Fourier-Mukai transform on complex tori, revisited

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Abstract

We study the Fourier-Mukai transform on complex tori. An inversion formula is given for good sheaves (defined by Kashiwara), which are replacements of quasi-coherent sheaves on algebraic varieties. We explain why goodness is necessary for the inversion formula.

Contents

1 Introduction

For a ringed space (Z, O_Z), let $D(Z)$ be the derived category of the abelian category of O_Z-modules. A scheme of finite type and separated over a field is called an algebraic variety. For two algebraic varieties (resp. complex analytic spaces) M, N, let $p_M : M \times N \to M$ and $p_N : M \times N \to N$ be the projections. For an object $K \in D(M \times N)$, the *integral transform* $\phi_K^{[M \to N]} : D(M) \to D(N)$ with integral kernel K is defined as

$$\phi_K^{[M\to N]}(\cdot) = Rp_{N,*}(K \otimes^L p_M^* \cdot). \tag{1}$$

When Z is a complex analytic space, let $D_{\mathrm{gd}}(Z) \subset D(Z)$ be the full subcategory consisting of complexes whose cohomology sheaves are good (Definition A.4.1). Roughly speaking, an analytic sheaf of modules is good if it can be approximated by coherent submodules. For a complex torus X of dimension g, let $\hat{X}$ be the dual complex torus. Let $\mathcal{P}$ be the normalized[1] Poincaré line bundle on $X \times \hat{X}$. Define functors $RS : D(\hat{X}) \to D(X)$ and $R\hat{S} : D(X) \to D(\hat{X})$ by $RS = \phi_{\mathcal{P}}^{[\hat{X}\to X]}$, $R\hat{S} = \phi_{\mathcal{P}}^{[X\to\hat{X}]}$. The pair $(RS, R\hat{S})$ is called the *Fourier-Mukai transform* of X. Theorem 1.0.1 establishes an analog of the Fourier inversion formula for this pair.

Theorem 1.0.1 (Theorem 4.1.1)**.** *The functor $R\hat{S}$ (resp. RS) restricts to a functor $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(\hat{X})$ (resp. $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X)$). Moreover, there are natural isomorphisms of functors*

$$\begin{aligned} RS \circ R\hat{S} &\cong [-1]_X^*[-g] : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X), \\ R\hat{S} \circ RS &\cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(\hat{X}), \end{aligned}$$

where $[-g]$ denotes degree shift.

Theorem 1.0.1 is a complex analytic variant of [Muk81, Thm. 2.2] (Statement 2.0.4, which has a minor problem for lack of quasi-coherence condition). For complex tori, a parallel false assertion is made as [BBBP07, Thm. 2.1] (Statement 2.0.5). Theorem 1.0.1 shows that "good sheaves" on complex manifolds serve as substitutes for "quasi-coherent sheaves" on algebraic varieties in this case. As an application, we recover Matsushima-Morimoto's classification of homogeneous vector bundles on complex tori.

Theorem (Theorem 5.3.6)**.** *A vector bundle F on the complex torus X is translation invariant if and only if there is an integer $n \geq 0$, unipotent vector bundles*[2] *$U_1, \ldots, U_n$ on X and $P_1, \ldots, P_n \in \mathrm{Pic}^0(X)$, such that F is isomorphic to $\oplus_{i=1}^n (P_i \otimes U_i)$.*

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[1] *i.e.*, both pullback *modules* $\mathcal{P}|_{X\times 0}$ and $\mathcal{P}|_{0\times\hat{X}}$ are trivial

[2] Definition 5.2.2

Notation and conventions

For a topological space M, the category of abelian sheaves on M is denoted by $\mathrm{Ab}(M)$. The category of ringed spaces is denoted by RingS. For a ringed space (X, O_X), let $\mathrm{Mod}(O_X)$ be the category of O_X-modules. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of quasi-coherent (resp. coherent) O_X-modules in the sense of Definition A.1.1 3 (resp. 6) is denoted by $\mathrm{Qch}(X)$ (resp. $\mathrm{Coh}(X)$). For a closed subset $Z \subset X$, let $\mathrm{Coh}_Z(X) \subset \mathrm{Coh}(X)$ be the full subcategory consisting of modules with support contained in Z.

Given a symbol $* \in \{\emptyset, +, -, b\}$, the notation $D^*(X)$ refers to the unbounded/bounded below/bounded above/bounded derived category of $\mathrm{Mod}(O_X)$ in order. The full subcategory of $D^*(X)$ consisting of the complexes whose cohomologies are coherent (resp. quasi-coherent) is denoted by $D^*_c(X)$ (resp. $D^*_{\mathrm{qc}}(X)$). Denote by $R\mathcal{H}om_X : D(X)^{\mathrm{op}} \times D(X) \to D(X)$ the internal hom bifunctor constructed in [Sta23, Tag 08DH].

For a locally ringed space X and $x \in X$, let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. For an $O_{X,x}$-module M, the O_X-module $(i_x)_*M$ is denoted by M_x.

All complex analytic spaces (in the sense of [KK83, Def. 43.2]) are assumed to be paracompact. Let An be the category of complex analytic spaces. The dimension of a complex manifold always refers to the complex dimension, which is assumed to be finite.

When X is an abelian variety (resp. complex torus), its dual abelian variety (resp. complex torus) is denoted by $\hat{X}$. The normalized Poincaré bundle on $X \times \hat{X}$ is denoted by $\mathcal{P}$. For $y \in \hat{X}$ (resp. $x \in X$), let P_y (resp. P_x) denote the line bundle $\mathcal{P}|_{X \times y}$ (resp. $\mathcal{P}|_{x \times \hat{X}}$).

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2 Fourier-Mukai transform

Complex tori are generalizations of complex abelian varieties. Every complex torus of dimension 1 is an abelian variety. By contrast, for every integer $g \geq 2$, a very general complex torus of dimension g is not[3] an abelian variety (see, *e.g.*, [BZ23, p.21]).

The Fourier-Mukai transform is an analog of the classical Fourier transform. It is proposed by Mukai [Muk81] on abelian varieties and complex tori. Let k be an algebraically closed field. Let X be an abelian variety over k (resp. a complex torus) of dimension g. Write RS and $R\hat{S}$ for $\phi_{\mathcal{P}}^{[\hat{X} \to X]}$ and $\phi_{\mathcal{P}}^{[X \to \hat{X}]}$ respectively. The pair $(RS, R\hat{S})$ is called the Fourier-Mukai transform of X. The functor RS (resp. $R\hat{S}$) restricts to a functor $D^b(\hat{X}) \to D^b(X)$ (resp. $D^b(X) \to D^b(\hat{X})$).

Let X be an abelian variety. The usual exchange of translation and time shifting (resp. multiplication and convolution) of Fourier transform finds analog for Fourier-Mukai transform, namely the exchange of translation and line bundle twisting (resp. tensor product and Pontrjagin product) in [Muk81, (3.1) (resp. (3.7))]. Moreover, Mukai proves a duality theorem similar to the classical Fourier inversion formula.

Fact 2.0.1. There are canonical isomorphisms of functors

$$RS \circ R\hat{S} \cong [-1]_X^*[-g] : D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(X);$$
$$R\hat{S} \circ RS \cong [-1]_{\hat{X}}^*[-g] : D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(\hat{X}).$$

In particular, the functor $RS : D_{\mathrm{qc}}(\hat{X}) \to D_{\mathrm{qc}}(X)$ is an equivalence of categories, with a quasi-inverse $[-1]_{\hat{X}}^* \circ R\hat{S}[g]$.

Example 2.0.2 ([Muk81, Eg. 2.6])**.** For every $y \in \hat{X}(k)$, one has $RS(k_y) = P_y$ and $R\hat{S}(P_y) = k_{-y}[-g]$.

Remark 2.0.3. Combining Fact 2.0.1, the natural equivalence $D(\mathrm{Qch}(X)) \to D_{\mathrm{qc}}(X)$ ([BN93, Cor. 5.5]) with the compatibility of derived direct images [TT07, Cor. B.9], one gets [Rot96, Mukai's Theorem, p.569] stated for $D^b(\mathrm{Qch}(*))$ instead of $D_{\mathrm{qc}}(*)$. The quasi-coherence restriction is essential for Čech resolution with respect to affine covers in [Rot96, p.571].

The proof of Fact 2.0.1 uses projection formula and the flat base change theorem ([Lip09, Prop. 3.9.4; Prop. 3.9.5]). Compared with Fact 2.0.1, the original statement (Statement 2.0.4) has no quasi-coherence restriction.

Statement 2.0.4 ([Muk81, Thm. 2.2]). The functor RS gives an equivalence of categories between $D(\hat{X})$ and $D(X)$, and its quasi-inverse is $[-1]_{\hat{X}}^* \circ R\hat{S}[g]$.

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[3] To the contrary, it is incorrectly implied in [BBR94, p.151] that every complex torus of dimension 2 admits a compatible structure of algebraic complex surface. In fact, it fails for each 2-dimensional complex torus X that is not a projective manifold. For otherwise, assume there is a complex algebraic surface V with $V^{\mathrm{an}} \cong X$. Then V is proper by [GR71, XII, Prop. 3.2 (v)]. In consequence, the algebraic variety V is projective by [Har77, p.357]. Thus, X is a projective manifold, a contradiction.

In [BBBP07, Thm. 2.1], an assertion similar to Statement 2.0.4 is made for complex tori.

Statement 2.0.5. Let X be a complex torus. Then the integral transform $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence of triangulated categories.

However, Lemma 2.0.6 shows that Statement 2.0.4 (resp. Statement 2.0.5) holds if and only if $g = 0$.

Lemma 2.0.6 ([th])**.** Let X be an abelian variety or a complex torus. If the functor $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence of categories, then $g = 0$.

Proof. When X is a complex torus, let $k = \mathbb{C}$. In both cases, let $F = k_0^{\mathbb{N}}$ be the product of a countable infinite family of k_0 in $\mathrm{Mod}(O_{\hat{X}})$. Since $k^{\mathbb{N}} = k^{\oplus I}$ as a k-module for some index set I, the direct sum sheaf $k_0^{\oplus I}$ is isomorphic to F. Therefore, by [Sta23, Tag 07D9 (2)], F is the direct sum of I copies of k_0 in $D^b(\hat{X})$. We claim that F is the product of $\mathbb{N}$ copies of k_0 in $D^b(\hat{X})$.

By [Gro57, p.129], the abelian category $\mathrm{Mod}(O_{\hat{X},0})$ satisfies the AB 4*) axiom. From [Sta23, Tag 07KC (2)], the inclusion $\mathrm{Mod}(O_{\hat{X},0}) \to D^b(\mathrm{Mod}(O_{\hat{X},0}))$ commutes with countable products. Let $i : 0 \to \hat{X}$ be the closed immersion. Since $i_* : \mathrm{Mod}(O_{\hat{X},0}) \to \mathrm{Mod}(O_{\hat{X}})$ is exact, there is a commutative square

$$\begin{tikzcd}
\mathrm{Mod}(O_{\hat{X},0}) \arrow[r, "i_*"] \arrow[d, hook] & \mathrm{Mod}(O_{\hat{X}}) \arrow[d, hook] \\
D^b(\mathrm{Mod}(O_{\hat{X},0})) \arrow[r, "Ri_*"] & D^b(\hat{X}).
\end{tikzcd}$$

Since $Ri_* : D^b(\mathrm{Mod}(O_{\hat{X},0})) \to D^b(\hat{X})$ has a left adjoint, it commutes with products. As $F = i_*(k^{\mathbb{N}})$, the claim is proved.

As $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence, inside $D^b(X)$, the object $RS(F)$ is the direct sum of I copies of $RS(k_0)$, as well as the product of $\mathbb{N}$ copies of $RS(k_0)$. By Example 2.0.2 (when X is an abelian variety) and Lemma 2.0.8 (when X is a complex torus), one has $RS(k_0) = O_X$. Therefore, $RS(F)$ is isomorphic to $O_X^{\oplus I}$ and to $O_X^{\mathbb{N}}$ in $\mathrm{Mod}(O_X)$.

Assume the contrary $g > 0$. Then there is a nonempty *connected* open subset $V \subset X$, such that $O_X(V)$ is an integral domain but not a field. In particular, the ring $O_X(V)$ is not Artinian. By [Har77, II, Exercise 1.11] (when X is an abelian variety) and Corollary A.5.4 (when X is a complex torus), the $O_X(V)$-module $\Gamma(V, RS(F))$ is isomorphic to $O_X(V)^{\oplus I}$ and to $O_X(V)^{\mathbb{N}}$. However, this contradicts Fact 2.0.7. □

Fact 2.0.7 ([Len68, Thm, p.211])**.** If A is a commutative ring such that $A^{\mathbb{N}}$ is a free A-module, then A is Artinian.

For algebraic varieties, the analog of Lemma 2.0.8 follows from the flat base change theorem and the projection formula.

Lemma 2.0.8. Let X, Y be two complex analytic spaces, let $K \in D(X \times Y)$, and let $x \in X$. Consider the closed embedding $h_x : Y \to X \times Y, \quad y \mapsto (x, y)$. Then $\phi_K^{[X\to Y]}(\mathbb{C}_x) = Lh_x^* K$.

Proof. Let $p : X \times Y \to X$, $q : X \times Y \to Y$ be the two projections. Denote the closed embedding of complex analytic spaces $x \to X$ by j_x. The cartesian square

$$\begin{tikzcd}
Y \arrow[r, "p_0"] \arrow[d, "h_x"] \arrow[dr, phantom, "\square"] & x \arrow[d, "j_x"] \\
X \times Y \arrow[r, "p"] & X
\end{tikzcd}$$

in the category An induces a natural morphism $\phi : p^* \mathbb{C}_x \to Rh_{x,*} O_Y$ in $\mathrm{Mod}(O_{X\times Y})$. Both sheaves are supported on $\{x\} \times Y$.

For two (Hausdorff) locally convex topological vector spaces E, F over $\mathbb{C}$, the completed projective topological tensor product $E \hat{\otimes}_{\mathbb{C}} F$ is defined in [Gro55, Ch. I, Déf. 2, p.32]. For every $y \in Y$, by [GR84, p.27], the stalk $O_{X\times Y,(x,y)} = O_{X,x} \hat{\otimes}_{\mathbb{C}} O_{Y,y}$. Then

$$(p^* \mathbb{C}_x)_{(x,y)} = \mathbb{C} \otimes_{O_{X,x}} O_{X\times Y,(x,y)} = O_{Y,y}.$$

Therefore, $\phi_{(x,y)} : (p^* \mathbb{C}_x)_{(x,y)} \to (h_{x,*} O_Y)_{(x,y)}$ is an isomorphism. Thus, ϕ is an isomorphism.

By [Sta23, Tag 0B55], the natural morphism $(Rh_{x,*} O_Y) \otimes^L K \to Rh_{x,*}(Lh_x^* K)$ is an isomorphism. Then

$$\begin{aligned} \phi_K^{[X\to Y]}(\mathbb{C}_x) &= Rq_*(p^* \mathbb{C}_x \otimes^L K) \cong Rq_*(Rh_{x,*} O_Y \otimes^L K) \\ \cong& Rq_* Rh_{x,*}(Lh_x^* K) \cong R(qh_x)_*(Lh_x^* K) = Lh_x^* K. \end{aligned}$$

□

The minor problem with Statement 2.0.4 occurs in the proof of [Muk81, Prop. 1.3], when the flat base change theorem [Har66, Prop. 5.12] stated for objects of $D_{\mathrm{qc}}(*)$ is applied to objects in $D^-(*)$. Similarly, the minor problem with Statement 2.0.5 originates from a lack of certain analytic quasi-coherence in the wrong Statement 2.0.9 (a counterpart of [Muk81, Prop. 1.3]). A modification of Statement 2.0.9 is Proposition 4.2.2.

Statement 2.0.9 ([BBBP07, p.427]). *If M, N, and P are compact complex manifolds and $K \in D^b(M \times N)$ and $L \in D^b(N \times P)$, then one has a natural isomorphism of functors from $D^b(M)$ to $D^b(P)$:*

$$\phi_L^{[N\to P]} \circ \phi_K^{[M\to N]} \cong \phi_{K*L}^{[M\to P]},$$

where

$$K * L = Rp_{M\times P*}(p_{M\times N}^* K \otimes^L p_{N\times P}^* L) \in D^b(M \times P),$$

and $p_{M\times N}$, $p_{M\times P}$, $p_{N\times P}$ are the natural projections $M \times N \times P \to M \times N$, etc.

3 Good modules

As Section 2 explains, to obtain an analytic analogue of Fact 2.0.1, it is necessary to find a substitute for quasi-coherence on complex manifolds. We show that goodness introduced by Kashiwara (Definition A.4.1) can be used as such.

3.1 Functoriality

In Corollary 3.1.14, we prove that goodness is preserved by integral transforms. To prove this, we show that goodness is preserved by the operations involved in (1).

Example 3.1.1. [Har66, Example 1., p.68] Let $f : X \to Y$ be a morphism of ringed spaces. Then the derived pullback $Lf^* : D(Y) \to D(X)$ (constructed in [Spa88, Prop. 6.7 (a)]) is bounded above (in the sense of [Lip09, 1.11.1]), and the derived pushout $Rf_* : D(X) \to D(Y)$ is bounded below.

Proposition 3.1.2 (Pullback)**.** *Let $f : X \to Y$ be a morphism of complex analytic spaces. Then $Lf^* : D(Y) \to D(X)$ restricts to a functor*

1. *$D^b_c(Y) \to D^b_c(X)$ when Y is a complex manifold or f is flat;*
2. *$D_{\mathrm{gd}}(Y) \to D_{\mathrm{gd}}(X)$.*

Proof.

1. Because Y is smooth or f is flat, by Lemma 3.1.3, the morphism f has finite tor-dimension. Thus, Lf^* restricts to a functor $D^b(Y) \to D^b(X)$.

   Consider $F \in D^b_c(Y)$. To prove that $Lf^*F \in D^b_c(X)$, by [Har66, I, Prop. 7.3 (i)], one may assume $F \in \mathrm{Coh}(Y)$. This case is proved by Lemma A.3.3.

2. (a) Let $G \in D^-_{\mathrm{gd}}(Y)$. By Example 3.1.1, Lemma A.4.3 3 and a dual of [Har66, Prop. 7.3 (ii)], to prove $Lf^*G \in D_{\mathrm{gd}}(X)$, one may assume $G \in \mathrm{Good}(Y)$. Let U be a relatively compact open subset of X. Then $f(\bar{U})$ is a compact subset of Y, so contained in a relatively compact open subset V of Y. Since G is good, its restriction $G|_V = \sum_{i\in I} G_i$ is the sum of a directed family of coherent O_V-submodules of $G|_V$. Let $g : f^{-1}(V) \to V$ be the base change of f along the inclusion $V \to Y$. As Lf^* commutes with colimits, one has

   $$(Lf^*G)|_{f^{-1}(V)} = \mathrm{colim}_{i\in I} Lg^* G_i.$$

   For every integer n, in $\mathrm{Mod}(O_{f^{-1}(V)})$ one has

   $$H^n(Lf^*G)|_{f^{-1}(V)} = H^n\big((Lf^*G)|_{f^{-1}(V)}\big)$$
   $$= H^n(\mathrm{colim}_{i\in I} Lg^*G_i) = \mathrm{colim}_{i\in I} H^n(Lg^*G_i).$$

Since G_i is coherent, by Lemma A.3.3, the $O_{f^{-1}(V)}$-module $H^n(Lg^*G_i)$ is coherent. By Lemma A.4.3 3, the $O_{f^{-1}(V)}$-module $H^n(Lf^*G)|_{f^{-1}(V)}$ is good. Since $\bar{U}$ is a compact subset of $f^{-1}(V)$, the subset U is relatively compact in $f^{-1}(V)$. Hence, $H^n(Lf^*G)|_U$ is the sum of a directed family of coherent submodules. Hence $Lf^*G \in D_{\mathrm{gd}}(X)$.

(b) Then consider the general case $C \in D_{\mathrm{gd}}(Y)$. For every integer $m \geq 0$, the m-th canonical truncation ([Sta23, Tag 0118 (4)]) $C_m := \tau^{\leq m}C$ is in $D^-_{\mathrm{gd}}(Y)$. From the proof of [Lip09, Prop. 2.5.5], there is a bounded above complex of flat O_Y-modules Q_m with a quasi-isomorphism $Q_m \to C_m$ that is functorial in C_m. Moreover, the complex $Q := \mathrm{colim}_m Q_m$ is K-flat (in the sense of [Spa88, Def. 5.1]), and the canonical morphism $Q \to C$ is a quasi-isomorphism. Because $Lf^* : D(Y) \to D(X)$ admits a right adjoint, it commutes with colimits. Thus, the resulting morphisms

$$\mathrm{colim}_m Lf^*Q_m \to Lf^*Q \to Lf^*C$$

are isomorphisms in $D(X)$.

Let $\mathrm{Ch}(\mathrm{Mod}(O_X))$ be the category of chain complexes over $\mathrm{Mod}(O_X)$. The directed set $\mathbb{N}$ can be seen naturally as a category. Define a functor $\mathbb{N} \to \mathrm{Ch}(\mathrm{Mod}(O_X)), \quad m \mapsto f^*Q_m$. Because $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, for every integer n, by [Hov99, Lem. 1.5], the natural morphism

$$\mathrm{colim}_m H^n(f^*Q_m) \to H^n(\mathrm{colim}_m f^*Q_m)$$

in $\mathrm{Mod}(O_X)$ is an isomorphism. Hence an isomorphism $H^n(Lf^*C) \cong \mathrm{colim}_m H^n(Lf^*Q_m)$ in $\mathrm{Mod}(O_X)$. Since $Q_m \in D^-_{\mathrm{gd}}(Y)$, by Case 2a, the O_X-module $H^n(Lf^*Q_m)$ is good. By Lemma A.4.3 3, so is the O_X-module $H^n(Lf^*C)$.

□

The tor-dimension tor-dim f of a morphism $f : X \to Y$ of ringed spaces is defined to be the lower dimension (in the sense of [Lip09, 1.11.1]) of the functor $Lf^* : D^-(Y) \to D(X)$. If f is flat, then tor-dim $f = 0$. If f has finite tor-dimension, then $Lf^* : D^-(Y) \to D(X)$ restricts to a functor $D^b(Y) \to D^b(X)$. The weak dimension $\mathrm{wgld}(R)$ of a commutative ring R is defined to be the supremum of flat dimension of all R-modules.

Lemma 3.1.3. Let $f : X \to Y$ be a morphism of complex analytic spaces, with Y a complex manifold. Then f has finite tor-dimension.

Proof. From [Lip09, (2.7.6.4)], one only needs to show that for every $x \in X$, the flat dimension of the $O_{Y,f(x)}$-module $O_{X,x}$ is uniformly bounded. By definition, the flat dimension of every $O_{Y,f(x)}$-module is bounded by the weak dimension of the ring $O_{Y,f(x)}$. Because Y is a complex manifold, the local ring $O_{Y,f(x)}$ is Noetherian regular. By Lemma 3.1.4, $\mathrm{wgld}\, O_{Y,f(x)}$ is the Krull dimension of

$O_{Y,f(x)}$, which coincides with the dimension of the complex manifold Y near $f(x)$. ☐

Lemma 3.1.4 (Serre)**.** Let R be a commutative, Noetherian, regular local ring. Then $\mathrm{wgld}(R)$ coincides with the Krull dimension of R, hence finite.

Proof. From [Osb12, Cor. 4.21], the weak dimension coincides with the global dimension of R. By Serre's theorem (see, *e.g.*, [Osb12, p.332]), the global dimension equals the Krull dimension, which is finite. ☐

Proposition 3.1.5 (Tensor product)**.** *Let X be a complex analytic space. Then the bifunctor* (constructed in [Spa88, Thm. A. (ii)]) $\otimes^L : D(X)\times D(X) \to D(X)$ *restricts to a bifunctor*

1. $D^b(X) \times D^b(X) \to D^b(X)$ *(resp.* $D^b_c(X) \times D^b_c(X) \to D^b_c(X)$*) when X is a complex manifold;*
2. $D_{\mathrm{gd}}(X) \times D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X)$.

Proof.

1. The weak dimension of a ringed space (M, O_M) is defined to be $\sup_{x\in M} \mathrm{wgld}(O_{M,x})$. By [HT07, (C.2.20)], to prove the statement for $D^b(X)$, it suffices to bound the weak dimension of X. As X is smooth, for every $x \in X$, the stalk $O_{X,x}$ is a Noetherian, regular local ring. Thus, by Lemma 3.1.4, its weak dimension $\mathrm{wgld}(O_{X,x})$ is equal to the dimension of the complex manifold X near x. Therefore, the weak dimension of X is at most $\dim X$

   Consider any $F, G \in D^b_c(X)$. To prove that $F \otimes^L G \in D^b_c(X)$, by [Har66, I, Prop. 7.3 (i)], one may assume $F, G \in \mathrm{Coh}(X)$. Then the conclusion follows from [GH78, 4., p.700].

2. Take $F, G \in D_{\mathrm{gd}}(X)$. To prove that $F \otimes^L G \in D_{\mathrm{gd}}(X)$, as in the proof of Proposition 3.1.2 2, one may assume that $F, G \in D^-_{\mathrm{gd}}(X)$. By a dual of [Har66, I, Prop. 7.3 (ii)], one may assume that $F, G \in \mathrm{Good}(X)$. Let U be a relatively compact open subset of X.

   For every integer n, we claim that the O_U-module $H^n(F \otimes^L_{O_X} G)|_U$ is good. By assumption, the restrictions $F|_U = \sum_{i\in I} F_i$ and $G|_U = \sum_{j\in J} G_j$ can be written as sums of directed families of coherent submodules. By [Sta23, Tag 08DJ], the functor $\otimes^L_{O_U} (G|_U) : D(U) \to D(U)$ has a right adjoint, so

   $$(F \otimes^L G)|_U = \mathrm{colim}_{i\in I}[F_i \otimes^L (G|_U)]. \tag{2}$$

   By [Sta23, Tag 05NI (2)], there exists a complex $C^\bullet$ of flat O_U-modules and a quasi-isomorphism $C^\bullet \to G|_U$. Then for every $i \in I$, in $D(U)$

   $$F_i \otimes_{O_U} C^\bullet \xrightarrow{\sim} F_i \otimes^L_{O_U} G|_U. \tag{3}$$

Define a functor $I \to \mathrm{Ch}(\mathrm{Mod}(O_X))$ by $i \mapsto F_i \otimes C^\bullet$. By [Hov99, Lem. 1.5], the natural morphism

$$\mathrm{colim}_{i\in I} H^n(F_i \otimes C^\bullet) \to H^n(\mathrm{colim}_{i\in I}(F_i \otimes C^\bullet))$$

in $\mathrm{Mod}(O_U)$ is an isomorphism. Combining it with (2) and (3), one gets an isomorphism in $\mathrm{Mod}(O_U)$

$$\mathrm{colim}_{i\in I} H^n(F_i \otimes^L_{O_U} G|_U) \to H^n(F \otimes^L_{O_X} G)|_U.$$

Because $\mathrm{Good}(U)$ is closed under colimits in $\mathrm{Mod}(O_U)$ by Lemma A.4.3 3, one may assume that $F|_U$ is coherent. Similarly, one may assume further that $G|_U$ is coherent. Then the claim follows from Lemma A.3.4.

□

As the proof of Theorem 3.1.6 is lengthy, we split it into a series of lemmas.

Theorem 3.1.6 (Pushout)**.** *Let $f : X \to Y$ be a* proper *morphism of complex analytic spaces. If $\dim X$ is finite, then $Rf_* : D(X) \to D(Y)$ restricts to a functor $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(Y)$ (resp. $D^b_{\mathrm{gd}}(X) \to D^b_{\mathrm{gd}}(Y)$).*

Proof. By Lemma 3.1.10, the functor Rf_* restricts to a functor $D^b(X) \to D^b(Y)$. We show that $Rf_*F \in D_{\mathrm{gd}}(Y)$ for every $F \in D_{\mathrm{gd}}(X)$. By [Har66, I, Prop. 7.3 (iii)], Lemmas 3.1.10 and A.4.3 3, one may assume that $F \in \mathrm{Good}(X)$. For every relatively compact open subset $V \subset Y$, its closure $\bar{V}$ is compact in Y. As f is proper, the preimage $f^{-1}(\bar{V})$ is compact. Thus, $U := f^{-1}(V)$ is a relatively compact open subset of X. Since F is good, $F|_U = \mathrm{colim}_{i\in I} F_i$, where $\{F_i\}_{i\in I}$ is a directed family of coherent O_U-submodules of $F|_U$. Let $g : U \to V$ be the base change of f. Fix an integer n. By Lemma 3.1.8, in $\mathrm{Mod}(O_V)$

$$(R^n f_* F)|_V = R^n g_*(F|_U) = \mathrm{colim}_{i\in I} R^n g_* F_i.$$

As a base change of f, the morphism g is proper. Then by Fact 3.1.7, for every $i \in I$, the O_V-module $R^n g_* F_i$ is coherent. By $\mathrm{Coh}(V) \subset \mathrm{Good}(V)$ and Lemma A.4.3 3, the O_V-module $(R^n f_* F)|_V$ is good. Therefore, $Rf_*F \in D_{\mathrm{gd}}(Y)$.

□

Fact 3.1.7 (Grauert direct image theorem, see *e.g.*, [GR84, p.207])**.** Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then $Rf_* : D(X) \to D(Y)$ restricts to a functor $\mathrm{Coh}(X) \to D_c(Y)$.

Lemma 3.1.8. Let $f : X \to Y$ be a *proper* map between locally compact, Hausdorff spaces. Then for every integer $n \geq 0$, the functor $R^n f_* : \mathrm{Ab}(X) \to \mathrm{Ab}(Y)$ commutes with filtrant colimits.

Proof. Let $(F_i, f_{ij})_{i\in I}$ be a filtrant inductive system with colimit F in $\mathrm{Ab}(X)$. Since the abelian category $\mathrm{Ab}(Y)$ is Grothendieck, the filtrant colimit $G = \mathrm{colim}_{i\in I} R^n f_* F_i$ exists and there is a canonical morphism $\phi : G \to R^n f_* F$ in

$\mathrm{Ab}(Y)$. For every $y \in Y$, the functor $\mathrm{Ab}(Y) \to \mathrm{Ab}$ taking the stalk at y commutes with colimits, so $G_y = \mathrm{colim}_{i \in I}(R^n f_* F_i)_y$. By [Mil13, Thm. 17.2], for every i the stalk $(R^n f_* F_i)_y = H^n(X_y, F_i|_{X_y})$. Then by [God58, Thm. 4.12.1], the morphism $\phi_y : G_y \to (R^n f_* F)_y$ is an isomorphism. Therefore, ϕ is an isomorphism. $\square$

The proof of Fact 3.1.9 is similar to that of [KS90, Prop. 3.2.2].

Fact 3.1.9. Let X be a locally compact, Hausdorff topological space which is countable at infinity. Suppose that there is an integer $n \geq 0$ such that every point of X has an open neighborhood homeomorphic to a locally closed subset of $\mathbb{R}^n$. Then for every abelian sheaf F on X and every integer $j > n$, one has $H^j(X, F) = 0$.

Lemma 3.1.10. Let X be a complex analytic space of finite dimension n. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then for an object $E \in D(X)$ with $H^m(E) = 0$ for every integer $m > 0$, one has $H^i(Rf_*E) = 0$ for every integer $i > 2n$. In particular, the functor $Rf_* : D(X) \to D(Y)$ is bounded.

Proof. For every open subset $V \subset Y$ and every O_X-module M, from $i > 2n$ and Fact 3.1.9, one has $H^i(f^{-1}(V), M) = 0$. Applying Lemma 3.1.12 to the functor $\Gamma(f^{-1}(V), \cdot) : \mathrm{Mod}(O_X) \to \mathrm{Ab}$, one gets

$$H^i(R\Gamma(f^{-1}(V), E)) = H^i(R\Gamma(f^{-1}(V), \tau^{\geq 1}E)) = 0.$$

By Lemma 3.1.11, the O_Y-module $H^i(Rf_*E) = 0$. $\square$

Lemma 3.1.11 is a derived version of [Har77, III, Prop. 8.1].

Lemma 3.1.11. Let $f : X \to Y$ be a continuous map of topological spaces. Then for every integer i and every $F \in D(\mathrm{Ab}(X))$, the sheaf $H^i(Rf_*F)$ on Y is the sheaf associated to the abelian presheaf $V \mapsto H^i R\Gamma(f^{-1}(V), F)$.

Proof. By [Spa88, Thm. D], there is a quasi-isomorphism $F \to I$, where I is a K-injective complex of abelian sheaves on X. Then the canonical morphism $Rf_*F \to f_*I$ is an isomorphism in $D(\mathrm{Ab}(Y))$. By [Mur06, Lem. 3], $H^i(Rf_*F)$ is the sheaf associated the presheaf

$$V \mapsto H^i\big(\Gamma(V, f_*I)\big) = H^i\big(\Gamma(f^{-1}(V), I)\big) = H^i\big(R\Gamma(f^{-1}(V), F)\big).$$

$\square$

Lemma 3.1.12. Let X be a ringed space as in Fact 3.1.9. Let $F : \mathrm{Mod}(O_X) \to \mathrm{Ab}$ be an additive functor. Assume that F commutes with countable products, and there is an integer $N \geq 0$ with $R^pF(M) = 0$ for every integer $p \geq N$ and every $M \in \mathrm{Mod}(O_X)$. Then the right derived functor $RF : D(X) \to D(\mathrm{Ab})$ exists. Moreover, for any integers $i \geq j$, the natural transformation

$$H^i(RF\cdot) \to H^i(RF(\tau^{\geq j-N+1}\cdot)) : D(X) \to \mathrm{Ab}$$

is an isomorphism.

Proof. The existence of N and [Wei95, Cor. 10.5.11] show that $RF : D^+(X) \to D^+(\mathrm{Ab})$ extends to a right derived functor $RF : D(X) \to D(\mathrm{Ab})$ of F.

For every integer m and every $E \in D(X)$, set $E_m := \tau^{\geq -m}E$. Then $\{E_m\}_{m\in\mathbb{Z}}$ forms an inverse system in $D(X)$. Let n be as in Fact 3.1.9. Then for every open subset $U \subset X$, any integers $p(> n)$ and q, one has $H^p(U, H^q(E)) = 0$. Then by [Sta23, Tag 0D64], the canonical morphism $E \to R\lim_m E_m$ is an isomorphism in $D(X)$. Since F commutes with countable products, from [Sta23, Tag 08U1], in $D(\mathrm{Ab})$ one has $RF(E) \xrightarrow{\sim} R\lim_m RF(E_m)$. For every integer i, by [Sta23, Tag 08U5], there is a short exact sequence in the category Ab

$$0 \to R^1 \lim_m H^{i-1}(RF(E_m)) \to H^i(RF(E)) \to \lim_m H^i(RF(E_m)) \to 0. \tag{4}$$

We claim that $R^1 \lim_m H^{i-1}(RF(E_m)) = 0$.

For every integer $m \geq N - i$, by [Sta23, Tag 08J5], there is an exact triangle

$$H^{-m}(E)[m] \to E_m \to E_{m-1} \xrightarrow{+1} H^{-m}(E)[m + 1] \tag{5}$$

in $D(X)$. By assumption, one has

$$H^i(RF(H^{-m}(E)[m])) = R^{i+m}F((H^{-m}(E)) = 0;$$
$$H^i(RF(H^{-m}(E)[m + 1])) = R^{i+m+1}F((H^{-m}(E)) = 0.$$

Taking the long exact sequence associated with (5), one concludes that the canonical morphism $H^i(RF(E_m)) \to H^i(RF(E_{m-1}))$ in Ab is an isomorphism. Since the inverse system $\{H^iRF(E_m)\}_{m\geq 1}$ is constant starting with $m = N - i - 1$, it satisfies the Mittag-Leffler condition in the sense of [Sta23, Tag 02N0]. From [Sta23, Tag 07KW (3)], one obtains

$$R^1 \lim_m H^i(RF(E_m)) = 0,$$

which proves the claim.

When $i \geq j$, as the inverse system is constant from $m = N - j - 1$, one has $\lim_m H^i(RF(E_m)) = H^i[RF(E_{N-j-1})]$. Then the sequence (4) induces an isomorphism $H^i(RF(E)) \to H^i(RF(\tau^{\geq j-N+1}E))$. □

Remark 3.1.13. In the statement of Lemma 3.1.12, because $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, it has enough injectives. By [Ver66, p.338], the total right derived functor $RF : D^+(X) \to D^+(\mathrm{Ab})$ exists (even if F may not be left exact).

Corollary 3.1.14. Let X, Y be complex manifolds (resp. complex analytic spaces), with X compact and Y finite dimensional. If F is an object of $D^b_c(X \times Y)$ (resp. $D_{\mathrm{gd}}(X \times Y)$), then $\phi_F^{[X\to Y]}$ restricts to a functor $D^b_c(X) \to D^b_c(Y)$ (resp. $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(Y)$).

Proof. Because X is compact, its dimension is finite and the projection $X\times Y \to Y$ is proper. Thus, $X \times Y$ is finite dimensional. The result is a combination of Proposition 3.1.2 1 (resp. 2), Proposition 3.1.5 1 (resp. 2), Fact 3.1.7 and Lemma 3.1.10 (resp. Theorem 3.1.6). □

Remark 3.1.15. Although we don't need the functors $R\mathcal{H}om$, $f_!$ and $f^!$, it is interesting to know whether they preserve goodness or not.

3.2 Base change theorems

As a replacement for the (algebraic) flat base change theorem (used in Mukai's proof of Fact 2.0.1), we give an analytic smooth base change theorem. It is a consequence of Theorem 3.2.3 and Fact 3.2.2.

Consider a cartesian square in the category An:

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ S' \arrow[r, "g"] & S. \end{tikzcd} \tag{6}$$

Then [Sta23, Tag 08HY] gives a natural transformation of functors $D(X) \to D(S')$

$$Lg^*Rf_* \to Rf'_*Lg'^*, \tag{7}$$

coming from the adjunction in [Sta23, Tag 079W].

Smooth base change

Definition 3.2.1. A morphism $g : S' \to S$ of complex analytic spaces is called locally product, if for every $s' \in S'$, there is an open neighborhood U of $s' \in S'$ and a complex analytic space Z, such that $g(U)$ is open in S and there is a $g(U)$-isomorphism $U \to g(U) \times Z$.

By [CD94, II, Cor. 2.7], a locally product morphism is flat.

Fact 3.2.2 ([Gro61b, Thm. 3.1])**.** A morphism of complex analytic spaces is smooth (in the sense of in the sense of [Gro61b, Déf. 3.2]) if and only if it is a submersion (in the sense of [Fis76, p.100]). In particular, a smooth morphism is locally product.

Theorem 3.2.3. *Consider the square (6) with both* $\dim X$ *and* $\dim X'$ *finite,* $f : X \to S$ *proper and* $g : S' \to S$ *locally product. Then (7) restricts to an isomorphism of functors* $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$.

We begin the proof with several lemmas.

Definition 3.2.4. A morphism of complex analytic spaces $g : S' \to S$ is said to satisfy property $\mathcal{Q}_S$ if for every *proper* morphism $f : X \to S$ of complex analytic spaces, every coherent O_X-module F and every integer $i \geq 0$, the base change morphism $g^*R^if_*F \to R^if'_*(g'^*F)$ induced by (6) is an isomorphism in $\mathrm{Mod}(O_{S'})$.

Lemma 3.2.5 shows that the property $\mathcal{Q}$ is local on the source and the target.

Lemma 3.2.5. Let $g : S' \to S$ and be a morphism of complex analytic spaces.

1. Let $h : S'' \to S'$ be another morphism of complex analytic spaces. If g and h satisfy $\mathcal{Q}_S$ and $\mathcal{Q}_{S'}$ respectively, then gh satisfies $\mathcal{Q}_S$.
2. Assume that $\{S'_i\}_{i\in I}$ (resp. $\{S_j\}_{j\in J}$) is an open covering of S' (resp. S) such that for every $i \in I$ (resp. $j \in J$), the morphism $g|_{S'_i} : S'_i \to S$ (resp. $g^{-1}(S_j) \to S_j$) satisfies $\mathcal{Q}_S$ (resp. $\mathcal{Q}_{S_j}$). Then g satisfies $\mathcal{Q}_S$.
3. If g is an open embedding of complex analytic spaces, then g satisfies $\mathcal{Q}_S$.

Proof.

1. The proof is similar to that of [Day23, Lem. 2.13 (2)].
2. It follows from the local nature of sheaves.
3. The proof is similar to that of [Har77, III, Cor. 8.2].

□

Lemma 3.2.6. Let $f : X \to S$ be a proper morphism of complex analytic spaces, with S Stein. Then for every coherent O_X-module F and every integer $n \geq 0$, one has $H^n(X, F) = H^0(S, R^n f_* F)$.

Proof. By properness of f and Fact 3.1.7, the O_S-module $R^n f_* F$ is coherent. As S is Stein, from Cartan's Theorem B (see, *e.g.*, [KK83, Sec. 52, Thm. B]), for every integer $m > 0$ one has $H^m(S, R^n f_* F) = 0$. The conclusion follows from [Sta23, Tag 01F4 (2)]. □

Lemma 3.2.7. Let X, Y be complex analytic spaces, with Y Stein. Let $p : X \times Y \to X$ be the projection. Then for every coherent O_X-module F and every integer $i \geq 0$, the natural morphism $H^i(X, F)\hat{\otimes}_{\mathbb{C}} O_Y(Y) \to H^i(X \times Y, p^* F)$ of locally convex topological vector spaces is an isomorphism.

Proof. Choose a Stein covering $\mathcal{U}$ of X. Let $C^\bullet$ be the Čech complex of F relative to $\mathcal{U}$. Then $H^i(C^\bullet) = H^i(X, F)$. By [EP$^+$96, Prop. 4.1.5], for every integer q, the q-th term C^q of the complex $C^\bullet$ is a Fréchet space. Moreover, $\{U \times Y : U \in \mathcal{U}\}$ forms a Stein covering of $X \times Y$. By [EP$^+$96, Prop. 4.2.3; Thm. 4.2.4], the Čech complex of $p^* F$ relative to this Stein covering is $C^\bullet\hat{\otimes}_{\mathbb{C}} O(Y)$. Therefore, $H^i(C^\bullet\hat{\otimes}_{\mathbb{C}} O(Y)) = H^i(X \times Y, p^* F)$. By [EP$^+$96, Prop. 4.1.5], $O(Y)$ is a unital Fréchet nuclear algebra, so from [EP$^+$96, Thm. A1.6 (d)], the functor $*\hat{\otimes}_{\mathbb{C}} O(Y)$ preserves exact sequences, hence commutes with taking cohomology groups of the Čech complexes. □

We consider the special case of products.

Corollary 3.2.8. Let S, Z be two complex analytic spaces. Then the projection $S \times Z \to S$ satisfies $\mathcal{Q}_S$.

Proof. Fix a proper morphism $X \to S$ of complex analytic spaces and a coherent O_X-module F. By Lemma 3.2.5, we may assume that S, Z are Stein spaces. Then the result follows from Lemma 3.2.6, Lemma 3.2.7 and [EP$^+$96, Prop. 4.2.3; Thm. 4.2.4]. □

Corollary 3.2.9. Every locally product morphism $g : S' \to S$ of complex analytic spaces satisfies $\mathcal{Q}_S$.

Proof. Fix $s' \in S'$, and let $s = g(s')$. Since g is locally product, there is an open neighborhood U (resp. V) of $s' \in S'$ (resp. $s \in S$), a complex analytic space Z and an isomorphism $\psi : U \to Z \times V$ of complex analytic spaces such that the diagram

$$\begin{tikzcd}
U \arrow[r, "\psi"] \arrow[d, "g|_U"] & Z \times V \arrow[dl, "p_2"] \\
V &
\end{tikzcd}$$

commutes, where p_2 is the projection to the second factor. By Corollary 3.2.8, $g|_U : U \to V$ satisfies $\mathcal{Q}_V$. By Lemma 3.2.5, the morphism $g : S' \to S$ satisfies $\mathcal{Q}_S$. □

Proof of Theorem 3.2.3. The morphism f' is a base change of f, hence a proper morphism. Because $\dim X, \dim X'$ are finite, by Theorem 3.1.6 and Proposition 3.1.2 2, the functors Lg^*Rf_* and $Rf'_*Lg'^*$ restrict to functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$.

For every $K \in D_{\mathrm{gd}}(X)$, we prove that the base change morphism $Lg^*Rf_*K \to Rf'_*Lg'^*K$ in $D(S')$ is an isomorphism. By Lemma 3.1.10, the functors $Rf_* : D(X) \to D(S)$ and $Rf'_* : D(X') \to D(S')$ are bounded. From [Har66, I, Prop. 7.1 (iii)] and Lemma A.4.3 3, one may assume that $K \in \mathrm{Good}(X)$. For every $s' \in S'$, there is a relatively compact open neighborhood $V \subset S$ of $g(s')$. The preimage $f^{-1}(V)$ is a relatively compact open subset of X. Consider the base change of the square (6) along the open embedding $V \to S$:

$$\begin{tikzcd}
f^{-1}(V) \times_V g^{-1}V \arrow[r, "v'"] \arrow[d, "u'"] \arrow[dr, phantom, "\square"] & f^{-1}(V) \arrow[d, "u"] \\
g^{-1}(V) \arrow[r, "v"] & V.
\end{tikzcd}$$

Because g is locally product, so is v. One can write $K|_{f^{-1}(V)} = \mathrm{colim}_{i \in I} G_i$, where $\{G_i\}_{i \in I}$ is a directed family of coherent submodules of $K|_{f^{-1}(V)}$. By Lemma 3.1.8, the natural morphism

$$(g^* R^i f_* K)|_{g^{-1}(V)} \to R^i f'_*(g'^* K)|_{g^{-1}(V)} \tag{8}$$

in $\mathrm{Mod}(O_{g^{-1}(V)})$ is the colimit of the morphisms

$$v^* R^i u_* G_i \to R^i u'_* v'^* G_i.$$

By Corollary 3.2.9, for all $i \in I$, they are isomorphisms. Then (8) is an isomorphism. □

Remark 3.2.10. In the proof of [BBR94, Lem. 5], an analytic flat base change result is applied without further justification. In [MS08, p.153], a flat base change theorem for cartesian squares in the category of complex manifolds is stated, referring to [Spa88] for the proof. However, the cited result [Spa88, Prop. 6.20] is for cartesian squares in the category RingS. In general, a cartesian square in the category of complex manifolds is not cartesian in RingS. For example, the complex vector space $\mathbb{C}^2$ is the product of two copies of $\mathbb{C}$ in the category of complex manifolds, but is not the product even in the subcategory LRS $\subset$ RingS of locally ringed space.[4]

In fact, by [Gil11, Cor. 5], the product E of two copies of $\mathbb{C}$ in LRS exists. By the universal property of E, there is a unique morphism $f : \mathbb{C}^2 \to E$ in LRS induced by the two projections $p_i : \mathbb{C}^2 \to \mathbb{C}$. Let $o = f(0) \in E$. We claim that the local ring $O_{E,o}$ is not Noetherian.

The local ring $A := O_{\mathbb{C},0} = \mathbb{C}\{z\}$ is the ring of convergent power series. Let $B = A \otimes_{\mathbb{C}} A$. Let $\epsilon : B \to A$ be the surjective (diagonal) morphism defined by $\epsilon(f \otimes g) = fg$. Set $I = \ker(\epsilon)$. Let $c : A \to \mathbb{C}$ be the ring map taking the constant term. Then $c\epsilon : B \to \mathbb{C}$ is surjective, so $m = \ker(c\epsilon)$ is a maximal ideal of B containing I. Set $S = B \setminus m$. Then $O_{E,o} = S^{-1}B$. From [Tu97, p.367], I/I^2 is a free B/I-module of *infinite* rank. Thus, $S^{-1}(I/I^2) = (S^{-1}I)/(S^{-1}I^2)$ is a free $S^{-1}(B/I) = (S^{-1}B)/(S^{-1}I)$-module of infinite rank. In particular, the ideal $S^{-1}I$ of the ring $S^{-1}B$ is not finitely generated. The claim is proved.

By [GH78, p.679], the ring $\mathbb{C}\{x, y\}$ is Noetherian. Thus, the local morphism $f_0^{\#} : O_{E,o} \to O_{\mathbb{C}^2,0} = \mathbb{C}\{x, y\}$ is not an isomorphism. Hence, f is not an isomorphism in LRS.

Non-smooth base change

Lemma 3.2.11 is used in the proof of Proposition 5.1.2.

Lemma 3.2.11 (Base change)**.** Consider the cartesian square (6) with $\dim X, \dim S'$ finite and f flat proper. Then (7) induces an isomorphism $Lg^*Rf_* \to Rf'_*Lg'^*$ of functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$.

Proof. Because $\dim X$ is finite, by Theorem 3.1.6 and Proposition 3.1.2 2, the functor $Lg^*Rf_* : D(X) \to D(S')$ restricts to a functor $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$. Consider the following commutative diagram

$$\begin{tikzcd}
X' \arrow[r, "i'"'] \arrow[d, "f'"] \arrow[rr, bend left, "g'"] & S' \times X \arrow[r, "p'"'] \arrow[d, "\mathrm{Id}_{S'} \times f"] & X \arrow[d, "f"] \\
S' \arrow[r, "i"] \arrow[rr, bend right, "g"'] & S' \times S \arrow[r, "p"] & S,
\end{tikzcd}$$

where the morphism $i : S' \to S' \times S$ is defined by $i(s') = (s', g(s'))$, and $p : S' \times S \to S$ is the projection. Then i is a closed embedding of complex analytic spaces.

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[4] By contrast, every cartesian square in the category of schemes remains cartesian in LRS ([Sta23, Tag 01JN]).

Because p is locally product, by Theorem 3.2.3, the natural transformation $Lp^*Rf_* \to R(\mathrm{Id}_{S'} \times f)_*Lp'^* : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S' \times S)$ is an isomorphism. Because f is flat proper, so is $\mathrm{Id}_{S'} \times f$. Moreover, $\dim(S' \times X) = \dim S' + \dim X$ is finite. Thus, there are isomorphism of functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(S')$

$$\begin{aligned} &Lg^*Rf_* \cong Li^*Lp^*Rf_* \xrightarrow{\sim} Li^*R(\mathrm{Id}_{S'} \times f)_*Lp'^* \\ &\overset{(a)}{\xrightarrow{\sim}} Rf'_*Li'^*Lp'^* \cong Rf'_*Lg'^*, \end{aligned} \tag{9}$$

where the isomorphism (a) uses Lemma 3.2.12 2. By [Sta23, Tag 0E47], the isomorphism (9) is induced by (7). ☐

Lemma 3.2.12. In the cartesian square (6), assume that g is a closed embedding of complex analytic spaces. Then:

1. The base change morphism $f^*g_*O_{S'} \to g'_*O_{X'}$ in $\mathrm{Mod}(O_X)$ is an isomorphism.
2. If f is flat proper and X has finite dimension, then (7) is an isomorphism.

Proof.

1. Let I be the kernel of the canonical surjection $O_S \to g_*O_{S'}$ in $\mathrm{Mod}(O_S)$. Since $f^* : \mathrm{Mod}(O_S) \to \mathrm{Mod}(O_X)$ is right exact, the sequence

$$f^*I \to O_X \to f^*g_*O_{S'} \to 0$$

is exact in $\mathrm{Mod}(O_X)$. Because g is a closed embedding, by [Gro61a, Remarque 2.10], the square (6) is cartesian in the category RingS. Then from [Gro61a, 9-05], the cokernel of the morphism $f^*I \to O_X$ in $\mathrm{Mod}(O_X)$ is $g'_*O_{X'}$. Therefore, the morphism $f^*g_*O_{S'} \to g'_*O_{X'}$ is an isomorphism.

2. As g is a closed embedding, the functor $g_* : \mathrm{Ab}(S') \to \mathrm{Ab}(S)$ is exact and $g^{-1}g_* = \mathrm{Id}_{\mathrm{Ab}(S')}$. Therefore, the functor $Rg_* = g_* : D(S') \to D(S)$ is conservative. Thus, it suffices to show that the natural transformation

$$Rg_*Lg^*Rf_*E \to Rg_*Rf'_*Lg'^*E \xrightarrow{\sim} Rf_*Rg'_*Lg'^*E \tag{10}$$

of functors $D(X) \to D(S)$ is an isomorphism. By [Sta23, Tag 0B55], the natural morphisms

$$(Rg_*O_{S'}) \otimes^L_{O_S} Rf_*E \to Rg_*Lg^*Rf_*E,$$
$$(Rg'_*O_{X'}) \otimes^L_{O_X} E \to Rg'_*Lg'^*E$$

are isomorphisms. One has

$$Rg'_*O_{X'} = g'_*O_{X'} \overset{(a)}{\xleftarrow{\sim}} f^*g_*O_{S'} \overset{(b)}{=} Lf^*Rg_*O_{S'},$$

where (a) uses Point 1, and (b) uses the flatness of f. Thus, the natural transformation (10) becomes

$$(Rg_*O_{S'}) \otimes^L_{O_S} Rf_*E \to Rf_*(Lf^*Rg_*O_{S'} \otimes^L_{O_X} E).$$

It is an isomorphism by the finiteness of $\dim X$, the properness of f and Fact 3.2.13.

☐

From Fact 3.1.9, one gets Fact 3.2.13 as a special case of [Spa88, Prop. 6.18].

Fact 3.2.13 (Projection formula)**.** Let $f : X \to Y$ be a morphism of complex analytic spaces. If $\dim X$ is finite, then there is a canonical isomorphism $(Rf_! -) \otimes^L_{O_Y} (+) \to Rf_!(- \otimes^L_{O_X} Lf^* +)$ of bifunctors $D(X) \times D(Y) \to D(Y)$.

3.3 Compatibility

For a complex algebraic variety X, let $\psi_X : X^{\mathrm{an}} \to X$ be its complex analytification. With quasi-coherence condition, the algebraic and analytic integral transforms are compatible.

Corollary 3.3.1. Let X, Y be two complex algebraic varieties, with X *proper*. Then for every $K \in D_{\mathrm{qc}}(X \times Y)$, the natural square

$$\begin{tikzcd}
D(X) \arrow[r, "\phi_K^{[X \to Y]}"] \arrow[d, "\psi_X^*"] & D(Y) \arrow[d, "\psi_Y^*"] \\
D(X^{\mathrm{an}}) \arrow[r, "\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}} \to Y^{\mathrm{an}}]}"'] & D(Y^{\mathrm{an}}),
\end{tikzcd}$$

restricts to a *commutative* square

$$\begin{tikzcd}
D_{\mathrm{qc}}(X) \arrow[r, "\phi_K^{[X \to Y]}"] \arrow[d, "\psi_X^*"] & D_{\mathrm{qc}}(Y) \arrow[d, "\psi_Y^*"] \\
D_{\mathrm{gd}}(X^{\mathrm{an}}) \arrow[r, "\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}} \to Y^{\mathrm{an}}]}"'] & D_{\mathrm{gd}}(Y^{\mathrm{an}}).
\end{tikzcd} \tag{11}$$

Proof. From [Sta23, Tag 08DW (1)], [Sta23, Tag 08DX (1)] and [Sta23, Tag 08D5 (1)], the functor $\phi_K^{[X \to Y]}$ restricts to a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{qc}}(Y)$. By Corollary 3.1.14 and compactness of X^{an}, the functor $\phi_{K^{\mathrm{an}}}^{[X^{\mathrm{an}} \to Y^{\mathrm{an}}]}$ restricts to a functor $D_{\mathrm{gd}}(X^{\mathrm{an}}) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$. By [Liu24, Lem. 2.3], the functor ψ_X^* (resp. ψ_Y^*) restricts to a functor $D_{\mathrm{qc}}(X) \to D_{\mathrm{gd}}(X^{\mathrm{an}})$ (resp. $D_{\mathrm{qc}}(Y) \to D_{\mathrm{gd}}(Y^{\mathrm{an}})$).

By [Sta23, Tag 0D5S] (resp. [Sta23, Tag 079U]), analytification commutes with derived pullback (resp. tensor product). As X is proper over $\mathbb{C}$, the projection $p_Y : X \times Y \to Y$ is proper. By [Liu24, Prop. 3.1], analytification commutes with derived direct image. Thus, the square (11) is commutative. □

4 Analytic Mukai duality

4.1 Statement

Let X be a complex torus of dimension g.

Theorem 4.1.1 (Mukai, Ben-Bassat, Block, Pantev)**.** *There are natural isomorphisms of functors*

$$RS \circ R\hat{S} \xrightarrow{\sim} [-1]^*_X[-g] : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X);$$
$$R\hat{S} \circ RS \xrightarrow{\sim} [-1]^*_{\hat{X}}[-g] : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(\hat{X}).$$

In particular, $RS : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X)$ *is an equivalence of categories, with a quasi-inverse* $[-1]^*_{\hat{X}}R\hat{S}[g]$*.*

Corollary 4.1.2. The functors $RS : D^b_c(\hat{X}) \to D^b_c(X)$ and $R\hat{S} : D^b_c(X) \to D^b_c(\hat{X})$ are equivalences of triangulated categories.

Proof. It follows from Corollary 3.1.14 and Theorem 4.1.1. □

Remark 4.1.3. A Mukai duality for complex tori similar to Corollary 4.1.2 is stated in [Blo10, p.314], with $D^b(\mathrm{Coh}(*))$ at the place of $D^b_c(*)$. However, Prof. Jonathan Block told the author that here we should stick to $D^b_c(*)$. In fact, in general the abelian category $\mathrm{Coh}(X)$ does not have enough injectives, so it is unclear how to define the derived direct image involved in [Blo10, p.314]. Moreover, recently Prof. Alexey Bondal announced[5] that for a generic complex torus X of dimension > 2, the natural functor $D^b(\mathrm{Coh}(X)) \to D^b_c(X)$ is *not* an equivalence.

4.2 Proof

We follow the strategy of [BBBP07, Thm. 2.1] to prove Theorem 4.1.1.

Preliminaries

Lemma 4.2.1, an analytic analog of [Muk81, Example 1.2], exhibits the derived pullback and direct image as particular examples of integral transforms.

Lemma 4.2.1. Let $f : X \to Y$ be a morphism of complex analytic spaces. Let $i : \Gamma_f \to X \times Y$ be the inclusion of the graph of f. Set $F = i_* O_{\Gamma_f} \in \mathrm{Mod}(O_{X\times Y})$. Then there are canonical isomorphism of functors

$$\phi_F^{[X\to Y]} \xrightarrow{\sim} Rf_* : D(X) \to D(Y); \tag{12}$$
$$\phi_F^{[Y\to X]} \xrightarrow{\sim} Lf^* : D(Y) \to D(X). \tag{13}$$

Proof. Let $g : \Gamma_f \to X$ be the projection. Since g is an isomorphism of complex analytic spaces, one has a canonical isomorphism

$$Lg^* \xrightarrow{\sim} R(g^{-1})^* \tag{14}$$

of functors $D(X) \to D(\Gamma_f)$. Consider the following diagram

---

[5] `https://www.mathnet.ru/eng/present35371`

$$\begin{tikzcd}
\Gamma_f \arrow[r, hook, "i"] \arrow[d, "g"] & X \times Y \arrow[dl, "p_X"] \arrow[dr, "p_Y"] & \\
X \arrow[rr, "f"] & & Y.
\end{tikzcd}$$

As i is a closed embedding of complex analytic spaces, by [Sta23, Tag 0B55], the natural transformation

$$Ri_*O_{\Gamma_f} \otimes^L Lp_X^*(\cdot) \to Ri_*Li^*Lp_X^*(\cdot) \tag{15}$$

is an isomorphism of functors $D(X) \to D(X \times Y)$. One has

$$\begin{aligned}
\phi_F^{[X\to Y]} :=& Rp_{Y*}(F \otimes^L p_X^* \cdot) = Rp_{Y*}(Ri_*O_{\Gamma_f} \otimes^L Lp_X^* \cdot) \\
\overset{\text{(a)}}{\xrightarrow{\sim}}& Rp_{Y*}Ri_*Li^*Lp_X^* \overset{\text{(b)}}{\xrightarrow{\sim}} Rp_{Y*}Ri_*Lg^* \\
\overset{\text{(c)}}{\xrightarrow{\sim}}& Rp_{Y*}Ri_*R(g^{-1})^* \overset{\text{(d)}}{\xrightarrow{\sim}} Rf_*,
\end{aligned}$$

where (a) (resp. (c)) uses (15) (resp. (14)), and (b), (d) are from [Spa88, Thm. A (iii)].

Thus, (12) is proved. The proof of (13) is similar. □

Proposition 4.2.2 is the first ingredient of the proof of Theorem 4.1.1, which expresses the composition of two integral transforms as another integral transform.

Proposition 4.2.2. *Let M, N, P be complex analytic spaces, with M, N compact and $\dim P$ finite. Let p_{ij} be the projections of the product $M \times N \times P$. For $K \in D_{\mathrm{gd}}(M \times N)$ and $L \in D(N \times P)$, set*

$$H = Rp_{13*}(p_{12}^*K \otimes^L p_{23}^*L)(\in D(M \times P)).$$

Then there is a natural isomorphism $\phi_L^{[N\to P]}\phi_K^{[M\to N]} \xrightarrow{\sim} \phi_H^{[M\to P]}$ of functors $D_{\mathrm{gd}}(M) \to D(P)$.

Proof. Let

$$\begin{aligned}
a &: M \times N \to M, & b &: N \times P \to P, \\
p &: M \times N \to N, & q &: N \times P \to N, \\
u &: M \times P \to M, & v &: M \times P \to P
\end{aligned}$$

be projections.

The morphism q is locally product. Properness of p follows from the compactness of M. By Propositions 3.1.2 2 and 3.1.5 2, the functor $K \otimes^L a^* \cdot : D(M) \to D(M \times N)$ restricts to a functor $D_{\mathrm{gd}}(M) \to D_{\mathrm{gd}}(M \times N)$. Then one can apply Theorem 3.2.3 to the cartesian square

$$\begin{tikzcd}
M \times N \times P \arrow[r, "p_{12}"] \arrow[d, "p_{23}"'] \arrow[dr, phantom, "\square"] & M \times N \arrow[d, "p"] \\
N \times P \arrow[r, "q"'] & N,
\end{tikzcd}$$

so the base change natural transformation induces an isomorphism

$$q^*Rp_*(K\otimes^L a^*\cdot)\to Rp_{23*}p_{12}^*(K\otimes^L a^*\cdot) \tag{16}$$

of functors $D_{\mathrm{gd}}(M)\to D_{\mathrm{gd}}(N\times P)$. Thus, one has isomorphisms

$$\begin{aligned}
\phi_L^{[N\to P]}\phi_K^{[M\to N]} &= Rb_*[L\otimes^L q^*Rp_*(K\otimes^L a^*\cdot)]\\
&\overset{(a)}{\xrightarrow{\sim}} Rb_*[L\otimes^L Rp_{23*}p_{12}^*(K\otimes^L a^*\cdot)]\\
&\overset{(b)}{\xrightarrow{\sim}} Rb_*Rp_{23*}[p_{23}^*L\otimes^L p_{12}^*(K\otimes^L a^*\cdot)]\\
&\cong Rp_{3*}[p_{23}^*L\otimes^L p_{12}^*(K\otimes^L a^*\cdot)]\\
&\cong Rv_*Rp_{13*}(p_{12}^*K\otimes^L p_{23}^*L\otimes^L p_1^*\cdot)\\
&\overset{(c)}{\xleftarrow{\sim}} Rv_*[H\otimes^L u^*\cdot]=\phi_H^{[M\to P]},
\end{aligned}$$

of functors $D_{\mathrm{gd}}(M)\to D(P)$ where (a) uses (16), and (b) (resp. (c)) is from the compactness of M (resp. N) and Fact 3.2.13. ☐

Fact 4.2.3, the other ingredient of the proof of Theorem 4.1.1, calculates the cohomology of the Poincaré bundle.

Fact 4.2.3 ([Kem91, Thm. 3.15]). Let X be a complex torus of dimension g. Let $p_X : X\times\hat{X}\to X$, $p_{\hat{X}} : X\times\hat{X}\to\hat{X}$ be the two projections. Then for the normalized Poincaré bundle $\mathcal{P}$, one has $Rp_{X*}\mathcal{P}=\mathbb{C}_0[-g]$ in $D^b(X)$ and $Rp_{\hat{X}*}\mathcal{P}=\mathbb{C}_0[-g]$ in $D^b(\hat{X})$.

Proof of Theorem 4.1.1

By Corollary 3.1.14, the functor RS (resp. $R\hat{S}$) restricts to a functor $D_{\mathrm{gd}}(\hat{X})\to D_{\mathrm{gd}}(X)$ (resp. $D_{\mathrm{gd}}(X)\to D_{\mathrm{gd}}(\hat{X})$). Let p_{ij} be the projections of $X\times X\times\hat{X}$. Set

$$H=Rp_{12,*}(p_{13}^*\mathcal{P}\otimes^L p_{23}^*\mathcal{P}).$$

By Propositions 3.1.2 1 and 3.1.5 1, Fact 3.1.7 and Lemma 3.1.10, one has $H\in D_c^b(X\times X)$. By Proposition 4.2.2, one has an isomorphism of $RS\circ R\hat{S}\xrightarrow{\sim}\phi_H^{[X\to X]}$ of functors $D_{\mathrm{gd}}(X)\to D_{\mathrm{gd}}(X)$. Let $m : X\times X\to X$ be the group law. Since the $O_{X\times X\times\hat{X}}$-module $p_{13}^*\mathcal{P}$ is flat, one has $p_{13}^*\mathcal{P}\otimes^L p_{23}^*\mathcal{P}=p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}$. By [BL04, Lem. 14.1.7],[6] the $O_{X\times X\times\hat{X}}$-module $p_{13}^*\mathcal{P}\otimes p_{23}^*\mathcal{P}$ is isomorphic to $(m\times\mathrm{Id}_{\hat{X}})^*\mathcal{P}$. Then $H\xrightarrow{\sim}Rp_{12,*}(m\times\mathrm{Id}_{\hat{X}})^*\mathcal{P}$.

Because the morphism m is smooth, applying Theorem 3.2.3 to the cartesian square

$$\begin{tikzcd}
X\times X\times\hat{X} \arrow[r, "m\times\mathrm{Id}_{\hat{X}}"] \arrow[d, "p_{12}"'] \arrow[dr, phantom, "\square"] & X\times\hat{X} \arrow[d, "p_X"] \\
X\times X \arrow[r, "m"'] & X
\end{tikzcd}$$

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[6] It is stated for abelian varieties, but its proof works for complex tori.

in the category An, one has an isomorphism $m^*Rp_{X,*}\mathcal{P} \to H$ in $D^b_c(X \times X)$. Let $i : \Gamma_{[-1]} \to X \times X$ be the inclusion of the graph of $[-1]_X : X \to X$. From Fact 4.2.3, one has $H \xrightarrow{\sim} m^*\mathbb{C}_0[-g] = i_*O_{\Gamma_{[-1]}}[-g]$. By Lemma 4.2.1, there is an isomorphism $\phi_H^{[X\to X]} \xrightarrow{\sim} [-1]_X^*[-g]$ of functors $D(X) \to D(X)$, which shows the isomorphism $RS \circ R\hat{S} \xrightarrow{\sim} [-1]_X^*[-g]$ of functors $D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(X)$. The proof of the second isomorphism is similar.

5 Properties of Fourier-Mukai transform

For later reference purposes, we check that each result starting from Theorem 2.2 to (3.12') in [Muk81] has an analytic version. We only indicate the necessary modifications in statements and proofs.

For a complex torus X, let g_X be its dimension. Let $(RS_X, R\hat{S}_X)$ be the Fourier-Mukai transform of X. The subscripts are omitted when there is only one complex torus in context. Let $p_X : X \times \hat{X} \to X$, $p_{\hat{X}} : X \times \hat{X} \to \hat{X}$ be the projections. For a morphism $\phi : X \to Y$ of complex tori, let $\hat{\phi} : \hat{Y} \to \hat{X}$ be the dual morphism.

5.1 Functoriality

Exchange of translations and twists

For every point x of the complex torus X, let $T_x : X \to X, \quad x' \mapsto x' + x$ be the translation by x.

Proposition 5.1.1. *For every $x \in X$ and every $\hat{x} \in \hat{X}$, there are canonical isomorphisms*

$$RS \circ T_{\hat{x}}^* \cong (\cdot \otimes_{O_X} P_{-\hat{x}}) \circ RS, \tag{17}$$

$$RS \circ (\cdot \otimes_{O_{\hat{X}}} P_x) \cong T_x^* \circ RS \tag{18}$$

of funtors $D(\hat{X}) \to D(X)$.

Proof. We prove (17). From [BL04, Cor. A.9], one gets

$$T_{(0,-\hat{x})}^*\mathcal{P} \xrightarrow{\sim} \mathcal{P} \otimes_{O_{X\times\hat{X}}} p_X^* P_{-\hat{x}}; \tag{19}$$

$$T_{(x,0)}^*\mathcal{P} \xrightarrow{\sim} \mathcal{P} \otimes_{O_{X\times\hat{X}}} p_{\hat{X}}^* P_x. \tag{20}$$

Then there are isomorphisms

$$\begin{aligned}
RS(T^*_{\hat{x}}\cdot) &= Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}} T^*_{\hat{x}}\cdot)\\
&= Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} T^*_{(0,\hat{x})} p^*_{\hat{X}}\cdot)\\
&= Rp_{X*} T^*_{(0,\hat{x})}(T^*_{(0,-\hat{x})}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&\xrightarrow{\sim} Rp_{X*} R(T_{(0,-\hat{x})})_*(T^*_{(0,-\hat{x})}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&\cong Rp_{X*}(T^*_{(0,-\hat{x})}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&\overset{\text{(a)}}{\xrightarrow{\sim}} Rp_{X*}(p^*_X P_{-\hat{x}} \otimes \mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&\overset{\text{(b)}}{\xleftarrow{\sim}} P_{-\hat{x}} \otimes Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&= P_{-\hat{x}} \otimes RS(\cdot)
\end{aligned}$$

of functors $D(\hat{X}) \to D(X)$, where (a) (resp. (b)) uses (19) (resp. Fact 3.2.13).

We prove (18) as follows:

$$\begin{aligned}
RS(P_x \otimes \cdot) &= Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}(P_x \otimes \cdot))\\
&= Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}} P_x \otimes p^*_{\hat{X}}\cdot))\\
&\overset{\text{(a)}}{\xrightarrow{\sim}} Rp_{X*}(T^*_{(x,0)}\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&= Rp_{X*} T^*_{(x,0)}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} T^*_{(-x,0)} p^*_{\hat{X}}\cdot)\\
&\xrightarrow{\sim} Rp_{X*} R(T_{(-x,0)})_*(\mathcal{P} \otimes_{O_{X\times\hat{X}}} T^*_{(-x,0)} p^*_{\hat{X}}\cdot)\\
&\cong R(T_{-x})_* Rp_{X*}(\mathcal{P} \otimes_{O_{X\times\hat{X}}} p^*_{\hat{X}}\cdot)\\
&\cong T^*_x RS(\cdot),
\end{aligned}$$

where (a) uses (20). □

Exchange of the direct image and the inverse image

The Fourier-Mukai transform is functorial.

Proposition 5.1.2. *For a morphism $\phi : Y \to X$ of complex tori, there are canonical isomorphisms of functors*

$$L\phi^* \circ RS_X \cong RS_Y \circ R\hat{\phi}_* : D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y), \tag{21}$$

$$R\phi_* \circ RS_Y \cong RS_X \circ L\hat{\phi}^*(\cdot)[g_X - g_Y] : D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(X). \tag{22}$$

Proof. The isomorphism (22) follows from (21) as follows. There are isomorphisms

$$\begin{aligned} R\phi_* RS_Y &\overset{\text{(a)}}{\xrightarrow{\sim}} [-1]_X^* RS_X R\hat{S}_X R\phi_* RS_Y(\cdot)[g_X] \\ &\overset{\text{(b)}}{\xrightarrow{\sim}} [-1]_X^* RS_X L\hat{\phi}^* R\hat{S}_Y RS_Y(\cdot)[g_X] \\ &\overset{\text{(c)}}{\xrightarrow{\sim}} [-1]_X^* RS_X L\hat{\phi}^* [-1]_Y^*(\cdot)[g_X - g_Y] \\ &= RS_X L\hat{\phi}^*(\cdot)[g_X - g_Y] \end{aligned}$$

of functors $D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(X)$, where (a) and (c) use Theorem 4.1.1, and (b) uses (21).

To prove (21), we show

$$(\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \cong (\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y. \tag{23}$$

Set $L := (\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \otimes_{O_{Y \times \hat{X}}} (\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y^{-1}$. By definition, on the one hand for every $\hat{x} \in \hat{X}$, one has $L|_{Y \times \hat{x}} \xrightarrow{\sim} \phi^* P_{\hat{x}} \otimes P^{-1}_{\hat{\phi}(\hat{x})} \xrightarrow{\sim} O_Y$; on the other hand, one has $L|_{0 \times \hat{X}} \xrightarrow{\sim} \hat{\phi}^* O_{\hat{Y}} \xrightarrow{\sim} O_{\hat{X}}$. By the seesaw principle [BL04, Cor. A.9], these imply $L \xrightarrow{\sim} O_{Y \times \hat{X}}$.

By applying Theorem 3.2.3 to the cartesian square

$$\begin{tikzcd} Y \times \hat{X} \arrow[r, "p_2"] \arrow[d, "\mathrm{Id}_Y \times \hat{\phi}"'] \arrow[dr, phantom, "\square"] & \hat{X} \arrow[d, "\hat{\phi}"] \\ Y \times \hat{Y} \arrow[r, "p_{\hat{Y}}"'] & \hat{Y} \end{tikzcd}$$

in the category An, the base change natural transformation

$$p_{\hat{Y}}^* R\hat{\phi}_* \to R(\mathrm{Id}_Y \times \hat{\phi})_* p_2^* \tag{24}$$

induces an isomorphism of functors $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y \times \hat{Y})$. By Propositions 3.1.2 2 and 3.1.5 2, the functor $\mathcal{P}_X \otimes p_{\hat{X}}^*(\cdot) : D(\hat{X}) \to D(X \times \hat{X})$ restricts to a functor $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(X \times \hat{X})$. Because p_X is smooth proper, by applying Lemma 3.2.11 to the cartesian square

$$\begin{tikzcd} Y \times \hat{X} \arrow[r, "\phi \times \mathrm{Id}_{\hat{X}}"] \arrow[d, "p_1"'] \arrow[dr, phantom, "\square"] & X \times \hat{X} \arrow[d, "p_X"] \\ Y \arrow[r, "\phi"'] & X \end{tikzcd}$$

in the category An, the base change natural transformation induces an isomorphism

$$L\phi^* Rp_{X*}(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \to Rp_{1*} L(\phi \times \mathrm{Id}_{\hat{X}})^* (\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \tag{25}$$

of functors $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y)$.

There are isomorphisms

$$\begin{aligned}
L\phi^* \circ RS_X =& L\phi^* Rp_{X*}(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \\
&\overset{(a)}{\xrightarrow{\sim}} Rp_{1*} L(\phi \times \mathrm{Id}_{\hat{X}})^*(\mathcal{P}_X \otimes p_{\hat{X}}^* \cdot) \\
&\cong Rp_{1*}[L(\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \otimes^L L(\phi \times \mathrm{Id}_{\hat{X}})^* p_{\hat{X}}^* \cdot] \\
&\cong Rp_{1*}[(\phi \times \mathrm{Id}_{\hat{X}})^* \mathcal{P}_X \otimes p_2^* \cdot] \\
&\overset{(b)}{\xrightarrow{\sim}} Rp_{1*}[(\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y \otimes p_2^* \cdot] \\
&\cong Rp_{Y*} R(\mathrm{Id}_Y \times \hat{\phi})_* [L(\mathrm{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y \otimes p_2^* \cdot] \\
&\overset{(c)}{\xleftarrow{\sim}} Rp_{Y*}[\mathcal{P}_Y \otimes R(\mathrm{Id}_Y \times \hat{\phi})_* p_2^* \cdot] \\
&\overset{(d)}{\xleftarrow{\sim}} Rp_{Y*}[\mathcal{P}_Y \otimes p_{\hat{Y}}^* R\hat{\phi}_* \cdot] \\
&= RS_Y R\hat{\phi}_*
\end{aligned}$$

of functors $D_{\mathrm{gd}}(\hat{X}) \to D_{\mathrm{gd}}(Y)$, where (a) (resp. (b), resp. (c), resp. (d)) uses (25) (resp. (23), resp. Fact 3.2.13, resp. (24)). This proves (21). □

5.1.1 Commutativity with external tensor product

Let M, N be two complex analytic spaces. Let $p : M \times N \to M$ and $q : M \times N \to N$ be the projections. The bifunctor $D(M) \times D(N) \to D(M \times N), \quad (-,+) \mapsto (p^* -) \otimes^L (q^* +)$ is denoted by $(\cdot) \boxtimes^L (\cdot)$.

Proposition 5.1.3. *Let X, Y be two complex tori and $Z = X \times Y$. Then there is a canonical isomorphism $RS_Z(- \boxtimes^L +) = RS_X(-) \boxtimes^L RS_Y(+)$ of bifunctors $D_{\mathrm{gd}}(\hat{X}) \times D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(Z)$.*

Proof. By the seesaw principle, one has $\mathcal{P}_Z \xrightarrow{\sim} \mathcal{P}_X \boxtimes^L \mathcal{P}_Y$. Then there are canonical isomorphisms

$$\begin{aligned}
RS_Z(- \boxtimes^L +) =& Rp_{Z*}[\mathcal{P}_Z \otimes^L Lp_{\hat{Z}}^*(- \boxtimes^L +)] \\
&\xrightarrow{\sim} Rp_{Z*}[(\mathcal{P}_X \boxtimes^L \mathcal{P}_Y) \otimes^L (Lp_{\hat{X}}^*(-) \boxtimes^L Lp_{\hat{Y}}^*(+))] \\
&\xrightarrow{\sim} R(p_X \times p_Y)_*[(\mathcal{P}_X \otimes^L Lp_{\hat{X}}^*(-)) \boxtimes^L (\mathcal{P}_Y \otimes^L Lp_{\hat{Y}}^*(+))] \\
&\overset{(a)}{\xleftarrow{\sim}} Rp_{X*}(\mathcal{P}_X \otimes^L Lp_{\hat{X}}^*(-)) \boxtimes^L Rp_{Y*}(\mathcal{P}_Y \otimes^L Lp_{\hat{Y}}^*(+)) \\
&= RS_X(-) \boxtimes^L RS_Y(+)
\end{aligned}$$

of bifunctors $D_{\mathrm{gd}}(\hat{X}) \times D_{\mathrm{gd}}(\hat{Y}) \to D_{\mathrm{gd}}(Z)$, where (a) uses Lemma 5.1.4 2. □

Lemma 5.1.4.

1. Let X, Y, T be complex analytic spaces, with X, T finite dimensional. Let $f : X \to Y$ be a *proper* morphism. Then there is a canonical isomorphism

$$Rf_*(-) \boxtimes^L (+) \to R(f \times \mathrm{Id}_T)_*(- \boxtimes^L +)$$

   of bifunctors $D_{\mathrm{gd}}(X) \times D(T) \to D(Y \times T)$.

2. Let $f_i : X_i \to Y_i$ $(i = 1, 2)$ be *proper* morphism of complex analytic spaces. If X_1, X_2 and Y_1 are finite dimensional, then there is a canonical isomorphism

$$(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(f_1 \times f_2)_*(- \boxtimes^L +)$$

   of bifunctors $D_{\mathrm{gd}}(X_1) \times D_{\mathrm{gd}}(X_2) \to D_{\mathrm{gd}}(Y_1 \times Y_2)$.

Proof.

1. Consider the notation in the commutative diagram

$$\begin{tikzcd}
 & X \times T \arrow[r, "u"] \arrow[dl, "v"'] \arrow[d, dashed, "f \times \mathrm{Id}_T"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
T & Y \times T \arrow[l, "q"] \arrow[r, "p"'] & Y,
\end{tikzcd}$$

   where u, v, p and q are projections. Since $v = q \circ (f \times \mathrm{Id}_T)$, there is a canonical isomorphism $v^* \xrightarrow{\sim} L(f \times \mathrm{Id}_T)^* q^*$ of functors $D(T) \to D(X \times T)$. As $f \times \mathrm{Id}_T$ is a base change of f, it is also proper. As $\dim(X \times T)$ is finite, by Fact 3.2.13, the canonical morphism

$$[R(f \times \mathrm{Id}_T)_* u^* -] \otimes^L q^* + \to R(f \times \mathrm{Id}_T)_*[u^* - \otimes^L v^* +] \tag{26}$$

   of bifunctors $D(X) \times D(T) \to D(Y \times T)$ is an isomorphism.

   By Theorem 3.2.3, one has a canonical isomorphism

$$p^* Rf_* \to R(f \times \mathrm{Id}_T)_* u^* : D_{\mathrm{gd}}(X) \to D_{\mathrm{gd}}(Y \times T). \tag{27}$$

   Therefore, there are canonical isomorphisms

$$\begin{aligned}
(Rf_*-) \boxtimes^L + &= (p^* Rf_* -) \otimes^L q^* + \\
&\overset{\text{(a)}}{\xrightarrow{\sim}} [R(f \times \mathrm{Id}_T)_* u^* -] \otimes^L q^* + \\
&\overset{\text{(b)}}{\xrightarrow{\sim}} R(f \times \mathrm{Id}_T)_*[u^* - \otimes v^* +] \\
&= R(f \times \mathrm{Id}_T)_*(- \boxtimes^L +),
\end{aligned}$$

   of bifunctors $D_{\mathrm{gd}}(X) \times D(T) \to D(Y \times T)$, where (a) (resp. (b)) uses (27) (resp. (26)).

2. Since $\dim(X_1 \times X_2)$ is finite, as in Corollary 3.1.14, the bifunctor $R(f_1 \times f_2)_*(- \boxtimes^L +)$ restricts to a bifunctor $D_{\mathrm{gd}}(X_1) \times D_{\mathrm{gd}}(X_2) \to D_{\mathrm{gd}}(Y_1 \times Y_2)$.

   As $\dim Y_1, \dim X_2$ are finite, by Point 1, there are canonical isomorphisms of bifunctors

$$(Rf_{1*}-) \boxtimes^L + \to R(f_1 \times \mathrm{Id}_{X_2})_*(- \boxtimes^L +) : D_{\mathrm{gd}}(X_1) \times D(X_2) \to D(Y_1 \times X_2),$$
$$(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(\mathrm{Id}_{Y_1} \times f_2)_*[(Rf_{1*}-) \boxtimes^L +] : D(X_1) \times D_{\mathrm{gd}}(X_2) \to D(Y_1 \times Y_2).$$

   Then there is a canonical isomorphism of bifunctors

$$\begin{aligned} &(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(\mathrm{Id}_{Y_1} \times f_2)_*[(Rf_{1*}-) \boxtimes^L +] \\ \to &R(\mathrm{Id}_{Y_1} \times f_2)_* R(f_1 \times \mathrm{Id}_{X_2})_*(- \boxtimes^L +) \\ \to &R(f_1 \times f_2)_*(- \boxtimes^L +) : D_{\mathrm{gd}}(X_1) \times D_{\mathrm{gd}}(X_2) \to D_{\mathrm{gd}}(Y_1 \times Y_2). \end{aligned}$$

□

5.1.2 Skew commutativity with duality

We summarize classical facts about the duality theory on complex manifolds.

Fact 5.1.5. Let X be a complex manifold of pure dimension n, and let $\omega_X = \bigwedge^n \Omega_X$ be the canonical line bundle.

1. ([RR70, p.81; p.90]) The dualizing functor $D_X = R\mathcal{H}om_X(\cdot, \omega_X)[n] : D(X) \to D(X)$ restricts to a functor $D_c(X) \to D_c(X)$ and the natural transformation $\mathrm{Id} \to D_X \circ D_X : D_c(X) \to D_c(X)$ is an isomorphism. If X is compact, then D_X exchanges[7] $D_c^+(X)$ with $D_c^-(X)$, and induces an equivalence $D_c^b(X) \to D_c^b(X)$.
2. ([RRV71, p.264]) There is a canonical isomorphism $R\mathcal{H}om_X(-, +) \to D_X(- \otimes^L D_X +)$ of bifunctors $D_c(X) \times D_c^+(X) \to D(X)$.
3. ([RRV71, p.264], [Bjö93, p.122]) Let $f : X \to Y$ be a proper morphism of complex manifolds. Then there is a canonical isomorphism of functors $Rf_* D_X \to D_Y Rf_* : D_c(X) \to D(Y)$.

Proposition 5.1.6 ([Muk81, (3.8)]). *There are canonical isomorphisms of functors*

$$D_X \circ RS \xrightarrow{\sim} ([-1]_X^* \circ RS \circ D_{\hat{X}})[g] : D_c^+(\hat{X}) \to D_c^-(X);$$
$$D_{\hat{X}} \circ R\hat{S} \xrightarrow{\sim} ([-1]_{\hat{X}}^* \circ R\hat{S} \circ D_X)[g] : D_c^+(X) \to D_c^-(\hat{X}).$$

We make some preparation for the proof of Proposition 5.1.6. Lemma 5.1.7 is an adaption of [Har66, Ch.II, Prop. 5.8] and [Sta23, Tag 0C6I].

---

[7] By [FS13, p.4971], in general the functor $R\mathcal{H}om_X(\cdot, \omega_X) : D(X) \to D(X)$ does not exchange $D_c^{b,\leq 0}(X)$ and $D_c^{b,\geq 0}(X)$.

Lemma 5.1.7. Let $f : X \to Y$ be a *flat* morphism of complex analytic spaces. Then:

1. There is a canonical natural transformation of bifunctors
$$f^* R\mathcal{H}om_Y(-,+) \to R\mathcal{H}om_X(f^*-, f^*+) : D(Y) \times D(Y) \to D(X). \quad (28)$$
2. The natural transformation (28) restricts to an isomorphism of bifunctors $D_c^-(Y) \times D(Y) \to D(X)$.

Proof. Set $G \in D(Y)$.

1. By [Spa88, Thm. D], there is a *functorial* quasi-isomorphism $G \to G'$, where G' is a K-injective complex over $\mathrm{Mod}(O_Y)$. There are natural transformations of functors $D(Y) \to D(X)$
$$\begin{aligned} f^* R\mathcal{H}om_Y(\cdot, G) &\to f^* \mathcal{H}om_Y(\cdot, G') \to \mathcal{H}om_X(f^*\cdot, f^*G') \\ \to & R\mathcal{H}om_X(f^*\cdot, f^*G') \xleftarrow{\sim} R\mathcal{H}om_X(f^*\cdot, f^*G). \end{aligned}$$
2. By [Har66, I, Examples 1], the (contravariant) functors
$$f^* R\mathcal{H}om_Y(\cdot, G), R\mathcal{H}om_X(f^*\cdot, f^*G) : D(Y) \to D(X)$$
are bounded below. Consider $F \in D_c^-(Y)$. To show the natural morphism $f^* R\mathcal{H}om_Y(F, G) \to R\mathcal{H}om_X(f^*F, f^*G) : D_c^-(Y) \to D(X)$ is an isomorphism, by [Har66, I, Prop. 7.1 (ii)], one may assume $F \in \mathrm{Coh}(Y)$. By [Sta23, Tag 08DL], one may shrink Y to open subsets. Thus, from Lemma A.3.1, one may assume that there is a quasi-isomorphism $K \to F$, where K is a complex of finite free O_Y-modules. The morphism f is flat, so $f^*K \to f^*F \to 0$ is a globally free resolution of f^*F. The morphism (28) is identified with $f^* \mathcal{H}om_Y(K, G) \to \mathcal{H}om_X(f^*K, f^*G)$, which is an isomorphism.

□

Lemma 5.1.8. Let $E \to X$ be a holomorphic vector bundle on a complex manifold, and let $E^\vee$ be the dual vector bundle. Then there is an isomorphism of functors $E^\vee \otimes D_X \cdot \to D_X(E \otimes \cdot) : D(X) \to D(X)$.

Proof. Since E is a vector bundle, one has isomorphisms
$$E \otimes \cdot \xrightarrow{\sim} \mathcal{H}om_X(E^\vee, \cdot) \xrightarrow{\sim} R\mathcal{H}om_X(E^\vee, \cdot)$$
of functors $D(X) \to D(X)$. Then
$$D_X(E \otimes \cdot) = R\mathcal{H}om_X(R\mathcal{H}om_X(E^\vee, \cdot), \omega_X)[\dim X].$$
As $E^\vee$ is a perfect object of $D(X)$ (in the sense of [Sta23, Tag 08CM]), by [Sta23, Tag 0G40], one has $D_X(E \otimes \cdot) = R\mathcal{H}om_X(\cdot, \omega_X)[\dim X] \otimes^L E^\vee = E^\vee \otimes D_X \cdot$. □

Corollary 5.1.9. Let $f : X \to Y$ be a flat morphism of complex manifolds of relative dimension n. Write $\omega_f = \omega_X \otimes_{O_X} f^*\omega_Y^\vee$ for the relative dualizing line bundle. Then there is a canonical isomorphism of functors $D_X f^* D_Y \to \omega_f \otimes_{O_X} f^*(\cdot)[n] : D_c^-(Y) \to D_c^-(X)$.

Proof. One has

$$\begin{aligned} D_X f^* D_Y O_Y &= D_X\big(f^* R\mathcal{H}om_Y(O_Y, \omega_Y)[\dim Y]\big) = D_X(f^*\omega_Y[\dim Y]) \\ &= R\mathcal{H}om_X(f^*\omega_Y, \omega_X)[\dim X - \dim Y] \overset{\text{(a)}}{=} \mathcal{H}om_X(f^*\omega_Y, \omega_X)[n] \\ &= f^*\omega_Y^\vee \otimes_{O_X} \omega_X[n] = \omega_f[n], \end{aligned} \tag{29}$$

where (a) uses that $f^*\omega_Y$ is a line bundle on X.

By Fact 5.1.5 1 and 2, there is an isomorphism $D_Y \xrightarrow{\sim} R\mathcal{H}om_Y(\cdot, D_Y O_Y)$ of functors $D_c^-(Y) \to D_c^+(Y)$. From Lemma 5.1.7 2, there are isomorphisms

$$f^* D_Y \xrightarrow{\sim} f^* R\mathcal{H}om_Y(\cdot, D_Y O_Y) \xrightarrow{\sim} R\mathcal{H}om_X(f^*\cdot, f^* D_Y O_Y)$$

of functors $D_c^-(Y) \to D_c^+(X)$. Then by Fact 5.1.5 1 and 2 again, there are isomorphisms

$$\begin{aligned} D_X f^* D_Y &\xrightarrow{\sim} f^*(\cdot) \otimes^L D_X f^* D_Y O_Y \\ &\overset{\text{(a)}}{=} f^*(\cdot) \otimes^L_{O_X} \omega_f[n] \overset{\text{(b)}}{=} f^*(\cdot) \otimes_{O_X} \omega_f[n] \end{aligned}$$

of functors $D_c^-(Y) \to D_c^-(X)$, where (a) (resp. (b)) equality uses (29) (resp. local freeness of ω_f). ☐

Lemma 5.1.10. There is an isomorphism $Rp_{X*}(\mathcal{P}^{-1} \otimes^L p_{\hat{X}}^* \cdot) = [-1]_X^* RS$ of functors $D(\hat{X}) \to D(X)$.

Proof. By [BL04, Cor. A.9], one has $\mathcal{P}^{-1} \xrightarrow{\sim} ([-1]_X \times [1]_{\hat{X}})^* \mathcal{P}$. Since $p_{\hat{X}} \circ ([-1]_X \times [1]_{\hat{X}}) = p_{\hat{X}}$, there are isomorphisms

$$\begin{aligned} &Rp_{X*}(\mathcal{P}^{-1} \otimes^L p_{\hat{X}}^* \cdot) \xrightarrow{\sim} Rp_{X*}([-1]_X \times [1]_{\hat{X}})^*(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) \\ \xleftarrow{\sim} &[-1]_X^* Rp_{X,*}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) = [-1]_X^* RS \end{aligned}$$

of functors $D(\hat{X}) \to D(X)$. ☐

Proof of Proposition 5.1.6. By Fact 5.1.5 1 and 3, There are isomorphisms

$$D_X \circ RS = D_X Rp_{X,*}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) \xrightarrow{\sim} Rp_{X,*} D_{X \times \hat{X}}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot)$$

of functors $D_c^+(\hat{X}) \to D_c^-(X)$. From Lemma 5.1.8, there is an isomorphism $D_{X \times \hat{X}}(\mathcal{P} \otimes^L p_{\hat{X}}^* \cdot) \xrightarrow{\sim} \mathcal{P}^{-1} \otimes^L D_{X \times \hat{X}} p_{\hat{X}}^* \cdot$ of functors $D(\hat{X}) \to D(X \times \hat{X})$. By Fact 5.1.5 1, the functor $D_{\hat{X}}$ restricts to a functor $D_c^+(\hat{X}) \to D_c^-(\hat{X})$, whence

Corollary 5.1.9 yields an isomorphism $D_{X\times\hat{X}}p_{\hat{X}}^* = (p_{\hat{X}}^* D_{\hat{X}}\cdot)[g]$ of functors $D_c^+(\hat{X}) \to D_c^-(X\times\hat{X})$. Therefore, there are isomorphisms

$$D_X \circ RS \xrightarrow{\sim} Rp_{X,*}(\mathcal{P}^{-1} \otimes^L p_{\hat{X}}^* D_{\hat{X}}\cdot)[g] \overset{\text{(a)}}{\xrightarrow{\sim}} [-1]_X^* RS(D_{\hat{X}}\cdot)[g]$$

of functors $D_c^+(\hat{X}) \to D_c^-(X)$, where (a) uses Lemma 5.1.10.

The second isomorphism follows from the first by swapping X and $\hat{X}$. $\square$

5.2 Unipotent vector bundles

Definition 5.2.1 ([Muk81, Def. 2.3])**.** We say that W.I.T. (weak index theorem) holds for a coherent module F on the complex torus X if there is an integer $i(F)$ such that $H^i R\hat{S}(F) = 0$ for every integer $i \neq i(F)$. In that case, the integer $i(F)$ is called the *index* of F and the coherent module $\hat{F} := H^{i(F)} R\hat{S}(F)$ on $\hat{X}$ is called the Fourier transform of F. We say that I.T. (index theorem) holds for F if there is an integer i_0 such that for every $L \in \mathrm{Pic}^0(X)$ and every integer $i \neq i_0$, one has $H^i(X, F \otimes_{O_X} L) = 0$.

Definition 5.2.2. A vector bundle U on a complex analytic space M is called unipotent if it has a filtration by vector subbundles

$$0 = U_0 \subset U_1 \subset \cdots \subset U_{n-1} \subset U_n = U$$

such that $U_i/U_{i-1} \cong O_M$ for all $1 \leq i \leq n$. Denote the full subcategory of $\mathrm{Coh}(M)$ consisting of unipotent vector bundles by $\mathrm{Uni}(M)$.

Proposition 5.2.3. *1. W.I.T. with index g holds for every unipotent vector bundle on X.*

2. The functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to an equivalence $\mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$, with a quasi-inverse $H^0 RS = RS : \mathrm{Coh}_0(\hat{X}) \to \mathrm{Uni}(X)$.

Proof. 1. Because $R\hat{S}$ is a triangulated functor, the full subcategory of $\mathrm{Coh}(X)$ comprised of modules satisfying W.I.T. of a fixed index is closed under extensions. By Lemma 2.0.8 and Theorem 4.1.1, one has $R\hat{S}(O_X) = R\hat{S}RS(\mathbb{C}_0) \xrightarrow{\sim} \mathbb{C}_0[-g]$. Then W.I.T. with index g holds for O_X, so it holds for every unipotent vector bundle on X.

2. By Point 1, one has an isomorphism of functors $H^g R\hat{S} \xrightarrow{\sim} R\hat{S}[g] : \mathrm{Uni}(X) \to \mathrm{Mod}(O_{\hat{X}})$. The full subcategory of $\mathrm{Mod}(O_X)$ comprised of modules F with $\mathrm{Supp}\big(H^g R\hat{S}(F)\big) \subset \{0\}$ is closed under extensions and contains O_X, so it contains Uni_X. Since $\mathrm{Uni}(X) \subset \mathrm{Coh}(X)$, the functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $\mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$.

For every $F \in \mathrm{Coh}_0(\hat{X})$, the restriction $\mathrm{Supp}(p_{\hat{X}}^* F \otimes \mathcal{P}) \to X$ of p_X is finite. By [GR04, Thm. 4, p.47], one has $RS(F) = H^0 RS(F)$. By

Lemma 5.2.4 3, the $O_{\hat{X}}$-module F has a filtration with successive quotients isomorphic to $\mathbb{C}_0$. Then $RS(F)$ has a filtration with successive quotients isomorphic to $RS(\mathbb{C}_0) = O_X$. By [Gro60, Ch. 0, 5.4.9], every term of this filtration is finite locally free. Therefore, $RS(F) \in \mathrm{Uni}(X)$ and RS restricts to a functor $\mathrm{Coh}_0(\hat{X}) \to \mathrm{Uni}(X)$. By Theorem 4.1.1, the functor $H^g R\hat{S} : \mathrm{Uni}(X) \to \mathrm{Coh}_0(\hat{X})$ is an equivalence with a quasi-inverse RS.

□

For a commutative ring R, let $\mathrm{Mod}_f(R) \subset \mathrm{Mod}(R)$ be the full subcategory comprised of R-modules of finite length. Lemma 5.2.4 1 confirms a guess in [Gro61a, 9-12] for complex field.

Lemma 5.2.4. Let X be a complex analytic space. Let $x \in X$.

1. The functor $i_x^{-1} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{X,x})$ taking the stalk at x restricts to a functor $\mathrm{Coh}_x(X) \to \mathrm{Mod}_f(O_{X,x})$. In particular, if X is a singleton, then $\dim_{\mathbb{C}} O_X$ is finite.
2. The functor $i_{x,*} : D(O_{X,x}) \to D(O_X)$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(X)$.
3. The functor $i_x^{-1} : \mathrm{Coh}_x(X) \to \mathrm{Mod}_f(O_{X,x})$ is an equivalence.

Proof.

1. For every $F \in \mathrm{Coh}_x(X)$, to prove that F_x is a finite length $O_{X,x}$-module, one may assume that $F_x \neq 0$. As F is a finite type O_X-module, F_x is a finite $O_{X,x}$-module. Then $\mathrm{Supp}_{O_{X,x}}(F_x)$ is nonempty. Let m_x be the maximal ideal of $O_{X,x}$. For every $f \in m_x$, there is an open neighborhood U of $x \in X$ such that f is the stalk of some $\bar{f} \in O_X(U)$. Then $\bar{f}$ vanishes on $\mathrm{Supp}(F)$. By the Rückert Nullstellensatz (see, *e.g.*, [GR84, p.67]), there is an integer $n \geq 1$ such that $\bar{f}^n F = 0$ near x. In particular, $f \in \sqrt{\mathrm{Ann}_{O_{X,x}}(F_x)}$. Therefore,

   $$m_x \subset \sqrt{\mathrm{Ann}_{O_{X,x}}(F_x)}.$$

   By [GR84, Corollary, p.44], the ideal m_x is finitely generated, so there is an integer $N \geq 1$ with $m_x^N \subset \mathrm{Ann}_{O_{X,x}}(F_x)$. By [Sta23, Tag 00L6], $\mathrm{Supp}_{O_{X,x}}(F_x)$ is the unique closed point of $\mathrm{Spec}(O_{X,x})$. By [Sta23, Tag 00L5], the $O_{X,x}$-module F_x has finite length. The second statement follows from Lemma 5.2.5.

2. Up to isomorphism, the only simple $O_{X,x}$-module is the residue field $\mathbb{C}$. Every $M \in \mathrm{Mod}_f(O_{X,x})$ has a composite series with successive quotients isomorphic to $\mathbb{C}$. Thus, M_x has a filtration with successive quotients isomorphic to $\mathbb{C}_x$. Since $\mathbb{C}_x$ is coherent, by [Sta23, Tag 01BY (4)], M_x is coherent. Therefore, $i_{x,*}$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(X)$.

3. Let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. There is a canonical isomorphism $i_x^*(i_x)_* \xrightarrow{\sim} \mathrm{Id}_{\mathrm{Mod}(O_{X,x})}$ of functors

$\mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_{X,x})$. By adjunction, $(i_x)_* : \mathrm{Mod}(O_{X,x}) \to \mathrm{Mod}(O_X)$ is fully faithful. By Point 2, pushout $(i_x)_*$ restricts to a functor $\mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(O_X)$. For every object F of $\mathrm{Coh}_x(O_X)$, by Point 1, F_x is an object of $\mathrm{Mod}_f(O_{X,x})$. The adjunction morphism $F \to (i_x)_*(F_x)$ is an isomorphism. Thus, $(i_x)_* : \mathrm{Mod}_f(O_{X,x}) \to \mathrm{Coh}_x(O_X)$ is essentially surjective and hence an equivalence. Therefore, the functor $i_x^* : \mathrm{Coh}_x(O_X) \to \mathrm{Mod}_f(O_{X,x})$ (taking the stalk at x) is an equivalence.

□

Lemma 5.2.5. Let $F \to A$ be a ring map, with F a field and (A, m) an Artinian local ring. If $\dim_F A/m$ is finite, then $\dim_F A$ is finite.

Proof. Because A is an Artinian local ring ring, by [Ati69, Prop. 8.4], there is an integer $n > 0$ with $m^n = 0$. For every integer $i \geq 0$, the A-module m^i is finitely generated, so the A/m-module m^i/m^{i+1} is finitely generated. Thus, $\dim_F m^i/m^{i+1} = \dim_F A/m \cdot \dim_{A/m} m^i/m^{i+1}$ is finite. Then $\dim_F A = \sum_{i=0}^{n} \dim_F m^i/m^{i+1}$ is finite. □

5.3 Homogeneous vector bundles

Definition 5.3.1. A vector bundle E on the complex torus X is called homogeneous if for every $x \in X$, one has $T_x^* E \cong E$. Let $H(X) \subset \mathrm{Coh}(X)$ be the full subcategory comprised of homogeneous vector bundles.

For a complex analytic space M, let $\mathrm{Coh}_f(M) \subset \mathrm{Coh}(M)$ be the full subcategory consisting of objects with finite support.

Proposition 5.3.2.

1. *For every integer i, the functor $H^i R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \mathrm{Coh}_f(\hat{X})$.*
2. *W.I.T. holds for every homogeneous vector bundle on X with index g.*
3. *The functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to an equivalence of categories $H(X) \to \mathrm{Coh}_f(\hat{X})$.*

Proof. 1. Let E be a homogeneous vector bundle on X. By Corollary 3.1.14, the $O_{\hat{X}}$-module $H^i R\hat{S}(E)$ is coherent. For every $x \in X$, by Proposition 5.1.1, one has $R\hat{S}(E) \xrightarrow{\sim} R\hat{S}(T_{-x}^* E) \xrightarrow{\sim} P_x^* \otimes R\hat{S}(E)$, so $H^i R\hat{S}(E) \xrightarrow{\sim} P_x^* \otimes H^i R\hat{S}(E)$. From Lemma 5.3.4, the support of $H^i R\hat{S}(E)$ is finite.

2. For every integer $i \neq g$, by Point 1, one has $H^i R\hat{S}(E) \in \mathrm{Coh}_f(\hat{X})$ and

$$\begin{aligned}
0 =& H^{i-g}([-1]_X^* E) \\
=& H^i([-1]_X^* E[-g]) \\
\overset{(a)}{\xrightarrow{\sim}}& H^i RS \circ R\hat{S}(E) \\
=& H^i Rp_{X*}(\mathcal{P} \otimes^L p_{\hat{X}}^* R\hat{S}(E)) \\
\overset{(b)}{\xrightarrow{\sim}}& H^0 Rp_{X*}(\mathcal{P} \otimes^L p_{\hat{X}}^* H^i R\hat{S}(E)) \\
=& H^0 RS(H^i R\hat{S}(E)),
\end{aligned}$$

   where (a) (resp. (b)) uses Theorem 4.1.1 (resp. [GR04, Thm. 4, p.47]).

   It remains to prove that for every $F \in \mathrm{Coh}_f(\hat{X})$ with $H^0 RS(F) = 0$, one has $F = 0$. Since F is the direct sum of finitely many coherent submodules whose supports are singletons, one may assume that $\mathrm{Supp}(F)$ is a singleton. By Proposition 5.1.1, one may assume that $F \in \mathrm{Coh}_0(\hat{X})$. From Proposition 5.2.3 2, one has $F = 0$.

3. By Point 1, the functor $H^g R\hat{S} : \mathrm{Mod}(O_X) \to \mathrm{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \mathrm{Coh}_f(\hat{X})$. From Point 2, one has an isomorphism of functors $H^g R\hat{S} \cong R\hat{S}[g] : H(X) \to \mathrm{Coh}_f(\hat{X})$.

   By Propositions 5.1.1 and 5.2.3, the functor $H^0 RS : \mathrm{Mod}(O_{\hat{X}}) \to \mathrm{Mod}(O_X)$ restricts to a functor $H^0 RS = RS : \mathrm{Coh}_f(\hat{X}) \to H(X)$. By Theorem 4.1.1, the functor $H^g R\hat{S} : H(X) \to \mathrm{Coh}_f(\hat{X})$ is an equivalence with a quasi-inverse $H^0 RS$.

□

For a sheaf of module F on a complex analytic space, denote the torsion part of F (in the sense of [CD94, p.60]) by $T(F)$.

Lemma 5.3.3. Let X be a compact Kähler manifold. Let F be a coherent O_X-module. Then for every irreducible component $C \subset \mathrm{Supp}(F)$, there is a connected compact Kähler manifold Z and a morphism $h : Z \to X$, such that $h(Z) = C$ and $h^*F/T(h^*F)$ is a vector bundle on Z of positive rank.

Proof. By [GR84, p.76], $\mathrm{Supp}(F)$ is an analytic subset of X. Because X is a Kähler manifold, with the induced reduced complex structure, the subspace C is a Kähler space in the sense of [Var89, II, 1.3]. Let $i : C \to X$ be the inclusion. Set

$$D = \{x \in C : i^*F \text{ is not locally free at } x\}.$$

From [Ros68, Prop. 3.1], D is a strict analytic subset of C. By Rossi's theorem (see, *e.g.* [Rie71, Thm. 2]), there is a reduced irreducible complex analytic space W and a proper modification $f : W \to C$, such that $W \setminus f^{-1}(D) \to C \setminus D$ is biholomorphic and $E := N/T(N)$ is a *vector bundle* on W, where $N = f^*i^*F$.

From [GD71, Cor. 5.2.4.1], one has $\mathrm{Supp}(N) = W$. From [CD94, I, Thm. 9.12], one gets $\mathrm{Supp}(T(N)) \neq W$. Therefore, the rank r of the vector bundle E is positive.

Since $f : W \to C$ is bimeromorphic, the space W is in the Fujiki class $\mathcal{C}$ (defined in [Fuj78, p.34]). By [Fuj78, Lem. 4.6, 1)], there is a connected compact Kähler manifold Z with a surjective morphism $g : Z \to W$. Denote the composition $Z \overset{g}{\to} W \overset{f}{\to} C \overset{i}{\to} X$ by h. Then $h(Z) = C$. As E is flat over O_W, by [Sta23, Tag 05NJ], applying g^* to the natural short exact sequence

$$0 \to T(N) \to N \to E \to 0$$

in $\mathrm{Mod}(O_W)$, one gets a short exact sequence in $\mathrm{Mod}(O_Z)$:

$$0 \to g^*T(N) \to h^*F \to g^*E \to 0.$$

As g^*E is torsion free, $g^*T(N) \supset T(h^*F)$. One has $g^*T(N) \subset T(g^*N) = T(h^*F)$. Therefore, $T(h^*F) = g^*T(N)$ and $h^*F/T(h^*F) = g^*E$ is a vector bundle on Z of rank $r > 0$. □

Lemma 5.3.4. *Let M be a coherent sheaf on the complex torus X. If $M \otimes P \cong M$ for all $P \in \mathrm{Pic}^0(X)$, then $\mathrm{Supp}(M)$ is finite.*

Proof. Suppose the contrary that $\mathrm{Supp}(M)$ is infinite. With the reduced induced complex structure, the complex subspace $\mathrm{Supp}(M)$ has positive dimension. Let C be an irreducible component of $\mathrm{Supp}(M)$ of maximal dimension. Take a morphism $h : Z \to X$ provided by Lemma 5.3.3. Then the rank r of the vector bundle $E := h^*M/T(h^*M)$ is positive. As $h(Z) = C$, the morphism of complex tori $h^* : \mathrm{Pic}^0(X) \to \mathrm{Pic}^0(Z)$ is nonzero. In particular, there is $L \in \mathrm{Pic}^0(X)$ such that the line bundle $(h^*L)^{\otimes r}$ is nontrivial.

On the other hand, we claim that the line bundle $(h^*L)^{\otimes r}$ is trivial. Indeed, by assumption $M \otimes L \cong M$, so $h^*M \otimes h^*L \cong h^*M$. Since $T(h^*M \otimes h^*L) = T(h^*M) \otimes h^*L$, one gets $E \otimes h^*L \cong E$. Taking the determinant of both sides, one has $\det(E) \otimes (h^*L)^{\otimes r} \cong \det(E)$. As $\det(E)$ is an invertible sheaf, the line bundle $(h^*L)^{\otimes r}$ on Z is trivial. The claim is proved, which gives a contradiction. □

Remark 5.3.5. The proof of [Muk81, Lem. 3.3] (the algebraic counterpart of Lemma 5.3.4) relies on the following fact: Every positive dimensional projective variety contains a projective curve. By contrast, every simple non-algebraic complex torus contains no 1-dimensional analytic subset ([Pil00, Lem. 4.3]).

The classification of homogeneous vector bundles on complex tori is due to Matsushima [Mat59] and Morimoto [Mor59]. Using the Fourier-Mukai transform, Mukai [Muk81, p.159] proves an analog for abelian varieties. We can similarly recover Matsushima-Morimoto's theorem.

Theorem 5.3.6. *A vector bundle F on the complex torus X is homogeneous if and only if there is an integer $n \geq 0$, unipotent vector bundles $U_1, \ldots, U_n$ on X and $P_1, \ldots, P_n \in \mathrm{Pic}^0(X)$, such that F is isomorphic to $\oplus_{i=1}^n P_i \otimes U_i$.*

Proof. It follows from Propositions 5.1.1, 5.2.3 2 and 5.3.2 3. □

A Sheaves of modules

We recall some facts about sheaves of modules. Let (X, O_X) be a ringed space.

A.1 Generalities

Definition A.1.1. An O_X-module F is called

1. ([Sta23, Tag 01B5]) of *finite type* if every $x \in X$ admits an open neighborhood U such that $F|_U$ is generated by finitely many sections;
2. ([Sta23, Tag 01BN]) of *finite presentation* if for every $x \in X$, there is an open neighborhood $U \subset X$, integers $n, m \geq 0$ and an exact sequence of O_U-modules
$$O_U^m \to O_U^n \to F|_U \to 0;$$
3. ([Gro60, 5.1.3]) *quasi-coherent* if for every $x \in X$, there is an open neighborhood $U \subset X$, two sets I, J and a morphism $O_U^{\oplus J} \to O_U^{\oplus I}$ whose cokernel is isomorphic to $F|_U$;
4. ([Kas03, Def. A.5 (1)]) *pseudo-coherent* if for every open subset $U \subset X$, every finite type O_U-submodule of $F|_U$ is of finite presentation. Let $\mathrm{PCoh}(X) \subset \mathrm{Mod}(O_X)$ be full subcategory of pseudo-coherent modules;
5. ([Kas03, Def. A.5 (2)]) *K-coherent* if F is pseudo-coherent and of finite type;
6. ([Sta23, Tag 01BV]) *coherent* if F is of finite type and for every open subset $U \subset X$ and every finite collection $\{s_i\}_{1 \leq i \leq n}$ in $F(U)$, the kernel of the associated morphism $O_U^n \to F|_U$ is of finite type over O_U.

Every property in Definition A.1.1 is local, in the sense that it restricts to every open subset, and if it holds on each member of an open covering of X, then it holds on X.

Lemma A.1.2. Let $0 \to F \xrightarrow{i} G \xrightarrow{r} H \to 0$ be a short exact sequence in $\mathrm{Mod}(O_X)$. If F, H are of finite presentation, then so is G.

Proof. For every $x \in X$, by [Sta23, Tag 01B8], there is an open neighborhood U of x such that the sequence $G(U) \xrightarrow{r_U} H(U) \to 0$ is exact. Up to shrinking U, there exist integers $m, n, p, q \geq 0$ and two exact sequences

$$O_U^m \to O_U^n \xrightarrow{f} F|_U \to 0, \quad O_U^p \to O_U^q \xrightarrow{h} H|_U \to 0.$$

The morphism h is defined by q elements $s_1, \dots, s_q$ of $H(U)$. For each $1 \leq i \leq q$, choose a preimage $t_i \in G(U)$ of s_i. Consider the morphism $\phi : O_U^{n+q} \to G|_U$ determined by $if(e_1), \dots, if(e_n), t_1, \dots, t_q \in G(U)$. Hence a commutative diagram with two exact middle rows

$$\begin{tikzcd}
0 \arrow[r] & O_U^m \arrow[r] \arrow[d] & \ker(\phi) \arrow[r] \arrow[d] & O_U^p \arrow[d] \arrow[dddll, dashed] & \\
0 \arrow[r] & O_U^n \arrow[r] \arrow[d, "f"] & O_U^{n+q} \arrow[r] \arrow[d, "\phi"] & O_U^q \arrow[r] \arrow[d, "g"] & 0 \\
0 \arrow[r] & F|_U \arrow[r] \arrow[d] & G|_U \arrow[r] \arrow[d] & H|_U \arrow[r] \arrow[d] & 0 \\
 & 0 \arrow[r] & \operatorname{coker}(\phi) \arrow[r] & 0. &
\end{tikzcd}$$

By the snake lemma, ϕ is surjective and $\ker(\phi)$ is finite type. Shrinking U again, one may find an integer $k \geq 0$ and a surjection $O_U^k \to \ker(\phi)$. The induced sequence $O_U^k \to O_U^{n+q} \to G|_U \to 0$ is exact. Therefore, G is of finite presentation. ☐

A.2 Pseudo-coherent modules

Lemma A.2.1.

1. Let $0 \to F \xrightarrow{i} G \xrightarrow{r} H \to 0$ be a short exact sequence in $\mathrm{Mod}(O_X)$. If F, H are pseudo-coherent, then so is G.

2. Let I be a directed set. Let (M_i, f_{ij}) be a direct system over I consisting of pseudo-coherent O_X-modules. Then $M := \mathrm{colim}_{i\in I} M_i$ in $\mathrm{Mod}(O_X)$ is pseudo-coherent.

3. If $\{M_\alpha\}_{\alpha\in A}$ is a family of pseudo-coherent O_X-modules, then $S := \oplus_{\alpha\in A} M_\alpha$ is also pseudo-coherent.

Proof. Let U be an open subset of X.

1. Let M be a finite type submodule of $G|_U$. Then the kernel of $r|_M : M \to H|_U$ is $(F|_U) \cap M$. Thus, $r|_M$ induces an injection $M/(F|_U \cap M) \to H|_U$. As H is pseudo-coherent, the finite type O_U-submodule $M/(F|_U \cap M)$ is of finite presentation. By [Sta23, Tag 01BP (2)], $F|_U \cap M$ is of finite type. As F is pseudo-coherent, $F|_U \cap M$ is of finite presentation. By Lemma A.1.2 applied to the exact sequence $0 \to F|_U \cap M \to M \to M/(F|_U \cap M) \to 0$, the O_U-module M is of finite presentation. Thus, G is pseudo-coherent.

2. Let N be a finite type submodule of $M|_U$. For every $x \in U$, from the first three lines of the proof of [Sta23, Tag 01BB], there is an open neighborhood $V \subset U$ of x and $i \in I$ such that $N|_V \subset F_i|_V$. Since F_i is pseudo-coherent, $N|_V$ is of finite presentation. As finite presentation is a local property, N is of finite presentation. Thus, M is pseudo-coherent.

3. Let I be the set of all finite subsets of A with the inclusion order. Then I is a directed set. For $B \in I$, set $F_B = \oplus_{\alpha\in B} M_\alpha$. By Point 1, F_B is pseudo-coherent. For $B \leq B'$ in I, set $f_{B,B'} : F_B \to F_{B'}$ to be the inclusion.

Hence a direct system $(F_B, f_{B,B'})$ over I. By Point 2, the O_X-module $S = \mathrm{colim}_{B \in I} F_B$ is pseudo-coherent.

$\square$

Lemma A.2.2. An O_X-module is K-coherent if and only if it is coherent.

Proof. Let $U \subset X$ be an open subset. Assume that F is a K-coherent module. Let $\{s_i\}_{1 \le i \le n}$ be a finite collection in $F(U)$, and let $f : O_U^n \to F|_U$ be the associated morphism. Then $\operatorname{im} f$ is a finite type submodule of $F|_U$. Because F is pseudo-coherent, $\operatorname{im} f$ is of finite presentation over O_U. From [Sta23, Tag 01BP (2)], $\ker f$ is of finite type over O_U. Therefore, F is coherent.

Conversely, assume that F is a coherent O_X-module. Let M be a finite type submodule of $F|_U$. By [Sta23, Tag 01BY (1)], M is coherent over O_U. From [Sta23, Tag 01BW], M is of finite presentation. Thus, F is pseudo-coherent and hence K-coherent. $\square$

The module O_X is quasi-coherent, but in general not pseudo-coherent. If it is pseudo-coherent, then O_X is called a coherent sheaf of rings ([Kas03, p.214], [Bjö93, A:II, Def. 6.29]).

Lemma A.2.3. If X is a locally Noetherian scheme, then every quasi-coherent module is pseudo-coherent.

Proof. By [Gro60, Cor. 9.4.9], a quasi-coherent module is a directed limit of coherent modules, hence pseudo-coherent by Lemma A.2.1 2. $\square$

Example A.2.4. Let $X = \mathbf{A}^1$ be the affine line over a field. Let $U = X \setminus \{0\}$, and let $j : U \to X$ be the inclusion. By [Har77, II, Example 5.2.3], the O_X-module $j_! O_U$ is not quasi-coherent. From [Har77, II, Exercise 1.19 (c)], it is a submodule of the coherent module O_X. Hence, $j_! O_U$ is pseudo-coherent.

Definition A.2.5 defines a local property. It is weaker than [Bjö93, A:III, 2.24] and [Kas03, Def. A.7].

Definition A.2.5. Assume that O_X is a coherent sheaf of rings. If for every open subset $U \subset X$, every family of coherent ideal sheaves $\{I_i\}_i$ in O_U, the ideal sheaf $\sum_i I_i$ is O_U-coherent, then O_X is called a quasi-Noetherian sheaf of rings.

Example A.2.6.

1. If (X, O_X) is a locally Noetherian scheme, then O_X is quasi-Noetherian.
2. If (X, O_X) is a complex analytic space, then by the Oka-Cartan theorem (see, *e.g.*, [Kas03, Thm. A.12]), O_X is quasi-Noetherian.

A.3 Analytic coherent modules

Let X be a complex analytic space. We show that a coherent O_X-module admits a local free resolution, from which we deduce that coherence is preserved by derived pullbacks and tensor products. An analog of Lemma A.3.1 for algebraic varieties is [Har77, III, Example 6.5.1]. By local syzygies [GH78, p.696], on complex manifolds, every coherent module local admits a finite-length, finite free resolution.

Lemma A.3.1. Every $x \in X$ admits an open neighborhood U, such that for every coherent O_X-module F, there is a (possibly infinite-length) resolution

$$\cdots \to O_U^{n_1} \to O_U^{n_0} \to F|_U \to 0,$$

where $n_i \geq 0$ are integers.

Proof. Shrinking X to an open neighborhood of x, one may assume that X is Stein. By [GR04, Thm. 8, p.108], there is a compact neighborhood $K \subset X$ of x, such that Theorem B is valid on K in the sense of [GR04, Def. 1, p.100]. Let $U = K^\circ$.

For a coherent O_X-module F, we construct inductively a sequence of morphisms. From [GR04, Cor. p.101], there is an integer $n_0 \geq 0$, an open neighborhood U_0 of $K \subset X$ and a morphism $f_0 : O_{U_0}^{n_0} \to F|_{U_0}$ in $\mathrm{Mod}(O_{U_0})$ such that $f_0|_U$ is an epimorphism in $\mathrm{Mod}(O_U)$. Set $\ker(f_{-1})|_{U_0} = F|_{U_0}$. Given such a morphism $f_j : O_{U_j}^{n_j} \to \ker(f_{j-1})|_{U_j}$ for an integer $j \geq 0$ and an open neighborhood $U_j \subset X$ of K, by [Sta23, Tag 01BY (3)], the O_{U_j}-module $\ker(f_j)$ is coherent. By [GR04, Cor. p.101], there is an open neighborhood $U_{j+1} \subset U_j$ of K, an integer $n_{j+1} \geq 0$ and a morphism $f_{j+1} : O_{U_{j+1}}^{n_{j+1}} \to \ker(f_j)|_{U_{j+1}}$ in $\mathrm{Mod}(O_{U_{j+1}})$ such that $f_{j+1}|_U$ is an epimorphism. Thus, one gets a sequence

$$\cdots \to O_U^{n_2} \overset{f_2|_U}{\to} O_U^{n_1} \overset{f_1|_U}{\to} O_U^{n_0} \overset{f_0|_U}{\to} F|_U \to 0$$

in $\mathrm{Mod}(O_U)$. By construction, it is exact, hence a resolution of $F|_U$. $\square$

Example A.3.2. Assume that $x \in X$ is a singular point. Then $F := \mathbb{C}_x$ is a coherent O_X-module, but for every open neighborhood $U \subset X$ of x, there is no finite-length resolution of $F|_U$ by finite locally free O_U-modules. (Otherwise, such a resolution induces a finite-length free resolution of the $O_{X,x}$-module $F_x = \mathbb{C} = O_{X,x}/m_x$. From [Osb12, Ch. 4, Prop. 4.4], the projective dimension $\mathrm{pd}_{O_{X,x}} O_{X,x}/m_x$ is finite. By [Mat87, Lem. 1, p.154] and [Osb12, Prop. 4.9], the global dimension of the ring $O_{X,x}$ is finite. By Serre's theorem (see, *e.g.*, [Osb12, p.332]), the local ring $O_{X,x}$ is regular. From [Ser56, p.6], x is a smooth point of X, a contradiction.)

Therefore, Lemma A.3.1 fails if one consider only finite-length resolutions. See also [EP$^+$96, Thm. 4.1.2].

Lemma A.3.3. Let $f : X \to Y$ be a morphism of complex analytic spaces. Then for every coherent O_Y-module F, the derived pullback $Lf^*F \in D_c(X)$.

Proof. For every $x \in X$, by Lemma A.3.1, there is an open neighborhood V of $f(x) \in Y$, such that there is a resolution $E_\bullet \to F|_V \to 0$ by finite free O_V-modules. Let $g : f^{-1}(V) \to V$ be the base change of f along the inclusion $V \to Y$. Then the morphism $g^*E_\bullet \to (Lf^*F)|_{f^{-1}(V)}$ in $D(f^{-1}(V))$ is an isomorphism. For every integer $j \geq 0$, the $O_{f^{-1}(V)}$-module g^*E_j is finite free. Thus, the $O_{f^{-1}(V)}$-module $(H^{-j}Lf^*F)|_{f^{-1}(V)}$ is coherent. Since coherence is a local property, the O_X-module $H^{-j}(Lf^*F)$ is coherent. ☐

Lemma A.3.4. For any coherent O_X-modules F and G, one has $F \otimes^L_{O_X} G \in D_c(X)$.

Proof. For every $x \in X$, by Lemma A.3.1, there is an open neighborhood $U \subset X$ of x and a resolution $E_\bullet \to F|_U \to 0$ by finite free O_U-modules. The natural morphism $E_\bullet \otimes_{O_U} G|_U \to F|_U \otimes^L_{O_U} G|_U$ in $D(U)$ is an isomorphism. For every integer n, the O_U-module $H^n(E_\bullet \otimes^L_{O_U} G|_U) = H^n(E_\bullet \otimes_{O_U} G|_U)$ is coherent. Therefore, the O_U-module $H^n(F \otimes^L_{O_X} G)|_U = H^n(F|_U \otimes^L_{O_U} G|_U)$ is coherent. Since coherence is a local property, the O_X-module $H^n(F \otimes^L_{O_X} G)$ is coherent. ☐

A.4 Good modules

Assume that the ringed space X is locally compact Hausdorff.

Definition A.4.1. [Kas03, Def. 4.22] An O_X-module F is called *good* if for every relatively compact open subset $U \subset X$, there exists a directed family $\{G_i\}_{i\in I}$ of coherent O_U-submodules of $F|_U$ such that $F|_U = \sum_{i\in I} G_i$, where $\{G_i\}_{i\in I}$ being a directed family means that for any $i, i' \in I$, there is $i'' \in I$ with $G_i + G_{i'} \subset G_{i''}$ (and hence $F|_U = \mathrm{colim}_{i\in I} G_i$). The full subcategory of $\mathrm{Mod}(O_X)$ consisting of good O_X-modules is denoted by $\mathrm{Good}(X)$.

Lemma A.4.2 (Goodness *vs.* pseudo-coherence)**.**

1. ([Kas03, p.77]) One has $\mathrm{Coh}(X) \subset \mathrm{Good}(X) \subset \mathrm{PCoh}(X)$.
2. Let E be a pseudo-coherent O_X-module. If on every relatively compact open subset $U \subset X$, the O_U-module $E|_U$ is the sum of its finite type submodules, then E is good.

Proof.

1. By definition, every coherent O_X-module is good. Let E be a good O_X-module. Let W be an open subset of X, and let $F \subset E|_W$ be a finite type O_W-submodule. We show that F is of finite presentation over O_W. Replacing (X, E) with $(W, E|_W)$, one may assume that $W = X$. Because X is locally compact, for every $x \in X$, there exists a *relatively compact* open neighborhood $U \subset X$ of x and finitely many sections $s_1, \ldots, s_n \in F(U)$ generating $F|_U$. As E is good, $E|_U = \sum_{i\in I} G_i$ is the sum of a directed family of coherent submodules. There exists $i_0 \in I$ and an open

neighborhood V of $x \in U$ with $s_i|_V \in G_{i_0}(V)$ for all $1 \leq i \leq n$. Then $F|_V$ is a finite type submodule of $G_{i_0}|_V$. By [Sta23, Tag 01BY (1)], $F|_V$ is O_V-coherent. As coherence is a local property, F is coherent. From [Sta23, Tag 01BW], F is of finite presentation.

2. The family of finite type submodules of $E|_U$ is directed, since the sum of two finite type submodules is of finite type. For every relatively compact open subset $U \subset X$, as E is pseudo-coherent, every finite type submodule of $E|_U$ is pseudo-coherent and hence coherent. Thus, E is good.

□

Basic properties of good modules (similar to those of quasi-coherent modules on algebraic varieties) are recapped in Lemma A.4.3.

Lemma A.4.3.

1. For every family of objects $\{F_i\}_{i \in I}$ in $\mathrm{Good}(X)$, the direct sum $\oplus_{i \in I} F_i$ in $\mathrm{Mod}(O_X)$ is good.
2. The subcategory $D_{\mathrm{gd}}(X)$ is closed under direct sums in $D(X)$. Moreover, the inclusion functor $\mathrm{Good}(X) \to D_{\mathrm{gd}}(X)$ commutes with direct sums.

Suppose that O_X is quasi-Noetherian. Then:

3. The subcategory $\mathrm{Good}(X) \subset \mathrm{Mod}(O_X)$ is weak Serre and closed under filtered colimits in $\mathrm{Mod}(O_X)$. In particular, $\mathrm{Good}(X)$ is a locally Noetherian category (in the sense of [Gab62, p.356]).
4. The inclusion functor $D_{\mathrm{gd}}(X) \to D(X)$ is a triangulated subcategory.

Proof.

1. Over every relatively compact open subset U of X, the direct sum $(\oplus_{i \in I} F_i)|_U$ is the sum of its coherent O_U-submodules. By Lemma A.2.1 3, the O_X-module $\oplus_{i \in I} F_i$ is pseudo-coherent. By Lemma A.4.2 2, it is good.
2. Since $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category, by [Sta23, Tag 07D9], the category $D(X)$ has arbitrary direct sums and they are computed by taking termwise direct sums of any representative complexes. Then by [Wei95, Exercise 1.2.1], for every integer q, the functor $H^q : D(X) \to \mathrm{Mod}(O_X)$ commutes with direct sums. The result follows from Point 1.
3. As O_X is quasi-Noetherian, by [Sta23, Tag 0754] and the proof of [Kas03, Prop. 4.23], $\mathrm{Good}(X)$ is a weak Serre subcategory of $\mathrm{Mod}(O_X)$. From [KS06, Thm. 18.1.6 (v)], the category $\mathrm{Mod}(O_X)$ is a Grothendieck abelian category. By Point 1 and [Sta23, Tag 002P], the filtered colimits in $\mathrm{Good}(X)$ exist and agree with the filtered colimits in $\mathrm{Mod}(O_X)$. Thus, filtered colimits in $\mathrm{Good}(X)$ are exact.

   Because of [Sta23, Tag 01BC], there is a set of coherent O_X-modules $\{F_i\}_{i \in I}$ such that each coherent O_X-module is isomorphic to exactly one of the F_i. Then $\{F_i\}$ is a family of Noetherian generators of $\mathrm{Good}(X)$. Therefore, the category $\mathrm{Good}(X)$ is locally Noetherian.

4. It follows from [Yek19, Prop. 7.4.5] and Point 3.

$\square$

Lemma A.4.4. A good module on a complex analytic space is quasi-coherent.

Proof. Let F be a good module on a complex analytic space X. From [Fri67, Thm. I, 9; Rem. I, 10], every $x \in X$ admits a neighborhood K that is a Noetherian Stein compactum. There is a relative compact open subset U of X containing K. As F is good, the O_U-module $F|_U = \sum_{i\in I} F_i$ is the sum of a directed family of coherent subsheaves. Applying the functor $\Gamma(K, \cdot)$ to the directed family $\{F_i\}_{i\in I}$ in $\mathrm{Coh}(U)$, by [Tay02, Prop. 11.9.2], one gets a directed family of finitely generated $\Gamma(K, O_K)$-submodule $\{M_i\}_{i\in I}$ of $\Gamma(K, F)$, whose associated family in $\mathrm{Mod}(O_K)$ is $\{F_i|_K\}_{i\in I}$. Let M be $\mathrm{colim}_{i\in I} M_i$ in $\mathrm{Mod}(\Gamma(K, O_K))$. Since the localization functor $\mathrm{Mod}(\Gamma(K, O_K)) \to \mathrm{Mod}(O_K)$ is left adjoint to $\Gamma(K, \cdot) : \mathrm{Mod}(O_K) \to \mathrm{Mod}(\Gamma(K, O_K))$, the localization preserves colimits. Then $F|_K$ is associated to M. By [Liu23, Lem. 2.5], F is quasi-coherent. $\square$

Remark A.4.5. The restriction of a good O_X-module to an open subset U is a good O_U-module. Unlike quasi-coherence on schemes, goodness is not a local property. In fact, by Lemma A.4.3 3, every free module on a complex manifold is good, while Gabber [Con06, Eg. 2.1.6] gives a locally free (hence quasi-coherent and pseudo-coherent), but not good module on the unit open disk in $\mathbb{C}$. (In particular, the converse of Lemma A.4.4 is wrong for noncompact complex manifolds.) Still, given an O_X-module F, if for *every* relatively compact open subset $U \subset X$, the O_U-module $F|_U$ is good, then F is good.

A.5 Sections of direct sum of sheaves

By [Har77, II, Exercise 1.11], on a Noetherian topological space, taking section commutes with (possibly infinite) direct sum of sheaves. This fails on complex manifolds, as Example A.5.1 shows.

Example A.5.1. Let $X = \mathbb{C}$. Let F be the O_X-module $\oplus_{n\geq 0}\mathbb{C}_n$. There is a section $s \in \Gamma(X, F^{\oplus\mathbb{N}})$, such that for every integer $n \geq 0$, the stalk $s_n \in (F^{\oplus\mathbb{N}})_n = (F_n)^{\oplus\mathbb{N}} = \mathbb{C}^{\oplus\mathbb{N}}$ is $(1, 1, \dots, 1, 0, 0, \dots)$, where the first $n+1$ entries are 1 and all the other entries are 0. Then s has no preimage under the canonical map $\Gamma(X, F)^{\oplus\mathbb{N}} \to \Gamma(X, F^{\oplus\mathbb{N}})$. For otherwise, let $(t^n)_{n\geq 0} \in \Gamma(X, F)^{\oplus\mathbb{N}}$ be a preimage of s. Then there are only finitely many integers $n \geq 0$ with $t^n \neq 0$. Every t^n has only finitely many nonzero stalks. However, s has infinitely many nonzero stalks, which is a contradiction.

Let X be a complex manifold. An O_X-module is called *privileged* if for every connected open subset $U \subset X$ and every $x \in U$, the map $\Gamma(U, F) \to F_x$ taking the stalk at x is injective. By the identity theorem (see, *e.g.*, [GH78, p.7]), O_X is privileged.

Lemma A.5.2. Assume that X is connected. Let $\{F_i\}_{i\in I}$ be a family of privileged O_X-modules. Then the canonical map $\oplus_{i\in I}\Gamma(X,F_i)\to\Gamma(X,\oplus_{i\in I}F_i)$ is bijective.

Proof. Let P be the presheaf direct sum of $\{F_i\}_{i\in I}$. Let $\theta : P\to\oplus_{i\in I}F_i$ be the sheafification morphism. Then $P(X)=\oplus_{i\in I}\Gamma(X,F_i)$ and $\theta_X : \oplus_{i\in I}\Gamma(X,F_i)\to\Gamma(X,\oplus_{i\in I}F_i)$ is the colimit of

$$\theta_X^{(J)} : \oplus_{i\in J}\Gamma(X,F_i)\to\Gamma(X,\oplus_{i\in I}F_i),$$

where J runs through the finite subsets of I. For every such J, by [Sta23, Tag 01AH (4)], the presheaf direct sum of $\{F_i\}_{i\in J}$ is a subsheaf of $\oplus_{i\in I}F_i$, so the map $\theta_X^{(J)}$ is injective. Therefore, their limit map θ_X is also injective. We prove that θ_X is surjective.

By construction of sheafification in [Har77, p.64], for every $s\in\Gamma(X,\oplus_{i\in I}F_i)$, there is a covering $\{U_\alpha\}_{\alpha\in A}$ of X by nonempty *connected* open subsets and an element $t_\alpha\in\Gamma(U_\alpha,P)$ for each $\alpha\in A$ such that $s_x=t_{\alpha,x}$ in $(\oplus_{i\in I}F_i)_x=\oplus_{i\in I}F_{i,x}$ for every $x\in U_\alpha$.

Fix $x_0\in X$ and $\alpha_0\in A$ with $x_0\in U_{\alpha_0}$. Then there is a finite subset $I_0\subset I$ such that $t_{\alpha_0}\in\Gamma(X,\oplus_{i\in I_0}F_i)\subset\Gamma(X,P)$. Let $B\subset A$ be the subset of indices α with $t_\alpha\notin\Gamma(U_\alpha,\oplus_{i\in I_0}F_i)$. Set $V=\cup_{\alpha\in B}U_\alpha$. Then V is open in X and its complement

$$X\setminus V\subset\cup_{\alpha\in A\setminus B}U_\alpha. \tag{30}$$

For every $\alpha\in A\setminus B$, we claim that $U_\alpha\subset X\setminus V$.

In fact, for every $y\in U_\alpha$, every $\beta\in A$ with $y\in U_\beta$ and every $i\in I\setminus I_0$, the stalk $t^i_{\beta,y}=s^i_y=t^i_{\alpha,y}=0$ in $F_{i,y}$. Since F_i is privileged and U_β is connected, the map $\Gamma(U_\beta,F_i)\to F_{i,y}$ is injective. Thus, $t^i_\beta=0$ in $\Gamma(U_\beta,F_i)$. Therefore, $t_\beta\in\Gamma(X,\oplus_{i\in I_0}F_i)$, *i.e.*, $\beta\notin B$. Hence $y\notin V$.

From the claim and (30), the subset $X\setminus V=\cup_{\alpha\in A\setminus B}U_\alpha$ is also open in X and contains U_{α_0}. Since X is connected, one has $V=B=\emptyset$. Consequently, $t_\alpha\in\Gamma(X,\oplus_{i\in I_0}F_i)$ for every $\alpha\in A$. Then the family $\{t_\alpha\}_{\alpha\in A}$ glues to a preimage of s in $\Gamma(X,\oplus_{i\in I_0}F_i)\subset\Gamma(X,P)$. Thus, θ_X is surjective and hence a group isomorphism. ☐

Corollary A.5.3. If F is a locally free (possibly of infinite rank) O_X-module, then F is privileged.

Proof. Let U be a connected open subset of X. Fix $x_0\in U$. We prove that the map $\Gamma(U,F)\to F_{x_0}$ is injective. Take $s\in\Gamma(U,F)$ with $s_{x_0}=0$. By [Har77, II, Exercise 1.14], the set $Z:=\{x\in U : s_x=0\}$ is open in U.

We claim that Z is closed in U. Let $\{x_n\}_{n\geq 1}$ be a sequence of points in Z converging to $y\in U$. Because F is locally free, there is a connected open neighborhood $V\subset U$ of y, a set I and an isomorphism $\phi : F|_V\xrightarrow{\sim}O_V^{\oplus I}$ of O_V-modules. There is an integer $N>0$ with $x_N\in V$. Because O_V is privileged, from Lemma A.5.2, the map on the bottom of the commutative square

$$\begin{tikzcd}
\Gamma(V,F) \arrow[r] \arrow[d, "\phi_V"] & F_{x_N} \arrow[d, "\phi_{x_N}"] \\
\Gamma(V,O_V^{\oplus I}) \arrow[r] & O_{V,x_N}^{\oplus I}
\end{tikzcd}$$

is injective. Then so is the map on the top. Since $s_{X_N} = 0$, one has $s|_V = 0$ and $s_y = 0$. Hence $y \in Z$. The claim is proved.

Because U is connected and $x_0 \in Z$, by claim one has $Z = U$. Therefore, $s = 0$ in $\Gamma(U,F)$. ☐

Corollary A.5.4. Let X be a connected complex manifold. Let $\{F_i\}_{i\in I}$ be a family of locally free O_X-modules. Then the canonical map $\oplus_{i\in I}\Gamma(X,F_i) \to \Gamma(X,\oplus_{i\in I}F_i)$ is bijective.

Proof. It follows from Lemma A.5.2 and Corollary A.5.3. ☐

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