Group extensions of complex Lie groups

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1 Introduction

In the history of cohomology theory of abelian varieties over positive characteristic fields, the study of group extension problem played an important role. For instance, Rosenlicht obtains Fact 1.0.1 through considering vectorial extensions of abelian varieties. Let k be an algebraically closed field and A/k be an abelian variety with $\dim A = g$. The dual abelian variety of A is denoted by $A^\vee$.

Fact 1.0.1. *[Ros58, Theorem 1 and 2] The dimension of the k-vector space $H^1(A, O_A)$ is g.*

A notable byproduct of Rosenlicht's work is the existence of the following object, the so-called *universal vectorial extension*.

Fact 1.0.2. *[Ros58, Prop. 11] There is a short exact sequence[1] of commutative algebraic groups over k: $0 \to \mathbb{G}_a^g \to A^\natural \to A \to 0$, where $A^\natural$ is the moduli space of line bundles equipped with an integrable connection on $A^\vee$.*

In [Rot96, (1.17)] and [Lau96, Thm. 3.2.1], it is proved that the Fourier-Mukai transform $D^b(\mathrm{Qch}(O_A)) \to D^b(\mathrm{Qch}(O_{A^\vee}))$ lifts to an equivalence $D^b(\mathrm{Qch}(O_{A^\natural})) \to D^b(\mathrm{Qch}(D_{A^\vee}))$, where for a smooth algebraic variety M/k, $\mathrm{Qch}(O_M)$ (resp. $\mathrm{Qch}(D_M)$) refers to the category of O_M (resp. left D_M) modules that are O_M-quasi-coherent.

The cohomology theory of complex analytic analogue of abelian varieties, namely complex tori, is elementary. By contrast, as far as we know, the existence of universal vectorial extension in the analytic setting is not covered in the literature, though admittedly easier and should be known. The main results are summarized imprecisely in Proposition 1.0.3 and Theorem 1.0.4.

Proposition 1.0.3 (Proposition 4.3.1)**.** *For two commutative complex Lie groups A, B, the* commutative *extensions of A by B are classified by the abelian group*

$$\mathrm{Ext}^1_{\mathbb{Z}}(\pi_0(A), \pi_0(B)) \oplus \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), \pi_0(B)) \oplus \mathrm{coker}(s).$$

Here s is the restriction morphism $\mathrm{Hom}_{\mathrm{Vec}}(L(A), L(B_0)) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B_0)$, A_0 (resp. B_0) signifies the identity component of A (resp. B), the notation $\pi_1(*)$ refers to the fundamental group, and $A/A_0 = \pi_0(A)$ denotes the 0-th homotopy group of A and similar for B.

Theorem 1.0.4. *Let A be a complex torus of dimension g. Then:*

- (Theorem 5.2.4 (resp. Theorem 5.3.2)) *The dual torus* $\mathrm{Pic}^0(A)$ *(resp. tangent space $T_0A = H^1(A, O_A)$) naturally classifies the extensions of A by the multiplicative group $\mathbb{C}^*$ (resp. additive group $\mathbb{C}$).*
- (Proposition 5.4.5 1, Proposition 5.4.7) *There is an extension*

$$0 \to H^0(A^\vee, \Omega^1_{A^\vee}) \to (\mathbb{C}^*)^{2g} \to A \to 0$$

that is universal among all vectorial extensions of A.

[1] in the sense of [Ros58, Sec. 2, p.691]

We emphasis some differences between the analytic case and the algebraic case. For a complex torus A, let $\mathrm{Div}(A)$ be the group of analytic divisors on A modulo linear equivalence. Let $\mathrm{Pic}(A)$ be the group of isomorphic classes of line bundles on A. The natural map $\mathrm{Div}(A) \to \mathrm{Pic}(A)$ is surjective if and only if A is an abelian variety ([Deb05, Sec. 4.3, Cor. 4]). This is why the Picard group is used in Theorem 5.2.4 while divisor group appears in its algebraic analogue ([Wei49, no. 2], [Ser88, Thm. 6]). Discrete groups like $\mathbb{Z}$ are not (finite type) algebraic groups, but there is no reason to exclude them as complex Lie groups. Plenty of important analytic morphisms are not algebraic, like the universal covering (exponential map) $\exp : \mathbb{C} \to \mathbb{C}^*$.

The organization is as follows. The main goal of this text is to classify extensions of complex Lie groups. Section 2 contains preliminaries about complex Lie groups. In Section 3 we define complex Lie group extensions and give several first results about the classification. Then we focus on commutative extensions in Section 4. Commutative extensions of complex tori deserve extra attention, and they are discussed in Section 5. Some extensions with complex-tori base are automatically commutative, as Section 6 shows. Noncommutative extensions are treated superficially in Section 7.

Convention and notation

A statement about Lie groups is understood to hold for both real and complex Lie groups. The topology underlying a Lie group is always assumed to be second countable.[2]

For every Lie group G, the identity component of G is denoted by G_0. The Lie algebra of G is written as $L(G)$. And $Z(G)$ denotes the center of G. The automorphism group of G is denoted by $\mathrm{Aut}(G)$. Let $\mathrm{Inn} : G \to \mathrm{Aut}(G)$ be the group morphism defined by taking conjugation $g \mapsto g \bullet g^{-1}$. Then the subgroup $\mathrm{Inn}(G)$ of inner automorphisms is normal in $\mathrm{Aut}(G)$. Let $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ be the group of outer automorphisms. Let G^{op} be the Lie group opposite to G. (If G is complex, then so is G^{op}.) There is a natural identification of real/complex manifolds $G \to G^{\mathrm{op}}$ denoted by $g \mapsto g^*$. If G is connected, then the universal covering group of G is denoted by $\tilde{G}$ and the fundamental group of G with the identity e_G as the base point is denoted by $\pi_1(G)$.

Complex Lie subgroups refer to embedded closed complex Lie subgroups. If G is a complex Lie group and $S \subset G$ is a subset, by [HN11, Exercise 15.1.3 (b)] there is a smallest complex Lie subgroup of G containing S, called the complex Lie subgroup generated by S.

Let Vec (resp. Ab, resp. $\mathcal{C}$, resp. Set) be the category of finite dimensional complex vector spaces (resp. abelian groups, resp. commutative complex Lie groups, resp. sets). For a complex manifold X and a commutative complex Lie group B, let $\mathcal{B}_X$ be the abelian sheaf on X of germs of holomorphic maps from X to B.

[2] A partial reason for such restriction is that, in this case, Condition (2) of [Hoc51b, Definition 1.1] is implied by Condition (1), showed in p.542 *loc.cit.*

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2 Generalities on complex Lie groups

Two fundamental facts about complex Lie groups are recalled.

Fact 2.0.1 ([Bou72, Ch. III, §3, no.8, Prop. 28]). *Let $f : G \to H$ be a morphism of complex Lie groups. Then:*

1. $\ker(f)$ *is a normal complex Lie subgroup of G and* $L(\ker(f)) = \ker(d_e f : L(G) \to L(H))$.

2. *If $f(G)$ is* closed *in H, then $f(G)$ is a complex Lie subgroup of H, and f induces a complex Lie group isomorphism $G/\ker(f) \to f(G)$. In particular, if f is surjective, then $d_e f : L(G) \to L(H)$ is surjective. If f bijective, then f is an isomorphism.*

Remark 2.0.2. Fact 2.0.1 2 fails if the topology of G is not assumed to be second countable. For example, let τ (resp. τ') be the discrete topology (resp. the Euclidean topology) of $\mathbb{C}$, then $\mathrm{Id} : (\mathbb{C}, \tau) \to (\mathbb{C}, \tau')$ is a bijective morphism but not open.

Right principal bundle is defined in [Bou07, 6.2.1]. Left principal bundle can be defined similarly.

Fact 2.0.3 ([HBS66, Thm. 3.4.3], [Bou72, Ch. III, §1, Propositions 10 and 11]). *Suppose G is a complex Lie group and K is a normal complex Lie subgroup of G. Then the group G/K has a unique structure of complex manifold, such that the quotient map $\pi : G \to G/K$ is a submersion.*[3] *With this structure, G/K is a complex Lie group and p is a left principal K-bundle under the natural left group action $K \times G \to G$ defined by $(k, g) \mapsto kg$. In particular, every surjective morphism of complex Lie groups is open.*

We recall that principal bundles are classified by the first sheaf cohomology, in the following way. Let X (resp. B) be a complex manifold (resp. commutative complex Lie group). Let S be the set of isomorphism classes of principal B-bundles[4] over X. Define a map

$$\Psi : S \to H^1(X, \mathcal{B}_X) \tag{1}$$

as follows. For every $[p : P \to X] \in S$, there exists an open cover $\{U_i\}_{i \in I}$ of X and a family of local trivializations $f_i : U_i \times B \to p^{-1}(U_i)$ for every $i \in I$.

[3] in the sense of [Bou07, 5.9.1]

[4] Here B is commutative, so it is unnecessary to specify the principal bundle to be left or right.

For any indices $i, j \in I$ and every $x \in U_i \cap U_j$, there exists a unique element $b_{ij}(x) \in B$ such that $b_{ij}(x) \cdot f_i(y) = f_j(y)$ for all $y \in p^{-1}(x)$. Hence a morphism $b_{ij} : U_i \cap U_j \to B$ of complex manifolds. Moreover, for any indices $i, j, k \in I$ and every $x \in U_i \cap U_j \cap U_k$, they satisfy the 1-cocycle relation $b_{ij}(x) + b_{jk}(x) + b_{ki}(x) = 0$. Thus, the family $\{b_{ij}\}_{i,j \in I}$ defines an element $\Psi(p)$ of $H^1(X, \mathcal{B}_X)$.

As per [HBS66, 3.2 b), p.41], the map Ψ is bijective. The structure of abelian group on $H^1(X, \mathcal{B}_X)$ is translated to S *via* Ψ. The zero element of S is the class of the trivial principal B-bundle. For every pair $[p_1 : P_1 \to X]$ and $[p_2 : P_2 \to X]$ in S, by taking a family of trivialization for each p_i, we can define a morphism $\phi : P_1 \times_X P_2 \to P_1 + P_2$ of principal B-bundles on X such that or every $b, b' \in B$, $u \in P_1$, $v \in P_2$ with $p_1(u) = p_2(v)$, one has

$$\phi(b \cdot u, b' \cdot v) = (b + b') \cdot \phi(u, v). \tag{2}$$

In particular, ϕ is surjective. Restricted to the fiber at some $x \in X$, ϕ is induced by the group law of B and the chosen trivializations.

We need a complex version of Cartan's subgroup theorem. Notice that a real analytic closed subgroup of a complex Lie group may not be a complex analytic subset. Lemma 2.0.4 is mentioned in [Bjö13, p.513].

Lemma 2.0.4. *Let X be a complex manifold, $Y \subset X$ be a complex analytic subset. If $p \in Y$ is a smooth point of Y, then near p, the subset Y is an embedded complex submanifold of X.*

Proof. As the problem is local, we may assume that X is an open subset $\mathbb{C}^n$, there exist $f_1, \ldots, f_m \in O_X(X)$ with $O_{X,p}/(f_1, \ldots, f_m) = O_{Y,p}$ and $Y = Z(f_1, \ldots, f_m)$. Let $r = \mathrm{rank}_p(f_1, \ldots, f_m)$. By reordering subscripts, one may assume

$$\det(\frac{\partial f_i}{\partial z_j})_{1 \le i,j \le r} \neq 0.$$

Then $(f_1, \ldots, f_r) : X \to \mathbb{C}^r$ is a holomorphic submersion near p. Therefore, near p, the subset $Z(f_1, \ldots, f_r)$ is an embedded complex submanifold of X of dimension $n - r$. By the Jacobian criterion (see, *e.g.*, [GR12, p.114]), $\mathrm{emb}_p Y = n - r$. By the criterion of smoothness ([GR12, p.116]), $\dim_p Y = n - r$. Now that $Y \subset Z(f_1, \ldots, f_r)$, near p the subset Y is an irreducible component of $Z(f_1, \ldots, f_r)$, hence also an embedded complex submanifold of X. $\square$

Corollary 2.0.5 contains [Lee01, Prop. 1.23] as a special case.

Corollary 2.0.5 (Complex Cartan subgroup theorem)**.** *Let G be a complex Lie group, and let H be a subgroup that is a complex analytic subset of G. Then H is a complex Lie subgroup of G.*

Proof. Endow H with the induced structure of reduced complex analytic space. By [GR12, p.117], the complex analytic space H has a smooth point p. For every $q \in H$, the left multiplication by qp^{-1} gives a biholomorphic map $G \to G$, which sends H to H and maps p to q. Therefore, q is also a smooth point of H. By Lemma 2.0.4, H is a complex submanifold of G near q for all $q \in H$. Thus, H is a complex submanifold of G and hence a complex Lie subgroup. $\square$

In Lemma 2.0.6, if G is furthermore connected, then the result of is contained in [Bou72, Ch.III, Sec. 6, no. 4, Cor. 4].

Lemma 2.0.6. *Let G be a complex Lie group. Then the center $Z(G)$ is a complex Lie subgroup of G.*

Proof. The holomorphic map $G \times G \to G$ defined by $(x, y) \mapsto yxy^{-1}$ is a group action of G on itself. By [Bou72, Ch. III, Sec. 1, no. 7, Prop. 14], for every $x \in G$, the stabilizer $C_G(x)$ of $x \in G$ is a complex Lie subgroup of G. Therefore, so is $Z(G) = \cap_{x\in G} C_G(x)$ by [HN11, Exercise 15.1.3 (a)]. □

A complex Lie group isomorphic to a complex Lie subgroup of $\mathrm{GL}_n(\mathbb{C})$ for some integer $n \geq 1$ is called *linear*. Proposition 2.0.7, due to Matsushima and Morimoto, is a characterization of commutative linear complex Lie groups.

Proposition 2.0.7. *Let B be a connected commutative complex Lie group. Then the following conditions are equivalent:*

1. *B is isomorphic to $\mathbb{C}^m \times (\mathbb{C}^*)^n$ for some integers $m, n \geq 0$;*
2. *the complex Lie group B is linear;*
3. *B is a Stein group (i.e., the underlying complex manifold is a Stein manifold).*

In that case, the pair (m, n) is unique.

Proof. See [HN11, Exercise 15.3.1] for the fact that 1 implies 2. Since $\mathrm{GL}_n(\mathbb{C})$ is a Stein manifold, 2 implies 3. As per [MM60, Proposition 4], 3 implies 1. The uniqueness is contained in the Remmert-Morimoto decomposition (see, *e.g.*, [AK01, Thm. 1.1.5]). □

Remark 2.0.8. The commutativity of B in Proposition 2.0.7 is important. In fact, there is a connected Stein group that is not linear ([Ari19, Sec.1]). This differs from the algebraic case where every algebraic group that is an affine variety is linear ([Mil17, Cor. 4.10]).

In some sense, Definition 2.0.9 is an antipode to Stein groups.

Definition 2.0.9. A connected complex Lie group on which every holomorphic function is constant is called a toroidal group.[5]

Complex tori are toroidal groups, but there exist toroidal groups that are not compact ([AK01, p.1]). Every toroidal group is a semi-torus in the sense of [NW13, Def. 5.1.5].

By [AK01, 1.1.5], every connected commutative complex Lie group G is uniquely isomorphic to $\mathbb{C}^l \times (\mathbb{C}^*)^m \times X$ with a toroidal group X. In particular, G can be presented as an extension of a complex torus by a connected linear group. (From [NW13, pp.169-170], a semi-torus can admit nonequivalent presentations, while semiabelian varieties admit exactly one algebraic presentation.)

[5] also known as a Cousin group

3 Group extensions

Given a surjective Lie group morphism $p : E \to Q$, by Fact 2.0.1, $K := \ker(p)$ is a normal Lie subgroup of E and the induced morphism $E/K \to Q$ is an isomorphism. We write it as

$$1 \to K \xrightarrow{i} E \xrightarrow{p} Q \to 1 \tag{3}$$

and call it a short exact sequence. In that case, E is called an *extension* of the base Q by the extension kernel K. Moreover, $d_e p : L(E) \to L(Q)$ is surjective of kernel $L(K)$, hence an extension of Lie algebras

$$0 \to L(K) \to L(E) \xrightarrow{d_e p} L(Q) \to 0.$$

When $K \subset Z(E)$, such an extension is called *central*. If (3) is a central extension with Q commutative, as in [MRM74, p.222], using Fact 2.0.3 one can construct a skew-symmetric bimorphism

$$e : Q \times Q \to K, \tag{4}$$

to measure the deviation of E from commutativity. Indeed, the group E is commutative if and only if e is constant.

Several topological properties of Lie groups are preserved by extensions.

Fact 3.0.1. *If K, Q in (3) are compact (resp.connected, resp. discrete), then so is E.*

Proof. The statement concerning connectedness is in [Che46, Prop. 2, p.36]. The others are consequences of Fact 2.0.3. □

Fact 3.0.2 ([HN11, Cor. 16.3.9])**.** *If (3) is a central extension of complex Lie groups, where K is finite and E is connected, then Q is linear if and only if E is linear.*

The finiteness of K in Fact 3.0.2 is necessary. Consider the exact sequence $0 \to \mathbb{Z}^2 \to \mathbb{C} \to A \to 0$ defining a complex torus A. Here $\mathbb{Z}^2$ and $\mathbb{C}$ are linear, while A is not.

Similarly, an extension E of a finite group Q by a linear group K is linear. Indeed, let $\rho : K \to \mathrm{GL}_n(\mathbb{C})$ be a faithful representation, then the induced representation $\mathrm{Ind}_K^E \rho : E \to \mathrm{GL}_{mn}(\mathbb{C})$ is also faithful, where $m = \#Q$. Again, the finiteness of Q is essential here. Example 3.0.3 shows the statement fails when Q is only discrete and linear but infinite.

Example 3.0.3. Work of Deligne [Del78] (see also [KRW20, p.470]) shows that for any integers $g \geq 2, n \geq 3$, there is a central extension $1 \to \mathbb{Z}/n \to G \to \mathrm{Sp}_{2g}(\mathbb{Z}) \to 1$ for which G is not residually finite. By Malcev's theorem ([Mal40, Thm. VII]; see also [Nic13, p.1]), the discrete complex Lie group G is not linear, even though $\mathbb{Z}/n$ and $\mathrm{Sp}_{2g}(\mathbb{Z})$ are linear.

We turn to the classification of extensions. Two Lie group extensions C and C' of B by A are called equivalent if there exists a morphism $f : C \to C'$ making a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & B & \longrightarrow & C & \longrightarrow & A & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \mathrm{Id}} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{Id}} & & \\ 0 & \longrightarrow & B & \longrightarrow & C' & \longrightarrow & A & \longrightarrow & 0. \end{array}$$

In this case, f is bijective, hence an isomorphism by Fact 2.0.1. The *trivial* extension of Q by K refers to the equivalence class of the obvious sequence

$$1 \to K \to K \times Q \to Q \to 1.$$

Fact 3.0.4 ([Bou72, Ch.III, no.4, Prop. 8]). *The Lie group extension (3) is trivial if and only if there is a morphism $r : E \to K$ with $ri = \mathrm{Id}_K$. The extension is a semidirect product if and only if there is a morphism $s : Q \to E$ with $ps = \mathrm{Id}_Q$.*

The extension (3) defines a group morphism $\psi : Q \to \mathrm{Out}(K)$, called the *outer action* corresponding to the extension. We call (K, ψ) the extension kernel of (3). Equivalent extensions induce the same outer action. For two complex Lie groups Q, K and a group morphism $\psi : Q \to \mathrm{Out}(K)$, denote by $\mathrm{Ext}(Q, K, \psi)$ the set of equivalence classes of extensions of Q by K with outer action ψ.

Since the center $Z(K)$ is a characteristic complex Lie subgroup of K by Lemma 2.0.6, there is a canonical group morphism $\mathrm{Aut}(K) \to \mathrm{Aut}(Z(K))$ which passes to another group morphism $\mathrm{Out}(K) \to \mathrm{Aut}(Z(K))$. Hence a group morphism

$$\psi_0 : Q \to \mathrm{Aut}(Z(K)) \tag{5}$$

induced by ψ. When K is *commutative*, $\psi = \psi_0$ and the construction of Baer sum ((42) and [FLA19, p.444]) makes $\mathrm{Ext}(Q, K, \psi)$ an abelian group.

3.1 Pullback and pushout

Extensions can be pulled back.

Example 3.1.1 (Pullback). Given a morphism $g : Q' \to Q$ of complex Lie groups, pulling (3) back along g gives an extension of Q' by K as follows.

The map $E \times Q' \to Q$ defined by $(x, h') \mapsto p(x)^{-1} g(h')$ is holomorphic, so the preimage E' of the identity element $e_Q \in Q$ is an analytic subset of $E \times Q'$. As $E' = \{(x, h') \in E \times Q' : p(x) = g(h')\}$ is a subgroup of $E \times Q'$, by Corollary 2.0.5, E' is a complex Lie subgroup of $E \times Q'$ (which is the extension group). Let $p' : E' \to Q'$ and $\epsilon : E' \to E$ be the projections. Then the triple (E', ϵ, p') is the fiber product $E \times_Q Q'$ in the category of complex Lie groups.

For every $h' \in Q'$, by surjectivity of p, there is $x \in E$ with $p(x) = g(h')$. Then $(x, h') \in E'$ with $p'(x, h') = h'$. Hence p' is surjective.

Define a morphism $i' : K \to E'$ by $i'(k) = (k, e_{Q'})$. Then i' is injective. Since $p'i'$ is trivial, $i'(K) \subset \ker(p')$. Conversely, for every $(x, h') \in \ker(p')$, $h' = e_{Q'}$

and $p(x) = g(e_{Q'}) = e_Q$. Thus, $x \in K$ and $(x, h') = i'(x) \in i'(K)$. Hence a commutative diagram with exact rows

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r, "i'"] \arrow[d, "\mathrm{Id}"] & E' \arrow[r, "p'"] \arrow[d, "\epsilon"] & Q' \arrow[r] \arrow[d, "g"] & 1 \\
1 \arrow[r] & K \arrow[r, "i"] & E \arrow[r, "p"'] & Q \arrow[r] & 1.
\end{tikzcd}$$

The first row is called the pullback extension of (3) along g. Its outer action is $\psi g : Q' \to \mathrm{Out}(K)$. Hence a map $\mathrm{Ext}(Q, K, \psi) \to \mathrm{Ext}(Q', K, \psi g)$. It is a group morphism when K is commutative ([Hoc51a, p.99]).

The universal property of pullback shows that the first row of every such commutative diagram is determined by the second row and $g : Q' \to Q$. By construction, the pullback of a central extension is also central.

A pushout extension along a morphism $f : K \to K'$ of complex Lie groups may not exist. When it exists, it satisfies a universal property.

Lemma 3.1.2. *Consider a commutative diagram of complex Lie groups, where each row is exact*

$$\begin{tikzcd}
1 \arrow[r] & K \arrow[r] \arrow[d, "f"] & E \arrow[r, "p"] \arrow[d, "m"] & Q \arrow[r] \arrow[d, "\mathrm{Id}"] & \text{‘} \\
1 \arrow[r] & K' \arrow[r, "\iota"] & E' \arrow[r, "\pi"] & Q \arrow[r] & 1.
\end{tikzcd} \tag{6}$$

Then the triple (E', m, ι) has the following universal property: For every commutative diagram of complex Lie groups

$$\begin{tikzcd}
K \arrow[r, "i"] \arrow[d, "f"] & E \arrow[d, "m"'] \arrow[ddr, "\phi"] & \\
K' \arrow[r, "\iota"] \arrow[drr, "\psi"'] & E' \arrow[dr, dashed, "\eta"] & \\
 & & H
\end{tikzcd}$$

with $\psi(m(c)^{-1} b m(c)) = \phi(c)^{-1} \psi(b) \phi(c)$ for every $c \in E$ and $b \in K'$, there exists a unique morphism $\eta : E' \to H$ keeping the diagram commutative.

In particular, up to a unique equivalence, the second row of (6) has at most one choice when the first row and $f : K \to K'$ are given.

Proof. We construct a map $\eta : E' \to H$ as follows. For every $c' \in E'$, there exists $c \in E$ with $p(c) = \pi(c')$. Let $b' = m(c)^{-1} c'$. Then $\pi(b') = p(c)^{-1} \pi(c') = e_Q$, so $b' \in K'$. Define $\eta(c') = \phi(c) \psi(b')$.

To show that η is well-defined, we claim that $\eta(c')$ is independent of the choice of c. Indeed, take another $c_1 \in E$ with $p(c_1) = \pi(c')$, then $p(c^{-1} c_1) = e_Q$, hence $c^{-1} c_1 \in K$. This time the element in K' is $b'_1 = m(c_1)^{-1} c'$, so

$b' = f(c^{-1}c_1)b'_1$ in K' and hence $\psi(b') = \phi(c^{-1}c_1)\psi(b'_1)$. Therefore, $\phi(c)\psi(b') = \phi(c_1)\psi(b'_1)$ in H as claimed.

We check that η is holomorphic near $c' \in E'$. Indeed, by Fact 2.0.3, there is an open neighborhood U of $\pi(c') \in Q$, and a holomorphic map $s : U \to E$ with $ps = \mathrm{Id}_U$. The map

$$\pi^{-1}(U) \to U \times K', \quad x \mapsto (\pi(x), [ms\pi(x)]^{-1}x)$$

is biholomorphic. The map

$$U \times K' \to H, \quad (u, b') \mapsto \phi(s(u))\psi(b')$$

is holomorphic. The composition is exactly $\eta|_{\pi^{-1}(U)}$.

We check that η is a group morphism. For $c'_i \in E'$ $(i = 1, 2)$, choose $c_i \in E$ with $p(c_i) = \pi(c'_i)$. Then for $c'_1c'_2$ we can choose c_1c_2. Let $b'_1 = m(c_1)^{-1}c'_1$ and $b'_2 = m(c_2)^{-1}c'_2$. Then

$$b' := m(c_1c_2)^{-1}c'_1c'_2 = m(c_2)^{-1}b'_1m(c_2)b'_2.$$

By the construction of η, one has

$$\begin{aligned} &\eta(c'_1c'_2) = \phi(c_1c_2)\psi(b') \\ =&\phi(c_1)\phi(c_2)\psi[m(c_2)^{-1}b'_1m(c_2)]\psi(b'_2) \\ =&\phi(c_1)\psi(b'_1)\phi(c_2)\psi(b'_2) = \eta(c'_1)\eta(c'_2) \end{aligned}$$

Then η is a morphism of complex Lie groups. By construction, η is the unique group morphism keeping the diagram commutative. □

Example 3.1.3. Assume that Q is connected. As the map $p : E \to Q$ in (3) is open by Fact 2.0.3, $p(E_0)$ is a nonempty open subgroup of Q and hence $p(E_0) = Q$ by the connectedness of Q. Then the following diagram is commutative and each row is exact

$$\begin{tikzcd} 1 \arrow[r] & K \cap E_0 \arrow[r] \arrow[d] & E_0 \arrow[r] \arrow[d, hook] & Q \arrow[r] \arrow[d, "\mathrm{Id}"] & 1 \\ 1 \arrow[r] & K \arrow[r] & E \arrow[r] & Q \arrow[r] & 1 \end{tikzcd}$$

By Lemma 3.1.2, the second row is determined by the inclusion $K \cap E_0 \to K$ (an open normal subgroup) and the first row.

3.2 Rudimentary classification

Let K, Q be complex Lie groups, where Q is discrete. Consider an abstract group extension $1 \to K \to E \to Q \to 1$. Then as a set $E = \sqcup_x xK$, where x runs through a set of left representatives of E/K. Thus E admits a unique complex manifold structure making the maps holomorphic. However, the group law of E needs not to be holomorphic in this complex structure. The semidirect product sequence $1 \to \mathbb{C} \to \mathbb{C} \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1$ serves as an example, where $\mathbb{Z}/2$ acts on $\mathbb{C}$ by complex conjugation. But when the base is discrete and the outer action is trivial, the Lie group extension problem reduces to the abstract group extension problem.

Proposition 3.2.1. *Let K, Q be complex Lie groups. If Q is discrete, then the natural forgetful map $\phi : \mathrm{Ext}(Q, K, 1) \to \mathrm{Ext}_{\mathrm{Abs}}(Q, K, 1)$ is bijective, where $\mathrm{Ext}_{\mathrm{Abs}}(Q, K, 1)$ denotes the set of isomorphism classes of* abstract *group extensions of Q by K with trivial outer action. In fact, for every abstract group extension $1 \to K \to E \to Q \to 1$, E admits a unique complex manifold structure making the sequence an extension of complex Lie groups.*

Proof. We prove that ϕ is injective. Consider $E_1, E_2 \in \mathrm{Ext}(Q, K, 1)$ with $\phi(E_1) = \phi(E_2)$. Then there is an abstract group isomorphism $f : E_1 \to E_2$ making a commutative diagram

$$\begin{tikzcd} K \arrow[r, hook] \arrow[dr] & E_1 \arrow[d, "f"] \\ & E_2. \end{tikzcd}$$

For every $x \in E_1$, the restriction $xK \to f(x)K$ of f is holomorphic, since the left multiplication $K \to xK$ (resp. $K \to f(x)K$) by x (resp. $f(x)$) in E_1 (resp. E_2) is biholomorphic. Thus, f is holomorphic and hence an equivalence of complex Lie group extensions.

We prove that ϕ is surjective. Given an abstract group extension $1 \to K \to E \to Q \to 1$ in $\mathrm{Ext}_{\mathrm{Abs}}(Q, K, 1)$, we endow E with the complex structure making the maps holomorphic. We show the group law $m : E \times E \to E$ is holomorphic. Choose a set-theoretic section $s : Q \to E$. Then the map $K \times Q \to E$ defined by $(a, b) \mapsto as(b)$ is biholomorphic. With this identification, m becomes the map

$$\mu : K \times Q \times K \times Q \to K \times Q, \quad (a, b, a', b') \mapsto (as(b)a's(b')s(bb')^{-1}, bb') = (a\rho(a')s(b)s(b'))s(bb')^{-1}, bb'),$$

where $\rho : K \to K$ is $x \mapsto s(b)xs(b)^{-1}$. Since the outer action is trivial, $\rho \in \mathrm{Inn}(K)$. Therefore, the map $K \times K \to K$ defined by $(a, a') \mapsto a\rho(a')$ is holomorphic. Because Q is discrete, μ (and hence m) is holomorphic. Then E is a complex Lie group and the abstract extension lifts to $\mathrm{Ext}(Q, K, 1)$. $\square$

Corollary 7.2.6 below is a result about discrete base with nontrivial outer action. We turn to two other simple cases.

Proposition 3.2.2. *Every extension of $\mathbb{C}$ is a semidirect product. In particular, every central extension of $\mathbb{C}$ trivial.*

Proof. Let $0 \to B \to C \xrightarrow{p} \mathbb{C} \to 0$ be an extension. Then $0 \to L(B) \to L(C) \xrightarrow{d_e p} L(\mathbb{C}) \to 0$ is an exact sequence of Lie algebras. Take a $\mathbb{C}$-linear map $ds : L(\mathbb{C}) \to L(C)$ with $d_e p \circ ds = \mathrm{Id}_{L(\mathbb{C})}$. Because $\dim_{\mathbb{C}} L(\mathbb{C}) = 1$, ds is a Lie algebra morphism. As $\mathbb{C}$ is simply connected, there is a unique morphism $s : \mathbb{C} \to C$ with $d_e s = ds$. Since $d_e(ps) = \mathrm{Id}_{L(\mathbb{C})}$, one has $ps = \mathrm{Id}_{\mathbb{C}}$. Therefore, this extension is a semidirect product by Fact 3.0.4. $\square$

Proposition 3.2.3. *Let B be a connected commutative complex Lie group. Then every central extension of $\mathbb{C}^*$ by B is trivial.*

Proof. Let C be a central extension of $\mathbb{C}^*$ by B. Consider the pullback extension along $\exp(2\pi i\bullet) : \mathbb{C} \to \mathbb{C}^*$. By Proposition 3.2.2, there is a morphism $\rho : \mathbb{C} \to C'$ with $p'\rho = \mathrm{Id}_{\mathbb{C}}$. Then $p\epsilon\rho(1) = \exp(2\pi i) = 1$, so $\epsilon\rho(1) \in B$. As B is connected commutative, its exponential map $\exp_B : L(B) \to B$ is surjective. Take $v \in L(B)$ with $\exp_B(-v) = \epsilon\rho(1)$.

$$\begin{tikzcd}
 & & & \mathbb{Z} \arrow[d, hook] & \\
1 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & C' \arrow[r, "p'"'] \arrow[d, "\epsilon"] & \mathbb{C} \arrow[r] \arrow[l, bend right, "\rho"'] \arrow[d, "\exp(2\pi i\bullet)"] & 0 \\
1 \arrow[r] & B \arrow[r] & C \arrow[r, "p"] & \mathbb{C}^* \arrow[r] & 1
\end{tikzcd}$$

Define a holomorphic map

$$\rho' : \mathbb{C} \to C', \quad \rho'(z) = \exp_B(zv)\rho(v).$$

We check that ρ' is a group morphism. For every $z, w \in \mathbb{C}$,

$$\begin{aligned}\rho'(z+w) &= \exp_B((z+w)v)\rho(z+w) = \exp_B(zv)\exp_B(wv)\rho(z)\rho(w) \\ &= \exp_B(zv)\rho(z)\exp_B(wv)\rho(w) = \rho'(z)\rho'(w),\end{aligned}$$

where the last but one equality uses $B \subset Z(C)$.

Therefore, ρ' is a complex Lie group morphism. Moreover, $\rho'(1) = \exp_B(v)\rho(1) = \epsilon\rho(-1)\rho(1)$. Then $\epsilon\rho'(1) = e_C$. Therefore, $\rho'(\mathbb{Z}) \subset \ker(\epsilon)$. Thus, ρ' induces a morphism $s : \mathbb{C}^* \to C$ making a commutative diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "\rho'"] \arrow[d, "\exp(2\pi i\bullet)"] & C' \arrow[r, "p'"] \arrow[d, "\epsilon"] & C \arrow[d, "\exp(2\pi i\bullet)"] \\
\mathbb{C}^* \arrow[r, dashed, "s"] & C \arrow[r, "p"] & \mathbb{C}^*
\end{tikzcd}$$

Since $p'\rho' = \mathrm{Id}_{\mathbb{C}}$ and $\exp(2\pi i\bullet) : \mathbb{C} \to \mathbb{C}^*$ is surjective, $ps = \mathrm{Id}_{\mathbb{C}^*}$. From Fact 3.0.4, the extension C is trivial. $\square$

Example 7.1.7 gives a result about non-central extensions of $\mathbb{C}^*$.

Now assume that the Lie group K is discrete and *commutative*. We recall results[6] from [Hoc51b, Sec. 3].

Fact 3.2.4 ([Hoc51b, p.545])**.** *Let K, Q be Lie groups. If K is discrete commutative and Q is connected, then the extension (3) of Lie groups is central.*

Corollary 3.2.5. *Let K, Q be commutative Lie groups. If Q is connected and K is discrete, then every extension of Q by K is commutative.*

[6] They are stated for real Lie groups, but the proofs extend to the complex setting.

Proof. Let (3) be such an extension. By Fact 3.2.4, this extension is central. Then consider the induced continuous map (4). Since Q is connected and K is discrete, this map is constant, or equivalently, E is commutative. $\square$

Let Ab_c be the abelian category of abelian groups that are at most countable. Fact 3.2.6 shows that the universal cover of a connected Lie group is "universal" among all the extensions with discrete commutative kernels.

Fact 3.2.6 (Hochschild, [Hoc51b, Thm. 3.2 and Cor.])**.** *Let Q be a connected Lie group. Then the functor $\mathrm{Ext}(Q, \cdot, 1) : \mathrm{Ab}_c \to \mathrm{Ab}$ is represented by $\pi_1(Q)$ and the class of the universal cover sequence $1 \to \pi_1(Q) \to \tilde{Q} \to Q \to 1$ in $\mathrm{Ext}(Q, \pi_1(Q), 1)$. Hence an isomorphism $\Gamma_K : \mathrm{Ext}(Q, K, 1) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(Q), K)$ functorial in $K \in \mathrm{Ab}_c$. Moreover, $E \in \mathrm{Ext}(Q, K, 1)$ is connected if and only if $\Gamma_K(E)$ is surjective.*

4 Commutative Extensions

4.1 Generalities

Lemma 4.1.1. *The category $\mathcal{C}$ is naturally additive with finite direct products.*

Proof. The Hom sets are commutative groups, and composition of morphisms is bilinear. Moreover, the product $G_1 \times G_2$ of two commutative complex Lie groups is both a product and a coproduct of G_1 and G_2 in $\mathcal{C}$. $\square$

Although the category Alg of commutative complex algebraic groups is an abelian category ([Mil17, Thm. 5.62]), as Example 4.1.2 and Example 4.1.3 show, $\mathcal{C}$ is NOT an abelian category.

Example 4.1.2. The map $i : \mathbb{Z}^2 \to \mathbb{C}$ defined by $(a, b) \mapsto a + b\sqrt{2}$ is injective. The image is not closed in $\mathbb{C}$ as it is dense in $\mathbb{R}$. For every morphism $f : \mathbb{C} \to X$ in $\mathcal{C}$, with $fi = 0$, we have $f = 0$ by identity theorem for holomorphic maps. Thus i is a monomorphism and epimorphism in $\mathcal{C}$, but not an isomorphism.

Example 4.1.3. Let $p : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^4$ be the natural projection. Let $i : \mathbb{C} \to \mathbb{C}^2$ be the closed embedding defined by $z \mapsto (z, \sqrt{2}z)$. Then the composition $pi : \mathbb{C} \to \mathbb{C}^2/\mathbb{Z}^4$ is an injective morphism (hence a monomorphism) in $\mathcal{C}$. By [Lee13, Example 7.19], $pi(C)$ is a connected dense subset of $\mathbb{C}^2/\mathbb{Z}^4$. In particular, pi is an epimorphism in $\mathcal{C}$. The cokernel of pi is the zero morphism $\mathbb{C}^2/\mathbb{Z}^4 \to 0$. However, pi is not an isomorphism in $\mathcal{C}$.

Proposition 4.1.4 3 is a special case of [Con14, Prop. D.2.1]. An elementary proof is given.

Proposition 4.1.4.

1. $\mathrm{Hom}_{\mathcal{C}}(\mathbb{C}^*, \mathbb{C}) = 0$.

2. *For $A \in \mathcal{C}$, the map*

$$\mathrm{Hom}_{\mathcal{C}}(\mathbb{C}^n, A) \to \mathrm{Hom}_{\mathrm{Vec}}(L(\mathbb{C}^n), L(A)), \quad f \mapsto d_e f$$

is a group isomorphism.

3. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a morphism in $\mathcal{C}$. Then there is an integer k such that $f(z) = z^k$ for every $z \in \mathbb{C}^*$. Hence an isomorphism $\mathbb{Z} = \mathrm{Hom}_{\mathcal{C}}(\mathbb{C}^*, \mathbb{C}^*)$.*

Proof. The Lie algebra of $\mathbb{C}^*$ is $\mathbb{C}$. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is normalized as $w \mapsto e^{2\pi i w}$.

1. Let $f : \mathbb{C}^* \to \mathbb{C}$ be a morphism. Then $d_e f : \mathbb{C} \to \mathbb{C}$ is linear. There is $a \in \mathbb{C}$ with $d_e f(v) = av$ for all $v \in \mathbb{C}$. Since $1 \in \mathbb{C} = L(\mathbb{C}^*)$ is mapped to $1 \in \mathbb{C}^*$ under the exponential map $\exp(2\pi i \bullet)$, one has $0 = f(1) = d_e f(1) = a$. Then $d_e f = 0$ and $f = 0$.

2. It follows from the fact that $\mathbb{C}^n$ is simply connected and both groups are commutative.

3. Consider the induced linear map on Lie algebras $df : \mathbb{C} \to \mathbb{C}$. There is a unique complex number k such that $df(w) = kw$ for all $w \in \mathbb{C}$. Then

$$e^{2\pi i k} = \exp(df(1)) = f \exp(1) = f(1) = 1.$$

Therefore, k is an integer. For every $z \in \mathbb{C}^*$, there is $w \in \mathbb{C}$ with $\exp(w) = z$. Then $f(z) = f(\exp(w)) = \exp df(w) = \exp(kw) = z^k$.

□

For $A, B \in \mathcal{C}$, the set of isomorphism classes of *commutative* extensions of A by B is denoted by $\mathrm{Ext}(A, B)$.

Proposition 4.1.5.

1. $\mathrm{Ext}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Set}$ *is a covariant functor.*

2. *Let $\mathcal{E}$ be the collection of extensions in $\mathcal{C}$. Then the pair $(\mathcal{C}, \mathcal{E})$ is an exact category.*[7]

Proof. 1. Fix $A, B \in \mathcal{C}$ and an element of $\mathrm{Ext}(A, B)$: $0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0$.

(a) If $f : B \to B'$ is a morphism in $\mathcal{C}$, then

$$g : B \to C \times B', \quad b \mapsto (-b, f(b))$$

is a morphism in $\mathcal{C}$. It is injective and the (set-theoretic) image is *closed* in $C \times B'$. By Fact 2.0.1 2, g identifies B as a complex Lie subgroup of $C \times B'$. Let $f_* C$ be the quotient $(C \times B')/B$ provided by Fact 2.0.3. The canonical map $B' \to C \times B'$ induces an injective

[7] see [Sta22, Tag 05SF]

morphism $f_*i : B' \to f_*C$ since $B \cap (\{0\} \times B') = \{0\}$. Moreover, B is in the kernel of the composition $C \times B' \to A$ by $(c, \beta) \mapsto p(c)$, hence a surjective morphism $f_*p : f_*C \to A$.

Note that $f_*p \circ f_*i = 0$, so $f_*i(B') \subset \ker(f_*p)$. For every element x of $\ker(f_*p)$, take a representative $(c, \beta) \in C \times B'$. As $p(c) = 0$, $c \in B$. Then $(0, \beta + f(c)) - (c, \beta) = g(c)$. Therefore,

$$x = [(0, \beta + f(c))] = f_*i(\beta + f(c)) \in f_*(B').$$

Thus, $f_*i(B') = \ker(f_*p)$

Therefore, the sequence

$$0 \to B' \overset{f_*i}{\to} f_*C \overset{f_*p}{\to} A \to 0$$

is exact and $f_*C \in \mathrm{Ext}(A, B')$. Hence a morphism $f_* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B')$ in the category Set.

Let F be the canonical morphism $C \to f_*C$. By construction, the extension $f_*C \in \mathrm{Ext}(A, B')$ has the following universal property: for every morphism $h : A \to A'$ in $\mathcal{C}$, every $C' \in \mathrm{Ext}(A', B')$, every morphism $G : C \to C'$ making the diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "F"'] \arrow[dd, bend left, "G"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B' \arrow[r] \arrow[d, "\mathrm{Id}"] & f_*C \arrow[r] \arrow[d, dashed, "u"'] & A \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B' \arrow[r] & C' \arrow[r] & A' \arrow[r] & 0,
\end{tikzcd} \tag{7}$$

there exists a unique morphism $u : f_*C \to C'$ keeping the diagram commutative.

(b) If $h : A' \to A$ is a morphism in $\mathcal{C}$, by Example 3.1.1, we get a morphism $h^* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ in the category Set. Let F be the canonical projection $h^*C \to C$. By construction, the extension g^*C has the following universal property: for every morphism $g : B' \to B$, every extension $C' \in \mathrm{Ext}(A', B')$, every morphism $G : C' \to C$ making the following diagram commutative

$$\begin{tikzcd}
0 \arrow[r] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, dashed, "v"] \arrow[dd, bend left, "G"] & A' \arrow[r] \arrow[d, "\mathrm{Id}"'] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & h^*C \arrow[r] \arrow[d, "F"'] & A' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & B \arrow[r] & C \arrow[r] & A \arrow[r] & 0,
\end{tikzcd} \tag{8}$$

there exits a unique morphism $v : C' \to h^*C$ keeping the diagram commutative.

(c) Let $f : B \to B'$, $g : A \to A'$ be morphisms in $\mathcal{C}$, $C \in \operatorname{Ext}(A, B)$, and $C' \in \operatorname{Ext}(A', B')$. Then the relation $f_* C = g^* C'$ in $\operatorname{Ext}(A, B')$ is equivalent to the existence of a morphism $F : C \to C'$ making a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] & C \arrow[r] \arrow[d, "F"] & A \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & B' \arrow[r] & C' \arrow[r] & A' \arrow[r] & 0.
\end{tikzcd}$$

Indeed, it follows from the universal properties in Points (1a) and (1b). For every $X \in \operatorname{Ext}(A', B)$, in view of the diagram

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & g^* X \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[d, "f"] & X \arrow[r] \arrow[d] & A' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B' \arrow[r] & f_* X \arrow[r] & A' \arrow[r] & 0,
\end{tikzcd}$$

one has $f_* g^* X = g^* f_* X$. This completes the proof.

2. It follows from Point 1 and Lemma 4.1.1.

$\square$

Example 4.1.6. If A is a complex torus with $\dim A = g$, B is the discrete group $\mathbb{Q}/\mathbb{Z}$, then $\operatorname{Ext}(A, B)$ is isomorphic to B^{2g} by Fact 3.2.6. Even though B is an injective object of Ab, the functor $\operatorname{Ext}(\cdot, B) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is nonzero.

Example 4.1.7. For an extension $0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0$ in $\mathcal{C}$, the pushout $i_* C \in \operatorname{Ext}(A, C)$ is the trivial extension. In fact, $i_* C = C \times C/B$ with the embedding

$$B \to C \times C, \quad b \mapsto (-b, b).$$

The group law $C \times C \to C$ descents to a morphism $r : i_* C \to C$. Then $r \circ i_*(i) = \mathrm{Id}_C$. By Fact 3.0.4, $i_* C$ is trivial.

Similarly, as the diagonal inclusion $C \to C \times C$ factors through a morphism $s : C \to p^* C$ and $p^*(p) \circ s = \mathrm{Id}_C$, the pullback $p^* C \in \operatorname{Ext}(C, B)$ is also trivial.

Fact 4.1.8 follows from Proposition 4.1.5.

Fact 4.1.8 ([Ros58, Prop. 5], [Ser88, Prop. 1, p.163])**.** *1. For every $A, B \in \mathcal{C}$, under the Baer sum* $\operatorname{Ext}(A, B)$ *is an abelian subgroup of* $\operatorname{Ext}(A, B, 1)$.

2. The functor $\operatorname{Ext}(\bullet, \bullet) : \mathcal{C}^{\mathrm{op}} \times \mathcal{C} \to \mathrm{Ab}$ *is an additive bifunctor.*

3. For any $C, C' \in \operatorname{Ext}(A, B)$, the product $C \times C'$ is naturally an element of $\operatorname{Ext}(A \times A, B \times B)$.

4. Let $d : A \to A \times A$ the diagonal map of A and $s : B \times B \to B$ the group law of B. Then $C + C' = d^* s_*(C \times C')$ in $\operatorname{Ext}(A, B)$.

Corollary 4.1.9. *For every commutative complex Lie group A, the restriction* $\mathrm{Ext}(A, \cdot) : \mathrm{Vec} \to \mathrm{Ab}$ *factors through a functor from* Vec *to the category of all complex vector spaces.*

By Example 4.3.3 below, for every $V \in \mathrm{Vec}$, $\dim_{\mathbb{C}} \mathrm{Ext}(A, V)$ is finite. Hence an additive functor $\mathrm{Ext}(A, \cdot) : \mathrm{Vec} \to \mathrm{Vec}$.

Example 4.1.10. Endowing each object of Ab_c the discrete topology gives a functor $\mathrm{Ab}_c \to \mathcal{C}$, identifying Ab_c as a full subcategory of $\mathcal{C}$. The subcategory Ab_c is closed under extension by Fact 3.0.1. From Proposition 3.2.1, the forgetful morphism $\mathrm{Ext}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A, B)$ is an isomorphism for every $A \in \mathrm{Ab}_c$ and every $B \in \mathcal{C}$.

Example 4.1.11. Analytification functor $(\bullet)^{\mathrm{an}} : \mathrm{Alg} \to \mathcal{C}$ identifies Alg as a subcategory of $\mathcal{C}$ (which is not full). The extension problem within the subcategory Alg is discussed by Rosenlicht [Ros58] and Serre [Ser88, Ch. VII]. They define a similar additive functor $\mathrm{Ext}_{\mathrm{Alg}} : \mathrm{Alg}^{\mathrm{op}} \times \mathrm{Alg} \to \mathrm{Ab}$. For every $A, B \in \mathrm{Alg}$, there is a natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(A, B) \to \mathrm{Ext}(A^{\mathrm{an}}, B^{\mathrm{an}})$. In general, this morphism is neither injective nor surjective.

For example, when $A/\mathbb{C}$ is an abelian variety, $\mathrm{Ext}_{\mathrm{Alg}}(\mathbb{G}_a, A) = 0$ while $\mathrm{Ext}_{\mathrm{Alg}}(\mathbb{G}_m, A)$ is non-canonically isomorphic to the torsion subgroup A_{tor} of A ([MM66, Introduction, 1.]). But $\mathrm{Ext}(\mathbb{C}^*, A^{\mathrm{an}}) = 0$ by Proposition 3.2.3, so the natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(\mathbb{G}_m, A) \to \mathrm{Ext}(\mathbb{C}^*, A^{\mathrm{an}})$ is not injective.

For any two abelian varieties $X_i/\mathbb{C}$ $(i = 1, 2)$ of positive dimension, $\mathrm{Ext}_{\mathrm{Alg}}(X_2, X_1)$ is countable while $\mathrm{Ext}(X_2^{\mathrm{an}}, X_1^{\mathrm{an}})$ is uncountable. In fact, the natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(X_2, X_1) \to \mathrm{Ext}(X_2^{\mathrm{an}}, X_1^{\mathrm{an}})$ is an embedding onto the torsion subgroup of $\mathrm{Ext}(X_2^{\mathrm{an}}, X_1^{\mathrm{an}})$ ([BL99, Ch. 1; Prop. 6.1, Cor. 6.3]).

Lemma 4.1.12 is mentioned at the bottom of [Hoc51b, p.546].

Lemma 4.1.12. *If G is a commutative connected Lie group, then G is a divisible $\mathbb{Z}$-module.*

Proof. The exponential map $\exp : L(G) \to G$ is surjective. For every $x \in G$, there is $v \in L(G)$ with $\exp(v) = x$. For every integer $n \geq 1$, $\exp(v/n) \in G$ and $n(\exp(v/n)) = x$. □

Corollary 4.1.13. *An extension $0 \to B \to C \to A \to 0$ in $\mathcal{C}$ with B connected and A discrete is trivial. In particular, for every $G \in \mathcal{C}$, the natural exact sequence*

$$0 \to G_0 \to G \to G/G_0 \to 0$$

is a trivial extension, hence a non-canonical isomorphism $G \to G_0 \times G/G_0$ in $\mathcal{C}$.

Proof. By Lemma 4.1.12, the $\mathbb{Z}$-module B is divisible, so the functor $\mathrm{Ext}^1_{\mathbb{Z}}(\cdot, B) : \mathrm{Ab} \to \mathrm{Ab}$ is zero. Since A is discrete, the result follows from Example 4.1.10. □

Example 4.1.14. The abelian group underlying a complex torus B is divisible by Lemma 4.1.12, hence an injective object of Ab and $\mathrm{Ext}^1_{\mathbb{Z}}(\bullet, B) : \mathrm{Ab} \to \mathrm{Ab}$ is zero. However, $\mathrm{Ext}(\bullet, B) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ can be nonzero. In fact, [BL04, (8) b), p.68] gives an example of a nontrivial exact sequence of complex tori

$$0 \to B \to C \to A \to 0$$

with $\dim A = \dim B = 1$.

4.2 Exact sequences of Ext

Let $0 \to A' \xrightarrow{i} A \xrightarrow{p} A'' \to 0$ be an exact sequence in $\mathcal{C}$, *i.e.*, $A \in \mathrm{Ext}(A'', A')$. For $f \in \mathrm{Hom}(A', B)$, there is $f_*A \in \mathrm{Ext}(A'', B)$. Hence a map

$$d : \mathrm{Hom}(A', B) \to \mathrm{Ext}(A'', B), \quad d(f) = f_*A.$$

Then d is a group morphism. The formation of d is functorial in B.

Proposition 4.2.1. *Let $B \in \mathcal{C}$. The sequence in* Ab *with obvious morphisms*

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A'', B) \to \mathrm{Hom}_{\mathcal{C}}(A, B) \to \mathrm{Hom}_{\mathcal{C}}(A', B) \xrightarrow{d} \mathrm{Ext}(A'', B) \xrightarrow{p^*} \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$$

is exact and functorial in B.

Proof.

- Exactness at $\mathrm{Hom}(A, B)$ follows from Fact 2.0.1.
- Exactness at $\mathrm{Hom}(A', B)$: By Example 4.1.7, the composition

  $$\mathrm{Hom}(A, B) \xrightarrow{i_*} \mathrm{Hom}(A', B) \to \mathrm{Ext}(A'', B)$$

  is zero. Now take $\phi \in \ker(d)$. By Fact 3.0.4, there is a morphism $r : \phi_*A \to B$ with $r\phi_*(i) = \mathrm{Id}_B$. Let $F : A \to \phi_*A$ be the canonical morphism. Then $rFi = r\phi_*(i)\phi = \phi$. Hence $\phi \in \mathrm{im}(i_*)$.
- Exactness at $\mathrm{Ext}(A'', B)$: By Example 4.1.7, for every $\phi \in \mathrm{Hom}(A', B)$, $p^*d\phi = p^*\phi_*A = \phi_*p^*A = 0$, *i.e.*, the composition

  $$\mathrm{Hom}(A', B) \xrightarrow{d} \mathrm{Ext}(A'', B) \xrightarrow{p^*} \mathrm{Ext}(A, B)$$

  is zero.

  Now take $C \in \ker(p^*) \subset \mathrm{Ext}(A'', B)$ with connecting morphisms $f : B \to C$ and $g : C \to A''$. By Fact 3.0.4, there is a morphism $s : A \to p^*C$ with $p^*(p) \circ s = \mathrm{Id}_A$. For every $a' \in A'$, the image of $s(a')$ in A'' is $p(a') = 0$, so the image of $s(a')$ in C lies in B. Thus, the restriction of s to A' is a morphism $\phi : A' \to B$. By construction, the extension group of $d(\phi) = \phi_*A \in \mathrm{Ext}(A'', B)$ is $A \times B/D$, where $D = \{(-a', \phi(a')) : a' \in A'\}$.

Define $\psi : A \to C$ by $\psi = F \circ s$. Define

$$A \times B \to C, \quad (a, b) \mapsto \psi(a) + f(b).$$

For every $a' \in A'$, $\psi(-a') + f(s(a')) = 0$, hence a factorization $\phi_* A \to C$ in the middle keeping the diagram commutative:

$$\begin{tikzcd}
0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, dashed, "\phi"] & A \arrow[r, "p"] \arrow[d] & A'' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "\phi_* i"] \arrow[d, "\mathrm{Id}"] & \phi_* A \arrow[r] \arrow[d, dashed] & A'' \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] & C \arrow[r, "g"'] & A'' \arrow[r] & 0 \\
0 \arrow[r] & B \arrow[r] \arrow[u, "\mathrm{Id}"] & p^* C \arrow[r, "p^* p"] \arrow[u, "F"] & A \arrow[r] \arrow[u, "p"] \arrow[l, dashed, bend left, "s"] & 0.
\end{tikzcd}$$

Then $C = \phi_* A = d\phi$ in $\mathrm{Ext}(A'', B)$. Therefore, $\ker(p^*) = \mathrm{im}(d)$.

- Exactness at $\mathrm{Ext}(A, B)$: As the composition $A' \to A \to A''$ is zero and $\mathrm{Ext}(\bullet, B) : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ is an additive functor, the composition $\mathrm{Ext}(A'', B) \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A', B)$ is zero.

  Conversely, if $C_1 \in \mathrm{Ext}(A, B)$ with $i^* C_1 = 0$ in $\mathrm{Ext}(A', B)$, then there is a morphism $s : A' \to i^* C_1$ with $i^* g \circ s = \mathrm{Id}_{A'}$. The composition $\phi : A' \to C_1$ is injective. Indeed, if $a' \in \ker(\phi)$, then $s(a') = (a', 0)$ in $A' \times C_1$. Thus, $i(a') = 0$ by the construction of pullback extension. Since i is injective, $a' = 0$.

  Let $C_1 \to C = C_1/\phi(A')$ be the quotient morphism. Let $f_0 : B \to C$ be the induced morphism. Then f_0 is injective. Indeed, if $b \in \ker(f_0)$, then $f(b) = \phi(a')$ for some $a' \in A'$. Then $(a', f(b)) \in i^* C_1$, so $i(a') = gf(b) = 0$. Hence $a' = 0$ and $f(b) = 0$. Therefore, $b = 0$.

  Because $pg\phi = p \circ i = 0$, the morphism $pg : C_1 \to A''$ descends to a surjective morphism $g_0 : C \to A''$. We prove that the bottom row of the following diagram is exact:

$$\begin{tikzcd}
0 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & i^* C_1 \arrow[r, "i^* g"'] \arrow[d] & A' \arrow[r] \arrow[d, "i"] \arrow[l, dashed, bend right, "s"'] \arrow[dl, dashed, "\phi"] & 0 \\
0 \arrow[r] & B \arrow[r, "f"] \arrow[d, "\mathrm{Id}"] & C_1 \arrow[r, "g"] \arrow[d] & A \arrow[r] \arrow[d, "p"] & 0 \\
0 \arrow[r] & B \arrow[r, "f_0"] & C \arrow[r, dashed, "g_0"] & A'' \arrow[r] & 0.
\end{tikzcd}$$

  Since $gf = 0$, one has $g_0 f_0 = 0$. Therefore, $f_0(B) \subset \ker(g_0)$. Conversely, for $c \in \ker(g_0)$, there is $c_1 \in C_1$ with $[c_1] = c$. Since $pg(c_1) = g_0(c) = 0$, one gets $g(c_1) \in A'$. Then $g\phi g(c_1) = gc_1$. So $c_1 - \phi g(c_1) \in \ker(g) = B$ and

$$f_0(c_1 - \phi g(c_1)) = [c_1 - \phi g(c_1)] = c.$$

Therefore, $\ker(g_0) = f_0(B)$. In particular, the bottom row is exact, *i.e.*, $C \in \mathrm{Ext}(A'', B)$. By the universal property showed in the diagram (8), $C_1 = p^*C$.

$\square$

EXAMPLE 4.2.2. Let A be a complex torus, and let B be a finite abelian group. Then $\mathrm{Hom}_{\mathcal{C}}(A, B) = 0$. Let integer $n(\geq 1)$ be a multiple of $\#B$. Applying Proposition 4.2.1 to the exact sequence in $\mathcal{C}$

$$0 \to A[n] \to A \xrightarrow{[n]_A} A \to 0,$$

one gets an exact sequence in Ab:

$$0 \to \mathrm{Hom}(A[n], B) \to \mathrm{Ext}(A, B) \xrightarrow{f} \mathrm{Ext}(A, B).$$

Since the morphism $[n]_B : B \to B$ is zero in $\mathcal{C}$, by Fact 4.1.8, $f = ([n]_B)_* = 0$. Hence an isomorphism $\mathrm{Hom}(A[n], B) \to \mathrm{Ext}(A, B)$ that is functorial in B, which is also confirmed by Fact 3.2.6.

Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{C}$. If $A \in \mathcal{C}$ and $\phi \in \mathrm{Hom}(A, B'')$, then $\phi^*B \in \mathrm{Ext}(A', B)$. Define a map $d : \mathrm{Hom}(A, B'') \to \mathrm{Ext}(A, B')$ by $d(\phi) = \phi^*B$.

Proposition 4.2.3. *Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{C}$ and $A \in \mathcal{C}$. Then the sequence*

$$0 \to \mathrm{Hom}(A, B') \to \mathrm{Hom}(A, B) \to \mathrm{Hom}(A, B'') \xrightarrow{d} \mathrm{Ext}(A, B') \to \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B'')$$

in Ab *is exact and functorial in A.*

The proof is analogous to that of Proposition 4.2.1 and is thereby omitted.

Consider the extension problem with connected bases. Corollary 4.2.4 should be compared to [Sha49, Thm. 1]: for two compact connected real Lie groups G, H, the cokernel of the restriction morphism $\mathrm{Hom}(\tilde{H}, Z(G)) \to \mathrm{Hom}(\pi_1(H), Z(G))$ is isomorphic to the group of extensions of H by G.

Corollary 4.2.4. *Let A, B be commutative complex Lie groups. Assume that A is connected with universal cover $\omega : \tilde{A} \to A$. Then there is a canonical exact sequence in* Ab*:*

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{\cdot\circ\omega} \mathrm{Hom}_{\mathcal{C}}(\tilde{A}, B) \xrightarrow{r} \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), B) \to \mathrm{Ext}(A, B) \to 0, \quad (9)$$

where r is induced by restriction.

Proof. By Proposition 3.2.2, Fact 3.2.6 and Corollary 4.1.13, the functor $\mathrm{Ext}(\mathbb{C}, \bullet) : \mathcal{C} \to \mathrm{Ab}$ is zero. By Fact 4.1.8,

$$\mathrm{Ext}(\mathbb{C}^n, \bullet) = 0. \tag{10}$$

The proof is concluded by Proposition 4.2.1. □

Example 4.2.5. In Corollary 4.2.4, if B discrete, then $\mathrm{Hom}_{\mathcal{C}}(\tilde{A}, B) = 0$ and the natural morphism $\mathrm{Hom}(\pi_1(A), B) \to \mathrm{Ext}(A, B)$ is an isomorphism, which agrees with Fact 3.2.6.

4.3 Determination of commutative extension group

The commutative extension problem of complex Lie groups is answered by Proposition 4.3.1. Fix two commutative complex Lie groups A, B.

Proposition 4.3.1. *There is a non-canonical isomorphism in* Ab*:*

$$\mathrm{Ext}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0) \oplus \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0) \oplus \mathrm{Ext}(A_0, B_0),$$

and $\mathrm{Ext}(A_0, B_0)$ *is the cokernel of the natural restriction morphism*

$$s : \mathrm{Hom}_{\mathrm{Vec}}(L(A), L(B)) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B_0).$$

Proof. By Corollary 4.1.13, there are non-canonical isomorphisms in $\mathcal{C}$: $A \to A/A_0 \times A_0$ and $B \to B/B_0 \times B_0$. Using Fact 4.1.8, one gets an isomorphism in Ab:

$$\mathrm{Ext}(A, B) \to \mathrm{Ext}(A/A_0, B_0) \oplus \mathrm{Ext}(A/A_0, B/B_0) \oplus \mathrm{Ext}(A_0, B/B_0) \oplus \mathrm{Ext}(A_0, B_0).$$

The first factor $\mathrm{Ext}(A/A_0, B_0) = 0$ by Corollary 4.1.13. By Example 4.1.10, the natural morphism $\mathrm{Ext}(A/A_0, B/B_0) \to \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0)$ is an isomorphism. Fact 3.2.6 gives a natural isomorphism $\mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0) \to \mathrm{Ext}(A_0, B/B_0)$. Corollary 4.2.4 identifies $\mathrm{Ext}(A_0, B_0)$ with the cokernel of the restriction map $r : \mathrm{Hom}_{\mathcal{C}}(\tilde{A}_0, B_0) \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B_0)$. By Proposition 4.1.4 2, the group morphism

$$t : \mathrm{Hom}_{\mathcal{C}}(\tilde{A}_0, B_0) \to \mathrm{Hom}_{\mathrm{Vec}}(L(A), L(B)), \quad \phi \mapsto d_e\phi$$

is an isomorphism. The proof is finished by setting $s = rt^{-1}$. □

For every $C \in \mathrm{Ext}(A, B)$, by Fact 2.0.3, the morphism $C \to A$ is a principal B-bundle. The bijection (1) gives rise to a canonical map

$$\pi : \mathrm{Ext}(A, B) \to H^1(A, \mathcal{B}_A). \tag{11}$$

Fact 4.3.2 is taken from [Ros58, pp.698-699] and the proof of [Ser88, Ch. VII, no. 5, Prop. 5].

Fact 4.3.2. *The map (11) is a group morphism and the formation of π is functorial, in the sense that it commutes with the morphisms $f_* : \mathrm{Ext}(A,B) \to \mathrm{Ext}(A,B')$ defined by $f : B \to B'$ and $g^* : \mathrm{Ext}(A,B) \to \mathrm{Ext}(A',B)$ defined by $g : A' \to A$. When B is a vector group, the map π is $\mathbb{C}$-linear.*

Example 4.3.3. Let X be a toroidal group, and let $\omega : \tilde{X} \to X$ be the universal covering of kernel F. Then F is a discrete subgroup of the vector space $\tilde{X}$. By Proposition 4.2.1,

$$\mathrm{Hom}_{\mathcal{C}}(X,\mathbb{C}) \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X},\mathbb{C}) \to \mathrm{Hom}_{\mathcal{C}}(F,\mathbb{C}) \to \mathrm{Ext}(X,\mathbb{C}) \to \mathrm{Ext}(\tilde{X},\mathbb{C})$$

is an exact sequence in Ab. From Definition 2.0.9, $\mathrm{Hom}_{\mathcal{C}}(X,\mathbb{C}) = 0$. By Proposition 10, $\mathrm{Ext}(\tilde{X},\mathbb{C}) = 0$. Hence the first exact row of Diagram (12).

According to [AK01, p.48], there is a $\mathbb{C}$-linear isomorphism $\mathrm{Hom}_{\mathcal{C}}(\tilde{X},\mathbb{C}) \to H^0(X,\Omega^1_X)$ and every global holomorphic 1-form on X is d-closed. So taking de Rham cohomology class results in a linear map $H^0(X,\Omega^1_X) \to H^1(X,\mathbb{C})$. The inclusion $\mathbb{C}_X \to O_X$ induces a linear map $H^1(X,\mathbb{C}) \to H^1(X,O_X)$. By universal coefficient theorem (see, *e.g.*, [Hat05, Thm. 3.2]), the natural morphism $\mathrm{Hom}_{\mathcal{C}}(F,\mathbb{C}) \to H^1(X,\mathbb{C})$ is an isomorphism. Hence a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\tilde{X},\mathbb{C}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Hom}_{\mathcal{C}}(F,\mathbb{C}) \arrow[r] \arrow[d, "\simeq"] & \mathrm{Ext}(X,\mathbb{C}) \arrow[r] \arrow[d, hook, "(11)"] & 0 \\
0 \arrow[r] & H^0(X,\Omega^1_X) \arrow[r] & H^1(X,\mathbb{C}) \arrow[r] & H^1(X,O_X).
\end{tikzcd} \tag{12}$$

Let $b_1(X) := \dim_{\mathbb{C}} H^1(X,\mathbb{C})$ be the first Betti number of X, *i.e.*, the $\mathbb{Z}$-rank of F. From [AK01, p.48], as a $\mathbb{C}$-vector space

$$\mathrm{Ext}(X,\mathbb{C}) = \frac{H^1(X,\mathbb{C})}{H^0(X,\Omega^1_X)} \tag{13}$$

is of dimension $b_1(X) - \dim X$.

If X is a toroidal theta group,[8] then $\pi : \mathrm{Ext}(X,\mathbb{C}) \to H^1(X,O_X)$ is a $\mathbb{C}$-linear isomorphism by [AK01, Thm. 2.2.6 b)]. Otherwise, X is a toroidal wild group[8] and $H^1(X,O_X)$ is infinite dimensional by [AK01, Prop. 2.2.7].

A seemingly different way to compute the last factor in Proposition 4.3.1, *i.e.*, the group of commutative extensions of two connected commutative complex Lie groups, is given in Example 4.3.4.

Example 4.3.4. Start by the special case that X is a toroidal group and B is a connected commutative complex Lie group. Denote the kernel of the universal cover of B (resp. X) by $\iota : K \to \tilde{B}$ (resp. $F \to \tilde{X}$). By (10) and Proposition 4.1.4 2, the sequence

$$0 \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X},K) \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X},\tilde{B}) \to \mathrm{Hom}_{\mathcal{C}}(\tilde{X},B) \to 0$$

[8] in the sense of [AK01, Def. 2.2.1]

is exact in Ab. As F is a free $\mathbb{Z}$-module,

$$0 \to \mathrm{Hom}_{\mathcal{C}}(F, K) \to \mathrm{Hom}_{\mathcal{C}}(F, \tilde{B}) \to \mathrm{Hom}_{\mathcal{C}}(F, B) \to 0$$

in Ab is also exact. Applying Proposition 4.2.1 and the snake lemma to the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, K) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, \tilde{B}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\tilde{X}, B) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(F, K) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(F, \tilde{B}) & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(F, B) & \longrightarrow & 0, \end{array}$$

one gets an exact sequence in Ab:

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, B) \xrightarrow{j} \mathrm{Ext}(X, K) \xrightarrow{\iota_*} \mathrm{Ext}(X, \tilde{B}) \to \mathrm{Ext}(X, B) \to 0. \tag{14}$$

Since K is a free $\mathbb{Z}$-module, by Fact 3.2.6, $\mathrm{Ext}(X, K) = H^1(X, \mathbb{Z}) \otimes_{\mathbb{Z}} K$. By Fact 4.1.8 and (13), one has

$$\mathrm{Ext}(X, \tilde{B}) = \frac{H^1(X, \mathbb{C})}{H^0(X, \Omega_X^1)} \otimes_{\mathbb{C}} \tilde{B}.$$

The group morphism ι_* is induced by the $\mathbb{Z}$-bilinear map

$$H^1(X, \mathbb{Z}) \times K \to (\frac{H^1(X, \mathbb{C})}{H^0(X, \Omega_X^1)}) \otimes_{\mathbb{C}} \tilde{B}, \quad (\eta, x) \mapsto [\eta] \otimes \iota(x).$$

Thus we can compute $\mathrm{Ext}(X, B)$ from (14).

For a general connected commutative complex Lie group A, by [AK01, 1.1.5], $A = \mathbb{C}^l \times (\mathbb{C}^*)^m \times X_0$ for some integers $l, m \geq 0$ and a toroidal group X_0. By Fact 4.1.8, Proposition 3.2.2 and Proposition 3.2.3, $\mathrm{Ext}(A, B) = \mathrm{Ext}(X_0, B)$, reducing to the previous case.

5 Commutative extensions of complex tori

5.1 Primitive cohomology classes

Every central extension of a compact real Lie group by a vector group is trivial, shown by Fact 5.1.1.

Fact 5.1.1 (Iwasawa, [Iwa49, Lem. 3.7], [Hoc51a, Footnote 10, p.107]). *Let (3) be an exact sequence of* real *Lie groups. If K is a vector group and Q is compact, then this extension is a semidirect product. In particular, if this extension is central, then it is trivial.*

Contrary to the real case, Example 5.1.2 shows a commutative extension of a complex torus by a vector group can be nontrivial.

Example 5.1.2 ([MM60, p.145, Exemple], [LH76, Sec. I.3]). Set $C = \mathbb{C}^* \times \mathbb{C}^*$. Then $B = \{(e^z, e^{iz}) : z \in \mathbb{C}\}$ is a complex Lie subgroup of C (but not an algebraic subgroup of $\mathbb{G}_m \times \mathbb{G}_m$) isomorphic to $\mathbb{C}$. The quotient $A = C/B$ is an elliptic curve. The exact sequence $0 \to B \to C \to A \to 0$ is a nontrivial extension, as C is not biholomorphic to $B \times A$.

In the remainder of Section 5, unless otherwise specified, let A be a complex torus of dimension g and B be a commutative complex Lie group. Let $s_A : A \times A \to A$ be the group law of A. The dual of A is $A^\vee = \mathrm{Pic}^0(A)$.

The analogue of Proposition 5.1.3 for abelian varieties is [Ros58, Prop. 9].

Proposition 5.1.3. *The morphism (11) is injective.*

Proof. Let $C \in \ker(\pi)$. The principal bundle $C \to A$ is trivial, so there is a morphism $s : A \to C$ of complex manifolds with $ps = \mathrm{Id}_A$. Then there exists a unique $b \in B$ with $b \cdot s(e_A) = e_C$, where dot signifies the action of B on the fiber $p^{-1}(e_A)$. Define

$$s' : A \to C, \quad s(a) = b \cdot s(a).$$

Then s' is a complex manifold morphism with $ps' = \mathrm{Id}_A$. Replacing s by s', we may suppose that $s(e_A) = e_C$. By [NW13, Thm. 5.1.36], s is a morphism in $\mathcal{C}$. By Fact 3.0.4, $C = 0$ in $\mathrm{Ext}(A, B)$. Therefore, π is injective. □

We propose to determine the image of (11). Let Mfd be the category of complex manifolds. Define a functor

$$T : \mathrm{Mfd}^{\mathrm{op}} \to \mathrm{Ab}, \quad T(X) = H^1(X, \mathcal{B}_X).$$

When X is a point, $T(X) = 0$. Let $X_1, X_2 \in \mathrm{Mfd}$, and let $p_i : X_1 \times X_2 \to X_i$ $(i = 1, 2)$ be the projection to the i-th factor. There is a morphism $p_1^* \oplus p_2^* : T(X_1) \times T(X_2) \to T(X_1 \times X_2)$.

Definition 5.1.4. [Ser88, (29), no.14, Ch. VII] For $A \in \mathcal{C}$, an element $x \in T(A) = H^1(A, \mathcal{B}_A)$ is called primitive if $s_A^*(x) = p_1^*(x) + p_2^*(x)$ in $T(A \times A)$. Denote by $\mathrm{PT}(A)$ the subgroup of $T(A)$ formed by the primitive elements.

Fact 5.1.5. *[Ser88, Lem. 8, p.181] The functor* $\mathrm{PT} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Ab}$ *is additive.*

Theorem 5.1.6 is an analytic analog of [Ser88, Thm. 5, p.181].

Theorem 5.1.6. *Assume that B_0 is linear. Then the image of the morphism (11) is the set of primitive elements of $H^1(A, \mathcal{B}_A)$.*

Proof. Take $C \in \mathrm{Ext}(A, B)$ and put $x = \pi(C)$. By Facts 4.1.8 and 4.3.2,

$$s_A^*(x) = s_A^*\pi(C) = \pi s_A^*(C) = \pi(p_1^*C + p_2^*C) = p_1^*\pi(C) + p_2^*\pi(C) = p_1^*x + p_2^*x,$$

so x is primitive.

Conversely, let $x \in H^1(A, \mathcal{B}_A)$ be a primitive element and let $p : C \to A$ be the corresponding principal B-bundle. We show that there exists a structure of commutative complex Lie group on C which makes it an extension of A by B.

By Corollary 4.1.13, every morphism of complex manifolds $A \to B$ is constant. Let $C' \to A \times A$ be the pull-back of $C \to A$ along $s_A : A \times A \to A$. As x is primitive, $C' = p_1^*C + p_2^*C$ in $T(A \times A)$. Choose a surjection $p_1^*C \times_{A\times A} p_2^*C \to C'$ satisfying (2). Since $p_1^*C = C \times A$ and $p_2^*C = A \times C$, as a complex manifold $p_1^*C \times_{A\times A} p_2^*C$ is isomorphic to $C \times C$. Hence a morphism $g : C \times C \to C$ of complex manifolds:

$$\begin{tikzcd} C \times C = p_1^*C \times_{A\times A} p_2^*C \arrow[r] \arrow[rr, bend left, "g"] & p_1^*C + p_2^*C = C' \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ & A \times A \arrow[r, "s_A"] & A. \end{tikzcd} \tag{15}$$

By construction, it satisfies

$$g(b \cdot c, b' \cdot c') = (b + b') \cdot g(c, c') \tag{16}$$

for every $c, c' \in C$ and $b, b' \in B$.

Choose a point $e \in p^{-1}(e_A)$. Since $p(g(e, e)) = s_A(e_A, e_A) = e_A$, there exists a unique $b \in B$ with $b \cdot g(e, e) = e$. Replacing e by $b \cdot e$, we can suppose that

$$g(e, e) = e. \tag{17}$$

We verify that (C, e, g) is a group.

Identity According to (15), there is a morphism $h : C \to B$ of complex manifolds with $g(c, e) = h(c) \cdot c$ for all $c \in C$. By (17), $h(e) = e_B$. Furthermore, (16) shows that $h(b \cdot c) = h(c)$ for all $b \in B$. Therefore, h factors as $C \xrightarrow{p} A \xrightarrow{\bar{h}} B$. The morphism $\bar{h}$ of complex manifolds is constant, so $g(c, e) = c$ for all $c \in C$. The formula $g(e, c) = c$ is proved similarly.

Associativity According to (15), there is a complex manifold morphism $u : C \times C \times C \to B$ with

$$g(c, g(c', c'')) = u(c, c', c'') \cdot g(g(c, c'), c'')$$

for all $c, c', c'' \in C$. Then $u(e, e, e) = e_B$. Equation (16) shows that u factors through a morphism $\bar{u} : A \times A \times A \to B$ of complex manifolds. Then $\bar{u}$ is of constant value e_B. Therefore, $g(c, g(c', c'')) = g(g(c, c'), c'')$ for all $c, c', c'' \in C$.

Inverse Denote by $i_A : A \to A$ (resp. $i_B : B \to B$) the inverse map of A (resp. B). Let $C^- \to A$ be the principal B-bundle corresponding to $-x \in H^1(A, \mathcal{B}_A)$. There is a morphism $f : C \to C^-$ of principal B-bundles over A, such that for every $b \in B$, $c \in C$, $f(b \cdot c) = (-b) \cdot c$. Since $0_A = i_A + \mathrm{Id}_A$, by Fact 5.1.5, $0 = 0_A^* x = i_A^* x + x$, hence $i_A^* x = -x$. In other words, the pullback of $p : C \to A$ along i_A is $C^- \to A$.

$$\begin{tikzcd} C \arrow[r, "f"] \arrow[rr, bend left, "i"] & i_A^* C \arrow[r] \arrow[d] & C \arrow[d, "p"] \\ & A \arrow[r, "i_A"] & A \end{tikzcd}$$

The induced morphism $i : C \to C$ of complex manifolds is such that for every $c \in C$, $b \in B$,

$$i(b \cdot c) = (-b) \cdot i(c). \tag{18}$$

Since $i(e) \in p^{-1}(e_A)$, there is $b \in B$ with $b \cdot i(e) = e$. Define $i' : C \to C$ by $i'(x) = b \cdot i(x)$ and replace i by i'. Then we may further assume that $i(e) = e$. Because

$$p(g(c, i(c))) = s_A(p(c), pi(c)) = s_A(p(c), i_A(p(c))) = e_A,$$

there exists a morphism $v : C \to B$ of complex manifolds such that $g(c, i(c)) = v(c) \cdot e$ and $v(e) = e_B$. By (16) and (18), v factors through $\bar{v} : A \to B$, which is of constant value e_B. Therefore, $g(c, i(c)) = e$ for all $c \in C$.

In conclusion, (C, e, g, i) is a complex Lie group and (15) shows that $p : C \to A$ is a morphism. Define an injective map $\iota : B \to C$ by $b \mapsto b \cdot e$. By (16), then ι is a morphism. Since $\iota(B) = p^{-1}(e)$, the sequence

$$0 \to B \xrightarrow{\iota} C \xrightarrow{p} A \to 0$$

is exact. By Proposition 6.0.2 2 below, C is commutative and hence $C \in \mathrm{Ext}(A, B)$. (The commutativity of C can also be proved using an argument of similar type.) Therefore, $x = \pi(C)$ is in the image of π. $\square$

5.2 The case $B = \mathbb{C}^*$

We review some basics about (holomorphic) line bundles on complex tori.

Definition 5.2.1. [Wei48, Ch.VIII, n.58] Let $L \to A$ be a line bundle on a complex torus. If for every $a \in A$, the pullback line bundle $T_a^* L$ is isomorphic to L, then we write $L \equiv O_A$. Here $T_a : A \to A$ is defined by $T_a(x) = x + a$.

By [BL04, p.36], L induces a morphism

$$\phi_L : A \to A^\vee, \quad a \mapsto T_a^* L \otimes L^{-1}.$$

Then $L \equiv O_A$ is equivalent to $\phi_L = 0$. Then [BL04, Prop. 2.5.3] becomes Fact 5.2.2.

Fact 5.2.2. *Let $L \to A$ be a line bundle on a complex torus. The following conditions are equivalent:*

1. *L is analytically equivalent to O_A;*

2. $L \in \mathrm{Pic}^0(A)$;

3. $L \equiv O_A$.

Proposition 5.2.3. *Let $L \to A$ be a line bundle on complex torus. Then $L \equiv O_A$ if and only if $s_A^* L = p_1^* L \otimes p_2^* L$.*

Proof. If $s_A^* L = p_1^* L \otimes p_2^* L$, then for every $a \in A$, the line bundle $T_a^* L = (s_A^* L)|_{A \times a} = (p_1^* L \otimes p_2^* L)|_{A \times a} = L$, *i.e.*, $L \equiv O_A$.

Conversely, if $L \equiv O_A$, then for every $a \in A$, $(s_A^* L)|_{A\times a} = T_a^* L = L = (p_1^* L)|_{A \times a}$. Therefore, $s^* L \otimes p_1^* L^{-1} \to A \times A$ is a line bundle, whose restriction to $A \times a$ is trivial for all $a \in A$. By seesaw theorem [BL04, A.8], there is a line bundle $M \to A$ such that $s^* L \otimes p_1^* L^{-1} = p_2^* M$. Then $s^* L = p_1^* L \otimes p_2^* M$. Hence, $L = s^* L|_{0 \times A} = (p_1^* L \otimes p_2^* M)|_{0 \times A} = M$. Therefore, $s^* L = p_1^* L \otimes p_2^* L$. □

Theorem 5.2.4 is mentioned without proof in [KKN08, Sec. 1.2]. The analogue for abelian varieties is in [Wei49, no. 2].

Theorem 5.2.4 (Weil)**.** *If A is a complex torus, then $\pi : \mathrm{Ext}(A, \mathbb{C}^*) \to \mathrm{Pic}^0(A)$ is an isomorphism.*

Proof. For $B = \mathbb{C}^*$, the sheaf $\mathcal{B}_A = O_A^*$ and $H^1(A, \mathcal{B}_A) = \mathrm{Pic}(A)$. The class of a line bundle $L \to A$ is primitive means the line bundle $s_A^* L$ is isomorphic to $p_1^* L \otimes p_2^* L$ on $A \times A$. By Proposition 5.2.3 and Fact 5.2.2, it is equivalent to $[L] \in \mathrm{Pic}^0(A)$. Then Proposition 5.1.3 and Theorem 5.1.6 complete the proof. □

With the identifications provided by Theorem 5.2.4 and Proposition 4.1.4 3, [AK01, Remark 1.1.16] can be rephrased in a coordinate-free way as follows. It is a criterion telling whether a semi-torus is a toroidal group.

Fact 5.2.5. *Let $r \geq 1$ be an integer, and let $0 \to (\mathbb{C}^*)^r \to X \to A \to 0$ be an extension in $\mathcal{C}$. Denote by $(L_1, \ldots, L_r) \in (A^\vee)^r$ the point corresponding to the equivalent class $[X] \in \mathrm{Ext}(A, (\mathbb{C}^*)^r)$. Then the following are equivalent:*

- *X is a toroidal group;*
- *for all $\sigma \in \mathbb{Z}^r \setminus \{0\}$, $\sum_{i=1}^r \sigma_i L_i \neq 0$ in $A^\vee$;*
- *for every nontrivial morphism $f : (\mathbb{C}^*)^r \to \mathbb{C}^*$, the pushout extension $f_* X$ of A by $\mathbb{C}^*$ is nontrivial.*

5.3 The case $B = \mathbb{C}$

When $B = \mathbb{C}$, the sheaf $\mathcal{B}_A = O_A$.

Fact 5.3.1 (Künneth formula, [Men20, (3.1)])**.** *Let X, Y be connected complex manifolds. Assume that Y is compact. Then there is a canonical decomposition $H^1(X \times Y, O_{X\times Y}) = H^1(X, O_X) \oplus H^1(Y, O_Y)$.*

The analogue of Theorem 5.3.2 for abelian varieties is [Ros58, Theorem 1].

Theorem 5.3.2 (Rosenlicht, Serre). *If A is a complex torus, then the canonical morphism $\pi : \mathrm{Ext}(A, \mathbb{C}) \to H^1(A, O_A)$ is a $\mathbb{C}$-linear isomorphism. In particular,* $\dim_{\mathbb{C}} \mathrm{Ext}(A, \mathbb{C}) = \dim A$.

Proof. Let m_1 (resp. m_2) be the injection $A \to A \times A$ defined by $a \mapsto (a, 0)$ (resp. $a \mapsto (0, a)$). Let $p_i : A \times A \to A$ $(u = 1, 2)$ be the two projections. By Fact 5.3.1, p_1^* and p_2^* identify $T(A \times A)$ as the direct sum $T(A) \oplus T(A)$. The projection to ith factor is m_i^*. Because $s_A \circ m_i = \mathrm{Id}_A$, one has $s_A^*(x) = p_1^* x + p_2^* x$ for every $x \in T(A)$, *i.e.*, x is primitive. Then Proposition 5.1.3 and Theorem 5.1.6 conclude the proof. □

Remark 5.3.3. Another way to prove Theorem 5.3.2 is to use (13). In this case, the diagram (12) can be completed into a commutative diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Hom}_{\mathcal{C}}(\tilde{A}, \mathbb{C}) & \longrightarrow & \mathrm{Hom}(\pi_1(A), \mathbb{C}) & \longrightarrow & \mathrm{Ext}(A, \mathbb{C}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow{\scriptstyle \pi} & & \\ 0 & \longrightarrow & H^0(A, \Omega_A^1) & \longrightarrow & H^1(A, \mathbb{C}) & \longrightarrow & H^1(A, O_A) & \longrightarrow & 0. \end{array} \tag{19}$$

The bottom row comes from the Hodge structure on $H^1(A, \mathbb{C})$ ([Huy05, Lem. 3.3.1]).

Corollary 5.3.4. *Let A be a complex abelian variety, and let $n (\geq 0)$ be an integer. Then the natural morphism $\mathrm{Ext}_{\mathrm{Alg}}(A, \mathbb{G}_a^n) \to \mathrm{Ext}(A^{\mathrm{an}}, \mathbb{C}^n)$ is an isomorphism.*

Proof. It is a combination of [Ser88, Thm. 7, p.185], Theorem 5.3.2 and [Ser56, Thm. 1]. □

5.4 Universal vectorial extension

Definition 5.4.1. [Ros58, p.705] Let H be a vector group. An extension

$$0 \to H \to G \to A \to 0 \tag{20}$$

in $\mathcal{C}$ is called decomposable if there exists an extension

$$0 \to H_1 \to G_1 \to A \to 0$$

in $\mathcal{C}$ of A by a vector subgroup H_1 of H, and H' is a vector subgroup of H of positive dimension with an isomorphism $f : G_1 \oplus H' \to G$ such that the maps $H_1 \to H \to G$ and $H_1 \to G_1 \xrightarrow{f|_{G_1}} G$ coincide. Otherwise, the extension G is called indecomposable.

Proposition 5.4.2. *The extension (20) is decomposable if and only if there is a strict vector subgroup H_1 of H and an extension $0 \to H_1 \to G_1 \xrightarrow{p_1} A \to 0$ with $\iota_* G_1 = G$, where $\iota : H_1 \to H$ is the inclusion.*

Proof. If G is decomposable, by definition, we can write $G = G_1 \oplus H'$, where $H' \subset H$ is a positive-dimensional vector subgroup and $0 \to H_1 \to G_1 \to A \to 0$ is an extension in $\mathcal{C}$ of A by a vector subgroup $H_1 \subset H$ making a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1 \arrow[r] \arrow[d, hook, "\iota"] & G_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & H \arrow[r] & G \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

By the universal property (7), $G = \iota_* G_1$. Moreover,

$$\dim H_1 = \dim G_1 - \dim A = \dim G - \dim H' - \dim A = \dim H - \dim H' < \dim H.$$

Conversely, assume that $\iota_* G_1 = G$. Choose a vector subspace H' of H with $H = H' \oplus H_1$, then $\dim H' = \dim H - \dim H_1 > 0$. The composed morphism $G_1 \oplus H' \stackrel{pr_1}{\to} G_1 \stackrel{p_1}{\to} A$ is surjective of kernel $H_1 \oplus H' = H$, hence a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & H_1 \arrow[r] \arrow[d, hook, "\iota"] & G_1 \arrow[r] \arrow[d] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & H \arrow[r] & G_1 \oplus H' \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

with exact rows. By the universal property (7), $G = \iota_* G_1 = G_1 \oplus H'$. This identification makes the maps $H_1 \to H \to G$ and $H_1 \to G_1 \to G$ coincide. Therefore, G is decomposable. $\square$

Proposition 5.4.3. *Let $0 \to \mathbb{C}^n \to G \to A \to 0$ be an extension in $\mathcal{C}$. Let $q_i : \mathbb{C}^n \to \mathbb{C}$ be the i-th coordinate function. Then G is indecomposable if and only if the family $\{q_{i,*}G\}_{1 \le i \le n}$ of vectors in* $\mathrm{Ext}(A, \mathbb{C})$ *is linearly independent.*

Proof. Assume that $\{q_{i,*}G\}$ is linearly dependent. By changing of coordinate, one may assume that $q_{n,*}G = 0$ in $\mathrm{Ext}(A, \mathbb{C})$. By Fact 3.0.4, there is a morphism $r : q_{n,*}G \to \mathbb{C}$ with $i_n r = \mathrm{Id}$ on $q_{n,*}G$.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}^n \arrow[r, "i"] \arrow[d, "q_n"] & G \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C} \arrow[r, "i_n"'] & q_{n,*}G \arrow[r] \arrow[l, dashed, bend right, "r"'] & A \arrow[r] & 0 \end{tikzcd}$$

Then $i_n r \alpha i = \alpha i = i_n q_n$. Since i_n is injective, one has

$$r\alpha i = q_n. \tag{21}$$

Let $q : \mathbb{C}^n \to \mathbb{C}^{n-1}$ be the projection to the first $(n-1)$ coordinates. Let $\beta : G \to q_* G$ be the canonical morphism. Define a morphism

$$\epsilon : G \to q_* G \oplus \mathbb{C}, \quad g \mapsto (\beta(g), r\alpha(g)).$$

Then the right square of the following diagram is commutative.

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}^n \arrow[r, "i"] \arrow[d, "q \oplus q_n"] & G \arrow[r] \arrow[d, "\epsilon"] & A \arrow[r] \arrow[d, "\mathrm{Id}"] & 0 \\ 0 \arrow[r] & \mathbb{C}^{n-1} \oplus \mathbb{C} \arrow[r] & q_* G \oplus \mathbb{C} \arrow[r] & A \arrow[r] & 0 \end{tikzcd}$$

By (21), the left square of the above diagram is commutative. Therefore, ϵ is an equivalence of extensions and $G = q_* G \oplus \mathbb{C}$ is decomposable.

Conversely, assume that G is decomposable. By Proposition 5.4.2, there is a vector subgroup $\iota : H_1 \to \mathbb{C}^n$ with $\dim H_1 < n$ and an extension $0 \to H_1 \to G_1 \to A \to 0$ with $\iota_* G_1 = G$. There is a linear combination $f = \sum_{i=1}^n a_i q_i : \mathbb{C}^n \to \mathbb{C}$, where $a_1, \dots, a_n \in \mathbb{C}$ are not all zero, such that $f\iota = 0$. Then $\sum_{i=1}^m a_i q_{i,*} G = f_* G = (f\iota)_* G_1 = 0$. Thus, the family $\{q_{i,*}G\}_i$ is linearly dependent. □

Corollary 5.4.4 follows from Proposition 5.4.3 and Theorem 5.3.2.

Corollary 5.4.4. *Let $0 \to V \to G \to A \to 0$ be an extension in $\mathcal{C}$ by a vector group V. If $\dim_{\mathbb{C}} V > g$, then G is decomposable.*

Proposition 5.4.5 is an analytic analogue of [Ros58, Prop. 11].

Proposition 5.4.5.

1. *There is a $\mathbb{C}$-vector group H with $\dim_{\mathbb{C}} H = g$ and an indecomposable extension*
$$0 \to H \to G \to A \to 0 \tag{22}$$
such that for every $V \in \mathrm{Vec}$, the map
$$\phi_V : \mathrm{Hom}_{\mathrm{Vec}}(H, V) \to \mathrm{Ext}(A, V), \quad l \mapsto l_* G \tag{23}$$
is a linear isomorphism. In other words, H together with the extension (22) represents the functor $\mathrm{Ext}(A, \bullet) : \mathrm{Vec} \to \mathrm{Vec}$.

2. *A $G' \in \mathrm{Ext}(A, V)$ is indecomposable if and only if the corresponding linear map $\phi_V^{-1}(G') : H \to V$ is surjective.*

Proof.

1. By Theorem 5.3.2, $\dim_{\mathbb{C}} \mathrm{Ext}(A, \mathbb{C}) = g$. Take a $\mathbb{C}$-basis $\{G_1, \dots, G_g\}$ of $\mathrm{Ext}(A, \mathbb{C})$. By Fact 4.1.8, $\mathrm{Ext}(A, \mathbb{C}^g) = \oplus_{i=1}^g \mathrm{Ext}(A, \mathbb{C})$, so there is an element $G \in \mathrm{Ext}(A, \mathbb{C}^g)$ corresponding to $(G_1, \dots, G_g) \in \oplus_{i=1}^g \mathrm{Ext}(A, \mathbb{C})$. Hence an extension $0 \to H \to G \to A \to 0$, where $H = \mathbb{C}^g$. By Proposition 5.4.3, G is indecomposable.

   When $l \in H^\vee$ is taking the i-th coordinate of $H = \mathbb{C}^g$, $l_* G = G_i$. Therefore, the image of the linear map $\phi_{\mathbb{C}}$ contains a basis of $\mathrm{Ext}(A, \mathbb{C})$. Thus, $\phi_{\mathbb{C}}$ is surjective. Since $\dim_{\mathbb{C}} H^\vee = \dim_{\mathbb{C}} \mathrm{Ext}(A, \mathbb{C})$, $\phi_{\mathbb{C}}$ is a linear isomorphism. Since every $V \in \mathrm{Vec}$ is the direct sum of finitely many copies of $\mathbb{C}$ and the formation of ϕ_V is functorial in V, ϕ_V is also a linear isomorphism.

2. By Proposition 5.4.2, G' is decomposable iff there is a proper linear subspace $\iota : V_1 \to V$ with G' in the image of the map $\iota_* : \mathrm{Ext}(A, V_1) \to \mathrm{Ext}(A, V)$ iff there is a proper linear subspace $\iota : V_1 \to V$ with $\phi_V^{-1}(G')$ in the image of the map $\iota_* : \mathrm{Hom}_{\mathrm{Vec}}(H, V_1) \to \mathrm{Hom}_{\mathrm{Vec}}(H, V)$ iff $\phi_V^{-1}(G') : H \to V$ factors through a proper linear subspace $\iota : V_1 \to V$ iff $\phi_V^{-1}(G') : H \to V$ is not surjective.

$\square$

The extension (22) is called the *universal vectorial extension* of A. (As a representing object, such an extension is unique up to equivalence.) By (23) and Theorem 5.3.2, $H = H^0(A^\vee, \Omega^1_{A^\vee})$.

Example 5.1.2 (continued). Since $\dim \mathrm{Ext}(A, C) = 1$, this nontrivial extension is equivalent to the universal vectorial extension.

We proceed to give an explicit construction of the universal vectorial extension.

Proposition 5.4.6. *Let $B^{\natural 1}$ be the group of isomorphic classes of rank 1 local systems on A. Let $B^\natural$ be the group of isomorphic classes of pairs (L, ∇), where $L \to A$ is a holomorphic line bundle and ∇ is a flat holomorphic connection on L. Then there exist natural identifications of groups*

$$B^\natural = B^{\natural 1} = \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \mathbb{C}^*) = H^1(A, \mathbb{C}^*) = \frac{H^1(A, \mathbb{C})}{H^1(A, \mathbb{Z})}.$$

They are isomorphic to $(\mathbb{C}^*)^{2g}$.

Proof. By the Riemann-Hilbert correspondence [Del06, Théorème 2.17, p.12], the map $B^\natural \to B^{\natural 1}$ defined by $(L, \nabla) \mapsto \ker(\nabla)$ is a group isomorphism. By [Del06, Corollaire 1.4, p.4], there is an isomorphism $B^{\natural 1} \to \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \mathbb{C}^*)$. By the universal coefficient theorem [Hat05, Thm. 3.2], there is a natural isomorphism $H^1(A, \mathbb{C}^*) \to \mathrm{Hom}(\pi_1(A), \mathbb{C}^*)$. The exact sequences $0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp(2\pi i \bullet)}{\to} \mathbb{C}^* \to 0$ of constant sheaves on A gives rise to an exact sequence

$$H^0(A, \mathbb{C}) \to H^0(A, \mathbb{C}^*) \to H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{C}) \to H^1(A, \mathbb{C}^*) \to H^2(A, \mathbb{Z}) \to H^2(A, \mathbb{C}).$$

Since the first map is surjective and the last map is injective, it breaks into a short exact sequence

$$0 \to H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{C}) \to H^1(A, \mathbb{C}^*) \to 0$$

and hence an isomorphism $H^1(A, \mathbb{C})/H^1(A, \mathbb{Z}) \to H^1(A, \mathbb{C}^*)$ functorial in A. Moreover, there is a non-canonical isomorphism $H^1(A, \mathbb{C}^*) \to (\mathbb{C}^*)^{2g}$. $\square$

For every $(L, \nabla) \in B^\natural$, the line bundle $L \in \mathrm{Pic}^0(A) = A^\vee$ by [Dem12, Ch. V, §9]. The bottom row of (19) induces an exact sequence in $\mathcal{C}$:

$$0 \to H^0(A, \Omega^1_A) \to \frac{H^1(A, \mathbb{C})}{H^1(A, \mathbb{Z})} \to \frac{H^1(A, O_A)}{H^1(A, \mathbb{Z})} \to 0. \tag{24}$$

Using the identifications $B^\natural \cong \frac{H^1(A,\mathbb{C})}{H^1(A,\mathbb{Z})}$ from Proposition 5.4.6 and $A^\vee = \mathrm{Pic}^0(A) = H^1(A, O_A)/H^1(A, \mathbb{Z})$, (24) is an extension of $A^\vee$ by $H^0(A, \Omega^1_A)$ and gives a morphism $B^\natural \to \mathrm{Pic}^0(A)$, which sends (L, ∇) to L. Hence a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_{\mathcal{C}}(\tilde{A}, \mathbb{C}^*) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \mathbb{C}^*) \arrow[r] & \mathrm{Ext}(A, \mathbb{C}^*) \arrow[r] \arrow[d, "\pi"] & 0 \\
0 \arrow[r] & H^0(A, \Omega^1_A) \arrow[r] & B^\natural \arrow[r, "u"] \arrow[u] & \mathrm{Pic}^0(A) \arrow[r] & 0,
\end{tikzcd}
$$

where the first exact row is (9) and the second comes from (24). The left vertical isomorphism uses Proposition 4.1.4 2 and the isomorphism $L(A)^\vee \to H^0(A, \Omega^1_A)$ given by [BL04, Thm. 1.4.1 b)]. The middle vertical isomorphism is contained in Proposition 5.4.6.

When A is an abelian variety, it is proved in [Mes73, p.260] that (24) is the universal vectorial extension of $A^\vee$. The proof is based on [Ros58, Thm. 1]. In a similar manner, Proposition 5.4.7 follows from Theorem 5.3.2.

Proposition 5.4.7. *The extension (24) is the universal vectorial extension of* $A^\vee = \mathrm{Pic}^0(A)$*. In particular, the extension group is isomorphic to* $(\mathbb{C}^*)^{2g}$ *(as a complex Lie group).*

Proof. Let $U = H^0(A, \Omega^1_A)$. Pushing out the extension (24) defines a natural transformation $\psi : \mathrm{Hom}_{\mathrm{Vec}}(U, \bullet) \to \mathrm{Ext}(A^\vee, \bullet)$ between two functors on Vec.

We claim that $\psi_{\mathbb{C}}$ is an isomorphism. Choose $u \in \ker(\psi_{\mathbb{C}}) \subset \mathrm{Hom}_{\mathrm{Vec}}(U, \mathbb{C})$. As the push-out along u is trivial, by Fact 3.0.4, there is a morphism $r : E \to \mathbb{C}$ with $ir = \mathrm{Id}_E$. Let $u' : H^1(A, \mathbb{C}) \to \mathbb{C}$ be the morphism in $\mathcal{C}$ induced by r. Then $u' = d_e u'$ is $\mathbb{C}$-linear. Now that $u'(H^1(A, \mathbb{Z})) = 0$ and $H^1(A, \mathbb{Z})$ contains a $\mathbb{C}$-basis of $H^1(A, \mathbb{C})$, one has $u' = 0$. As the diagram commutes, $u = 0$.

$$
\begin{tikzcd}
 & & H^1(A, \mathbb{C}) \arrow[d] \arrow[ddl, dashed, "u'"'] & & \\
0 \arrow[r] & U \arrow[r] \arrow[d, "u"'] & \frac{H^1(A,\mathbb{C})}{H^1(A,\mathbb{Z})} \arrow[r] \arrow[d] & A^\vee \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathbb{C} \arrow[r, "i"'] & E \arrow[r] \arrow[l, dashed, bend right, "r"'] & A^\vee \arrow[r] & 0
\end{tikzcd}
$$

Therefore, $\psi_{\mathbb{C}}$ is injective. By Theorem 5.3.2, $\dim_{\mathbb{C}} \mathrm{Ext}(A^\vee, \mathbb{C}) = \dim_{\mathbb{C}} \mathrm{Hom}_{\mathrm{Vec}}(U, \mathbb{C})$. Therefore, $\psi_{\mathbb{C}}$ is a linear isomorphism. Similar to the proof of Proposition 5.4.5 1, ψ is a natural isomorphism of the two functors. $\square$

Another construction of the universal vectorial extension is in [Nak94, Prop. 2.4][9].

Remark 5.4.8. The real Lie group extension underlying (24) is trivial by Fact 5.1.1. Indeed, consider the real analytic group morphism $A^\vee \to B^\natural$ defined by $L \mapsto (L, \nabla^L)$, where ∇^L is the unique flat Chern connection on L given by This map is a real Lie group section to (24), but not holomorphic.

[9] stated for complex abelian varieties but the proof extends to complex tori.

Remark 5.4.9. Let A be a complex abelian variety of dimension g. By Corollary 5.3.4, the extension (22) is equivalent to an algebraic one. Thus, the analytification of the algebraic universal vectorial extension $0 \to \mathbb{G}_a^g \to E \to A \to 0$ is exactly the analytic universal vectorial extension . From [Bri09, Prop. 2.3 (i)] and the footnote in [MRM74, p.34], the algebraic variety E is anti-affine, *i.e.*, every morphism $E \to A^1_{\mathbb{C}}$ of algebraic varieties is constant. On the other hand, by Proposition 5.4.7, E^{an} is isomorphic to $(\mathbb{C}^*)^{2g}$ as a complex Lie group, so E^{an} is not a toroidal group. Although E is not an affine variety, E^{an} is a Stein manifold. See also Serre's example [Har70, Exampe 3.2, p.232].

Remark 5.4.10. Universal vectorial extensions can be defined for not only complex tori but also toroidal groups. Consider a toroidal group X of dimension n. Similar to Proposition 5.4.5, the functor $\mathrm{Ext}(X, \cdot) : \mathrm{Vec} \to \mathrm{Vec}$ is represented by $\mathrm{Ext}(X, \mathbb{C})^\vee$, which is the kernel of the natural linear map $H_1(X, \mathbb{C}) \to H^0(X, \Omega^1_X)^\vee$ by (13).

An extrinsic description is possible. Choose a presentation

$$0 \to (\mathbb{C}^*)^{n-q} \to X \to T \to 0 \tag{25}$$

according to [AK01, 1.1.14], where T is a complex torus of dimension q. For every $V \in \mathrm{Vec}$, by Proposition 4.2.1, the induced sequence

$$\mathrm{Hom}_{\mathcal{C}}((\mathbb{C}^*)^{n-q}, V) \to \mathrm{Ext}(T, V) \to \mathrm{Ext}(X, V) \to \mathrm{Ext}((\mathbb{C}^*)^{n-q}, V)$$

is exact in Vec. By Proposition 4.1.4 1, $\mathrm{Hom}_{\mathcal{C}}((\mathbb{C}^*)^{n-q}, V) = 0$. By Proposition 3.2.3, $\mathrm{Ext}((\mathbb{C}^*)^{n-q}, V) = 0$. Thus, the morphism $\mathrm{Ext}(T, V) \to \mathrm{Ext}(X, V)$ is a $\mathbb{C}$-linear isomorphism. In other words, the natural transformation $\mathrm{Ext}(T, \cdot) \to \mathrm{Ext}(X, \cdot)$ between the two functors on Vec is an isomorphism. In this way, the case of toroidal groups is reduced to the case of complex tori.

5.5 Application to the functor $\mathrm{Ext}(A, \bullet)$

Analogue of Proposition 5.5.1 for abelian varieties is [Ros58, Cor., p.711].

Proposition 5.5.1. *If B is a complex Lie subgroup (not necessarily connected) of A, then there is a natural exact sequence in* Ab*:*

$$0 \to \mathrm{Ext}(A/B, \mathbb{C}) \to \mathrm{Ext}(A, \mathbb{C}) \to \mathrm{Ext}(B, \mathbb{C}) \to 0.$$

Proof. By Corollary 4.1.13, there is an isomorphism $B \to B_0 \times B/B_0$ in $\mathcal{C}$ and $\mathrm{Ext}(B/B_0, \mathbb{C}) = 0$. By Fact 4.1.8, $\mathrm{Ext}(B, \mathbb{C}) = \mathrm{Ext}(B_0, \mathbb{C})$. Since B is compact and B_0 is open in B, the quotient B/B_0 is finite, thus $\mathrm{Hom}_{\mathrm{Ab}}(B/B_0, \mathbb{C}) = 0$. By the compactness of B_0, $\mathrm{Hom}_{\mathcal{C}}(B_0, \mathbb{C}) = 0$. Then $\mathrm{Hom}(B, \mathbb{C}) = 0$. Now that $A, B_0, A/B$ are complex tori, Theorem 5.3.2 implies $\dim \mathrm{Ext}(A, \mathbb{C}) = \dim \mathrm{Ext}(A/B, \mathbb{C}) + \dim \mathrm{Ext}(B, \mathbb{C})$. This together with Proposition 4.2.1 proves the stated exactness. ☐

The proof of Theorem 5.5.2 is shorter than that of its algebraic analogue [Ser88, Thm. 12, p.195].

Theorem 5.5.2. *If $0 \to B' \to B \xrightarrow{\phi} B'' \to 0$ is an exact sequence in $\mathcal{C}$, then the sequence*[10] *in* Ab

$$\mathrm{Ext}(A, B') \to \mathrm{Ext}(A, B) \xrightarrow{\phi_*} \mathrm{Ext}(A, B'') \to 0 \tag{26}$$

is exact. If B''_0 is linear, then the first map in (26) is injective.

Proof. By Proposition 4.2.3, it suffices to prove that $\phi_* : \mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B'')$ is surjective. From (10) and Proposition 4.2.1, one obtains a commutative square

$$\begin{array}{ccc}
\mathrm{Hom}(\pi_1(A), B) & \longrightarrow & \mathrm{Hom}(\pi_1(A), B'') \\
\downarrow & & \downarrow \\
\mathrm{Ext}(A, B) & \xrightarrow{\phi_*} & \mathrm{Ext}(A, B''),
\end{array}$$

where the vertical maps are surjective. Since $\pi_1(A)$ is a free $\mathbb{Z}$-module, the top row is surjective, then so is the bottom.

Now assume that B''_0 is linear, then $\mathrm{Hom}_{\mathcal{C}}(A, B'') = 0$. By Proposition 4.2.3, the first map is injective. □

Remark 5.5.3. The linearity of B''_0 is necessary to guarantee the injectivity in Theorem 5.5.2. For instance, let $0 \to \mathbb{C}^g \to (\mathbb{C}^*)^{2g} \to A \to 0$ be the universal vectorial extension of A and assume $g \geq 1$. By Proposition 4.2.3, the natural sequence $0 \to \mathrm{Hom}_{\mathcal{C}}(A, A) \to \mathrm{Ext}(A, \mathbb{C}^g) \to \mathrm{Ext}(A, (\mathbb{C}^*)^{2g})$ is exact. Thus, Id_A is a nonzero element in the kernel of the first map of (26).

Example 5.5.4. Applying Theorem 5.5.2 to the exact sequence $0 \to \mathbb{Z} \to \mathbb{C} \overset{\exp(2\pi i \bullet)}{\to} \mathbb{C}^* \to 1$, and using Fact 3.2.6, Theorems 5.2.4 and 5.3.2, one gets an exact sequence

$$0 \to \mathrm{Hom}(\pi_1(A), \mathbb{Z}) \to H^1(A, O_A) \to \mathrm{Pic}^0(A) \to 0. \tag{27}$$

In particular, $\mathrm{Ext}(A, \cdot)$ tuns the exponential map to the universal cover of the complex torus $A^\vee$. Identifying $\mathrm{Hom}(\pi_1(A), \mathbb{Z})$ with the sheaf cohomology $H^1(A, \mathbb{Z})$, th sequence (27) is also induced by the exponential sequence of sheaves on A:

$$0 \to \mathbb{Z}_A \to O_A \overset{\exp(2\pi i)}{\to} O_A^* \to 1.$$

Theorem 5.5.5 is an analytic version of [Ser88, Thm. 13, p.196]

Theorem 5.5.5. *If $0 \to L \xrightarrow{i} C \to A \to 0$ is an exact sequence in $\mathcal{C}$ with L connected and $G \in \mathrm{Ab}_c$. Then there is a natural exact sequence*

$$0 \to \mathrm{Ext}(A, G) \to \mathrm{Ext}(C, G) \xrightarrow{i^*} \mathrm{Ext}(L, G) \to 0.$$

[10] induced by Proposition 4.2.3

Proof. As L is connected and G is discrete, $\mathrm{Hom}_{\mathcal{C}}(L, G) = 0$. By Proposition 4.2.1, it suffices to show that $i^* : \mathrm{Ext}(C, G) \to \mathrm{Ext}(L, G)$ is surjective. For every $L' \in \mathrm{Ext}(L, G)$, by Theorem 5.5.2, the map $\mathrm{Ext}(A, L') \to \mathrm{Ext}(A, L)$ is surjective. Thus, there exists $C' \in \mathrm{Ext}(A, L')$ having image $C \in \mathrm{Ext}(A, L)$.

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & G \arrow[r, "\beta"] \arrow[d] & \ker(\alpha) \arrow[r] \arrow[d] & 0 \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & L' \arrow[r] \arrow[d] & C' \arrow[r] \arrow[d, "\alpha"] & A \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L \arrow[r, "i"] \arrow[d] & C \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & 0 \arrow[r] & 0 &
\end{tikzcd}
$$

By the snake lemma, α is surjective and β is an isomorphism. Therefore, $C' \in \mathrm{Ext}(C, G)$ and $i^*C' = L'$ in $\mathrm{Ext}(L, G)$. $\square$

In Example 5.5.6, we give another proof of [BL99, Prop. 5.7, p.21], which computes the extension group of two complex tori.

Example 5.5.6. Let $X_i = \mathbb{C}^{g_i}/\Pi_i\mathbb{Z}^{2g_i}$ $(i = 1, 2)$ be two complex tori, where the chosen period matrix is of the form $\Pi_i = (\tau_i, I_{g_i})$ with $\tau_i \in M_{g_i}(\mathbb{C})$ and $\det(\mathrm{Im}(\tau_i)) \neq 0$. Define $\xi : M(2g_1 \times 2g_2, \mathbb{Z}) \to M(g_1 \times g_2, \mathbb{C})$ by $\xi(P) = \Pi_1 P\binom{I_{g_2}}{\tau_2}$.

Define a map $\rho : M(g_1 \times g_2, \mathbb{C}) \to \mathrm{Ext}(X_2, \tilde{X_1})$ as follows. For every $\alpha \in M(g_1 \times g_2, \mathbb{C})$, let $\alpha' = (\alpha, 0) \in M(g_1 \times 2g_2, \mathbb{C})$. Consider the sequence

$$0 \to \mathbb{C}^{g_1} \xrightarrow{i} \frac{\mathbb{C}^{g_1+g_2}}{\{(\alpha' v, \Pi_2 v) : v \in \mathbb{Z}^{2g_2}\}} \xrightarrow{p} X_2 \to 0,$$

where i is induced by $\mathbb{C}^{g_1} \to \mathbb{C}^{g_1+g_2}$ defined by $x \mapsto (x, 0)$ and p is induced by the second projection $\mathbb{C}^{g_1+g_2} \to \mathbb{C}^{g_2}$. It is an exact sequence. Denote its class by $\rho(M) \in \mathrm{Ext}(X_2, \tilde{X_1})$. This sequence fits into a commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \tilde{X_1} \arrow[r] \arrow[d] & \frac{\mathbb{C}^{g_1+g_2}}{\{(\alpha' v, \Pi_2 v) : v \in \mathbb{Z}^{2g_2}\}} \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & X_1 \arrow[r] & X \arrow[r] & X_2 \arrow[r] & 0,
\end{tikzcd}
$$

where the second row is $\psi_{\Pi_1,\Pi_2}(\alpha') \in \mathrm{Ext}(X_2, X_1)$ defined in [BL99, p.20], and

$$X = \frac{\mathbb{C}^{g_1+g_2}}{\{(\Pi_1 u + \alpha' v, \Pi_2 v) : u \in \mathbb{Z}^{2g_1}, v \in \mathbb{Z}^{2g_2}\}}.$$

Then ρ is a linear isomorphism by Theorem 5.3.2.

Define a map $\phi : M(2g_1 \times 2g_2, \mathbb{Z}) \to \operatorname{Ext}(X_2, \pi_1(X_1))$ as follows. Given $P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in M(2g_1 \times 2g_2, \mathbb{Z})$, with each $P_i \in M(g_1 \times g_2, \mathbb{Z})$, we set $A = \tau_1 P_2 + P_4 \in M(g_1 \times g_2, \mathbb{C})$ and $\alpha = \xi(P)$. The linear map $\mathbb{C}^{g_1+g_2} \stackrel{(I,-A)}{\to} \mathbb{C}^{g_1}$ sends $(u, 0)$ to u for all $u \in \mathbb{C}^{g_1}$ and sends $(\alpha' v, \Pi_2 v)$ to $\Pi_1 \begin{pmatrix} P_1 & -P_2 \\ P_3 & -P_4 \end{pmatrix} v \in \Pi_1 \mathbb{Z}^{2g_1}$ for all $v \in \mathbb{Z}^{2g_2}$. Thus it descents to the vertical morphism in the middle of the following commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{C}^{g_1} \arrow[r] \arrow[d] & \frac{\mathbb{C}^{g_1+g_2}}{\{(\alpha' v, \Pi_2 v) : v \in \mathbb{Z}^{2g_2}\}} \arrow[r] \arrow[d, "(I_{g_1}, -A)"] & X_2 \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & X_1 \arrow[r] & X_1 \arrow[r] & 0 \arrow[r] & 0, \end{tikzcd} \tag{28}$$

where the first row is of class $\rho(\alpha) = \rho(\xi(P))$. The snake lemma gives an extension of X_2 by $\pi_1(X_1)$, whose class is denoted by $\phi(P)$.

The image of $\phi(P)$ under the pushout map $\operatorname{Ext}(X_2, \pi_1(X_1)) \to \operatorname{Ext}(X_2, \tilde{X}_1)$ is exactly the first row of (28), *i.e.*, $\rho(\xi(P))$. Then ϕ is a group isomorphism by Fact 3.2.6. And there is a commutative diagram

$$\begin{tikzcd} & & & M(g_1 \times 2g_2, \mathbb{C}) \arrow[ddl, "\psi_{\Pi_1,\Pi_2}"] \\ M(2g_1 \times 2g_2, \mathbb{Z}) \arrow[r, "\xi"] \arrow[d, "\phi"] & M(g_1 \times g_2, \mathbb{C}) \arrow[r] \arrow[d, "\rho"] \arrow[urr, hook, "\alpha \mapsto \alpha'"] & \frac{M(g_1 \times g_2, \mathbb{C})}{\operatorname{Im}(\xi)} \arrow[d, dashed] & \\ \operatorname{Ext}(X_2, \pi_1(X_1)) \arrow[r] & \operatorname{Ext}(X_2, \tilde{X}_1) \arrow[r] & \operatorname{Ext}(X_2, X_1) \arrow[r] & 0 \end{tikzcd}$$

where the second row is from (14) and the induced dotted isomorphism is exactly the content of [BL99, Proposition 5.7, p.21].

To conclude Section 5.5, we show that the groups of commutative extensions of complex tori by linear groups are naturally complex Lie groups. Let $\mathcal{T}$ (resp. $\mathcal{S}$) be the full subcategory of $\mathcal{C}$ comprised of complex tori (resp. objects whose identity component is linear). Then $\operatorname{Ext} : \mathcal{T}^{\mathrm{op}} \times \mathcal{S} \to \mathrm{Ab}$ is an additive functor by Fact 4.1.8. Theorem 5.5.7, an analytic analogue of [Wu86, Theorem 5], lifts this functor.

Theorem 5.5.7 (Wu). *There is a natural way to lift* $\operatorname{Ext} : \mathcal{T}^{\mathrm{op}} \times \mathcal{S} \to \mathrm{Ab}$ *to an additive functor* $\operatorname{Ext} : \mathcal{T}^{\mathrm{op}} \times \mathcal{S} \to \mathcal{C}$.

Proof. First we define a complex Lie group structure on $\operatorname{Ext}(A, H)$, where $A \in \mathcal{T}$ and $H \in \mathcal{S}$. Let $g = \dim A$.

If there is an isomorphism $f : H \to (\mathbb{C}^*)^n$ in $\mathcal{S}$, then by Theorem 5.2.4, f gives rise to an isomorphism $\operatorname{Ext}(A, H) \to (A^\vee)^n$ making $\operatorname{Ext}(A, H)$ a complex torus. The complex structure on $\mathrm{Ext}(A, H)$ is independent of the choice of the isomorphism f.

If H is connected, by Proposition 2.0.7, there is an isomorphism $u : H \to V \times H_m$, where $V \in \mathrm{Vec}$ and H_m is a power of $\mathbb{C}^*$. Then $u_* : \mathrm{Ext}(A, H) \to \mathrm{Ext}(A, V) \times \mathrm{Ext}(A, H_m)$ is an isomorphism. By Theorem 5.3.2, the vector space $\mathrm{Ext}(A, V)$ is *finite dimensional*. Together with last paragraph, $\mathrm{Ext}(A, H)$ inherits a complex Lie group structure, which is independent of the choice of u.

For a general object $H \in \mathcal{S}$, the natural exact sequence $0 \to H_0 \to H \to H/H_0 \to 0$ in $\mathcal{C}$ is trivial by Corollary 4.1.13. Thus, the resulting exact sequence $0 \to \mathrm{Ext}(A, H_0) \to \mathrm{Ext}(A, H) \to \mathrm{Ext}(A, H/H_0) \to 0$ in Ab is also trivial. Now that $\mathrm{Ext}(A, H/H_0) = \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), H/H_0)$ by Fact 3.2.6, one regards it as a discrete group. From the complex structure on $\mathrm{Ext}(A, H_0)$, the group $\mathrm{Ext}(A, H)$ has a unique complex Lie group structure, such that the identity component is $\mathrm{Ext}(A, H_0)$.

It remains to show:

1. If $A \in \mathcal{T}$ is fixed, then $\mathrm{Ext}(A, \cdot)$ sends morphisms in $\mathcal{S}$ to morphisms in $\mathcal{C}$.

2. If $H \in \mathcal{S}$ is fixed, then $\mathrm{Ext}(\cdot, H)$ sends morphisms in $\mathcal{T}$ to morphisms in $\mathcal{C}$.

To show 1, let $h : H \to H'$ be a morphism in $\mathcal{S}$. By decomposing H, H' according to Corollary 4.1.13 and Proposition 2.0.7, one may assume that each of H and H' is either discrete, $\mathbb{C}$ or $\mathbb{C}^*$.

- If H is discrete, then so is $\mathrm{Ext}(A, H)$, hence $\mathrm{Ext}(A, h)$ is a morphism in $\mathcal{C}$.

- If $H = H' = \mathbb{C}$, by Proposition 4.1.4 2, h is a linear map. By Corollary 4.1.9, so is $\mathrm{Ext}(A, h)$.

- If $H = \mathbb{C}$, $H' = \mathbb{C}^*$. By Proposition 4.1.4 2, h is the composition of a linear map $\mathbb{C} \to \mathbb{C}$ followed by the exponential map $\exp(2\pi i \cdot) : \mathbb{C} \to \mathbb{C}^*$. By Example 5.5.4, $\mathrm{Ext}(A, h)$ is the composition of a linear map $H^1(A, O_A) \to H^1(A, O_A)$ followed by the universal cover $H^1(A, O_A) \to A^\vee$. Thus, $\mathrm{Ext}(A, h)$ is a morphism in $\mathcal{C}$.

- If H' is discrete and H is connected, then h is trivial and so is $\mathrm{Ext}(A, h)$.

- If $H = \mathbb{C}^*$ and $H' = \mathbb{C}$, then h is trivial by Proposition 4.1.4 1 and so is $\mathrm{Ext}(A, h)$.

- If $H = H' = \mathbb{C}^*$, then h is a power map by Proposition 4.1.4 3. Then $\mathrm{Ext}(A, h)$ is a power map of $A^\vee$, hence a morphism in $\mathcal{C}$.

This proves 1.

To show 2, let $g : A \to A'$ be a morphism in $\mathcal{T}$. By decomposing H again, we may divide the proof into three cases.

- $H = \mathbb{C}^*$. By pulling back line bundles, g induces the dual morphism $g^* : \mathrm{Pic}^0(A') \to \mathrm{Pic}^0(A)$. It is identified with $\mathrm{Ext}(g, H)$ by Fact 4.3.2 and Theorem 5.2.4.
- H is discrete. Then so is $\mathrm{Ext}(A', H)$ and thus $\mathrm{Ext}(g, H)$ is a morphism in $\mathcal{C}$.
- $H = \mathbb{C}$. By pulling back, g induces a $\mathbb{C}$-linear map $H^1(A', O_{A'}) \to H^1(A, O_A)$. It is identified with $\mathrm{Ext}(g, H)$ by Fact 4.3.2 and Theorem 5.3.2.

This proves 2. □

Remark 5.5.8. In Theorem 5.5.7, we cannot generalize from complex tori to toroidal groups, nor remove the linear restriction.

Let X be a toroidal group. Then $\mathrm{Hom}_{\mathcal{C}}(X, \mathbb{C}^*) = 0$, hence (14) specializes to

$$0 \to \mathrm{Ext}(X, \mathbb{Z}) \xrightarrow{i} \mathrm{Ext}(X, \mathbb{C}) \to \mathrm{Ext}(X, \mathbb{C}^*) \to 0. \tag{29}$$

Note that $\mathrm{Ext}(X, \mathbb{Z}) = H^1(X, \mathbb{Z})$ (Fact 3.2.6), and by (13) the injection i is the composition of the inclusion $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C})$ with the projection $H^1(X, \mathbb{C}) \to \frac{H^1(X,\mathbb{C})}{H^0(X,\Omega^1_X)}$.

When X is compact, the sequence (29) lifts to an exact sequence in $\mathcal{C}$ by Theorem 5.5.7. As opposed to the compact case, when X is not compact and consider the presentation (25), one has $1 \leq q < n$, so

$$\mathrm{rank}_Z \mathrm{Ext}(X, \mathbb{Z}) = n + q > 2q = \dim_{\mathbb{R}} \mathrm{Ext}(X, \mathbb{C}).$$

Therefore, the image of i is not closed in the vector space $\mathrm{Ext}(X, \mathbb{C})$ (a phenomenon seen in Example 4.1.2). In particular, the sequence (29) has no lift to an exact sequence in $\mathcal{C}$.

Let A, B be two complex tori, $g = \dim A$, $g' = \dim B$ and reconsider (14):

$$0 \to \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{j} \mathrm{Ext}(A, \pi_1(B)) \to \mathrm{Ext}(A, \tilde{B}) \to \mathrm{Ext}(A, B) \to 0.$$

Here, $\mathrm{Ext}(A, \tilde{B})$ is a $\mathbb{C}$-vector space of dimension gg' by Theorem 5.3.2. Identifying $\mathrm{Ext}(A, \pi_1(B))$ with $\mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A), \pi_1(B))$ *via* Fact 3.2.6, j is the map ρ_r in [BL04, p.10]. The quotient $\frac{\mathrm{Ext}(A,\pi_1(B))}{\mathrm{Hom}_{\mathcal{C}}(A,B)}$ is a free abelian group of rank $4gg' - \mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A, B)$. As long as $\mathrm{rank}_{\mathbb{Z}} \mathrm{Hom}_{\mathcal{C}}(A, B) < 2gg'$ (say, when $A = B$ is an elliptic curve without complex multiplication, then $\mathbb{Z} = \mathrm{Hom}_{\mathcal{C}}(A, B)$), the image of the induced injection $\frac{\mathrm{Ext}(A,\pi_1(B))}{\mathrm{Hom}_{\mathcal{C}}(A,B)} \to \mathrm{Ext}(A, \tilde{B})$ is not closed. In particular, $\mathrm{Ext}(A, B)$ has no structure of complex Lie group making this sequence exact in $\mathcal{C}$.

6 Extensions of complex tori are often commutative

In Section 6, we prove that under suitable hypotheses, an extension of a complex torus is commutative.

Proposition 6.0.1. *If* $1 \to B \to C \xrightarrow{p} A \to 1$ *is a* central *extension of complex Lie groups, where A is a toroidal group, then C is commutative. Or equivalently, for every $B \in \mathcal{C}$, the natural injection* $\mathrm{Ext}(A, B) \to \mathrm{Ext}(A, B, 1)$ *is an isomorphism.*

Proof. Consider the holomorphic map $A \times A \to B$ given by (4). By [NW13, Thm. 5.1.36], it is a group morphism, so constant. Thus, C is commutative. □

An algebraic analogue of Proposition 6.0.2 is [Wu86, Cor. 2, p.370].

Proposition 6.0.2. *Let $1 \to K \to E \to A \to 1$ be an extension of complex Lie groups, where A is a complex torus.*

1. *If $Z(K)_0$ is Stein, then $Z(K) = Z(E) \cap K$.*
2. *If K is commutative and K_0 is Stein, then E is commutative.*

Proof.

1. Since $Z(E) \cap K \subset Z(K)$, it suffices to prove that $Z(K) \subset Z(E)$. Consider the group morphism (5): $\theta : A \to \mathrm{Aut}(Z(K))$. For every $x \in Z(K)$, the map
$$\phi : A \to Z(K), \quad a \mapsto \theta_a(x)x^{-1}$$
is continuous. Moreover, $\phi(0) = e_K$. By the connectedness of A, $\phi(A) \subset Z(K)_0$. As $Z(K)_0$ is Stein and A is compact, $\phi(A)$ is the singleton $\{e_K\}$. Therefore, $\theta_a(x) = x$ for every $x \in Z(K)$, which proves $Z(K) \subset Z(E)$.
2. By 1, $K \subset Z(E)$. By Proposition 6.0.1, E is commutative.

□

In Proposition 6.0.3, when B is isomorphic to $\mathbb{C}^n$ for some integer $n \geq 0$ or to $\mathbb{C}^*$, we recover [BZ21, Lem. 2.10].

Proposition 6.0.3. *Let $1 \to B \to C \xrightarrow{p} A \to 1$ be an exact sequence of complex Lie groups, where A is a complex torus and B is commutative. If the group B/B_0 is torsion (i.e., every element of B/B_0 has finite order), then C is commutative.*

Proof. Let Z be the center of C. By Proposition 6.0.1, it suffices to check $B \subset Z$.

The outer action induces a morphism $A \to \mathrm{Aut}(B_0)(\leq \mathrm{GL}(L(B)))$. It is trivial by the compactness of A, *i.e.*, $B_0 \leq Z$. By Corollary 4.1.13, one may assume $B = B_0 \times D$, where D is a discrete subgroup of B isomorphic to B/B_0 and $D \cap B_0 = \{e_B\}$. Let $q : B \to D$ and $r : B \to B_0$ be the corresponding projections.

It remains to show that $0 \times D(\leq B)$ is contained in Z. Fix $d \in D$ and put $b = (0, d) \in B$. The map
$$\nu : C \to C, \quad c \mapsto cbc^{-1}$$

is holomorphic and $\nu(e) = b$. For every $b' \in B$, one has

$$\nu(cb') = cb'bb'^{-1}c^{-1} = cbc^{-1} = \nu(c).$$

The right multiplication action of B on the complex manifold C has quotient A by Fact 2.0.3, so ν factors through a morphism $u : A \to B$ of complex manifolds. Then $qu : A \to D$ is continuous. Since A is connected, qu is constant. Since $qu(e_A) = d$, one gets $qu \equiv d$.

On the other hand, the map $ru : A \to B_0$ is holomorphic. By assumption, there is an integer $n \geq 1$ (depending on d) such that $d^n = e_D$ in D. Thus, $b^n = e_B$. For every $c \in C$, one has $\nu(c)^n = (cbc^{-1})^n = cb^nc^{-1} = e_B$. Therefore, $ru(A)$ is contained in the torsion subgroup $B_{0,\mathrm{tor}}$ of B_0. In view of [AK01, Prop. 1.1.2], $B_{0,\mathrm{tor}}$ is totally disconnected. Since A is connected, ru is constant.

Since $ru(e_A) = 0$, one has $ru \equiv 0$. Therefore, $u \equiv b$, *i.e.*, $b \in Z$. Therefore, $0 \times D \subset Z$ and the proof is completed. ☐

Corollary 6.0.4 follows immediately from Proposition 6.0.3.

Corollary 6.0.4. *Given an extension*

$$0 \to (\mathbb{C}^*)^n \to G \to A \to 0 \tag{30}$$

of complex Lie groups, where A is a complex tours and $n(\geq 1)$ is an integer, then G is a semi-torus.

Corollary 6.0.5. *In Corollary 6.0.4, if A is algebraic, then G admits a unique structure of semiabelian variety such that (30) defines a commutative extension of algebraic groups.*

Proof. From Corollary 6.0.4, (30) defines an element of $\mathrm{Ext}(A^{\mathrm{an}}, (\mathbb{C}^*)^n)$. By [Ser88, Thm. 6, p.184] and Theorem 5.2.4, the natural map $\mathrm{Ext}_{\mathrm{Alg}}(A, \mathbb{G}_m^n) \to \mathrm{Ext}(A^{\mathrm{an}}, (\mathbb{C}^*)^n)$ is identified with the analytification map $[\mathrm{Pic}^0(A)]^n \to [\mathrm{Pic}^0(A^{\mathrm{an}})]^n$, hence a group isomorphism. In particular, there is a unique exact sequence $0 \to \mathbb{G}_m^n \to C \to A \to 0$ in Alg whose analytification is equivalent to (30). ☐

Lemma 6.0.6 is used in the proof of Proposition 6.0.7.

Lemma 6.0.6. *Let G be a real Lie group with Lie algebra $\mathfrak{g}$.*

1. *If $X, Y \in \mathfrak{g}$ are such that $[X, [X, Y]] = 0$ and $[Y, [X, Y]] = 0$, then*

$$\exp(X)\exp(Y)\exp(-X)\exp(-Y) = \exp([X, Y]). \tag{31}$$

2. *If $X \in \mathfrak{g}$ satisfies that $\exp(X)$ commutes with every element of G_0 and $[X, \mathfrak{g}] \subset Z(\mathfrak{g})$, then $X \in Z(\mathfrak{g})$.*

Proof.

1. According to Baker-Campbell-Hausdorff formula (see, *e.g.*, [Far08, Cor. 3.4.5]), there is a symmetric open neighborhood U of $0 \in \mathfrak{g}$ such that for every $A, B \in U$, $\exp(A)\exp(B) = \exp(Z)$, where

$$Z = Z(A, B) = A + B + [A, B]/2 + \dots$$

and "..." indicates terms involving higher commutators of A and B. There is a symmetric open neighborhood V of $0 \in U$ such that $Z(A, B) \in U$ for every $A, B \in V$.

Define $f : \mathbb{R} \to G$ by

$$f(t) = \exp(tX)\exp(tY)\exp(-tX)\exp(-tY)\exp(-t^2[X, Y]).$$

Then f is real analytic. There is $\epsilon > 0$ such that $tX, tY \in V$ for all $t \in (-\epsilon, \epsilon)$. By assumption, $[Z(tX, tY), Z(-tX, -tY)] = 0$ and $Z(tX, tY) + Z(-tX, -tY) = t^2[X, Y]$. Then

$$f(t) = \exp(Z(tX, tY))\exp(Z(-tX, -tY))\exp(-t^2[X, Y]) = e_G$$

for all $t \in (-\epsilon, \epsilon)$ (see [Laz54, p.144]). By [ADGK23, Cor. A.5], $f(1) = e_G$.

2. Let $D = \exp^{-1}(e_G)$. There is an open neighborhood W of $0 \in \mathfrak{g}$ such that $\exp(W)$ is open in G and $\exp : W \to \exp(W)$ is a diffeomorphism. Then $D \cap W = \{0\}$. For every $Y \in \mathfrak{g}$, there is $k > 0$ with $[X, Y/k] \in W$. By assumption, $[X, Y/k] \in Z(\mathfrak{g})$, so $[X, [X, Y/k]] = 0$ and $[Y/k, [X, Y/k]] = 0$. Since $\exp(Y/k) \in G_0$, it commutes with $\exp(X)$. By 1, $\exp([X, Y/k]) = e_G$. Then $[X, Y/k] \in D \cap W$. Therefore, $[X, Y] = 0$. Thus, $X \in Z(\mathfrak{g})$.

□

An algebraic analogue of Proposition 6.0.7 is [Ros56, Cor. 2, p.433].

Proposition 6.0.7. *If $1 \to B \to C \xrightarrow{p} A \to 1$ is an exact sequence of complex Lie groups, where A is a complex torus and B is commutative, then C_0 is commutative.*

Proof. We may assume that C is connected by replacing C (resp. B) with C_0 (resp. $B \cap C_0$). Let $\omega : \mathbb{C}^g \to A$ be the universal covering of A. Denote by $\mathfrak{b}$ (resp. $\mathfrak{c}$) the Lie algebra of B (resp. C). Let $\eta : A \to \mathrm{Aut}(B)$ be the outer action. Then η induces a holomorphic morphism $\eta_0 : A \to \mathrm{Aut}(B_0)$. Because $\mathrm{Aut}(B_0)$ is complex Lie subgroup of $\mathrm{GL}(\mathfrak{b})$, η_0 is trivial.

Consider the pullback extension along ω.

$$
\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & \ker(\epsilon) \arrow[r, "\pi|_{\ker(\epsilon)}"] \arrow[d] & \ker(\omega) \arrow[d] \arrow[dddll, dashed] & \\
1 \arrow[r] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & E \arrow[r, "\pi"'] \arrow[d, "\epsilon"'] & \mathbb{C}^g \arrow[r] \arrow[d, "\omega"] & 1 \\
1 \arrow[r] & B \arrow[r] \arrow[d] & C \arrow[r, "p"] & A \arrow[r] & 1 \\
 & 0 & & &
\end{tikzcd}
$$

By the snake lemma, ϵ is surjective and π restricts to an isomorphism $\ker(\pi) \to \ker(\omega)$. In particular, $d_e\epsilon : L(E) \to L(C)$ is an isomorphism. By Fact 2.0.3, the morphism ϵ is open. Since E_0 is open in E, $\epsilon(E_0)$ is an open subgroup of C. By connectedness of C, $\epsilon(E_0) = C$. Similarly, $\pi(E_0) = \mathbb{C}^g$. By Fact 7.2.7 1 below, $B \cap E_0$ is connected. Therefore, $B \cap E_0 \subset B_0$. Since $B_0 \subset B \cap E_0$, one has $B_0 = B \cap E_0$. Hence an extension $1 \to B_0 \to E_0 \to \mathbb{C}^g \to 1$. The outer action is $\eta_0\omega : \mathbb{C}^g \to \mathrm{Aut}(B_0)$, so it is a central extension. Then

$$0 \to \mathfrak{b} \to \mathfrak{c} \to \mathbb{C}^g \to 0 \tag{32}$$

is a central extension of Lie algebras. In particular, $\mathfrak{b} \subset Z(\mathfrak{c})$. We shall prove the extension (32) is trivial.

We show that $\exp_E : \mathfrak{c} \to E_0$ is surjective. Indeed, for every $x \in E_0$, there is $v \in \mathfrak{c}$ with $d_e p(v) = \pi(x)$. Then $\pi(\exp_E(v)) = \pi(x)$, so $\pi(x \exp_E(-v)) = 0$ and hence $x \exp_E(-v) \in B_0$. As B_0 is connected commutative, there is $u \in \mathfrak{b}$ with $\exp_B(u) = x \exp_E(-v)$. Since $u \in Z(\mathfrak{c})$, one gets $x = \exp_E(u)\exp_E(v) = \exp_E(u+v)$.

By Corollary 4.1.13, there is a decomposition $B = B_0 \times D$, where $D \in \mathrm{Ab}_c$ is discrete. The natural morphism $E_0 \times D \to E_0 \to \mathbb{C}^g$ is surjective of kernel $B_0 \times D$, hence the first row of the diagram

$$
\begin{tikzcd}
1 \arrow[r] & B \arrow[r] & E_0 \times D \arrow[r] & \mathbb{C}^g \arrow[r] & 1 \\
1 \arrow[r] & B_0 \arrow[r] \arrow[u, hook] \arrow[d, hook] & E_0 \arrow[r] \arrow[u] \arrow[d] & \mathbb{C}^g \arrow[r] \arrow[u, "\mathrm{Id}"] \arrow[d, "\mathrm{Id}"] & 1 \\
1 \arrow[r] & B \arrow[r] & E \arrow[r] \arrow[uu, dashed, bend right, "\phi"'] & \mathbb{C}^g \arrow[r] & 1
\end{tikzcd}
$$

By Lemma 3.1.2, there is an equivalence of extensions $\phi : E \to E_0 \times D$.

Fix $x \in \ker(\epsilon)$, let $\phi(x) = (\phi_1(x), \phi_2(x)) \in E_0 \times D$. For every $y \in E_0$,

$$(y, 1)\phi(x)(y, 1)^{-1} = (y\phi_1(x)y^{-1}, \phi_2(x)) \in \phi(\ker(\epsilon)).$$

Hence, $\phi^{-1}((y\phi_1(x)y^{-1}, \phi_2(x))) \in \ker(\epsilon)$. The map

$$E_0 \to \ker(\epsilon), \quad y \mapsto \phi^{-1}((y\phi_1(x)y^{-1}, \phi_2(x)))$$

is continuous. As E_0 is connected and $\ker(\epsilon)$ is discrete, this map is constantly x. Thus, $y\phi_1(x)y^{-1} = \phi_1(x)$. Therefore, $\phi_1(x)$ commutes with every element of E_0. As $\exp_E : \mathfrak{c} \to E_0$ is surjective, there is $X \in \mathfrak{c}$ with $\exp_E(X) = \phi_1(x)$. Since $\mathbb{C}^g$ is an abelian Lie algebra, $[\mathfrak{c}, \mathfrak{c}]$ is contained in the kernel of $d_e p : \mathfrak{c} \to \mathbb{C}^g$, which is $\mathfrak{b}$. Then $[\mathfrak{c}, \mathfrak{c}] \subset Z(\mathfrak{c})$, *i.e.*, $[\mathfrak{c}, [\mathfrak{c}, \mathfrak{c}]] = 0$. By Lemma 6.0.6 2, $X \in Z(\mathfrak{c})$.

Consider the commutative diagram

$$\begin{tikzcd} \mathfrak{c} \arrow[r, "d_e p"'] \arrow[d, "\exp_E"] & \mathbb{C}^g \arrow[d, "\mathrm{Id}"] \\ E \arrow[r, "\pi"] & \mathbb{C}^g \end{tikzcd}$$

Then $\pi(x) = \pi(\phi_1(x)) = d_e p(X) \in d_e p(Z(\mathfrak{c}))$. Therefore, $\ker(\omega) = \pi(\ker(\epsilon)) \subset d_e p(Z(\mathfrak{c}))$. Since $d_e p$ is $\mathbb{C}$-linear and $\ker(\omega)$ contains a $\mathbb{C}$-basis of $\mathbb{C}^g$, one has $d_e p(Z(\mathfrak{c})) = \mathbb{C}^g$. Consequently, there is a $\mathbb{C}$-linear map $s : \mathbb{C}^g \to Z(\mathfrak{c})$ with $d_e p \circ s = \mathrm{Id}_{\mathbb{C}^g}$. As $s : \mathbb{C}^g \to \mathfrak{c}$ is a Lie algebra morphism, the central extension (32) is trivial and $\mathfrak{c}$ is the direct sum of $\mathfrak{b}$ and $\mathbb{C}^g$. In particular, $\mathfrak{c}$ is abelian. As C is connected and its Lie algebra is abelian, C is commutative. $\square$

Example 6.0.8 shows that the the condition that B/B_0 is torsion (resp. K_0 is Stein) in Proposition 6.0.3 (resp. Proposition 6.0.2 2) is necessary. Moreover, in Proposition 6.0.7, the commutativity of C fails in general.

Example 6.0.8. Let A be a complex torus and $B = A \times \mathbb{Z}$ be the product group. Consider the complex manifold morphism $A \times B \to B$ defined by $(a, a', k) \mapsto (a' + ka, k)$. It is a non trivial group action of A on B. Let C be the corresponding semidirect product (see [Bou72, Ch.III, no. 4, Prop. 7]), then the resulting complex Lie group extension $1 \to B \to C \to A \to 1$ is not central.

7 Noncommutative extensions

7.1 Lifted extensions

The real Lie group extension problem is studied by G. Hochschild in [Hoc51a] and [Hoc51b]. As Example 7.1.1 shows, the case of real Lie groups is different from the case of complex Lie groups.

Example 7.1.1. Let $G = \mathbb{C}$. The morphism of real Lie groups $\rho : \mathbb{C} \to \mathbb{C}^* = \mathrm{Aut}(G)$ defined by $z \mapsto e^{\bar{z}}$ is an action of G on itself which is real analytic but not holomorphic. Hence an exact sequence of real Lie groups $1 \to G \to G \rtimes_\rho G \to G \to 1$ by [Bou72, Ch. III, no. 4, Prop. 7]. However, the middle term has no structure of complex Lie group making the maps holomorphic. Therefore, [Iwa49, Theorem 7] fails for complex Lie groups. Besides, this shows that the real Lie group extension problem and the complex one are different.

In Section 7, we review Hochschild's work, but in the context of complex Lie groups. References to the original statement are given when the proofs are similar modulo slight modifications. All results in the sequel are essentially known.

In Section 7.1, the goal is to derive Corollary 7.1.6, a result about the extensions of a commutative group by a connected group.

Let L be a complex Lie group and $K \in \mathcal{C}$. For a fixed holomorphic group action $L \times K \to K$, let $\phi : L \to \mathrm{Aut}(K)$ denote the induced group morphism. Let $Z(L, K, \phi)$ denote the set of crossed morphisms, *i.e.*, morphisms $\rho : L \to K$ of complex manifolds such that $\rho(l_1 l_2) = \rho(l_1)\phi_{l_1}(\rho(l_2))$ for all $l_1, l_2 \in L$. Then $Z(L, K, \phi)$ is an abelian group under addition. (When ϕ is trivial, $Z(L, K, \phi) = \mathrm{Hom}(L, K)$.)

For a *normal* complex Lie subgroup H of L, define

$$\mathrm{Ophom}_L(H, K, \phi) = \{\psi \in \mathrm{Hom}(H, K) : \psi(lhl^{-1}) = \phi_l(\psi(h)), \forall l \in L, h \in H\}.$$

Then $\mathrm{Ophom}_L(H, K, \phi)$ is a subgroup of $\mathrm{Hom}(H, K)$. When $H \subset Z(L)$, one has

$$\mathrm{Ophom}_L(H, K, \phi) = \mathrm{Hom}_{\mathcal{C}}(H, K^{\phi(L)}), \tag{33}$$

where $K^{\phi(L)} = \cap_{l \in L}\{x \in K : \phi_l(x) = x\}$ is the set of elements fixed by $\phi(L)(\leq \mathrm{Aut}(K))$. Here $K^{\phi(L)}$ is indeed a complex Lie subgroup of K by Corollary 2.0.5. When ϕ is trivial, $\mathrm{Ophom}_L(H, K, \phi)$ is the set of morphisms $H \to K$ invariant under the conjugation action of L.

Proposition 7.1.2. *Assume that H is a normal complex Lie subgroup of L contained in* $\ker(\phi)$. *For every $\rho \in Z(L, K, \phi)$, $\rho|_H \in \mathrm{Ophom}_L(H, K, \phi)$, hence a group morphism $Z(L, K, \phi) \to \mathrm{Ophom}_L(H, K, \phi)$, whose image is denoted by $Z_H(L, K, \phi)$.*

Proof. For every $h, h' \in H$, $\rho(hh') = \rho(h)\phi_h(\rho(h')) = \rho(h)\rho(h')$ since $h \in \ker(\phi)$. Thus $\rho|_H \in \mathrm{Hom}(H, K)$. In particular, $\rho(e_L) = e_K$. For every $l \in L$,

$$e_K = \rho(e_L) = \rho(ll^{-1}) = \rho(l)\phi_l(\rho(l^{-1})),$$

so $\rho(l)^{-1} = \phi_l(\rho(l^{-1}))$. Then

$$\begin{aligned} \rho(lhl^{-1}) &= \rho(lh)\phi_{lh}(\rho(l^{-1})) \\ &= \rho(lh)\phi_l(\rho(l^{-1})) = \rho(lh)\rho(l)^{-1} \\ &= \rho(l)\phi_l(\rho(h))\rho(l)^{-1} = \phi_l(\rho(h)). \end{aligned}$$

The last equality uses the commutativity of K. Therefore, $\rho|_H \in \mathrm{Ophom}_L(H, K, \phi)$. $\square$

Let $\omega : Q' \to Q$ be a surjective morphism of connected complex Lie groups with kernel F. Let $\eta : Q \to \mathrm{Aut}(K)$ be a group morphism such that the induced group action $Q \times K \to K$ is *holomorphic*. As K is commutative, the pulling back map $\omega^* : \mathrm{Ext}(Q, K, \eta) \to \mathrm{Ext}(Q', K, \eta\omega)$ is a group morphism. Fact 7.1.3 gives a description of $\ker(\omega^*)$.

Define a map $\sigma : \mathrm{Ophom}_{Q'}(F, K, \eta\omega) \to \mathrm{Ext}(K, Q, \eta\omega)$ as follows. As the group action defined by η is holomorphic, the semidirect complex Lie group $K \rtimes_{\eta\omega} Q'$ exists by [Bou72, Ch.III, no.4, Prop. 7]. For $\psi \in \mathrm{Ophom}_{Q'}(F, K, \eta\omega)$, the morphism $F \to K \rtimes_{\eta\omega} Q'$ defined by $k \mapsto (\psi(k), k)$ identifies F as a normal complex Lie subgroup of $K \rtimes_{\eta\omega} Q'$. Let $E = K \rtimes_{\eta\omega} Q'/F$. The projection $K \rtimes_{\eta\omega} Q' \to Q'$ descends to a morphism $E \to Q$. The injection $K \to K \rtimes_{\eta\omega} Q'$ induces a morphism $K \to E$. Then the resulting sequence $1 \to K \to E \to Q \to 1$ is exact with outer action $\eta\omega$, whose equivalence class is denoted by $\sigma(\psi)$.

Fact 7.1.3. *[Hoc51a, Thm. 1.1] The map σ is a group morphism and the sequence*

$$Z(Q', K, \eta\omega) \to \mathrm{Ophom}_{Q'}(F, K, \eta\omega) \xrightarrow{\sigma} \mathrm{Ext}(Q, K, \eta) \xrightarrow{\omega^*} \mathrm{Ext}(Q', K, \eta\omega)$$

is exact.

The use of Fact 7.1.3 is based on the existence of $\omega : Q' \to Q$ such that every extension in $\mathrm{Ext}(Q, K, \eta)$ becomes a semidirect product when pulled back to $\mathrm{Ext}(Q', K, \eta\omega)$ along ω.

Fact 7.1.4. *[Hoc51a, Thm. 2.1] Let Q be a connected complex Lie group. Assume that $\eta : Q \to \mathrm{Aut}(K)$ is a group morphism such that the induced group action is holomorphic. Then there exists a simply connected complex Lie group Q' and a surjective morphism $\omega : Q' \to Q$ such that the pullback morphism $\omega^* : \mathrm{Ext}(Q, K, \eta) \to \mathrm{Ext}(Q', K, \eta\omega)$ is zero.*

Remark 7.1.5. The connectedness condition of the extension kernel in [Hoc51a, Theorems 1.1 and 2.1] is in fact unnecessary.

Corollary 7.1.6 follows from Fact 7.1.3 and Fact 7.1.4.

Corollary 7.1.6 ([Hoc51a, Corollary 2.1])**.** *In the notation of Fact 7.1.4,* $\mathrm{Ext}(Q, K, \eta) = \mathrm{Ophom}_{Q'}(F, K, \eta\omega)/Z_F(Q', K, \eta\omega)$*, where* $F = \ker(\omega)$.

Example 7.1.7. Let $Q = \mathbb{C}^*$, $L = \mathbb{C}$ and $\omega : L \to \mathbb{Q}$ be defined by $\omega(z) = e^{2\pi i z}$. Then $F = \ker(\omega) = \mathbb{Z}$. Let $\mathbb{C}^* \times K \to K$ be a holomorphic group action and $\eta : \mathbb{C}^* \to \mathrm{Aut}(K)$ be the induced group morphism. Then $\mathrm{Ophom}_L(F, K, \eta\omega) = \mathrm{Hom}(\mathbb{Z}, K^{\eta(\mathbb{C}^*)}) = K^{\eta(\mathbb{C}^*)}$. By Proposition 3.2.2 and Corollary 7.1.6, one has $\mathrm{Ext}(\mathbb{C}^* K, \eta) = K^{\eta(\mathbb{C}^*)}/Z_{\mathbb{Z}}(\mathbb{C}, K, \eta\omega)$.

7.2 Factor systems

It is well-known that extensions of abstract groups can be classified in terms of factor systems, see [CE56, Ch. XIV, Sec. 4]. This description relies on the existence of set-theoretical cross sections. In general, nevertheless, it is not possible to find a continuous cross section to a surjective morphism of topological groups.

Consider the extension (3) of complex Lie groups with outer action $\psi : Q \to \mathrm{Out}(K)$.

Example 7.2.1. Assume that there is a cross section to (3), *i.e.*, a morphism $s : Q \to E$ of complex manifolds with $ps = \mathrm{Id}_Q$. Replacing s by $s(e_Q)^{-1}s$ when necessary, one may assume that s is normalized as $s(e_Q) = e_E$. Define

$$f : Q \times Q \to E, \quad f(g, h) = s(g)s(h)s(gh)^{-1}.$$

Then f is holomorphic. Since $p(f(g, h)) = e_Q$, $f(g, h) \in K$, so f factors through K. The map f measures the failure of s to be a morphism. If E is commutative, then additionally f is symmetric in the sense of [Ser88, (16), p.166]:

$$f(x, y) = f(y, x) \quad \forall x, y \in Q. \tag{34}$$

Define $\phi : Q \to \mathrm{Aut}(K)$ by $\phi_g = \mathrm{Inn}_{s(g)}|_K$. Then ϕ is a map (but not necessarily a group morphism) lifting ψ, and the induced map

$$Q \times K \to K, \quad (g, x) \mapsto \phi_g(x) \tag{35}$$

is holomorphic. When K is commutative, $\phi = \psi$ is a group morphism independent of the choice of s. When (3) is a central extension, ϕ is constantly Id_K.

Moreover, f and ϕ satisfy the following relations:

$$\begin{aligned} &f(e_Q, h) = f(g, e_Q) = e_K; \\ &\phi_e = \mathrm{Id}_K; \\ &\phi_g\phi_h = \mathrm{Inn}_{f(g,h)}\phi_{gh}; \\ &f(g, h)f(gh, k) = \phi_g(f(h, k))f(g, hk). \end{aligned} \tag{36}$$

Example 7.2.1 motivates Definition 7.2.2.

Definition 7.2.2 (Factor system)**.** If a morphism $f : Q \times Q \to K$ of complex manifolds and a map $\phi : Q \to \mathrm{Aut}(K)$ making (35) holomorphic satisfy the relations (36), then f is called a ϕ-factor system (and simply a factor system when ϕ is trivial, in which case the last relation in (36) is $f(g, h)f(gh, k) = f(h, k)f(g, hk)$.) A factor system f is called symmetric if (34) holds.

When K is commutative, the set of ϕ-factor systems is an abelian group under addition.

We examine how the ϕ-factor system f induced by s in Example 7.2.1 depends on the choice of the cross section s.

Example 7.2.3. Let $s' : Q \to E$ be another normalized cross section still inducing ϕ. Define

$$g : Q \to E, \quad g(x) = s(x)^{-1}s'(x).$$

Then $g(e_Q) = e_E$ as s, s' are normalized and g is holomorphic. For every $x \in Q$, $p(g(x)) = e_Q$, so $g(x) \in K$. For every $k \in K$, $\mathrm{Inn}_{s(x)}k = \phi_x(k) = \mathrm{Inn}_{s'(x)}k$, so

$g(x) \in Z(K)$, *i.e.*, g factors through $Z(K)$. Then $s'(x) = s(x)g(x)$. Let f' be the factor system induced by s'. Then

$$\begin{aligned}
&f'(x,y) = s'(x)s'(y)s'(xy)^{-1}\\
=&s(x)g(x)s(y)g(y)[s(xy)g(xy)]^{-1}\\
=&\phi_x(g(x))s(x)s(y)g(y)g(xy)^{-1}s(xy)^{-1}\\
=&\phi_x(g(x))f(x,y)s(xy)g(y)g(xy)^{-1}s(xy)^{-1}\\
=&\phi_x(g(x))f(x,y)\phi_{xy}(g(y)g(xy)^{-1})\\
=&g^{\phi}(x,y)f(x,y),
\end{aligned}$$

where $g^{\phi} : Q \times Q \to K$ is a morphism of complex manifolds defined by

$$g^{\phi}(x,y) = \phi_x(g(x))\phi_{xy}(g(y)g(xy)^{-1}). \tag{37}$$

When (3) is a central extension, ϕ is trivial, then (37) reduces to [Ser88, (15), p.166]: $g^{\phi}(x,y) = g(x)g(y)g(xy)^{-1}$.

Example 7.2.3 motivates Definition 7.2.4.

Definition 7.2.4. Let f, f' be two ϕ-factors systems. If there is a holomorphic map $g : Q \to Z(K)$ with $g(e_Q) = e_E$ such that $f' = g^{\phi}f$ with g^{ϕ} defined by (37), then f and f' are called ϕ-equivalent, denoted by $f \sim_{\phi} f'$.

In Definition 7.2.4, $\sim_{\phi}$ is an equivalent relation on the set of ϕ-factor systems. When K is commutative, inside the group of all ϕ-factor systems, the elements ϕ-equivalent to the zero form a subgroup. A result similar to Proposition 7.2.5 for algebraic groups is in [Ser88, Ch. VII, Sec. 1, no.4].

Proposition 7.2.5. *Let K, Q be complex Lie groups with a map $\phi : Q \to \mathrm{Aut}(K)$ such that (35) is holomorphic and the induced map $\psi : Q \to \mathrm{Out}(K)$ is a group morphism. Then:*

1. *The set $\mathcal{F}$ of $\sim_{\phi}$-equivalence classes of ϕ-factor systems is canonically identified with the subset $\mathcal{E} \subset \mathrm{Ext}(Q, K, \psi)$ of equivalence classes of extensions of Q by K which admit at least one normalized cross section inducing ϕ.*
2. *When K is commutative, the identification in 1 is a group isomorphism.*
3. *If further Q is also commutative and $\phi = \psi = 1$ is trivial, then the subgroup of equivalence classes of symmetric factor systems corresponds to the subgroup of equivalence classes of* commutative *extensions.*

Proof. We only prove 1. Examples 7.2.1 and 7.2.3 construct a map $\Phi : \mathcal{E} \to \mathcal{F}$. (Note that equivalent extensions induces the same ϕ-equivalence class.)

Conversely, we define a map $\Psi : \mathcal{F} \to \mathcal{E}$ by the following construction. Given a ϕ-factor system f, one can construct an exact sequence $1 \to K \to E_{f,\phi} \to Q \to 1$ of complex Lie groups with a (holomorphic) normalized cross section

$s : Q \to E_{f,\phi}$ as follows. Let $E_{f,\phi} = K \times Q$ as a complex manifold. Define a map

$$g : E_{f,\phi} \times E_{f,\phi} \to E_{f,\phi}, \quad g((k,x),(l,y)) = (k\phi_x(l)f(x,y), xy).$$

As f and the map (35) are holomorphic, so is g. Moreover, (36) shows g defines an associative multiplication. The pair $(1,1) \in E_{f,\phi}$ is the identity, and the inverse of (k,x) is

$$(\phi_x^{-1}[k^{-1}f(x,x^{-1})^{-1}], x^{-1}).$$

Hence $(E_{f,\phi}, g)$ is a complex Lie group. The projection $p : E_{f,\phi} \to Q$ is a surjective morphism. The map $i : K \to E_{f,\phi}$ by $k \mapsto (k,1)$ is the kernel of p. Moreover, define $s : Q \to E_{f,\phi}$ by $s(g) = (1,g)$, then s is normalized cross section. Put $\Psi(f) = E_{f,\phi}$.

We check that $\Psi\Phi = \mathrm{Id}_{\mathcal{E}}$. Indeed, the map $E_{f,\phi} \to E$ defined by $(k,x) \mapsto ks(x)$ is an equivalence of extensions. We check that $\Phi\Psi = \mathrm{Id}_{\mathcal{F}}$, or equivalently s induces f and ϕ. In fact, for every $x \in Q$, $k \in K$, one has

$$\phi_x(k)s(x) = (\phi_x(k),1)(1,x) = (\phi_x(k),x) = (1,x)(k,1) = s(x)k,$$

so $\phi_x = \mathrm{Inn}_{s(x)}|_K$, *i.e.*, s induces ϕ. For every $y \in Q$,

$$\begin{aligned}
&s(x)s(y)s(xy)^{-1} = (1,x)(1,y)(1,xy)^{-1} \\
=&(f(x,y),xy)(\phi_{xy}^{-1}[f(xy,y^{-1}x^{-1})^{-1}], y^{-1}x^{-1}) \\
=&(f(x,y)\phi_{xy}\phi_{xy}^{-1}(f(xy,y^{-1}x^{-1})^{-1})f(xy,y^{-1}x^{-1}),1) \\
=&(f(x,y),1).
\end{aligned}$$

Therefore, s induces f. □

When the base Q of (3) is discrete, then a set-theoretic cross section is automatically holomorphic.

Corollary 7.2.6. *Let Q be a discrete complex Lie groups, and let $\eta : Q \to \mathrm{Aut}(K)$ be a group morphism. Then the group $\mathrm{Ext}(Q,K,\eta)$ is isomorphic to the group of $\sim_\eta$-equivalence classes of η-factor systems. Furthermore, if Q is also commutative, then $\mathrm{Ext}(Q,K)$ is isomorphic to the group of $\sim$-equivalence classes of symmetric factor systems.*

Proof. Since Q is discrete, the group action $Q \times K \to K$ induced by η is holomorphic. The first (resp. second) half follows from Proposition 7.2.5 2 (resp. 3). □

Another important case where a cross section exists is with simply connected bases. For this, we need a holomorphic version of Malcev's theorem ([Mal42, (E), p.12], [Hoc51a, Lemma 3.1], [Mac60, Theorem 3.2]).

Fact 7.2.7 (Malcev, [Bou72, Ch. III, sec 6, no.6, Prop. 14; Cor. 2])**.** *Let L be a connected complex Lie group, N be a normal immersed complex Lie subgroup of L.*

1. *If N is closed in L and L/N is simply connected, then N is connected.*
2. *If L is simply connected, N is connected, then N is closed in L and there exists a biholomorphic map $f : L \to N \times L/N$ making a commutative diagram*

$$\begin{tikzcd} L \arrow[r, "f"] \arrow[dr, "q"'] & N \times L/N \arrow[d, "p_2"] \\ & L/N, \end{tikzcd}$$

where p_2 is the projection to the second factor and $q : L \to L/N$ is the quotient morphism.

In the same way that [Hoc51a, Theorem 3.1] follows from [Hoc51a, Lemma 3.1], Fact 7.2.8 can be deduced from Fact 7.2.7.

Fact 7.2.8. *Let (3) be an exact sequence of complex Lie groups, where E is connected and Q is simply connected. Then there exists a cross section, i.e., a holomorphic map $s : Q \to E$ with $ps = \mathrm{Id}_Q$. In particular, the principal K-bundle $p : E \to Q$ is trivial.*

Example 7.2.9. Let A be a complex elliptic curve. Take a nonzero element of $A^\vee$, which induces a nontrivial extension E of A by $\mathbb{C}^*$ *via* Theorem 5.2.4. By Proposition 5.1.3, the principal $\mathbb{C}^*$-bundle $E \to A$ is nontrivial. Therefore, Fact 7.2.8 fails if the base is not simply connected.

Corollary 7.2.10 follows immediately from Fact 7.2.8 and Proposition 7.2.5.

Corollary 7.2.10. *Let K, Q be complex Lie groups, where K is connected commutative and Q is simply connected. Let $\eta : Q \to \mathrm{Aut}(K)$ be a complex Lie group morphism*[11]*. Then $\mathrm{Ext}(Q, K, \eta)$ is isomorphic to the group of $\sim_\eta$-equivalence classes of η-factor systems.*

Similar to [Hoc51a, Theorem 3.2], Fact 7.2.11 can be proved using Fact 7.2.7 and Fact 7.2.8,

Fact 7.2.11. *Let K, Q be complex Lie groups, where K is connected and Q is simply connected. Then the map (on the set of equivalence classes) which associates with each extension of Q by K the induced extension of $L(Q)$ by $L(K)$ is injective. The image is the set of classes of those extensions $0 \to L(K) \to \mathfrak{E} \to L(Q) \to 0$ in which the derivation*

$$[x, \bullet]_{\mathfrak{E}}|_{L(K)} \in \mathrm{Der}(L(K)) = L(\mathrm{Aut}(L(K)))$$

belongs to $L(\mathrm{Aut}(K))$ for every $x \in \mathfrak{E}$. Furthermore, if K is commutative and $\eta : Q \to \mathrm{Aut}(K)$ is a morphism, then the resulting map

$$\mathrm{Ext}(Q, K, \eta) \to \mathrm{Ext}(L(Q), L(K), d_e\eta)$$

is a group isomorphism.

[11] Here $\mathrm{Aut}(K)$ is a complex Lie subgroup of $\mathrm{GL}(L(K))$ by [Lee01, Propositions 1.26 and 1.27].

A connected Lie group is called semisimple if its Lie algebra is semisimple. Analogue of Fact 7.2.12 for semisimple real Lie groups H and real vector groups G is contained in the proof of [Hoc51b, Theorem 5.1]. Fact 7.2.12 can be proved in a similar way.

Fact 7.2.12. *Let G, H be connected complex Lie groups, where G is commutative and H is* semisimple. *Let $\eta : H \to \mathrm{Aut}(G)$ be a morphism of complex Lie groups. If $\phi \in Z(H, G, \eta)$ is a crossed morphism, then there exists $g \in G$ such that $\phi(x) = \eta_x(g)g^{-1}$ for all $x \in H$. In particular, $\phi \equiv e_G$ on $\ker(\eta)$.*

Theorem 7.2.13 is a complex version of [Hoc51a, Theorem 4.4].

Theorem 7.2.13. *In Fact 7.2.12, $\mathrm{Ext}(H, G, \eta)$ is canonically isomorphic to $\mathrm{Hom}_{\mathrm{Ab}}(\pi_1(H), G^{\eta(H)})$.*

Proof. Let $\omega : \tilde{H} \to H$ be the universal covering of H. Then $\ker(\omega) = \pi_1(H)$ is a discrete subgroup of $\tilde{H}$. By Fact 3.2.4, $\pi_1(H) \subset Z(\tilde{H})$. Then (33) gives

$$\mathrm{Ophom}_{\tilde{H}}(\ker(\omega), G, \eta\omega) = \mathrm{Hom}(\pi_1(H), G^{\eta(H)}).$$

By Fact 7.2.12, for every $\rho \in Z(\tilde{H}, G, \eta\omega)$, $\rho|_{\ker \omega} = 1$, *i.e.*, $Z_{\ker(\omega)}(\tilde{H}, G, \eta\omega) = 0$. By Fact 7.2.11, the natural map $\mathrm{Ext}(\tilde{H}, G, \eta\omega) \to \mathrm{Ext}(L(H), L(G), d_e\eta)$ is a group isomorphism. Since $L(H)$ is a semisimple complex Lie algebra, Levi's theorem [Ser64, Theorem 4.1, p.48] affirms that $\mathrm{Ext}(L(H), L(G), d_e\eta) = 0$. By Fact 7.1.3, $\mathrm{Ext}(H, G, \eta) = \mathrm{Hom}(\pi_1(H), G^{\eta(H)})$. ☐

7.3 Non-abelian kernels and extensions of the center

For two complex Lie groups K, Q and a group morphism $\theta : Q \to \mathrm{Out}(K)$, if θ is induced by some extension of Q by K, then the extension kernel (K, θ) is called *extendible*. The problem to determine the extendibility of a given extension kernel is more difficult than that for abstract groups treated in [EM47, Theorem 8.1], because of the obstruction to the existence of a cross section. For extendible kernels, Corollary 7.3.8 shows that the problem for extensions by K can be reduced to that with an *abelian* kernel, namely $Z(K)$.

Let $1 \to K \to E \xrightarrow{p} Q \to 1$ and $1 \to K' \to E' \xrightarrow{p'} Q \to 1$ be two extension of complex Lie groups. Denote their outer action by $\theta : Q \to \mathrm{Out}(K)$ and $\theta' : Q \to \mathrm{Out}(K')$ respectively. Assume that $Z(K) = Z(K') := C$ and θ, θ' induce a common center action[12] $\theta_0 : Q \to \mathrm{Aut}(C)$. Hence a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Out}(K) \arrow[dr] & \\
Q \arrow[ur, "\theta"] \arrow[rr, "\theta_0"] \arrow[dr, "\theta'"'] & & \mathrm{Aut}(C) \\
 & \mathrm{Out}(K') \arrow[ur] &
\end{tikzcd} \qquad (38)$$

[12] see (5)

We recall the multiplication of kernels defined in [EM47, Sec. 4]. The group law $C \times C \to C$ is holomorphic, so the subset

$$C^* := \{(x, x^{-1}) : x \in C\} \tag{39}$$

is analytic in $C \times C$. By Lemma 2.0.6, $C \times C$ is an analytic subset of $K \times K'$. As C^* is a central subgroup of $K \times K'$, it is also a complex Lie subgroup of $K \times K'$ by Corollary 2.0.5. Let $K'' = K \times K'/C^*$. From [EM47, p.328], the morphism $C \to K''$ by $g \mapsto [(g, 1)]$ identifies C as the center of K''.

For every $x \in Q$, select automorphisms $\alpha \in \theta(x)(\subset \mathrm{Aut}(K))$ and $\alpha' \in \theta'(x)(\subset \mathrm{Aut}(K'))$. Because the diagram (38) is commutative, $\alpha \times \alpha'$ is an automorphism of $K \times K'$ sending C^* into itself. It thus determines an automorphism α'' of K''. The class $[\alpha''] \in \mathrm{Out}(K'')$ depends only on θ, θ', but not the choices of α, α'. Hence a group morphism

$$\theta'' : Q \to \mathrm{Out}(K'') \tag{40}$$

that also induces $\theta_0 : Q \to \mathrm{Aut}(C)$.

Definition 7.3.1. The pair (K'', θ'') constructed above is called the C-product of the two given extension kernels (K, θ) and (K', θ').

Example 7.3.2. If $K' = C$ is commutative, it is asserted in [EM47, (4.4)] that K' acts as an identity for the C-product. To make it explicit, we define a surjective morphism $\phi : K \times C \to K$ of complex manifolds by $\phi(k, k') = k'k$. Then ϕ is a morphism and $C^* = \ker(\phi)$. Thus, ϕ induces an isomorphism $\sigma : K'' \to K$ satisfying [EM47, (4.2), (4.3)].

Then we review the multiplication of the given two extensions, contained the proof of [EM47, Lem. 5.1].

As the map $E \times E' \to Q$ by $(x, x') \mapsto p'(x')p(x)^{-1}$ is holomorphic, the preimage of e_Q

$$D = D_{p,p'}(E, E') = \{(x, x') \in E \times E' : p(x) = p'(x')\}, \tag{41}$$

is analytic in $E \times E'$. Since D is a subgroup of $E \times E'$, by Corollary 2.0.5, D is a complex Lie subgroup of $E \times E'$.

For every $(x, x') \in D$ with $y = p(x) = p(x')$, every $g \in C$, the element

$$(x, x')(g, g^{-1})(x^{-1}, x'^{-1}) = (\theta_0(y)(g), \theta_0(y)(g)^{-1})$$

is in C^*. Therefore, C^* defined by (39) is normal in D.

As C^* is a normal complex Lie subgroup of D, we can set $E'' = D/C^*$. The inclusion $K \times K' \to D$ descends to an injective morphism $K'' \to E''$. The map $D \to Q$ defined by $(x, x') \mapsto p(x)$ induces a surjective morphism $p'' : E'' \to Q$ whose kernel is K''. Hence an extension $1 \to K'' \to E'' \to Q \to 1$. The induced outer action $Q \to \mathrm{Out}(K'')$ is (40). We call (E'', p'') the C-product of the two given extensions (E, p) and (E', p'), written as $(E'', p'') = (E, p) \otimes (E', p')$. Thus, [EM47, Lemmas 5.1 and 5.2] hold for complex Lie groups.

Fact 7.3.3. *The C-product of two extendible kernels is extendible. The kernel of the C-product $(E,p)\otimes(E',p')$ of two extensions is the C-product of the two kernels.*

Proposition 7.3.4. *When $K'=C$, (E',p') is the semidirect product $C\rtimes_{\theta_0}Q$, then (E'',p'') is naturally equivalent to (E,p).*

Proof. Consider the subgroup $D\leq E\times E'=E\times(C\rtimes_{\theta_0}Q)$ defined in (41). Define a map $\psi:D\to E$ by $(x,c,q)\mapsto cx$ for $x\in E$ and $(c,q)\in C\rtimes_{\theta_0}Q$. Then ψ is holomorphic.

We check that ψ is a group morphism. Take another $(x,c',q')\in D$. Since $\theta_{0,q}(c')=\theta_{p(x)}(c')=xc'x^{-1}$, one has

$$\begin{aligned}&\psi((x,c,q)(x',c',q'))=\psi(xx',c\theta_{0,q}(c'),qq')\\=&c\theta_{0,q}(c')xx'=cxc'x'=\psi(x,c,q)\psi(x',c',q').\end{aligned}$$

For every $g\in C$, $\psi(g,g^{-1})=e_E$, so $C^*\subset\ker\psi$. Thus, ψ induces a morphism $\epsilon:E''\to E$. Together with σ defined in Example 7.3.2, ϵ fits into a commutative diagram.

$$\begin{tikzcd}1 \arrow[r] & K'' \arrow[r] \arrow[d, "\sigma"] & E'' \arrow[r] \arrow[d, "\epsilon"] & Q \arrow[r] \arrow[d, "\mathrm{Id}"] & 1\\ 1 \arrow[r] & K \arrow[r] & E \arrow[r] & Q \arrow[r] & 1\end{tikzcd}$$

Therefore, ϵ is an equivalence of extensions. $\square$

By construction, C-product defines a map $\mathrm{Ext}(Q,K,\theta)\times\mathrm{Ext}(Q,K',\theta')\to\mathrm{Ext}(Q,K'',\theta'')$. When $K'=C$, it specializes to

$$\mathrm{Ext}(Q,K,\theta)\times\mathrm{Ext}(Q,C,\theta_0)\to\mathrm{Ext}(Q,K,\theta),\tag{42}$$

which defines an action of the abelian group $\mathrm{Ext}(Q,C,\theta_0)$ on the set $\mathrm{Ext}(Q,K,\theta)$. If further K is also commutative, by [Hoc51a, p.97], (42) is exactly the group law defined by the Baer sum on $\mathrm{Ext}(Q,C,\theta_0)$.

Definition 7.3.5. [EM47, p.329] For every extension kernel (K,θ), let θ^* be the composition of $\theta:Q\to\mathrm{Out}(K)$ with the natural group isomorphism $\mathrm{Out}(K)\to\mathrm{Out}(K^{\mathrm{op}})$. Then the extension kernel $(K^{\mathrm{op}},\theta^*)$ is called the inverse of (K,θ).

For every $(E,p)\in\mathrm{Ext}(Q,K,\theta)$, define $p^*:E^{\mathrm{op}}\to Q$ by $p^*(x^*)=p(x^{-1})$, then it is a surjective morphism. Since $\ker(p^*)=K^{\mathrm{op}}$, $1\to K^{\mathrm{op}}\to E^{\mathrm{op}}\xrightarrow{p^*}Q\to 1$ is an extension. The associated outer action is θ^*. Thus, we get an element $(E^{\mathrm{op}},p^*)\in\mathrm{Ext}(Q,K^{\mathrm{op}},\theta^*)$ of (E,p). It is called the inverse of (E,p) and its extension kernel is the inverse of (K,θ).

It is a classical result that the group action (42) is simple transitive. For abstract groups, see [EM47, Lem. 11.2 and 11.3]. For algebraic groups, see [FLA19, Thm. 1.1]. It remains true for complex Lie groups. The first half, Fact 7.3.6, can be proved in the same way as in [Hoc51b, Thm. 1.1], using the inverse in the group $\mathrm{Ext}(Q,C,\theta_0)$ and Proposition 7.3.4.

Fact 7.3.6. *Let K, Q be complex Lie groups, $C = Z(K)$. Let $\theta : Q \to \mathrm{Out}(K)$ be a group morphism that induces $\theta_0 : Q \to \mathrm{Aut}(C)$. Then the action of $\mathrm{Ext}(Q, C, \theta_0)$ on $\mathrm{Ext}(Q, K, \theta)$ defined by (42) is free.*

Theorem 7.3.7 is analogue to [EM47, Lemma 11.2].

Theorem 7.3.7. *In the notation of Fact 7.3.6, if $\mathrm{Ext}(Q, K, \theta)$ is nonempty (i.e., the extension kernel (K, θ) is extendible), then its $\mathrm{Ext}(Q, C, \theta_0)$-action defined by (42) is transitive. Equivalently, for every $(E, p), (E_1, p_1) \in \mathrm{Ext}(Q, K, \theta)$, there exits $F \in \mathrm{Ext}(Q, C, \theta_0)$ with $F \otimes E$ equivalent to E_1.*

Proof. Define $D_{p_1,p^*}(E_1, E^{\mathrm{op}})$ like (41). Set

$$S = \{(x_1^{-1}, x^*) \in D_{p_1,p^*}(E_1, E^{\mathrm{op}}) : x_1 k x_1^{-1} = x k x^{-1}, \forall k \in K\}.$$

Then S is a subgroup of $E_1 \times E^{\mathrm{op}}$. For every $k \in K$, the map

$$\phi_k : E_1 \times E^{\mathrm{op}} \to K \quad (x_1, x^*) \mapsto x_1^{-1} k x_1 x k^{-1} x^{-1}$$

is holomorphic, so $\phi_k^{-1}(e_K)$ is analytic in $E_1 \times E^{\mathrm{op}}$. Then $S = D_{p_1,p^*}(E_1, E^{\mathrm{op}}) \cap \cap_{k \in K} \phi_k^{-1}(e_K)$ is analytic in $E_1 \times E^{\mathrm{op}}$, by [Whi72, Theorem 9C, p.100]. By Corollary 2.0.5, S is a complex Lie subgroup of $E_1 \times E^{\mathrm{op}}$.

The map $K \times K^{\mathrm{op}} \to K$ by $(k, k'^*) \mapsto kk'$ is holomorphic, so $K^* = \{(k^{-1}, k^*) : k \in K\}$ is an analytic subset of $K \times K^{\mathrm{op}}$. It is a subgroup of S, hence a complex Lie subgroup of S by Corollary 2.0.5.

For every $(x_1^{-1}, x^*) \in S$, $k \in K$, one has

$$\begin{aligned} &(x_1^{-1}, x^*)(k^{-1}, k^*)(x_1, (x^*)^{-1}) = (x_1^{-1} k^{-1} x_1, x^* k^* (x^{-1})^*) \\ =&(x^{-1} k^{-1} x, (x^{-1} k x)^*) \in K^*, \end{aligned}$$

so K^* is a normal subgroup of S. Let $F = S/K^*$ and $\nu : S \to F$ be the quotient morphism. The map $i : C \to F$ defined by $c \mapsto [(c, 1)]$ is an injective morphism.

The map $\bar{\phi} : S \to Q$ defined by $\bar{\phi}(x_1^{-1}, x^*) = p(x^{-1})$ is a morphism with K^* contained in the kernel. We check that $\bar{\phi}$ is surjective. For every $h \in Q$, there exist $x \in E$ and $x_1 \in E_1$ with $p(x) = p_1(x_1) = h^{-1}$. Since the two automorphisms of K, $\mathrm{Inn}_x|_K$ and $\mathrm{Inn}_{x_1}|_K$ have the same class $\theta_{h^{-1}}$ in $\mathrm{Out}(K)$, there exists $k_0 \in K$ such that $\mathrm{Inn}_{x_1}|_K = \mathrm{Inn}_x|_K \mathrm{Inn}_{k_0}$. Then $(x_1^{-1}, (xk_0)^*) \in S$ and $\bar{\phi}(x_1^{-1}, (xk_0)^*) = h$.

If $(x_1^{-1}, x^*) \in \ker \bar{\phi}$, then $p_1(x_1) = p(x_1) = e_Q$, so $x_1, x \in K$. Moreover, $x_1 k x_1^{-1} = x k x^{-1}$ for all $k \in K$. Then $x_1^{-1} x \in C$, so $(x_1^{-1}, x^*) = (x_1^{-1} x, 1^*)(x^{-1}, x^*)$. Thus, $[(x_1^{-1}, x^*)] = i(x_1^{-1} x) \in i(C)$.

Thus $\bar{\phi}$ induces a surjective morphism $\phi : F \to Q$ with $i(C) \supset \ker \phi$. In addition, ϕi is trivial, so $i(C) \subset \ker(\phi)$. Hence an extension $1 \to C \xrightarrow{i} F \xrightarrow{\phi} Q \to 1$ with the induced action $Q \to \mathrm{Aut}(C)$ coinciding with θ_0.

It remains to show that the C-product extension $F \otimes E$ is equivalent to E_1. By construction, $F \otimes E$ is represented by $G = D_{\phi,p}(F, E)/C^*$, where $C^* = \{(c, c^{-1}) \in F \times E : c \in C\}$. The pullback of $D_{\phi,p}(F, E)$ along the natural surjection $S \times E \to F \times E$ is $D_{\phi\nu,p}(S, E)$.

For every $(a, b^*, x) \in D_{\phi\nu,p}(S, E) \subset E_1 \times E^{\mathrm{op}} \times E$, one has $p_1(a) = p(b^{-1}) = p(x)$, whence $bx \in K$ and $a \cdot (bx) \in E_1$. Define a holomorphic map $\tau : D_{\phi\nu,p}(S, E) \to E_1$ by $\tau(a, b^*, x) = a \cdot (bx)$.

$$\begin{tikzcd}
E_1 & & \\
D_{\phi\nu,p}(S,E) \arrow[u, "\tau"] \arrow[r] \arrow[d, hook] \arrow[rr, bend left, "\nu^*"] & D_{\phi,p}(F,E) \arrow[r, two heads] \arrow[d, hook] & G \arrow[llu, dashed] \\
S \times E \arrow[r, "\nu \times \mathrm{Id}_E"] \arrow[d, hook] & F \times E & \\
E_1 \times E^{\mathrm{op}} \times E & &
\end{tikzcd}$$

We check that τ is a group morphism. For every $(a, b^*, x), (a', b'^*, x') \in D_{\phi\nu,p}(S, E)$, since $(a', b'^*) \in S$ and $bx \in K$, one has $a'^{-1}(bx)a' = b'(bx)b'^{-1}$. Hence,

$$\begin{aligned}
&\tau(a, b^*, x)\tau(a', b'^*, x') = [a(bx)][a'(b'x')] \\
=&aa'[a'^{-1}(bx)a'](b'x') = aa'[b'(bx)b'^{-1}](b'x') \\
=&aa'(b'bxx') = \tau(aa', (b'b)^*, xx') = \tau(aa', b^*b'^*, xx').
\end{aligned}$$

We check that τ is surjective. For every $x_1 \in E_1$, $p_1(x_1) \in Q$. As $\phi\nu : S \to Q$ is surjective, there is $(a, b^*) \in S$ with $\phi\nu(a, b^*) = p_1(x_1)$. Then $p_1(a) = p_1(x_1)$. Thus, $a^{-1}x_1 \in K$. Let $x = b^{-1}(a^{-1}x_1) \in E$. Then $p(x) = p(b^{-1}) = \phi\nu(a, b^*)$, so $(a, b^*, x) \in D_{\phi\nu,p}(S, E)$ and $\tau(a, b^*, x) = a(bx) = a(a^{-1}x_1) = x_1$.

We check that $\ker(\nu^*) \subset \ker(\tau)$. For every $(x_1, x^*, y) \in \ker(\nu^*) \subset E_1 \times E^{\mathrm{op}} \times E$, there is $c \in C$ with $([(x_1, x^*)], y) = (c, c^{-1})$ in $F \times E$. Equivalently, $y = c^{-1}$ in E and $[(x_1, x^*)] = [(c, 1^*)]$ in $F = S/K^*$. Whence, $(x_1c^{-1}, x^*) \in K^*$, *i.e.*, $x \in K$ and $x_1 = x^{-1}c$. Therefore, $(x_1, x^*, y) = (x^{-1}c, x^*, c^{-1})$ with $x \in K, c \in C$. Thus, $\tau(x_1, x^*, y) = x^{-1}c(xc^{-1}) = e_{E_1}$ and $(x_1, x^*, y) \in \ker(\tau)$.

Conversely, we check $\ker(\tau) \subset \ker(\nu^*)$. For every $(a, b^*, x) \in \ker(\tau)$, one has $a(bx) = e_{E_1}$, so $a \in K$. Because $(a, b^*) \in D_{p_1,p^*}(E_1, E^{\mathrm{op}})$, we obtain $p(b^{-1}) = p(a) = e_Q$ and hence $b \in K$. Since $\mathrm{Inn}_{a^{-1}} = \mathrm{Inn}_b \in \mathrm{Aut}(K)$, one has $ab \in C$. Therefore, $[(a, b^*)] = [(ab, 1^*)] = i(ab)$ in $F = S/K^*$ and $(a, b^*, x) = (ab, (ab)^{-1}) \in C^* \leq F \times E$. Then $(a, b^*, x) \in \ker(\nu^*)$.

Therefore, $\ker(\tau) = \ker(\nu^*)$, so τ induces an isomorphism $G \to E_1$ that establishes an equivalence between the two elements of $\mathrm{Ext}(Q, K, \theta)$. $\square$

Fact 7.3.6 and Theorem 7.3.7 yield Corollary 7.3.8.

Corollary 7.3.8. *Let K, Q be complex Lie groups, $C = Z(K)$, $\theta : Q \to \mathrm{Out}(K)$ be a group morphism. Let $\theta_0 : Q \to \mathrm{Aut}(C)$ be the induced group morphism. If $\mathrm{Ext}(Q, K, \theta)$ is nonempty, then $\mathrm{Ext}(Q, K, \theta)$ is in (non-canonical) bijection with $\mathrm{Ext}(Q, C, \theta_0)$.*

A Maximal morphisms

A result stronger than Proposition 5.1.3 holds.

Definition A.0.1. [Ser88, Definition 1, p.125]. Let X be a complex manifold, A be a complex torus. A morphism $f : X \to A$ is called maximal if whenever f factors as $X \xrightarrow{g} A' \xrightarrow{h} A$, where $A' \in \mathcal{C}$ is connected and $h - h(0) : A' \to A$ is a finite morphism, it holds that $h - h(0)$ is an isomorphism.

Proposition A.0.2. *If X is a regular manifold*[13]*, then the Albanese morphism $f : X \to \mathrm{Alb}(X)$ associated to some base point $x \in X$ is maximal.*

Proof. Assume that f factors as $X \xrightarrow{g} A' \xrightarrow{h} \mathrm{Alb}(X)$, where $A' \in \mathcal{C}$ is a connected and $h - h(0)$ is a finite morphism. Then A' is compact, hence a complex torus. Choosing $g(x)$ as the new zero element of A', we get a new structure of complex torus on A', to which we stick from now on. Then h is a finite morphism. By [Liu23, Proposition 4.1.2 3], there is a morphism $\phi : \mathrm{Alb}(X) \to A'$ with $\phi f = g$ and the complex Lie subgroup of $\mathrm{Alb}(X)$ generated by $f(X)$ is $\mathrm{Alb}(X)$ itself. Then $h\phi f = f$ and hence $h\phi = \mathrm{Id}_{\mathrm{Alb}(X)}$. In particular, h is surjective. By Fact 3.0.4, the exact sequence $0 \to \ker(h) \to A' \xrightarrow{h} A \to 0$ defines a trivial extension, so A' is isomorphic to $\ker(h) \times A$. By connectedness of A', $\ker(h) = 0$ and h is an isomorphism. $\square$

When $f = \mathrm{Id}_A$, Proposition A.0.3 reduces to Proposition 5.1.3.

Proposition A.0.3 ([Ser88, Prop. 14, p.188])**.** *Let X be a connected compact complex manifold, A be a complex torus, $B \in \mathcal{C}$. Let $f : X \to A$ be a maximal morphism. If B_0 is linear, then the composed morphism*

$$\mathrm{Ext}(A, B) \xrightarrow{\pi} H^1(A, \mathcal{B}_A) \xrightarrow{f^*} H^1(X, \mathcal{B}_X) \tag{43}$$

is injective.

Proof. Let $C \in \ker(f^* \circ \pi)$. Then the principal fiber bundle $f^*p : f^*C \to X$ is trivial. Fix a point $c \in f^*C$ lying over $0 \in C$. Then there is a morphism $s : X \to f^*C$ with $f^*p \circ s = \mathrm{Id}_X$ and $s(f^*p(c)) = c$. Let $t : X \to C$ be the morphism induced by s.

$$\begin{tikzcd}
& & f^*C \arrow[r, dashed, "f^*p"] \arrow[d] & X \arrow[l, dashed, bend right, "s"'] \arrow[d, "f"] \arrow[dl, dashed, "t"'] \arrow[ddl, dashed] & \\
0 \arrow[r] & B \arrow[r] & C \arrow[r, "p"] & A \arrow[r] \arrow[l, dashed, bend left, "h^{-1}"] & 0 \\
0 \arrow[r] & B \cap A' \arrow[r] \arrow[u] & A' \arrow[r, "h"] \arrow[u, hook] & A \arrow[r] \arrow[u, "\mathrm{Id}"'] & 0
\end{tikzcd}$$

[13] in the sense of [Var86, p.233]

By Remmert's theorem [Whi72, Theorem 4A, p.150], $t(X)$ is an analytic subset of C. By [CD94, (14.14), p.89], the analytic space $t(X)$ is irreducible. Moreover, $t(X)$ is compact and $0 = t(f^*p(c)) \in t(X)$. Let A' be the complex Lie subgroup of C generated by $t(X)$. By [Liu23, Lemma A.3.5], A' is a complex torus. Then $(A' \cap B)_0$ is a compact. As a closed complex submanifold of B_0, $(A' \cap B)_0$ is also a Stein manifold, hence a point. Thus, $A' \cap B$ is discrete and compact, hence finite. Therefore, $h : A' \to A$ is a finite morphism. As the maximal morphism f factors as $X \xrightarrow{t} A' \xrightarrow{h} A$, h is an isomorphism. Then $h^{-1} : A \to C$ is a morphism and $ph^{-1} = \mathrm{Id}_A$. By Fact 3.0.4, $C = 0$ in $\mathrm{Ext}(A, B)$. □

Example A.0.4. Let X be a regular manifold, $f : X \to A$ be the Albanese morphism associated to some base point $x \in X$. When $B = \mathbb{C}$, the composed morphism (43) is a linear isomorphism $f^* : H^1(A, O_A) \to H^1(X, O_X)$. When $B = \mathbb{C}^*$, it is the inclusion of the identity component $\mathrm{Pic}^0(A) \to \mathrm{Pic}(X)$.

B Commutative extensions of real Lie groups

Let $\mathcal{R}$ be the category of commutative real Lie groups. The solution to the extension problem within $\mathcal{R}$ is summarized in Proposition B.0.2. Similar to Lemma 4.1.1, the category $\mathcal{R}$ is additive but not abelian. Parallel to the construction in Section 4, we can define an additive functor $\mathrm{Ext}_{\mathcal{R}} : \mathcal{R}^{\mathrm{op}} \times \mathcal{R} \to \mathrm{Ab}$ by considering commutative extensions.

Proposition B.0.1 generalizes [LH76, Proposition 5, p.110] (which says that C is isomorphic to $A \times B$) and [HN11, Lemma 15.3.2] (which is for real tori). The similar statement for complex tori is false, shown by Example 4.1.14.

Proposition B.0.1. *Let $0 \to B \to C \to A \to 0$ be an extension of commutative real Lie groups. If A, B are connected, this extension is trivial.*

Proof. Similar to Proposition 3.2.2, every extension of $\mathbb{R}$ is a semidirect product, hence $\mathrm{Ext}_{\mathcal{R}}(\mathbb{R}, \bullet) = 0$ on $\mathcal{R}$. Similar to Proposition 3.2.3, $\mathrm{Ext}_{\mathcal{R}}(S^1, B) = 0$. According to [LH76, Proposition 4, p.109], A is isomorphic to $(S^1)^n \times \mathbb{R}^m$ for some $m, n \in \mathbb{N}$. As the functor $\mathrm{Ext}_{\mathcal{R}}(\bullet, B) : \mathcal{R} \to \mathrm{Ab}$ is additive, we get $\mathrm{Ext}_{\mathcal{R}}(A, B) = 0$. □

Proposition B.0.2. *For every $A, B \in \mathcal{R}$, there is a non canonical isomorphism in* Ab*:*

$$\mathrm{Ext}_{\mathcal{R}}(A, B) \to \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0) \oplus \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0).$$

Proof. By a real version of Corollary 4.1.13, there are non canonical isomorphisms in $\mathcal{R}$: $A \to A/A_0 \times A_0$ and $B \to B/B_0 \times B_0$. By additivity of the bifunctor $\mathrm{Ext}_{\mathcal{R}}$, we get an isomorphism in Ab:

$$\mathrm{Ext}_{\mathcal{R}}(A, B) \to \mathrm{Ext}_{\mathcal{R}}(A/A_0, B_0) \oplus \mathrm{Ext}_{\mathcal{R}}(A/A_0, B/B_0) \oplus \mathrm{Ext}_{\mathcal{R}}(A_0, B/B_0) \oplus \mathrm{Ext}_{\mathcal{R}}(A_0, B_0).$$

Using Lemma 4.1.12, one can prove that $\mathrm{Ext}_{\mathcal{R}}(A/A_0, B_0) = 0$. Identical to Example 4.1.10, $\mathrm{Ext}_{\mathcal{R}}(A/A_0, B/B_0) = \mathrm{Ext}^1_{\mathbb{Z}}(A/A_0, B/B_0)$. Similar to Corollary

3.2.5 and [Hoc51b, Thm. 3.2], $\mathrm{Ext}_{\mathcal{R}}(A_0, B/B_0) = \mathrm{Hom}_{\mathrm{Ab}}(\pi_1(A_0), B/B_0)$. By Proposition B.0.1, $\mathrm{Ext}_{\mathcal{R}}(A_0, B_0) = 0$. The proof is completed. □

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