheme, then $H^j_{\text{ét}}(X_{k^s}, \mathbb{Q}_p)$ is a p-adic representation of G_k. It is an important origin of Galois representations.

Then we review étale fundamental groups.

Definition 3.3.2 (geometric point)**.** [Sta23, Tag 03PO]Let X be a scheme. A geometric point of X is a morphism $\text{Spec}(k) \to X$ where k is an algebraically closed field (separably closed fields also work). It is usually denoted by $\bar{x}$.

Definition 3.3.3 (étale fundamental group)**.** [Sta23, Tag 0BNC]Let X be a connected scheme. Let $\bar{x}$ be a geometric point of X. Let $F_{\bar{x}}$ be the fiber functor $FEt/X \to Set$. Define

$$\pi_1^{\text{ét}}(X, \bar{x}) = Aut(F_{\bar{x}}).$$

The $\pi_1^{\text{ét}}(X, \bar{x})$ is naturally a profinite group. Another choice of geometric point leads to isomorphic fundamental group, although the isomorphism is not canonical. So, sometimes we omit the geometric point in the notation.

Example 3.3.4. Let $\bar{x}$ be the natural geometric point of $X = \text{Spec}(k)$, then $\pi_1^{\text{ét}}(X, \bar{x}) = G_k$.

In general, for a connected scheme X/k and a geometric point $\bar{x}$ of $X_{\bar{k}}$ (or X, they are the same), we put $\pi_1^{geom}(X, \bar{x}) = \pi_1^{\text{ét}}(X_{\bar{k}}, \bar{x})$.

Lemma 3.3.5 (fundamental exact sequence)**.** *[Sta23, Tag 0BTX] Let X/k be a geometrically connected quasi-compact quasi-separated scheme, then there is a natural exact sequence*

$$1 \to \pi_1^{geom}(X) \to \pi_1^{ét}(X) \to G_k \to 1.$$

Theorem 3.3.6 (Grauert-Remmert)**.** *[Sza09, Theorem 5.7.4][Ray71, Exp. XII, Corollaire 5.2] Let $X/\mathbb{C}$ be a connected scheme locally of finite type. Then the functor $(Y \to X) \mapsto (Y^{an} \to X^{an})$ induces an equivalence of the category FEt/X of finite étale covers of X with that of finite-sheeted covers of X^{an}. Consequently, for every $x \in X(\mathbb{C})$ this functor induces an isomorphism*

$$\pi_1\widehat{(X^{an}, x)} \to \pi_1^{ét}(X, x)$$

where $\hat{\cdot}$ on the left hand side means profinite completion.

Theorem 3.3.7. *[Ser07, Theorem 6.3.3] Let L/k be an extension of two algebraically closed fields of characteristic 0 and V/k be an algebraic variety. Then the base change functor $FEt/V \to FEt/V_L$ is an equivalence of categories. In particular, for any geometric point $\bar{v}'$ of V_L, let $\bar{a}$ be the corresponding geometric point of V, then the natural map $\pi_1^{ét}(V_L, \bar{v}') \to \pi_1^{ét}(V, \bar{v})$ is an isomorphism.*

Proposition 3.3.8. *[EVdGM12, Corollary 10.39] Let k be a field, A/k be an abelian variety, l be a rational prime invertible in k. Then $H_{ét}1(A_{k^s}, \mathbb{Z}_l) = Hom(T_lA, \mathbb{Z}_l)$ as $\mathbb{Z}_l$-modules with continuous G_k-action.*

Theorem 3.3.9 is part of Tate's conjecture, proved by Zarhin (positive characteristic) and Faltings (zero characteristic).

Theorem 3.3.9. *Let K be a finitely generated field, X/K a smooth proper integral variety with a geometric point $\bar{x}$. If l is a rational prime different from the characteristic of K and $\pi_1^{geom}(X, \bar{x})$ has no l-torsion, then the representation $G_K \to GL(H^1_{ét}(X_{K^s}, \mathbb{Q}_l))$ is semisimple. In particular, the identity component of the Zariski closure of the image is a reducitve algebraic group.*

It is conjectured to hold for the Galois representation on $H^k_{\text{ét}}(X_{K^s}, \mathbb{Q}_l)$ for each $k \in \mathbb{N}$.

Proof. We may enlarge K to a finite extension. Let $x \in X$ be the image of $\bar{x}$. We may assume that x is a closed point of X, then $k(x)/K$ is a finite extension. By enlarging K, we may assume $x \in X(K)$. Consder the Albanese map $\alpha_x : (X, x) \to (Alb(X), 0)$ defined over K. It induces an morphism $\pi_1^{geom}(X, \bar{x}) \to \pi_1^{geom}(Alb(X), 0)$ identifying the latter as the free part of the abelianization of the former. It induces further an isomorphism of G_K-modules $H^1_{\text{ét}}(Alb(X)_{K^s}, \mathbb{Q}_l) \to H^1_{\text{ét}}(X_{K^s}, \mathbb{Q}_l)$. By Proposition 3.3.8 the first as a G_K-module is dual to the Tate moduel $V_l Alb(X)$. The result follows from [Fal83, Satz 3] in the number field case and [MB85, Théorème 2.5, pp.244–245, Ch XII] in positive characteristic case. $\square$

3.4 Crystalline cohomology

Assume that p is a rational prime and $(S, \mathcal{I}, \gamma)$ is a divided power scheme over $\mathbb{Z}_{(p)}$. Set $S_0 = V(\mathcal{I}) \subset S$. Suppose that p is locally nilpotent on an S_0-scheme X.

Definition 3.4.1 (divided power thickening)**.** [Sta23, Tag 07I4]Let $U \to T$ be a thickening of schemes and let $\mathcal{J}$ be the corresponding ideal sheaf. If $(T, \mathcal{J}, \gamma)$ is a divided power scheme, we call the triple (U, T, γ) a divided power thickening.

Definition 3.4.2 (big crystalline site)**.** [Sta23, Tag 07IB]A divided power thickening of X relative to $(S, \mathcal{I}, \gamma)$ is a divided power thickening (U, T, δ) with an S-morphism $U \to X$. All of them form a category, denoted by $CRIS(X/S, \mathcal{I}, \gamma) = CRIS(X/S)$. A family $\{(U_i, T_i, \delta_i) \to (U, T, \delta)\}$ is a covering if $U_i = T_i \times_T U$ for all i and $\{T_i \to T\}$ is a Zariski covering.

Definition 3.4.3 (small crystalline site)**.** [Sta23, Tag 07IG]The full subcategory $Cris(X/S)$ of $CRIS(X/S)$ consisting of those $(U, T, \delta) \in CRIS(X/S)$ such that $U \to X$ is an open immersion, endowed with the induced (Zariski) topology.

Let $\mathbb{F}$ be a *perfect* field with char($\mathbb{F}$) = p > 0. Let $W = W(\mathbb{F})$ be the ring of Witt vectors of $\mathbb{F}$ and $W_n = W/p^n$. For an $\mathbb{F}$-scheme X, we denote $H^j_{\text{cris}}(X/W_n) = H^j(Cris(X/W_n))$ and $H^j_{\text{cris}}(X/W) = \varprojlim_n H^j_{\text{cris}}(X/W_n)$. This is a graded W-module depending functorially on X.

Lemma 3.4.4. *If $X/\mathbb{F}$ is a smooth proper scheme, then $H^j_{\text{cris}}(X/W(\mathbb{F}))$ is a finite $W(\mathbb{F})$-module. When $j > 2\dim X$, $H^j_{\text{cris}}(X/W(\mathbb{F})) = 0$.*

Let E be a field of characteristic 0 with a complete discrete valuation. Assume its residue field $\mathbb{F}$ is *perfect* and $\mathrm{char}(\mathbb{F}) = \mathrm{p} > 0$. Let φ be the Frobenius endomorphism of W. Denote E_0 the fraction field of W. (In application, we will take E to be a finite unramified extension of $\mathbb{Q}_p$, then $W = O_E$, $\varphi = \sigma_{E/\mathbb{Q}_p}|_{O_E}$ and $E_0 = E$.) Let X/E be a proper smooth variety.

Definition 3.4.5 (Good reduction, [GM87, B.1.1], [Dal06, p.1])**.** The variety X/E has good reduction provided there exists a smooth proper O_E-scheme $\mathcal{X}$ such that the generic fiber $\mathcal{X}_E$ is E-isomorphic to X. In that case, the special fiber $\bar{\mathcal{X}}$ is a smooth proper variety over $\mathbb{F}$, called a reduction of X. Similarly, given a smooth proper variety Y over a number field K and a finite place v of K, we call Y has good reduction at v if $Y \otimes_K K_v$ has good reduction in the above sense.

We consider two particular cases. When A/E is an abelian variety, then it has good reduction if and only if its the identity component of its Néron model is an abelian scheme over O_E. When C/E is a smooth projective curve with $H^0(C, O_C) = E$ and positive genus, then it has good reduction if and only if its minimal model (provided by [Sta23, Tag 0C6B]) is smooth over O_E.

Proposition 3.4.6. *If X has good reduction over E, then so does its Albanese variety.*

Proof. It follows from Fact 2.3.2 and the Néron–Ogg–Shafarevich criterion. (But the converse is not true.) □

Theorem 3.4.7 (finiteness of bad reduction)**.** *If Y is a smooth proper variety over a number field K, then there is a finite set of (finite) places of K, outside which Y has good reduction.*

Theorem 3.4.8 (Fontaine, Abrashkin)**.** *[Fon85, Corollaire, p.517][Abr87] There is no abelian variety over $\mathbb{Q}$ with everywhere good reduction.*

Assume X has good reduction. We define its crystalline cohomology to be $H^j_{\mathrm{cris}}(X) := E_0 \otimes_W H^j_{cris}(\bar{\mathcal{X}}/W)$. Then $H^j_{\mathrm{cris}}(X)$ is equipped with a bijective φ-semilinear endomorphism ϕ induced by the absolute Frobenius of $\bar{\mathcal{X}}$ and functoriality of H^j_{cris}. (By [GM87, Corollary B.3.6], $H^j_{\mathrm{cris}}(X)$, together with the Gauss-Manin connection and the crystalline Frobenius, is independent of the model $\mathcal{X}$.)

3.5 Comparison theorems

Theorem 3.5.1. *[Gro66a, Theorem 1][Gro68, Theorem 1.2, p.310][HMS17, Proposition 4.1.7, p.101] Let X be an algebraic variety over $\mathbb{C}$ (neither smoothness nor properness are required), then analytification induces a linear isomorphism $H^*_{dR}(X/\mathbb{C}) \to H^*_{\mathrm{dR}}(X^{an})$.*

The singular cohomology groups are usually called Betti cohomology.

Lemma 3.5.2 (Relative holomorphic Poincaré lemma)**.** *Let $f : X \to Y$ be a smooth morphism of complex analytic spaces. Then the kernel of $d_{X/Y} : O_X \to \Omega^1_{X/Y}$ is the sheaf theoretic inverse image $f^{-1}O_Y$ and the complex $\Omega^*_{X/Y}$ is exact in all higher degrees.*

In other words, $f^{-1}O_Y$ is quasi-isomorphic to the de Rham complex $(\Omega^*_{X/Y}, d)$.

Corollary 3.5.3 (analytic de Rham)**.** *Let M be a complex manifold, then $H^q_{\mathrm{dR}}(M)$ is canonically isomorphic to $H^q(M;\mathbb{C})$ (the sheaf cohomology or singular cohomology) for all $q \geq 0$.*

Theorem 3.5.4 (Artin)**.** *[AGV73, Théorème 4.4, Exposé XI] Let $X/\mathbb{C}$ be a smooth scheme, then the canonical morphism $H^q_{ét}(X, \mathbb{Q}_l) \to H^q_{sing}(X^{an}, \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}_l$ is an isomorphism for all $q \geq 0$.*

Theorem 3.5.5 (Comparison of Hodge structures, [Del82, Theorem 1.4])**.** *If $X/\mathbb{C}$ is a smooth proper variety, then the canonical morphism $X^{\mathrm{an}} \to X_{\mathrm{Zar}}$ of ringed spaces induces an isomorphism $H^*_{dR}(X/\mathbb{C}) \to H^*_{sing}(X^{an};\mathbb{C})$ under which the Hodge filtration $Fil^i H^n_{\mathrm{dR}}(X/\mathbb{C})$ corresponds to $\oplus_{p\geq i, p+q=n} H^{p,q}(X^{an};\mathbb{C})$.*

Proof. By Lemma 3.5.2, the complex

$$0 \to \underline{\mathbb{C}}_{X^{an}} \to \Omega^*_{X^{an}}$$

is exact. $H^j_{dR}(X^{an}, \mathbb{C}) = H^j(X^{an}, \underline{\mathbb{C}}_{X^{an}})$ equals the hypercohomology $\mathbb{H}*(X^{an}, \Omega^*_{X^{an}})$ and the latter equals $\mathbb{H}^j(X, \Omega^*_{X/\mathbb{C}}) = H^j_{dR}(X/\mathbb{C})$ by GAGA. $\square$

Theorem 3.5.6. *[BO83, Corollary 2.5][Ber06, p.24] Let V be a complete discrete valuation ring of mixed characteristic $(0,p)$ with fraction field K and perfect residue field k. Let $\mathcal{X}$ be a smooth proper V-scheme, with generic* *fiber X/K and special fiber X_0/k. Then there exists a canonical K-linear isomorphism*

$$\sigma_{\text{cris}} : H^i_{dR}(X/K) = H^i_{dR}(\mathcal{X}/V) \otimes_V K \to H^i_{\text{cris}}(X_0/W) \otimes_W K,$$

where $W = W(k)$ is a subring of K.

By Lemma 3.1.10, $H^i_{dR}(\mathcal{X}/V) \otimes_V K = H^i_{dR}(X/K)$. Thus, if one takes $E/\mathbb{Q}_p$ a finite unramified extension and $V = O_E$, then $H^i_{dR}(X/E)$ is equipped with a $\sigma_{E/\mathbb{Q}_p}$-semilinear operator ϕ (called the *crystalline Frobenius operator*). Note that $\phi^{[E:\mathbb{Q}_p]}$ is E-linear.

3.6 Variation of Hodge structures and Period maps

Definition 3.6.1 (relative algebraic (holomorphic) connection)**.** [BE13, Definition 1.17 (i)] Let $f : X \to Y$ be a morphism of schemes (complex analytic spaces), and let $\mathcal{E}$ be a vector bundle on X. A *connection* on $\mathcal{E}$ relative to Y is an $f^{-1}O_Y$-linear map of sheaves $\nabla : \mathcal{E} \to \Omega^1_{X/Y} \otimes_{O_X} \mathcal{E}$ satisfying Leibniz rule: for g a section of O_X and s a section of $\mathcal{E}$,

$$\nabla(gs) = d_{X/Y}(g) \otimes s + g \cdot \nabla(s).$$

A relative connection ∇ is called integrable/flat if its relative curvature form R^∇ vanishes.

When Y is a point, we recover Definition 2.1.4.

Example 3.6.2 (Analytic GM connection)**.** Let $f : X \to Y$ be a smooth proper morphism of complex analytic spaces. Then $R^j\pi_*\underline{\mathbb{C}}_X$ is a $\mathbb{C}$-local system on Y (*cf.* [Vir21, Proposition 2]).

$$\begin{tikzcd}
O_Y \otimes_{\mathbb{C}} R^j f_*\underline{\mathbb{C}} \arrow[r, "d_Y \otimes Id"] \arrow[d, "\simeq"] & \Omega^1_Y \otimes_{\mathbb{C}} R^j f_*\underline{\mathbb{C}} \arrow[d, "\simeq"] \\
\mathcal{H}^j_{dR}(X/Y) \arrow[r, "\nabla_j"] & \Omega^1_Y \otimes_{O_Y} (O_Y \otimes_{\mathbb{C}} R^j f_*\underline{\mathbb{C}})
\end{tikzcd}$$

The vertical isomorphism on the left is given by Proposition 3.1.14. The *integrable* connection on $\mathcal{H}^j_{dR}(X/Y)$ corresponding to $d_Y \otimes Id$ is called the *complex analytic Gauss-Manin connection.* By Lemma 3.5.2, the local system $R^j\pi_*\underline{\mathbb{C}}_X = \ker(\nabla_j)$ is the sheaf of germs of horizontal sections of $\mathcal{H}^j_{dR}(X/Y)$. When Y is a complex manifold, this is an example of Theorem 2.1.7.

Definition 3.6.3 (Variation of complex Hodge structures(=$\mathbb{C}$-VHS), [Gri70, sec 2,p.232], [CMSP17, Definition 4.6.1, p.156], [PS03, p.189], [SS22, Definition 4.1.4, 4.1.5])**.** Let S be a connected complex manifold and $w \in \mathbb{Z}$. A variation of $\mathbb{C}$-Hodge structure of weight w on S refers to the following data:

1. a $\mathbb{C}$-local system $\mathcal{L}$ on S with the corresponding object $(E, \nabla) \in DE(S)$ given by Theorem 2.1.7;

2. opposite filtrations: two finite decreasing filtrations F' (called the Hodge filtration) and $\overline{F''}$ of E, such that for each $k \in \mathbb{Z}$, F'^k and $\overline{F''^k}$ are (holomorphic) vector subbundles of E (i.e., of holomorphic variation) and such that at each $s \in S$, the $\mathbb{C}$-vector space E_s with the filtrations $F_s'^*, F_s''^*$ is a $\mathbb{C}$-Hodge structure of weight w;

3. Griffiths transversality: $\nabla(F'^p) \subset F'^{p-1} \otimes \Omega_S^1$ and anti-Griffiths transversality: $\nabla(\bar{F''}^p) \subset \bar{F''}^{p-1} \otimes \Omega_S^1$ for all $p \in \mathbb{Z}$.

The $\mathbb{C}$-VHS above is called real, an $\mathbb{R}$-VHS for short, if there is an $\mathbb{R}$-local system $\mathcal{L}_{\mathbb{R}}$ on S such that $\mathcal{L}_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \mathcal{L}$ and we have Hodge symmetry: for all $k \in \mathbb{Z}$, $F''^k = \bar{F'}^k$. Then the fiber at each point $x \in S$ is an $\mathbb{R}$-Hodge structure.

The $\mathbb{R}$-VHS above is called rational, a $\mathbb{Q}$-VHS for short, if there is a $\mathbb{Q}$-local system $\mathcal{L}_{\mathbb{Q}}$ on S such that $\mathcal{L}_{\mathbb{Q}} \otimes_{\mathbb{Q}} \mathbb{R} = \mathcal{L}_{\mathbb{R}}$.

For a $\mathbb{C}$-VHS, by condition 2 in Definition 3.6.3, when p large enough, $F'^p = 0$, $F''^p = 0$; when $-p$ is large enough, then $F'^p = E$ and $F''^p = \bar{E}$. The natural map from

$$E^{p,w-p} := F'^p \cap F''^{w-p} \tag{4}$$

to the graded piece $Gr^p E = \frac{F'^p}{F'^{p+1}}$ is an isomorphism of smooth vector subbundles of $E \otimes_{O_S} \Omega_S^1$. And we have a global Hodge decomposition

$$E \otimes_{O_S} \Omega_S^1 = \oplus_{p \in \mathbb{Z}} E^{p,w-p}. \tag{5}$$

Remark 3.6.4. One can also use smooth vector bundle with integrable connection to define $\mathbb{C}$-VHS, see [SS22, Lem. 4.1.6]

We proceed to define period mappings, central objects in this note.

Given a $\mathbb{C}$-VHS $(\mathcal{L}, E, F'^*)$ on a connected complex manifold S with a base point p, we take a simply connected open neighborhood $\Omega \subset S$ of p. As the connection ∇ is integrable, by Theorem 2.1.2 for any $x \in \Omega$ we have a parallel transport ($\mathbb{C}$-linear) isomoprhism $E(x) \to E(p)$ between *fibers* of E. (Since Ω is simply connected, this isomorphism does not depend on the chosen curve from x to p used in the parallel transport.) The Hodge filtration of the fiber $E(x)$ is sent to a filtration of $E(p)$, which has the same dimension data as (but in general is different from) the Hodge filtration of $E(p)$. This (local) variation of Hodge filtrations is encoded in the period map.

We define period maps as follows. Consider the smooth projective variety $H/\mathbb{C}$ (so-called flag variety) parametrizing the filtrations of the $\mathbb{C}$-vector space $\mathcal{L}_p = \mathcal{H}_p$ with same dimension data as its Hodge filtration. (Explicitly, the $\mathbb{C}$-linear algebraic group $GL(\mathcal{L}_p)$ acts transitively on the set of such filtrations. Let $St \leq GL(\mathcal{L}_p)$ be the closed subgroup of the stabilizer of the Hodge filtration of $\mathcal{L}_p$. Then St is a parabolic subgroup of $GL(\mathcal{L}_p)$ and $H = GL(\mathcal{L}_p)/St$.) Then we get a *holomorphic* map

$$\Phi : \Omega \to H^{\mathrm{an}},$$

where $\Phi(x) \in H^{\mathrm{an}}$ represents the filtration corresponding to the Hodge filtration on $E(x)$.

To globalize, consider the universal cover $\pi : \tilde{S} \to S$ and pull the $\mathbb{C}$-VHS back over $\tilde{S}$, then we get a holomorphic map

$$\tilde{\Phi} : \tilde{S} \to H^{\mathrm{an}} \tag{6}$$

fitting to a commutative square

$$\begin{tikzcd} p^{-1}(\Omega) \arrow[r, hook] \arrow[d, "p"] & \tilde{S} \arrow[d, "\tilde{\Phi}"] \\ \Omega \arrow[r, "\Phi"] & H^{\mathrm{an}}_{\mathbb{C}}. \end{tikzcd}$$

We call Φ a local period map and $\tilde{\Phi}$ the global period map. Recall that the monodromy ($\mathbb{C}$-)representation

$$\pi_1(S, p) \to GL(\mathcal{L}_p)$$

given by Proposition 2.1.1. Let Γ be the Zariski closure of the image, which is a $\mathbb{C}$-linear algebraic group. Then there is a natural morphism $\Gamma \times H \to H$ defining a group action. Moreover, $\tilde{\Phi}$ is $\pi_1(S, p)$-equivariant ([Sch73, (3.25)]) and we get the Griffiths' period mapping [Sch73, (3.26)] $S \to \pi_1(S, p)\backslash H^{an}$. Therefore, the image of the global period map is bounded below by monodromy.

Lemma 3.6.5. *[LV20, Lemma 3.1]For every $\mathbb{C}$-VHS on a complex manifold S, The analytic Zariski closure Z of $\Phi(\Omega)$ inside $H_{\mathbb{C}}$ coincides with that of $\tilde{\Phi}(\tilde{S})$, hence independent of choice of Ω. Moreover, Z contains $\Gamma \cdot \Phi(p)$.*

Proof. As $\tilde{\Phi}$ is holomorphic, $\tilde{\Phi}^{-1}(Z)$ is an analytic subset of $\tilde{S}$ containing the non-empty open $\pi^{-1}(\Omega)$. In particular, $\tilde{\Phi}^{-1}(Z)$ is not rare in $\tilde{S}$. By Proposition 2.0.1 $\tilde{\Phi}^{-1}(Z) = \tilde{S}$. Therefore, Z contains $\tilde{\Phi}(\tilde{S})$. Note that $\tilde{\Phi}(\tilde{S}) \supset \Phi(\Omega)$, so Z is the analytic Zariski closure of $\tilde{\Phi}(\tilde{S})$. As the global period map (6) is $\pi_1(S,p)$-equivariant, the image $\tilde{\Phi}(\tilde{S})$ contains $\pi_1(S,p)\cdot\Phi(p)$. The preimage of Z under the morphism $\mathrm{GL}(\mathcal{L}_p) \to H_{\mathbb{C}}$ of algebraic varieties contains the image of $\pi_1(S,p)$ and is Zariski closed in $\mathrm{GL}(\mathcal{L}_p)$. Thus, Z contains $\Gamma \cdot \Phi(p)$. □

An important class of VHS comes from geometry.

Theorem 3.6.6 (Griffiths)**.** *[Dem96, Theorem 10.9][Voi02, Theorem 10.3] Let $\pi : X \to Y$ be a proper, smooth morphism of complex manifolds. Suppose that for each $t \in Y$, the Hodge to de Rham spectral sequence (3) degenerates at page E_1. For each $k \in \mathbb{N}$, the following data: $\mathbb{Q}$-local system $\mathcal{L}_{\mathbb{Q}} := R^k\pi_*\mathbb{Q}$ inducing the holomorphic vector bundle $E = \mathcal{H}^k_{\mathrm{dR}}(X/Y)$ with the (analytic) Gauss-Manin connection $\nabla : \mathcal{H}^k_{\mathrm{dR}}(X/Y) \to \mathcal{H}^k_{\mathrm{dR}}(X/Y) \otimes \Omega^1_Y$ as well as the Hodge filtration $Fil^*H^k(X_t,\mathbb{C}) \subset H^k(X_t,\mathbb{C})$ (at all $t \in Y$) form a $\mathbb{Q}$-VHS of weight k on Y. (Modulo torsion, $R^kf_*\mathbb{Z}$ gives integral VHS.) The smooth subbundle $E^{p,k-p}$ in (4) underlies the holomorphic vector bundle $R^{k-p}\pi_*\Omega^p_{X/Y}$.*

Remark 3.6.7. The decomposition (5) induces a decomposition $\mathcal{H}_{dR}(X/Y) = \oplus_{0\le p\le k} R^{k-p}\pi_*\Omega^p_{X/Y}$ of smooth vector bundles. In general, this decomposition does not hold in the sense of holomorphic vector bundles, see [Dem96, p.59].

Now we turn to algebraic Gauss-Manin connection.

Theorem 3.6.8 (Katz-Oda, existence of Gauss-Manin connection)**.** *Let S be a scheme and $\pi : X \to Y$ be a* smooth *S-morphism of* smooth *S-schemes. Then for each $q \in \mathbb{N}$, there exists a canonical integrable connection relative to S*

$$\nabla = \nabla_q : \mathcal{H}^q_{dR}(X/Y) \to \Omega^1_{Y/S} \otimes_{O_Y} \mathcal{H}^q_{dR}(X/Y).$$

The connection ∇ is compatible with the cup product in the sense that

$$\nabla(e \cdot e') = \nabla(e)\cdot e' + (-1)^q e \cdot \nabla(e'),$$

where e and e' are sections of $\mathcal{H}^q_{dR}(X/Y)$ and $\mathcal{H}^{q'}_{dR}(X/Y)$ respectively over an open subset of Y.

The above theorem is a slight generalization of [KO68, Thm. 1] and the proof therein extends to our situation. The complex Ω_X^* has a filtration $L^r\Omega_X^* = \pi^*\Omega_Y^r \otimes \Omega_X^{*-r}$. The short exact sequence of O_X-modules

$$0 \to \pi^*\Omega_Y^1 \otimes F^{p-1}\Omega_X^{*-1} \to F^p\Omega_X^* \to F^p\Omega_{X/Y}^* \to 0$$

induces a long exact sequnece of O_Y-modules with connected morphims $R^q\pi_*F^p\Omega_{X/Y}^* \to \Omega_Y^1 \otimes F^{p-1}R^q\pi_*\Omega_{X/Y}^*$ is the connection. See also [Sta23, Tag 0FMN] and [BP96, Section 2.C]. A result of GAGA type.

Fact 3.6.9 (Deligne, [HT07, Corollary 5.3.9, p.156])**.** *Let $X/\mathbb{C}$ be a smooth integral variety. Let $\mathrm{DE}(X/\mathbb{C})$ be the category of locally free finite rank O_X-modules (i.e., vector bundles) with integrable connection. Let $\mathrm{DE}_r(X/\mathbb{C})$ be the full subcategory of $\mathrm{DE}(X/\mathbb{C})$ comprised of object with regular singularity. Then the analytification functor induces an equivalence of neutralized Tannakian categories*

$$\mathrm{DE}_r(X/\mathbb{C}) \to \mathrm{DE}(X^{\mathrm{an}}).$$

If X is proper, then $\mathrm{DE}_r(X/\mathbb{C}) = \mathrm{DE}(X/\mathbb{C})$.

If $S = \mathrm{Spec}(\mathbb{C})$ and π is further proper, we see that $\mathcal{H}_{dR}^q(X/Y)$ with algebraic Gauss-Manin connection is naturally a D_Y-module by Theorem 3.1.11 and [HT07, Theorem 1.4.10].

By [BP96, Theorem 2.1], the analytic and algebraic Gauss-Manin connections are compatible when π is furthermore proper.

Corollary 3.6.10. *Let $\pi : X \to Y$ be a smooth proper morphism of smooth $\mathbb{C}$-algebraic varieties with Y irreducible, then for each $k \in \mathbb{N}$, $R^k\pi_*\mathbb{Q}_{X^{an}}$ carries a natural $\mathbb{Q}$-VHS of weight k on Y^{an}.*

Proof. Combine Theorem 3.1.11 and Theorem 3.6.6. □

3.7 p-adic Hodge theory

Let $E/\mathbb{Q}_p$ be a finite unramified extension and $\sigma = \sigma_{E/\mathbb{Q}_p}$ the Frobenius on E. See [BC09, Def. 9.1.4] for the crystalline period ring B_{cris} of E. It is an integral domain containing the maxima unramified extension of $\mathbb{Q}_p$ and carries an action of G_E, an injective Frobenius operator $\phi : B_{\mathrm{cris}} \to B_{\mathrm{cris}}$ ([BC09, Thm. 9.1.8]) and a non-increasing, exhausting and separated filtration $(\mathrm{Fil}^i B_{\mathrm{cris}})_{i\in\mathbb{Z}}$ stable under G_E.

Definition 3.7.1 (Filtered ϕ-module, [BC09, 6.2; 7.3.1; 7.3.4], [Dia17, p.11])**.** If V is a finite dimensional E-vector space equipped with $\phi : V \to V$ a σ-semilinear *bijection* and a decreasing exhaustive and separated filtration[2] Fil^*V, then we call the triple (V, ϕ, Fil^*V) a filtered ϕ-module. A morphism is required to preserve ϕ and compatible with the filtrations. Let MF_E^{ϕ} be the category of filtered ϕ-modules over E.

Note here we require no compatibility of Fil^*V with ϕ. Given two objects $(V, \phi, F^*V), (V', \phi', F^*V')$, we define their tensor product to be $(V \otimes_E V', \phi \otimes \phi', G^*(V \otimes_E V'))$, where $G^k(V \otimes_E V')$ is $\sum_{p+q=k} W_{p,q}$, where $W_{p,q}$ is the image of the natural map $F^pV \otimes_E F^qE' \to V \otimes_E V'$. The forget functor $\mathrm{MF}_E^{\phi} \to \mathrm{Vec}_E$ is a fibre functor. We see that MF_E^{ϕ} is a symmetric monoidal category. The pair (V, ϕ) is an isocrystal over E in the sense of [BC09, Def. 7.3.1], and the triple (V, ϕ, Fil^*V) is called a filtered isocrystal in [BC09, Section 7].

Define a functor $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}(G_E) \to \mathrm{MF}_E^{\phi}$ by

$$D_{\mathrm{cris}}(V) := (B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_E},$$

and for every integer i, $\mathrm{Fil}^i D_{\mathrm{cris}}(V) = (\mathrm{Fil}^i B_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V)^{G_E}$.

Definition 3.7.2 (Crystalline representation)**.** [BC09, p.133] A p-adic representation V of G_E is called crystalline if $\dim_E D_{\mathrm{cris}}(V) = \dim_{\mathbb{Q}_p}(V)$.

Denote the full subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_E)$ comprised of crystalline ones by $\mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_E)$.

Proposition 3.7.3 ([BC09, Prop. 9.1.11])**.** *The functor $D_{\mathrm{cris}} : \mathrm{Rep}_{\mathbb{Q}_p}^{\mathrm{cris}}(G_E) \to \mathrm{MF}_E^{\phi}$ is an exact fully faithful tensor functor.*

Theorem 3.7.4 (Fontaine's C_{cris} conjecture, Faltings' crystalline comparison theorem, [Fon82, A.11, p.573], [Fal88, Cor. p.69], [Dal06, p.2], [Hon, Example 3.2.2 (2); Remark to Thm. 1.2.4, p.10])**.** *If $\mathcal{X}/\mathcal{O}_E$ is a smooth proper scheme, let $X = \mathcal{X}_E$, then $H^i_{\acute{e}t}(X_{\bar{E}}, \mathbb{Q}_p) \in Rep_{\mathbb{Q}_p}^{\mathrm{cris}}(G_E)$ and its image under D_{cris} is*

$$(H^i_{dR}(X/E), \phi, \textit{Hodge filtration}),$$

where the crystalline Frobenius ϕ on $H^i_{dR}(X/E)$ is given by Theorem 3.5.6.

[2]Here the dimension data is changing.

4 A family of varieties with good reduction

We assemble some ingredients needed for the method of Lawrence and Venkatesh. The following lemma relates rational points to integral points. It relies on [Gro61, Proposition 7.3.3].

Lemma 4.0.1. *Let Y be an integral scheme of generic point b and function field K. Assume that for any $y \neq b \in Y$, the stalk $O_{Y,y}$ is a valuation ring. Let $f : X \to Y$ be a separated closed morphism.*

1. *If X is an integral scheme, the generic fiber $f^{-1}(b) = \{x\}$ is a singleton and the corresponding homomorphism $k(b) \to k(x)$ is bijective, then f is an isomorphism.*

2. *The natural map $\iota : X(Y) \to X_K(K)$ is a bijection.*

Proof. 1.For any $y \neq b \in Y$, the localization $\mathrm{Spec}(O_{Y,y}) \to Y$ is a topological embedding, so the base change $f_y : X_y := X \times_Y \mathrm{Spec}(O_{Y,y}) \to \mathrm{Spec}(O_{Y,y})$ is closed. The morphism $i_y : X_y \to X$ is also a localization of ring when restricted to affine opens of X, hence i_y is a topological embedding and X_y is reduced. As $x \in X$ is the generic point and $x \in X_y$, the scheme X_y is irreducible. Thus by [Gro61, Propositon 7.3.1], f_y is an isomorphism. Therefore f is an isomorphism.

2.By [Gro60, Ch I, Corollaire 5.4.7], ι is injective. Let's show ι is surjective.

Take any $x \in X_K(K)$. Since f is separated, so is $X_K \to \mathrm{Spec}(K)$. Then $x \in X_K$ is a closed point by [Gro60, Ch I, Corollaire 5.4.6]. Let X' be the closure of $x \in X$ with reduced induced scheme structure. Then X' is integral. The restricted morphism $X' \to Y$ is closed and separated, which is an isomorphism by 1. The inverse $Y \to X'$ followed by $X' \to X$ is in $\iota^{-1}(x)$. We conclude that ι is bijective. $\square$

A rough picture for the presented proof of Mordell's conjecture is summarized below: Given a smooth $\mathcal{O}_S$-scheme $\mathcal{Y}$, our aim is to show the finiteness of $\mathcal{Y}(\mathcal{O}_S)$ in view of Lemma 4.0.1. In stages:

- Construct a smooth proper morphism $\pi : \mathcal{X} \to \mathcal{Y}$.

- Show the semisimplicity of the global Galois representation $\rho_y : G_K \to GL\big(H^q_{\text{ét}}((X_y)_{\overline{K}}, \mathbb{Q}_p)\big)$ for "most" $y \in \mathcal{Y}(\mathcal{O}_S)$. Lemma 2.3.6 ensures that for such y, ρ_y lies in only finitely many isomorphism classes. We are going to show there are only finitely many y for each class.

- Choose a place v and use Gauss-Manin connection to construct a v-adic period map which encodes the variation of the Galois representations ρ_y.
- Calculate monodromy to give a lower bound of the image of v-adic period map, i.e. show that Hodge structure indeed varies.
- Control the centralizer of Frobenius operator to give an upper bound of the image of integral points under the v-adic period map.

In this section we consider a proper smooth morphism $\Pi : \mathcal{X} \to \mathcal{Y}$ of smooth separated $\mathcal{O}_S$ schemes, whose base change to K is denoted by $\pi : X \to Y$. Fix $y_0 \in \mathcal{Y}(\mathcal{O}_S) \subset Y(K)$. Let X_{y_0} be the fiber above y_0, which is a smooth proper K-variety that has good reduction at every finite place of K outsider S. Let $\mathcal{V} := H^q_{dR}(\mathcal{X}_{y_0}/\mathcal{O}_S)$ and $V := H^q_{dR}(X_{y_0}/K) = \mathcal{V} \otimes_{\mathcal{O}_S} K$. Since $\mathcal{Y}$ is separated over $\mathcal{O}_S$, the map $y_0 : \mathrm{Spec}(\mathcal{O}_S) \to \mathcal{Y}$ is a closed immersion. Let $\overline{y_0}(\in \mathcal{Y}(\mathbb{F}_v)) : \mathrm{Spec}\mathbb{F}_v \to \mathrm{Spec}(\mathcal{O}_S) \to \mathcal{Y}$ denote the reduction modulo v and $y_0^{\#} : \mathcal{O}_{\mathcal{Y},\overline{y}} \to \mathcal{O}_{(v)}$ the local surjection.

4.1 Comparison of connections

By Theorem 3.1.11, $R^q\pi_*\Omega^p_{\mathcal{X}/\mathcal{Y}}$ and $\mathcal{H}^q_{dR}(\mathcal{X}/\mathcal{Y})$ are finite locally free $O_{\mathcal{Y}}$-modules. As the morphism Π is smooth of smooth $\mathcal{O}_S$ schemes, Theorem 3.6.8 provides us with the Gauss-Manin connection

$$\nabla : \mathcal{H}^q_{dR}(\mathcal{X}/\mathcal{Y}) \to \mathcal{H}^q_{dR}(\mathcal{X}/\mathcal{Y}) \otimes_{O_{\mathcal{Y}}} \Omega^1_{\mathcal{Y}/\mathcal{O}_S}.$$

Fix an archimedean place $\iota : K \to \mathbb{C}$ and a finite place v of K satisfying:

- if p is the rational prime below v, then $p > 2$;
- v is unramified in $K/\mathbb{Q}$;
- no place of K above p lies in S.

In fact, the condition $p > 2$ is here just to simplify notation. Once for all, we fix an isomorphism $\mathbb{C}_{K_v} \to \mathbb{C}$. For any K-scheme Z, we denote interchangeably by $Z_{\mathbb{C}}$ or Z_ι its base change to $\mathbb{C}$ via ι.

Let's show that the Gauss-Manin connection is defined by power series on K.

Take a basis $\{v_1, \ldots, v_r\}$ of the free $O_{\mathcal{Y},\overline{y_0}}$-module $\mathcal{H}^q_{dR}(\mathcal{X}/\mathcal{Y})_{\overline{y_0}}$. On some open neighborhood of $\overline{y_0} \in \mathcal{Y}$, $\{v_1, \ldots, v_r\}$ is defined and form a local basis. Write $\nabla v_i = \sum_j A_i^j v_j$, where the A_i^j are local sections of $\Omega^1_{\mathcal{Y}/\mathcal{O}_S}$ near $\bar{y}_0$.

Since $\mathcal{Y}$ is smooth over $\mathcal{O}_S$, $O_{\mathcal{Y},\overline{y_0}}$ is a Noetherian regular local ring. Also $\mathcal{O}_{(v)}$ is a discrete valuation ring. There exists a regular system of parameters $\{z_0, z_1, \ldots, z_m\}$ of $O_{\mathcal{Y},\overline{y_0}}$ such that $\{z_1, \ldots, z_m\}$ generate the kernel of $y_0^{\#}$. The completion of $O_{\mathcal{Y},\overline{y_0}}$ is $\mathcal{O}_v[[z_1, \ldots, z_m]]$ in which $O_{\mathcal{Y},\overline{y_0}}$ is included in $\mathcal{O}_{(v)}[[z_1, \ldots, z_m]]$.

$$\begin{tikzcd}
\mathcal{O}_{\mathcal{Y},\overline{y_0}} \arrow[r, hook] \arrow[d, hook] & \mathcal{O}_{(v)}[[z_1, \ldots, z_m]] \arrow[r, hook] \arrow[d, hook] & K[[z_1, \ldots, z_m]] \arrow[d, hook] & \\
\widehat{\mathcal{O}_{\mathcal{Y},\overline{y_0}}} \arrow[r, "="] & \mathcal{O}_v[[z_1, \ldots, z_m]] \arrow[r, hook] & K_v[[z_1, \ldots, z_m]] \arrow[r, hook] & \mathbb{C}[[z_1, \ldots, z_m]]
\end{tikzcd}$$

The stalk $A_i^j(\overline{y_0}) = \sum_k a_i^{jk} dz_k$ with

$$a_i^{jk} \in \mathcal{O}_{\mathcal{Y},\overline{y_0}} \subset \mathcal{O}_{(v)}[[z_1, \ldots, z_m]]. \tag{7}$$

(The stalk $\Omega^1_{\mathcal{Y}/\mathcal{O}_S,\overline{y_0}} = \Omega^1_{\mathcal{O}_{\mathcal{Y},\overline{y_0}}/\mathcal{O}_{(v)}}$.) Note that the a_i^{jk} are power series with coefficients in K!

Then we consider the existence of horizontal sections.

A local section of $\mathcal{H}^q_{dR}(\mathcal{X}/\mathcal{Y})$ near $\bar{y}_0$ writes as $f = \sum_\alpha f^\alpha v_\alpha$, where f^α are local sections of $O_{\mathcal{Y}}$. Recall that a local section f satisfying the flat equation $\nabla f = 0$ is called horizontal. In local basis, the equation expand as a linear system of differential equations

$$df^\alpha + \sum_\beta A^\alpha_\beta f^\beta = 0; \quad \alpha = 1, 2, \ldots, r. \tag{8}$$

By integrability of the Gauss-Manin connection, for any K-initial condition we obtain a formal solution $(f^1, \ldots, f^r) \in K[[z_1, \ldots, z_m]]$ to (8) (compare to Theorem 2.1.5).

Lemma 4.1.1 (Picard-Lindelöf method)**.** *[LV20, p.915] Any K_v-formal solution $\underline{f}$ is v-adically absolutely convergent on $\{\underline{z} \in K_v^m : |z_i|_v < |p|_v^{1/(p-1)}\}$. If the initial value is in O_L, then it remains true over this disk.*

Proof. For any given initial condition $(f^1(0), \ldots, f^r(0)) = (a^1, \ldots, a^r) \in K_v^r$, let $f^\alpha = \sum_{I \in \mathbb{N}^m} a^\alpha_I z^I$ be the unique *formal* solution with coefficients in K_v.

We show that $\max_\alpha |a_I^\alpha| \leq \frac{C}{|I!|}$, where C is shorthand for $\max_\alpha |a^\alpha|$. by induction on I. It is trivial if $I = \vec{0}$. Assume the inequality for $J \leq I = (i_1, \ldots, i_m)$. We proceed to show that for $I + 1_k = (i_1, \ldots, i_{k-1}, i_k + 1, i_{k+1}, \ldots, i_m)$. Consider equation (8) with $\frac{\partial}{\partial z_k}$, then

$$(1 + i_k) f_{I+1_k}^\alpha + \sum_\beta \sum_{J+L=I, J, L \in \mathbb{N}^m} a_{\beta,J}^\alpha f_L^\beta = 0.$$

Note that for $a_{\beta,J}^\alpha \in O_{(v)}$ by (7), so for each $L \leq I$,

$$|a_{\beta,J}^\alpha f_L^\beta| \leq |f_L^\beta| \leq \frac{C}{|L!|} \leq \frac{C}{|I!|}$$

by induction hypothesis. As $(K_v, |\cdot|)$ is non-archimedean,

$$|f_{I+1_k}^\alpha| \leq \frac{C}{|I!|(i_k + 1)} = \frac{C}{|(I + 1_k)!|}.$$

The induction is completed. Recall that for any $n \in \mathbb{N}^+$, $|n|_v > |p|_v^{-\frac{n}{p-1}}$, so $\frac{1}{|I!|} < |p|_v^{-\frac{|I|}{p-1}}$. Now given such a point $\underline{z}$, for any α,

$$|f_I^\alpha z^I| < C\left(\frac{\max_{1 \leq i \leq m} |z_i|_v}{|p|_v^{\frac{1}{p-1}}}\right)^{|I|}.$$

□

Since $a_i^{jk} \in \mathbb{C}\{z_1, \ldots, z_m\}$, the formal $\mathbb{C}$-solution converges near the origin of $\mathbb{C}^m$ for any initial $\mathbb{C}$-condition.

For the base change $\pi_{\mathbb{C}} : X_{\mathbb{C}} \to Y_{\mathbb{C}}$, the pullback of $\mathcal{H}_{dR}^q(X/Y)$ to $Y_{\mathbb{C}}$ is $\mathcal{H}_{dR}^q(X_{\mathbb{C}}/Y_{\mathbb{C}})$. Its analytification is $\mathcal{H}_{dR}^q(X_{\mathbb{C}}^{an}/Y_{\mathbb{C}}^{an})$. Thus, the pullback of $\{v_1, \ldots, v_r\}$ to $Y_{\mathbb{C}}^{an}$ is a local basis of $\mathcal{H}_{dR}^q(X_{\mathbb{C}}^{an}/Y_{\mathbb{C}}^{an})$ around $y_0 \in Y^h$. The analytification of the algebraic Gauss-Manin connection ${}_{\mathbb{C}}\nabla : \mathcal{H}_{dR}^q(X_{\mathbb{C}}/Y_{\mathbb{C}}) \to \mathcal{H}_{dR}^q(X_{\mathbb{C}}/Y_{\mathbb{C}}) \otimes_{O_Y} \Omega_{Y_{\mathbb{C}}/\mathbb{C}}^1$ is the analytic counterpart $\nabla^{an} : \mathcal{H}_{dR}^q(X^{an}/Y^{an}) \to \mathcal{H}_{dR}^q(X^{an}/Y^{an}) \otimes_{O_{Y^{an}}} \Omega_{Y^{an}}^1$. Therefore, the formal solution $(f^1, \ldots, f^r)$ induces a local basis of $\mathcal{H}_{dR}^q(X_{\mathbb{C}}^{an}/Y_{\mathbb{C}}^{an})$ near y_0 consisting of horizontal sections. By Lemma 4.1.1, the argument carries over mutatis mutandis to the K_v-analytic setting and results in a local basis of $\mathcal{H}_{dR}^q(X_v^{an}/Y_v^{an})$ on the chart disk $\{\underline{z} \in K_v^m : |z_i|_v < |p|_v^{1/(p-1)}\}$ centered at y_0 (the chart induced by $y_0 \in$

$\mathcal{Y}(O_v)$) consisting of horizontal sections. The ι-adic and v-adic local bases are *represented by the same power series* in K with respect to $\{v_1, \dots, v_r\}$.

Now we use the horizontal sections to identify nearby fibers in both v-adic and ι-adic settings via parallel transport.

On $Y_{\mathbb{C}}^{an}$, $R^q\pi_*^h\underline{\mathbb{C}}_{X^h}$ is a local system (Proposition 3.1.16). The fiber at $y \in Y(\mathbb{C})$ is $H^q_{\mathrm{sing}}(X_y^{an}, \mathbb{C})$ (singular cohomology) by proper base change theorem [Har11, Theorem 4.4.17]. When $y_a, y_b \in Y(\mathbb{C})$ are sufficiently close in the analytic topology, the complex analytic Gauss-Manin connection induces an identification

$$GM : H^q_{sing}(X^{an}_{y_a}; \mathbb{C}) \to H^q_{sing}(X^{an}_{y_b}; \mathbb{C}). \tag{9}$$

Consider the v-adic analogue. On Y_{K_v}, $R^q\pi_*\underline{K_v}$ is a local system. The fiber at $y \in Y(K_v)$ is $H^q_{\mathrm{dR}}(X_y/K_v)$. By the choice of v, $p > 2$ and $|p|_v^{1/(p-1)} > |p|_v$. Thus by Lemma 4.1.1, for any $y_1, y_2 \in \mathcal{Y}(\mathcal{O}_v)$ with $y_1 \equiv y_2$ modulo v, we obtain an identification [LV20, (3.7)]:

$$GM : H^q_{dR}(X_{y_1}/K_v) \to H^q_{dR}(X_{y_2}/K_v).$$

The fiber over $y \in \mathcal{Y}(\mathcal{O}_v)$ is a smooth proper $\mathcal{O}_v$-model $\mathcal{X}_y$ for X_y/K_v.

$$\begin{tikzcd}
X_y \arrow[r] \arrow[d] & \mathcal{X}_y \arrow[d] & \overline{\mathcal{X}_y} \arrow[l] \arrow[d] \\
\mathrm{Spec}(K_v) \arrow[r] & \mathrm{Spec}(\mathcal{O}_v) & \mathrm{Spec}(F_v). \arrow[l]
\end{tikzcd}$$

By Theorem 3.5.6, the Frobenius operator ϕ_v on $H^q_{dR}(X_y/K_v)$ is induced by the identification with $H^q_{cris}(\overline{\mathcal{X}_y}) \otimes_{\mathcal{O}_v} K_v$. As $y_1 \equiv y_2$ modulo v, $\overline{\mathcal{X}_{y_1}} = \overline{\mathcal{X}_{y_2}}$.

$$\begin{tikzcd}
H^q_{dR}(X_{y_1}/K_v) \arrow[dd, "\mathrm{GM}"] \arrow[dr, "\sim"] & \\
& H^q_{cris}(\overline{\mathcal{X}_{y_1}}) \otimes_{\mathcal{O}_v} K_v \\
H^q_{dR}(X_{y_2}/K_v) \arrow[ur, "\sim"'] &
\end{tikzcd} \tag{10}$$

The commutativity of Diagram (10) follows from [Ber06, Prop.3.6.1]. Thus the identification GM is a morphism of filtered modules.

The GM identifications in general do not preserve the Hodge filtration. The period maps are tools to study the variation of Hodge structures.

4.2 Monodromy and the period mappings

By Theorem 3.5.5, we get an isomorphism $H^q_{dR}(X_y/\mathbb{C}) \to H^q_{sing}(X_y^{an}, \mathbb{C})$ for $y \in Y(\mathbb{C})$. In particular, $V_{\mathbb{C}} = H^q_{dR}(X_{y_0}/\mathbb{C}) = H^q_{sing}(X^h_{y_0}, \mathbb{C})$. First recall the monodromy representation

$$\mu : \pi_1(Y_{\mathbb{C}}^{an}, y_0) \to GL(V_{\mathbb{C}}). \tag{11}$$

Let Γ be the Zariski closure of the image, which is an algebraic subgroup of $GL(V_{\mathbb{C}})$.

Form the flag scheme $\mathcal{H}/\mathcal{O}_S$ such that $\mathcal{H}(\mathcal{O}_S)$ is the set of flags in $\mathcal{V}$ with the same dimension data as the Hodge filtration defined by fiber over y_0. Note that $\mathcal{H}/\mathcal{O}_S$ is proper and its generic fiber H/K is the usual (projective) flag variety.

We start by the complex case. Let Ω_ι be a simply connected open neighborhood of $y_0 \in Y_{\mathbb{C}}^{an}$. Recall the analytic period maps defined in Section 3.6

$$\Phi_\iota : \Omega_\iota \to \mathcal{H}_\iota^{an} \tag{12}$$

and

$$\tilde{\Phi} : \tilde{Y_{\mathbb{C}}^{an}} \to \mathcal{H}_{\mathbb{C}}(\mathbb{C}), \tag{13}$$

where $p : \tilde{Y^{an}} \to Y^{an}$ denotes the universal cover of Y^{an}.

Put

$$h_0^\iota = \Phi_\iota(y_0). \tag{14}$$

Consider a v-adic analogue. Let $\Omega_v = \{y \in \mathcal{Y}(\mathcal{O}_v) : y \equiv y_0 \text{ modulo } v\}$ be the residue disk centered at a base point $y_0 \in \mathcal{Y}(O_v)$. Then Ω_v is a K_v-analytic manifold in the sense of [Ser09, Part II, Ch. III.2] since $\mathcal{Y}(\mathcal{O}_S) \subset Y_v^{an}$ is open. For any $y \in \Omega_v$, the Gauss-Manin connection (10) induces a K_v-linear isomorphism $H^q_{dR}(X_y/K_v) \xrightarrow{\sim} H^q_{dR}(X_{y_0}/K_v)$ translating the Hodge filtration at y to a filtration with *same* dimension date on V_v (by Theorem.3.1.11). There is a v-adic period map

$$\Phi_v : \Omega_v \to \mathcal{H}(O_v) = H(K_v).$$

The map Φ_v is K_v-analytic. We record a simple observation.

Lemma 4.2.1. *[LV20, Lem. 3.3] If $\mathcal{H}_v^{bad} \subset \mathcal{H}_v$ is a Zariski closed subset and*

$$\dim \mathcal{H}_v^{bad} < \dim \overline{\Phi_v(\Omega_v)},$$

then $\Phi_v^{-1}(\mathcal{H}_v^{bad})$ is a proper K_v-analytic subset of Ω_v. Here $\overline{\Phi_v(\Omega_v)}$ denotes the Zariski closure of $\Phi_v(\Omega_v)$ inside $\mathcal{H}_v$.

When Ω_v is one-dimensional, the result implies further the finiteness of $\Phi_v^{-1}(H_v^{bad})$. In the next subsection we will get lower bounds of $\dim \overline{\Phi_v(\Omega_v)}$.

4.3 v-period$\geq$ $\mathbb{C}$-period $\geq$ monodromy

The title of is an informal way to put the results. The second inequality refers to Lemma 3.6.5.

To compare complex and v-adic period maps, the key is that the flat variety is over K and the two Gauss-Manin connections are defined by common power series with K coefficients (see Section 4.1).

Lemma 4.3.1. *Given a field K of characteristic 0 and power series $B_1, \dots, B_N \in K[[z_1, \dots, z_m]]$, then there exists a closed subvariety $Z \subset A_K^N$ with the following property: For every local field L containing K and $\epsilon > 0$ such that all the B_i are absolutely convergent on $U = U_{\epsilon,L} = \{\underline{z} \in L^m : |z_i| < \epsilon, \forall i\}$, the base change Z_L is the Zariski closure of $\underline{B}(U)$ inside A_L^N, where $\underline{B} : U \to A_L^N$ is the map associated with $B_1, \dots, B_N$.*

Proof. Let $I \subset K[x_1, \dots, x_N]$ be the ideal generated by all polynomials Q such that $Q(B_0, \dots, B_N) = 0$ in $K[[z_1, \dots, z_m]]$ and let $Z \subset A_K^N$ be the corresponding subvariety.

Claim: the vanishing ideal of $L[x_1, \dots, x_N]$ associated to $\underline{B}(U)(\subset A_L^N)$ is
$I \cdot L[x_1, \dots, x_N]$.

In fact, if a polynomial $P \in L[x_1, \dots, x_N]$ vanishes on $\underline{B}(U)$, then $P(B_0, \dots, B_N)$ as a function on U is L-analytic and identically vanishes, so $P(B_0, \dots, B_N) = 0$ in $L[[z_1, \dots, z_m]]$. This is equivalent to an infinite system of linear equation with coefficients in K (the finitely many variables being the coefficients of P). Any L-solution of this linear system is a L-linear combination of K-solutions. The Claim is proved.

So, the Zariski closure of $\underline{B}(U)$ contains Z_L. The reverse inclusion is clear. □

The following lemma links v-adic period map to its complex counterpart.

Lemma 4.3.2. *There is a closed subvariety $W \subset \mathcal{H}$ defined over K such that $W_{\mathbb{C}}$ is the Zariski closure of $\Phi_\iota(\Omega_\iota)$ in $H_{\mathbb{C}}$ and W_{K_v} is contained in the Zariski closure $\overline{\Phi_v(\Omega_v)}$ of $\Phi_v(\Omega_v)$. In particular,*

$$\dim \overline{\Phi_v(\Omega_v)} \geq \dim \Gamma \cdot h_0^\iota.$$

Proof. Take $\epsilon > 0$ small enough such that a formal solution to the flat equation (8) converges absolutely on both $U_{\epsilon,K_v} \subset \Omega_v$ and $U_{\epsilon,\mathbb{C}}$. Recall that the v-adic and ι-adic Gauss-Manin connections are determined by same power series over K. Lemma 4.3.1 implies that the Zariski closures of $\Phi_v(U_{\epsilon,K_v})$ and that of $\Phi_\iota(U_{\epsilon,\mathbb{C}})$ are base changes of a same K-closed subvariety $W \subset \mathcal{H}$. By Lemma 3.6.5, $W_{\mathbb{C}}$ is the Zariski closure of $\Phi_\iota(\Omega_\iota)$ and $\dim W_{\mathbb{C}} \geq \dim \Gamma \cdot h_0^\iota$. $\square$

Note that we may use some variant of the flag variety in Lemma 4.2.1, 3.6.5 and 4.3.2. Below we will use some Lagrangian Grassmannian in the place of $\mathcal{H}$.

4.4 A prototype of arguments

Fix $q \geq 0$. For $y \in \mathcal{Y}(\mathcal{O}_S)$, we denote by ρ_y the global p-adic Galois representation

$$\rho_y : G_K \to GL\big(H^q_{\text{ét}}(X_{y,\overline{K}}, \mathbb{Q}_p)\big).$$

Restricted to G_{K_v}, ρ_y is crystalline by Theorem 3.7.4 (as a good model $\mathcal{X}_y$ exists) and

$$D_{cris}(\rho|_{G_{K_v}}) = (H_{dR}(X_y/K_v), \phi_v, \text{Hodge filtration}).$$

Then Gauss-Manin connection identifies the last triple with $(V_v, \phi_v, \Phi_v(y))$ where $V_v := V \otimes_K K_v = H^q_{dR}(X_{y_0}/K_v)$. (Recall that a morphism in MF_K^ϕ is compatible with ϕ but may not preserve the filtration.)

Here is a sample result.

Lemma 4.4.1. *[LV20, Prop. 3.4] Suppose that*

$$\dim_{K_v}(Z(\phi_v^{[K_v:\mathbb{Q}_p]})) < \dim \Gamma \cdot h_0^\iota,$$

where $Z(-)$ denotes the centralizer of the K_v-linear operator $\phi_v^{[K_v:\mathbb{Q}_p]}$ in $\mathrm{GL}_{K_v}(V_v)$ and h_0^ι is defined in (14). Then the set

$$U_{ss} = \{y \in \mathcal{Y}(\mathcal{O}_S) : y \equiv y_0 \textit{ modulo } v, \rho_y \textit{ semisimple}\}$$

is contained in a proper K_v-analytic subvariety of Ω_v. (Here $\Gamma \cdot h_0^\iota$ is locally closed in the Zariski topology of $H_{\mathbb{C}}$ by [Mil17, p.27].)

Proof. Let S' be the union of S with the places above p of K. For $y \in U_{ss}$ and at a place outside S', the p-adic representation ρ_y of Γ_K is unramified by Fact 2.3.2, pure of weight q with integral characteristic polynomial by Theorem 2.3.5. By Lemma 2.3.6, ρ_y belongs to finitely many isomorphism classes. We may consider $y \in U_{ss}$ with ρ_y in a fixed isomorphism class. Take a representative (V_v, ϕ_v, h) of the D_{cris} image of this fixed class. By Proposition 3.7.3, the triple $(V_v, \phi_v, \Phi_v(y))$ is isomorphic to (V_v, ϕ_v, h) in $\mathrm{MF}^{\phi}_{K_v}$, so $\Phi_v(y) \in Z(\phi_v) \cdot h$. Note that $Z(\phi_v) \subset Z(\phi_v^{[K_v:\mathbb{Q}_p]})$ and $Z(\phi_v^{[K_v:\mathbb{Q}_p]})$ is the K_v-points of a K_v-algebraic subgroup of $GL_{K_v}(V_v)$. Therefore $\Phi_v(U_{ss})$ is contained in a finite union of subsets of H_{K_v} of the form $Z(\phi_v^{[K_v:Q_p]}) \cdot h$, each of which is Zariski-closed subset of $\mathcal{H}_v$ having dimension no greater than $\dim_{K_v}(Z(\phi_v^{[K_v:Q_p]}))$. We conclude by applying Lemma 4.2.1. □

Lemma 4.4.1 appeals us to find upper bound of the centralizer of the Frobenius ϕ_v. To this end, we record the following linear algebra result.

Lemma 4.4.2. *[LV20, Lemma 2.1]Let E be a field and $\sigma : E \to E$ a field automorphism of finite order e, with fixed subfield F. Assume that V is a E-vector space of dimension d with $\phi : V \to V$ is a σ-semilinear[3] bijection. Define $Z_{end}(\phi) = \{f \in \mathfrak{gl}(V) : f\phi = \phi f\}$. It's an F vector space of dimension $\dim_E Z_{end}(\phi^e)$. In particular, $\dim_F Z_{end}(\phi) \leq d^2$.*

4.5 Abelian-by-finite family

We concentrate on the specific type of morphism $\Pi : \mathcal{X} \to \mathcal{Y}$ to be used.

Definition 4.5.1 (relative (smooth proper) curve)**.** Let S be a scheme. A relative curve over S is defined to be a smooth proper morphism $X \to S$ of relative dimension 1 whose geometric fibers are connected curves.

Definition 4.5.2 (abelian scheme)**.** [FP19, Definition 1.1]A smooth proper group scheme $X \to S$ is called an abelian scheme if the geometric fibers are connected.

We call an abelian scheme of relative dimension one an elliptic scheme.

Example 4.5.3 (relative Jacobian)**.** If $C \to S$ is a relative curve, then $Pic^0_{C/S} \to S$ is an abelian scheme.

[3] *i.e.*, ϕ is F-linear and for every $\lambda \in E$, $v \in V$, one has $\phi(\lambda v) = \sigma(\lambda)\phi(v)$

Proposition 4.5.4. *[FGI05, Remark 9.6.22]If $A \to S$ is an abelian scheme, then $Pic^0_{A/S} = Pic^\tau_{A/S}$.*

Let $\pi : A \to S$ be an abelian scheme, then $\hat{\pi} : Pic^0_{A/S} \to S$ is also an abelian scheme, called the dual of π. If π is projective, then so is $\hat{\pi}$ (cf.[MFK94, Corollary 6.8]). Denote $Pic^0_{A/S}$ by $\hat{A}$.

Theorem 4.5.5. *[BBM06, Theorem 5.1.6]Let $\pi : A \to S$ be an abelian scheme. Then there is a canonical isomorphism*

$$\Phi_A : \mathcal{H}^1_{dR}(A/S)^\vee \to \mathcal{H}^1_{dR}(\hat{A}/S).$$

Corollary 4.5.6. *[Wed08, Section 5.1]A polarization $\lambda : X \to \hat{X}$ of an abelian variety X/K induces a symplectic pairing $H^1_{dR}(X/K) \times H^1_{dR}(X/K) \to K$.*

Definition 4.5.7 (polarization)**.** [MFK94, Definition 6.3]Let $\pi : X \to S$ be a projective abelian scheme. A polarization of X is an S-homomorphism $\lambda : X \to \hat{X}$ such that for all geometric points $\bar{s} \in S$, then induced $\overline{\lambda} : X_{\bar{s}} \to \hat{X}\bar{s}$ is a polarization of abelian variety. If λ is further an isomorphism, we call it a principal polarization.

Definition 4.5.8. [LV20, Definition 5.1] An *abelian-by-finite family* over a scheme Y is a sequence of morphisms

$$X \longrightarrow Y' \xrightarrow{\pi} Y$$

where π is finite étale, and $X \to Y'$ is a polarized abelian scheme. A good $\mathcal{O}_S$-model for such a family is an abelian-by-finite family $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ of smooth finite type separated $\mathcal{O}_S$-scheme whose base change to K is $X \to Y' \to Y$ and satisfies the assumptions in the start of Section 4.1.

Remark 4.5.9. In our application, K is a number field, then for any $y \in Y(K)$, ρ_y on $H^1_{\text{ét}}$ is semisimple by Theorem 3.3.9. This deep result is avoided in [LV20] intentionally.

Given an abelian-by-finite family $X \to Y' \to Y$ over K with a good $\mathcal{O}_S$-model $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$, let $E_y = O(Y'_y)$ for $y \in Y(K)$, then the finite étale K-scheme $Y'_y = \mathrm{Spec}E_y$ and $E_y = \prod_{\tilde{y} \in \pi^{-1}(y)} k(\tilde{y})$ is an étale K-algebra. The fiber X_y is a polarized abelian scheme over E_y. Let d be the relative dimension of the abelian scheme $X \to Y'$. For any $\tilde{y} \in \pi^{-1}(y)$, $X_{\tilde{y}}/k(\tilde{y})$

is an abelian variety of dimension d. As a result, $V_y := H^1_{dR}(X_y/K) = H^1_{dR}(X_y/E_y) = \oplus_{\tilde{y}\in\pi^{-1}(y)} H^1_{dR}(X_{\tilde{y}}/k(\tilde{y}))$ is actually a free E_y-module of rank $2d$. The polarization on X_y gives an E_y-bilinear symplectic pairing

$$\omega_y : H^1_{dR}(X_y/K) \times H^1_{dR}(X_y/K) \to E_y.$$

Also $k(\tilde{y})/K$ is unramified outside S of bounded degree (see identification (20) below). By Theorem 2.3.7, up to K-isomorphism, there are finitely many extensions $k(\tilde{y})/K$. There are only finitely many possibilities for the algebras E_y up to isomorphism, or equivalently, for the finite G_K-sets $Y'_y(\bar{K})$. The Frob_v-orbits of $Y'_y(\bar{K})$ are in bijection with pairs (y', w) where $y' \in \pi^{-1}(y)$ and $w|v$ a place of $k(y')$. The (y', w) orbit has $[k(y')_w : K_v]$ elements.

Let v be a place of K as in the start of Section 4.1. Fix $y_0 \in \mathcal{Y}(\mathcal{O}_S)$ and define Ω_v as before. Write $E_{y,v} = E_y \otimes_K K_v$ then

$$E_{y,v} = \prod_{\tilde{y},w} k(\tilde{y})_w \tag{15}$$

where $\tilde{y} \in \pi^{-1}(y)$ and $w|v$ is a place of $k(\tilde{y})$. Now $k(\tilde{y})_w/\mathbb{Q}_p$ is unramified, hence a Frobenius $\sigma_{k(\tilde{y})_w/\mathbb{Q}_p} \in G(k(\tilde{y})_w/\mathbb{Q}_p)$ is available.

Write $V_{y,v} = V_y \otimes_K K_v$. Then

$$V_{y,v} = \oplus_{\tilde{y},w} V_{\tilde{y},w}, \tag{16}$$

where $V_{\tilde{y},w} = H^1_{dR}(X_{\tilde{y}}/k(\tilde{y})_w)$. The two decompositions (15), (16) and module structures are compatible.

Write $\rho_{\tilde{y}}$ for the $2d$ dimensional p-adic representation

$$\rho_{\tilde{y}} : G_{k(\tilde{y})} \to H^1_{\text{ét}}(X_{\tilde{y},\overline{K}}, \mathbb{Q}_p). \tag{17}$$

For $y' \in \pi^{-1}(y_0)$, define the flag variety by Weil restriction: $H_{y'} = \mathrm{Res}_{\mathrm{K}}^{\mathrm{k(y')}} \mathrm{LGr}(\mathrm{V_{y'}}, \omega_{\mathrm{y'}})$ and

$$H = \prod_{y'\in\pi^{-1}(y_0)} \mathcal{H}_{y'}. \tag{18}$$

Here Lagrangian Grassmann LGr classifies $k(y')$-Lagrangian subspaces of $V_{y'}$. The period map

$$\Phi_v : \Omega_v \to \mathcal{H}_v \tag{19}$$

by $y \mapsto Fil^1 H^1_{dR}(X_y/K_v)$ is K_v-analytic. In fact, we have

$$\begin{tikzcd}
\Omega_v \arrow[r, dotted] \arrow[dr] & H_v \arrow[d] \\
& LGr_{K_v}(V_{y_0,v}, \omega)
\end{tikzcd}$$

For $y \in \Omega_v \cap \mathcal{Y}(\mathcal{O}_S)$, the v-adic Gauss-Manin connection of $Y' \to Y$ on H^0_{dR} induces an isomorphism

$$E_{y,v} \to E_{y_0,v} \tag{20}$$

which is compatible with the identification

$$H^1_{dR}(X_{y,K_v}/K_v) \to H^1_{dR}(X_{y_0,K_v}/K_v)$$

and gives a bijection between pairs

$$(\tilde{y}, w) \text{ above } (y, v) \to (\tilde{y_0}, w_0) \text{ above } (y_0, v).$$

This bijection gives further identifications $V_{\tilde{y},w} \to V_{\tilde{y_0},w_0}$. Recall that the Lagrangian Grassmannian $LGr_{k(\tilde{y_0})}(V_{\tilde{y_0}}, \omega)$ is a projective variety of dimension $\frac{(d+1)d}{2}$ carrying a transitive action of symplectic group, so $\dim \mathcal{H}_{\tilde{y_0}} = [k(\tilde{y_0})_{w_0} : K_v]\frac{(d+1)d}{2}$.

Projecting the period map further to each factor, we have period maps

$$\Phi_{\tilde{y_0},w_0} : \Omega_v \to \mathcal{H}_{\tilde{y_0},w_0} \tag{21}$$

where $\Phi_{\tilde{y_0},w_0}(y) = Fil^1 H^1_{dR}(X_{\tilde{y},w}/k(\tilde{y})_w)$. Here $Fil^1 H^1$ is embedded into $V_{\tilde{y_0},w_0}$ via Gauss-Manin connection.

In application we hope that the monodromy is "big" in some sense in view of Lemma 4.4.1. We make a precise definition.

Definition 4.5.10 (full monodromy)**.** Let Y/K be a variety and $X \to Y' \xrightarrow{\pi} Y$ be an abelian-by-finite family. This family is said to have full monodromy if for a base point $y_0 \in Y(\mathbb{C})$ the Zariski closure of the image of monodromy representation

$$\pi_1(Y^{an}_{\mathbb{C}}, y_0) \to GL(H^1_{sing}(X^{an}_{y_0}, \mathbb{Q}))$$

contains $\prod_{\pi(\tilde{y})=y_0} Sp(H^1_{sing}(X_{\tilde{y}}, \mathbb{Q}), \omega)$. Here we use the decomposition

$$H^1_{sing}(X^{an}_{y_0}, \mathbb{Q}) = \oplus_{\tilde{y}\in\pi^{-1}(y_0)} H^1_{sing}(X^{an}_{\tilde{y}}, \mathbb{Q})$$

and the symplectic form ω comes from the polarization.

5 Mordell's conjecture

Theorem 5.0.1 (Mordell's conjecture, Faltings)**.** *If Y/K is an integral smooth projective curve of genus $g \geq 2$, then $Y(K)$ is finite.*

The proof relies on Proposition 5.0.4 below and a specific construction of family of curves. Note that in the case of S-unit equation, we may use the group law of the base variety $G_{m/\mathcal{O}_S}$ to twist the Legendre family, and hence a large field extension. But here such a twist is absent. In Proposition 5.0.4, we need a size condition to get large field extension, which is checked using Weil pairing in the proof of Theorem 5.0.1.

Start by some properties of the constructed family. For a rational prime q, let

$$\mathrm{Aff}(q) = \{\begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{F}_q^*, b \in \mathbb{F}_q\} = \mathbb{F}_q \rtimes \mathbb{F}_q^*.$$

Given Y as in Theorem 5.0.1, the abelian-by-finite family $X_q \to Y'_q \to Y$ in Definition 6.0.5 for each prime $q \geq 3$ has the following properties:

1. It has full monodromy.
2. $d_q = (q-1)(g-1/2)$, where d_q is the relative dimension of $X_q \to Y'_q$.
3. For each $y_0 \in Y(K)$, there is a G_K-equivariant identification of $\pi^{-1}(y_0)(\overline{K})$ with the conjugacy classes of surjections $\pi_1^{\text{ét}}(Y - y_0, *) \twoheadrightarrow \mathrm{Aff}(q)$ that are nontrivial on a loop around y_0.

We state it in a theorem, whose proof is in Section 7.2.

Theorem 5.0.2. *The Kodaira-Parshin family for the group* $\mathrm{Aff}(q)$ *with* $q \geq 3$ *a rational prime has full monodromy. The relative dimension of $X_q \to Y'_q$ is $(q-1)(g-1/2)$.*

Fix a geometric symplectic basis of $H_1(Y^{an}, \mathbb{Z})$, then by Theorem 3.3.6, $\pi_1^{geom}(Y - y_0, *)$ is the profinite completion of the free group on $2g$ generators $x_1, x'_1, \ldots, x_g, x'_g$ and the loop around y_0 corresponds to $[x_1, x'_1][x_2, x'_2] \ldots [x_g, x'_g]$, so $\pi_1^{geom}(Y - y_0, *)^{ab} = \pi_1^{geom}(Y, *)^{ab}$.

$$H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{Z}/(q-1)) = \mathrm{Hom}(\pi_1^{geom}(Y, *), \mathbb{Z}/(q-1)) = \mathrm{Hom}(\pi_1^{geom}(Y - y_0, *), \mathbb{Z}/(q-1)).$$

Note that a conjugacy classes of group morphisms $H \to \mathrm{Aff}(q)$ induces a unique morphism $H \to \mathbb{F}_q^*$. Thus the third property of Kodaira-Parshin family gives a G_K-equivariant

$$\pi^{-1}(y_0)(\overline{K}) \to H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{Z}/(q-1)), \tag{22}$$

which is the map [LV20, (2.3)] with regard to the fixed basis.

Definition 5.0.3. Let K be a number field and E be a finite set with G_K action. Fix a positive integer d (it will be the relative dimension of $X \to Y'$ when used). If v is a place of K such that v unramified in $K/\mathbb{Q}$ and the G_K action on E is *unramified* at v, then we define

$$\mathrm{size_v(E) = |A|/|E|},$$

where $A \subset E$ is the subset of elements whose $\mathrm{Frob_v}(\in \mathrm{G_K})$ orbit has size $\leq \frac{8d}{d+1}$.

See [LV20, Definition 2.7] for the term "friendly place" appearing below. At friendly places, we have control of the local behavior of global Galois representations (that is the restriction to G_{K_v} of a representation of G_K).

Proposition 5.0.4. *Let Y/K be a smooth curve, with an abelian-by-finite family $X \to Y' \xrightarrow{\pi} Y$ of full monodromy and having a good model $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ over $\mathcal{O}_S$. Let d be the relative dimension of $X \to Y'$. Let $v \notin S$ be a friendly place of K. Then*

$$Y(K)^* = \{y \in \mathcal{Y}(\mathcal{O}_S) : \mathrm{size_v}(\pi^{-1}(\mathrm{y})(\overline{\mathrm{K}})) < \frac{1}{\mathrm{d}+1}\}$$

is finite.

Example 5.0.5. The variant Legendre family in [LV20, Section 4.2] satisfies the conditions. Let v be as in Proposition 5.0.4. For $t \in Y(K)$, $Y'_t = \mathrm{Spec}(K(t^{1/m}))$ and $\pi^{-1}(t)(\overline{K}) = \{z \in \overline{K} : z^m = t\}$. The cardinality of each $\mathrm{Frob_v}$ orbit is the order of *the* element $\mathrm{Frob_{v,K(t^{1/m})/K} \in G(K(t^{1/m})/K)}$.

Assuming Proposition 5.0.4, we finish the proof of Theorem 5.0.1 .

Proof. We are going to find a suitable prime $q \geq 3$, a finite set S of places and a friendly place v such that for the family $X_q \to Y'_q \to Y$, the conditions of Proposition 5.0.4 are satisfied (relatively easy) and $Y(K)^* = Y(K)$ (main part of arguments). The finiteness follows.

We can enlarge K freely, so assume that $K/\mathbb{Q}$ is Galois. First, choose a prime q such that

1. $q \equiv 3(4)$ and $q \equiv 2(l)$ for any odd prime $l|\mathrm{disc(K)}$ or $l \leq 8[K:\mathbb{Q}]$
2. K is linearly disjoint from $\mathbb{Q}(\zeta_{q-1})$ over $\mathbb{Q}$
3. $\frac{7\cdot 2^{g+1}}{(q-1)^g} < \frac{1}{(g-1/2)(q-1)+1}$.

By Dirichlet's theorem on arithmetic progressions, we choose q satisfying the first and last condition (since $g \geq 2$). If a rational prime p ramifies in $\mathbb{Q}(\zeta_{q-1}) = \mathbb{Q}(\zeta_{(q-1)/2})$ and in K, then $p|\frac{q-1}{2}$ and $p|\mathrm{disc(K)}$, so $p \neq 2$ and this contradicts the first condition on q. The second condition follows and

$$G(K(\zeta_{q-1})/\mathbb{Q}) \to G(K/\mathbb{Q}) \times G(\mathbb{Q}(\zeta_{q-1})/\mathbb{Q})$$

is an isomorphism. (That any odd prime factor of $q-1$ must $> 8[K:\mathbb{Q}]$ is used when we choose v.) Choose S such that the family $X_q \to Y'_q \to Y$ has a good model over $\mathcal{O}_S$. We admit the existence of v satisfying the following conditions (proved in [LV20, p. 27] and relying essentially on Chebotarev density theorem):

1. $(q_v, q-1) = 1$
2. For any odd prime $l|(q-1)$, $q_v \in \mathbb{F}_l^*$ has order > 8.

We are going to bound size of $\pi^{-1}(y)(\overline{K})$ to show $Y(K) = Y(K)^*$.

Let E be the image of (22). By [LV20, Lemma 2.11], each nonempty fiber of the map have the same cardinality and $\#E = (q-1)^{2g}\prod_{p|(q-1)}(1-p^{-2g}) \geq (q-1)^{2g}/2$, in particular $\mathrm{size}_v(\pi^{-1}(y)(\overline{K})) \leq \mathrm{size}_v(E)$. Let J be the Jacobian variety of Y. Recall $M := H^1_{\text{ét}}(Y_{\overline{K}}, \mathbb{Z}/(q-1)) = J[q-1]$ is a free $\mathbb{Z}/(q-1)$-module of rank $2g$ equipped with a perfect G_K-equivariant Weil pairing ([Mum74, p.183])

$$M \times M \to \mathrm{Hom}(\mu_{q-1}(\overline{K}), \mathbb{Z}/(q-1)).$$

Also $\mathrm{Frob_v} \in \mathrm{G_{K_v}} \leq \mathrm{G_K}$ induces a $\mathbb{Z}/(q-1)$-linear automorphism T of M. As $\mathrm{Frob_v}$ acts on $\mu_{q-1}(\overline{K})$ by q_v-power and the Weil pairing is G_K-equivariant,

$$\langle Tv_1, Tv_2 \rangle = q_v^{-1}\langle v_1, v_2 \rangle.$$

We estimate $\mathrm{size}_v(E)$. An element of M of $\mathrm{Frob_v}$-orbit < 8 is in $\cup_{i=1}^{7} \ker(T^i - 1)$. If $m_1, m_2 \in \ker(T^i-1)$, then $(q_v^{-i}-1)\langle m_1, m_2\rangle = 0$. By the second condition on v, $2\langle m_1, m_2\rangle = 0$. The pairing restricted to $2M$ is nondegenerate and vanishes on $2\ker(T^i - 1)$, so $\#[2\ker(T^i-1)] \leq \sqrt{\#2M} = (\frac{q-1}{2})^g$. Hence,

$$\#\ker(T^i - 1) \leq 2^g(q-1)^g.$$

$$\mathrm{size}_v(E) \leq \frac{7 \cdot 2^g (q-1)^g}{\frac{1}{2}(q-1)^{2g}} < \frac{1}{d_q + 1}.$$

The proof of Theorem 5.0.1 is completed. $\square$

By the existence of a good model $\mathcal{X} \to \mathcal{Y}' \to \mathcal{Y}$ over $\mathcal{O}_S$, the G_K action on $\pi^{-1}(y)(\overline{K})$ is unramified at v (which justifies the notation $\mathrm{size}_v(\pi^{-1}(y)(\overline{K}))$). As in the proof of S-unit equation, we hope that for most $y \in Y(K)$, there is a pair (y', w) above (y, v) such that $[k(y')_w : K_v]$ is large (to obtain better control when applying Lemma 4.4.2). Lemma 5.0.6 and 5.0.8 show that the condition on $\mathrm{size}_{\mathrm{v}}$ gives the existence of such pairs. To prove Proposition 5.0.4, we need Lemmas 5.0.6 and 5.0.7.

Lemma 5.0.6. *[LV20, Lemma 6.1] Setting as in Proposition 5.0.4. There is a finite subset $F \subset \Omega_v \cap Y(K)^*$ such that for $y \in \Omega_v \cap Y(K)^* - F$, there exists $(\tilde{y}, w)$ above (y, v) such that $[k(\tilde{y})_w : K_v] > \frac{8d}{d+1}$ and $\rho_{\tilde{y}}$ (see (17)) is a simple $G_{k(\tilde{y})}$-representation.*

Lemma 5.0.6 is analogue to Lemma 8.0.2 but more complicated. It takes care of failure of simplicity and its proof requires friendliness of the place v. We advice the reader to skip it and use Theorem 3.3.9 instead in the proof of Proposition 5.0.4 below at first reading.

Lemma 5.0.7. *[LV20, Lemma 6.2]Setting as in Proposition 5.0.4 but $v \notin S$ is allowed to be unfriendly. Fix a finite extension E/K_v with $[E : K_v] > \frac{8d}{d+1}$ and a semisimple p-adic representation ρ of G_E. There are only finitely many $y \in \Omega_v \cap Y(K)$ for which there exists a pair $(\tilde{y}, w)$ above (y, v) such that $(k(\tilde{y})_w, \rho_{\tilde{y},w})$ is isomorphic to (E, ρ).*

Assuming Lemma 5.0.6 and 5.0.7 temporarily, we prove Proposition 5.0.4.

Proof. Fix $y_0 \in Y(K)^*$. It suffices to show the finiteness of $Y(K)^* \cap \Omega_v$. There are only finitely many possibilities for $k(\tilde{y})$ when $y \in \mathcal{Y}(\mathcal{O}_S)$. Consider the finite subset F in Lemma 5.0.6. When $y \in (\Omega_v \cap Y(K)^*) \setminus F$, let $(\tilde{y}, w)$ be as in Lemma 5.0.6. By Lemma 2.3.6, there are only finitely many possibilities for $(k(\tilde{y}), \rho_{\tilde{y}})$ and also for $(k(\tilde{y})_w, \rho_{\tilde{y}}|_{G_{k(\tilde{y})_w}})$. The conclusion then follows from Lemma 5.0.7. $\square$

Proof of Lemma 5.0.7. It's a repeat of the proof of Lemma 4.4.1.

By Theorem 3.7.4,

$$D_{cris}(\rho_{\tilde{y},w}) = (H^1_{dR}(X_{\tilde{y}}/k(\tilde{y})_w), \phi, \text{Hodge filtration}). \tag{23}$$

Here ϕ is the $\sigma_{k(\tilde{y})_w/\mathbb{Q}_p}$-semilinear Frobenius operator and the Hodge filtration is determined by $Fil^1 H^1_{dR}(X_{\tilde{y}}/k(\tilde{y})_w)$. Using the Gauss-Manin connection and the period map (21) we identify the tripe with

$$(H^1_{dR}(X_{\tilde{y_0}}/k(\tilde{y_0})_{w_0}), \phi, \Phi_{\tilde{y_0},w_0}(y)). \tag{24}$$

It remains to show the finiteness of points $y \in \Omega_v \cap Y(K)$ such that the triple (24) belongs to a fixed isomorphism class, i.e. $\Phi_{\tilde{y_0},w_0}(y)$ lies in a fixed orbit for the action of centralizer $Z(\phi)$ on $\mathcal{H}_{\tilde{y_0},w_0}$. Here $Z(-)$ is taken inside $GL_{k(\tilde{y_0})_{w_0}}(V_{\tilde{y_0},w_0})$. note that $Z(\phi) \subset Z(\phi^{[K_v:\mathbb{Q}_p]})$ and $\phi^{[K_v:\mathbb{Q}_p]}$ is $\sigma_{k(\tilde{y})_w/K_v} = \sigma^{[K_v:\mathbb{Q}_p]}_{k(\tilde{y})_w/\mathbb{Q}_p}$-semilinear. By Lemma 4.4.2,

$$\dim_{K_v} Z_{end}(\phi^{[K_v:\mathbb{Q}_p]}) \leq (\dim_{k(\tilde{y_0})_{w_0}} V_{\tilde{y_0},w_0}) = 4d^2.$$

The assumption of full monodromy implies that $\Phi_v(\Omega_v) \subset \mathcal{H}_v$ is Zariski-dense, so is each $\Phi_{\tilde{y_0},w_0}(\Omega_v) \subset \mathcal{H}_{\tilde{y_0},w_0}$. Now that $[E : K_v] > \frac{8d}{d+1}$, we find

$$\dim(\Gamma \cdot h_0^\iota) = \dim \mathcal{H}_{\tilde{y_0},w_0} = [E : K_v]\frac{(d+1)d}{2} > 4d^2.$$

Lemma 4.2.1 terminates the proof. □

We proceed to the proof of Lemma 5.0.6. It follows lines similar to the proof of Lemma 8.0.2. We call $y \in \Omega_v \cap Y(K)^*$ "bad" if no $(\tilde{y}, w)$ above (y, v) is such that $[k(\tilde{y})_w : K_v] > \frac{8d}{d+1}$ and $\rho_{\tilde{y}}$ is simple simultaneously. We will take F to be the set of bad points.

Lemma 5.0.8. *Setting as in Lemma 5.0.6. If $y \in \Omega_v \cap Y(K)^*$ is bad, then there exists a pair $(\tilde{y}, w)$ above (y, v) with $[k(\tilde{y})_w : K_v] > \frac{8d}{d+1}$ and a nonzero proper ϕ-stable subspace $W_{dR} \leq H^1_{dR}(X_{\tilde{y}}/k(\tilde{y})_w) = V_{\tilde{y},w}$ such that $\dim_{k(\tilde{y})_w} Fil^1 W_{dR} \geq \frac{1}{2}\dim_{k(\tilde{y})_w} W_{dR}$. (Here ϕ is as in (23).)*

Proof. Assume the contrary of Lemma 5.0.8. For each $y' \in \pi^{-1}(y)$, choose $W_{y'}$ a minimal nonzero subrepresentation of $\rho_{y'}$. For each $w|v$ a place of $k(y')$, we have $D_{dR,w} : \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_{k(y')_w}) \to Fil_{k(y')_w}$. The subspace $D_{dR,w}(W_{y'}) \leq$

$D_{dR,w}(\rho_{y'}) = H^1_{dR}(X_{y'}/k(y')_w)$ is ϕ-stable and equipped with induced Hodge filtration. The representation $G_{k(y')} \to GL(W_{y'})$ is crystalline at all primes above v (by Theorem 3.7.4) and pure of weight 1,

$$Fil^2 D_{dR,w}(W_{y'}) \leq Fil^2 H^1_{dR}(X_{y'}/k(y')_w) = 0.$$

Therefore $\dim D_{dR,w}(W_{y'})$ is even and $2{+}2\dim Fil^1 D_{dR,w}(W_{y'}) \leq \dim D_{dR,w}(W_{y'})$. Applying [LV20, Lemma 2.10] to the *friendly* place v, we find

$$\sum_{w|v} [k(y')_w : K_v] \frac{\dim Fil^1 D_{dR,w}(W_{y'})}{\dim D_{dR,w}(W_{y'})} = \frac{[k(y') : K]}{2}.$$

If $[k(y')_w : K_v] > \frac{8d}{d+1}$, then $\rho_{y'}$ is not simple as y is bad. As $X_{y',w}/k(y')_w$ is a polarized abelian variety,

$$\dim_{k(y')_w} D_{dR,w}(W_{y'}) \leq \frac{1}{2} \dim_{k(y')_w} H^1_{dR}(X_{y'}/k(y')_w) = d, \tag{25}$$

$$\frac{\dim Fil^1 D_{dR,w}(W_{y'})}{\dim D_{dR,w}(W_{y'})} \leq \frac{d-1}{2d}. \tag{26}$$

Sum over all $y' \in \pi^{-1}(y)$,

$$\begin{aligned} \frac{1}{2} \sum_{(y',w)} [k(y')_w : K_v] &= \sum_{y' \in \pi^{-1}(y)} \frac{1}{2} [k(y') : K] \\ &= \sum_{(y',w)} [k(y')_w : K_v] \frac{\dim Fil^1 D_{dR,w}(W_{y'})}{\dim D_{dR,w}(W_{y'})} \\ &\leq \sum_{[k(y')_w : K_v] > \frac{8d}{d+1}} \frac{d-1}{2d} [k(y')_w : K_v] + \sum_{[k(y')_w : K_v] \leq \frac{8d}{d+1}} [k(y')_w : K_v]. \end{aligned}$$

We deduce that

$$d \sum_{[k(y')_w : K_v] \leq \frac{8d}{d+1}} [k(y')_w : K_v] \geq \sum_{[k(y')_w : K_v] > \frac{8d}{d+1}} [k(y')_w : K_v].$$

Let $f_1, \ldots, f_k$ be the cardinal of each $\mathrm{Frob_v}$-orbit. Then

$$d \sum_{i : f_i \leq \frac{8d}{d+1}} f_i \geq \sum_{i : f_i > \frac{8d}{d+1}} f_i,$$

so $\mathrm{size}_v(\pi^{-1}(y)(\overline{K})) \geq \frac{1}{d+1}$. This contradicts the assumption that $y \in Y(K)^*$.

$\square$

Proof of Lemma 5.0.6. Fix (y', w) above (y_0, v). Define $A_{y',w} \subset LGr_{k(y')_w}(V_{y',w}, \omega)$ (closed sub $k(y')_w$-variety) as the set of Lagrangian $k(y')_w$-subspaces $F \subset V_{y',w}$ for which there exists a ϕ-stable subspace $W \subset V_{y',w}$ satisfying $\dim_{k(y')_w}(F \cap W) \geq \frac{1}{2}\dim_{k(y')_w} W$. Put $\mathcal{H}^{bad}_{y',w} = Res^{k(y')_w}_{K_v} A_{y',w}$. By [LV20, Lemma 6.3,6.4], $\mathcal{H}^{bad}_{y',w} \subset \mathcal{H}_{y',w}$ is a proper closed K_v-subvariety. The assumption of full monodromy and Lemma 4.2.1 show that $\Phi^{-1}_{y',w}(\mathcal{H}^{bad}_{y',w})$ is finite. Combined with Lemma 5.0.8 this implies the finiteness of bad points. Take F to be the set of bad points. $\square$

6 Construction of Kodaira-Parshin family

We present the construction of the family used in the proof of Theorem 5.0.1. Readers can admit Proposition 6.0.1 and jump to Definition 6.0.4.

Proposition 6.0.1. *[LV20, Proposition 7.1]Let Y be a smooth projective curve over a number field K of genus $g \geq 1$, and let G be a center-free finite group. Then there is a smooth projective K-curve Y' with a finite étale morphism $\pi : Y' \to Y$ and a relative curve $Z \to Y'$ with the following properties:*

1. *For $y \in Y(\overline{K})$, $\pi^{-1}(y)$ is in bijection with the set of G-conjugacy classes of surjections $\pi_1^{geom}(Y - y, *) \to G$ nontrivial on a loop around y. Moreover, if $y \in Y(K)$, this identification is G_K-equivariant.*
2. *There's a finite Y'-morphism $f : Z \to Y' \times Y$, where G acts on Z covering the trivial action on $Y' \times Y$ and making $Z - f^{-1}(\Gamma_\pi) \to Y' \times Y - \Gamma_\pi$ into a G-Galois cover, where $\Gamma_\pi \subset Y' \times_K Y$ is the graph of π. For $y' \in Y'(\overline{K})$, the base change $Z_{y'} \to Y$ is branched exactly at $y = \pi(y')$ and the induced morphism $\pi_1^{geom}(Y^{an} - y, *) \to G$ is in the conjugacy class from 1.*

The strategy of proof is to construct it analytically over $\mathbb{C}$, where the properties are easily verified. Then show it is algebraic over $\mathbb{C}$ and use GAGA to translate the properties. The difficulty is to descent to K (i.e., show that it is algebraic over K).

Proof of Proposition 6.0.1. Lemma 6.0.3 gives $Z^\circ \to Y' \times_K Y \to Y^2$ where everything is algebraic over K. Let $Z \to Y^2$ be the normalization of Y^2 inside the function field of Z°. Then Z is normal, and finite over Y^2. The base extension $Z_{\mathbb{C}}$ is normal, so coincides with the normalization of $Y^2_{\mathbb{C}}$ inside the function field of $\mathcal{Z}^\circ$. The desired properties can be verified since they are true over $\mathbb{C}$ by Proposition 6.0.2. □

Proposition 6.0.2. *Let Y be a compact Riemann surface and let G be a finite group. Then there exists a compact Riemann surface Y' with a finite sheeted cover map $\pi : Y' \to Y$, and an algebraic relative curve $Z \to Y'$ with the following properties:*

1. *For $y \in Y$, $\pi^{-1}(y)$ is in bijection with the finite set $S(y)$ of G-conjugacy classes of surjections $\pi_1(Y^{an} - y, *) \to G$ nontrivial on a loop around y.*

2. *There is an algebraic finite Y'-morphism $f : Z \to Y' \times Y$. And G acts on Z covering the trivial action on $Y' \times Y$ making $Z - f^{-1}(\Gamma_\pi) \to Y' \times Y - \Gamma_\pi$ into a G-Galois cover, where $\Gamma_\pi \subset Y' \times Y$ is the graph of π. For $y' \in Y'$, the base change $Z_{y'} \to Y$ is branched exactly at $y = \pi(y')$ and the induced morphism $\pi_1(Y^{an} - y, *) \to G$ is in the conjugacy class from 1.*

Proof. For $y \in Y$ we identify $S(y)$ to the set of isomorphism classes of connected branched coverings of Y branched precisely at y, whose restriction to $Y - y$ is a G-Galois finite sheeted cover.

Put a set $Y' = \sqcup_{y \in Y} S(y)$ and $\pi : Y' \to Y$ the natural map. Fix a small open neighborhood U of $y \in Y$, then $Y - U \subset Y - y$ is a deformation retract, which induces an isomorphism $\pi_1(Y - U) \to \pi_1(Y - y)$ and hence a way to identify the various groups $\pi_1(Y - y')$ for $y' \in U$. Another choice U' leads to same identification over $y' \in U \cap U'$ (cf.[Ful69, Section 1.3]). Therefore, locally on Y, e is a in the form of the projection $Y \times S \to Y$, where S is a finite set. So, Y' has a unique topology making $\pi : Y' \to Y$ a finite sheeted covering. Then Y' becomes a compact Riemann surface.

Let $Z_{y'} \to Y$ be the branched cover represented by $y' \in Y$. Put $Z = \sqcup_{y' \in Y'} Z_{y'}$ and $f : Z \to Y' \times Y$ the natural map. Then Z is a complex manifold and f is holomorphic. The group G acts holomorphically on Z (leaving the ramification locus of each $Z_{y'} \to Y$ invariant) and f is G-equivariant where the action on $Y' \times Y$ is trivial. Restriction $Z - f^{-1}(\Gamma_\pi) \to Y' \times Y - \Gamma_\pi$ is a G-Galois cover.

We have explicitly constructed what we need in the analytic setting. Now note that π is algebraic as Y', Y are projective. As f is a finite analytic map, the manifold Z and the map f are algebraic by [Har77, Appendix B, Theorem 3.2]. (Note that the cited result is false if the map is only proper instead of being finite, as there is a compact complex manifold not algebraic.) $\square$

The following lemma descends from $\mathbb{C}$ to K.

Lemma 6.0.3. *[LV20, Lemma 7.4]Settings as in Proposition 6.0.1. Let $\Delta \subset Y \times_K Y = Y^2$ be the diagonal. Recall the embedding $\iota : K \to \mathbb{C}$ fixed in Section 4.1. Denote by $\mathscr{Z}/\mathbb{C}$ the algebraic variety and by $f : \mathscr{Z} \to (Y' \times_K Y)_{\mathbb{C}}$ the finite morphism given by Proposition 6.0.2. Put $\mathscr{Z}^\circ = f^{-1}((Y' \times_K Y)_{\mathbb{C}} - \Gamma_\pi)$. Then*

1. *There exists a unique finite étale cover $F : Z^\circ \to Y^2 - \Delta$ over K such that its base extension to $\mathbb{C}$ is $(\pi \times Id) \circ f : \mathscr{Z}^\circ \to (Y^2 - \Delta)_{\mathbb{C}}$. This Z°/K is a smooth integral variety equipped with a G-action and F is G-equivariant, where $Y^2 - \Delta$ is with trivial G-action.*

2. *Let $(y_1, y_0) \in Y(\overline{K})^2$ with $y_1 \neq y_0$, the categorical quotient $F^{-1}(y_1, y_0)/G$ is identified with $S(y_0)$ (notation from of Proposition 6.0.2, using $\pi_1^{geom}(Y - y_0, y_1)$ instead of topological π_1 to define it). If furthermore $(y_1, y_0) \in Y(K)$, then this identification is G_K-equivariant.*

3. *There is a smooth projective curve Y'/K and an étale cover $\pi : Y' \to Y$ such that $\pi \times Id : Y' \times_K Y \to Y^2$ extends $Z^\circ/G \to Y^2 - \Delta$.*

Proof. (I can only prove a weaker version, allowing finite extension of K, which suffices for our purpose.) Firstly, we descend from $\mathbb{C}$ to $\bar{\mathbb{Q}}$. By Proposition 6.0.1, there is a finite étale morphism $\pi : Y'_{\mathbb{C}} \to Y_{\mathbb{C}}$ over $\mathbb{C}$. By Theorem 3.3.7, π and Y' are defined over $\bar{\mathbb{Q}}$. Idem, the finite étale morphism $\mathscr{Z}^\circ \to (Y' \times_K Y)_{\mathbb{C}} - \Gamma_\pi$ is defined over $(Y' \times_K Y)_{\mathbb{C}} - \Gamma_\pi$. By enlarging K we find F, Z° in 1. $\square$

The construction uses also Prym variety to get abelian scheme from relative curve.

Definition 6.0.4 (Prym variety)**.** [LV20, Section 7.2]Let k be an algebraically closed field. For $f : C_1 \to C_2$ a nonconstant morphism between smooth projective curves over k, the Prym variety is defined to be $\mathrm{Prym(C_1/C_2)} = \mathrm{coker(f^* : J_2 \to J_1)}$, where J_i is the Jacobian of C_i and the norm map $J_2 \to J_1$ is by pulling back a divisor.

Note that the so defined Prym variety is isogenous to the connected component of $\ker(N : J_1 \to J_2)$, where N is the norm map. If the cover $f : C_1 \to C_2$ is Galois of group G, then $J_2 \to J_1$ is of finite kernel and the image is the connected component of the G-invariants. For a subgroup H of G, let $C_1 \to M$ be the corresponding Galois cover. Then $\mathrm{Prym(M/C_2)}$ is isogenous to cokernel of the map

$$\text{connected component of } J_1^G \to \text{connected component of } J_1^H.$$

Form the idempotent $e = \frac{1}{\#H}\sum_{h\in H} h - \frac{1}{\#G}\sum_{g\in G} g \in \mathbb{Q}[G]$ and let $e' = 1-e$ be the complementary idempotent. Then $e'' = \#G \cdot e' \in \mathbb{Z}[G]$ acts on J_1. The connected component of the kernel $J_1[e'']$ is isogenous to $\mathrm{Prym(M/C_2)}$.

Let Y/K be a smooth projective curve of genus $g \geq 1$ and let $q \geq 3$ be a rational prime, consider the sequence $Z_q \to Y'_q \to Y$ be the sequence given by Proposition 6.0.1 with $G = \mathrm{Aff}(q)$. Let $H(\leq)G$ be the stabilizer of $0 \in \mathbb{F}_q$ and form e'' as above. For each $y' \in Y'_q(\overline{K})$. Define X_q to be the relative identity component (cf. [Sta23, Tag 055K]) of $Pic^0_{Z_q\to Y'_q}[e'']$. Then $X_q \to Y'_q$ is a polarized abelian scheme. Its fiber over $y' \in Y'_q(\overline{K})$ is isogenous to $\mathrm{Prym(Z^{red}_{y'}/Y)}$, where

$$Z^{red}_{y'} := Z_{y'} \times_G \mathbb{F}_q \to Y \tag{27}$$

is a cover of degree q and $\mathbb{F}_q$ is viewed as a G-set.

Definition 6.0.5 (Kodaira-Parshin family)**.** Notations as above. We call $X_q \to Y'_q \to Y$ the Kodaira-Parshin family over Y associated to $\mathrm{Aff}(q)$.

7 The monodromy of Kodaira-Parshin families

A few words about the proof of Theorem 5.0.2. We relate the fundamental group to the mapping class group using Birman exact sequence. When working with the latter we have access to Dehn twists, which is amenable to explicit calculation.

From now on, by a "surface" we mean a connected orientable closed surface with finitely many punctures. For such a surface Y, $MCG(Y)$ denotes the mapping class group of Y. When we discuss homology or cohomology, the coefficients are assumed to be $\mathbb{Q}$ unless stated otherwise. An embedding $e : S^1 \to \Sigma$ to a surface Σ is called a simple closed curve, which is oriented by a fixed orientation of S^1. Quite often we use the image of e to refer to e by abusing language.

7.1 Covers and their homology

Definition 7.1.1 (Primitive homology)**.** For a map $\pi : Z \to Y$ between surfaces, define the primitive homology to be $H_1^{Pr}(Z,Y) = \ker(\pi_* : H_1(Z,\mathbb{Q}) \to H_1(Y,\mathbb{Q}))$.

Let S be a compact Riemann surface with Jacobian J. Fix a basepoint $p \in S$ and let $\Psi : S \to J$ be the corresponding Abel-Jacobi map, then $\Psi_* : H_1(S,\mathbb{Z}) \to H_1(J,\mathbb{Z})$ is an isomorphism.

Lemma 7.1.2. *If $\pi : Z \to Y$ is a non-constant morphism of compact Riemann surfaces, then the primitive homology equals the homology of the Prym variety, i.e.* $H_1(\mathrm{Prym(Z/Y)},\mathbb{Q}) = \mathrm{H_1^{Pr}(Z,Y;\mathbb{Q})}$.

Proof. Let $J_Y \to J_Z$ be the induced map on the Jacobians. Note that J_Z is isogenous to $J_Y \times \mathrm{Prym(Z/Y)}$, so $H_1(J_Z,\mathbb{Q}) = H_1(J_Y,\mathbb{Q}) \oplus H_1(\mathrm{Prym(Z/Y)},\mathbb{Q})$. Choose base point $z_0 \in Z$ and $\pi(z_0) \in Y$ and form the corresponding Abel-Jacobian map.

$$\begin{tikzcd}
0 \arrow[r] & H_1^{Pr}(Z,Y) \arrow[r] \arrow[d, dotted] & H_1(Z) \arrow[r] \arrow[d, "="] & H_1(Y) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H_1(\mathrm{Prym(Z/Y)}) \arrow[r] & H_1(J_Z) \arrow[r] & H_1(J_Y) \arrow[r] & 0
\end{tikzcd}$$

Therefore, $H_1(\mathrm{Prym(Z/Y)},\mathbb{Q}) = \mathrm{H_1^{Pr}(Z,Y,\mathbb{Q})}$. □

Given a finite-sheeted topological covering $\pi : Z \to Y$ of surfaces, we have an exact sequence

$$0 \to H_1^{Pr}(Z,Y,\mathbb{Q}) \to H_1(Z,\mathbb{Q}) \xrightarrow{\pi_*} H_1(Y,\mathbb{Q}) \to 0.$$

By path lifting, we have a splitting of this sequence $\pi^* : H_1(Y,\mathbb{Q}) \to H_1(Z,\mathbb{Q})$, hence a decomposition

$$H_1(Z,\mathbb{Q}) = \pi^* H_1(Y,\mathbb{Q}) \oplus H_1^{Pr}(Z,Y,\mathbb{Q}). \tag{28}$$

If furthermore the intersection pairing on $H_1(Z,\mathbb{Q})$ is nondegenerate, then it's an orthogonal direct sum, making $H_1^{Pr}(Z,Y;\mathbb{Q})$ a symplectic $\mathbb{Q}$-vector space.

Definition 7.1.3 (Aff(q)-cover)**.** A q-sheeted covering $Z \to Y$ between surfaces whose monodromy representation on a general fiber is equivalent to the action of Aff(q) on $\mathbb{F}_q$ (i.e. we can label the points in the fiber by $\mathbb{F}_q$ such that $\pi_1(Y, y) \to \mathrm{Sym}(\mathbb{F}_q)$ has image Aff(q)) is called an Aff(q)-cover.

Do not confuse it with Galois cover of group Aff(q)!

Fix a basepoint $y_0 \in Y$. For an Aff(q)-cover $\pi : Z \to Y$, its monodromy representation $\pi_1(Y, y_0) \twoheadrightarrow \mathrm{Aff}(q)$ is well-defined up to conjugation by the normalizer of Aff(q) in $\mathrm{Sym}(\mathbb{F}_q)$ (due to various ways to label a fiber in Definition 7.1.3). But the normalizer is Aff(q) itself. As a result, the isomorphism classes of Aff(q)-covers over Y is in bijection with the set of Aff(q)-conjugacy classes of surjections $\pi_1(Y, y_0) \to \mathrm{Aff}(q)$.

Define the type that the cover (27) belongs to.

Definition 7.1.4 (singly ramified Aff(q)-cover)**.** Let $f : Z \to Y$ be a branched cover between compact Riemann surfaces. If its ramification locus is a singleton $z \in Z$, and $Z^\circ = Z - z \to Y - f(z)$ is an Aff(q)-cover, then $Z \to Y$ is called a singly ramified Aff(q)-cover.

For a singly ramified Aff(q)-cover, Riemann-Hurwitz formula implies that $g(Z) = g(Y)q - \frac{q-1}{2}$. Note that the intersection pairing on $H_1(Z^\circ, \mathbb{Q}) = H_1(Z)$ is perfect, and

$$
\begin{tikzcd}
0 \arrow[r] & H_1^{Pr}(Z^\circ, Y - y) \arrow[r] \arrow[d, dashed, "="] & H_1(Z - z, \mathbb{Q}) \arrow[r] \arrow[d, "="] & H_1(Y - y, \mathbb{Q}) \arrow[r] \arrow[d, "="] & 0 \\
0 \arrow[r] & H_1^{Pr}(Z, Y) \arrow[r] & H_1(Z, \mathbb{Q}) \arrow[r] & H_1(Y, \mathbb{Q}) \arrow[r] & 0
\end{tikzcd}
$$

We find that $H_1^{Pr}(Z^\circ, Y - y) = H_1^{Pr}(Z, Y)$ (with subspace intersection pairings). To emphasis, the importance is that each branch point has only one preimage. See also Remark 7.3.4.

7.2 Mapping class group

For a surface Y, recall the natural morphism $MCG(Y) \to Out(\pi_1(Y, y_0))$ and that $Out(\pi_1(Y, y_0))$ acts on the set of Aff(q)-conjugacy classes of surjections $\pi_1(Y, y_0) \to \mathrm{Aff}(q)$, or rather isomorphism classes of Aff(q)-covers over Y.

Given an $\mathrm{Aff}(q)$-cover $\pi : Z \to Y$, let $MCG(Y)_Z \leq MCG(Y)$ be the stabilizer of the isomorphism class of π. There's a homomorphism

$$MCG(Y)_Z \to MCG(Z). \tag{29}$$

Consider the symplectic representation $MCG(Z) \to Sp(H_1(Z, \mathbb{Q}))$. The action on $H_1(Z, \mathbb{Q})$ of an element of $MCG(Y)_Z$ preserves the decomposition (28). If the intersection pairing on $H_1(Z, \mathbb{Q})$ is nondegenerate, then we get

$$Mon : MCG(Y)_Z \to Sp(H_1^{Pr}(Z, Y; \mathbb{Q})).$$

Fix Y a closed surface of genus $g \geq 2$, a basepoint $y \in Y$, a prime $q \geq 3$. There are only finitely many isomorphism classes for singly ramified $\mathrm{Aff}(q)$-cover of Y branched at y. Choose a representative system $Z_1, \ldots, Z_N$. $MCG(Y - y)_0 := \cap_i MCG(Y - y)_{Z_i^\circ}$. We have a combined monodromy map

$$MCG(Y - y)_0 \to \prod_{i=1}^{N} Sp(H_1^{Pr}(Z_i, Y)). \tag{30}$$

Theorem 7.2.1 (Birman exact sequence)**.** *[FM11, Theorem 4.6] Let S be a surface with $\chi(S) < 0$ and $x \in S$. Then the following sequence is exact:*

$$1 \to \pi_1(S, x) \to MCG(S, x) \to MCG(S) \to 1.$$

As $\chi(Y) = 2 - 2g < 0$, Theorem 7.2.1 applies and we have a morphism $\pi_1(Y, y) \to MCG(Y - y)$. Let $\pi_1(Y, y)_0 \leq \pi_1(Y, y)$ be the preimage of $MCG(Y - y)_0$.

Theorem 7.2.2. *Let notation be as above. The restriction of (30)*

$$Mon : \pi_1(Y, y)_0 \to \prod_{i=1}^{N} Sp(H_1^{Pr}(Z_i, Y)) \tag{31}$$

has Zariski-dense image.

By definition, each lift in Y' of a simple closed curve representing a class of $\pi_1(Y, y)_0$ is a simple closed curve.

Proof of Theorem 5.0.2 assuming Theorem 7.2.2. In the analytic setting, fix $y \in Y$ and for $y' \in \pi^{-1}(y)$, the cover $Z_{y'} \to Y$ is singly branched and outside the branch locus is an $\mathrm{Aff}(q)$-Galois cover. The induced homomorphism $\pi_1(Y - y, *) \to G$ maps a loop around y to a q-cycle by the discussion following Theorem 5.0.2. The induced morphism $Z_{y'}^{red} \to Y$ is therefore a singly ramified $\mathrm{Aff}(q)$-cover branched at y, hence isomorphic to a unique $Z_i \to Y$. By the construction of Kodaira-Parshin family, $X_{y'}$ is isogenous to $\mathrm{Prym(Z_{y'}^{red}/Y)}$, so $\dim X_{y'} = g(Z_{y'}) - 1 = (q-1)(g-1/2)$ and $H_1(X_{y'}, \mathbb{Q}) = H_1^{Pr}(Z_{y'}^{red}, Y; \mathbb{Q})$ by Lemma 7.1.2. The space $H^1(X_{y'}, \mathbb{Q})$ is dual to $H_1(X_{y'}, \mathbb{Q})$. As $\{Z_{y'}^{red} : y' \in \pi^{-1}(y)\}$ is in bijection with $\{Z_i : 1 \leq i \leq N\}$, Theorem 7.2.2 shows that the monodromy is full. $\square$

Lemma 7.2.3. *[LV20, Lemma 8.7]*

$$MCG(Y - y)_0 \to \prod_{i=1}^{N} SpH_1^{Pr}(Z_i, Y)$$

has Zariski dense image.

Lemma 7.2.3 is proved by Lemma 7.2.4, 2.2.5 and [LV20, Lemma 8.8].

Proof of Theorem 7.2.2, assuming Lemma 7.2.3. As $\pi_1(Y, y)_0 \leq MCG(Y - y)_0$ is a normal subgroup and the symplectic group is almost simple, $\pi_1(Y, y)_0 \to SpH_1^{Pr}(Z_i, Y)$ has Zariski dense image for each i and the Zariski-closure of the image of (31) is normal subgroup. Lemma 2.2.4 readily implies the desired result. $\square$

Lemma 7.2.4. *Let $Z \to Y$ be a singly ramified* $\mathrm{Aff}(q)$*-cover branched at y. The monodromy map $Mon : MCG(Y - y)_{Z^\circ} \to SpH_1^{Pr}(Z, Y)$ has Zariski-dense image.*

We prove Lemma 7.2.4 in the next section by showing that there are enough Dehn twists.

7.3 Dehn twist

Consider an $\mathrm{Aff}(q)$-cover $f : M \to N$, or rather a conjugacy class $\pi_1(N, *) \twoheadrightarrow \mathrm{Aff}(q)$. For e a simple closed curve in N, let n_e be the order of the "image" in $\mathrm{Aff}(q)$ of $[e] \in \pi_1(N, *)$ (It is well-defined.) The "image" in $\mathrm{Sym(\mathbb{F}_q)}$ has a well-defined cycle type $(d_1, \dots, d_k)$, then $n_e = \mathrm{lcm(d_1, \dots, d_k)}$. The Dehn twist about e is denoted by $D_e \in MCG(N)$. Then $D_e^{n_e} \in MCG(N)_M$. The lift of e in M is $\sqcup_i e_i$, where e_i are closed curves in M. In fact, e_i is the concatenate of d_i pieces of lift of e. The image of $D_e^{n_e}$ in $MCG(N)_M \to MCG(M)$ is $\prod_i D_{e_i}^{n_e/d_i}$. (As e_i are disjoint, the elements $D_{e_i} \in MCG(M)$ are commuting.)

Lemma 7.3.1. *[LV20, Lemma 8.2]Notations as above. Then $[e_1], \dots, [e_k] \in H_1(M, \mathbb{Q})$ are linearly independent over $\mathbb{Q}$. When projected to $H_1^{Pr}(M, N; \mathbb{Q})$, their span has dimension $k-1$.*

Definition 7.3.2 (liftable curve)**.** For a finite sheeted covering of surfaces $X \to Y$, a simple closed curve e on Y whose lift to X consists of two disjoint closed curves is called liftable.

For a liftable simple closed curve e on N, D_e^{q-1} is mapped to a transvection (cf. Definition 2.2.1) under the map $MCG(N)_M \to SpH_1^{Pr}(M, N; \mathbb{Q})$. Let $f^{-1}(e) = e^+ \sqcup e^-$, where e^+ is of degree 1 over e and e^- of degree $q-1$. Write $\tilde{e}$ for the projection of $[e^+] \in H_1(M)$ to $H_1^{Pr}(M, N)$. We admit the following lemma.

Lemma 7.3.3. *[LV20, Lemma 8.10]For a singly ramified* $\mathrm{Aff}(q)$*-cover $Z \to Y$ branched at y with $g(Y) \geq 1$, there exists a collection of liftable curves $A_1, \dots, A_N$ on $Y - y$ such that*

- *the $\tilde{A}_i$ span $H_1^{Pr}(Z, Y)$*
- *the graph obtained by connecting A_i, A_j when $\tilde{A}_i \cdot \tilde{A}_j \neq 0$ is connected.*

Lemma 2.2.4 and [LV20, Lemma 2.14] prove Lemma 7.2.4.

Remark 7.3.4. For $Y = P^1_K - \{0, 1, \infty\}$, the Legendre family $L \to Y$ (which is an abelian scheme by [MFK94, Theorem 6.14]) defined by $y^2 = x(x-1)(x-\lambda)$ can be viewed as the Kodaira-Parshin family associated to $G = \mathbb{Z}/2\mathbb{Z}$. It's monodromy group is $\Gamma(2)$ by [CMSP17, Theorem 1.1.7], which is Zariski dense inside $\mathrm{SL}_{2/Q}$. The strategy of Section 7 gives another proof to the full monodromy of Legendre family. In the analytic setting, given $y \in \mathbb{C} - \{0, 1\}$, there is only one (up to isomorphism) degree-2 morphism $E \to P^1$ with branch locus $\{0, 1, \infty, y\}$. Each branch point has only one preimage, so $H_1^{Pr}(E - R, P^1 - B) = H_1^{Pr}(E, P^1)$ where R, B stand for ramification and branch locus respectively. The analogue of Lemma 7.3.3 is that there exists two split curves A, B on $P^1 - \{0, 1, \infty, y\}$ such that $\tilde{A} \cdot \tilde{B} \neq 0$, which is clear from the picture. (Note that $H_1^{Pr}(E, P^1) = H_1(E)$ is of dimension two.)

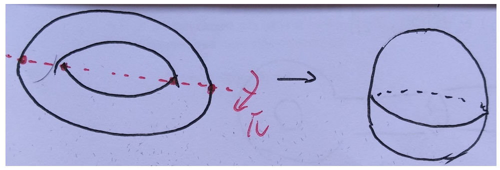

Figure 1: The unique double cover

8 Siegel's theorem

The following theorem of Siegel is an immediate corollary of Faltings' theorem and Chevalley-Weil theorem on integral points ([Fuc14, p.140]). Still, we present a proof base on the method of [LV20].

Theorem 8.0.1. *Let $\mathcal{Y}$ be a smooth separated $\mathcal{O}_S$-scheme whose base change to K is $Y = E - O$, where E/K is an elliptic curve with $O \in E(K)$ the zero element of the group law. Then $\mathcal{Y}(\mathcal{O}_S)$ is a finite set.*

If we start with an affine embedding $Y \to A^n_K$, then $Y(K) \cap A^n(O_S)$ is finite. In fact, let $\mathcal{Y}$ be the scheme theoretic image of $Y \to A^n_K \to A^n_{O_K}$, then the generic fiber $\mathcal{Y} \otimes_{O_K} K$ is the smooth curve Y. By [Gro66b, Théorème 12.2.4 (iii)] by enlarging S we may assume $\mathcal{Y}/\mathcal{O}_S$ is smooth. Note that $Y(K) \cap A^n_{\mathcal{O}_S} \subset \mathcal{Y}(\mathcal{O}_S)$, we deduce finiteness of the former from the latter.

To prove Theorem 8.0.1, we make use of $X \to Y' \xrightarrow{[l]} Y$ the l-twist of KP family, where $\mathbf{X} \to Y$ is the KP family (see Definition 9.0.2 and 9.0.3). Some ideas are borrowed from [Che19, Chapter 3], where finiteness is established for modular curves $Y = Y_1(N)$ when $X_1(N)$ is of genus 1.

In the following proof the reader will recognize similarity to that of S-unit equation in [LV20, Section 4], parallel to the similarity between Dirichlet's unit theorem and Mordell-Weil theorem.

Proof. Choose a prime $l > 4$ and enlarge K such that $E[l] \subset E(K)$, so $E[l]$ is isomorphic to $(\mathbb{Z}/l\mathbb{Z})^2$. Let $\mathcal{E}$ be a good $\mathcal{O}_S$-model of E and $\mathcal{X} \to \mathcal{Y}' \xrightarrow{[l]} \mathcal{Y}$ a good $\mathcal{O}_S$-model of the l-twist of KP family $X \to Y' \xrightarrow{[l]} Y$, where $\mathcal{Y}' \subset \mathcal{Y}$ is a Zariski open. By Lemma 4.0.1 and Mordell-Weil theorem, $\mathcal{E}(\mathcal{O}_S) = E(K)$ is a finitely generated abelian group. Let m be the largest integer such that $\mathcal{E}(\mathcal{O}_S)$ has an element of order l^m.

$$\textbf{Claim}: \mathcal{Y}(\mathcal{O}_S) \subset \cup_{i=0}^m [l^i]U$$

If $y' \in \mathcal{E}(\mathcal{O}_S)$ is such that $ly' \in \mathcal{Y}(\mathcal{O}_S)$, then $y' \in \mathcal{Y}'(\mathcal{O}_S) \subset \mathcal{Y}(\mathcal{O}_S)$. Let $U = \{y \in \mathcal{Y}(\mathcal{O}_S) : y \notin [l]\mathcal{E}(\mathcal{O}_S)\}$. Assume $y \in \mathcal{Y}(\mathcal{O}_S) - \cup_{i=0}^{m}[l^i]U$, $y \notin U$, so $y = ly_1$ with $y_1 \in \mathcal{Y}(O_S)$. $y_1 \notin U$, so $y_1 = ly_2$ with $y_2 \in \mathcal{Y}(O_S)$ till then $y_m = ly_{m+1}$ with $y_{m+1} \in \mathcal{Y}(O_S)$. $y = l^{m+1}y_{m+1}$. By maximality of m, the torsion group $\mathcal{E}(\mathcal{O}_S)[l^m]$ has an element $r \notin [l]\mathcal{E}(\mathcal{O}_S)$. $y = l^m(ly_{m+1} + r)$, so $r + ly_{m+1} \in \mathcal{Y}(O_S)$ and further $r + ly_{m+1} \in U$. Claim is proved.

Recall the Kummer map

$$E(K)/lE(K) \to H^1(G_K, E[l]), x \mapsto (g \mapsto gx' - x'),$$

where x' is any point with $lx' = x$. Since $E[l] \subset E(K)$, the field $K(x')$ is independent of x', written as $K(l^{-1}x)$, and $K(x')/K$ is Galois. When $y \in U$, $K(l^{-1}y)/K$ is nontrivial. The Kummer pairing embeds $G(K(y')/K)$ into $(\mathbb{Z}/l\mathbb{Z})^2$, so it is isomorphic to either $\mathbb{Z}/l\mathbb{Z}$ or $(\mathbb{Z}/l\mathbb{Z})^2$. As discussed in Section 4.5, there are only finitely many isomorphism classes of $K(l^{-1}y)/K$ when y varies over U. Therefore, it suffices to show finiteness of

$$U_L = \{y \in U : K(l^{-1}y)/K \text{ isomorphic to } L/K\}$$

for a fixed extension L/K. We may apply the Chebotarev density theorem to choose a prime v of K such that

1. $$\begin{cases} v \text{ inert in } L & \text{if } G(L/K) = \mathbb{Z}/l\mathbb{Z} \\ v = w_1 \dots w_l & \text{if } G(L/K) = (\mathbb{Z}/l\mathbb{Z})^2 \end{cases} \tag{32}$$

2. the rational prime $p > 2$ below v is unramified in L.

3. no prime of S lies above p.

We leave the rest part of proof to Lemma 8.0.3 below. $\square$

For the following, the numbering with prime in parentheses indicates to which result in [LV20] it should be compared.

Lemma 8.0.2 (4.4')**.** *Let $p > 2$ a rational prime unramified in K such that no element of S is above p. Assume that $\mathcal{E} \to \mathcal{X}$ is an elliptic scheme defined over $\mathcal{O}_S$ such that $E \to X$, the base change to K, has nontrivial monodromy. Then there are only finitely many $x \in \mathcal{X}(\mathcal{O}_T)$ such that the elliptic curve E_x/K has good reduction at all places above p but $H^1_{\acute{e}t}(E_{x,\overline{K}}, \mathbb{Q}_p)$ is not simple representation of G_K.*

Proof. Fix $x_0 \in \mathcal{X}(\mathcal{O}_S)$ such that E_{x_0} has good reductions above p. We just need to prove the finiteness of such z (i.e. those with non-simple representation) in the residue disk $\Omega_p = \{x \in \mathcal{X}(\mathcal{O}_S) : x \equiv x_0 \text{ modulo } v, \forall v|p\}$.

Consider the situation where ρ_z is not simple. Say ρ_z has a one-dimensional subrepresentation W_z, which is pure of weight 1 and crystalline at all places above p by Theorem 2.3.5 and 3.7.4. By [LV20, Lemma 2.10] (with notation loc.cit.),

$$\sum_{u|p} [K_u : \mathbb{Q}_p] a_u(W) = [K : \mathbb{Q}]/2.$$

Since $\dim W_z = 1$, the weights of Hodge filtration $a_u(W)$ are integers, and in particular, $a_w(W_z) \geq 1$ for some $w|p$.

Recall $Fil^2 H^1_{dR}(E_{z,K_u}/K_u) = 0$, so

$$Fil^1 D_{dR,w}(W_z) = D_{dR,w}(W_z) = Fil^1 H^1_{dR}(E_{z,K_w}/K_w),$$

where $D_{dR,w} : \mathrm{Rep}^{dR}_{\mathbb{Q}_p}(G_{K_w}) \to Fil_{K_w}$ is the functor in p-adic Hodge theory.

Since $\dim D_{dR,w}(W_z) = 1$, the Hodge polygon of $D_{dR,w}(W_z)$ is a slope one line, so is its Newton polygon. The slope of the Frobenius on $D_{dR,w}(W_z)$ is 1. The Hodge polygon of $H^1_{dR}(E_{z,K_w}/K_w)$ consists of a slope zero line followed by a slope one line of length 1, so the other slope of the Frobenius ϕ_w on $H^1_{dR}(E_{z,K_w}/K_w)$ is 0. The K_w-linear operator $\phi_w^{[K_w:\mathbb{Q}_p]}$ has *distinct* eigenvalues and $D_{dR,w}(W_z)$ is *the* eigenspace of slope one. (We remind that the identification $GM : H^1_{dR}(E_{z,K_w}/K_w) \to H^1_{dR}(E_{z_0,K_w}/K_w)$ is compatible with ϕ_w.)

To sum up, if $z \in \Omega_p$ and ρ_z is not simple, then there exists a place $w|p$, such that $\Phi_w(z)$ is the unique slope-1 ϕ_w-eigenline inside $H^1_{dR}(E_{z_0,K_w}/K_w)$.

The w-adic period map $\Phi_w : \{z \in \mathcal{O}_w : z \equiv x_0 \text{ modulo } w\} \to PH^1_{dR}(E_{x_0}/L_w)$ is K_w-analytic and non-constant by the results of Section 4.3 and the assumption that the monodromy is non-trivial. The conclusion follows. □

Lemma 8.0.3. *Adopt the notation of proof of Theorem 8.0.1. For fixed extension L/K and $y_0 \in U_L$, the set*

$$U_{0,L} = \{y \in U_L : y \equiv y_0 \pmod{v}\}$$

is finite.

Proof. We may enlarge S such that the twisted KP family has a good model over $\mathcal{O}_S$. By Lemma 8.0.2 and Theorem 10.0.1, there are only finitely many

$y \in \mathcal{Y}(\mathcal{O}_S)$ such that there exist $y' \in Y'_y$ with non-simple G_L-representation $V_p X_{y'}$ (Tate module). In virtue of this and [LV20, Lemma 2.2], we may assume that ρ_y is semisimple. Therefore, $\rho_y|_{G_L}$ lies in finitely many isomorphism class by Lemma 2.3.6.

The K_v-analytic period maps (21) in this case are $\Phi_{(y',u)} : \Omega_v \to \mathcal{H}_{y',u}$, where $\Omega_v = \{y \in \mathcal{Y}(\mathcal{O}_v) : y \equiv y_0 \pmod{v}\}$ and $\mathcal{H}_{y',u} = Res^{L_u}_{K_v}\mathbb{P}^1_{L_u}$, for $y' \in Y'_{y_0}$ and $u|v$ a place of L.

The functor $D_{cris,u} : \mathrm{Rep}^{cris}_{\mathbb{Q}_p}(G_{L_u}) \to \mathrm{MF}^{\phi}_{L_u}$ maps $\rho_b|_{G_{L_u}}$ to

$$(H^1_{dR}(X_b/L_u), \phi_u, \text{Hodge filtration}).$$

Here ϕ_u (acting on $H^1_{dR}(X_{y'}/L_u)$) is $\sigma_{L_u/\mathbb{Q}_p}$-semilinear and $\phi_u^{[K_v:\mathbb{Q}_p]}$ is σ_{L_u/K_v}-semilinear.

When $y \in \mathcal{Y}(\mathcal{O}_S)$, $\Phi_{(y',u)}(y)$ is in a finite union of orbits $\cup_i Z(\phi_u)h_i$, where $Z(\phi_u) \leq GL_{L_u}(H^1_{dR}(X_{y'}/L_u))$ denotes the centralizer. By [LV20, Lemma 2.1],

$$\dim_{K_v} Z_{end}(\phi^{[K_v:\mathbb{Q}_p]}) \leq (\dim_{L_u} H^1_{dR}(X_{y'}/L_u))^2 = 4.$$

By Section 4.3 and Theorem 10.0.2, $\Phi_{y',u}$ is of Zariski-dense image. Note that $\dim \mathcal{H}_{(y',u)} = [L_u : K_v] = l > 4$. Apply Lemma 4.2.1 to the period map $\Phi_{y',u}$ to finish the proof. $\square$

9 Construction of KP family

Proposition 9.0.1 (7.1'). *Let $Y = E - O$ be the K-curve under consideration and let $G = S_3 = \mathrm{Aff}(\mathbb{F}_3)$. Then there is a relative curve Z over Y with a finite morphism $Z \to Y \times E$ such that for each $y \in Y(\overline{K})$, the base extension $Z_y \to E_{\overline{K}}$ along $y : \mathrm{Spec}(\overline{K}) \to Y$ is a branched Galois cover of group G corresponding to S the G-conjugacy class of surjections $\pi_1^{geom}(Y - y, *) \to S_3$ sending σ_1, σ_2 to the same 3-cycle and τ to a transposition. Here $\{\sigma_1, \sigma_2\}$ are loops of $Y - y$ forming a geometric symplectic basis of E and τ is a loop around y.*

Note that a loop around O is necessarily sent to a transposition.

Proof. By Lemma 9.0.5, there is a finite étale morphism $Z^\circ \to Y^2 - \Delta$, so Z° is normal integral. Let Z be the normalization of Y^2 inside the function field of Z°. Then Z is normal, and finite over Y^2. The base extension $Z_{\mathbb{C}}$ is therefore normal, and finite over $Y^2_{\mathbb{C}}$. Consequently, $Z_{\mathbb{C}}$ coincides with $\mathcal{Z}$, the normalization of $Y^2_{\mathbb{C}}$ inside the function field of $\mathcal{Z}^\circ$. The proposition follows from the construction of $\mathcal{Z}$. $\square$

Definition 9.0.2 (KP family)**.** Let $Z \to Y$ be the family given by Proposition 9.0.1. Define $\mathbf{X}$ to be the relative identify component of $Pic^0_{Z\to Y}[e'']$, where e'' is that in the discussion following Definition 6.0.4 with $H = \mathbb{Z}/3\mathbb{Z}(\leq G = S_3)$. We call $\mathbf{X} \to Y$ the KP family over Y.

Then $\mathbf{X} \to Y$ is a polarized elliptic scheme. Its fiber over $y \in Y(\overline{K})$ is isogenous to $Prym(Z_y^{red}/E)$, where $Z_y^{red} \to E_{\overline{K}}$ is the subcover of degree two of $Z_y \to E_{\overline{K}}$ corresponding to $\mathbb{Z}/3\mathbb{Z} \leq S_3$,. As $Z_y \to E$ is unramified outside $\{O, y\}$, so is $Z_y^{red} \to E$. In Proposition 9.0.1 we require that a loop around y is sent to a transposition, so Z_y^{red}/E is branched at $\{y, O\}$. By Riemann-Hurwitz formula, $g(Z_y^{red}) = 2$ and $\dim Prym(Z_y^{red}/E) = 1$.

As in the S-unit equation case, we use the group law of E to twist the constructed KP family.

Definition 9.0.3. Let Y' be the pullback of Y under the isogeny $[l] : E \to E$ and $j : Y' \to Y$ be the natural open immersion. Given an abelian family $\mathbf{X} \to Y$, call the abelian-by-finite family $X \to Y' \xrightarrow{[l]} Y$ obtained by restriction to Y' its l-twist.

$$\begin{array}{ccccc} \mathbf{X} & \longrightarrow & Y & & \\ \uparrow & & \uparrow{\scriptstyle j} & & \\ X & \longrightarrow & Y' & \xrightarrow{[l]} & Y \end{array}$$

We complete the proof of Proposition 9.0.1.

Proposition 9.0.4. *Proposition 9.0.1 remains true if we replace both K and $\overline{K}$ by $\mathbb{C}$ and replace π_1^{geom} by topological π_1.*

Proof. We start by working in the analytic topology. For each $y \in Y^{an}$, let $Z_y \to E$ be the branched cover of Riemann surfaces corresponding to the said homomorphism (with $\pi(Y^{an} - y, *)$ in the place of π_1^{geom}). Define $Z = \cup_{y \in Y^{an}} Z_y$ and $f : Z \to Y \times E$ to be the natural map. Then Z has a structure of complex manifold such that f is a finite sheeted cover over $Y^2 - \Delta$, where $\Delta \subset Y^2$ is the diagonal. With this topology, f is a closed map (verified directly) of finite fiber, hence a finite (analytic) morphism. By [Har77, Appendix B, Theorem 3.2], both Z and f are algebraic over $\mathbb{C}$. $\square$

Lemma 9.0.5 (7.4'). *Settings as in Proposition 9.0.1. Let $\mathcal{Z}$ and $f : \mathcal{Z} \to (Y' \times_K E)_{\mathbb{C}}$ be as given by Proposition 9.0.4 . Let $\mathcal{Z}^\circ = f^{-1}((Y^2 - \Delta)_{\mathbb{C}})$. Then there exists a unique finite étale morphism $F : Z^\circ \to Y^2 - \Delta$ over K whose base change to $\mathbb{C}$ is $\mathcal{Z}^\circ \to Y^2_{\mathbb{C}} - \Delta$.*

Proof. Fix $y_0 \neq y_1 \in Y(\mathbb{C})$. Then y_0, y_1 are geometric points of Y. Write $y = (y_1, y_0) \in Y^2$. Let $\Gamma = \pi_1^{\text{ét}}(Y_{\overline{K}} - y_0, y_1)$, $\tilde{\Gamma}^{geom} = \pi_1^{\text{ét}}((Y^2 - \Delta)_{\overline{K}}, y)$ and $\tilde{\Gamma} = \pi_1^{\text{ét}}(Y^2 - \Delta, y)$.The universal cover of Y^{an} is biholomorphic to the unit disk, so $\pi_2(Y^{an}, y_0) = 0$. Use the terminology of [Sch78]: $\mathbb{Z}$ is locally free of type F, so belongs to $F*$. By Theorem 2 loc.cit, so is $\pi_1(Y^{an}, y_0)$, hence the sequence

$$1 \to \pi_1^{geom}(Y - y_0, y_1) \to \pi_1^{geom}(Y^2 - \Delta, y) \to \pi_1^{geom}(Y, y_0) \to 1$$

is exact by Proposition 2 loc.cit. The rest is the same as the proof of Lemma 6.0.3. □

10 The monodromy of KP family

Theorem 10.0.1. *The KP family $\Psi : \mathbf{X} \to Y$ has full topological monodromy. Explicitly, fix $y_0 \in Y(\mathbb{Q}^a) \subset Y(\mathbb{C})$,*

$$\rho_{top} : \pi_1(Y^{an}, y_0) \to SpH^1_{sing}(\mathbf{X}_{y_0}, \mathbb{Q}) \tag{33}$$

has Zariski-dense image.

Theorem 10.0.2. *For a positive integer $l \geq 1$, the l-twisted KP family $X \to Y' \to Y$ (9.0.3) is also of full monodromy.*

Proof assuming Theorem 10.0.1. The proof parallels that of [LV20, Lemma 4.3]. To fix ideas, take $l = 2$. Fix a basepoint $y_0 \in Y$ and denote by $y'_1, \dots, y'_4 \in Y'$ its preimages under $[l]$. The situation is depicted in Figure 2. The round holes are punctures, the blacks being those only for Y' but not Y. The monodromy action is

$$\pi_1(Y^{an}, y_0) \to \mathrm{GL}(\oplus_{i=1}^4 H^1_{\mathrm{sing}}(X_{y'_i}, \mathbb{Q})).$$

Let G be the Zariski closure of the image. Fix an element $(v_1, \dots, v_4) \in \oplus_{i=1}^4 H^1_{sing}(X_{y'_i})$.

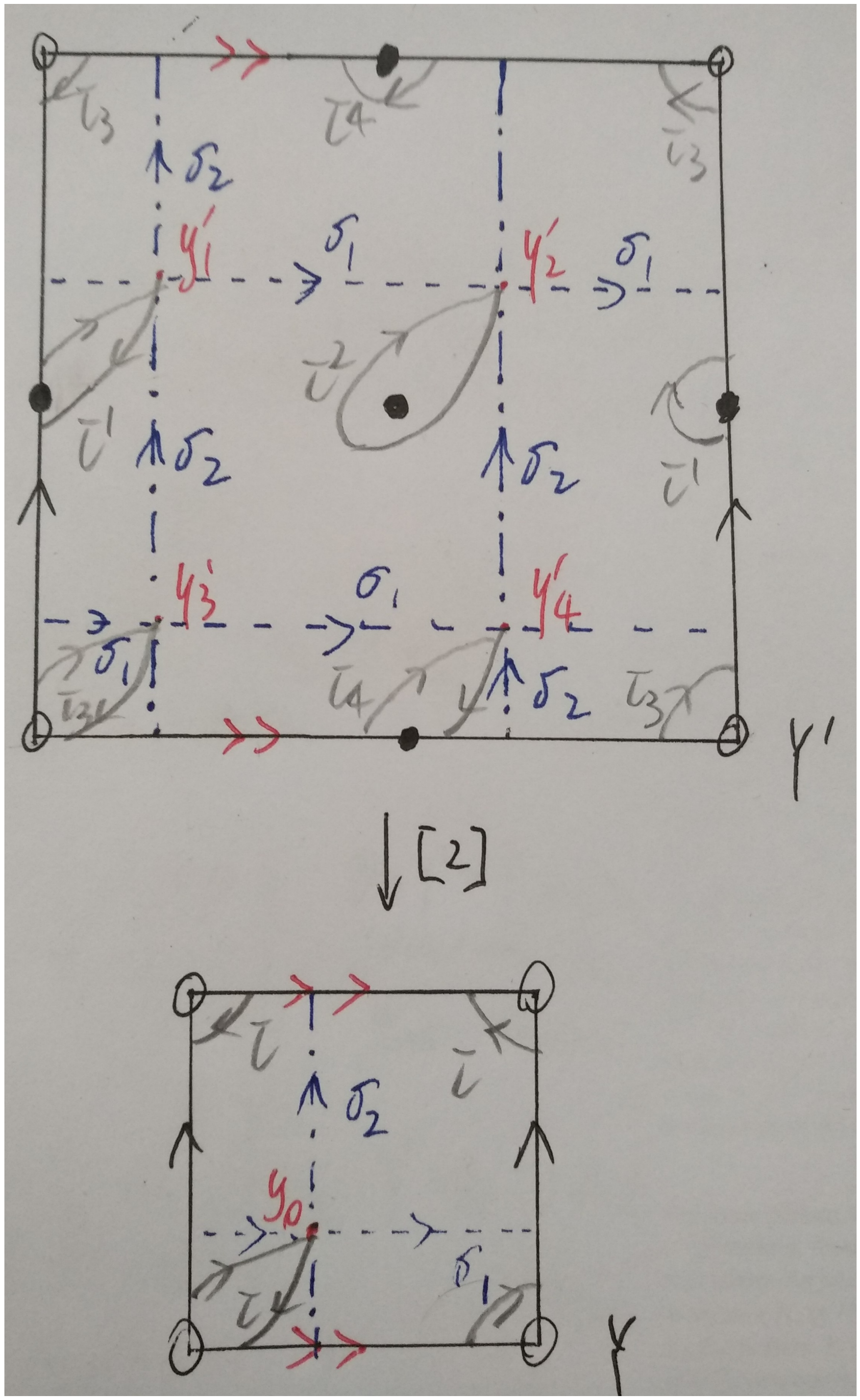

Figure 2: $Y' \xrightarrow{[l]} Y$

Take two simple closed curves $\{\sigma_1, \sigma_2\}$ of Y representing a geometric symplectic basis of $H_1(E, \mathbb{Z})$ and τ a small loop around y, all of which are of base point y_0. Each σ lifts to four paths who form two disjoint loops.

The τ-monodromy is of matrix $(Id, Id, T, Id) \in \prod_i SpH^1(X_*)$. The picked element is mapped to $(A_{12}v_2, A_{21}v_1, A_{34}v_4, A_{43}v_3)$ under the σ_1-monodromy, and to $(B_{13}v_3, B_{24}v_4, B_{31}v_1, B_{42}v_2)$ under the σ_2-monodromy.

We check the two conditions of Lemma 2.2.5 for $G \cap \prod_{i=1}^4 Sp(H^1_{\mathrm{sing}}(X_{y_i'}, \mathbb{Q}))$. As in the proof of [LV20, Lemma 4.3], we recognize a transitive action on indices $i = 1, \ldots, 4$. Iterate twice the two σ-monodromies we get the monodromies of the untwisted KP family $\mathbf{X} \to Y$. By Theorem 10.0.1, the projection to $SpH^1(X_3)$ is surjective. By transitive action on i, it remains true if we replace 3 by other indices i. The first condition is thus fulfilled.

By Borel's monodromy theorem (cf.[Kat70, Section 0.2]), T is quasi-unipotent. Recall that T is the commutator of the two σ monodromies and that $PSL_2(\mathbb{Q})$ is not abelian, so $T \notin \{\pm I_2\}$. By transitive action on i, the second condition is fulfilled. We conclude by Lemma 2.2.5. $\square$

10.1 Topological proof of Theorem 10.0.1

We make some preparation. Throughout this section we use analytic topology. Fix $y \in Y$.

Lemma 10.1.1. *For a branched cover $p : Z \to E$ of degree two with exactly two branch points $\{O, y\}$, $H_1^{Pr}(Z - p^{-1}(O, y), E - \{O, y\}) \to H_1^{Pr}(Z/E)$ is an isomorphism preserving their subspace intersection pairing.*

Proof. Let τ be a loop around $y \in E$, then its lift in Z is a simple closed curve $\tilde{\tau}$ surrounding the ramification point above y and is of degree 2 over τ.

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \mathbb{Q} \cdot [\tilde{\tau}] \arrow[r, "\times 2"] \arrow[d] & \mathbb{Q} \cdot [\tau] \arrow[d] \arrow[dddll, dotted, no head] & \\
0 \arrow[r] & H_1^{Pr}(Z - p^{-1}(O, y), Y - y) \arrow[r] \arrow[d] & H_1(Z - p^{-1}(O, y)) \arrow[r] \arrow[d] & H_1(E - \{O, y\}) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & H_1^{Pr}(Z/E) \arrow[r] \arrow[d] & H_1(Z) \arrow[r, "p_*"] \arrow[d] & H_1(E) \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}$$

We conclude by Snake Lemma. $\square$

By Lemma 10.1.1, the morphism (29) (now $MCG(Y-y)_{Z-p^{-1}(O,y)} \to MCG\big(Z - p^{-1}(O,y)\big)$) induces a monodromy map

$$MCG(Y-y)_{Z-p^{-1}(O,y)} \to Sp(H_1^{Pr}(Z,E)). \tag{34}$$

Let $Z_1, \ldots, Z_N$ be a (finite) representative system of twice-ramified double covers of E branched at $\{O,y\}$. In fact, $N=4$. Let $MCG(Y-y)_0$ denote the intersection of $MCG(Y-y)_{Z_i - p_i^{-1}(O,y)}$. Combine the maps (34) to be

$$MCG(Y-y)_0 \to \prod_{i=1}^{N} Sp(H_1^{Pr}(Z_i,E)).$$

Theorem 7.2.1 gives an exact sequence

$$1 \to \pi_1(Y,y) \to MCG(Y,y) \to MCG(Y) \to 1,$$

where $MCG(Y,y)(\leq MCG(Y-y))$ is the subgroup preserving the marking point $y \in Y$. Let $\pi_1(Y,y)_0$ and $MCG(Y,y)_0$ be the pullbacks of $MCG(Y-y)_0$ respectively.

Theorem 10.1.2 (8.1'). *Notation as above. The map*

$$\pi_1(Y,y)_0 \to \prod_{i=1}^{N} SpH_1^{Pr}(Z_i,E) \tag{35}$$

has Zariski-dense image.

Proof of Theorem 10.0.1. By the discussion following Definition 9.0.2, the degree two cover $Z_y^{red} \to E$ is branched at $\{O,y\}$, so isomorphic to one $Z_i \to E$. It follows that the fiber $\mathbf{X}_y$ is isogenous to $Prym(Z_i/E)$, hence an isomorphism

$$H_1(\mathbf{X}_y,\mathbb{Q}) \to H_1(Prym(Z_i/E),\mathbb{Q}) \to H_1^{Pr}(Z_i,E).$$

This identification is compatible with monodromy, so the conclusion follows form Theorem 10.1.2. $\square$

Note that Theorem 10.1.2 is stronger than what we need. The rest part is devoted to the proof of Theorem 10.1.2.

Proof of Theorem 10.1.2. By Lemma 10.1.3, $MCG(Y,y)_0 \to SpH_1^{Pr}(Z_i,E)$ has Zariski-dense image for each i. Because $\pi_1(Y,y)_0 \leq MCG(Y,y)_0$ is a normal subgroup of finite index, the Zariski closure of the image of (35) is a normal subgroup of finite index of the right hand side and for each i, $\pi_1(Y,y)_0 \to SpH_1^{Pr}(Z_i,E)$ has Zariski-dense image. The result follows from Lemma 2.2.4. □

Lemma 10.1.3 (8.7'). *The monodromy map*

$$MCG(Y,y)_0 \to \prod_{i=1}^{N} SpH_1^{Pr}(Z_i,E)$$

has Zariski-dense image.

Proof. It follows from Lemma 10.1.4, 10.1.5 and [LV20, Lemma 2.12]. □

Lemma 10.1.4 (8.8'). *For two non-isomorphic two-sheeted covers $Z_1,Z_2 \to E-\{O,y\}$, there exists a simple closed curve e in $E-\{O,y\}$ such that the cycle decomposition of the monodromy along e in Z_1,Z_2 are different.*

Proof. Obvious since the induced morphisms $\pi_1(Y-y,*) \to \mathbb{Z}/2\mathbb{Z}$ are different. □

Lemma 10.1.5 (8.9'). *For a double cover $p: Z \to E$ branched at $\{O,y\}$, the monodromy map $MCG(Y-y)_{Z-p^{-1}(O,y)} \to SpH_1^{Pr}(Z,Y)$ has Zariski dense image.*

Proof. By the construction of Dehn twists from liftable curves (consult the paragraph following Definition 7.3.2), as well as [LV20, Lemma 2.14], the desired result follows from Lemma 10.1.6 below. □

For a two-sheeted cover $\Sigma \to S$ of surfaces, the preimage of a liftable curve e is e^+,e^- each of degree 1 over e and $\tilde{e} = \frac{e^+-e^-}{2}$ is the projection of $[e^+] \in H_1(S,\mathbb{Q})$ to $H_1^{Pr}(\Sigma,S)$. Given a pair of liftable curves A,B, we have

$$\tilde{A}\cdot\tilde{B} = (A^+\cdot B^+) - \frac{1}{2}A\cdot B. \tag{36}$$

(Compare this to [LV20, (8.6)])

Lemma 10.1.6 (8.10'). *Given a double cover $Z \to E$ branched at $\{O,y\}$ with Z closed, there exists a pair of liftable curves (A,B) on $E-\{O,y\}$ such that $\tilde{A}\cdot\tilde{B} \neq 0$.*

Proof. Adopt the notation in proof of Theorem 10.0.2 but the loops are forbidden to pass y. Note that $[\sigma_i\tau] \in \pi_1(E - \{O, y\}, y_0)$ is represented by a simple closed curve e_i. Consider the image of $[\sigma_i] \in \pi_1(E - \{O, y\}, x)$ in $\mathbb{Z}/2\mathbb{Z}$. If both of them are trivial, then take the pair (σ_1, σ_2). If σ_1 is trivial but σ_2 not, take (σ_1, e_2). The case that σ_2 is trivial but σ_1 not follows by symmetry. If both are nontrivial, take (e_1, e_2). In each case, the pairing is nonzero by (36) since $A \cdot B = 1$. $\square$

10.2 Algebraic proof of Theorem 10.0.1

Denote by $\mathcal{M}_g$ the moduli space of genus g compact Riemann surfaces.

Lemma 10.2.1. *The map*

$$H : Y(\mathbb{C}) \to \mathcal{M}_2 \quad y \mapsto [Z_y^{red}]$$

is of finite fiber.

We introduce some notation: Let A be the set of isomorphism classes of double covers of E by genus two surface, branched at O. (Recall that two E-covers $Z_1 \to E$ and $Z_2 \to E$ are called isomorphism if there exists an E-isomorphism $Z_1 \to Z_2$.)

Let D be the underlying scheme of E, i.e., the curve forgetting the group law. Let B be the set of isomorphism classes of double covers by genus two surface over some elliptic curve, whose underlying scheme is D, branched at the origin. In the definition of B, an isomorphism between two covers p_1, p_2 signifies an isomorphism Φ and an isomorphism of elliptic curves ϕ such that the following diagram commutes.

$$\begin{tikzcd} Z_1 \arrow[r, "\Phi"] \arrow[d, "p_1"] & Z_2 \arrow[d, "p_2"] \\ E_1 \arrow[r, "\phi"] & E_2 \end{tikzcd}$$

Let C be the set of isomorphism classes of double covers of D by genus two surface. Let $N = \{(Z, \iota) : Z \in \mathcal{M}_2, \iota \in Aut(Z)\}$.

Proof. Let $F : Y(\mathbb{C}) \to A$ be the map $y \mapsto (Z_y^{red} \to E)$. Let $E^{(2)}$ be the two-fold symmetric product of E. The following diagram shows that F is injective.

$$\begin{tikzcd} Y(\mathbb{C}) \arrow[rr, "F"] \arrow[dr, hook, "y\mapsto\{O,y\}"'] & & A \arrow[dl, "\text{branch locus}"] \\ & E^{(2)} & \end{tikzcd}$$

We have forgetful maps $A \to B$ and $B \to C$. The automorphism group of a *complex elliptic curve* is finite, so $A \to B$ is of finite fiber. Since the origin is in the finite branch locus, $B \to C$ is of finite fiber. The map $C \to N$ by $(Z \to D) \mapsto (Z, \iota_{Z/D})$ is injective, where $\iota_{Z/D} \in Aut(Z/D)$ is the nontrivial element. The projection $N \to \mathcal{M}_2$ is of finite fiber by Hurwitz's automorphisms theorem. The map H is the composition of all the morphisms above, so of finite fiber. $\square$

We give another proof of Theorem 10.0.1. **Assuming the contrary from now on**, we are going to derive a contradiction by cardinality argument.

The following lemma is a first consequence.

Lemma 10.2.2. *For $y \in Y(\mathbb{C})$, the fibers $\mathbf{X}_y$ are isomorphic complex elliptic curves.*

Let Γ be the Zariski closure of the image of monodromy (33). By [Del71, Corollary 4.2.9], Γ is a semisimple proper closed subgroup of $SL_{2/\mathbb{Q}}$. Therefore Γ is a finite $\mathbb{Q}$-group scheme. Denote the generic point of Y by ξ. $R^1\Psi_*\underline{\mathbb{Q}_l}_X$ is a lisse $\mathbb{Q}_l$-sheaf on Y, and its fiber over y_0 is $H^1_{\text{ét}}(\mathbf{X}_{y_0}, \mathbb{Q}_l)$ which as a $\mathbb{Q}_l$-module is canonically isomorphic to $H^1_{sing}(\mathbf{X}_{y_0}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{Q}_l$ by Theorem 3.5.4. This isomorphism induces an injection of (abstract) groups $i : SL(H^1_{sing}(\mathbf{X}_{y_0}, \mathbb{Q})) \to SL(H^1_{\text{ét}}(\mathbf{X}_{y_0}, \mathbb{Q}_l))$. Consider the geometrical monodromy representation (cf.[CT12, Section 5.1.1])

$$\rho_{geo} : \pi_1^{\text{ét}}(Y_{\mathbb{C}}, y_0) \to SL(H^1_{\text{ét}}(\mathbf{X}_{y_0}, \mathbb{Q}_l)) \tag{37}$$

which is compatible with (33) in the sense that the following diagram is commutative:

$$\begin{tikzcd} \pi_1(Y^{an}, y_0) \arrow[r, "\rho_{top}"] \arrow[d, "j"] & SL(H^1_{sing}(\mathbf{X}_{y_0}, \mathbb{Q})) \arrow[d, hook, "i"] \\ \pi_1^{\text{ét}}(Y_{\mathbb{C}}, y_0) \arrow[r, "\rho_{geo}"] & SL(H^1_{\text{ét}}(\mathbf{X}_{y_0}, \mathbb{Q}_l)) \end{tikzcd}$$

where j is the map given by Theorem 3.3.6. By Theorem 3.3.6, $\Gamma_{\mathbb{Q}_l}$ is the Zariski closure of the image of (37). Let $W \to Y_{\mathbb{C}}$ be the connected finite étale cover corresponding to the kernel of (37) (which is an open subgroup of $\pi_1^{\text{ét}}(Y_{\mathbb{C}}, y_0)$). The morphism $W \to Y_{\mathbb{C}}$ and the scheme $Y_{\mathbb{C}}$ are affine, so $W = \mathrm{Spec}(R)$ is also affine. By [Sta23, Tag 0BQL], W is normal integral. As $W/\mathbb{C}$ is a smooth curve, R is a Dedekind domain.

Remark 10.2.3. By Lemma 2.2.6, Γ is cyclic, so the morphism (37) factors through the natural map $\pi_1^{\text{ét}}(Y_{\mathbb{C}}, y_0) \to \pi_1^{\text{ét}}(E_{\mathbb{C}}, y_0)$. The kernel of $\pi_1^{\text{ét}}(E_{\mathbb{C}}, y_0) \to SL_2(\mathbb{Q}_l)$ corresponds to a finite étale cover $p : \tilde{E} \to E_{\mathbb{C}}$. A finite étale cover of an elliptic curve is still an elliptic curve. Therefore, $W = p^{-1}(Y_{\mathbb{C}})$ is an elliptic curve with finitely many punctures. This provides another way to see that $W = \mathrm{Spec}(R)$ for some Dedekind domain R.

Let $\mathbb{C}(W) = \mathrm{Frac(R)}$ be the function field of W. Let $\eta = \mathrm{Spec}(\mathbb{C}(W)) \in W$ be the generic point. Let $P = \mathbf{X} \times_Y \mathrm{Spec}(\mathbb{C}(W))$ be the generic fiber of the abelian scheme $\mathbf{X} \times_Y W \to W$.

We have a canonical surjection $G_{\mathbb{C}(W)} \to \pi_1^{\text{ét}}(W, \bar{\eta})$ given by [Sta23, Tag 0BQM]. By the choice of W and Proposition 3.3.8, the $G_{\mathbb{C}(W)}$ action on the Tate module $T_l P$ is trivial, or equivalently, $P[l^n] \subset P(\mathbb{C}(W))$ for each $n \geq 1$. In particular, the $\mathbb{Z}$-module $P(\mathbb{C}(W))$ has infinite torsion thus is not finitely generated.

By [Con06, Example 2.2], the $\mathbb{C}(W)$-elliptic curve P is defined over $\mathbb{C}$. More precisely, there exists an elliptic curve $E'/\mathbb{C}$ such that P is isomorphic to $E'_{\mathbb{C}(W)}$.

Proof of Lemma 10.2.2. An abelian scheme over R is a Néron model of its generic fiber. By uniqueness of Néron model, $W \times_Y X = W \times_{\mathbb{C}} E'$ as W-abelian scheme. By GAGA, $W^{an} \to Y_{\mathbb{C}}^{an}$ is a finite sheeted cover. Therefore, $\mathbf{X}_y = E'$ as $\mathbb{C}$-elliptic curve for each $y \in Y(\mathbb{C})$. □

From now on, the base field is $\mathbb{C}$ unless otherwise specified.

Definition 10.2.4 (optimal cover)**.** [Dju17, Definition 1.1]Let C be a curve of genus 2 and E an elliptic curve. A covering map $\phi : C \to E$ is called optimal if whenever ϕ factors through an isogeny $\psi : E_1 \to E$ with E_1 being another elliptic curve, then ψ is an isomorphism.

For example, $\phi : C \to E$ is optimal if $\deg(\phi)$ is a prime.

Lemma 10.2.5. *[Dju17, Lemma 1.6]Let C be a curve of genus* 2 *and let $\phi : C \to E$ be an optimal cover of an elliptic curve E with* $\deg(\phi) = n$. *Then* $E' = \ker(\phi_* : J_C \to E)$ *is an elliptic curve. There exists an isogeny*

$\varphi : E \times E' \to J_C$ such that $\deg(\varphi) = n^2$ and $\ker(\varphi) \subset (E \times E')[n]$ is the graph of an isomorphism $E[n] \to E'[n]$.

Lemma 10.2.6. *Given an elliptic curve $E/\mathbb{C}$, its isogeny class is at most countable.*

It is a special case of [Har77, Ch.IV, Exercise 4.9].

Proof. For τ in $\mathcal{H}$ the upper half plane, write $\Lambda_\tau = \mathbb{Z} \oplus \mathbb{Z}\tau$ and $E_\tau = \mathbb{C}/\Lambda_\tau$. We may assume that $E = E_{\tau_0}$. Assume that $\phi : E_\tau \to E_{\tau_0}$ is a (nonzero) isogeny. Then there exists $a \in \mathbb{C}^*$ such that $a\Lambda_\tau \subset \Lambda_{\tau_0}$ and the following diagram commutes:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{a} & \mathbb{C} \\ \downarrow & & \downarrow \\ \mathbb{C}/\Lambda_\tau & \xrightarrow{\phi} & \mathbb{C}/\Lambda_{\tau_0} \end{array}$$

So, there exists integers (p, q, r, s) such that $\tau = \frac{p+q\tau_0}{r+s\tau_0}$. In particular, τ has at most countably many choices. $\square$

Lemma 10.2.7. *The image of the map H is at most countable.*

Proof. The elliptic curve $E'_y := \ker(J_{Z_y^{red}} \to E)$ is isogenous to $Prym(Z_y^{red}/E)$, so isogenous to $\mathbf{X}_y$ by the construction of KP family. By Lemma 10.2.2, $\mathbf{X}_y = E'$. In virtue of Lemma 10.2.6, there are at most countably many E'_y up to isomorphism. We fix one E'_y. By Lemma 10.2.5, $(E \times_{\mathbb{C}} E'_y)/\ker(\varphi) \to J_{Z_y^{red}}$ is an isomorphism. There are only finitely many isomorphisms $E[n] \to E'[n]$, so up to isomorphism there are only finitely many $J_{Z_y^{red}}$. By [NN81, Theorem 1.1], there are only finitely many principally polarized abelian variety $(J_{Z_y^{red}}, \lambda)$. By Torelli theorem, there are only finitely many $[Z_y^{red}] \in \mathcal{M}_2$ (and at most countably many $[Z_y^{red}]$ when E'_y is allowed to vary). $\square$

Lemma 10.2.7 and Lemma 10.2.1 forces $Y(\mathbb{C})$ to be countable. This contradiction completes the proof of Theorem 10.0.1.

11 MPN family

A far simpler abelian-by-finite family is used by Marc Paul Noordman to demonstrate Siegel' theorem in [Noo21]. We briefly present that construction.

By enlarging K suitably, we may assume that E is given by Legendre form $y^2 = x(x-1)(x-\lambda)$ for some $\lambda \in K - \{0,1\}$. Then $E[2] \subset E(K)$. There is a K-morphism $x : E \to P^1$ of degree 2 whose ramification locus $E[2]$ and branch locus is $\{0,1,\infty,\lambda\} \subset P^1(K)$. Denote $T = E - E[2]$. Define an elliptic scheme $A' \to T$, which we call MPN family, by pullback:

$$\begin{tikzcd} A' \arrow[r] \arrow[d] & T \arrow[d, "x"] \\ L \arrow[r] & P^1 - \{0,1,\infty\} \end{tikzcd}$$

where $L \to P^1 - \{0,1,\infty\}$ is the Legendre family. For a positive integer $m \geq 1$, consider the twisted MPN family

$$\begin{tikzcd} A_m \arrow[r] \arrow[d] & E - E[2m] \arrow[r, "{[2m]}"] \arrow[d, hook] & Y \\ A' \arrow[r] & T & \end{tikzcd}$$

Note that everything is defined over K. Analogue to Theorem 10.0.1, the MPN family has full monodromy.

Theorem 11.0.1. *Fix a base point $t_0 \in T(\mathbb{C})$, the monodromy representation*

$$\phi : \pi_1(T^{an}, t_0) \to \mathrm{Sp}H^1_{sing}(A'_{t_0}; \mathbb{Q})$$

has Zariski-dense image.

Proof. The morphism $x : T \to P^1 - \{0,1,\infty\}$ induces

$$\pi_1(x) : \pi_1(T^{an}, t_0) \to \pi_1(\mathbb{CP}^1 - \{0,1,\infty\}, x(t_0))$$

The two-sheeted covering $T^{an} \to \mathbb{CP}^1 - \{0,1,\infty,\lambda\}$ identifies $\pi_1(T^{an}, t_0)$ as an index 2 subgroup of $\pi_1(\mathbb{CP}^1 - \{0,1,\infty,\lambda\}, x(t_0))$.

$$\begin{tikzcd} \pi_1(T^{an}, t_0) \arrow[r, "\phi"] \arrow[d, "\pi_1(x)"] & \mathrm{Sp}H^1_{sing}(A'_{t_0}; \mathbb{Q}) \arrow[d, "="] \\ \pi_1(\mathbb{CP}^1 - \{0,1,\infty\}), x(t_0) \arrow[r, "\psi"] & \mathrm{Sp}H^1_{sing}(L_{x(t_0)}; \mathbb{Q}) \end{tikzcd}$$

Because Legendre family is of full monodromy, and the natural morphism

$$\pi_1(\mathbb{CP}^1 - \{0, 1, \infty, \lambda\}, x(t_0)) \to \pi_1(\mathbb{CP}^1 - \{0, 1, \infty\}, x(t_0))$$

is surjective, the index of the Zariski closure G of the image of ϕ is at most 2 in SL_2. Recall that SL_2 is Zariski connected, so $G = SL_2$. □

A counterpart of Theorem 10.0.2 follows by parallel proof. (We can also cite [LV20, Lemma 2.12] in stead of Lemma 2.2.5 in the end.)

Theorem 11.0.2. *For $m \geq 2$, the twisted MPN family $A_m \to E - E[2m] \to Y$ is of full monodromy.*

Given these two theorems, Lemma 8.0.3 and hence Siegel's theorem (Theorem 8.0.1) follow immediately.

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