Around Lawrence-Venkatesh method

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Chapter 1

Introduction

1.1 Rational points

Intuitively, given an algebraic variety over a number field, the complexity of its geometry affects how many rational points (over finite extensions of the base field) it can possess. In Chapter 1, by an algebraic variety, we mean a geometrically integral, finite type, separated scheme over a field. An algebraic variety of dimension 1 is called a curve.

1.1.1 Mordell conjecture

Let $K$ be a number field. Let $S$ be a finite subset of places of $K$ containing all infinite ones. By Siegel’s theorem [Sie29, p.252], on the projective line $P_K^1$ with at least three punctures, or on a genus 1 curve over $K$ with at least one puncture, there are at most finitely many $O_{K,S}$-integral points. For curves of higher genus, Faltings’s theorem (Fact 1.1.1.1) was conjectured by Mordell [Mor22, (5), p.192].

Fact 1.1.1.1 (Faltings, [Fal83, Satz 7]). Let $Y$ be a smooth projective curve over $K$ of genus $\geq 2$. Then $Y(K)$ is finite.

A sample application of Faltings’s theorem is a partial solution to Fermat’s Last Theorem: for every integer $n \geq 4$, there are only finitely many pairwise coprime integer solutions to the equation $x^n + y^n = z^n$. Indeed, the projective plane curve (known as the $n$-th Fermat curve) in $P_K^2$ cut out by this equation has genus $(n - 1)(n - 2)/2 \geq 2$. It is more than a decade earlier than Andrew Wiles’s complete solution in 1994 to Fermat’s conjecture.

Parshin [Par68] constructed a family of curves over $Y$, with which he showed that Fact 1.1.1.1 is a consequence of Shafarevich’s conjecture for curves. This conjecture in turn follows from Shafarevich’s conjecture for abelian varieties and Torelli’s theorem.

We recall the statement of Shafarevich’s conjecture. A smooth proper variety (resp. abelian variety) over a discrete valuation field $E$ is said to have
good reduction if it is isomorphic to the generic fiber of a smooth proper scheme (resp. abelian scheme) over the integer ring $O_E$ of $E$. By [Mil20, Prop. 6.4], there is at most one such abelian scheme. A smooth proper variety (resp. abelian variety) over $K$ is said to have good reduction at a finite place $v$ of the number field $K$ if its base change to $K_v$ has good reduction.

**Fact 1.1.1.2** (Shafarevich conjecture, [Fal83, Korollar 1, p.365 (resp. Satz 6)]). For every integer $g$ at least 2 (resp. 1), up to $K$-isomorphism there are only finitely many smooth projective curves (resp. abelian varieties) defined over $K$ of genus (resp. dimension) $g$, with good reduction outside $S$.

In 1983, Faltings proved Shafarevich’s conjecture for abelian varieties and hence Mordell’s conjecture, “opening thereby a new chapter in number theory.” Faltings’s proof can be decomposed into two parts, Facts 1.1.1.3 and 1.1.1.4.

**Fact 1.1.1.3** ([Fal83, Satz 5]). For every integer $g > 0$, up to $K$-isogeny there are only finitely many abelian varieties over $K$ of dimension $g$, with good reduction outside $S$.

Fact 1.1.1.3 is weaker than Shafarevich’s conjecture for abelian varieties. Its proof is to consider the representations of the absolute Galois group $\Gamma_K$ of $K$ on the Tate modules of $K$-abelian varieties. For one thing, by Tate’s conjecture over number fields [Fal83, Korollar 2], the Galois representation on the Tate module determines the abelian variety up to $K$-isogeny. For another, by Weil’s conjecture proved by Deligne [Del74, Thm. 1.6], there are only finitely many such representations up to isomorphism.

**Fact 1.1.1.4.** Let $A$ be an abelian variety over $K$. Then up to $K$-isomorphism, there are only finitely many abelian varieties over $K$ which are $K$-isogenous to $A$.

Faltings introduced a differential height function, now known as Faltings’s height, to measure the “complexity” of abelian varieties. Height function is a tool of global nature, as it collects the information at every place of the base number field. The core of the proof of Fact 1.1.1.4 is that Faltings’s height does not change much under isogeny ([Fal83, Lem. 5]).

### 1.1.2 Lang conjectures

“One natural generalization to higher dimensions of the notion of ‘curve of geometric genus $g \geq 2$’ is ‘variety of general type’.” ([CHM97, p.2]). For a smooth projective variety $X$ over a field, let $\omega_X$ be its canonical line bundle. For an integer $d \geq 0$, let $P_d(X) = h^0(X, \omega_X^d)$ be the $d$-th plurigenus of $X$. The Kodaira dimension $\kappa(X)$ is defined to be $-\infty$ (or $-1$ depending on the convention) if $P_d(X) = 0$ for every integer $d > 0$; otherwise, it is the minimum real number $r$ such that the sequence $\{P_d(X)/d^r\}_{d>0}$ is bounded. Then Kodaira dimension is the “most basic” ([Laz04, Eg. 2.1.5]) integer birational invariant of

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1 quotation from [Blo84, p.41]
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X. If $\kappa(X) = \dim X$, then X is called of general type. For instance, a smooth projective curve is of general type if and only if its genus is at least 2.

A high-dimensional analog of Fact 1.1.1.1 is conjectured by Lang (see, e.g., [CHM97, Conjecture A]).

**Conjecture 1.1.2.1.** Let X be a positive dimensional smooth projective variety of general type over a number field K. Then $X(K)$ is not Zariski dense in X.

Using techniques from Diophantine approximation, Faltings proves Conjecture 1.1.2.1 for subvarieties of abelian varieties, which gives a second proof of Fact 1.1.1.1. From [Hin98, p.95], a subvariety of an abelian variety is of general type if and only if its stabilizer is finite.

**Fact 1.1.2.2 ([Fal91, Thm. 1]).** Let A be an abelian variety over a number field K. Let $X \subset A$ be a subvariety of general type. Then $X(K)$ is finite.

Based on Faltings’s work [Fal94], Moriwaki proves another particular case of Conjecture 1.1.2.1. A smooth projective variety with ample cotangent bundle is of general type.

**Fact 1.1.2.3 ([Mor95, p.114]).** Let X be a smooth projective variety over a number field K. If the cotangent bundle $\Omega^1_{X/K}$ is ample and generated by global sections, then $X(K)$ is finite.

Conjecture 1.1.2.1 is stronger than the uniformity conjecture.

**Fact 1.1.2.4 ([CHM97, Thm. 1.1]).** Assume Conjecture 1.1.2.1 over every number field. Then for every number L and every integer $g \geq 2$, there is an integer $B(L, g)$ such that every smooth curve C over L of genus g, one has $\#C(L) \leq B(L, g)$.

Conjecture 1.1.2.1 for algebraic surfaces was independently raised by Bombieri, so also known as the Bombieri-Lang conjecture. It gives a conditional solution to the Erdös-Ulam problem.

A rational distance set in $\mathbb{R}^2$ is a subset such that every pairwise distance between its points is rational. Erdös and Ulam conjectured in 1945 that there is no dense rational distance set in $\mathbb{R}^2$.

**Fact 1.1.2.5 ([Sha18, Cor. 1.4]).** Assume Conjecture 1.1.2.1 for algebraic surfaces over all number fields. Let S be an infinite rational distance set. Then either all but at most 4 points of S are on a line, or all but at most 3 points of S are on a circle.

A complex manifold $M$ is called Brody hyperbolic if every morphism $\mathbb{C} \to M$ of complex manifolds is constant. For example, a compact Riemann surface is Brody hyperbolic if and only if its genus is at least 2.

**Conjecture 1.1.2.6 ([Lan86, Conjecture 5.6], see also [BD21, Conjecture, p.2]).** A complex smooth projective variety is hyperbolic if and only if every subvariety is of general type.

---

2Subvarieties are assumed to be closed.
Conjecture 1.1.2.6 is known as the geometric Lang conjecture. It lies between algebraic geometry and complex analytic geometry. Both directions of it are unknown till now. For subvarieties of abelian varieties, Conjecture 1.1.2.6 is confirmed by [Yam19, Cor. 1.3] (and Brody’s theorem [Bro78, p.213] that Brody hyperbolicity agrees with Kobayashi hyperbolicity for compact complex manifolds).

Conjecture 1.1.2.7 would follow from Conjectures 1.1.2.1 and 1.1.2.6.

**Conjecture 1.1.2.7 (Lan74, (1.3)).** Let \(V\) be a smooth projective variety over a number field \(F\). If a complex analytification of \(V\) is Brody hyperbolic, then \(V(F)\) is finite.

### 1.1.3 Lang conjecture for Shimura varieties

Shimura varieties are higher-dimensional analogs of modular curves. As Alex Youcis puts it, the reason to study Shimura varieties is multiple: They are highly symmetrical objects with rich actions of various Lie groups; Thy are moduli spaces of abelian varieties (with extra structures); They are moduli spaces of motives; They are objects conjectured to realize the global Langlands correspondence, etc. However, to define Shimura varieties requires an exceptional amount of technical sophistication. See [Mil17b] for a reference.

Let \((G, X)\) be a Shimura datum, and let \(K \leq G(A_f)\) be a sufficiently small, neat, compact open subgroup. Let \(S\) be a connected component of the complex manifold \(S_{\text{K}}(G, X)\). From Nadel’s work [Nad89, Thm. 0.2], the Baily-Borel compactification \(S^*\) of \(S\) is Brody hyperbolic. As the canonical model of \(S_{\text{K}}(G, X)\) exists (see, e.g., [Mil17b, p.128]), \(S\) is naturally a smooth quasi-projective variety defined over a number field \(F\). Then Conjecture 1.1.2.7 predicts that \(S(F')\) is finite for every finite extension \(F'/F\). Similar speculation for integral points on Shimura varieties of abelian type is confirmed by Ullmo. His proof relies on Faltings’s solution to Shafarevich’s conjecture for abelian varieties (Fact 1.1.1.2).

**Fact 1.1.3.1** ([Ull04, Thm. 3.2 (a)]). Suppose that the Shimura datum \((G, X)\) is of adjoint abelian type. Let \(\Gamma \leq G(\mathbb{Q})\) be a net arithmetic lattice. Then for every number field \(F\), every finite set of places \(\Sigma\) of \(F\) and every model \(M\) of \(X^+/\Gamma\) over \(O_{F,\Sigma}\), the set \(M(O_{F,\Sigma})\) is finite.

Concerning the rational points on general Shimura varieties, Lang’s conjecture (Conjecture 1.1.2.1) is related to an alternative principle [UY10, Thm. 1.1]. For a projective variety \(Z\) over a number field, Ullmo and Yafaev [UY10, (1)] define its **Lang locus** \(Z^L\) to be the Zariski closure of \(\bigcup_M \overline{Z(M)}^{>0}\), where \(M\) runs through finite extensions of the definition field of \(Z\) (inside a fixed algebraic closure), and \(\overline{Z(M)}^{>0}\) is the union of positive-dimensional irreducible components of the Zariski closure \(\overline{Z(M)}\).

The Lang locus measures the failure of Lang’s conjecture, since \(Z^L = \emptyset\) if and only if \(Z\) satisfies Conjecture 1.1.2.1. For Shimura varieties, Fact 1.1.3.2 shows that Lang’s conjecture is either true or very false.
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Fact 1.1.3.2 (Ullmo-Yafaev’s all-or-nothing principle, [UY10, Thm. 1.1]). Let $S$ be a (connected) Shimura variety of sufficiently high level. Then $S \cap (S^*)^L$ is either $\emptyset$ or $S$.

As Shimura varieties are not proper in general, it is equally natural to consider integral points instead of rational points. For quasi-projective varieties over $\overline{\mathbb{Q}}$, we define an “integral Lang locus” measuring the infiniteness of integral points by choosing an integral model. This locus is independent of the choice of the model. We give a result for integral points parallel to Fact 1.1.3.2.

Theorem (Theorem 2.5.0.12). The integral Lang locus of a (connected) Shimura variety $S$ is either $\emptyset$ or $S$.

In fact, we form several axioms for an abstract locus formation, and prove that such an alternative principle results from the axioms. Both Lang locus and integral Lang locus satisfy the axioms.

1.2 Lawrence-Venkatesh technique

Lawrence-Venkatesh’s new proof ([LV20]) of Faltings’s theorem (Fact 1.1.1.1) sheds light on Conjecture 1.1.2.1. This technique, compared with Faltings’s strategy, is of local nature. We give a highly sketchy review, and refer the reader to [LV20] for more details.

1.2.1 Setting

Let $K, S$ be as in Section 1.1.1. Let $f : X \to Y$ be a smooth proper morphism of smooth algebraic varieties over $K$. By enlarging $S$, one may choose a smooth proper morphism $\tilde{f} : \tilde{X} \to \tilde{Y}$ between smooth $O_{K,S}$-schemes whose base change to $K$ is exactly $f$. Lawrence-Venkatesh’s idea uses the induced variation of local Galois representations, to prove that $\mathcal{Y}(O_{K,S})$ is not Zariski dense in $Y$.

Remark 1.2.1.1. If $Y$ is as in Fact 1.1.1.1, then by properness of $Y$ over $K$ and [Poo17, Thm. 3.2.13 (ii)], the natural map $\mathcal{Y}(O_{K,S}) \to Y(K)$ is bijective. By $\dim Y = 1$, a subset of $Y$ which is not Zariski dense is necessarily finite. That is why one only needs nondensity of integral points in this case. Lawrence and Venkatesh [LV20] apply the following machinery to a sophisticated variant of the relative curve constructed by Parshin (and of a construction due to Kodaira).

1.2.2 Galois representations

Choose a finite place $v$ of $K$ with underlying rational prime $p$, such that $p$ is unramified in $K$ and no place dividing $p$ is in $S$. Let $K_v$ be the completion of $K$ at $v$. Let $O_v \subset K_v$ be the integer ring. There is a natural inclusion $\mathcal{Y}(O_{K,S}) \subset \mathcal{Y}(O_v)$. For every $y \in \mathcal{Y}(O_v)$, the fiber $X_y$ is a smooth proper
scheme over $O_v$ with generic fiber $X_y$:

\[
\begin{array}{ccc}
X_y & \to & y \\
\downarrow & & \downarrow \\
\mathcal{X} & \to & \mathcal{Y} \\
\downarrow & & \downarrow \\
\text{Spec } O_v & \to & \text{Spec } K_v
\end{array}
\]

Let $\text{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})$ the category of (continuous) $\mathbb{Q}_p$-representations of the absolute Galois group $\Gamma_{K_v}$. For every integer $d \geq 0$, there is a local Galois representation $\rho_y^d : \Gamma_{K_v} \to \text{GL}(H^d_{dR}(X_y/K_v, \mathbb{Q}_p))$ on the $d$-th étale cohomology group. For a locally small category $\mathcal{C}$, let $\mathcal{C}/\sim$ be the set of isomorphism classes of objects of $\mathcal{C}$. Hence, one gets a map $\rho : \mathcal{Y}(O_v) \to \text{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})/\sim$. Representations are more or less “linear” data.

1.2.3 $p$-adic Hodge theory

The functor $D_{\text{cris}}$ in $p$-adic Hodge theory induces a functor from the category $\text{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})$ to the category $\text{FVec}_{K_v}$ of filtered vector spaces over $K_v$. Because the $K_v$-algebraic variety $X_y$ has a smooth proper model $\mathcal{X}$ over $O_v$, the $p$-adic Galois representation $\rho_y^d$ is crystalline in the sense of [BC09, p.133]. By Fontaine’s conjecture proved by Faltings [Fal88, Cor., p.69], this functor sends $\rho_y^d$ to the $d$-th de Rham cohomology $H^d_{dR}(X_y/K_v)$ equipped with its Hodge filtration ([Sta23, Tag 0FM8]). The step is informally depicted below.

Locally, one can interpret the map

\[
\mathcal{Y}(O_S) \choosing \text{ a suitable place } v|p \subset \mathcal{Y}(O_v) \xrightarrow{\rho} \text{Rep}_{\mathbb{Q}_p}(\Gamma_{K_v})/\sim \xrightarrow{D_{\text{cris}}} \text{FVec}_{K_v}/\sim.
\]

as a period map. There is a vector bundle $V = H^d_{dR}(X/Y)$ on $Y$, and a decreasing Hodge filtration $F^*V$ by vector subbundles, whose fiber at every $y \in Y(K_v)$ is $H^d_{dR}(X_y/K_v)$ with its Hodge filtration. There is a natural flat connection $\nabla_{\text{GM}}$ on $V$, the Gauss-Manin connection.

1.2.4 Complex period map

We begin with the complex analytic analog. Consider a variation of Hodge structure $(V, F^*V, \nabla)$ on a connected complex manifold $Y$, where $V$ is a vector bundle, $F^*V$ is a decreasing filtration of $V$ by vector subbundles, and $\nabla$ is a flat connection on $V$. (On a complex manifold, by connection we mean a holomorphic connection in the sense of [Huy05, Def. 4.2.17].) Take a base point
$y_0 \in Y$ and a small open disk $\Omega \subset Y$ around $y_0$. As $\nabla$ is flat, for $y \in \Omega$, the parallel transport induces a $\mathbb{C}$-linear isomorphism $V_y \to V_{y_0}$. In general, the connection $\nabla$ does not respect the filtration. Still, the fiberwise filtration $F^*V_y$ is transported to a filtration on the fiber $V_{y_0}$, which has the same dimensional data as the filtration $F^*V_{y_0}$. Let $\text{Flag}/\mathbb{C}$ be the projective variety parameterizing the filtrations of $V_{y_0}$ of this common dimension data. In this way, one gets a holomorphic map (only locally defined on $Y$), called a period map,

$$\Phi_\mathbb{C} : \Omega \to \text{Flag}, \quad y \mapsto \text{transport of } F^*V_y \text{ to } V_{y_0}.$$

### 1.2.5 $p$-adic period map

Let $k/\mathbb{Q}_p$ be a finite extension of $\mathbb{Q}_p$. Let $O_k$ be the integer ring of $k$. Let $m_k$ be the maximal ideal of $O_k$. Let $Y$ be a smooth $O_k$-scheme with generic fiber $Y$. Let $\mathcal{Y}$ be the special fiber of $Y$:

$$
\begin{array}{ccc}
Y & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\text{Spec } k & \longrightarrow & \text{Spec } O_k \\
\end{array}
\quad \longleftrightarrow 
\begin{array}{ccc}
\text{Spec } O_k/m_k & \longrightarrow & \mathcal{Y} \\
\downarrow & & \downarrow \\
\text{Spec } O_k & \longrightarrow & \text{Spec } k \\
\end{array}
$$

Fix a base point $y_0 \in \mathcal{Y}(O_k)$. Denote by $\Omega$ the fiber passing $y_0$ of the reduction map $\mathcal{Y}(O_k) \to \mathcal{Y}(O_k/m_k)$ to reside field, and call it the residue disk around $y_0$. Then $\Omega$ is an open neighborhood of $y_0$ in the $k$-analytic manifold $Y^{an}$. Consider a triple $(V, F^*V, \nabla)$, where $V$ is a vector bundle on $Y$, $F^*V$ be a decreasing filtration on $V$ by vector subbundles, and $\nabla$ a flat connection. As in Section 1.2.4, one can similarly define a flag variety $\text{Flag}$ over $k$, and a $p$-adic period map $\Phi_p : \Omega \to \text{Flag}$ which is $k$-analytic.

### 1.2.6 Ax-Schanuel property of period map

In the notation of Section 1.2.1, take $k = K_v$. Take the triple $(V, F^*V, \nabla)$ to be $(\mathcal{H}^d_{dR}(X/Y) \otimes_K K_v, \text{Hodge filtration}, \nabla_{GM})$ on $Y_{K_v}$. When the fibers of $V$ on a residue disk $\Omega$ are identified by the Gauss-Manin connection $\nabla_{GM}$, the restriction of the map (1.1) to $\Omega$ coincides with the $p$-adic period map $\Phi_p$. Because $\mathcal{Y}(O_v)$ is covered by finitely many residue disks, to prove that $\mathcal{Y}(O_{K,v})$ is not Zariski dense in $Y$, it suffices to prove the nondensity of $\Omega \cap \mathcal{Y}(O_{K,v})$.

Fact 1.2.6.1 counts essentially on Bakker-Tsimerman’s Ax-Schanuel type result [BT19]. Let $\mathfrak{y}_p$ be the Zariski closure of $\Phi_p(\Omega)$ in $\text{Flag}$.

**Fact 1.2.6.1** ([KM23, Prop. 7.10 (4)]). Let $Z \subset \mathfrak{y}_p$ be a subvariety with $\dim \mathfrak{y}_p \geq \dim Z + \dim Y$. Then $\Phi_p^{-1}(Z)$ is not Zariski dense in $Y_{K_v}$.

The situation is summarized as follows.
Take $Z$ to be the Zariski closure of $\Phi_p(\Omega(O_{K,S}))$ in $\mathcal{H}_p$. If $\dim \mathcal{H}_p \geq \dim Z + \dim Y$, then by Fact 1.2.6.1, the subset $\Omega \cap Y(O_{K,S}) \subset Y$ is not Zariski dense.

### 1.2.7 Summary

To show nondensity of integral points in Lawrence-Venkatesh’s method, one needs to show that the dimension of the image $\mathcal{H}_p$ of the $p$-adic period map $\Phi_p$ is “large” when compared with that of $O_{K,S}$-points. Let $\mathcal{H}_C$ be the image of the complex period map $\Phi_C$ induced by $f_C : X_C \to Y_C$. For one thing, as the Gauss-Manin connection $\nabla_{GM}$ is defined on $K$, one gets $\dim \mathcal{H}_p \geq \dim \mathcal{H}_C$. Using the corresponding variation of Hodge structures, one proves that $\mathcal{H}_C$ contains the orbit of the base point under the monodromy action. For another thing, using Faltings’s finiteness theorem (see, e.g., [LV20, Lem. 2.3]), one gets an upper bound (involving the centralizer of the crystalline Frobenius operator arising from the comparison of de Rham cohomology and crystalline cohomology) on $\Phi_p(\Omega(O_{K,S}))$.

### 1.3 Lawrence-Sawin technique

The technique of Lawrence-Venkatesh is a promising approach to Conjecture 1.1.2.1, because it is successfully applied in higher dimension. For example, based on this technique, Lawrence and Sawin establish in an innovatory way the Shafarevich conjecture for hypersurfaces in abelian varieties. Let $K, S$ be as in Section 1.1.1. Let $A$ be an abelian variety over $K$ of dimension $g$ with good reduction outside $S$. A subvariety $V \subseteq A$ is said to have good reduction at a place $v \notin S$ of $K$ if the Zariski closure of $V$ in the unique abelian scheme $A/O_{K_v}$ with generic fiber $A_{K_v}$ is smooth.

**Fact 1.3.0.1** (Lawrence-Sawin, [LS20, Thm. 1.1]). Suppose $\dim A \geq 4$. Fix an ample class $\phi$ in the Néron-Severi group of $A$. Then there are only finitely many hypersurfaces $H \subseteq A$ over $K$ representing $\phi$, with good reduction outside $S$, up to translation by points in $A(K)$.

Using a similar technique, Krämer and Maculan obtained an analog for subvarieties of dimension less than half the dimension of the ambient abelian variety.

**Fact 1.3.0.2** (Krämer-Maculan, [KM23, Cor., p.3]). Fix a polynomial $P \in \mathbb{Q}[z]$ of degree $d < (g-1)/2$ and an ample line bundle $L$ on $A$. Then up to translation
by points in \( A(K) \), there are only finitely many nondivisible geometric complete intersections of ample divisors \( X \subset A \) over \( K \), with good reduction outside \( S \), that have Hilbert polynomial \( P \) with respect to \( L \) and satisfy \( 2\chi(X \times X, \Omega^2_{X \times X}) \leq \chi_{\text{top}}(X \times X) \).

In both cases of Facts 1.3.0.1 and 1.3.0.2, the dimension of the base algebraic variety \( Y \) in Section 1.2 is greater than 1. So nondensity of the set of integral points is strictly weaker than finiteness. An idea suggested in [LV20, Sec. 10.2] is to iterate the Lawrence-Venkatesh argument by replacing \( Y \) with the Zariski closure of integral points. In this manner, an estimate of topological monodromy group that is uniform for subvarieties of the variety under consideration is needed. This uniform estimate, "a major technical difficulty" as [LV20, p.4] describes, is achieved by comparison with a Tannakian group. This Tannakian group arises from sheaf convolution developed fundamentally by Krämer-Weissauer [KW15b]. The comparison of the monodromy group and the Tannakian group involved in the proof of Fact 1.3.0.2 leans on [JKLM23]. We give a cursory review of the comparison.

### 1.3.1 Tannakian theory of sheaf convolution

Tannakian formalism is a way to reconstruct a group from its representation theory. By the Tannaka-Krein duality, a compact group can be recovered from the abelian category of its complex representations together with the tensor product operation.

**Definition 1.3.1.1** ([DM22, Def. 2.19]). A rigid, symmetric, monoidal abelian category \((C, \otimes)\) of unit object \(1\) is a neutral Tannakian category over a field \(k\) if it admits an exact faithful \(k\)-linear tensor functor \(\omega: C \to \text{Vec}_k\) (called a fiber functor) and if \(\text{End}(1) = k\).

**Fact 1.3.1.2** ([DM22, Thm. 2.11], [Del90, Sec. 9.2]). Let \((C, \otimes)\) be a neutral Tannakian category over a field \(k\) with a fiber functor \(\omega: C \to \text{Vec}_k\). Then there is a natural affine group scheme \(\text{Aut}^\otimes(C, \omega)\) over \(k\) (called the Tannakian group of \((C, \otimes)\) at \(\omega\)), such that \(\omega\) factors through an equivalence \(C \to \text{Rep}_k(\text{Aut}^\otimes(C, \omega))\) of symmetric monoidal categories. If \(k\) is algebraically closed, then \(\text{Aut}^\otimes(C, \omega)\) is independent of the choice of \(\omega\) up to \(k\)-isomorphism.

We review the work of Krämer and Weissauer. Let \(A\) be a complex abelian variety. Perverse sheaves on algebraic variety are the singular version of local systems. They form a full, abelian subcategory \(\text{Perv}(A)\) of the triangulated category \(D^b(A)\) of complexes of sheaves with bounded, constructible cohomologies. This abelian category is Noetherian and Artinian. For every smooth subvariety \(X \subset A\), the complex of sheaves \(\mathbb{C}X[\dim X]\) is a perverse sheaf on \(A\).

Let \(a: A \times A \to A\) be the group law. Let \(p_i: A \times A \to A\) \((i = 1, 2)\) be the projection to the \(i\)-th factor. The bifunctor

\[
(\cdot) \ast (\cdot): D^b_c(A) \times D^b_c(A) \to D^b_c(A), \quad (-, +) \mapsto Ra_*(p_1^* - \otimes L p_2^*)
\]
is called the convolution on $A$. In general, $\text{Perv}(A)$ is not stable under the convolution. Still, its quotient modulo the subcategory of "negligible objects" is stable under the convolution. Let $N(A) \subset \text{Perv}(A)$ be the full subcategory comprised of (so-called negligible) objects with Euler characteristic 0.

**Fact 1.3.1.3** ([Krä22, 1.b]). *The subcategory $N(A)$ is Serre (in the sense of [Sta23, Tag 02MO (1)]) in $\text{Perv}(A)$. Let $\bar{P}(A)$ be the quotient abelian category (in the sense of [Sta23, Tag 02MS]). Then the convolution descends to a bifunctor $\ast : \bar{P}(A) \times \bar{P}(A) \to \bar{P}(A)$. Moreover, $(\bar{P}(A), \ast)$ is a neutral Tannakian category.*

Every object $F \in \text{Perv}(A)$ generates a Tannakian subcategory $\langle F \rangle$ of $\bar{P}(A)$. Let $G(F)$ be the (unique up to isomorphism) Tannakian group of $\langle F \rangle$. The computation of the Tannakian group in [LS20] follows essentially the general approach in Krämer’s work [Krä22, Krä21].

### 1.3.2 Monodromy group and generic Tannakian group

In Lawrence-Sawin [LS20, Sec. 11], the strategy of Lawrence-Venkatesh is applied to the universal family of hypersurfaces over the corresponding Hilbert scheme:

$$
\begin{array}{ccc}
U & \xleftarrow{f} & \text{Hilb} \times A \\
\downarrow & & \downarrow \\
& & \text{Hilb}
\end{array}
$$

More generally, Krämer and Maculan [KM23, Sec. 1.4] consider the following setting. We work over $\mathbb{C}$. (One can similarly work over a base field of characteristic 0 and use perverse sheaves with $\ell$-adic coefficients.) Let $Y$ be a smooth integral variety. Let $X \subset A \times_{\mathbb{C}} Y$ be a subvariety, such that the projection $f : X \to Y$ is smooth of relative dimension $d$:

$$
\begin{array}{ccc}
X & \xleftarrow{f} & Y \times A \\
\downarrow & & \downarrow \\
A & \xrightarrow{\pi} & Y.
\end{array}
$$

Then the flat vector bundle $(\mathcal{H}_{\text{dR}}^{d}(X/Y), \nabla_{\text{GM}})$ on the complex manifold $Y_{\text{an}}$ is induced by the local system $R^{d}f_{\ast}\mathbb{C}_{X}$. To get “big monodromy", we “twist" this local system as follows. Let

$$
\Pi(A) := \text{Hom}(\pi_{1}(A, 0), \mathbb{C}^{\ast})
$$

denote the algebraic torus of characters of the fundamental group. For every character $\chi \in \Pi(A)$, let $L_{\chi}$ be the corresponding rank 1 local system on $A_{\text{an}}$. Consider the local system $V_{\chi} := R^{d}f_{\ast}\pi^{\ast}L_{\chi}$ on $Y_{\text{an}}$. Then $V_{1} = R^{d}f_{\ast}\mathbb{C}_{X}$. Let $\text{Mon}(\chi)$ be the Zariski closure of the image of the monodromy representation of $V_{\chi}$. We need to find $\chi$ such that the monodromy group $\text{Mon}(\chi)$ is big enough to carry out Lawrence-Venkatesh-Sawin’s method.
Let η ∈ Y be the generic point. Denote the perverse sheaf C_{X,η}[d] on A_η by P_η. By Krämer-Weissauer’s theorem (Fact 1.4.0.1), for a generic χ ∈ Π(A), the functor

\[ \omega_\chi: (P_\eta)(\subset P(A_\eta)) \to \text{Vec}_C, \quad F \mapsto H^0(A_\eta; F \otimes^L L_\chi) \]

is a fiber functor. Using \( \omega_\chi(P_\eta) = V_{X,0} \), one can compare \( \text{Mon}(\chi) \) with \( G(P_\eta) \).

**Lemma 1.3.2.1** ([JKLM23, p.28]). For a generic character \( \chi \in \Pi(A) \), the monodromy group \( \text{Mon}(\chi) \) is a closed subgroup of \( G(P_\eta) \).

To apply Bakker-Tsimerman’s Theorem (Fact 1.2.6.1), we need a lower bound on the monodromy group. Its proof uses the normality of the geometric Tannakian group inside the generic Tannakian group.

### 1.3.3 Normality of geometric generic Tannakian group

Let \( k \) be an algebraically closed field of characteristic 0 with an algebraic closure \( \bar{k} \). Let \( A \) be an abelian variety over \( k \). Let \( \ell \) be a prime number. Let \( \Lambda \) be an algebraic closure of \( \mathbb{Q}_\ell \).

**Fact 1.3.3.1** ([LS20, Lem. 3.7]). Let \( G_k \) (resp. \( G_k' \)) be the Tannakian fundamental group of the category of geometrically semisimple perverse sheaves on \( A \) with coefficients in \( \Lambda \) (resp. summands of the pullbacks to \( A_k \) of geometrically semisimple perverse sheaves on \( A \)), modulo the full subcategory of “negligible objects”. Then \( G_k' \) is naturally a normal closed subgroup of \( G_k \), with quotient isomorphic to the Tannakian group of the neutral Tannakian category \( (\text{Rep}_\Lambda(\Gamma_k), \otimes) \).

The assumption of geometric semisimplicity in Fact 1.3.3.1 is removed in [JKLM23]. Let \( K/k \) be a field extension. Let \( k' \) be the algebraic closure of \( k \) in \( K \). Assume that \( k'/k \) is Galois. Let \( \mathcal{C} \subset \bar{P}(A) \) be a full abelian tensor subcategory. Let \( \mathcal{C}_K \subset \text{Perv}(A_K)/\text{N}(A_K) \) be the full subcategory of subquotients of the pullbacks to \( A_K \) of perverse sheaves on \( A \). Fix a fiber functor \( \omega: \mathcal{C}_K \to \text{Vec}_\Lambda \).

**Fact 1.3.3.2** ([JKLM23, Thm. 4.3]). There is a short exact sequence of proalgebraic groups

\[ 1 \to \text{Aut}^{\otimes}(\mathcal{C}_K, \omega) \to \text{Aut}^{\otimes}(\mathcal{C}, \omega) \to \text{Aut}^{\otimes}(\mathcal{C} \cap \text{Rep}_\Lambda(\text{Gal}(k'/k)), \omega) \to 1. \]

### 1.3.4 Monodromy group and geometric generic Tannakian group

Fact 1.3.3.1 and an analog of Larsen’s alternative ([LS20, Lem. 5.4]) permit one to get a lower bound on the monodromy group. For a generic \( \chi \in \Pi(A) \), by [LS20, Thm. 5.6] and [JKLM23, Thm. 4.10], under certain geometric condition on the family \( f: X \to Y \), one has inclusions as follows:

\[
\begin{array}{ccc}
\text{Mon}(\chi) & \hookrightarrow & G(P_\eta) \\
\uparrow & & \downarrow \text{normal} \\
& & G(P_\eta).
\end{array}
\]
1.3.5 Summary

In brief, in the work of Lawrence-Sawin and that of Krämer-Maculan, the crucial uniform estimation on the monodromy group follows from a comparison to the Tannakian group (of perverse sheaves on the geometric generic fiber). In fact, both the monodromy group and the Tannakian group on the geometric generic fiber are embedded as closed subgroups in the Tannakian group on the generic fiber. The geometric generic Tannakian group is normal in the generic Tannakian group. This normality is used to prove that for most characters, the corresponding monodromy group contains the geometric generic Tannakian group.

1.3.6 Normality of monodromy group

Complementing Facts 1.3.3.1 and 1.3.3.2, we prove that for many characters, the associated monodromy group is also normal in the generic Tannakian group. This result poses a restriction on what the monodromy group can be.

One uses perverse sheaves on the generic fiber of an abelian scheme in Section 1.3.2. Hansen and Scholze’s work [HS23] on relative perverse sheaves provides a way to study a family of perverse sheaves. Let \( f : X \to Y \) be a morphism of algebraic varieties. Assume that the prime \( \ell \) is invertible in the base field. By [HS23, Thm. 1.1], the category \( D_{c}^{b}(X, \Lambda) \) has a unique t-structure, called the relative perverse t-structure, which restricts to the perverse t-structure on every geometric fiber of \( f \). The heart \( \text{Perv}(X/Y) \) is called the category of relative perverse sheaves. For every \( y \in Y \), restricting to the fiber over \( y \) induces a functor \( \text{Perv}(X/Y) \to \text{Perv}(X_{y}) \). An object of \( \text{Perv}(X/Y) \) should be thought as a family of perverse sheaves. However, in general the abelian category \( \text{Perv}(X/Y) \) is not Artinian.

To get an abelian category with many of the same properties familiar in the absolute setting, Hansen and Scholze add a condition, the so-called universal local acyclicity. Roughly, an object of \( D_{c}^{b}(X, \Lambda) \) is universally locally acyclic if it satisfies the base change theorem. (For precision, see Definition 3.3.2.1.) The relative perverse t-structure preserves universally locally acyclic complexes. The resulting abelian subcategory \( \text{Perv}^{\text{ULA}}(X/Y) \subset \text{Perv}(X/Y) \) is Noetherian, Artinian and compatible with Verdier duality. Moreover, if \( Y \) is smooth integral with generic point \( \eta \), then the functor \( \text{Perv}^{\text{ULA}}(X/Y) \to \text{Perv}(X_{\eta}) \) exhibits a Serre subcategory. In this sense, a universally locally acyclic relative perverse sheaf is determined by the perverse sheaf on the generic fiber.

Let \( k \) be an algebraically closed field of characteristic 0. Let \( A/k \) be an abelian variety. Let \( Y \) be an integral algebraic variety over \( k \) with generic point \( \eta \). Let \( \text{Perv}^{\text{ULA}}(A \times Y/Y) \) be the abelian category of universally locally acyclic relative perverse sheaves ([HS23]) with coefficients in \( \Lambda \).

**Theorem 1.3.6.1** (Theorem 3.1.0.2). Assume \( \dim A > 0 \). Then there are uncountably many characters \( \chi : \pi_{1}^{\text{ét}}(A) \to \Lambda^{*} \), such that the corresponding generic Tannaka group is a reductive group containing the monodromy group corresponding to \( \chi \) as a closed reductive normal subgroup.
1.4 GENERIC VANISHING

In spirit, Theorem 1.3.6.1 is similar to that of André’s normality theorem.

**Fact 1.3.6.2** ([And92, Thm. 1]). For a polarizable good variation of mixed Hodge structure over a smooth, connected, complex algebraic variety \( X \) and every \( x \) in the complement of some meager subset of \( X \), the corresponding connected monodromy group is a normal subgroup of the corresponding derived Mumford-Tate group.

Fact 1.3.6.2 is proved via the theorem of the fixed part due to Griffiths-Schmid-Steenbrink-Zucker. In our case, an analog of the fixed part theorem is Theorem 3.1.0.3.

### 1.4 Generic vanishing

In the proof of Fact 1.3.1.3, the existence of a fiber functor on the quotient category is deduced from Krämer-Weissauer’s generic vanishing theorem.

**Fact 1.4.0.1** (Krämer-Weissauer, [KW15b, Thm. 1.1]). Let \( P \) be a perverse sheaf on a complex abelian variety \( A \). Then there is a finite union \( S(P) \) of translates of strict algebraic subtori of \( \Pi(A) \), such that for every character \( \chi \in \Pi(A) \setminus S(P) \) and every integer \( i \neq 0 \), one has \( H^i(A, P \otimes L^\chi) = 0 \).

The proof of Fact 1.4.0.1 relies on the proofs of Kashiwara’s conjecture for semisimple perverse sheaves ([Dri01]) and de Jong’s conjecture ([BK06, Gai07]). As [KW15b, Sec. 3] explains, Krämer-Weissauer’s theorem is a (partial) generalization of Green-Lazarsfeld’s generic vanishing theorem.

**Fact 1.4.0.2** ([GL87, Thm. 2]). Let \( X \) be a compact Kähler manifold. Let

\[
w(X) = \max\{ \text{codim}_X Z(\omega) : \omega \in H^0(X, \Omega^1_X) \setminus \{0\} \},
\]

where \( Z(\omega) \) denotes the zero-locus of a holomorphic 1-form \( \omega \). Then for any integers \( i, j \geq 0 \) with \( i + j < w(X) \) and a generic line bundle \( L \in \text{Pic}^0(X) \), one has \( H^i(X, \Omega^j_X \otimes L) = 0 \).

Green-Lazarsfeld’s theorem is an analog of the Kodaira-Nakano vanishing theorem and answers a problem of Beauville [Ue83, Problem 8, p.620] affirmatively. Fact 1.4.0.1 implies generic vanishing theorem for compact Kähler manifolds whose Albanese manifolds are abelian varieties (for example, projective manifolds). In this sense, it generalizes Fact 1.4.0.2 partially. The reason is that Fact 1.4.0.1 is stated for abelian varieties. To recover generic vanishing for all compact Kähler manifolds, one needs a generalization of Fact 1.4.0.1 for all complex tori.

**Fact 1.4.0.3** (Bhatt-Schnell-Scholze, [BSS18, Thm. 1.1]). Let \( P \) be a perverse sheaf on a complex torus \( A \). Then there is a strict Zariski closed subset \( S(P) \) of the algebraic torus \( \Pi(A) \) such that for every character \( \chi \in \Pi(A) \setminus S(P) \) and every integer \( i \neq 0 \), one has \( H^i(A, P \otimes L^\chi) = 0 \).
CHAPTER 1. INTRODUCTION

Existing vanishing results are mainly stated for Kähler manifolds. Deligne [Del68] shows that parallel to the Kähler setting, every complex smooth proper algebraic variety (not necessarily Kähler) admits a Hodge theory. We show that generic vanishing results hold for such varieties. Instead of giving a demonstration parallel to the Kähler situation, one can give a uniform proof in Fujiki class $C$.

A compact complex manifold is called in Fujiki class $C$ if it is the meromorphic image of a compact Kähler manifold. Compact Kähler manifolds and smooth proper complex algebraic varieties are in this class. Fujiki class $C$ admits a Hodge theory. We give a generic vanishing theorem for Fujiki class $C$.

**Theorem 1.4.0.4** (Theorem 4.7.1.3). Let $X$ be a complex manifold in Fujiki class $C$ with a flat unitary vector bundle $F$. Then for any two integers $p, q \geq 0$ with $\dim X - p - q$ larger than the defect of semismallness of an Albanese morphism of $X$, for a generic line bundle $L \in \text{Pic}^0(X)$, one has

$$H^q(X, \Omega^p_X \otimes_{\mathcal{O}_X} F \otimes_{\mathcal{O}_X} L) = 0.$$ 

**Corollary 1.4.0.5** (Corollary 4.7.2.5). Let $X/C$ be a smooth proper algebraic variety with a flat unitary vector bundle $F$. Then for any two integers $p, q \geq 0$ with $\dim X - p - q$ larger than the defect of semismallness of an Albanese morphism of $X$, the locus

$$\{L \in \text{Pic}^0(X) : H^q(X, \Omega^p_X \otimes_{\mathcal{O}_X} F \otimes_{\mathcal{O}_X} L) \neq 0\}$$

is contained in a finite union of translates of strict abelian subvarieties of the Picard variety $\text{Pic}^0_{X/C}$.

The strategy of the proof of Theorem 1.4.0.4 is considering the unitary local system corresponding to $F$ (provided by the Riemann-Hilbert correspondence). Its derived pushout along the Albanese morphism is a complex of constructible sheaves on a complex torus. For this complex of abelian sheaves, by estimating the perverse sheaf cohomologies, one deduces a generic vanishing result from Fact 1.4.0.3. This result proves Theorem 1.4.0.4 for vector bundles.

Fact 1.4.0.3 generalizes Krämer-Weissauer's theorem (Fact 1.4.0.1) to complex tori, but with a coarser control of the jump locus $S(P)$ of a perverse sheaf $P$. The finer control in Krämer-Weissauer’s theorem results from the classification of simple perverse sheaves of Euler characteristic zero ([KW15b, Prop. 10.1 (a)])..

A question is that if Krämer-Weissauer’s theorem (Fact 1.4.0.1) and the classification have generalizations to all complex tori. A positive answer would allow one to describe the failure locus of generic vanishing theorem for all compact Kähler manifolds. The proof of [KW15b, Prop. 10.1 (a)] uses Poincaré’s reducibility theorem for abelian varieties, which fails for complex tori. Still, there is an independent proof due to Schnell [Sch15, Thm. 7.6] of Fact 1.4.0.1 as well as the classification. Schnell’s proof is relatively elementary and makes profound use of a lift of Fourier-Mukai transform to $D$-modules.
1.5 Fourier-Mukai transform

Fourier-Mukai transform on abelian varieties, initiated by Mukai [Muk81], is an analog of the classical Fourier transform. For a ringed space $(X, O_X)$, let $\text{Mod}(O_X)$ be the category of $O_X$-modules. Let $D(O_X)$ be the derived category of the abelian category $\text{Mod}(O_X)$.

1.5.1 Construction

Let $k$ be an algebraically closed field. Let $A$ be an abelian variety over $k$ with dual abelian variety $B$. Let $p_A : A \times_k B \to A$ (resp. $p_B : A \times_k B \to B$) be the projection to $A$ (resp. $B$). Denote the Poincaré bundle on $A \times_k B$ by $P$.

**Definition 1.5.1.1.** The pair of functors

$$R\hat{S} : D(O_A) \to D(O_B), \quad \bullet \mapsto R_{PB*}(P \otimes^L p_A^* \bullet),$$

$$RS : D(O_B) \to D(O_A), \quad \bullet \mapsto R_{PA*}(P \otimes^L p_B^* \bullet)$$

is called the Fourier-Mukai transform between $A$ and $B$.

The Fourier-Mukai transform has found many applications in algebraic geometry: the Künneth decomposition for Chow motives [MD91], a new proof of Torelli’s theorem [BP01], the study of stable bundles on elliptic surfaces [Bri98], etc. Motivated by noncommutative geometry, Ben-Bassat, Block and Panetev [BBBP07] study the Fourier-Mukai transform on complex tori. Similar to the classical Fourier inversion, a duality result for the Fourier-Mukai transform is stated in [Muk81, Thm. 2.2] (resp. [BBBP07, Thm. 2.1]) in the algebraic (resp. analytic) case. However, both statements are imprecise (Lemma 5.2.0.1). In the algebraic case, the minor problem is bypassed by adding a quasi-coherence condition. Let $D_{qc}(O_A) \subset D(O_A)$ be the full subcategory of objects with quasi-coherent cohomologies.

**Fact 1.5.1.2** (Mukai). The functor $R\hat{S}$ (resp. $RS$) restricts to a functor $D_{qc}(O_A) \to D_{qc}(O_B)$ (resp. $D_{qc}(O_B) \to D_{qc}(O_A)$). Moreover, there are canonical isomorphisms of functors

$$RS \circ R\hat{S} \cong T^{-g}[-1]_A^* : D_{qc}(O_A) \to D_{good}(O_A);$$

$$R\hat{S} \circ RS \cong T^{-g}[-1]^*_B : D_{qc}(O_B) \to D_{good}(O_B),$$

where $T$ denotes the degree shift. In particular, $R\hat{S} : D_{qc}(O_A) \to D_{qc}(O_B)$ is an equivalence with a quasi-inverse $T^g[-1]^*_A RS$.

1.5.2 Complex tori

In the analytic setting, there are several competing definitions of “quasi-coherent” sheaves. A choice is the so-called good sheaves proposed by Kashiwara [Kas03, Def. 4.22]. With good sheaves, we give a way to correct [BBBP07, Thm. 2.1] in Chapter 5. First, we show that good sheaves are analytic analogs of quasi-coherent sheaves.
Proposition 1.5.2.1 (GAGA, Proposition 5.3.3.4). Let $X/\mathbb{C}$ be a smooth proper algebraic variety. Then analytification induces an equivalence from the category of quasi-coherent $O_X$-module to the category of good $O_{X^{an}}$-modules.

Let $A$ be a complex torus of dimension $g$. Let $B$ be its dual torus. Let $D_{\text{good}}(O_A)$ be the full subcategory of $D(O_A)$ comprised of objects with good cohomologies. Notation for $B$ are similarly understood. Set $RS : D(O_B) \to D(O_A)$ and $R\hat{S} : D(O_A) \to D(O_B)$ for the corresponding Fourier-Mukai transform.

Theorem 1.5.2.2 (Mukai, Ben-Bassat, Block, Panter, Theorem 5.4.1.1). The functor $R\hat{S}$ (resp. $RS$) restricts to a functor $D_{\text{good}}(O_A) \to D_{\text{good}}(O_B)$ (resp. $D_{\text{good}}(O_B) \to D_{\text{good}}(O_A)$). Moreover, there are canonical isomorphisms of functors

\[ RS \circ R\hat{S} \cong T^{-g}\cdot[-1]_A^*: D_{\text{good}}(O_A) \to D_{\text{good}}(O_A); \]
\[ R\hat{S} \circ RS \cong T^{-g}\cdot[-1]_B^*: D_{\text{good}}(O_B) \to D_{\text{good}}(O_B). \]

In particular, $R\hat{S} : D_{\text{good}}(O_A) \to D_{\text{good}}(O_B)$ is an equivalence with a quasi-inverse $T^g\cdot[-1]_A^*RS$.

Mukai’s proof for abelian varieties uses the flat base change theorem, of which we need an analytic analogue to prove Theorem 1.5.2.2. Our analytic replacement (Theorem 5.3.2.3) concerns only smooth base changes, but this weak version suffices for our purpose.

1.5.3 Homogeneous vector bundles

As an application of the analytic Fourier-Mukai transform, we recover Matsushima-Morimoto’s classification of homogeneous vector bundles on complex tori ([Mat59, Mor50], see also [FL14, Thm. 7.1]).

The classification of vector bundles on a complex manifold $X$ is completely worked out by Grothendieck [Gro57a] if $X$ is the Riemann sphere $P^1_\mathbb{C}$, and by Atiyah [Ati57b] if $X$ is an elliptic curve. When $X$ is an abelian variety of higher dimension, “there are ‘too’ many vector bundles on $X^n$” ([Muk78, p.239]). Still, there are classification results for some special classes of vector bundles.

A vector bundle on a complex torus is called homogeneous if it is invariant under all translations. For example, a line bundle on $A$ is homogeneous if and only if its isomorphism class is in $\text{Pic}^0(A)$.

An extension of finitely many $O_A$ is called a unipotent vector bundle on $A$. By [FL14, Lem. 5.1], for every unipotent vector bundle $U$ on $A$ of rank $r$, there is a unipotent representation $\rho : \pi_1(A) \to \text{GL}_r(\mathbb{C})$ inducing $U$. More precisely, let $L_\rho$ be the local system on $A$ corresponding to $\rho$. Then $L_\rho \otimes_{\mathbb{C}} O_A$ is isomorphic to $U$. The extension of two homogeneous vector bundles is still homogeneous, so every unipotent vector bundle is homogeneous.

Theorem 1.5.3.1 (Matsushima-Morimoto, Theorem 5.5.3.6). A vector bundle $F$ on the complex torus $A$ is homogeneous if and only if there is an integer $n \geq 0$, unipotent vector bundles $U_1, \ldots, U_n$ on $A$ and $P_1, \ldots, P_n \in \text{Pic}^0(A)$, such that $F$ is isomorphic to $\bigoplus_{i=1}^n P_i \otimes U_i$. 

1.6 Laumon-Rothstein transform

Laumon and Rothstein independently lift the Fourier-Mukai transform to \( D \)-modules and establish a duality result similar to [Muk81, Thm. 2.2]. The lift is referred to as the Laumon-Rothstein transform.

1.6.1 \( D \)-modules

On a complex manifold \( X \), an \( O_X \)-module with a flat connection is called a \( D_X \)-module. A \( D_X \)-module is a flat vector bundle if and only if it is \( O_X \)-coherent. The reason that we need \( D \)-modules is twofold. For one thing, by the Riemann-Hilbert correspondence (see, e.g., [HT07, Thm. 7.2.1]), perverse sheaves on \( X \) are equivalent to regular holonomic \( D_X \)-modules. For another, Krämer-Weissauer’s convolution theory relies on [KW15b, Prop. 10.1 (a)]. Its proof (as well as an independent proof of Schnell [Sch15]) uses \( D \)-modules.

1.6.2 Construction

Let \( A \) be an abelian variety over an algebraically closed field. Let \( B \) be the abelian variety dual to \( A \). Set \( g = \dim A \). By independent work of Rosenlicht [Ros58] and Serre [Ser12, Ch. VII], there is a universal vectorial extension \( \pi : B^g \to B \), where \( B^g \) is a connected commutative algebraic group of dimension \( 2g \).

Let \( D_b(D_A) \) be the bounded derived category of the category of left \( D_A \)-modules, and let \( D_{qc}^b(D_A) \) (resp. \( D_b^c(D_A) \)) be the full subcategory of \( D_b(D_A) \) of objects with \( O \)-quasi-coherent (resp. \( D \)-coherent) cohomologies. Let \( D_{qc}^b(O_{B^g}) \) (resp. \( D_b^c(O_{B^g}) \)) be the full subcategory of \( D_b(O_{B^g}) \) of objects with quasi-coherent (resp. coherent) cohomologies.

Let \( \mathcal{P}^g \) be the pullback of the Poincaré bundle \( \mathcal{P} \) along the morphism \( \pi \times \text{Id}_A : B^g \times A \to B \times A \). By Mazur-Messing’s theorem (see, e.g., [Lau96, Thm. 2.1.2]), \( B^g \) is the moduli space of flat line bundles on \( A \). Thus, there is a flat connection \( \nabla^g \) relative \( B^g \) on \( \mathcal{P}^g \). Then the pair \((\mathcal{P}^g, \nabla^g)\) is naturally a \( D_{B^g \times A/B^g} \)-module.

Let \( \widetilde{\mathcal{P}}^g : B^g \times A \to A \) and \( \widetilde{\mathcal{P}}^g : B^g \times A \to B^g \) be the projections.

**Definition 1.6.2.1** ([Lau96, p.14]). The Laumon-Rothstein transform between \( A \) and \( B \) is a pair of functors

\[
\mathcal{F} : D_{qc}^b(D_A) \to D_{qc}^b(O_{B^g}), \quad \bullet \mapsto R\widetilde{\mathcal{P}}^g_\ast DR_{B^g \times A/B^g}((\mathcal{P}^g, \nabla^g) \otimes_{O_{B^g \times A}} \widetilde{\mathcal{P}}^g_\ast \bullet),
\]

\[
\mathcal{F}^* : D_b^c(O_{B^g}) \to D_b^c(D_A), \quad \bullet \mapsto R\widetilde{\mathcal{P}}_\ast((\mathcal{P}^g, \nabla^g) \otimes_{O_{B^g \times A}} L\widetilde{\mathcal{P}}^g_\ast \bullet).
\]

The Laumon-Rothstein transform turns noncommutative \( D_A \)-modules to modules over the commutative ringed space \( B^g \).

**Fact 1.6.2.2** (Laumon-Rothstein, [Lau96, Thm. 3.2.1; Cor. 3.1.3], [Rot96, Thm. 4.5; Thm. 6.2], [Rot97]). One has \( \mathcal{F}^\ast \mathcal{F} \cong T^{-g}[-1]_A \) on \( D_{qc}^b(D_A) \) and \( \mathcal{F} \mathcal{F}^\ast \cong T^{-g}[-1]_{B^g} \) on \( D_{qc}^b(O_{B^g}) \). In particular, the functor \( \mathcal{F} : D_b^c(D_A) \to D_b^c(O_{B^g}) \) is an equivalence of categories. Moreover, it restricts to an equivalence \( D_b^c(D_A) \to D_b^c(O_{B^g}) \).
1.6.3 Schnell’s proof of Fact 1.4.0.1

The Laumon–Rothstein transform is a geometric tool to study generic vanishing theorems. The link is as follows. The Riemann–Hilbert correspondence induces an isomorphism \( \Phi : (B^2)^{an} \rightarrow \Pi(A)^{an} \) of complex manifolds. For a holonomic \( D_A \)-module \( M \), by [Sch15, Sec. 3], the support of \( \mathcal{F}(M) \) in \( B^2 \) is identified via \( \Phi \) with the failure locus in \( \Pi(A) \) of generic vanishing for \( M \). Schnell “deforms” the Laumon–Rothstein transform to a transform for Higgs bundles.

On a complex manifold \( X \), in general a connection on a vector bundle is not \( O_X \)-linear. Higgs bundles can be regarded as degenerations of vector bundles with flat “linear” connection.

**Definition 1.6.3.1.** A Higgs bundle is a vector bundle \( E \) with a holomorphic 1-form \( \phi \in \Gamma(X, \mathcal{E}nd(E)) \) taking values in the bundle of endomorphisms of \( E \) such that \( \phi \wedge \phi = 0 \).

Deligne’s \( \lambda \)-connection is a notion interpolating between flat bundles and Higgs bundles.

**Definition 1.6.3.2.** For \( \lambda \in \mathbb{C} \), a \( \lambda \)-connection of a vector bundle \( E \) on \( X \) is a \( \mathbb{C} \)-linear morphism of sheaves \( \nabla : E \rightarrow \Omega^1_X \otimes_{O_X} E \) such that for every open subset \( U \subset X \), every \( f \in O_X(U) \) and every \( s \in \Gamma(U, E) \), one has \( \nabla(f \cdot s) = f \cdot \nabla s + \lambda df \otimes s \). A \( \lambda \)-connection is called flat if its \( O_X \)-linear curvature operator \( \nabla \circ \nabla : E \rightarrow \Omega^2_X \otimes_{O_X} E \) is equal to 0.

Then 1-connection is exactly a connection, and a vector bundle with a flat 0-connection is the same as a Higgs bundle. The moduli spaces of \( \lambda \)-connections on projective varieties are studied in Simpson’s work [Sim97].

For a complex abelian variety \( A \), Schnell [Sch15, Sec. 10] analyzes the moduli space \( E(A) \) of line bundles on \( A \) with \( \lambda \)-connections. Let \( \lambda : E(A) \rightarrow A^1_\mathbb{C} \) be the morphism taking the parameter of generalized connections. By [Sch15, Lem. 10.7, 10.9], one has \( \lambda^{-1}(1) = B^2 \) and \( \lambda^{-1}(0) \) is the moduli space (more precisely, \( M_{Dol}(A) \)) of rank 1 Higgs bundles on \( A \). The morphism \( \lambda \) is real-analytically trivial, recovering the isomorphism \( M_{Dol}(A) \rightarrow B^2 \) of real Lie groups in nonabelian Hodge theory ([Sim93, p.364]).

Schnell [Sch15, Sec. 11] introduces an “extended Fourier–Mukai transform” taking values in \( D(O_{E(A)}) \). Restricting to \( \lambda^{-1}(1) \), it coincides with the Laumon–Rothstein transform. Restricting to \( \lambda^{-1}(0) \), it is essentially the Fourier transform for Higgs bundles ([Bon06, Bon10]).

Schnell deforms the holonomic \( D_A \)-module \( M \) to an \( O_{T^* A} \)-module \( M' \), which is a “generalized” Higgs bundle (more precisely, a holonomic Higgs module as [Sab07, Example 5.1.6 (1)] shows). By definition, \( \text{Supp} \ M' \) in \( T^* A \) has pure dimension \( g \). Therefore, the support of the Fourier transform of \( M' \) is a strict subset of \( \lambda^{-1}(0) \). It is the intersection of the support of the extended Fourier–Mukai transform of \( M \) in \( E(A) \) with \( \lambda^{-1}(0) \). Using the real analytic isomorphism \( \lambda^{-1}(0) \rightarrow \lambda^{-1}(1) \), Schnell proves that the support of \( \mathcal{F}(M) \) is also a strict subset of \( B^2 \). Fact 1.4.0.1 follows from this strictness. Details can be found in [Sch15, Prop. 18.2].
1.6.4 Complex tori

An analytic Laumon-Rothstein transform may help to extend Schnell’s method to all complex tori. Let \(A, B\) be complex tori dual to each other, of dimension \(g\). By Proposition E.5.4.5, the universal vectorial extension \(\pi: B^\natural \to B\) still exists. Contrary to the algebraic case, the complex Lie group \(B^\natural = (\mathbb{C}^*)^{2g}\). In light of \([Fav12, Thm. 3]\), an analytic version of Fact 1.6.2.2 needs a modification. In fact, we construct an \(O_B\)-subalgebra \(A_B\) of \(\pi_* O_B\), and define a pair of functors
\[
\tilde{\mathcal{F}}: \mathcal{D}(\mathcal{D} A) \to \mathcal{D}(A_B), \quad \tilde{\mathcal{F}}^\natural: \mathcal{D}(A_B) \to \mathcal{D}(D_A).
\]
A coherent \(\mathcal{D} A\)-module is called good if it admits global good filtration. (In the algebraic case, every coherent \(\mathcal{D}\)-module admits a global good filtration. The complex analytic analog is false.) Let \(D^b_{\text{good}}(\mathcal{D} A)\) (resp. \(D_{O^*\text{-good}}(\mathcal{D} A)\)) be the full subcategory of \(D^b(\mathcal{D} A)\) (resp. \(D(\mathcal{D} A)\)) of objects with good (resp. \(O_A\)-good) cohomology. Theorem 1.6.4.1 is a “lift” of Theorem 1.5.2.2.

**Theorem 1.6.4.1** (Theorem 6.1.0.6). 1. The pair \((\tilde{\mathcal{F}}, \tilde{\mathcal{F}}^\natural)\) is a lift of Fourier-Mukai transform in the sense that the following squares are commutative:
\[
\begin{array}{ccc}
\mathcal{D}(\mathcal{D} A) & \xrightarrow{\tilde{\mathcal{F}}} & \mathcal{D}(A_B) \\
\downarrow & & \downarrow \\
\mathcal{D}(O_A) & \xrightarrow{\tilde{\mathcal{F}}^\natural} & \mathcal{D}(O_B),
\end{array}
\]
\[
\begin{array}{ccc}
\mathcal{D}(\mathcal{D} A) & \xleftarrow{\tilde{\mathcal{F}}^\natural} & \mathcal{D}(A_B) \\
\downarrow & & \downarrow \\
\mathcal{D}(O_A) & \xleftarrow{\tilde{\mathcal{F}}} & \mathcal{D}(O_B),
\end{array}
\]
where the vertical functors are forgetful.

2. One has \(\tilde{\mathcal{F}}^\natural \tilde{\mathcal{F}} \cong T^{-g}[-1]_{B^\natural}\) on \(D_{O^*\text{-good}}(A_B)\) and \(\tilde{\mathcal{F}}^\natural \tilde{\mathcal{F}} \cong T^{-g}[-1]^*_A\) on \(D_{O^*\text{-good}}(D_A)\). Moreover, \(\tilde{\mathcal{F}}^\natural \tilde{\mathcal{F}}\) preserves \(D^b_{\text{good}}(D_A)\).

1.6.5 Vector bundles with connection

A smooth vector bundle on a smooth manifold always admits a smooth connection. In the complex analytic case, a vector bundle many not admit any connection.

**Fact 1.6.5.1** (Atiyah, [Ati57a, Theorems 2, 5, 6]). Let \(X\) be a compact Kähler manifold. Let \(E\) be a vector bundle on \(X\) admitting a connection. Then for every integer \(k > 0\), the \(k\)-th Chern class \(c_k(E) = 0\) in \(H^{2k}(X, \mathbb{R})\).

Fact 1.6.5.1 leads to Question 1.6.5.2, which is attributed to Atiyah in [BD24, p.1].

**Question 1.6.5.2.** Does every vector bundle on a compact Kähler manifold admitting a connection also admit a flat connection?

Using the analytic Laumon-Rothstein transform, we recover a result of Matsushima [Mat59, Thm. 1] and Morimoto [Mor59, Thm. 2], which answers Question 1.6.5.2 affirmatively for complex tori.
**Theorem.** 1. (Theorem 6.3.0.6) Let $E$ be a coherent module on a complex torus with a connection $\nabla$. Then $E$ is a homogeneous vector bundle and the pair $(E, \nabla)$ is translation invariant.

2. (Proposition 6.5.0.6) A homogeneous vector bundle on a complex torus admits a flat connection.

### 1.7 Future directions

Several possible topics for further research are as follows. Depending on the limit author's knowledge, they vary from a vague idea to a relatively concrete plan.

#### 1.7.1 Six-functor formalism of analytic quasi-coherent sheaves

As the proof of Theorem 1.5.2.2 needs a bit six-functor formalism in complex analytic geometry, there are a few natural questions: Does Theorem 1.5.2.2 have an analog for analytic quasi-coherent sheaves in Scholze and Clausen's sense ([Sch19] and [Sch22])? What is the relation between the notions of analytic quasi-coherence existing in the literature: the one of Scholze and Clausen, good sheaves proposed by Kashiwara (Definition A.1.4.1) and quasi-coherent sheaves in the sense of [RR74, p.100]?

#### 1.7.2 Analytic Krämer-Weissauer’s vanishing theorem

With the analytic Laumon-Rothstein transform and Theorem 1.6.4.1 at our disposal, we can study holonomic $D$-modules (instead of perverse sheaves only) following Schnell [Sch15]. This shall lead to a convolution theory on complex tori, extending that on abelian varieties. The resulting analytic Krämer-Weissauer theorem would hopefully give a finer control of the loci (1.2) for not only projective manifolds but also compact Kähler manifolds.

#### 1.7.3 Lawrence-Venkatesh’s method

Faltings [Fal83] deduced Mordell’s conjecture (Fact 1.1.1.1) from Shafarevich’s conjecture [Fal83, Satz 6]. By Shafarevich’s conjecture, for any integers $g \geq 1$ and $n \geq 3$, the Siegel variety $A_{g,n}$ (a Shimura variety parametrizing principally polarized abelian varieties of dimension $g$ with a level $n$-structure) has only finitely many integral points ([Ull04, Prop. 3.1 (a)]). Now that Lawrence-Venkatesh’s method [LV20] can recover Faltings’s theorem, a natural question is if it can also prove the finiteness of integral points of $A_{g,n}$.

The situation should be compared to that in [LS20, p.7], where the authors considered the universal hypersurface inside a constant abelian scheme and compared its Tannakian group with the monodromy group. Similarly, to study Shafarevich’s conjecture, we can consider the convolution of relative perverse sheaves ([HS23]) on the universal abelian variety over $A_{g,n}$. Then we may
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calculate the Tannakian group associated with the universal theta divisor and try to relate it to the corresponding monodromy group.

1.8 Overview

The thesis consists of several independent chapters. Chapter 2 contains an arithmetic result related to Conjecture 1.1.2.1. The geometric foundation of the work [LS20, KM23] has inspired the study in Chapters 3, 4 and 6. Chapter 3 is related to the monodromy comparison part of [LS20, KM23]. Chapters 4, 5 and 6 are of complex analytic nature, completely independent of arithmetic. Appendix A reviews generalities of sheaves of modules over ringed spaces and supplements Chapter 5. Appendix B compares two notions of quasi-coherent sheaves on complex analytic spaces. Appendix C complements Chapter 4 by giving more details. Appendix D concerns basics of $D$-modules and adds a detail on the Laumon-Rothstein theorem (Fact 1.6.2.2). Appendix E details a construction used in Chapter 6 and investigates related group-theoretic problems.

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Chapter 2

Integral points of Shimura varieties: an “all or nothing" principle

2.1 Introduction

A complex space $X$ is called Brody hyperbolic if every morphism $\mathbb{C} \to X$ is constant (see, e.g., [Liu18, p.5341]). For example, a genus $g$ compact Riemann surface is Brody hyperbolic if and only if $g \geq 2$ ([Cos05, p.78]). Conjecture 2.1.0.1 predicts that a geometric property, hyperbolicity, restricts the behavior of rational points.

Conjecture 2.1.0.1 (Lang, [Lan74, (1.3)], [Lan86, p.160]). Let $X$ be a projective variety over a number field $k(\subset \mathbb{C})$. If $X^{\text{an}}$ is Brody hyperbolic, then the set of rational points $X(k)$ is finite.

Fact 2.1.0.2 is derived from [Nad89, Thm. 0.2] in the paragraph following [UY10, Thm. 2.1]

Fact 2.1.0.2 (Nadel). Let $S$ be a Shimura variety associated with a triple $(G, X, K)$. Then there is an open subgroup $K' \leq K$ such that the analyticification $S'^{\text{an}}$ of the Baily-Borel compactification $S'^{\text{c}}$ of the corresponding finite étale cover $S$ is Brody hyperbolic.

As $S'^{\text{c}}$ is defined on a number field $F$ (depending on $K'$), Conjecture 2.1.0.1 predicts $S'(F')$ to be finite for every finite extension $F'/F$. Ullmo and Yafaev [UY10] introduced the notion of “Lang locus" (Example 2.2.0.1) for algebraic varieties over $\mathbb{Q}^a$ that measures the failure of Conjecture 2.1.0.1. For Shimura varieties, they proved an alternative principle.

\footnote{in the sense of page 34}
CHAPTER 2. INTEGRAL POINTS OF SHIMURA VARIETIES: AN "ALL OR NOTHING" PRINCIPLE

Fact 2.1.0.3 ([UY10, Thm. 1.1]). Let $S$ be a Shimura variety of sufficiently high level. Then its Lang locus is either full $S$ or empty.

As Ullmo and Yafaev put it, Fact 2.1.0.3 roughly means that for Shimura varieties, Conjecture 2.1.0.1 is either true or very false.

As Shimura varieties are not proper in general, it is equally natural to consider integral points. Conjecture 2.1.0.1 predicts that $S'$ has only finitely many integral points (compare [Lan91, Conjecture 5.1]). This note aims at deriving an analogue of Fact 2.1.0.3, with integral points at the place of rational points. We define a notion of "integral Lang locus" (Definition 2.5.0.1) for algebraic varieties over $\mathbb{Q}^a$ that measures the failure of finiteness of integral points.

Theorem (Theorem 2.5.0.12). The integral Lang locus of a Shimura variety $S$ (associated with a triple $(G, X, K)$) is either empty or full $S$.

2.2 Preliminaries

Let $\mathbb{Q}^a$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Unless otherwise specified, an (algebraic) variety means a finite type separated (geometrically) reduced scheme over a field. A component of the variety refers to an irreducible component. The closure of a subset of the variety is taken in the Zariski topology. A subvariety is closed unless otherwise specified. A Zariski-closed subset of a variety is endowed with the reduced scheme structure, hence a subvariety.

the field of definition of such a variety or a morphism between varieties is a number field (subfield of $\mathbb{Q}^a$). By an étale cover $X \to Y$, we mean a finite étale morphism between integral varieties (hence a surjection). It is called Galois if $\text{Aut}(X/Y)$ acts transitively on each fiber. For a topological space $X$, we write $X^{>0}$ for the union of components of positive Krull dimension. Note that if $X \subset Y$ is a subspace then $X^{>0} \subset Y^{>0}$.

We shall show that an alternative principle (Corollary 2.4.0.4) for an abstract locus is a formal consequence of some axioms. We begin with these axioms.

Suppose that for each integral variety $X/\mathbb{Q}^a$, we define a subvariety $X^L \subset X$. For the decomposition into components $Z = \cup_{i=1}^n Z_i$ of a reducible variety $Z/\mathbb{Q}^a$, we define $Z^L = \cup_{i=1}^n Z_i^L$. Assume that the formation satisfies the following four axioms: For any integral varieties $X, Y/\mathbb{Q}^a$:

1. (Dimension) If $X^L \neq \emptyset$, then every component of $X^L$ has positive dimension;
2. (Inheritance) If $i : X \to Y$ is a closed immersion over $\mathbb{Q}^a$, then $i(X^L) \subset Y^L$;
3. (Cover) For any étale cover $f : X \to Y$ over $\mathbb{Q}^a$, we have $f(X^L) \subset Y^L$;
4. (Iteration) $X^L \subset (X^L)^L$.

By Axiom 2, $X^L \supset (X^L)^L$. From Axiom 4, $X^L = (X^L)^L$. 
Example 2.2.0.1. The Lang locus defined in [UY10, Sec. 2.2] satisfy the four axioms, which we recall now. For each integral variety $X/\mathbb{Q}^a$, by [Gro65, Prop. 4.8.13] there exist a number field $F$ and an integral variety $X_F/F$ with an isomorphism $X_F \otimes_F \mathbb{Q}^a \to X$. For each finite extension $M/F$, let $X(X_F, M)$ be the image of the natural injection$^2$ $X_F(M) \to X(\mathbb{Q}^a)$. The Lang locus of $X$ relative to $X_F$ is defined to be the Zariski closure of

$$\bigcup_M X(X_F, M)^{>0}$$

in $X$, where $M$ runs through all finite extensions of $F$. (Notice that the closures are taken inside $X$ rather than $X_F$!) The Lang locus depends only on $X$ by Lemma 2.2.0.2 and satisfies the axioms above by [UY10, Lemmas 2.3, 2.5]. It measures the failure of finiteness of rational points, since $X^L = \emptyset$ if and only if $X_F(M)$ is finite for every finite extension $M/F$.

Lemma 2.2.0.2. The Lang locus of $X$ is independent of the choice of $X_F$.

Proof. Take another such $X_{F'}/F'$. There is a $\mathbb{Q}^a$-isomorphism $X_F \otimes_F \mathbb{Q}^a \to X_{F'} \otimes_{F'} \mathbb{Q}^a$. By [Gro65, Prop. 4.8.13], it is defined over a number field $F''$ including both $F$ and $F'$. For any finite extension $M/F$, there is a number field $M'$ containing $M$ and $F''$. Then $X(X_{F'}, M) \subset X(X_F, M')$, so $X(X_F, M)^{>0} \subset X(X_{F'}, M')^{>0}$ and hence the Lang locus relative to $X_F$ is contained in that relative to $X_{F'}$. The reverse inclusion follows by symmetry.

Remark 2.2.0.3. The Lang locus $X^L$ in Example 2.2.0.1 is slightly different from the "lieu de Lang" $X^L_F$ (a closed subset of $X_F$) defined by [UY10, (1)]. Let $\phi : X \to X_F$ be the natural morphism of schemes. For every finite extension $M/F$, let $X_F[M]$ be the image of the natural map $X_F(M) \to X_F$, then $\phi(X(X_F, M)) = X_F$. Because $\phi$ is integral and surjective integral morphisms preserve the dimension, $\phi(X(X_F, M)) = X_F[M]$ and hence $\phi(X(X_F, M)^{>0}) = X_F[M]^{>0}$. Therefore, $\phi(X^L) = X^L_F$.

We gather some properties of the so defined locus.

Lemma 2.2.0.4. Let $X/\mathbb{Q}^a$ be a variety.

1. (Union) If $X \supseteq \bigcup_{i=1}^r Z_i$, where each $Z_i$ is a subvariety of $X$, then $X^L = \bigcup_{i=1}^r Z_i^L$.

2. (Component) If $Z$ is a component of $X^L$, then $Z^L = Z$.

3. (Galois) If $f : X \to Y$ is Galois cover over $\mathbb{Q}^a$, then $f^{-1}(f(X^L)) = X^L$. If $Z \subseteq Y$ is an irreducible subvariety, and $Z'$ is a component of $f^{-1}(Z)$, then $f(f^{-1}(Z)^L) = f(Z'^L)$.

Proof.

$^2$The natural map $X_F(M) \to X_F$ is not injective in general.
CHAPTER 2. INTEGRAL POINTS OF SHIMURA VARIETIES: AN “ALL OR NOTHING” PRINCIPLE

- By Axiom 2, $\cup_{i=1}^n Z_i^L \subset X^L$. If $Y$ is a component of $X$, then there exists an index $i$ such that $Y \subset Z_i$. By Axiom 2, $Y^L \subset Z_i^L$. By definition, $X^L \subset \cup_{i=1}^n Z_i^L$.

- Write $X^L = \cup_{i=1}^n Z_i$ for the decomposition into components with $Z_1 = Z$.

- By Axiom 4,

$$Z \subset X^L = (X^L)_L = \cup_{i=1}^n Z_i^L.$$  

As $Z$ is irreducible, there is an index $i$ such that $Z \subset Z_i^L \subset Z_i$. Then $i = 1$ as $Z = Z_1$ is a component, so $Z = Z^L$.

- If $x \in f^{-1}(f(X^L))$, there is $x' \in X^L$ such that $f(x') = f(x)$. Let $\Theta$ be the Galois group of $f$. There is $\theta \in \Theta$ with $\theta(x') = x$, so $x \in X^L$ by Axiom 2. Therefore, $f^{-1}(f(X^L)) = X^L$. Since $\Theta$ permutes transitively the components of $f^{-1}(Z)$, one has $f^{-1}(Z) = \Theta \cdot Z'$, so $f^{-1}(Z)^L = \Theta \cdot Z'^L$ by 1. Then apply $f$ to both sides.

$\square$

In general, given an étale cover $f : X \rightarrow Y$, the induced map $X^L \rightarrow Y^L$ is not surjective. We introduce a sublocus that lifts along any étale cover.

For an integral variety $X/\mathbb{Q}$, define its \textit{locus at infinite level} by

$$X^{L_{\infty}} := \cap_{f:T \rightarrow X} f(T^L),$$

where $f : T \rightarrow X$ runs through all étale covers of $X$. By Axiom 3, $X^{L_{\infty}}$ is a subvariety of $X^L$. As $X$ is topologically noetherian, and $f(T^L) \subset X$ is closed for each $f : T \rightarrow X$, there exists a particular cover $f_1 : X_1 \rightarrow X$ such that $f_1(X_1^L) = X^{L_{\infty}}$. For any étale cover $X_2 \rightarrow X_1$, the composition $X_2^L \rightarrow X_1 \rightarrow X^{L_{\infty}}$ is still surjective. If $X^{L_{\infty}} \neq \emptyset$, then its components are \textit{positive dimensional} by Axiom 1. For a reducible $\mathbb{Q}$-variety $Y = \cup_{i=1}^n Y_i$ decomposed into components, define $Y^{L_{\infty}} = \cup_{i=1}^n Y_i^{L_{\infty}}$, which is a subvariety of $Y^L$.

\textbf{Lemma 2.2.0.5.}

1. If $f : T \rightarrow S$ is an étale cover over $\mathbb{Q}$, then $f^{-1}(S^{L_{\infty}}) = T^{L_{\infty}}$. In particular, $T^{L_{\infty}} = T$ if and only if $S^{L_{\infty}} = S$.

2. Let $X/\mathbb{Q}$ be a variety with $X^{L_{\infty}} = X$, $Y \subset X$ be a component. Then $Y^{L_{\infty}} = Y$.

\textbf{Proof.}

- We show $T^{L_{\infty}} \subset f^{-1}(S^{L_{\infty}})$. For $t \in T^{L_{\infty}}$, set $s = f(t)$. For any étale cover $g : S' \rightarrow S$, there is a diagram

$$
\begin{array}{ccc}
T & \xrightarrow{f} & S' \\
\downarrow{g'} & & \downarrow{g} \\
T' & \xrightarrow{f'} & S'
\end{array}
$$

- \textbullet\ We show $T^{L_{\infty}} \subset f^{-1}(S^{L_{\infty}})$. For $t \in T^{L_{\infty}}$, set $s = f(t)$. For any étale cover $g : S' \rightarrow S$, there is a diagram

$$
\begin{array}{ccc}
T & \xrightarrow{f} & S' \\
\downarrow{g'} & & \downarrow{g} \\
T' & \xrightarrow{f'} & S'
\end{array}
$$
where each arrow is an étale cover. There is $t' \in T'\!\!L$ lying above $t$, then $s' = f(t') \in S'\!\!L$ by Axiom 3, so $s = g(s') \in \text{Im}(S'\!\!L \to S)$. Thus, $s \in S^{L\infty}$.

Conversely, if $t \in f^{-1}(S^{L\infty})$, then $s = f(t) \in S^{L\infty}$. For any étale cover $Z \to T$, there is $N \to Z$ an étale cover such that $N \to S$ is a Galois cover:

$$N \to Z \to T \to S.$$ 

One has

$$(N \to T)^{-1}(t) \subset (N \to S)^{-1}(s) \subset (N \to S)^{-1}(S^{L\infty}) \subset (N \to S)^{-1}((N \to S)(N^L)) = N^L.$$ 

Therefore, $(Z \to T)^{-1}(t) \subset (N \to Z)(N^L) \subset Z^L$ and $t \in \text{Im}(Z^L \to T)$, so $t \in T^{L\infty}$. This shows that $f^{-1}(S^{L\infty}) = T^{L\infty}$.

- Let $X = \bigcup_{i=1}^n Y_i$ be the decomposition into components, where $Y = Y_1$. Then

  $Y \subset X = X^{L\infty} = \bigcup_{i=1}^n Y_i^{L\infty}.$

  There is $1 \leq i \leq n$ with $Y \subset Y_i^{L\infty} \subset Y_i$. As $Y$ is a component of $X$, $i = 1$ and $Y = Y^{L\infty}$.

\[\square\]

### 2.3 Shimura varieties and special subvarieties

We review some basic facts about Shimura varieties, the main objects of interest in this note.

Let $G/\mathbb{Q}$ be an affine algebraic group.

**Definition 2.3.0.1** ([Pin90, Sec. 0.1, p.13]). For every prime $p$, choose an embedding $\mathbb{Q} \to \overline{\mathbb{Q}}_p$.

- For an element $g = (g_p)_p \in \text{GL}_n(\mathbb{A}_f)$, let $\Gamma_p \leq \overline{\mathbb{Q}}_p^*$ be the subgroup generated by all eigenvalues of $g_p \in \text{GL}_n(\mathbb{Q}_p)$. If

  $$\cap_p (\overline{\mathbb{Q}}_p^* \cap \Gamma_p)_{\text{tor}} = \{1\},$$

  then $g$ is called *neat*.

- An element of $G(\mathbb{A}_f)$ is called neat if its image in some faithful algebraic representation of $G \to \text{GL}_n/\mathbb{Q}$ is neat.

- A subgroup of $G(\mathbb{A}_f)$ is called neat if all its elements are neat.

**Fact 2.3.0.2** ([Pin90, p.13]).
1. Let $K \leq G(\mathbb{A}_f)$ be a compact open subgroup. Then there is an open normal subgroup $K' \leq K$ that is neat.

2. If $K \leq G(\mathbb{A}_f)$ is a neat subgroup, then $K \cap G(\mathbb{Q})$ is a neat subgroup of $G(\mathbb{Q})$ in the sense of [Mil17b, p.34].

From now on, let $(G, X)$ be a Shimura datum in the sense of [Mil17b, Def. 3.3], $K \leq G(\mathbb{A}_f)$ be a compact open subgroup. Then $X$ is naturally a finite disjoint union of hermitian symmetric domains by [Mil17b, Prop. 5.9].

In view of Fact 2.3.0.2 1, we always make the mild assumption that $K$ is neat. Then by [Pin90, Prop. 3.3 (b)], $\text{Sh}_K(G, X)$ is naturally a complex manifold. Let $X^+$ be a connected component of $X$. The set $G(\mathbb{R})$ is naturally a (real) Lie group. For a Lie group $L$, let $L^+$ be its identity component. Let $G^{\text{ad}}$ be the quotient of $G$ by its center. Set $G(\mathbb{R})_+$ to be the preimage of $G^{\text{ad}}(\mathbb{R})^+$ under the natural morphism $G(\mathbb{R}) \to G^{\text{ad}}(\mathbb{R})$ of Lie groups, and

$$G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+.$$

Then $G(\mathbb{Q})_+$ is a finite-index subgroup of $G(\mathbb{Q})$. For each $g \in G(\mathbb{A}_f)$, put $\Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+$ and $S_g := \Gamma_g \backslash X^+$. By Fact 2.3.0.2 2 and [Mil17b, Prop. 4.1], $\Gamma_g$ is a neat (hence torsion-free) arithmetic subgroup of $G(\mathbb{Q})$ in the sense of [Mil17b, p.33]. From [Mil17b, Prop. 3.1], $S_g$ is naturally a connected complex manifold. Let $C$ be a set of representatives for the double coset space $G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K$. From [Mil17b, Lem. 5.13], the complex manifold $\text{Sh}_K(G, X) = \bigsqcup_{g \in C} S_g$.

By [Mil17b, Thm. 3.12; Cor. 3.16], the complex manifold $S_g$ has a unique canonical structure of a complex algebraic variety. The algebraic variety $S_g$ is an irreducible, smooth arithmetic locally symmetric variety ([Mil17b, p.58]). It is Zariski-open in its Baily-Borel compactification ([Mil17b, p.40]), which is a projective variety. Thus, $\text{Sh}_K(G, X)$ is also a smooth quasi-projective (reducible) complex algebraic variety. Let $E(G, X) \subset \mathbb{Q}^a$ be the reflex field of the Shimura datum $(G, X)$ ([Mil17b, Def. 12.2]). By [Mil17b, Rk. 12.3 (a)], $E(G, X)$ is a number field. From [Mil99, Rk. 2.4 (b)] and [Mil17b, p.128], $\text{Sh}_K(G, X)$ admits a unique (up to a unique isomorphism) canonical model over $E(G, X)$ in the sense of [Mil17b, Def. 12.8]. Hence a smooth quasi-projective variety over the $E(G, X)$.

Each connected component of $\text{Sh}_K(G, X)$ and the inclusion to the Baily-Borel compactification is defined over a finite abelian extension of $E(G, X)$ ([GN20, Remark (3), p.56], [Moo98b, p.282]). A component algebraic variety is called a Shimura variety associated with $(G, X, K)$.

For each open subgroup $K' \leq K$, the natural morphism $\text{Sh}_{K'}(G, X) \to \text{Sh}_K(G, X)$ is finite étale and defined over $E(G, X)$. There is a component $S'$ of $\text{Sh}_{K'}(G, X)$ such that $S' \to S$ is an étale cover defined over a finite extension of $E(G, X)$. When $K'$ is normal in $K$, this étale cover is Galois of group $K/K'$ ([CK16, p.1901]).

\[\text{It is a connected Shimura variety in the sense of [Mil17b, Def. 4.10] when $K$ is sufficiently small by [Moo98b, Prop. 2.9] and [Mil17b, Thm. 5.17].}\]
Definition 2.3.0.3. [Moc98a, Definition 2.5] An irreducible subvariety $Z \subset \text{Sh}_K(G, X)_C$ is called special if there exist an algebraic subgroup $H \leq G$ defined over $\mathbb{Q}$, an element $g \in G(A_f)$ and a connected component $D_H^+$ of

$$D_H := \{ x \in X | h_x : \text{Res}_{\mathbb{C}/\mathbb{R}}(G_m) \to G_R \text{ factors through } H_R \},$$

such that $Z(\mathbb{C})$ is the image of $D_H^+ \times gK$ in $\text{Sh}_K(G, X)(\mathbb{C}) = G(\mathbb{Q}) \backslash X \times G(A_f)/K$. (Notice that $D_H$ is a finite union of $H(\mathbb{R})$-conjugacy classes.)

A $\mathbb{C}$-point that is a special subvariety of $\text{Sh}_K(G, X)$ is called a special point of it. In the proof of Fact 2.1.0.3, Ullmo and Yafaev show that each component of the locus under consideration is a special subvariety. A component of any intersection of special subvarieties of $\text{Sh}_K(G, X)$ is again special ([Moc98a, Sec. 2.9]). Therefore, for an irreducible subvariety $Y \subset \text{Sh}_K(G, X)_C$, there is a smallest special subvariety $Z_Y \subset \text{Sh}_K(G, X)$ containing $Y$. If $Y$ is defined over $\mathbb{Q}^+$, then so is $Z_Y$. We say that $Y$ is Hodge generic in $Z_Y$. The set of $\mathbb{C}$-points that are not Hodge generic in $\text{Sh}_K(X, G)_C$ is a strict countable union of analytic subspaces ([CDK95]). Fact 2.3.0.4 is a characterization of special subvarieties.

Fact 2.3.0.4 ([UY10, Lem. 2.7], [UY14, Lem. 2.1]). Let $(H, X_H) \subset (G, X)$ be a Shimura subdatum with $H$ being the generic Mumford-Tate group on $X_H$. Let $X_H^+$ be a connected component of $X_H$ contained in $X^+$. Set $\Gamma_{H,g} = gK_g^{-1} \cap H(\mathbb{Q})_+$ and $Z = Z_g := \Gamma_{H,g} \backslash X_H^+$ (a component of $\text{Sh}_K(gK_g^{-1} \cap H(\mathbb{A}_f))(H, X_H)$).

Then the image $Z_g$ of $Z_g$ under the $\mathbb{C}$-morphism

$$\text{Sh}_{gK_g^{-1} \cap H(\mathbb{A}_f)}(H, X_H) \to \text{Sh}_K(G, X), \quad [x, h] \mapsto [x, hg]$$

is a special subvariety of $S_C$ and the induced morphism $\pi : \tilde{Z} \to Z$ is finite birational. Conversely, every special subvariety of $S_C$ arises in this way, so is defined over a number field.

Given an open subgroup $K'$ of $K$, let $f : S' \to S$ be the corresponding étale cover of Shimura varieties over $\mathbb{C}$. Then any component $Z'$ of $Z \times_S S'$ is an étale cover of $Z$. The image $Z'$ of $Z'$ in $S'$ is a special subvariety which is an étale cover of $Z$ and a component of $f^{-1}(Z)$.

We may replace $S$ by an étale cover for our purpose. Consider the open subgroup $K' := K \cap gK_g^{-1}$ of $K$, and set $\Gamma' = K' \cap G(\mathbb{Q})_+$ which induces an étale cover $S' \to S$ with $S' = \Gamma'/X^+$. Thus up to replace $S$ (resp. $K$, resp. $\Gamma$) by $S'$ (resp. $K'$, resp. $\Gamma'$), we may assume $g = 1$. Then $Z_g$ is a connected component of $\text{Sh}_{K(\mathbb{A}_f)}(H, X_H)$.

Lemma 2.3.0.5 is a synthesis of [UY10, Lemmas 2.8–2.10]. In Section 2.4, Lemma 2.3.0.5 forces certain subset of the original locus to live inside the infinite level locus, i.e., can lift along covers.

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4in the sense of [CLZ16, p.804]

5in the sense of [Moo17, p.286] and [Moo04, p.16]
Lemma 2.3.0.5. Let $Z \subset S$ be a subvariety over $\mathbb{Q}^a$ such that $Z_C$ is a finite union of special subvarieties, then:

1. There is a Galois cover $f : S' \to S$ over $\mathbb{Q}^a$ such that $f(f^{-1}(Z)^L) = Z^{L_\infty}$ and $f(S'^L) = S^{L_\infty}$.

2. If $Z$ is irreducible, one may further require that any component $Z' \subset f^{-1}(Z)$ is an étale cover of $Z$ and satisfies $Z'^L = Z'^{L_\infty}$.

3. If $S^{L_\infty} \subset Z$, then $S^{L_\infty} = Z^{L_\infty}$.

Proof.

We prove 1 and 2. By Lemma 2.2.0.4 1, we may replace $Z$ by one of its components and take a normal closure of the finitely many covers corresponding to each component of $Z$. Now that $Z$ is irreducible, $Z_C$ is a special subvariety of $S_C$. We may assume that $g = 1$.

As $S$ is topologically noetherian, there is an open subgroup $K' \subseteq K$ such that the induced étale cover $f_1 : S_1 \to S$ over $\mathbb{Q}^a$ satisfies $f(S_1^L) = S^{L_\infty}$. Similarly, there is an étale cover $g : Z_1 \to Z$ over $\mathbb{Q}^a$ such that $g(Z_1^L) = Z^{L_\infty}$.

Let $(H, X_H) \subset (G, X)$ be a Shimura subdatum and a component $\tilde{Z} \subset \text{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H)$ corresponding to $Z$ via Fact 2.3.0.4. Replacing $Z_1$ by an étale cover of $Z_1$, one may assume that there is an open subgroup $K'_H \subseteq K \cap H(\mathbb{A}_f)$ such that the induced morphism

\[ \text{Sh}_{K'_H}(H, X_H) \to \text{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H) \]

makes a component $\tilde{Z}' \subset \text{Sh}_{K'_H}(H, X_H)$ an étale cover of $\tilde{Z}$, and the morphism $\tilde{Z}' \to Z_1$ given by [Sza09, Prop. 5.5.5] is birational. Hence a commutative diagram of algebraic varieties over $\mathbb{Q}^a$:

\[
\begin{array}{ccc}
\tilde{Z}' & \longrightarrow & \text{Sh}_{K'_H}(H, X_H) \\
\downarrow & & \downarrow \\
Z_1 & \longrightarrow & \text{Sh}_{K' \cap H(\mathbb{A}_f)}(H, X_H) \\
\downarrow & & \downarrow \\
Z & \longrightarrow & S & \longrightarrow & \text{Sh}_K(G, X).
\end{array}
\]

Take an open normal subgroup $K_N \leq K$ such that $K_N \subset K'$ and $K_N \cap H(\mathbb{A}_f) \subset K'_H$. It induces an étale cover $S' \to S_1$ over $\mathbb{Q}^a$ such that the composition $f : S' \to S$ is a Galois cover of group $\Theta := K/K_N$. Here $S'$ is a quotient of $X^+$ and a component of $\text{Sh}_{K_N}(G, X)$. Then $f(S'^L) = S^{L_\infty}$.

Put $\Gamma' = K_N \cap H(\mathbb{Q})_+$. Then $\tilde{Z}_N := \Gamma' \setminus X^+_H$ is a component of $\text{Sh}_{K_N \cap H(\mathbb{A}_f)}(H, X_H)$ mapped inside $\tilde{Z}'$ under the morphism

\[ \text{Sh}_{K_N \cap H(\mathbb{A}_f)}(H, X_H) \to \text{Sh}_{K'_H}(H, X_H). \]
2.4. PROOF OF THE FORMAL ALTERNATIVE

We show that an alternative (Theorem 2.4.0.3) results from the formal axioms. Let $S$ be a Shimura variety associated with $(G, X, K)$ as defined in Section 2.3. Proposition 2.4.0.1 is related to [UY10, Prop. 3.4].

**Proposition 2.4.0.1.** If $S_{L\infty} = \emptyset$ and contains a component $Z$ that is Hodge generic in $S$, then $S_{L\infty} = S$.

**Proof.** We may replace $S$ by an étale cover induced by an open subgroup of $K$. Indeed, for an étale cover $f : S_1 \to S$, by Lemma 2.2.0.5 1, $S_{L\infty}^{f^{-1}(Z)} = S_{L\infty}^{f^{-1}(Z)} \supset f^{-1}(Z)$, and $f^{-1}(Z)$ has a component that is Hodge generic in $S_1$. Thus, the condition holds for $S_1$. If the statement $S_{L\infty} = S_1$ is true, then $S = S_{L\infty}$. Thus, one may assume $g = 1$.

Consider the Hecke correspondence. For $q \in G(\mathbb{Q})^+$, put $\Gamma_q = \Gamma_1 \cap q^{-1}\Gamma_1 q$ and $S_q = \Gamma_q \setminus X^+$. Then the morphisms $\alpha_q : S_q \to S$ (resp. $\beta_q : S_q \to S$) induced by $\text{Id}_{X^+}$ (resp. $X^+ \to X^+, \ x \mapsto q \cdot x$) are étale covers over $\mathbb{Q}^a$ ([UY10, p.700]):

---

Its image $Z'$ under the morphism

$$\text{Sh}_{K \cap H(A_f)}(H, X_H) \to \text{Sh}_{K}(G, X)$$

is a component of $f^{-1}(Z)$, which is an étale cover of $Z_1$. By Lemma 2.2.0.5 1, $Z''_L = Z'^{L\infty}$. Since $\Theta$ permutes transitively the components of $f^{-1}(Z)$, it remains true for any component. Point 2 is proved.

Now $f(Z''_L) = Z^{L\infty}$. By Lemma 2.2.0.4 3, $f(f^{-1}(Z)^L) = Z^{L\infty}$. Therefore, 1 is proved.

- By hypothesis and 1, $S'^L \subset f^{-1}(S^{L\infty}) \subset f^{-1}(Z) \subset S'$, so $S'^L = f^{-1}(Z)^L$ by Axioms 4 and 2. Thus, $S^{L\infty} = f(S'^L) = f(f^{-1}(Z)^L) = Z^{L\infty}$ by 1.

\[\square\]
Theorem 2.4.0.3 (Ullmo-Yafaev alternative)

Either $S^{L\infty} = \emptyset$ or $S^{L\infty} = S$.

**Proof.** By Lemma 2.2.0.5, we may replace $S$ by its étale cover induced by an open subgroup of $K$. We may therefore assume $S^L = S^{L\infty} \neq \emptyset$. For each component $Z \subset S^L$, dim$(Z) > 0$ and $Z^L = Z$.

We prove that $Z_C$ is a special subvariety of $S_C$.

Let $S_{M,C}$ be the smallest special subvariety of $S_C$ containing $Z_C$. By Fact 2.3.0.4, there is a subvariety $S_M \subset S$ containing $Z$ whose base change to $\mathbb{C}$ is $S_{M,C}$. We claim that $Z \subset S_{M,C}^L$. In fact, take $f : S^L \to S$ given by Lemma 2.3.0.5 2 for $S_M \subset S$. Since $Z$ is a component of $S^L$, any component $T \subset f^{-1}(Z)$ is a component of

$$f^{-1}(S^L) = f^{-1}(S^{L\infty}) = S_{M,C}^{L\infty} = S_{M}^{L\infty}$$

by Axiom 3. By Lemma 2.2.0.4 2, $T^L = T$. There is a component $S_M' \subset f^{-1}(S_M)$ containing $T$ and

$$f|_{S_M'} : S_M' \to S_M$$

is an étale cover. From the choice of $f$,

$$T = T^L \subset S_M^{L} = S_{M}^{L\infty}.$$

Therefore, $f(T) \subset f(S_{M}^{L\infty}) \subset S_{M}^{L\infty}$ and $Z = f(f^{-1}(Z)) \subset S_{M}^{L\infty}$ follows. The claims is proved.

The claim shows that $Z$ is a component of $S_{M}^{L\infty}$. By Proposition 2.4.0.1, $S_M = S_{M}^{L\infty} = S_{M}^{L} \subset S^L$. The irreducible subset $S_M$ of $S^L$ contains one component $Z$ of $S^L$, so $Z = S_M$ is a special subvariety of $S$. 

The Hecke operator $T_q = \beta_q \alpha_q^*$ acting on cycles of $S$. By Lemma 2.2.0.5 1, $\alpha_q(S^{L\infty}) = \beta_q(S^{L\infty}) = S^{L\infty}$, so

$$S^{L\infty} \subset T_q S^{L\infty}.$$

As noted in Section 2.2, dim$(Z) > 0$. The proof is completed by Theorem 2.4.0.2 below with $Y = S^{L\infty}$. 

**Theorem 2.4.0.2 (Ullmo-Yafaev).** Let $Y \subset S_C$ be a subvariety with one positive dimensional component that is Hodge generic in $S$. If $Y \subset T_q Y$ for all $q \in G(\mathbb{Q})^+$, then $Y = S$. 

**Proof.** Write $Y = Y_1 \cup Y_2$, where $Y_1$ is the union of components of $Y$ that are Hodge generic in $S$, and $Y_2$ is the union of other components. By assumption, dim$(Y_1) > 0$ and $Y_1 \subset T_q Y = T_q Y_1 \cup T_q Y_2$. Each component of $T_q Y_2$ is not Hodge generic in $S$, so $Y_1 \subset T_q Y_1$. By [UY10, Théorème 3.6], $Y_1 = S$ and $Y = S$. 

}\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure}
\caption{Diagram of the Hecke operator $T_q$ acting on cycles of $S$.}
\end{figure}
2.5. "ALL OR NOTHING" PRINCIPLE FOR INTEGRAL POINTS

Now that \( S^L \) is a finite union of special subvarieties, by Lemma 2.3.0.5.3, \((S^L)^{L_{\infty}} = S^{L_{\infty}} = S^L\). By Lemma 2.2.0.5.2, \( Z^{L_{\infty}} = Z \). This property is preserved up and down along étale covers (Lemma 2.2.0.5.1), so also preserved by Hecke correspondences. Explicitly, let \( Z' \subset T_q Z \) be a component, then there is a diagram

\[
\begin{array}{ccc}
Z_1 & \xrightarrow{\alpha_q} & Z \\
& \beta_q \swarrow & \\
Z & & Z'
\end{array}
\]

where \( Z_1 \subset \alpha_q^{-1}(Z) \) is a component and both arrows are étale covers. In particular, \( Z' = (Z')^{L_{\infty}} = Z'^{L} \subset S^L \), so \( T_q Z \subset S^L \) and hence \( T_q S^L \subset S^L \).

From the last paragraph in [UY10, p.705], \( S^L = S \).

Corollary 2.4.0.4 ([UY10, Thm. 1.1]). If a Shimura variety \( S/\mathbb{Q}^a \) is of sufficiently high level, then either \( S^L = \emptyset \) or \( S^L = S \).

Proof. As the level is high, we have \( S^L = S^{L_{\infty}} \). The proof is completed by Theorem 2.4.0.3. \( \square \)

2.5 “All or nothing” principle for integral points

We define an locus concerning integral points, an analog of Lang locus. We check the axioms for this locus and then an alternative principle follows.

Let \( X/\mathbb{Q}^a \) be an integral variety. From Section 2.2, there is a number field \( F \subset \mathbb{Q}^a \) and an integral variety \( X_F/F \) and an isomorphism \( X_F \otimes_F \mathbb{Q}^a \rightarrow X \). There exists a finite set \( \Sigma \) of places of \( F \) including all archimedean ones and an integral scheme \( X \) that is finite type and separated (hence flat) over the ring \( O_{F,\Sigma} \) of \( \Sigma \)-integers, whose generic fiber is \( X_F \). We call \( X \) an integral model for \( X \) relative to \((F, \Sigma)\). For any finite extension \( M/F \) and a finite set \( \Omega \) of places of \( M \) containing all the places above \( \Sigma \), which we denote by \((M, \Omega)/(F, \Sigma)\) informally, we define \( X(X, M, \Omega) \) to be the image of the injection \( X(O_{M, \Omega}) \rightarrow X(\mathbb{Q}^a) \).

Definition 2.5.0.1. Let \( X' \) be the Zariski closure of

\[
\cup_{(M, \Omega)/(F, \Sigma)} X(X, M, \Omega)^{>0}
\]

inside \( X \) and call it the integral Lang locus of \( X \) relative to \( (X, F, \Sigma) \).

The integral Lang locus \( X' \) is a subvariety of the Lang locus of \( X \).

Lemma 2.5.0.2. Given two models \( X_1/O_{F_1, \Sigma_1} \) and \( X_2/O_{F_2, \Sigma_2} \) for \( X \), we have \( X_1' = X_2' \).

Proof. By [Gro66, Corollaire 8.8.2.5], there is a number field \( F_3 \) containing \( F_1, F_2 \) with a finite set of places \( \Sigma_3 \supset \Sigma_1, \Sigma_2 \) such that there is an \( O_{F_3, \Sigma_3} \)-isomorphism

\[
X_1 \otimes_{O_{F_1, \Sigma_1}} O_{F_3, \Sigma_3} \rightarrow X_2 \otimes_{O_{F_2, \Sigma_2}} O_{F_3, \Sigma_3}
\]
extending the isomorphism between the generic fibers. For any \((M_1, \Omega_1)/(F_1, \Sigma_1)\), there is a pair \((M_2, \Omega_2)\) over \((F_3, \Sigma_3)\) and \((M_1, \Omega_1)\). Then

\[ X_1(O_{M_1}, \Omega_1) \subset X_1(O_{M_2}, \Omega_2) = X_2(O_{M_2}, \Omega_2), \]

so \(X(M_1, \Omega_1) \subset X(M_2, \Omega_2, \Omega_3)\). Therefore,

\[ X(M_1, \Omega_1) > 0 \subset X(M_2, \Omega_2) > 0 \subset X^I, \]

so \(X_I \subset X^I\). The reverse inclusion is similar.

By Lemma 2.5.0.2, we may safely use the notation \(X^I\) for \(X_I\) and call it integral Lang locus of \(X\). We extend the definition to reducible varieties as in Section 2.2.

**Example 2.5.0.3.** Let \(X = P^1 - \{0, 1, \infty\} = Y(2)\) be a modular curve over \(\mathbb{Q}\). Its compactification is \(X^* = P^1\) and the Lang locus of \(X\) is full but \(X\) is arithmetically hyperbolic by S-unit theorem. An elliptic curve with one puncture provides another such example by Siegel’s theorem on integral points.

**Remark 2.5.0.4.** If \(X\) is proper over \(\mathbb{Q}\), then there is an integral model \((X, F, \Sigma)\) of \(X\) such that \(X\) is proper over \(O_{F, \Sigma}\). By [Pool7, Thm. 3.2.13 (ii)], \(X^I\) coincides with the Lang locus of \(X\).

**Definition 2.5.0.5.** [Ull04, Definition 2.3] A variety \(X/\mathbb{Q}\) with \(X_I = \emptyset\) is called arithmetically hyperbolic.

A variety \(X/\mathbb{Q}\) is arithmetically hyperbolic if and only if for one (hence for every by Lemma 2.5.0.2) model \(X/O_{F, \Sigma}\), the set of integral points \(X(O_{M, \Omega})\) is finite for any \((M, \Omega)/(F, \Sigma)\), so [Ull04, Lem. 2.4] follows from Lemma 2.5.0.2.

**Conjecture 2.5.0.6** ([Ull04, Conjecture 2.5]). If \(X/\mathbb{Q}\) is a irreducible quasi-projective variety such that \(X^an\) is hyperbolic, then \(X\) is arithmetically hyperbolic.

Fact 2.5.0.7 is an evidence of Conjecture 2.5.0.6 and relies on Faltings’ solution [Fal83, Satz 6] to Shafarevich’s conjecture.

**Fact 2.5.0.7** ([Ull04, Thm. 3.2 (a)]). Let \((G, X)\) be an adjoint Shimura datum of abelian type, and let \(K \leq G(\mathbb{A}_f)\) be a neat compact open subgroup. Then each component of \(\text{Sh}_K(G, X)_{\mathbb{Q}}\) is arithmetically hyperbolic.\(^7\)

We prove that Ullmo-Yafaev’s alternative principle holds for integral points on general Shimura varieties. For this, we check the four axioms listed in Section 2.2 for integral Lang loci. The Axiom 1 holds, since a component of \(X^I\) with dimension 0 is an isolated point. Lemma 2.5.0.8 verifies Axioms 3 and 2.

**Lemma 2.5.0.8.** Let \(f : Z_1 \to Z_2\) be a \(\mathbb{Q}\)-morphism between integral varieties. If \(f\) has finite geometric fibers, then \(f(Z_1^I) \subset Z_2^I\).

\(^6\)In the sense of [Ull04, p.4118]

\(^7\)The model over \(\mathbb{Q}\) defined by Faltings used in [Ull04, Thm. 3.2 (a)] is the base change of the canonical model ([Moo98b, 2.17]).
2.5. “ALL OR NOTHING” PRINCIPLE FOR INTEGRAL POINTS

Proof. We may choose a finite set \( \Sigma \) of places of \( F \), a model \( \mathcal{X}/O_{F,\Sigma} \) for \( \mathcal{X} \), and an \( O_{F,\Sigma} \)-morphism \( f' : \mathcal{X}_1 \to \mathcal{X}_2 \) whose base change to \( F \) is \( f \). Then for any \( (M, \Omega)/(F, \Sigma), f'(\mathcal{X}_1(\Omega,M)) \subseteq \mathcal{X}_2(\Omega,M) \), so \( f(\mathcal{X}_1(\mathcal{X}_1, M, \Omega)) \subseteq \mathcal{X}_2(\mathcal{X}_2, M, \Omega) \).

Thus,

\[
f(\mathcal{X}_1(\mathcal{X}_1, M, \Omega)) \subseteq \mathcal{X}_2(\mathcal{X}_2, M, \Omega) .
\]

Let \( C \subseteq \mathcal{X}_1(\mathcal{X}_1, M, \Omega) \) be a component of positive dimension, then \( f(C) \) is irreducible but not a singleton (for otherwise, \( C \) is a finite set by assumption, which is a contradiction), so

\[
f(C) \subseteq \mathcal{X}_2(\mathcal{X}_2, M, \Omega) .
\]

Therefore, \( f(\mathcal{X}_1(\mathcal{X}_1, M, \Omega)) \subseteq \mathcal{X}_2(\mathcal{X}_2, M, \Omega) \) and \( f(\mathcal{X}_1') \subseteq \mathcal{X}_2' \). \( \square \)

Corollary 2.5.0.9 ([Ull04, Prop. 2.6]). A locally closed subvariety of an arithmetically hyperbolic variety is also arithmetically hyperbolic.

Proof. It follows from Lemma 2.5.0.8. \( \square \)

Lemma 2.5.0.10 verifies Axiom 4 for integral Lang loci.

Lemma 2.5.0.10. If \( X/Q^a \) is an integral variety, then \( X^l \subset (X^l)^l \).

Proof. Write \( X^l = \cup_{i=1}^n Y_i \) as the union of components. Take a model \( \mathcal{X}/O_{F,\Sigma} \) for \( X \) and let \( \mathcal{Y}_i \) be the scheme-theoretic image of the composition \( Y_i \to X \to \mathcal{X} \), which is model of \( Y_i \) relative to \( (F, \Sigma) \). For any \( (M, \Omega)/(F, \Sigma), \) \( x \in \mathcal{Y}(O_{M,\Omega}) \), i.e., a section \( x : \text{Spec}(O_{M,\Omega}) \to \mathcal{X} \), we have

\[
x|_{\text{Spec} M} \in X_M(M) \subseteq X(\mathcal{X}(\Omega,M,\Omega)) = X(\mathcal{X}(\Omega,M,\Omega))^{>0} \cup \{p_1, \ldots, p_t\},
\]

where \( p_i \in X(\mathcal{X}(\Omega,M,\Omega)) \) are isolated points. If \( x|_{\text{Spec} M} \notin \{p_1, \ldots, p_t\} \), then \( x|_{\text{Spec} M} \in X(\mathcal{X}(\Omega,M,\Omega))^{>0} \subseteq X^l \). Thus, there exists an index \( i \) such that \( x|_{\text{Spec} M} \in Y_i \). The section \( x \) factors through \( \mathcal{Y}_i \), i.e. \( x \in \mathcal{Y}_i(O_{M,\Omega}) \). Therefore, \( X(\mathcal{X}(\Omega,M,\Omega)) \subseteq \cup_{i=1}^n Y_i(O_{M,\Omega}) \cup \{p_1, \ldots, p_t\} \).

Then

\[
X(\mathcal{X}(\Omega,M,\Omega))^{>0} \subseteq \cup_{i=1}^n Y_i(O_{M,\Omega})^{>0} \subseteq \cup_{i=1}^n Y_i = (X^l)^l,
\]

so \( X^l \subset (X^l)^l \). \( \square \)

Proposition 2.5.0.11 implies [Ull04, Prop. 2.8].

Proposition 2.5.0.11 (Chevalley-Weil). If \( f : X \to Y \) is an étale cover over \( \mathbb{Q}^a \), then \( f(X^l) = Y^l \). In particular, \( X^l = X^l \) and the property that “integral Lang locus is full/empty” lifts and descends along étale covers.
**Proof.** By Lemma 2.5.0.8, \( f(X^I) \subset Y^I \). There is a number field \( F \), a finite set \( \Sigma \) of places of \( F \) containing all the archimedean ones and \( f' : \mathcal{X} \to \mathcal{Y} \) a finite étale \( O_{F,\Sigma} \)-morphism between models whose restriction to generic fiber recovers \( f \). From the proof in [SBW89, Section 4.2] we see that for any \( (\mathcal{M}, \Omega)/(F, \Sigma) \), there is \( (\mathcal{M}', \Omega')/(M, \Omega) \) such that \( Y(Y^I, \mathcal{M}, \Omega) \subset f(X(X^I, \mathcal{M}', \Omega')) \). Recall that zero dimensional scheme is discrete,

\[
Y(Y^I, \mathcal{M}, \Omega)^{>0} \subset f(X(X^I, \mathcal{M}', \Omega')^{>0}) \subset f(X^I)
\]

and we get \( Y^I \subset f(X^I) \).

**Theorem 2.5.0.12.** The integral Lang locus of a Shimura variety \( S \) is either empty or whole \( S \).

**Proof.** Because the formation of the integral Lang locus \((\cdot)^I\) satisfies the four axioms by Lemmas 2.5.0.8 and 2.5.0.10, the result is a combination of Theorem 2.4.0.3 and Proposition 2.5.0.11. \(\square\)
Chapter 3

Normality of monodromy group in generic Tannakian group

3.1 Introduction

In Lawrence-Sawin's work [LS20], the authors use the Lawrence-Venkatesh technique ([LV20]) to prove the Shafarevich conjecture for hypersurfaces in abelian varieties. Krämer-Maculan [KM23] apply roughly the same strategy to obtain an arithmetic finiteness result for very irregular varieties of dimension less than half the dimension of their Albanese variety. In both cases, a key is an estimation of monodromy groups of a moduli family.

In [LS20], the crucial estimation follows from a comparison to the Tannakian group arising from Krämer-Weissauer's convolution theory [KW15b] of perverse sheaves on the (geometric) generic fiber. We recall (roughly) their argument. Both the monodromy group and the Tannakian group on the geometric generic fiber are embedded as closed subgroups in the Tannakian group on the generic fiber. The geometric generic Tannakian group is normal in the generic Tannakian group ([LS20, Lem. 3.7], [JKLM23, Thm. 4.3]). This normality is used to prove that for most characters, the corresponding monodromy group contains the geometric generic Tannakian group. In the main result (Theorem 3.1.0.2), we prove that for many characters, the monodromy group is also normal in the generic Tannakian group.

Setting 3.1.0.1. Let \( k \) be an algebraically closed field of characteristic 0. Let \( X/k \) be an integral algebraic variety with generic point \( \eta \). Let \( A/k \) be an abelian variety. Denote by \( \rho : A \times X \rightarrow X \) and \( \pi : A \times X \rightarrow A \) the projections.

Let \( \ell \) be a prime number. Let \( \Lambda \) be an algebraic closure of \( \mathbb{Q}_\ell \). Let \( \text{Perv}^{\text{ULA}}(A \times X/X) \) be the abelian category of \( \rho \)-universally locally acyclic
(ULA) perverse sheaves,\(^1\) inside the triangulated category \(D^b_c(A \times X)\) of bounded constructible \(\Lambda\)-sheaves on \(A \times X\). Let \(\pi^\ell_1(A)\) be the étale fundamental group of \(A\) based at the geometric origin point. For every character\(^2\) \(\chi : \pi^\ell_1(A) \to \Lambda^*\), let \(\psi : \pi^\ell_1(A_\eta) \to \Lambda^*\) be the pullback of \(\chi\) along the morphism \(\pi|_{A_\eta} : A_\eta \to A\). For every \(K \in D^b_c(A \times X)\), every point \(x\) of \(X\), set \(K_x := K|_{A_x}\). Fix \(K \in \operatorname{Perv}^\text{ULA}(A \times X/X)\) a semisimple object\(^3\) in \(D^b_c(A \times X)\). The monodromy group \(G_{\text{mon}}(K, \psi)\) and the generic Tannakian group \(G_{\omega_\chi}(K_\eta)\) are defined in Section 3.5.

**Theorem 3.1.0.2** (Theorems 3.6.0.1, 3.6.0.6). Assume \(\dim A > 0\). Then there are at least uncountably many characters \(\chi : \pi^\ell_1(A) \to \Lambda^*\), such that the Tannaka group \(G_{\omega_\chi}(K_\eta)\) is a well-defined reductive group and contains the monodromy group \(G_{\text{mon}}(K, \psi)\) as a closed, reductive, normal subgroup.

The line of the proof of Theorem 3.1.0.2 is similar to that of André’s normality theorem [And92, Thm. 1]. It proves that for almost all stalks of a polarizable good variation of mixed Hodge structure, the connected monodromy group is a normal subgroup of the derived Mumford-Tate group. As [And92, p.10] explains, the normality is a consequence of the theorem of the fixed part due to Griffiths-Schmidt-Steinbrink-Zucker. In our case, an analog of the theorem of the fixed part is Theorem 3.1.0.3.

**Theorem 3.1.0.3** (Theorem 3.6.0.5). There is a subobject \(K^0 \subset K\) in \(\operatorname{Perv}^\text{ULA}(A \times X/X)\), such that for every character \(\chi_\ell : \pi^\ell_1(A) \to \Lambda^*\) of finite order prime to \(\ell\), there is a nonempty Zariski open subset \(U\) of the scheme \(C(A)_\ell\) of \(\ell\)-adic characters,\(^4\) such that for every \(\chi_\ell \in U\), the \(\Lambda\)-vector space \(H^0(A_\eta, K^0_\eta \otimes^L (L_\psi)_{\eta})\) is the subspace of \(\Gamma_{k(\eta)}\)-invariants of the \(\Gamma_{k(\eta)}\)-representation \(H^0(A_\eta, K^0_\eta \otimes^L (L_\psi)_{\eta})\), where \(\chi = \chi_\ell \chi_\ell\) and \(\psi : \pi^\ell_1(A_\eta) \to \Lambda^*\) is the pullback of \(\chi\) and \(L_\psi\) is the rank 1 lisse \(\Lambda\)-sheaf on \(A_\eta\) corresponding to \(\psi\).

By an algebraic variety, we mean a scheme of finite type, separated over a field. An algebraic group \(G\) is called **reductive** if its identity component \(G^0\) is reductive in the sense of [Mil7a, 6.46, p.135]. In an abelian category, an object is called semisimple if it is the direct sum of finitely many simple objects. The abelian category is called semisimple if every object is semisimple. For a field \(k\), its absolute Galois group is denoted by \(\Gamma_k\). Let \(\operatorname{Vec}_k\) be the category of finite dimensional \(k\)-vector spaces.

### 3.2 Cardinal argument

The main result is Lemma 3.2.2.5, used in the proof of Theorem 3.6.0.6. We show that over an uncountable algebraically closed field, a reasonable scheme has uncountably many rational points outside a countable union of strict closed subsets. For this, we need some elementary facts.

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\(^1\)Relative perverse sheaves and universal local acyclicity are reviewed in Section 3.3.

\(^2\)Characters are assumed to be connected.

\(^3\)In the sense of Definition 3.3.1.4

\(^4\)defined in Section 3.4
3.2. CARDINAL ARGUMENT

3.2.1 Elementary facts

**Lemma 3.2.1.1.** Let \( k \) be a field. Let \( X \) be a Noetherian \( k \)-scheme of dimension 0. Then the set underlying \( X \) is finite.

*Proof.* By assumption, the scheme \( X = \bigsqcup_{x \in X} \text{Spec}(O_{X,x}) \), where each \( O_{X,x} \) is an Artinian local ring. Then \( X \) is quasi-compact and discrete. Thus, the set \( X \) is finite. \( \square \)

**Lemma 3.2.1.2.** Let \( f : X \to Y \) be a dominant integral morphism of schemes. Then \( \dim X = \dim Y \). (It is possible that both sides are infinite.)

*Proof.* The induced morphism \( f_{\text{red}} : X_{\text{red}} \to Y_{\text{red}} \) between reductions is still dominant and integral. By [GD71, Prop. 5.4.3], for every affine open subset \( V = \text{Spec}(A) \subseteq Y \), \( f^{-1}(V) = \text{Spec}(B) \), the induced ring map \( A \to B \) is injective. It is also integral as \( f \) is integral. From [Sta23, Tag 00OK], one has \( \dim A = \dim B \), i.e., \( \dim V = \sup_{V} \dim f^{-1}(V) = \sup_{V} \dim V = \dim Y \). \( \square \)

**Lemma 3.2.1.3.** Let \( A \) be an integral domain that is not a field. Let \( n \geq 0 \) be an integer. If for every \( a \in A \setminus (A^* \cup \{0\}) \), one has \( \dim A/a = n \), then \( \dim A = n + 1 \).

*Proof.* Since \( A \) is not a field, there is \( a_0 \in A \setminus (A^* \cup \{0\}) \). Then \( Z(a_0) \) is a strict Zariski closed subset of the irreducible space \( \text{Spec}(A) \). Therefore, \( \dim A > \dim Z(a_0) = n \). Assume the contrary that \( \dim A \geq n + 2 \), then there is a chain

\[ 0 \subseteq p_1 \subseteq \cdots \subseteq p_{n+2} \]

of prime ideals of \( A \). Take \( f \in p_1 \setminus \{0\} \). Then

\[ p_1/(f) \subseteq \cdots \subseteq p_{n+2}/(f) \]

is a chain of prime ideals of \( A/f \), which contradicts the assumption that \( \dim A/f = n \). This contradiction completes the proof. \( \square \)

Fix an uncountable algebraically closed field \( k \). A \( k \)-scheme \( X \) is called "good", if \( \dim X \) is finite and \( X(k) \setminus (\bigcup_{i \geq 1} Z_i(k)) \) is uncountable for every sequence \( \{Z_i\}_{i \geq 1} \) of closed subschemes of \( X \) with \( \dim Z_i < \dim X \) for all \( i \). A \( k \)-algebra \( R \) is called good if the \( k \)-scheme \( \text{Spec}(R) \) is good.

**Lemma 3.2.1.4.** If \( X/k \) is a Noetherian scheme of dimension 1 with \( X(k) \) uncountable, then \( X \) is good.

*Proof.* Let \( \{Z_i\}_{i \geq 1} \) be a sequence of closed subschemes of \( X \) with \( \dim Z_i < \dim X \) for all \( i \). By Lemma 3.2.1.1, each \( Z_i(k) \) is finite. Therefore \( X(k) \setminus (\bigcup_{i \geq 1} Z_i(k)) \) is uncountable. \( \square \)

**Lemma 3.2.1.5.** If \( f : X \to Y \) is a finite surjective morphism of \( k \)-schemes and \( Y \) is good, then \( X \) is also good.
Proof. By Lemma 3.2.1.2, \( \dim X = \dim Y \) is finite. The induced map \( X(k) \to Y(k) \) is surjective. Indeed, for every \( y \in Y(k) \), \( X_y \) is a nonempty finite \( k \)-scheme. As \( k \) is algebraically closed, \( X_y(k) \) is nonempty, so there is \( x \in X(k) \) lying over \( y \).

Let \( \{Z_i\}_i \) be a sequence of closed subschemes of \( X \) with \( \dim Z_i < \dim X \). Then for every integer \( i \geq 1 \), since \( f \) is a closed morphism, \( Y_i := f(Z_i) \) is closed in \( Y \). Endow each \( Y_i \) with the reduced induced structure. Let \( Z'_i := f^{-1}(Y_i) = Y_i \times_Y X \). Then there is a canonical closed immersion \( Z_i \to Z'_i \) and \( \dim Z_i \to Y_i \) is a finite surjective morphism. By Lemma 3.2.1.2, one has \( \dim Y_i = \dim Z_i < \dim X = \dim Y \). Moreover, from last paragraph, the induced map

\[
X(k) \setminus (\cup_{i \geq 1} Z'_i(k)) \to Y(k) \setminus (\cup_i Y_i(k))
\]

is surjective. Because \( Y \) is good, the target is uncountable. Then \( X(k) \setminus (\cup_{i \geq 1} Z_i(k)) \) is also uncountable as it contains the source.

Proposition 3.2.1.6 should be well known.

**Proposition 3.2.1.6.** If \( X/k \) is a finite type scheme with \( \dim X > 0 \), then \( X \) is good.

**Proof.** Since \( X \) is of finite type over \( k \), its dimension \( m \) is finite and \( X \) has only finitely many irreducible components. Replacing \( X \) with an irreducible component of dimension \( m \), one may that assume \( X \) is irreducible. Then by [Har77, Exercise 3.20 (e), p.94], every nonempty open subset of \( X \) has dimension \( m \). Replacing \( X \) by an affine open, one may assume that \( X \) is affine. By Noether’s normalization lemma, there is a finite surjective \( k \)-morphism \( p : X \to A^m_k \). By Lemma 3.2.1.5 one may assume \( X = A^m_k \).

By induction on \( m \), one proves that \( A^m_k \) is good. The case \( m = 1 \) is treated by Lemma 3.2.1.4. Assume the statement is proved for \( m-1 \) with \( m \geq 2 \). Let \( \{Z_i\}_i \) be a sequence of closed subschemes of \( A^m_k \) with \( \dim Z_i < m \). Each \( Z_i \) is a Noetherian scheme, so it has only finitely many irreducible components. The set of irreducible components of the family \( \{Z_i\}_i \) is at most countable. Thus, one may assume that each \( Z_i \) is irreducible. Since \( Z_i(k) = Z_i,\text{red}(k) \), we may assume each \( Z_i \) is reduced.

The set of hyperplanes in \( A^m \) is uncountable, so there is a hyperplane \( H \subset A^m \) with \( H \neq Z_i \) for all integers \( i \geq 1 \). As the \( Z_i \) are irreducible and \( \dim H \geq \dim Z_i \), one gets \( H \nsubseteq Z_i \) for all \( i \), so \( H \cap Z_i \neq H \). Since \( H \) is irreducible, one gets \( \dim(H \cap Z_i) < \dim H \) for all \( i \). By the inductive hypothesis, the set \( H(k) \setminus (\cup_{i \geq 1} H \cap Z_i)(k) \) is uncountable, which is a subset of \( X(k) \setminus (\cup_{i \geq 1} Z_i(k)) \). The induction is completed.

**Lemma 3.2.1.7.** Let \( X/k \) be a Noetherian scheme. Then \( X \) is good if and only if \( X \) has a irreducible component \( C \) with \( \dim C = \dim X \) such that \( C \) is good in the reduced induced structure.

**Proof.** Endow every closed subset of \( X \) the reduced induced structure. Assume that there is such a component \( C \). Consider a sequence of closed subschemes
\{Z_i\} of \( X \) with \( \dim Z_i < \dim X \) for all \( i \). Then for every \( i \geq 1 \), \( \dim C \cap Z_i \leq \dim Z_i < \dim X = \dim C \). Since \( C \) is good, the set \( C(k) \setminus \bigcup_i(C \cap Z_i)(k) \) is uncountable. Therefore, \( X(k) \setminus \bigcup_i Z_i(k) \) is also uncountable.

Assume every component of \( X \) of maximum dimension is not good. As \( X \) is Noetherian, one can write \( X = \bigcup_{i=1}^n C_j \) as a finite union of the irreducible components. For every \( j \) with \( \dim C_j = \dim X \), the scheme \( C_j \) is not good. Therefore, there is a sequence \( \{Z_i^j\}_{i \geq 1} \) of closed subschemes of \( C_j \) such that \( \dim Z_i^j < \dim C_j \) for all \( i \) and \( C_j(k) \setminus \bigcup_i Z_i^j(k) \) is countable. The finite family of components \( C_k \) with \( \dim C_k < \dim X \), joint with the sequences \( \{Z_i^j\} \), for all \( j \) with \( \dim C_j = \dim X \), gives a countably family \( \{Z_s\} \) of closed subschemes of \( X \) with \( \dim Z_s < \dim X \) for all \( s \). Then \( X(k) \setminus (\bigcup_s Z_s(k)) \) is countable, so \( X \) is not good.

3.2.2 Cotorii are good

Similar to Proposition 3.2.1.6, we show every positive dimensional “cotorus” defined in Section 3.4 is good. By contrast, the proof is much longer, mainly due to the fact that the cotorus is not locally of finite type over the base field.

Definition 3.2.2.1. Let \( A \) be a \( k \)-algebra, and let \( A[X] \to B \) be an injective ring map. We say that \( B \) is Rücker over \( A/k \) if there is a nonempty family \( W \) of monic polynomials in \( A[X] \) such that the following axioms are fulfilled:

1. If \( f, g \in A[X] \) are monic polynomials with \( fg \in W \), then \( f, g \in W \).

2. For every \( \omega \in W \), the \( A \)-algebra \( B/\omega B \) is isomorphic to \( A[X]/\omega A[X] \).

3. For every \( f \in B \setminus \{0\} \), there is an automorphism \( \sigma \) of the \( k \)-algebra \( B \) and a unit \( u \in B^* \) such that \( u\sigma(f) \in W \).

Remark 3.2.2.2. From Axiom 1, one gets \( 1 \in W \). If \( W = \{1\} \), then by Axiom 3, for every \( f \in B \setminus \{0\} \), one has \( f \in B^* \), i.e., \( B \) is a field. Conversely, if \( B \) is a field, then \( B \) is Rücker over \( A/k \) with \( W = \{1\} \).

If \( W \neq \{1\} \), take \( w(\neq 1) \in W \). Then there is an \( A \)-isomorphism \( B/wB \to A[X]/w \), hence an isomorphism \( \Spec(A[X]/w) \to \Spec(B/w) \) of \( \Spec(A) \)-schemes. Because \( w \) is a monic polynomial different from 1, the ring map \( A \to A[X]/w \) is injective finite. The induced morphism \( \Spec(A[X]/w) \to \Spec(A) \) is surjective, and then \( \Spec(B/w) \to \Spec(A) \) is surjective. In particular, the morphism \( \Spec(B) \to \Spec(A) \) is surjective.

Lemma 3.2.2.3 is used in the proof of Lemma 3.2.2.5.

Lemma 3.2.2.3. Let \( A \) be Noetherian good \( k \)-algebra of Krull dimension \( n \). Let \( B \) be a domain but not a field containing \( A[X] \) and Rücker over \( A/k \). Assume that \( S \) is an uncountable subset of \( A \) such that for every \( f \in S \), the subset \( Z_A(f) \subset \Spec(A) \) is of dimension \( n - 1 \). Suppose that the family \( \{Z_A(f)f \in S\} \) is pairwise disjoint. Then \( B \) is a Noetherian good \( k \)-algebra of Krull dimension \( n + 1 \). Moreover, for every \( f \in S \), the subset \( Z_B(f) \) of \( \Spec(B) \) is of dimension \( n \), and the family \( \{Z_B(f)\}f \in S \) is pairwise disjoint.
By Lemma 3.2.1.5, the $k$-algebra $B/b$ is isomorphic to $\dim B/b$. By [Sta23, Tag 00OK], $\dim B/b$ is injective. For every integer $n$, the surjection $\mathbb{A}$ is injective. The proof is completed.

By Lemma 3.2.1.3, the Krull dimension $\dim B = n + 1$.

For every $f \in S$, $\dim Z_B(f) < \dim A$, so $f \neq 0$. The preimage of $Z_A(f)$ under the surjection $\text{Spec}(B) \rightarrow \text{Spec}(A)$ is $Z_B(f)$, so $Z_B(f)$ is nonempty. In particular, $f \notin B^*$. Moreover, the family $\{Z_B(f)\}_{f \in S}$ is pairwise disjoint. By (3.1), one gets $\dim Z_B(f) = n$ and $B/f$ is good.

According to the proof of Proposition 3.2.1.6, one may assume that each $Z_i$ is irreducible.

For each $f \in S$, by Lemma 3.2.1.7, there is a good irreducible component $C_f \subset Z_B(f)$ of dimension $n$. Then $\{C_f\}_{f \in S}$ is pairwise disjoint. As there are uncountably many pairwise different $C_f$, there is $f_0 \in S$ with $C_{f_0} \neq Z_j$ for all $j \geq 1$. For every $j \geq 1$, $Z_j$ is irreducible and $\dim C_{f_0} = n \geq \dim Z_j$, so $C_{f_0} \subsetneq Z_j$. Then $Z'_j := C_{f_0} \cap Z_j \neq C_{f_0}$. By the irreducibility of $C_{f_0}$, one has $\dim(Z'_j) < \dim C_{f_0}$. As $C_{f_0}$ is good, the subset $C_{f_0}(k) \setminus Z_{j \geq 1}(k)$ of $\text{Spec}(B)(k) \setminus (\cup_{j \geq 1}Z_j(k))$ is uncountable. The proof is completed.

Now we specialize to the case of a cotorus. For every integer $n \geq 0$, consider the $\Lambda$-algebra

$$A_n := (\mathbb{Z}_d[[X_1, \ldots, X_n]]) \otimes_{\mathbb{Z}_d} \Lambda. \quad (3.2)$$

By [GL96, Prop. 3.2.2 (1)], the natural morphism $A_n \rightarrow \Lambda[[X_1, \ldots, X_n]]$ is injective.

**Fact 3.2.2.4.** For every integer $n$,

1. ([GL96, Thm. A.2.1, Prop. A.2.2.1]) if $n \geq 0$, then the ring $A_n$ is a Noetherian, regular, Jacobson domain of Krull dimension $n$;

2. ([GL96, Prop A.2.2.2, proof of A.2.2.3 (ii)]) if $n \geq 1$, then $A_n$ is Rücker over $A_{n-1}/\Lambda$.

**Lemma 3.2.2.5.** For every integer $n \geq 1$, the $\Lambda$-algebra $A_n$ is good.
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Proof. Since $\Lambda$ is a flat $\mathbb{Z}_l$-module, the injection $\mathbb{Z}_l[X_1, \ldots, X_n] \to \Lambda[[X_1, \ldots, X_n]]$ induces an injection $\Lambda[X_1, \ldots, X_n] \to A_n$. The natural map

$$\text{Spec}(\Lambda[[X_1, \ldots, X_n]]) \to A_n^\Lambda$$

(3.3)

factors through a morphism $p_n : \text{Spec}(A_n) \to A_n^\Lambda$.

Let $\mathcal{M} = \bigcup m_E$, where $E$ runs through all finite extensions of $\mathbb{Q}_l$ inside $\Lambda$ and $m_E$ is the maximal ideal of the ring of integers of $E$. Then $\mathcal{M}$ is the maximal ideal of the integral closure $\mathbb{Z}_l$ of $\mathbb{Z}_l$ inside $\Lambda$. The residue field $\mathbb{Z}_l/\mathcal{M}$ is the algebraic closure of the finite field $F_\ell$, hence countable. As $\mathbb{Z}_l$ is uncountable, so is the set $\mathcal{M}$.

For every $(a_1, \ldots, a_n) \in \mathcal{M}^n$, there is a surjective morphism of $\Lambda$-algebras:

$$\Lambda[[X_1, \ldots, X_n]] \to \Lambda, \quad f \mapsto f(a_1, \ldots, a_n).$$

Its kernel is a $\Lambda$-point of $\text{Spec}(\Lambda[[X_1, \ldots, X_n]])$, whose image under the map (3.3) is $(a_1, \ldots, a_n) \in A_n^\Lambda(\Lambda)$. Therefore, $\mathcal{M}^n$ is contained in $p_n(\text{Spec}(A_n)(\Lambda))$. In particular, $\text{Spec}(A_n)(\Lambda)$ is uncountable.

By induction on $n$, we prove that $A_n$ is good and $\{Z_{A_n}(X_1 - a)\}_{a \in \mathcal{M}}$ is a pairwise disjoint closed subsets with dimension $n - 1$ of $\text{Spec}(A_n)$.

When $n = 1$, by Lemma 3.2.1.4 and Fact 3.2.2.1, $A_1$ is good. Moreover, $\{Z_{A_1}(X_1 - a)\}_{a \in \mathcal{M}}$ is a family of closed point in $\text{Spec}(A_1)$ and these points are pairwise different. The statement is proved for $n = 1$. Assume the statement for $n - 1$ with $n \geq 2$. By Fact 3.2.2.4 and Lemma 3.2.2.3, the statement holds for $n$. The induction is completed. 

\[\square\]

3.2.3 Representations of a reductive group

Let $k$ be a field. For a group scheme $G/k$, let $\text{Rep}(G)$ be the category of (rational) representations of $G$. Let $H \leq G$ be an open subgroup of finite index. Let $\rho : H \to \text{GL}(W)$ be an object of $\text{Rep}(H)$. Let $V$ be the set of functions $f : G \to W$ which satisfy $f(hg) = \rho(h)f(g)$ for all $h \in H, g \in G$. Then $V$ is naturally a $k$-vector space of dimension $[G : H] \dim_k W$.

Definition 3.2.3.1. Define a homomorphism $\sigma : G \to \text{GL}(V)$ by $\sigma(g)(f)(x) = f(xg)$ for all $g, x \in G, f \in V$. Then $\sigma$ is an object of $\text{Rep}(G)$, called the representation induced by $\rho$ and denoted by $\text{Ind}_H^G \rho$.

An equivalent reformulation is as follows. Take a set of left representatives $\{g_1, \ldots, g_n\}$ of $H$ in $G$. As a $k$-vector space $V = \bigoplus_{i=1}^n g_iW$. Here each $g_iW$ is an isomorphic copy of $W$ whose elements are written as $g_1w$ with $w \in W$. For every $g \in G$, there exist a permutation $\sigma$ in the symmetric group $S_n$ and a subset $\{h_1, \ldots, h_n\}$ of $H$ with $gg_i = g_{\sigma(i)}h_i$. The $G$-action on $V$ is

$$\sigma(g) \sum_{i=1}^n g_iw_i := \sum_{i=1}^n g_{\sigma(i)}\rho(h_i)w_i.$$

Thus we get an additive functor $\text{Ind}_H^G : \text{Rep}(H) \to \text{Rep}(G)$. 
Proposition 3.2.3.2 (Frobenius reciprocity). The functor $\text{Ind}_H^G : \text{Rep}(H) \rightarrow \text{Rep}(G)$ is right adjoint to the restriction functor $\text{Res}_H^G : \text{Rep}(G) \rightarrow \text{Rep}(H)$. For $W \in \text{Rep}(H)$, let
\[ \text{ev}_W : \text{Ind}_H^G W \rightarrow W, \quad f \mapsto f(1) \]
be the evaluation ($k$-linear) map. For $V \in \text{Rep}(G)$, consider the maps
\[ \text{Hom}_{\text{Rep}(G)}(V, \text{Ind}_H^G W) \rightarrow \text{Hom}_{\text{Rep}(H)}(\text{Res}_H^G V, W), \quad \phi \mapsto \text{ev}_W \circ \phi; \]
\[ \text{Hom}_{\text{Rep}(H)}(\text{Res}_H^G V, W) \rightarrow \text{Hom}_{\text{Rep}(G)}(V, \text{Ind}_H^G W), \quad \psi \mapsto \psi_* , \]
where $\psi_*(v)(g) = \psi(gv)$. The two maps are inverse to each other and functorial in $V$ and $W$.

Proof. By assumption, the map $\text{ev}_W$ is $H$-equivariant. Hence, it gives a natural transformation $\text{ev} : \text{Res}_H^G \text{Ind}_H^G \rightarrow \text{Id}_{\text{Rep}(H)}$. The two maps are functorial in $V$ and $W$.

For every $\phi \in \text{Hom}_{\text{Rep}(G)}(V, \text{Ind}_H^G W)$, every $v \in V$ and every $g \in G$, one has
\[ (\text{ev}_W \circ \phi)_*(v)(g) = (\text{ev}_W \circ \phi)(gv) = \phi(gv)(1) = [g\phi(v)](1) = \phi(v)(g), \]
so $(\text{ev}_W \circ \phi)_* (v) = \phi(v)$. Therefore, $(\text{ev}_W \circ \phi)_* = \phi$.

Conversely, for every $\psi \in \text{Hom}_{\text{Rep}(H)}(V, W)$, $(\text{ev}_W \circ \psi_*)(v) = \psi_*(v)(1) = \psi(v)$ for all $v \in V$. Therefore, $\text{ev}_W \circ \psi_* = \psi$. Therefore, the two maps are inverse to each other.

Lemma 3.2.3.3. Assume that $H$ is normal in $G$. Let $\pi : G \rightarrow \text{GL}(V)$ be a simple representation. Then there is a simple representation $\rho : H \rightarrow \text{GL}(W)$ such that $\pi$ is a $G$-subrepresentation of $\text{Ind}_H^G \rho$.

Proof. Since in the abelian category $\text{Rep}(H)$ is Noetherian and Artinian, by [Sta23, Tag 0FCJ], there is a simple quotient $\text{Res}_H^G V \rightarrow W$ in $\text{Rep}(H)$. By Proposition 3.2.3.2, it induces a nonzero morphism $V \rightarrow \text{Ind}_H^G(W)$ in $\text{Rep}(G)$. As $V$ is a simple object of $\text{Rep}(G)$, this morphism identifies $V$ as a $G$-subrepresentation of $\text{Ind}_H^G(W)$.

Lemma 3.2.3.4. Assume that $k$ is algebraically closed of characteristic 0. Let $G/k$ be a reductive algebraic group. Then up to isomorphism, $G$ has at most countably many representations.

Proof. By Lemma 3.2.3.3, for every simple representation $W$ of $G$, there is a simple representation $U$ of $G^c$ such that $W$ is isomorphic to a $G$-subrepresentation of $\text{Ind}_{G^c}^G U$. By [Mil17a, 22.3], up to isomorphism in $\text{Rep}(G^c)$, there are at most countably many such $U$. As the abelian category $\text{Rep}(G)$ is Noetherian and Artinian, by Lemma 3.2.3.5, each $\text{Ind}_{G^c}^G U$ has only finitely many simple subobjects up to isomorphism. Therefore, up to isomorphism $\text{Rep}(G)$ has at most countably many simple objects. By [Mil17a, Cor. 22.43], every object of $\text{Rep}(G)$ is semisimple. Consequently, $\text{Rep}(G)$ has at most countably many objects up to isomorphism.
In a Noetherian and Artinian abelian category, an object may have infinitely many distinct subobjects up to isomorphism.

**Lemma 3.2.3.5.** Let $A$ be an abelian category, and let $X \in A$ be a Noetherian and Artinian object.

1. Let $Y$ be a simple subquotient of $X$. Then there is a composite series of $X$ with one graded piece isomorphic to $Y$. In particular, up to isomorphism $X$ has only finitely many simple subquotients.

2. If every subobject of $X$ admits a direct complement, then $X$ is semisimple.

**Proof.**

1. There is a subobject $i : X_0 \subset X$ and a quotient $q : X_0 \to Y$ in $A$. Let $N = \ker(q)$. By [Sta23, Tag 0FCH, Tag 0FCJ], both $N$ and $X/X_0$ are Noetherian and Artinian. From [Sta23, Tag 0FCJ], they admit composite series. A composite series of $X/X_0$ is equivalent to a filtration $X_0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$ by subobjects such that $X_i/X_{i-1}$ is simple for $1 \leq i \leq n$. This filtration and every composite series of $N$ glue to a composite series of $X$ with a step $N \subset X_0$, whose factor is isomorphic to $Y$. By the Jordan-Hölder lemma [Sta23, Tag 0FCK], up to isomorphism $Y$ has finitely many choices.

2. One may assume that $X \neq 0$. Let $P$ be the family of nonzero semisimple subobjects of $X$. By [Sta23, Tag 0FCJ], $X$ has a nonzero simple subobject, so $P$ is nonempty. Since $X$ is Noetherian, the family $P$ has a maximal element $i : X_0 \to X$. By assumption, there is a subobject $F \subset X$ with $X_0 \oplus F = X$. We claim that $F = 0$. Otherwise, by [Sta23, Tag 0FCJ], $F$ has a nonzero simple subobject $F_0$, so $X_0 \oplus F_0 \in P$ is strictly larger than $X_0$, a contradiction. From the claim, $i$ is an isomorphism and $X$ is semisimple.

\[\square\]

### 3.3 Recollections on constructible sheaves

No originality is claimed in Section 3.3. Let $k$ be a field where $\ell$ is invertible. For every algebraic variety $X/k$, denote by $D^b_c(X) : = D^b_c(X, \Lambda)$ the triangulated category of complexes of $\Lambda$-sheaves on $X$ with bounded constructible cohomologies defined in [BBD82, p.74]. For a closed subvariety $i : Z \to X$ and $K \in D^b_c(X)$, set $K|_Z := i^*K$.

#### 3.3.1 Basics

As [KW01, p.110] explains, Fact 3.3.1.1 follows from [KW01, p.344].
Fact 3.3.1.1 (Projection formula). Let \( f : X \to Y \) be a morphism of algebraic varieties over an algebraically closed field where the prime \( \ell \) is invertible. Let \( L \) be a bounded complex of lisse \( \Lambda \)-sheaves on \( Y \). Then there is a natural isomorphism \((Rf_\ast \cdot) \otimes^L L \to Rf_\ast (\cdot \otimes^L f^* L)\) of functors \( D^b(X) \to D^b(Y)\).

Fact 3.3.1.2 ([FK13, Prop. 12.10]). For every constructible \( \Lambda \)-sheaf \( F \) on \( X \), there is a nonempty Zariski open subset \( U \subset X \) such that \( F|_U \) is a lisse sheaf. In particular, when \( X \) is integral with generic point \( \eta \), there is a natural \( \Lambda \)-representation \( \Gamma_{k(\eta)} \to \text{GL}(F_\eta)\).

The heart of the standard (resp. perverse) t-structure on \( D^b_c(X) \) is denoted by \( \text{Cons}(X) \) (resp. \( \text{Perv}(X) \)). The corresponding six-functor formalism is reviewed in [ES, 2.5].

Fact 3.3.1.3. Let \( E/F \) be an extension of algebraically closed fields of characteristic 0. Let \( X/F \) be an algebraic variety. Then:

   \[ (\cdot)_E : \text{Perv}(X) \to \text{Perv}(X_E) \]
   is exact, fully faithful.

2. [BBDG82, Thm. 4.3.1 (ii)] An object of \( \text{Perv}(X) \) is simple if and only if it is simple in \( \text{Perv}(X_E) \).

Definition 3.3.1.4. [BC18, Def. 78] An object \( K \in D^b_c(X) \) is called semisimple if it is isomorphic to a finite direct sum of degree shifts of semisimple objects of \( \text{Perv}(X) \).

If \( K \in D^b_c(X) \) is semisimple, then there is a non-canonical isomorphism \( K \to \oplus_{q \in \mathbb{Z}} \mathcal{H}^q(K)[-q] \) in \( D^b_c(X) \), where each \( \mathcal{H}^q(K) \) is a semisimple object of \( \text{Perv}(X) \). A degree shift of a semisimple object of \( D^b_c(X) \) is still semisimple.

Example 3.3.1.5. Every perverse cohomology sheaf of a semisimple object of \( D^b_c(X) \) is semisimple. By contrast, the cohomology sheaves may be no longer semisimple in \( D^b_c(X) \).

Consider \( k = \mathbb{C} \) and \( X = A^1 \). Let \( j : U = A^1 \setminus \{ 0, 1 \} \to X \) be the inclusion. Then \( \pi_1(U^\text{an}, -1) \) is the free group generated by two loops \( a \) and \( b \), surrounding 0 and 1 respectively. There is a unique morphism
\[ \pi_1(U^\text{an}, -1) \to \text{SL}_2(\mathbb{Z}) \] (3.4)
sending \( a, b \) to
\[
A = \begin{pmatrix} -1 & -2 \\ 0 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 0 \\ -2 & -1 \end{pmatrix}
\]
respectively. By Grauert-Remmert’s theorem (see, e.g., [Sza09, Thm. 5.7.4]), \( \pi_1^\text{et}(U, -1) \) is the profinite completion of \( \pi_1(U^\text{an}, -1) \). Since \( \text{SL}_2(\mathbb{Z}_\ell) \) is a profinite group, the morphism (3.4) extends to a continuous morphism
\[ \pi_1^\text{et}(U, -1) \to \text{SL}_2(\mathbb{Z}_\ell) \] (3.5)
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It defines a 2-dimensional $\Lambda$-representation of $\pi_1^\et(U, -1)$.

The representation (3.5) is irreducible. Otherwise, assume that $v := (x, y)^T \neq 0$ in $\Lambda^2$ generates a 1-dimensional subrepresentation. Then $Av = (-x - 2y, -y)^T$ is parallel to $v$. Therefore, $y = 0$. Similarly, $Bv = (-x, -2x - y)^T$ is parallel to $v$, then $x = 0$, a contradiction.

Let $L$ be the rank 2 simple lisse $\Lambda$-sheaf on $U$ corresponding to (3.5). Then $L^\an$ is the local system corresponding to (3.4). For every small open ball $B_0 \subset X^\an$ centered at 0, $H^0(B_0, j_-^\an L^\an)$ is the kernel of the linear operator $A - 1$ on the stalk $L^\an_{\mathfrak{m}}$. Since $A - 1$ is invertible, $H^0(B_0, j_-^\an L^\an) = 0$. Therefore, the stalk $(j_-^\an L^\an)_0 = 0$. Similarly, the stalk $(j_-^\an L^\an)_1 = 0$. In conclusion, the natural morphism $j_-^\an L^\an \to j_-^\an L^\an$ is an isomorphism in $\text{Cons}(X^\an)$.

We prove that $H^1(U^\an, L^\an) = H^1(\pi_1(U^\an, -1), L^\an)$ is nonzero. Define a map $f : \pi_1(U^\an, -1) \to \Lambda^2$ inductively. Set $f(e) = 0$, $f(a) = f(b) = (1, 0)^T$, $f(a^{-1}) = -A^{-1}f(a)$, and $f(b^{-1}) = -B^{-1}f(b)$. Once $f$ is defined for every element of length $n \geq 1$, we define it on elements of length $n + 1$ as follows. For every element $g \in \pi_1(U^\an, -1)$ of length $n$, set

$$f(ag) = Af(g) + f(a), \quad f(bg) = Bf(g) + f(b),$$

$$f(a^{-1}g) = A^{-1}f(g) + f(a^{-1}), \quad f(b^{-1}g) = B^{-1}f(g) + f(b^{-1}).$$

The map $f$ is a crossed homomorphism. It is not a boundary, because the equation $(A - 1)x = (B - 1)x = (1, 0)^T$ admits no solution in $\Lambda^2$.

Therefore, $L^\an$ is in the cohomology support loci of $U^\an$ (in the sense of [BLSW17, p.295]). From [HT07, Example 8.1.35 (ii)], $j_! L[1]$ is simple in $\text{Perv}(X)$. By [BLSW17, p.299], $j_-^\an L^\an[1]$ is not semisimple in $\text{Perv}(X^\an)$.

By [BBDG82, Thm. 4.3.1 (ii)], the intermediate extension $K := j_! L[1]$ is a simple object of $\text{Perv}(X)$. We claim that $H^{-1}K$ is not semisimple in $D^b_c(X)$. From [HT07, Prop. 8.2.11], $K$ is isomorphic to $\tau_{\leq -1} R j_! L[1]$, where $\tau_{\leq -1}$ is the truncation functor with respect to the standard $t$-structure on $D^b_c(X)$. Thus, $H^{-1}K$ is isomorphic to $H^{-1}(R j_! L[1]) = j_! L$ in $\text{Cons}(X)$.

Then $(H^{-1}K)^\an$ is isomorphic to $j_-^\an L^\an$ in $\text{Cons}(X^\an)$. From [Kat90, p.375], $(H^{-1}K)[1] \in \text{Perv}(X)$. Since $(H^{-1}K)^\an[1]$ is not semisimple in $\text{Perv}(X^\an)$, by [Kat90, Lem. 12.7.1.1], $(H^{-1}K)[1]$ is not semisimple in $\text{Perv}(X)$. The claim is proved.

**Lemma 3.3.1.6.** Let $U \subset X$ be an open subset of $X$. Then the restriction functor $\text{Perv}(X) \to \text{Perv}(U)$ sends every simple object of $\text{Perv}(X)$ to a simple or zero object of $\text{Perv}(U)$. In particular, the restriction function $D^b_c(X) \to D^b_c(U)$ preserves semisimplicity.

**Proof.** Let $K$ be a simple object of $\text{Perv}(X)$. By [BBDG82, Thm. 4.3.1 (ii)], there is an irreducible, locally closed and geometrically smooth subvariety $j : V \to X$ and a simple lisse $\Lambda$-sheaf on $V$, such that $K$ is isomorphic to $j_! L[\dim V]$. If $V$ is disjoint from $U$, then $K|_U = 0$. Otherwise, take a geometric point $\bar{x}$ of $V \cap U$. From [Sta23, Tag 0BQI], the morphism $\pi^\et_1(U \cap V, \bar{x}) \to \pi^\et_1(\bar{x})$ is surjective. Thus, the composite representation

$$\pi^\et_1(U \cap V, \bar{x}) \to \text{GL}(L_{\bar{x}})$$
is also irreducible, i.e., the lisse $\Lambda$-sheaf $L|_{U \cap V}$ is simple. Let $h : U \cap V \to U$ be the base change of $j$. Then $K|_U$ is isomorphic to $h_*L|_{U \cap V}[\dim U \cap V]$, hence simple in $\Perv(U)$.  

When $k = \mathbb{C}$, Fact 3.3.1.7 1 follows from Kashiwara’s conjecture for semisimple perverse sheaves\footnote{formulated in [Kas98, Sec. 1]; see also [Dri01, Sec. 1.2, 1]. It is reduced to de Jong’s conjecture by Drinfeld [Dri01] that is proved in [BK06] and [Gai07].} and the paragraph following [BBDG82, Thm. 6.2.5]. The case of general $k$ follows via Fact 3.3.1.3.

**Fact 3.3.1.7.** Let $k$ be an algebraically closed field of characteristic 0. Let $f : X \to Y$ be a proper morphism of algebraic varieties over $k$. Let $K$ be a semisimple object of $D^b_c(X)$.

1. (Decomposition theorem) Then $Rf_*K$ is a semisimple object of $D^b_c(Y)$.

2. (Global invariant cycle theorem, [BBDG82, Cor. 6.2.8]) Let $i$ be an integer. Let $V \subset Y$ be a nonempty connected open subset such that $H^iRf_*K|_V$ is a lisse sheaf. Then for every $y \in V(k)$, the canonical map

$$H^i(X, K) \to H^i(X_y, K|_{X_y})^{\pi^*(V,y)}$$

is surjective.

For $F \in \text{Cons}(X)$, set $\text{Supp} F := \{x \in X(k^a) : F_x \neq 0\}$ to be its support. Then $\text{Supp} F$ is a quasi-constructible subset of $X$ in the sense of [Gro66, 10.1.1]. For $K \in D^b_c(X)$, set $\text{Supp} K = \bigcup_{j \in \mathbb{Z}} \text{Supp} H^j K$.

**Lemma 3.3.1.8.** Let $L$ be a lisse $\Lambda$-sheaf of rank 1 on $X$. Then

$$\cdot \otimes^L L : D^b_c(X) \to D^b_c(X)$$ (3.6)

is an equivalence of categories that is $t$-exact relative to perverse $t$-structures, with a quasi-inverse $\cdot \otimes^L L'$.

**Proof.** By associativity of the derived tensor product $\otimes^L$, the pair of functors $(\cdot \otimes^L L, \cdot \otimes^L L')$ is an equivalence. The functor (3.6) is right $t$-exact relative to perverse $t$-structures. In fact, it is $t$-exact relative to the standard $t$-structures. For every $K \in pD^{\leq 0}(X)$, every integer $q$, one has $H^q(K \otimes^L L) = H^q(K) \otimes^L L$, so $\text{Supp} H^q(K \otimes^L L) = \text{Supp} H^q(K)$. Thus, $K \otimes^L L \in pD^{\leq 0}(X)$.

The functor (3.6) is also left $t$-exact. Indeed, by [KW01, II, Cor. 7.5 f)], the functor $\mathbb{D}_X(\cdot \otimes^L L) = R\text{Hom}(L, \mathbb{D}_X \cdot) = L' \otimes^L \mathbb{D}_X \cdot$ on $D^b_c(X)$. For every $K \in pD^{\geq 0}(K)$, $\mathbb{D}_X K \in pD^{\leq 0}(X)$. By last paragraph, $L' \otimes^L \mathbb{D}_X K \in pD^{\geq 0}(X)$. Therefore, $K \otimes^L L \in pD^{\geq 0}(X)$.  

\footnote{such $V$ exists by Fact 3.3.1.2}
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3.3.2 Universal local acyclicity

All schemes in Section 3.3.2 are assumed to be qcqs. For a scheme $X$ and a geometric point $x$ of it, set $X_x := \text{Spec}(O_{X,x})$ for the strict henselization of $X$ at $x$. Let $f : X \to S$ be a separated morphism having finite presentation between schemes over $\mathbb{Z}[1/\ell]$.

**Definition 3.3.2.1** ([Sta23, Tag 0GJM], [Bar23, Def. 1.2]). Let $K$ be an object of $\mathcal{D}^b_c(X)$.

- If for every geometric point $x$ of $X$ and every geometric point $t$ of $S$, with $s = f(x)$, the canonical morphism $R\Gamma(X_x, K) \to R\Gamma(X_x \times_S t, K)$ is an isomorphism, then $f$ is called locally acyclic relative to $K$.

- If for every morphism $S' \to S$ of schemes, the base change $f' : X' \to S'$ of $f$ is locally acyclic relative to the pullback of $K$ to $X'$, then $K$ is called $f$-universally locally acyclic ($f$-ULA). Let $\mathcal{D}_{ULA}(X/S)$ be the full subcategory of $\mathcal{D}^b_c(X)$ comprised of $f$-ULA objects.

By [HS23, Thm. 4.4], an object $K \in \mathcal{D}^b_c(X)$ is $f$-ULA if and only if $K$ is universally locally acyclic in the sense of [HS23, Def. 3.2]. Thus, the notation $\mathcal{D}_{ULA}(X/S)$ agrees with that in [HS23]. It is a triangulated subcategory of $\mathcal{D}^b_c(X)$. Let $\text{Loc}(X) \subset \text{Cons}(X)$ be the full subcategory of lisse $\Lambda$-sheaves on $X$.

**Fact 3.3.2.2.**

1. ([Bar23, Lem. 3.4]) Let $f : X \to \text{Spec}(k)$ be the structure morphism of an algebraic variety. Then every object of $\mathcal{D}^b_c(X)$ is $f$-ULA.

2. ([Bar23, Cor. 3.10 (i)]) $\mathcal{D}_{ULA}(X/X) = \text{Loc}(X)$.

3. ([HS23, Prop. 3.4 (i)]) Let $g : S' \to S$ be a morphism of algebraic varieties, and $f' : X' \to S'$ (resp. $g' : X' \to X$) be the base change of $f$ (resp. $g$).

Then $Lg^* : \mathcal{D}^b_c(X) \to \mathcal{D}^b_c(X')$ sends $\mathcal{D}_{ULA}(X/S)$ to $\mathcal{D}_{ULA}(X'/S')$.

4. ([Ric14, Lem. 3.15], [Bar23, Lem. 3.3 (i), (ii)]) Let $f : X \to Y$ be an $S$-morphism, where $X, Y, S$ are algebraic varieties over $k$. If $f$ is smooth (resp. proper), then $f^*$ sends $\mathcal{D}_{ULA}(Y/S)$ to $\mathcal{D}_{ULA}(X/S)$ (resp. $Rf_*$ sends $\mathcal{D}_{ULA}(X/S)$ to $\mathcal{D}_{ULA}(Y/S)$).

5. ([HS23, p.13]) Let $g : S \to S'$ be a smooth morphism of schemes. If $K \in \mathcal{D}^b_c(X)$ is $f$-ULA, then $K$ is $g \circ f$-ULA.
3.3.3 Relative perverse sheaves

Let $f : X \to S$ be a morphism of algebraic varieties over $k$. For every point $s$ of $S$ and $K \in D^b_c(X)$, set $K_s := K|_{X_s}$. Set $K_{X/S} := Rf^! A_S \in D^b_c(D_X)$ to be the relative dualizing complex. The contravariant functor

$$\mathbb{D}_{X/S} = R\text{Hom}_{A_S} (\cdot, K_{X/S}) : D^b_c(X) \to D^b_c(X)$$

is called the relative Verdier duality. There is a canonical natural transformation $\text{Id}_{D^b_c(X)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$ ([KL85, (1.1.5)]).

Fact 3.3.3.1 is stated for $\infty$-categories in [HS23], but holds for the underlying triangulated categories (described in [HRS23, Lem. 7.9]) by [HS23, Footnote 1, p.2].

Fact 3.3.3.1.

1. ([HS23, Thm. 1.1]) There is a unique $t$-structure $(p/S D^{\leq 0}, p/S D^{\geq 0})$ on $D^b_c(X)$, called the relative perverse $t$-structure, with the following property: An object $K \in D^b_c(X)$ lies in $p/S D^{\leq 0}$ (resp. $p/S D^{\geq 0}$) if and only if for all geometric points $\bar{s} \to S$, the restriction $K_{\bar{s}}$ lies in $p D^{\leq 0}$ (resp. $p D^{\geq 0}$), for the absolute perverse $t$-structure on $D^b_c(X_{\bar{s}})$. In particular, for every $s \in S$, the restriction functor $D^b_c(X) \to D^b_c(X_s)$ is $t$-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse $t$-structure. It induces an exact functor $\text{Perv}(X/S) \to \text{Perv}(X_s)$.

2. ([HS23, Thm. 1.9]) The relative perverse $t$-structure on $D^b_c(X)$ restricts to a $t$-structure $(p/S D^{\infty A,S,\leq 0}(X/S), p/S D^{\infty A,S,\geq 0}(X/S))$ on $D^{\infty A,S}(X/S)$.

3. ([HS23, Prop. 3.4]) The functor $\mathbb{D}_{X/S}$ preserves $D^{\infty A,S}(X/S)$, and the restriction $\text{Id}_{D^{\infty A,S}(X/S)} \to \mathbb{D}_{X/S} \circ \mathbb{D}_{X/S}$ is an isomorphism between functors on $D^{\infty A,S}(X/S)$. The formation of $\mathbb{D}_{X/S} : D^{\infty A,S}(X/S) \to D^{\infty A,S}(X/S)$ commutes with any base change in $S$, so $\mathbb{D}_{X/S}$ exchanges $p/S D^{\infty A,S,\leq 0}(X/S)$ with $p/S D^{\infty A,S,\geq 0}(X/S)$.

Let $\text{Perv}(X/S)$ (resp. $\text{Perv}^{\infty A,S}(X/S)$) be the heart of the relative perverse $t$-structure on $D^b_c(X)$ (resp. $D^{\infty A,S}(X/S)$). By Fact 3.3.3.1 1, an object $K \in D^b_c(X)$ lies in $\text{Perv}(X/S)$ if and only if for every geometric point $\bar{s} \to S$, one has $K_{\bar{s}} \in \text{Perv}(X_{\bar{s}})$.

Example 3.3.3.2.

1. [HS23, p.2] If $S = \text{Spec}(k)$, then Fact 3.3.3.1 1 gives the absolute perverse $t$-structure and $\text{Perv}(X/k) = \text{Perv}(X)$. If $f$ is universally injective, then it gives the standard $t$-structure and $\text{Perv}(X/S) = \text{Cons}(X)$.

2. Let $L$ be a lisse $A$-sheaf on $X$. If $f$ is smooth of relative dimension $r$, then by [Bar23, Cor. 3.10 (ii)], $L[r] \in \text{Perv}^{\infty A,S}(X/S)$. 
Example 3.3.3.3. Let \( i : Y \to X \) be a closed immersion of \( S \)-schemes. Assume that the morphism \( Y \to S \) is smooth of relative dimension \( d \) with geometrically connected fibers. If \( L \) is a lisse \( \Lambda \)-sheaf on \( Y \), then \( i_*Y[d] \in \text{Perv}^{\text{ULA}}(X/S) \).

Indeed, by Fact 3.3.2.2 2, \( L \in \text{D}^{\text{ULA}}(Y/Y) \). From the smoothness of \( Y/S \) and Fact 3.3.2.2 5, one has \( L \in \text{D}^{\text{ULA}}(Y/S) \). Using the properness of \( i \) and Fact 3.3.2.2 4, one gets \( i_*Y[d] \in \text{D}^{\text{ULA}}(X/S) \). For every closed point \( s \in S \), let \( i_s : Y_s \to X_s \) be the base change of \( i \). By the proper base change theorem, \( i_*Y[d] \in \text{Perv}(X_s) \). Therefore, \( i_*Y[d] \in \text{Perv}^{\text{ULA}}(X/S) \).

Lemma 3.3.3.4. If \( S \) is irreducible and geometrically unibranch, then the category \( \text{Perv}^{\text{ULA}}(X/S) \) is a Serre subcategory of \( \text{Perv}(X/S) \).

Proof. By [BBDG82, Thm. 1.3.6], \( \text{Perv}^{\text{ULA}}(X/S) \) is a fully faithful abelian subcategory of \( \text{Perv}(X/S) \) closed under extensions. From the proof of [HS23, Thm. 6.8 (ii)], it is closed under quotients. By [Sta23, Tag 02MP], it is a Serre subcategory. \( \square \)

Lemma 3.3.3.5 is stated more generally for regular \( S \) in [HS23, p.6].

Lemma 3.3.3.5. If \( S \) is smooth over \( k \) of equidimension \( d \), then the shifted inclusion

\[
\bullet [d] : \text{D}^{\text{ULA}}(X/S) \to \text{D}^b(X) \tag{3.7}
\]

is \( t \)-exact, where \( \text{D}^{\text{ULA}}(X/S) \) (resp. \( \text{D}^b(X) \)) is equipped with the relative (resp. absolute) perverse \( t \)-structure. It induces an exact functor

\[
\bullet [d] : \text{Perv}^{\text{ULA}}(X/S) \to \text{Perv}(X). \tag{3.8}
\]

Proof. We claim that the functor

\[
\bullet [d] : \text{D}^b(X) \to \text{D}^b(X) \tag{3.9}
\]

is right \( t \)-exact, where the source (resp. target) is equipped with the relative (resp. absolute) perverse \( t \)-structure. For every geometric point \( s \) of \( S \), let \( i_s : X_s \to X \) be the inclusion. For every integer \( q \), every \( K \in \text{D}^b(S) \), the functor \( i^*_s : \text{D}^b(X) \to \text{D}^b(X_s) \) is \( t \)-exact relative to the standard \( t \)-structures. Thus, \( H^q(K[d] \mid X_s = H^{q+d}(K_s) \). Then

\[
X_s \cap \text{Supp} H^q(K[d]) = \text{Supp} H^{q+d}(K_s).
\]

As \( K \in \text{D}^b(X_s) \), \( \dim \text{Supp} H^{q+d}(K_s) \leq -q - d \). By Lemma 3.3.3.8 3,

\[
\dim \text{Supp} H^q(K[d]) \leq -q.
\]

From Lemma 1, the Zariski closure of \( \text{Supp} H^q(K[d]) \) in \( X \) has dimension at most \(-q \). Therefore, \( K[d] \in \text{D}^b(X) \) (defined in [Max19, p.133]). The claim is proved.

It remains to show that the functor (3.7) is left \( t \)-exact. One may assume that \( k \) is algebraically closed. For every \( M \in \text{D}^b(X/S) \), by the proof of [Bar23, Cor. 3.8], \( \text{D}_X(M[d]) \) is (noncanonically) isomorphic to \( \text{D}^{\text{ULA}}(X/S)[d] \) in the category \( \text{D}^b(X) \). From Fact 3.3.3.1 3, \( \text{D}^{\text{ULA}}(X/S) \). By the claim, \( \text{D}^{\text{ULA}}(X/S)[d] \in \text{D}^b(X) \). Thus, \( M[d] \in \text{D}^b(X) \).
Remark 3.3.3.6. The functor (3.9) may not send $\text{Perv}(X/S)$ to $\text{Perv}(X)$. Indeed, let $f : X = 0 \to S = A^1_k$ be the inclusion of the origin. By Example 3.3.3.2, the relative perverse t-structure on $D^b_k(X)$ coincides with the standard one (which is also the absolute perverse t-structure). Then $\text{Perv}(X/S) = \text{Perv}(X)$.

Lemma 3.3.3.7 seems to be used in the proof of [SFFK23, Lem. 3.11].

Lemma 3.3.3.7. If $S$ is integral with generic point $\eta$ and of dimension $d$, then the functor
\[ D^b_k(X) \to D^b_k(X_\eta), \quad K \mapsto K_\eta[-d] \] (3.10)
is t-exact relative to absolute perverse structures, hence an exact functor
\[ \text{Perv}(X) \to \text{Perv}(X_\eta), \quad K \mapsto K_\eta[-d]. \] (3.11)

Proof. The functor (3.10) is right t-exact. Indeed, take $K \in pD^{\leq 0}(X)$. For every integer $q$, $\text{Supp} H^q(K_\eta[-d]) = \text{Supp} H^{q-d}(K_\eta) = X_\eta \cap \text{Supp} H^{q-d}(K)$.

By Lemma 3.3.3.8 4,
\[ \dim \text{Supp} H^q(K_\eta[-d]) \leq \dim \text{Supp}(H^{q-d}(K)) - d \leq -q. \]

From Lemma 3.3.3.8 1, $H^q(K_\eta[-d]) \in pD^{\leq 0}(X_\eta)$.

For every $M \in D^b_k(X)$, by [DGIV77, Thm. 2.13], there is a nonempty open subset $U \subset S$ with $M|_U \in D^{\text{UL}}(X_U/U)$. By the proof of [Bar23, Cor. 3.8], $\mathbb{D}_X M = (\mathbb{D}_X S M)(d)[2d]$. From Fact 3.3.3.1 3, $(\mathbb{D}_X M)_\eta[-d]$ is a Tate twist of $\mathbb{D}_X(M_\eta[-d])$. For every integer $q$,
\[ \text{Supp} H^q(\mathbb{D}_X(M_\eta[-d])) = \text{Supp} H^q((\mathbb{D}_X M)_\eta[-d]). \] (3.12)

The functor (3.10) is left t-exact. Indeed, take $L \in pD^{\geq 0}(X)$. Then $\mathbb{D}_X L \in pD^{\leq 0}(X)$. From the first paragraph, $(\mathbb{D}_X L)_\eta[-d] \in pD^{\leq 0}(X_\eta)$. By (3.12), $\mathbb{D}_X(L_\eta[-d]) \in pD^{\leq 0}(X_\eta)$. Therefore, $L_\eta[-d] \in pD^{\geq 0}(X_\eta)$.

Lemma 3.3.3.8. Let $f : X \to Y$ be an $F$-morphism between schemes of finite type over a field $F$. Let $A$ be a quasi-constructible subset of $X$. By convention, the dimension of an empty space is $-\infty$.

1. Then its Krull dimension $\dim A = \dim \bar{A}$.

2. Let $\{B_i\}_{i=1}^n$ be finitely many locally closed subsets of $X$ and $B = \bigcup_{i=1}^n B_i$. Then $\dim B = \max_{i=1}^n \dim B_i$.

3. Let $n \geq 0$ be an integer such that $\dim A \cap f^{-1}(y) \leq n$ for every $y \in Y$. Then $\dim A \leq \dim Y + n$.

4. Assume that $Y$ is integral with generic point $\eta$. Then $\dim Y + \dim A \cap X_\eta \leq \dim A$.

Proof.
3.3. RECOLLECTIONS ON CONSTRUCTIBLE SHEAVES

1. As $X$ is a Noetherian scheme, the topological space $A$ is Noetherian. Therefore, $A$ is the union of finitely many irreducible components. Thus, one may assume further that $A$ is nonempty and irreducible. Then the reduced induced closed subscheme $\overline{A}$ of $X$ is integral and of finite type over $F$. By [Bor91, AG, Prop. 1.3], $A$ contains a nonempty open subset of $\overline{A}$. By [Har77, II, Exercise 3.20 (e)], $\dim A = \dim \overline{A}$.

2. For every $1 \leq i \leq n$, since $B_i \subset B$, one has $\dim B_i \leq \dim B$. Then $\max_i \dim B_i = \dim \overline{B}$. As $\overline{B}_i$ is quasi-constructible in $X$, by 1, one has $\dim B \leq \dim \overline{B} + \max_i \dim B_i$.

3. By 2, one may assume that $A$ is locally closed in $X$. Taking irreducible components, one may assume further that $A$ is irreducible. Let $Z$ be the Zariski closure of $f(A)$ in $Y$. Then $Z$ is irreducible. With reduced induced subscheme structures, one views $A$ and $Z$ as integral schemes of finite type over $F$. Moreover, $f$ induces a dominant $F$-morphism $g : A \rightarrow Z$. By [Har77, II, Exercise 3.22 (b)], for every $y \in f(A) = g(A)$, one has

$$n \geq \dim A \cap f^{-1}(y) = \dim g^{-1}(y) \geq \dim A - \dim Z.$$ 

Thus, $\dim A \leq \dim Z + n \leq \dim Y + n$.

4. As in the proof of 3, one may assume that $A$ is an irreducible locally closed subset of $X$ and view $A$ as an integral scheme of finite type over $F$. One may assume that $A \cap X_\eta$ is nonempty. As $A_\eta$ is homeomorphic to $A \cap X_\eta$, the morphism $A \rightarrow Y$ induced by $f$ is dominant. Thus, by [Har77, II, Exercise 3.22 (c)], $\dim A \cap X_\eta = \dim A_\eta = \dim A - \dim Y$.

\[ \Box \]

**Lemma 3.3.3.9** (Scholze). Assume that $S$ is regular, integral with generic point $\eta$ and of dimension $d$. Then:

1. Let $A \in \Perv^{ULA}(X/S)$, and let $B[d]$ be a subquotient of $A[d] \in \Perv(X)$. If the image $B_\eta \in \Perv(X_\eta)$ of $B[d]$ under the functor (3.11) is zero, then $B[d] = 0$ in $\Perv(X)$.

2. The functor (3.8) identifies $\Perv^{ULA}(X/S)$ as a Serre subcategory of $\Perv(X)$.

**Proof.**

1. By [HS23, Cor. 1.12], $B \in D^{ULA}(X/S)$. Since $B_\eta = 0$, one has $B = 0$.

2. The functor (3.8) is exact, fully faithful of kernel 0. By Fact 3.3.3.12 and [BBDG82, Thm. 1.3.6], $\Perv^{ULA}(X/S)$ is closed under extensions in $D^{ULA}(X/S)$. Thus, the essential image of (3.8) is closed under taking subobjects.
Take $K \in \text{Perv}^{ULA}(X/S)$ and a subobject $L[d]$ of $K[d] \in \text{Perv}(X)$. Then $L_\eta$ is a subobject of $K_\eta \in \text{Perv}(X_\eta)$. By [HS23, Thm. 1.10 (ii)], there is a subobject $L'$ of $K$ in $\text{Perv}^{ULA}(X/S)$ with $L'_\eta = L_\eta$. Set $M = K/L' \in \text{Perv}^{ULA}(X/S)$. Let $N[d]$ be the image of $L[d]$ under the morphism $K[d] \to M[d]$ in $\text{Perv}(X)$. As the sequence

$$0 \to L'[d] \cap L[d] \to L[d] \to N[d] \to 0$$

is exact in $\text{Perv}(X)$, by Lemma 3.3.3.7, the sequence

$$0 \to L'_\eta \cap L_\eta \to L_\eta \to N_\eta = 0$$

is exact in $\text{Perv}(X_\eta)$. Thus, $N_\eta = 0$. Since $N[d]$ is a subobject of $M[d] \in \text{Perv}(X)$, by 1, one has $N[d] = 0$. Then $L[d] \subset L'[d]$ in $\text{Perv}(X)$. Since $(L'[d])/(L[d]) \in \text{Perv}(X)$ is a quotient of $L'[d]$ and $L'/L_\eta = 0$, by 1 again, one gets $(L'[d])/(L[d]) = 0$ in $\text{Perv}(X)$. Therefore, $L[d] = L'[d]$. The claim is proved.

Similarly, the essential image is closed under taking quotients. By [Sta23, Tag 02MP], the essential image is a Serre subcategory of $\text{Perv}(X)$.

$\square$

### 3.4 Cotori

We review the contents of [GL96, Sec. 3.2]. For a compact group $G$, let $C(G)(\Lambda)$ be the group of characters, i.e., continuous morphisms $G \to \Lambda^*$. Let $C(G)_f(\Lambda)$ (resp. $C(G)_r(\Lambda)$) be the subgroup of characters of finite order prime to $\ell$ (resp. that are pro-$\ell$). We shall review the $\Lambda$-scheme whose set of $\Lambda$-points is naturally identified with $C(G)_r(\Lambda)$. Fix an integer $n \geq 1$.

**Lemma 3.4.0.1.** Let $M$ be a finitely generated free module of rank $r$ over a commutative ring $A$. Let $B = A[[T_1, \ldots, T_n]]$. Then the canonical $B$-linear map $M \otimes_A B \to M[[T_1, \ldots, T_n]]$ is an isomorphism.

**Proof.** We claim that the map is surjective. Fix an $A$-basis $\{e_1, \ldots, e_r\}$ of $M$. Then for every $f = \sum_{\alpha \in \mathbb{N}^n} x_\alpha T^\alpha \in M[[T_1, \ldots, T_n]]$, with $x_\alpha \in M$, one may write $x_\alpha = \sum_{i=1}^r a_{\alpha,i} e_i$ with $a_{\alpha,i} \in A$. Define $g_i \in B$ by $g_i = \sum_{\alpha \in \mathbb{N}^n} a_{\alpha,i} x^\alpha$. Then

$$\sum_{i=1}^r e_i \otimes g_i \in M \otimes_A B$$

is mapped to $f$. The claim is proved.

The $B$-module $M \otimes_A B$ is free of rank $r$. Moreover, $M[[T_1, \ldots, T_n]]$ is also a free $B$-module of same rank. By [Vas69, Prop. 1.2, p.506] and the claim, the map is an isomorphism. $\square$
3.4. **COTORI**

**Remark 3.4.0.2.** Let \( \mathcal{R} = \{ O_E : E/Q_\ell \text{ is a finite subextension of } \Lambda \} \), which is a directed set under inclusion. We explain the isomorphism in the proof of [GL96, Prop. 3.2.2]. In fact, \( \Lambda = \lim_{\rightarrow} \mathcal{R} \) and direct limits commute with tensor product, so

\[
\Lambda \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[T_1, \ldots, T_n]] = \lim_{\rightarrow} (\mathcal{R} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[T_1, \ldots, T_n]])
\]

\[
= \lim_{\rightarrow} (\mathcal{R} \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[T_1, \ldots, T_n]])[\mathcal{R}^{-1}] = \lim_{\rightarrow} (\mathcal{R}[[T_1, \ldots, T_n]])[\mathcal{R}^{-1}],
\]

where the last equality uses Lemma 3.4.0.1.

For an integer \( m \geq 1 \), let \( \mu_{\ell^m} \) be the set of \( \ell^m \)-roots of unity in \( \Lambda \). Set \( \mu_{\ell^\infty} = \bigcup_{m \geq 1} \mu_{\ell^m} \).

**Lemma 3.4.0.3.** Let \( E/Q_\ell \) a finite extension. If \( f \in O_E[[X_1, \ldots, X_n]] \) satisfies \( f(\zeta - 1, \ldots, \zeta_n - 1) = 0 \) for all \( (\zeta_1, \ldots, \zeta_n) \in \mu_{\ell^\infty} \), then \( f = 0 \).

**Proof.** Induction on \( n \). When \( n = 1 \), for every integer \( m \geq 1 \), by Weierstrass’ division lemma (see, e.g., [Ouy, Lem. 2.1]), one may write

\[
f(t) = ((1 + t)^{\ell^m} - 1)g(t) + r(t)
\]

where \( r \in O_E[t] \) with \( \deg(r) < \ell^m \) and \( g \in O_E[[t]] \). For every \( \zeta \in \mu_{\ell^m} \), by assumption \( f(\zeta - 1) = 0 \), so \( r(\zeta) = 0 \). Then \( r \) has at least \( \ell^m \) distinct roots, so \( r = 0 \). Or equivalently, the element \( f \) is in the ideal \( ((1 + t)^{\ell^m} - 1) \subset O_E[[t]] \).

By the last paragraph of the proof of [Ouy, Thm. 2.8], the intersection of ideals \( \cap_{m \geq 1} ((1 + t)^{\ell^m} - 1) = 0 \), so \( f = 0 \). The case \( n = 1 \) is proved for all finite extensions \( E \) of \( Q_\ell \).

Now assume that \( n \geq 2 \) and the statement is proved for \( n - 1 \) for every finite extension \( E/Q_\ell \). One may write

\[
f(X_1, \ldots, X_n) = \sum_{i \geq 0} g_i(X_1, \ldots, X_{n-1})X_n^i,
\]

where \( g_i \in O_E[[X_1, \ldots, X_{n-1}]] \). For each \( (\zeta_1, \ldots, \zeta_n) \in \mu_{\ell^\infty}^{n-1} \), set

\[
h(X_n) := f(\zeta_1 - 1, \ldots, \zeta_{n-1} - 1, X_n) \in O_{E'}[[X_n]],
\]

where \( E' = E(\zeta_1, \ldots, \zeta_{n-1}) \) is another finite extension of \( Q_\ell \). For every \( \zeta_n \in \mu_{\ell^\infty} \), one has \( h(\zeta_n - 1) = 0 \). By the proved case of \( n = 1 \) for \( E' \), one has \( h = 0 \).

Therefore, for every integer \( i \geq 0 \), one has \( g_i(\zeta_1 - 1, \ldots, \zeta_{n-1} - 1) = 0 \) for all \( (\zeta_1, \ldots, \zeta_{n-1}) \in \mu_{\ell^\infty}^{n-1} \). By the inductive hypothesis (the case of \( n - 1 \) for \( E' \)), one has \( g_i = 0 \) for every \( i \), so \( f = 0 \). The induction is completed. \( \square \)

There is a unique absolute value on \( \Lambda \) extending the discrete absolute value \( | \cdot |_\ell \) on \( Q_\ell \). It induces a topology on \( \Lambda \) that is totally disconnected. A subset \( A \subset \Lambda \) is closed if and only if for every finite subextension \( E/Q_\ell \) of \( \Lambda \), \( A \cap E \) is closed in the discrete valuation field \( E \).
Lemma 3.4.0.4.

1. Let $C$ be a compact subset of $\Lambda$. Then there is a finite subextension $E$ of $\Lambda/\mathbb{Q}_l$ containing $C$.

2. Let $G \leq \Lambda^*$ be a compact subgroup. Then there is a finite subextension $E$ of $\Lambda/\mathbb{Q}_l$ with $G \subset O_E^*$. 

3. In 2, the topological group $G$ is isomorphic to the direct product of a finite group of order prime to $\ell$ with a pro-$\ell$ group.

Proof. 1. Otherwise, there is a sequence of elements $x_1, x_2, \ldots$ in $C$ with $[\mathbb{Q}_l(x_{n+1}) : \mathbb{Q}_l] > [\mathbb{Q}_l(x_n) : \mathbb{Q}_l]$ for every integer $n > 0$. Let $B \subset C$ be the infinite set of elements of this sequence. For every subset $S \subset B$, every finite subextension $F/\mathbb{Q}_l$, the set $S \cap F$ is finite, so closed in $F$. Therefore, $S$ is closed in $\Lambda$. In particular, the set $B$ is closed and hence compact in $C$. As every subset of $B$ is closed in $B$, $B$ is also discrete. Thus, $B$ is finite, a contradiction.

2. By 1, there is a finite subextension $E$ of $\Lambda/\mathbb{Q}_l$ containing $G$. By [Ser64, Thm. 1 2, p.122], $G \subset O_E^*$.

3. By 2 and [Ser64, Cor., p.155], $G$ is an $\ell$-adic Lie group. From Lazard's theorem (see, e.g., [GSK09, p.711]), there is a pro-$\ell$ open subgroup $U \leq G$. By [RV98, Thm. 1-23], there is an $\ell$-Sylow subgroup $H \leq G$ containing $U$. Since $G$ is compact, $[G : U]$ is finite. Thus, the group $G/H$ is finite of order prime to $\ell$. By [RV98, Cor. 1-24 (iii)], $G$ is isomorphic to $G/H \times H$. 

By Lemma 3.4.0.4 3, for every compact group $G$,

$$C(G)(\Lambda) = C(G)_f(\Lambda) \times C(G)_r(\Lambda), \quad C(G)_f(\Lambda) \cap C(G)_r(\Lambda) = \{1\}.$$ 

Fix $G = \mathbb{Z}_\ell^n$. For each $R \in \mathcal{R}$, the completed group ring $R[[G]]$ is a Noetherian regular complete local domain of Krull dimension $1 + n$, and there is a canonical injective morphism $G \to R[[G]]^*$ of groups. A $\mathbb{Z}_\ell$-basis 

$$\{\gamma_1, \ldots, \gamma_n\}$$ 

of $G$ defines an isomorphism of topological rings

$$R[[G]] \to R[[t_1, \ldots, t_n]], \quad \gamma_i \mapsto 1 + t_i.$$ 

and an injective morphism of group

$$\Gamma : C(G)_r(\Lambda) \to (\Lambda^*)^n, \quad \chi \mapsto (\chi(\gamma_1), \ldots, \chi(\gamma_n)).$$

Lemma 3.4.0.5. Under the map (3.15), the torsion subgroup $C(G)^{\mathrm{tor}}_\ell$ of $C(G)_r(\Lambda)$ is identified with $\mu^*_\ell$. 

}\end{lemma}
3.4. COTORI

Proof. If $\chi \in \mathcal{C}(G)_\ell,\text{tor}$, then there is an integer $m \geq 1$ with $\chi^m = 1$. So $\chi$ takes value in $\mu_{\ell^m}$ and $\Gamma(\chi) \in \mu_{\ell^m}^n$. Therefore, $\Gamma(\mathcal{C}(G)_\ell,\text{tor}) \subset \mu_{\ell^\infty}^n$.

For every $(a_1, \ldots, a_n) \in \mu_{\ell^\infty}^n$, there is an integer $m \geq 1$ with $a_i^m = 1$ for every $1 \leq i \leq n$. For every $g \in G$, there exists a unique $n$-tuple $(b_1, \ldots, b_n) \in \mathbb{Z}_\ell^n$ with $g = \sum_{i=1}^n b_i \gamma_i$. Define

$$\chi : G \to \Lambda^*, \quad g \mapsto \prod_{i=1}^n a_i^{b_i} \mod \ell^m.$$ 

Then $\chi \in \mathcal{C}(G)_\ell$ and $\Gamma(\chi) = (a_1, \ldots, a_n)$. Therefore, $\Gamma(\mathcal{C}(G, \Lambda^*)_\ell,\text{tor}) = \mu_{\ell^\infty}^n$. □

Define a $\Lambda$-algebra

$$S = S(G) = \Lambda \otimes_{\mathbb{Z}_\ell} Z_\ell[[G]].$$

Let $C_\ell$ be the affine scheme Spec$(S)$. It is called a “cotorus”. In general, $C_\ell$ is not locally of finite type over $\Lambda$. By [GL96, Prop. A.2.2.3 (ii)], the scheme $C_\ell$ is integral and regular, $C_\ell(\Lambda)$ coincides with the set of closed points, and $C_\ell(\Lambda)$ is dense in $C_\ell$.

Every character $\chi : G \to \Lambda^*$ defines a continuous morphism of $\mathbb{Z}_\ell$-algebras

$$\text{ev}_\chi : Z_\ell[[G]] \to \Lambda : \quad g \mapsto \chi(g), \quad \forall g \in G. \quad (3.16)$$

It extends to a surjective continuous morphism of $\Lambda$-algebras:

$$S \to \Lambda. \quad (3.17)$$

Let $m_\chi \subset S$ be the kernel of (3.17). Then $m_\chi$ is a maximal ideal of $S$ with residue field $\Lambda$. Let $\Psi(\chi)$ be the corresponding element of $C_\ell(\Lambda)$. Hence a map

$$\Psi : \mathcal{C}(G)_\ell(\Lambda) \to C_\ell(\Lambda). \quad (3.18)$$

Fact 3.4.0.6 ([GL96, p.519]). The map (3.18) is bijective.

Lemma 3.4.0.7. The subset $C_{\ell,\text{tor}} := \Psi(\mathcal{C}(G)_\ell(\Lambda)_{\text{tor}})$ is Zariski dense in $C_\ell$.

Proof. With the chosen basis (3.13), the morphism (3.16) induces a morphism $Z_\ell[[X_1, \ldots, X_n]] \to \Lambda$ given by

$$f \mapsto f(\chi(\gamma_1) - 1, \ldots, \chi(\gamma_n) - 1) \quad (3.19)$$

via the isomorphism (3.14). The isomorphism (3.14) identifies $S$ with the subalgebra $A_n$ of $\Lambda[[X_1, \ldots, X_n]]$ defined in (3.2). Under this identification, the map (3.17) is also given by (3.19).

Let $C \subset C_\ell$ be a Zariski closed subset containing $C_{\ell,\text{tor}}$. Then there is an ideal $I \subset S$ with $C = Z(I) \subset \text{Spec}(S)$. Fix $f \in I$. We show $f = 0$.

One can write

$$f = \sum_{i=1}^n a_i \otimes f_i, \quad a_i \in \Lambda, \quad f_i \in Z_\ell[[G]].$$
There is an integer \( N > 0 \) and a finite subextension \( E/\mathbb{Q}_\ell \) such that \( \ell^N a_i \in O_E \) for every \( 1 \leq i \leq n \). By [Bou89, Ch. III, § 3, n. 4, Cor. 3], the injection \( O_E \to \Lambda \) induces an injection \( O_E \otimes_{\mathbb{Z}_\ell} \mathbb{Z}_\ell[[G]] \to S \) of rings. By Lemma 3.4.0.1, its image is \( O_E[[G]] \). Since \( \ell^N \in \Lambda^* \subset S^* \), one may replace \( f \) by \( \ell^N f \) and then \( f \in O_E[[G]] \). One may view \( f \) as an element of \( O_E[[X_1, \ldots, X_n]] \) via the isomorphism (3.14).

The regular function \( f : C_\ell(\Lambda) \to \Lambda \) vanishes on \( C(\Lambda) \). From (3.19) and Lemma 3.4.0.5, one has \( f(\zeta_1 - 1, \ldots, \zeta_n - 1) = 0 \) for all \( (\zeta_1, \ldots, \zeta_n) \in \mu^n_{\ell^\infty} \). By Lemma 3.4.0.3, \( f = 0 \).

Therefore, the ideal \( I = 0 \) and whence \( C = C_\ell \). \( \square \)

### 3.5 Krämer-Weissauer’s vanishing theorem

Let \( k \) be a field of characteristic 0. Let \( A/k \) be an abelian variety. Choose an algebraic closure \( k^a \) of \( k \). For an abelian variety \( A/k \) set \( \mathcal{C}(A)(\Lambda) := \mathcal{C}(\pi^a(A_k^a, 0))(\Lambda), \mathcal{C}(A)(\Lambda) = \mathcal{C}(\pi^a(A_k^a, 0))f(\Lambda) \) and \( \mathcal{C}(A)_\ell(\Lambda) = \mathcal{C}(\pi^a(A_k^a, 0))\ell(\Lambda) \).

For every \( \chi \in \mathcal{C}(A)(\Lambda) \), let \( L_\chi \) be the corresponding rank 1 lisse sheaf on \( A_{k^a} \).

**Definition 3.5.0.1** ([KW15b, p.553], [Wei16, p.563]). For \( K \in \text{Perv}(A) \), the set

\[
\mathcal{S}(K) := \{ \chi \in \mathcal{C}(A)(\Lambda) : H^i(A_{k^a}, K_{k^a} \otimes L_\chi) \neq 0 \text{ for some integer } i \neq 0 \}
\]

is called the spectrum of \( K \).

**Fact 3.5.0.2** ([KW15b, Thm. 1.1], [Wei16, Vanishing Theorem, p.561; Thm. 2]).

For every \( K \in \text{Perv}(A) \), every character \( \chi_f : \pi^a_1(A_k^a) \to \Lambda^* \) of finite order prime to \( \ell \), the set \( \{ \chi \in \mathcal{C}(A)(\Lambda) : \chi_f \chi \in \mathcal{S}(K) \} \) is the set of \( \Lambda \)-points of a strict Zariski closed subset of \( \mathcal{C}(A)_\ell \).

Let \( m : A \times_k A \to A \) be the group law on \( A \). The bifunctor

\[
*: D^b_c(A) \times D^b_c(A) \to D^b_c(A), \quad K_1 \ast K_2 := Rm_*(K_1 \boxtimes K_2)
\]

is called the convolution on \( A \).

**Example 3.5.0.3.** For every closed reduced subvariety \( i : X \to A \), let \( \delta_X := i_*\Lambda_X \in D^b_c(A) \). Then for every closed point \( x \in A \), \( \delta_x \ast \delta_X = \delta_{x+X} \).

By [Wei11] and [JKLM23, Sec. 3.1], the pair \( (D^b_c(A), *) \) is a rigid symmetric monoidal category, with unit the skyscraper sheaf \( \delta_0 \) of rank 1 supported at the origin. For every \( K \in D^b_c(A) \), its adjoint dual is \( K^\vee := [-1]^*\Lambda D_A K \). Moreover, for every \( K \in \text{Perv}(A) \), the Euler characteristic

\[
\chi(A, K) := \sum_{i \in \mathbb{Z}} (-1)^i \dim_\Lambda H^i(A, K) \geq 0. \quad (3.20)
\]

Let \( N(A) \subset \text{Perv}(A) \) be the full subcategory of objects \( K \) with \( \chi(A, K) = 0 \). From (3.20) and the additivity of the function \( \chi(A, \cdot) : \text{Perv}(A) \to \mathbb{N} \),
the subcategory $N(A)$ is Serre in $\text{Perv}(A)$ ([KW15a, p.725]). Let $\bar{P}(A) := \text{Perv}(A)/N(A)$ be the quotient abelian category. Fix $\chi \in C(A)(\Lambda)$, and set

$$E^\chi(A_k) = \{ K \in \text{Perv}(A_k) : H^i(A_k, K \otimes L_\chi) = 0, \forall i \neq 0 \}.$$ 

Then $E^\chi(A_k)$ is closed under extensions in $\text{Perv}(A_k)$. Let $P^\chi(A) \subset \text{Perv}(A)$ be the full subcategory of objects $K$ with $Q \in E^\chi(A_k)$ for every simple subquotient $Q$ of $K_{k^n} \in \text{Perv}(A_{k^n})$.

By [BBDG82, Thm. 4.3.1 (i)], every $K \in \text{Perv}(A)$ is Noetherian and Artinian. For every character $\chi_f : \pi_1^\text{et}(A, 0) \rightarrow \Lambda^*$ of finite order prime to $\ell$, by Fact 3.5.0.2 and Lemma 3.2.3.51, the set $\{ \chi \in C(A)(\Lambda) : K \in P^{\chi_f}(A) \}$ is the set of $\Lambda$-points of a strict Zariski closed subset of $C(A)\ell$.

**Lemma 3.5.0.4.** Let $A$ be a Noetherian and Artinian abelian category. Let $E$ be a class of objects of $A$ closed under isomorphisms. Let $S \subset A$ be the full subcategory of objects whose every nonzero simple subquotient is in $E$. Then

1. $S$ is a Serre subcategory of $A$.

2. If $E$ is closed under extensions, then $S \subset E$.

**Proof.**

1. Let $X$ be an object of $S$ with a subquotient $Y$. Every simple subquotient of $Y$ is that of $X$, hence in $E$. Thus, $Y \in S$. Therefore, $S$ is closed under subquotients. Let $0 \rightarrow L \xrightarrow{f} M \xrightarrow{g} N \rightarrow 0$ be a short exact sequence in $A$ with $L, N \in S$.

Let $Q$ be a nonzero simple subquotient of $M$, we claim that $Q \in E$. First, assume that $Q$ is a quotient of $M$. The natural morphism $L \rightarrow Q$ is either an epimorphism or zero, in which case $Q$ is a simple quotient of $L$ or $N$ respectively. Thus, $Q \in E$.

For a general $Q$, there is a subobject $M_0 \subset M$ and an epimorphism $M_0 \rightarrow Q$. Then

$$0 \rightarrow f^{-1}(M_0) \rightarrow M_0 \rightarrow g(M_0) \rightarrow 0$$

is a short exact sequence in $A$ with $f^{-1}(M_0)$ (resp. $g(M_0)$) a subobject of $L$ (resp. $N$). From the first paragraph, $f^{-1}(M_0)$ and $g(M_0)$ are in $E$. From the second paragraph, $Q \in E$. The claim is proved.

From the claim, $M \in S$ and $S$ is closed under extensions. The result follows from [Sta23, Tag 02MP].

2. By [Sta23, Tag 0FCJ], every $X \in S$ admits a filtration in $A$

$$0 \subset X_1 \subset X_2 \subset \cdots \subset X_n = X$$

by subobjects such that each $X_i/X_{i-1}$ is a simple subquotient of $X$. Then $X_i/X_{i-1} \in E$. As $E$ is closed under extensions, $X \in E$. 

\qed
By Lemma 3.5.0.4.1, \( P^\chi(A) \) is a Serre subcategory of \( \text{Perv}(A) \). From Lemma 3.5.0.4.2, for every \( K \in P^\chi(A) \) and every integer \( i \neq 0 \),

\[
H^i(A_{k^\ast}, K_{k^\ast} \otimes^L L_\chi) = 0. \tag{3.21}
\]

From the proof of [LS20, Lem. 3.4 (3)], the functor

\[
\omega_\chi : P^\chi(A) \to \text{Vec}_A, K \mapsto H^0(A_{k^\ast}, K_{k^\ast} \otimes^L L_\chi) \tag{3.22}
\]

is exact. Let \( N^\chi(A) \) be the full subcategory of \( P^\chi(A) \) of objects in \( N(A) \). For every \( K \in N^\chi(A) \), by [KW15b, Cor. 4.2], \( \chi(A, K \otimes^L L_\chi) = 0 \). From (3.21), one has \( H^0(A_{k^\ast}, K_{k^\ast} \otimes^L L_\chi) = 0 \). By [Sta23, Tag 02MS], the functor \( \omega_\chi \) factors uniquely through an exact functor (still denoted by \( \omega_\chi \))

\[
P^\chi(A)/N^\chi(A) \to \text{Vec}_A. \tag{3.23}
\]

**Fact 3.5.0.5** ([KW15b, proof of Thm. 13.2], [JKLM23, Prop. 3.1]). The convolution on \( D^b_c(A) \) induces a bifunctor

\[
P(A) \times \overline{P}(A) \to \overline{P}(A), \quad (K, L) \mapsto \overline{P}(K \ast L).
\]

It defines the structure of a neutral Tannakian category over \( \Lambda \) on \( \overline{P}(A) \). The subcategory \( P^\chi(A)/N^\chi(A) \) is a Tannakian subcategory\(^7\) on which (3.23) is a fiber functor.

Let \((C, \otimes)\) a neutral Tannakian category over an algebraically closed field of characteristic 0 with a fiber functor \( \omega : C \to \text{Vec} \). Let \( \text{Aut}^\otimes(C, \omega) \) be the corresponding affine group scheme. By [Del90, Sec. 9.2, p.187], up to isomorphism of group schemes, \( \text{Aut}^\otimes(C, \omega) \) is independent of the choice of \( \omega \).

For an object \( K \in C \), let \( K^* \) be its adjoint dual, \( \iota : \langle K \rangle \to C \) be the full subcategory of subquotients of \( \{(K \oplus K^* \otimes^m)\}_{m \geq 1} \). Then \((\langle K \rangle, \otimes)\) is a neutral Tannakian subcategory with a fiber functor \( \omega_\iota \), with \( G_{\omega_\iota}(\langle K \rangle) = \text{im}[\text{Aut}^\otimes(C, \omega) \to \text{GL}(\omega(K))] \). It is called the Tannakian monodromy group of \( K \) at \( \omega \) and denoted by \( G_{\omega_\iota}(K) \). It is reductive if and only if \( K \) is semisimple in \( C \) ([Sim92, p.69]).

Let \( \text{Rep}_\Lambda(\Gamma_k) \) be the category of continuous finite dimensional \( \Lambda \)-representations of \( \Gamma_k \). Then with tensor product and the forgetful fiber functor \( \omega : \text{Rep}_\Lambda(\Gamma_k) \to \text{Vec}_A \), it is a neutral Tannakian category over \( \Lambda \). For an object \( \rho : \Gamma_k \to \text{GL}(V) \) in it, the corresponding Tannakian monodromy group is the Zariski closure of \( \rho(\Gamma_k) \) inside \( \text{GL}(V) \).

**Example 3.5.0.6.** [KW15a, Example 7.1] Fix a closed point \( x \in A \). Then \( \delta_x \in \text{Perv}(A) \), the spectrum \( S(\delta_x) \) is empty and for every \( \chi \in \mathcal{C}(A)(A) \), \( \delta_x \in P^\chi(A) \). If \( x \) is a torsion point of order \( n \), then its Tannakian monodromy group \( G(\delta_x) \) is isomorphic to \( \mathbb{Z}/n \). If \( x \) is not a torsion point, then \( G(\delta_x) \) is isomorphic to \( \mathbb{G}_m/\Lambda \).

---

\(^7\)in the sense of [DM22, Def. 2.19]

\(^8\)in the sense of [Mil07, 1.7]
3.6. MAIN RESULTS

Let $\psi : \pi^q_1(A) \to \Lambda^*$ be a continuous character and set $\chi = \psi|_{\pi^q_1(A kron)}$. Let $L_\psi$ denote the corresponding rank 1 lisse $\Lambda$-sheaf on $A$. The functor

$$\omega_\psi : \Perv(A) \to \Rep_\Lambda(\Gamma_k), \quad K \mapsto H^0(A kron, (K \otimes^L L_\psi)_{k^m})$$

restricts to an exact functor $P^A(A) \to \Rep_\Lambda(\Gamma_k)$. The composition of the restricted functor with $\omega$ is (3.22). It induces a morphism of affine groups schemes

$$\omega_\psi^* : \text{Aut}^\otimes(\Rep_\Lambda(\Gamma_k), \omega) \to \text{Aut}^*(P^A(A)/N^A(A), \omega_\chi). \quad (3.24)$$

For $K \in \Perv(A)$, let $G_{mon}(K, \psi)$ be the Tannakian monodromy group of $\omega_\psi(K)$ in $\Rep_\Lambda(\Gamma_k)$. The restricted functor $\omega_\psi : \langle K \rangle \to \langle \omega_\psi(K) \rangle$ induces an isometric immersion of algebraic groups $\omega_\psi^* : G_{mon}(K, \psi) \to G_{\omega_\psi}(K)$, which is taking the image of (3.24) in $\GL(\omega_\chi(K))$.

3.6 Main results

Consider Setting 3.1.0.1. For every $\chi \in \mathcal{C}(A)(\Lambda)$, denote the pullback of $\chi$ along $(\pi|_{A_q})_* : \pi^q_1(A_q) \to \pi^q_1(A)$ by $\psi : \pi^q_1(A_q) \to \Lambda^*$. Then the restriction $\psi|_{\pi^q_1(A_q)}$ is identified with $\chi$ via the isomorphism $(\pi|_{A_q})_* : \pi^q_1(A_q) \to \pi^q_1(A)$. Let $K \in \Perv(A \times X/X)$ be a semisimple object in $D^b_\Lambda(A \times X)$.

**Theorem 3.6.0.1.** For every $\chi \in \mathcal{C}(A)(\Lambda) \setminus S(K_q)$, the monodromy group $G_{mon}(K_q, \psi)$ is reductive.

**Proof.** If $X$ is replaced by a nonempty open subset, then by Lemma 3.3.1.6 the semisimplicity of $K$ in $D^b(A \times X)$ is preserved. Moreover, the representation $\omega_\psi(K_q)$ and hence the group $G_{mon}(K_q, \psi)$ remain unchanged. By [Sta23, Tag 056V], one may assume that $X$ is smooth. As $K$ is semisimple in $D^b(A \times X)$, from Lemma 3.3.1.8, so is $K \otimes^L \pi^* L_\chi$. By Fact 3.3.1.7 1, the object $R\rho_*(K \otimes^L \pi^* L_\chi)$ is semisimple in $D^b(X)$.

By proper base change theorem (see, e.g., [Sta23, Tag 095T]), for every integer $j$, the geometric generic stalk $[H^j R\rho_*(K \otimes^L \pi^* L_\chi)]_{\eta}$ is $H^j(A_{\bar{\eta}}, K_{\bar{\eta}} \otimes^L L_\chi)$. Since $\chi \notin S(K_q)$, for every integer $q \neq 0$,

$$H^q(A_{\bar{\eta}}, K_{\bar{\eta}} \otimes^L L_\chi) = 0.$$

By Fact 3.3.1.2, there is a nonempty open subset $U_0$ (resp. $U_q$ for every integer $q \neq 0$) of $X$ such that $[H^0 R\rho_*(K \otimes^L \pi^* L_\chi)]_{U_0}$ is a lisse $\Lambda$-sheaf (resp. $[H^q R\rho_*(K \otimes^L \pi^* L_\chi)]_{U_q} = 0$). The set

$$J := \{q \in \mathbb{Z} : H^q R\rho_*(K \otimes^L \pi^* L_\chi) \neq 0\}$$

is finite and $X$ is irreducible, so $U := U_0 \cap \bigcap_{q \in J} U_q$ is a nonempty open subset of $X$. Shrinking $X$ to $U$, one may assume further that $H^q R\rho_*(K \otimes^L \pi^* L_\chi) = 0$ for every integer $q \neq 0$ and $H^0 R\rho_*(K \otimes^L \pi^* L_\chi)$ is a lisse $\Lambda$-sheaf on $X$. 


Thus, the semisimple object $R\rho_*(K \otimes^L \pi^* L_\chi)[\dim X]$ of $D^b_c(X)$ lies in $\text{Perv}(X)$, so it is semisimple in $\text{Perv}(X)$. By [Ach21, Prop. 3.4.1], the object $R\rho_*(K \otimes^L \pi^* L_\chi)$ of $\text{Loc}(X)$ is semisimple. Therefore, the corresponding representation

$$\pi^\text{et}_1(X, \bar{\eta}) \rightarrow \text{GL}(H^0(A_{\bar{\eta}}, K_{\bar{\eta}} \otimes^L L_\chi))$$

is semisimple. Because $X$ is smooth, the natural morphism $\eta_* : \Gamma_{k(\eta)} \rightarrow \pi^\text{et}_1(X, \bar{\eta})$ is surjective, so the composition $\Gamma_{k(\eta)} \rightarrow \text{GL}(H^0(A_{\bar{\eta}}, K_{\bar{\eta}} \otimes^L L_\chi))$, i.e., the representation $\omega_\psi(K_\eta)$, is semisimple. By [Mil17a, Cor. 19.18], the algebraic group $G_{\text{mon}}(K_\eta, \psi)$ is reductive. □

**Example 3.6.0.2.** Let $X/k$ be a smooth projective integral curve of genus $1$. Then $\pi^\text{et}_1(X, \bar{\eta}) = \mathbb{Z}^2$. There exists a character $\sigma : \pi^\text{et}_1(X, \bar{\eta}) \rightarrow \Lambda^*$ of infinite order. Let $A = \text{Spec}(k)$ and $K \in \text{Loc}(X) = \text{Perv}_{\text{UL}(A \times X/X)}$ be the lisse $\Lambda$-sheaf of rank $1$ on $X$ corresponding to $\sigma$. Then $C(A)(\Lambda) = \{1\}$ and $G_{\text{mon}}(K_\eta, 1) = G_{m/\Lambda}$ is an algebraic torus that is not semisimple.

**Remark 3.6.0.3.** In view of Example 3.3.1.5, the semisimplicity of $H^0 R\rho_*(K \otimes^L \pi^* L_\chi)$ is not clear a priori. That is why we exclude characters in the spectrum $S(K_\eta)$ in Theorem 3.6.0.1.

**Remark 3.6.0.4.** Let $i : Y \rightarrow A \times X$ be a closed subvariety that is smooth over $X$ with connected fibers of dimension $d$.

\[
\begin{array}{ccc}
Y & \xrightarrow{i} & A \times X \\
\downarrow f & & \downarrow \rho \\
X & \xrightarrow{\pi} & A \\
\end{array}
\]

By Example 3.3.3.3, $K := i_* C_Y[d]$ belongs to $\text{Perv}_{\text{UL}(A \times X/X)}$. It is semisimple in $D^b_c(A \times X)$ by Fact 3.1.7.1. Assume that $X/\mathbb{C}$ is smooth. Then for every $\chi \in C(A)(\Lambda) \setminus S(K_\eta)$, the algebraic group $G_{\text{mon}}(K_\eta, \psi)$ coincides with the Zariski closure of the image of the monodromy representation of the lisse $\Lambda$-sheaf $R^d f_* \pi^* L_\chi$ on $X$, which is studied in [KM23, Sec. 1.4] (but with coefficient $\mathbb{C}$ instead of $\Lambda$).

Theorem 3.1.0.3 follows from Theorem 3.6.0.5 and Fact 3.5.0.2, because (3.25) is in fact a finite union.

**Theorem 3.6.0.5.** Assume $K \in \text{Perv}_{\text{UL}(A \times X/X)}$. Then there exists a subobject $K^0 \rightarrow K$ in $\text{Perv}_{\text{UL}(A \times X/X)}$ such that for every $\chi \in C(A)(\Lambda)$ satisfying the three conditions

- $\chi$ is not in $\bigcup_{j \in \mathbb{Z}} S(R^j(\rho \pi_*(K)))$, \hspace{1cm} (3.25)
- $K_\eta \in P^\chi(A_\eta)$ and
- $R^0(\rho \pi_*(K)) \in P^\chi(A)$,
one has
\[ \omega_{\psi}(K_{\eta}^0) = \omega_{\psi}(K_{\eta})^{\Gamma_{k(\eta)}}. \]

**Proof.** By Fact 3.3.1.2, there is a nonempty open subset \( V \subset X \) such that \((H^0 Rf_* K)|_V \) is a lisse sheaf. From [Har77, Lem. 10.5, p.271], one may assume that \( V \) is smooth. By [Sta23, Tag 0BQM], the canonical morphism \( \Gamma_{k(\eta)} \to \pi_1^et(V, \tilde{\eta}) \) is surjective. Thus, from Fact 3.3.1.7 2, the natural map
\[ H^0(A \times X, K \otimes^L \pi_* L_\chi) \to \omega_{\psi}(K_{\eta})^{\Gamma_{k(\eta)}} \quad (3.26) \]
is surjective.

By Fact 3.3.1.1, one has
\[ H^0(A \times X, K \otimes^L \pi_* L_\chi) = H^0(A, (R\pi_* K) \otimes^L L_\chi). \quad (3.27) \]
For any integers \( i \neq 0 \) and \( j \), as \( \chi \notin S(pH^i(R\pi_* K)) \), one has
\[ H^i(A, pH^j(R\pi_* K) \otimes^L L_\chi) = 0. \]
By Lemma 3.3.1.8, the spectral sequence in [Max19, Rk. 8.1.14 (6)] becomes
\[ E_2^{i,j} = H^i(A, pH^j(R\pi_* K) \otimes^L L_\chi) \Rightarrow H^{i+j}(A, (R\pi_* K) \otimes^L L_\chi) \]
and degenerates at page \( E_2 \). Hence
\[ H^0(A, (R\pi_* K) \otimes^L L_\chi) = H^0(A, (pH^0 R\pi_* K) \otimes^L L_\chi). \quad (3.28) \]

Set \( K_1 := \pi^* pH^0(R\pi_* K) \in D^b_{UL}(A \times X) \). By Fact 3.3.2.2 1, \( pH^0(R\pi_* K) \in D^{UL}(A/k) \). From Fact 3.3.2.2 3, \( K_1 \in D^{UL}(A \times X/X) \). For every \( x \in X(k) \), the restriction \( \pi|_{A_x} : A_x \to A \) is an isomorphism of \( k \)-abelian varieties, so the functor
\[ (\pi|_{A_x})^* : \text{Perv}(A) \to \text{Perv}(A_x) \quad (3.29) \]
is an isomorphism of abelian categories. It sends \( pH^0(R\pi_* K) \) to \( K_1^1 \), so \( K_1 \in \text{Perv}^{UL}(A \times X/X) \). Similarly,
\[ K_1^1 = (\pi|_{A_x})^* pH^0(R\pi_* K) \]
is the \( k(\eta)/k \)-scalar extension of \( pH^0(R\pi_* K) \), so
\[ \omega_{\chi}(K_1^1) = H^0(A, pH^0(R\pi_* K) \otimes^L L_\chi). \quad (3.30) \]

Each fiber of the morphism \( \pi \) is of dimension \( \dim X \), so by [BBDG82, 4.2.4], the functor
\[ R\pi_*[- \dim X] : D^b_{UL}(A \times X) \to D^b(A) \]
is left t-exact with respect to the absolute perverse t-structures. From Lemma 3.3.3.5, \( K[\dim X] \in \text{Perv}(A \times X) \) and so \( R\pi_* K \in pD^{\geq 0}(A) \). Then the perverse truncation \( p_{\tau \leq 0}(R\pi_* K) = pH^0(R\pi_* K) \). Via the adjunction formula (see, e.g., [KW01, p.107]), the natural morphism
\[ p_{\tau \leq 0}(R\pi_* K) \to R\pi_* K \]
in \( D^b_c(A) \) (from the definition of t-structure) induces a morphism \( h : K^1 \to K \) in \( D^b_c(A \times X) \). Then \( h \) is a morphism in \( \operatorname{Perv}^{ULA}(A \times X/X) \). Let \( K^0 \) be the image of \( h \) in the abelian category \( \operatorname{Perv}^{ULA}(A \times X/X) \). By Fact 3.3.3.1, the functor \( \operatorname{Perv}(A \times X/X) \to \operatorname{Perv}(A_{\eta}) \) is exact. Then \( K^0 \) is the image of \( h_{\eta} : K^1_{\eta} \to K_{\eta} \) in \( \operatorname{Perv}(A_{\eta}) \).

By assumption, both \( K_{\eta}, K^1_{\eta} \) are in \( P^{x}(A_{\eta}) \). Because \( P^{x}(A_{\eta}) \) is a Serre subcategory of \( \operatorname{Perv}(A_{\eta}) \), the image of \( h_{\eta} \) in \( P^{x}(A_{\eta}) \) is still \( K^0_{\eta} \). As the functor (3.22) is exact, the image of \( \omega_{\chi}(h_{\eta}) : \omega_{\chi}(K^1_{\eta}) \to \omega_{\chi}(K_{\eta}) \) is \( \omega_{\chi}(K^0_{\eta}) \). Combining (3.26), (3.27), (3.28) with (3.30), one gets \( \omega_{\chi}(K^0_{\eta}) = \omega_{\psi}(K^0_{\eta})_{\ell} \). □

If \( K \in \operatorname{Perv}(A \times X/X) \), then by [JKLM23, Thm. 4.3], for every character \( \chi \in C(A)(\Lambda) \), the geometric generic Tannakian group \( G_{\omega_{\chi}}(K_{\eta}) \) is a normal closed subgroup of the generic Tannakian group \( G_{\omega_{\chi}}(K_{\eta}) \). Theorem 3.6.0.6 shows that for uncountably many \( \chi \), the corresponding monodromy group is also a normal closed subgroup of the generic Tannakian group.

**Theorem 3.6.0.6.** Setting as in Theorem 3.6.0.5. Assume further that \( \dim A > 0 \). For every character \( \chi_\ell : \pi^{et}_{\ell}(A) \to \Lambda^* \) of finite order prime to \( \ell \), there is an uncountable subset \( E \subset C(A)(\Lambda) \) with the following property: For every \( \chi_\ell \in E \), set \( \chi = \chi_\ell \chi_\ell \) and denote the pullback of \( \chi \) by \( \psi : \pi^{et}_{\ell}(A_{\eta}) \to \Lambda^* \). Then \( K_{\eta} \in P^{x}(A_{\eta}) \), \( G_{\omega_{\chi}}(K_{\eta}) \) is reductive, and \( G_{\operatorname{mon}}(K_{\eta}, \psi) \) is a normal closed subgroup of \( G_{\omega_{\chi}}(K_{\eta}) \).

**Proof.** Both \( G_{\operatorname{mon}}(K_{\eta}, \psi) \) and \( G_{\omega_{\chi}}(K_{\eta}) \) depend only on the generic fiber of \( \rho \). Thus, shrinking \( X \) to a nonempty open subset does not change them. Therefore, one may assume that \( X \) is smooth. We claim that \( K_{\eta} \in \operatorname{Perv}(A_{\eta}) \) is a semisimple object.

For every subobject \( M \subset K_{\eta} \) in \( \operatorname{Perv}(A_{\eta}) \), by [HS23, Thm. 1.10 (ii)] and the smoothness of \( X \), there is a subobject \( K' \subset K \) in \( \operatorname{Perv}^{ULA}(A \times X/X) \) with \( K' = M \). By Lemma 3.3.3.5, the morphism \( K'[\dim X] \to K[\dim X] \) is a monomorphism in \( \operatorname{Perv}(A \times X) \). Because \( K \) is semisimple in \( D^b_c(A \times X) \), its shift \( K[\dim X] \) is semisimple in \( \operatorname{Perv}(A \times X) \). Thus, there is there is a subobject \( N \subset K[\dim X] \) in \( \operatorname{Perv}(A \times X) \) with

\[
K[\dim X] = (K'[\dim X]) \oplus N.
\]

Then \( K = K' \oplus (N[- \dim X]) \) in \( D^b_c(A \times X) \). For every integer \( j \neq 0 \), one has

\[
0 = p^{S} H^j(K) = p^{S} H^j(K') \oplus p^{S} H^j(N[- \dim X])
\]

in \( \operatorname{Perv}(A \times X/X) \), so \( p^{S} H^j(N[- \dim X]) = 0 \) and hence

\[
N[- \dim X] \in \operatorname{Perv}(A \times X/X).
\]

Consequently, \( K_{\eta} = M \oplus (N_{\eta}[- \dim X]) \) in \( \operatorname{Perv}(A_{\eta}) \). By [BBDG82, Thm. 4.3.1 (i)], the abelian category \( \operatorname{Perv}(A_{\eta}) \) is Noetherian and Artinian. As every subobject of \( K_{\eta} \) in \( \operatorname{Perv}(A_{\eta}) \) admits a direct complement, the claim follows from Lemma 3.2.3.5 2.
3.6. MAIN RESULTS

From the claim and Lemma 3.6.0.8.1, the object $K_\eta$ of $\tilde{P}(A_\eta)$ is also semisimple. Therefore, there is a reductive group $G/\Lambda$ and an equivalence of neutral Tamakian categories $\text{Rep}(G) \to \langle K_\eta \rangle (\subset \tilde{P}(A_\eta))$. By Lemma 3.2.3.4, there is a sequence $\{V_i\}_{i \geq 1}$ of objects in $\text{Rep}(G)$, such that every object of $\text{Rep}(G)$ is isomorphic to one $V_i$. For every integer $i \geq 1$, let $\tilde{K}_i$ be the object of $\langle K_\eta \rangle$ corresponding to $V_i$.

We affirm that for every object $N \in (K_\eta)$, there is $L \in \text{Perv}^{\text{ULA}}(A \times X/X)$ that is semisimple in $D^b_c(A \times X)$ with $L_\eta$ isomorphic to $N$ in $\tilde{P}(A_\eta)$.

From [Mil17a, Cor. 22.43], the abelian category $\text{Rep}(G)$ is semisimple, so $N$ is semisimple in $\tilde{P}(A_\eta)$. There is an integer $n \geq 0$ such that $N$ is a subquotient of $(K_\eta \oplus K_\eta')^n$ in $\tilde{P}(A_\eta)$. Define a bifunctor

$$*: D^b_c(A \times X) \times D^b_c(A \times X) \to D^b_c(A \times X), (K, L) \mapsto R(m \times \text{Id}_X)_*(p^*_{13}K \otimes \bigotimes_{i=1}^{23} p^*_i L),$$

where $p_{ij}$ are the projections on $A \times A \times X$. By the proper base change theorem, for every $x \in X(k)$, $(K *_X L)_x = (K_x)*_x(L_x)$ in $D^b_c(A_x)$. Therefore, $(K *_X L)_n = (K_\eta) * (L_\eta)_n$ in $D^b_c(A_\eta)$.

The bifunctor (3.31) preserves universal local acyclicity. Indeed, if $K, L \in D^{\text{ULA}}(A \times X/X)$, then by [Zhu17, Thm. A.2.5 (4)],

$$p^*_{13}K \otimes L \in D^{\text{ULA}}(A \times A \times X/X).$$

By Fact 3.3.2.2.4, $L \in D^{\text{ULA}}(A \times X/X)$.

Set $K^\vee := ([-1]_A \times \text{Id}_X)^*D_{A \times X/X}K$. By Fact 3.3.3.1.3, $K^\vee \in \text{Perv}^{\text{ULA}}(A \times X/X)$ and $(K^\vee)_n = (K_\eta)^\vee$. From last paragraph,

$$(K \oplus K^\vee)^{\times n} \in D^{\text{ULA}}(A \times X/X).$$

Set $M := n/X H^0((K \oplus K^\vee)^{\times n}) \in \text{Perv}^{\text{ULA}}(A \times X/X)$. Then $M_\eta = n \cdot H^0((K_\eta \oplus (K_\eta)^\vee)^{\times n})$ in $\text{Perv}(A_\eta)$. By Lemma 3.6.0.8.3, there is a semisimple subquotient $L'$ of $M_\eta$ in $\text{Perv}(A_\eta)$, whose image in $\tilde{P}(A_\eta)$ is $N$. By [HS23, Thm. 1.10 (ii)], there is a semisimple subquotient $L$ of $M$ in $\text{Perv}^{\text{ULA}}(A \times X/X)$ with $L_\eta = L'$. By Lemma 3.3.3.9.2, $L[\dim X]$ is a semisimple in $\text{Perv}(A \times X)$ and hence in $D^b_c(A \times X)$. The affirmation is proved.

From the affirmation, for every integer $i \geq 1$, there is $K_i \in \text{Perv}^{\text{ULA}}(A \times X/X)$ that is semisimple in $D^b_c(A \times X)$ with $K_i, \eta$ isomorphic to $K_i$ in $\tilde{P}(A_\eta)$. From Theorem 3.6.0.5 and Fact 3.5.0.2, there is a subobject $K^0_\eta \subset K_i$ in $\text{Perv}^{\text{ULA}}(A \times X/X)$ and a strict Zariski closed subset $B_i$ of the scheme $\mathcal{C}(A)_\ell$, such that for every $\chi_\ell \in (\mathcal{C}(A)_\ell \setminus \cup_{i \geq 1} B_i)(\Lambda)$, one has $K_{i, \eta} \in P^X(D_i)$. By [Gro06, Cor. 2.4], the algebraic subgroup $G_{\text{mon}}(K_\eta, \psi)$ of $G_{\omega_\psi}(K_\eta)$ is observable in the sense of [BBHM63, p.134]. Since $K^0_\eta, \eta$ is a subobject of $K_\eta$ in $(K_\eta)$, by [DE22, Prop. A.12], for every $\chi \in E$, the corresponding group $G_{\text{mon}}(K, \psi)$ is a normal closed subgroup of $G_{\omega_\psi}(K_\eta)$.\hfill \Box
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Remark 3.6.0.7. When $A = \text{Spec}(k)$, the bifunctor (3.31) becomes $\otimes^L : \mathcal{D}_c^b(X) \times \mathcal{D}_c^b(X) \to \mathcal{D}_c^b(X)$. The derived tensor product may not preserve semisimplicity in the category $\mathcal{D}_c^b(X)$. That is why we need semisimplicity in $\text{Perv}^{ULA}$ in the last paragraph of the proof of the affirmation.

In fact, consider $k = \mathbb{C}$, $X = A^1$, $U = X \setminus \{0\}$. Let $j : U \to X$ be the inclusion. Then $\pi_1^d(U, 1) = \hat{\mathbb{Z}}$. The unique surjective morphism $\pi_1^d(U, 1) \to \mathbb{Z}/2$ corresponds to a rank 1 lisse $\Lambda$-sheaf $L$ on $U$. Then $L \otimes^L L$ is the constant lisse $\Lambda$-sheaf and $L^{an}$ is a $\Lambda$-local system on $\mathbb{C}^\ast$. Let $U_0$ be a small punctured ball in $X^{an}$ centered at 0 containing 1.

One has $H^0(U_0, L^{an}) = (L^{an})_{\pi_1(U_0, 1)} = 0$, and $H^1(U_0, L^{an})$ coincides with the group cohomology $H^1(\pi_1(U_0, 1), L^{an})$, where the $\pi_1(U_0, 1) = \mathbb{Z}$ action on the fiber $L_1^{an}$ is the monodromy. For every crossed homomorphism $f : \mathbb{Z} \to L_1^{an}$, every integer $j$, one has $f(1 + j) = f(1) - f(j)$. Therefore, when $j$ is even (resp. odd), $f(j) = 0$ (resp. $f(1)$). In particular, $f$ is a boundary and hence $H^1(\pi_1(U_0, 1), L^{an}) = 0$. Thus, $L^{an}$ is not in the cohomology support loci of $U_0$. From [BLSW17, p.299], $j^{an}_1 L^{an}[1]$ is a simple object of $\text{Perv}(X^{an})$. By [BBDG82, p.150], $M := j_! L[1]$ is a simple object of $\text{Perv}(X)$.

From [KW01, II, Cor. 7.5 g)], one has
\[ N := M \otimes^L M = j_!(L \otimes^L j^* j_! L)[2] = j_! \Lambda_U[2]. \]

Thus, $N[-1] \in \text{Perv}(X)$ by [HT07, Example 8.1.35 (ii)]. Let $i : 0 \to A^1$ be the inclusion. From the short exact sequence
\[ 0 \to j_! \Lambda_U \to \Lambda_X \to i_* (\Lambda_0) \to 0 \]
in $\text{Cons}(X)$, one gets an exact sequence
\[ p H^0(\Lambda_X) \to p H^0(i_* (\Lambda_0)) \to p H^1(j_! \Lambda_U) \to p H^1(\Lambda_X) \to p H^1(i_* (\Lambda_0)) \]
in $\text{Perv}(X)$. Since $i_* (\Lambda_0), \Lambda_X[1] \in \text{Perv}(X)$, it gives a short exact sequence
\[ 0 \to i_* (\Lambda_0) \to N[-1] \to \Lambda_X[1] \to 0 \]
in $\text{Perv}(X)$. This sequence does not split as $N[-1]$ is supported on $U$. Therefore, $N[-1]$ is not a semisimple object of $\text{Perv}(X)$. It follows that $M \otimes^L M$ is not semisimple in $\mathcal{D}_c^b(X)$.

Lemma 3.6.0.8. Let $A$ be an abelian category. Let $B \subseteq A$ be a Serre subcategory. Consider the quotient functor $F : A \to A/B$. Let $X \in A$.

1. Let $i : Y \to F(X)$ be a monomorphism in $A/B$. Then there is a monomorphism $j : Z \to X$ in $A$ and an isomorphism $u : Y \to F(Z)$ in $A/B$ fitting to a commutative triangle in $A/B$

\[ \begin{array}{ccc}
  F(Z) & \xrightarrow{F(j)} & F(X) \\
  u & \downarrow & \\
  Y & \xrightarrow{i} & F(X)
\end{array} \]
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Dually, up to isomorphism every quotient in $A/B$ lifts to a quotient in $A$. In particular, if $X \in A$ is a simple object, then $F(X)$ is either simple or zero in $A/B$.

2. Let $V \in A$ be a Noetherian and Artinian object. If $F(V)$ is simple in $A/B$, then there is a simple subquotient $W$ of $V$ in $A$ such that $F(W)$ is isomorphic to $F(V)$ in $A/B$.

3. Assume that $A$ is Noetherian and Artinian. If $Y$ is a simple subquotient of $X$ in $A/B$, then there is a simple subquotient $W$ of $X$ such that $F(W)$ is isomorphic to $Y$ in $A/B$.

Proof.

1. By the construction in the proof of [Sta23, Tag 02MS] and the right calculus of fractions in [Sta23, Tag 04VB], there is a triangle

$$
\begin{array}{ccc}
M & \xrightarrow{g} & X \\
\downarrow{f} & & \downarrow{X}
\end{array}
$$

in $A$, such that $F(f)$ is an isomorphism and $F(g) = i \circ F(f)$ in $A/B$. Therefore, $F(g)$ is a monomorphism. Since $F$ is exact, $F(\ker(g)) = \ker(F(g)) = 0$, so $\ker(g) \in B$. Let $q : M \to M/\ker(g) \to X$ be the monomorphism, and let $j : M/\ker(g) \to X$ be the monomorphism in $A$ induced by $g$. Then $F(q)$ is an isomorphism in $A/B$. Set $u : Y \to F(M/\ker(g))$ to be the morphism $F(q) \circ F(f)^{-1}$ in $A/B$. Then $u$ is an isomorphism with the stated property.

2. Let $\mathcal{P}$ be the family of subobjects $V'$ of $V$ in $A$ with $V/V' \in B$. Then $\mathcal{P}$ is nonempty since $V \in \mathcal{P}$. Moreover, the morphism $F(U) \to F(V)$ is an isomorphism in $A/B$. Let $\mathcal{Q}$ be the family of subobjects of $U \in A$ lying in $B$. Then $\mathcal{Q}$ is nonempty since $0 \in \mathcal{Q}$. As $V$ is Noetherian in $A$, so is $U$. Thus, $\mathcal{Q}$ has a maximal object $U_0$. Then $W := U/U_0$ is a subquotient of $V \in A$ and the morphism $F(U) \to F(W)$ is an isomorphism in $A/B$. In particular, $W \neq 0$ in $A$.

We claim that $W$ is simple in $A$. Indeed, let $U' \to W$ be a subobject in $A$. Then there is a subobject $U''$ of $U$ in $B$ containing $U_0$ with $U''/U_0 = U'$. As $F(U'')$ is a subobject of a simple object $F(U)$ in $A/B$, either the morphism $F(U'') \to F(U)$ is an isomorphism or $F(U'') = 0$. If $F(U'') = 0$, then $U'' \in \mathcal{B}$ and $U'' \in \mathcal{Q}$. Since $U_0$ is maximal in $\mathcal{Q}$, one has $U_0 = U''$, so $U' = 0$. If $F(U'') \to F(U)$ is an isomorphism, then $U/U'' \in \mathcal{B}$. Since the sequence

$$0 \to U/U'' \to V/U'' \to V/U \to 0$$

is exact in $A$ and $B$ is closed under extensions, one gets $V/U'' \in \mathcal{B}$ and $U'' \in \mathcal{P}$. Since $U$ is minimal in $\mathcal{P}$, one has $U'' = U$. The morphism $U' \to W$ is thus an isomorphism in $A$. 
3. By 1, there is a subquotient $Z$ of $X$ in $A$ with $F(Z)$ isomorphic to $Y$. Then $F(Z)$ is simple in $A/B$. By assumption, $Z$ is Noetherian and Artinian in $A$. Thus from 2, there is a simple subquotient $W$ of $Z$ in $A$ with $F(W)$ isomorphic to $F(Z)$ and to $Y$ in $A/B$.

Example 3.6.0.9. Let $s : X \rightarrow A \times X$ be a section to $\rho$. Let $F$ be a lisse $A$-sheaf on $X$, and let $\rho : \pi_1^\text{ét}(X, \bar{\eta}) \rightarrow \text{GL}(F_{\bar{\eta}})$ be the corresponding representation. By Fact 3.3.2.2 2, $F \in D^{\text{ULA}}(X/X)$. Then from Fact 3.3.2.2 4, $K := Rs_* F \in D^{\text{ULA}}(A \times X/X)$. For every $x \in X(k)$, by proper base change theorem, $K_x$ is the skyscraper supported at the closed point $s(x) \in A_x$ with stalk $F_x$. Thus, $K_x \in \text{Perv}(A_x)$ and $K \in \text{Perv}^{\text{ULA}}(A \times X/X)$. Moreover, $K_{\bar{\eta}}$ is the skyscraper supported at $s(\bar{\eta}) \in A_{\bar{\eta}}$ with stalk $F_{\bar{\eta}}$. Therefore, the generic and geometric generic Tannakian groups agree and are computed in Example 3.5.0.6.

For every $\chi \in C(A)(\Lambda)$, by Fact 3.3.1.1, $K \otimes^L \pi^* L_\chi = Rs_*(F \otimes^L s^* \pi^* L_\chi)$. Thus, $R\rho_*(K \otimes^L \pi^* L_\chi) = F \otimes^L s^* \pi^* L_\chi$ is a lisse $A$-sheaf on $X$. The corresponding $\pi_1^\text{ét}(X, \bar{\eta})$-representation is the tensor product of $\rho$ with the character

$$\pi_1^\text{ét}(X, \bar{\eta}) \rightarrow \pi_1^\text{ét}(A, \pi s(\bar{\eta})) \sim \pi_1^\text{ét}(A) \otimes \Lambda^*.$$  

The induced pullback $\Gamma_{k(\eta)}$-representation along $\eta_* : \Gamma_{k(\eta)} \rightarrow \pi_1^\text{ét}(X, \bar{\eta})$ is $\omega_\psi(K_{\eta})$.

If $F$ is semisimple in $\text{Loc}(X)$, then $F[\dim X]$ is a semisimple object of $\text{Perv}(X)$. By Fact 3.3.1.7 1, $K$ is semisimple in $D^b_c(A \times X)$. 

\[
$$
\text{CHAPTER 3. NORMALITY OF MONODROMY GROUP IN GENERIC TANNAKIAN GROUP}$
\]

\[
3. \text{ By 1, there is a subquotient } Z \text{ of } X \text{ in } A \text{ with } F(Z) \text{ isomorphic to } Y. \text{ Then } F(Z) \text{ is simple in } A/B. \text{ By assumption, } Z \text{ is Noetherian and Artinian in } A. \text{ Thus from 2, there is a simple subquotient } W \text{ of } Z \text{ in } A \text{ with } F(W) \text{ isomorphic to } F(Z) \text{ and to } Y \text{ in } A/B.\]

\[
\text{Example 3.6.0.9. Let } s : X \rightarrow A \times X \text{ be a section to } \rho. \text{ Let } F \text{ be a lisse } A\text{-sheaf on } X, \text{ and let } \rho : \pi_1^\text{ét}(X, \bar{\eta}) \rightarrow \text{GL}(F_{\bar{\eta}}) \text{ be the corresponding representation. By Fact 3.3.2.2 2, } F \in D^{\text{ULA}}(X/X). \text{ Then from Fact 3.3.2.2 4, } K := Rs_* F \in D^{\text{ULA}}(A \times X/X). \text{ For every } x \in X(k), \text{ by proper base change theorem, } K_x \text{ is the skyscraper supported at the closed point } s(x) \in A_x \text{ with stalk } F_x. \text{ Thus, } K_x \in \text{Perv}(A_x) \text{ and } K \in \text{Perv}^{\text{ULA}}(A \times X/X). \text{ Moreover, } K_{\bar{\eta}} \text{ is the skyscraper supported at } s(\bar{\eta}) \in A_{\bar{\eta}} \text{ with stalk } F_{\bar{\eta}}. \text{ Therefore, the generic and geometric generic Tannakian groups agree and are computed in Example 3.5.0.6.}

\text{For every } \chi \in C(A)(\Lambda), \text{ by Fact 3.3.1.1, } K \otimes^L \pi^* L_\chi = Rs_*(F \otimes^L s^* \pi^* L_\chi). \text{ Thus, } R\rho_*(K \otimes^L \pi^* L_\chi) = F \otimes^L s^* \pi^* L_\chi \text{ is a lisse } A\text{-sheaf on } X. \text{ The corresponding } \pi_1^\text{ét}(X, \bar{\eta})\text{-representation is the tensor product of } \rho \text{ with the character}

\[
\pi_1^\text{ét}(X, \bar{\eta}) \rightarrow \pi_1^\text{ét}(A, \pi s(\bar{\eta})) \sim \pi_1^\text{ét}(A) \otimes \Lambda^*.
\]

\text{The induced pullback } \Gamma_{k(\eta)}\text{-representation along } \eta_* : \Gamma_{k(\eta)} \rightarrow \pi_1^\text{ét}(X, \bar{\eta}) \text{ is } \omega_\psi(K_{\eta}).

\text{If } F \text{ is semisimple in } \text{Loc}(X), \text{ then } F[\dim X] \text{ is a semisimple object of } \text{Perv}(X). \text{ By Fact 3.3.1.7 1, } K \text{ is semisimple in } D^b_c(A \times X).
Chapter 4

Generic vanishing theorem for Fujiki class $C$

4.1 Introduction

Recall the historical origin of generic vanishing results. In the last paragraph of [Enr39], Enriques gave an upper bound on the dimension of the paracanonical system of curves on some class of algebraic surfaces. However, in [Enr49, p.354] he pointed out a mistake in the proof of his result as well as a similar theorem by Severi [Sev42]. Catanese [Cat83, p.103] posed Conjecture 4.1.0.1.

Conjecture 4.1.0.1. For a smooth projective surface $S/C$ without irrational pencils, the dimension of the paracanonical system $\{K_S\}$ is at most the geometric genus $p_g(S)$.

In 1987, Green and Lazarsfeld [GL87, Theorem 4.2] provided a positive answer to Conjecture 4.1.0.1. Its proof uses a result ([GL87, Prop. 4.1]) of generic vanishing type.

As is explained in [Ue83, pp.619-620], the dimension of $\{K_S\}$ in Conjecture 4.1.0.1 is related to Conjecture 4.1.0.2, which is also of generic vanishing type.

Conjecture 4.1.0.2 ([Ue83, Problem 8, p.620]). Let $X$ be a projective manifold and $\alpha : X \to \text{Alb}(X)$ be an Albanese morphism. If $\dim \alpha(X) > 1$, then $H^1(X, L) = 0$ for generic $L \in \text{Pic}^0(X)$.

Green and Lazarsfeld [GL87] proved a strengthening of Conjecture 4.1.0.2. Since then, the theory of generic vanishing results has been very much investigated and numerous authors have contributed to its development, so the overview in Section 4.1.1 is by no means complete.

For a finitely generated $\mathbb{Z}$-module $H$, let $H_{\text{tor}}$ be the submodule of $H$ comprised of torsion elements and $H_{\text{free}} := H/H_{\text{tor}}$. Let $F \to X$ be a (holomorphic) vector bundle$^1$ on a complex manifold. The dimension of a complex space

$^1$We use the words “locally free sheaf” and “vector bundle” interchangeably.
always means the complex dimension. For any three integers \( p, q, m \geq 0 \), the corresponding jumping locus is defined as

\[
S_{p,q}^m(X, F) := \{ L \in \text{Pic}^0(X) : h^q(X, \Omega^p_X \otimes \mathcal{O}_X L \otimes \mathcal{O}_X F) \geq m \}.
\]

For simplicity, \( p \) (resp. \( m \), resp. \( F \)) is omitted when \( p = 0 \) (resp. \( m = 1 \), resp. \( F = \mathcal{O}_X \)). Roughly speaking, generic vanishing results show that these loci are small (in some sense) and study their structure when \( F \) is flat unitary (in the sense of Definition 4.2.2.2).

4.1.1 Known results

Let \( X \) be a connected compact Kähler manifold, \( \alpha : X \to \text{Alb}(X) \) be the Albanese map associated with some base point and \( F \to X \) be a flat unitary vector bundle. Each locus \( S_{p,q}^m(X, F) \) is an analytic subset of the complex torus \( \text{Pic}^0(X) \) (see the proof of Theorem 4.7.1.3 1) and “generic” means outside a strict analytic subset. In the literature, generic vanishing results concerning \( S^q(X, F) \) (resp. \( S^{p,q}(X, F) \)) are usually called of Kodaira type (resp. Nakano type). Such results typically involve the following invariants:

- \( \dim \alpha(X) \);
- \( w(X) := \max \{ \text{codim}(Z(\eta), X) : 0 \neq \eta \in H^0(X, \Omega^1_X) \} \), where \( Z(\eta) \) denotes the zero-locus of the 1-form \( \eta \);
- the defect of semismallness \( r(\alpha) \) of \( \alpha \) (Section 4.5.2).

Using deformation theory of cohomology groups, Green and Lazarsfeld [GL87, Remarks (1), p.401] proves Fact 4.1.1.1, which is of Kodaira type and implies Conjecture 4.1.0.2.

**Fact 4.1.1.1.** For every integer \( k \geq 0 \), one has

\[
\text{codim}_{\text{Pic}^0(X)}(S^k(X, F)) \geq \dim \alpha(X) - k.
\]

In particular, if \( k < \dim \alpha(X) \), then \( H^k(X, F \otimes \mathcal{O}_X \mathcal{L}) = 0 \) for a generic line bundle \( \mathcal{L} \in \text{Pic}^0(X) \).

Green and Lazarsfeld also give a Nakano-type generic vanishing theorem.

**Fact 4.1.1.2** ([GL87, Remarks (1), p.404]). For any integers \( i, j \geq 0 \) with \( i + j < w(X) \), one has \( S^{i,j}(X, F) \neq \text{Pic}^0(X) \).

In another direction, there are known results concerning the structure of the jumping loci.

**Fact 4.1.1.3** ([GL91, Thm. 0.1 (1)]). For any two integers \( k, m \geq 0 \), the subset \( S^k_m(X) \) is a finite union of translates of subtori of \( \text{Pic}^0(X) \).
4.1. INTRODUCTION

Beauville and Catanese conjectured that for every integer $q \geq 0$, $S^q(X)$ is a finite union of torsion translates of subtori of $\text{Pic}^0(X)$ ([Cat91, Problem 1.25] and [Bea92, p.1]). When $X$ is a projective manifold, this conjecture is proved by Simpson ([Sim93, Sec. 5]).

**Fact 4.1.1.4 (Simpson).** If $X$ is furthermore projective, then for any two integers $k, m \geq 0$, the locus $S^k_m(X)$ is a finite union of torsion translates of subtori of $\text{Pic}^0(X)$.

Some arguments of [Sim93] are of arithmetic nature, so they do not apply to the Kähler case. Campana [Cam01, Sec. 1.5.2] provided a partial answer for not only Kähler manifolds but also for Fujiki class $C$ (Definition 4.7.1.1).


**Fact 4.1.1.5 (Wang).** For any three integers $p, q, m \geq 0$, the subset $S^p_m^q(X)$ of $\text{Pic}^0(X)$ is a finite union of torsion translates of subtori.

Hacon [Hac04, Cor. 4.2] uses Fourier-Mukai transforms of coherent modules on complex abelian varieties to recover Fact 4.1.1.1 when $X$ is a projective manifold. This algebraic viewpoint sheds new insight on this topic. Similarly, as a byproduct of the theory on convolution of perverse sheaves on abelian varieties, Krämer and Weissauer obtain a Nakano-type generic vanishing theorem. The proof of [KW15b, Thm. 3.1] gives Fact 4.1.1.6.

**Fact 4.1.1.6.** If furthermore the Albanese torus $\text{Alb}(X)$ is algebraic, then for any two integers $p, q \geq 0$ with $p + q < \dim X - r(\alpha)$, the locus $S^p_q(X, F)$ is contained in a finite union of translates of strict subtori of $\text{Pic}^0(X)$.

Around the same time, by different methods Popa and Schnell [PS13, Thm. 1.2] obtained precise codimension bounds.

**Fact 4.1.1.7.** If furthermore $X$ is a projective manifold, then
\[
\text{codim}_{\text{Pic}^0(X)}(S^p_q(X)) \geq |p + q - \dim X| - r(\alpha)
\]
for any two integers $p, q \geq 0$. Moreover, for every $X$ there exist $p$ and $q$ for which the inequality becomes an equality.

4.1.2 The main result and a sketch

Even though not necessarily Kähler, a complex smooth proper algebraic variety\footnote{An algebraic variety means an integral scheme of finite type and separated over a field.} also admits Hodge theory ([Del68, Prop. 5.3], [Del71, Thm. 3.2.5]). It is natural to ask if generic vanishing results also hold for such varieties. The aim of this note is to show that generic vanishing result is not only true for Kähler manifolds, but also for complex manifolds in Fujiki class $C$. This class contains compact Kähler manifolds as well as smooth proper algebraic varieties.

Here is the main result that is of Nakano type.
Theorem (Theorem 4.7.1.3). Let $X$ be an $n$-dimensional complex manifold in Fujiki class $C$ with an Albanese morphism $\alpha : X \to \text{Alb}(X)$, and let $F$ be a flat unitary vector bundle on $X$. Then for any two integers $p, q \geq 0$ with $p + q < n - r(\alpha)$, the locus $S^{p,q}(X, F)$ is a strict analytic subset of the complex torus $\text{Pic}^0(X)$.

For smooth proper algebraic varieties, the following finer result follows from Corollary 4.7.2.5 and Lemma 4.6.1.2. It is not immediate from previously known generic vanishing results.

Corollary 4.1.2.1. Let $X/\mathbb{C}$ be an $n$-dimensional smooth proper algebraic variety with an algebraic Albanese morphism $\alpha : X \to \text{Alb}(X)$. Let $\mathcal{L}$ be a unitary local system on the analytification $X^{an}$, and let $F = \mathcal{L} \otimes_{\mathbb{C}} O_{X^{an}}$ be the corresponding holomorphic vector bundle. Then, for any two integers $p, q \geq 0$ with $p + q < n - r(\alpha)$, the subset $S^{p,q}(X, F)$ is contained in a finite union of translates (torsion translates if $\mathcal{L}$ is semisimple of geometric origin$^5$) of strict abelian subvarieties of the Picard variety$^4 \text{Pic}^0_{X/\mathbb{C}}$.

Here is the outline of the proof of Theorem 4.7.1.3. By the Riemann-Hilbert correspondence restricted to unitary objects, we pass from flat unitary vector bundles to unitary local systems. The corresponding cohomology groups are related by Hodge decomposition (Fact 4.7.1.2). In this way, the initial generic vanishing problem for a flat unitary vector bundle twisted by line bundles is reduced to a generic vanishing problem for a unitary local system twisted by rank 1 local systems.

By pushing forward along the Albanese map, the problem about the local system on a manifold in Fujiki class $C$ is converted to a problem about a complex of sheaves on a complex torus. The last problem is solved by Krämer and Weissauer [KW15b] for perverse sheaves (on complex abelian varieties) and by the subsequent generalization (to all complex tori) due to Bhatt, Schnell and Scholze [BSS18].

This text is organized as follows. Sections 4.2 reviews the unitary Riemann-Hilbert correspondence. Section 4.3 and 4.4 construct the Jacobian and the Albanese map for regular manifolds, relaxing the usual Kähler condition. Several definitions of defect of semi-smallness are proved to be equivalent in Section 4.5. The work of Krämer and Weissauer on generic vanishing for perverse sheaves is recalled in Section 4.6. Finally in Section 4.7, the previous results are applied to prove the main result, Theorem 4.7.1.3, for Fujiki class $C$.

4.2 Riemann-Hilbert correspondence

In Section 4.2, we review how the classical Riemann-Hilbert correspondence restricts to an equivalence between unitary local systems and flat unitary vector

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$^3$reviewed in (4.18)

$^4$recalled in Section 4.7.2

$^5$in the sense of [BBDG82, p.163]
bundles on complex manifolds. The reason to introduce this restricted equivalence is that unitary local systems on manifolds in Fujiki class admit Hodge decomposition (Fact 4.7.1.2).

4.2. Unitary local systems

Let $X$ be a path-connected, locally path-connected and locally simply connected topological space with a base point $x_0 \in X$. Let $\text{Loc}(X)$ be the category of local systems (of finite dimensional $\mathbb{C}$-vector spaces) on $X$. Let $\pi_1(X, x_0)$ be the fundamental group of $X$ at $x_0$ and $\text{Rep}_\mathbb{C}(\pi_1(X, x_0))$ be the category of its finite dimensional complex representations. By [Del70, Cor. 1.4, p.4], the functor taking the stalk at $x_0$ gives rise to an equivalence

$$\text{Loc}(X) \to \text{Rep}_\mathbb{C}(\pi_1(X, x_0))$$

(4.1)

compatible with tensor products. The image under (4.1) of a local system on $X$ is called the corresponding monodromy representation.

Let $\text{Rep}_\mathbb{C}^u(\pi_1(X, x_0)) \subset \text{Rep}_\mathbb{C}(\pi_1(X, x_0))$ be the full subcategory of unitary representations. That means representations $\rho : \pi_1(X, x_0) \to \text{GL}(V)$ satisfying the following equivalent conditions:

1. The closure of $\rho(\pi_1(X, x_0))$ inside $\text{GL}(V)$ is compact;
2. There is a hermitian inner product $h : V \otimes \mathbb{C} \overline{V} \to \mathbb{C}$ such that $\rho(\pi_1(X, x_0))$ is contained in the corresponding unitary group $U(V, h)$.

Let $\text{Loc}^u(X)$ be the full subcategory of $\text{Loc}(X)$ corresponding to $\text{Rep}_\mathbb{C}^u(\pi_1(X, x_0))$ via the equivalence (4.1). Its objects are called unitary local systems on $X$. Every unitary local system is semisimple, since every unitary representation is so.

4.2.2 Flat unitary bundles

Let $E \to X$ be a holomorphic vector bundle on a complex manifold with a hermitian metric $h$. By [Huy05, Prop. 4.2.14], there exists a unique hermitian connection $\nabla_h$ that is compatible with the holomorphic structure (in the sense of [Huy05, Def. 4.2.12], i.e., $\nabla^{0,1} = \partial\overline{\partial}$), which is called the Chern connection of $(E, h)$. The corresponding curvature form, called the Chern curvature, is an $\text{End}(E, h)$-valued $(1,1)$-form, (see, e.g., [Huy05, Prop. 4.3.8 iii]).

For every integer $k \geq 0$ (resp. any two integers $i, j \geq 0$), let $A^k_X$ (resp. $A^{i,j}_X$) be the sheaf of smooth $k$ (resp. $(i,j)$) forms on $X$. Then there is a direct sum decomposition $A^k_X = \oplus_{i+j=k} A^{i,j}_X$. In general, a (smooth) flat connection $\nabla$ on $E$ that is compatible with the holomorphic structure needs not to be a holomorphic connection (in the sense of [Huy05, Def. 4.2.17]).

\footnote{Since every compact subgroup of $\text{GL}_r(\mathbb{C})$ can be conjugated into the unitary subgroup $U_r(\mathbb{C})$.}
**Lemma 4.2.2.1.** Let $E \rightarrow X$ be a holomorphic vector bundle with a flat connection $\nabla : E \rightarrow E \otimes \mathcal{A}^0 X \mathcal{A}_X$. If $\nabla$ is compatible with the holomorphic structure, then $\nabla$ is a holomorphic connection.

**Proof.** Take a local holomorphic frame $\{e_1, \ldots, e_r\}$ of $E$, and denote the corresponding local smooth connection matrix 1-form by $\Omega$. As $\nabla^0 = \bar{\partial}E$, one has $\Omega^0 = 0$. By flatness, $d\Omega + \Omega \wedge \Omega = 0$. Taking the $(1,1)$ part of it, one gets $\bar{\partial}\Omega = 0$, i.e., $\Omega$ is a holomorphic form. This shows that $\nabla$ is holomorphic. \qed

Let $\text{Mod}(O_X)$ be the category of $O_X$-modules, and let $\text{VB}(X) \subset \text{Mod}(O_X)$ be the full subcategory of finite locally free $O_X$-modules. Let $\text{DE}(X)$ be the category of holomorphic vector bundles with a flat holomorphic connection. Forgetting the connection gives a functor $\text{DE}(X) \rightarrow \text{VB}(X)$. Let $\text{DE}^u(X) \subset \text{DE}(X)$ be the full subcategory comprised of objects $(F, \nabla)$ such that there exists a hermitian metric on $F$ whose Chern connection is $\nabla$.

**Definition 4.2.2.2.** An object in the essential image of $\text{DE}^u(X)$ under the forgetful functor $\text{DE}(X) \rightarrow \text{VB}(X)$ is called a flat unitary vector bundle on $X$.

From [Huy05, Eg. 4.2.15], the trivial line bundle $O_X$ is flat unitary. By Lemma 4.2.2.1, a holomorphic vector bundle is flat unitary if and only if it admits a hermitian metric whose Chern connection is flat.

### 4.2.3 An equivalence

Let $X$ be a connected complex manifold. By the Riemann-Hilbert correspondence ([Del70, Thm. 2.17, p.12]), the pair of functors

\begin{align}
\text{Loc}(X) &\rightarrow \text{DE}(X), \quad \mathcal{L} \mapsto (\mathcal{L} \otimes O_X, \text{Id}_\mathcal{L} \otimes d); \\
\text{DE}(X) &\rightarrow \text{Loc}(X), \quad (E, \nabla) \mapsto \ker(\nabla)
\end{align}

forms an equivalence of categories. It is compatible with tensor products and preserves the rank.

**Theorem 4.2.3.1 (Unitary Riemann-Hilbert correspondence).** The equivalence (4.2), (4.3) restricts to an equivalence between $\text{Loc}^u(X)$ and $\text{DE}^u(X)$.

**Proof.** First, we prove that the functor (4.2) sends $\text{Loc}^u(X)$ to $\text{DE}^u(X)$. Consider a unitary local system $\mathcal{L}$ on $X$. Since the corresponding monodromy representation is unitary, we may choose a hermitian inner product $h_{x_0}$ on the stalk $\mathcal{L}_{x_0}$ such that the representation factors through $U(\mathcal{L}_{x_0}, h_{x_0})$. For any $x \in X$, choose a path $\gamma$ from $x_0$ to $x$ and propagate $h_{x_0}$ along this curve, i.e., using the linear isomorphism $\gamma_* : \mathcal{L}_{x_0} \rightarrow \mathcal{L}_x$ induced by $\gamma$, we translate $h_{x_0}$ to a hermitian inner product $h_x$ of $\mathcal{L}_x$. This $h_x$ is independent of the choice of $\gamma$ by assumption. Hence a positive definite hermitian form $h$ on $\mathcal{L}$ that is invariant under the monodromy action. Then $h$ extends naturally to a (smooth) hermitian metric $h'$ on the associated holomorphic vector bundle $\mathcal{L} \otimes O_X$ on $X$ and the corresponding flat holomorphic connection $\text{Id}_\mathcal{L} \otimes d$ is a hermitian
connection. Therefore, \( \text{Id}_L \otimes d \) is the Chern connection of \((L \otimes \mathcal{O}_X, h')\) and \((L \otimes \mathcal{O}_X, \text{Id}_L \otimes d) \in \text{DE}^n(X)\).

Conversely, we prove that the functor (4.3) sends \( \text{DE}^n(X) \) to \( \text{Loc}^n(X) \). Consider a holomorphic hermitian vector bundle \((E, h)\) on \(X\) whose Chern connection \(\nabla_h\) is flat. Around every point we can find a local \(\nabla_h\)-horizontal holomorphic frame \(\{e_1, \ldots, e_r\}\) of \(E\). For any \(1 \leq i, j \leq r\), since the connection \(\nabla_h\) is compatible with \(h\), we have

\[
d[h(e_i, e_j)] = h(\nabla_h e_i, e_j) + h(e_i, \nabla_h e_j) = 0.
\]

Therefore, the local function \(h(e_i, e_j)\) is locally constant and the parallel transport along every closed path on \(X\) preserves the hermitian inner product on the fibers of \(E\). The sheaf \(\ker(\nabla_h)\) of horizontal sections of \(E\) forms a local system on \(X\), whose stalks are exactly the fibers of \(E\). Thus, it admits a monodromy-invariant positive definite hermitian form and is consequently unitary.\(^7\)

\[\text{4.3 Hodge theory and Jacobian}\]

In Section 4.3, we review the definition of Jacobian and show that for every complex manifold admitting Hodge theory (Definition 4.3.1.1), its Jacobian has nice expected properties.

\[\text{4.3.1 Regular manifolds}\]

Let \(X\) be a complex manifold. Let \(d : A^{p,q}_X \to A^{p+1,q}_X\) be the exterior derivative. Then \(d = \partial + \bar{\partial}\), where \(\partial : A^{p,q}_X \to A^{p+1,q}_X\) and \(\bar{\partial} : A^{p,q}_X \to A^{p,q+1}_X\) are the \((1, 0)\) and \((0, 1)\) part of \(d\) respectively. For every \(E \in \text{Loc}^n(X)\), every integer \(k \geq 0\), define a decreasing filtration of \(A^k_X \otimes \mathcal{E}\) by

\[
F_p = F^p(A^k_X \otimes \mathcal{E}) := \bigoplus_{i \geq p} A^{i-k, i}_X \otimes \mathcal{E}.
\]

Then \((d \otimes \text{Id}_\mathcal{E})(F^p) \subset F^p\). Therefore, this filtration induces a spectral sequence, called the Frölicher spectral sequence:

\[
E_{1}^{p,q} = H^q(X, \Omega^p_X \otimes \mathcal{E}) \Rightarrow H^{p+q}(X, \mathcal{E}),
\]

where the differential \(d_{1}^{p,q} : E_1^{p,q} \to E_1^{p+1,q}\) is induced by the operator \(\partial : A^{p,q}_X \to A^{p+1,q}_X\) on \(X\). It is the classical notion in [Voi02, Sec. 8.3.3] when \(\mathcal{E}\) is the constant sheaf \(\mathbb{C}_X\).

Although the Hodge theory for the first cohomology groups \(H^1\) suffices for most properties of the Jacobian and the Albanese, in the sequel we mainly work with manifolds admitting Hodge theory in all degrees. Such manifolds are called “regular” for convenience.

**Definition 4.3.1.1** (Regular manifold, [DGMS75, 5.21 (2)]). Assume that \(X\) is compact. Let \(\mathcal{E} \in \text{Loc}^n(X)\). If the following conditions are satisfied:

\[\text{The definition of unitary local system in [Tim87, p.152] seems to forget this invariance.}\]
1. The corresponding spectral sequence (4.5) degenerates at page $E_1$;

2. For every integer $k \geq 0$, the filtration induced by $F^* (A_X^\bullet \otimes \mathbb{C} \mathcal{E})$ on $H^k (X, \mathcal{E})$ gives a complex Hodge structure of weight $k$, in particular a Hodge decomposition

$$H^k (X, \mathcal{E}) = \oplus_{p+q=k} H^q (X, \Omega_X^p \otimes \mathbb{C} \mathcal{E}); \quad (4.6)$$

3. For any integers $p, q \geq 0$, the conjugation map induces a $\mathbb{C}$-anti-linear isomorphism

$$H^q (X, \Omega_X^p \otimes \mathbb{C} \mathcal{E}) \rightarrow H^p (X, \Omega_X^q \otimes \mathbb{C} \mathcal{E}^\vee),$$

where $\mathcal{E}^\vee = \mathcal{H}om (\mathcal{E}, \mathbb{C}_X)$ is the dual local system.

Then $X$ is called $\mathcal{E}$-regular (and simply regular when $\mathcal{E} = \mathbb{C}_X$).

For instance, classical Hodge theory asserts that compact Kähler manifolds are regular (see e.g., [Vois02, Sec. 6.1.3]). Because of Fact 4.3.1.2, regular manifolds are also called $\partial \bar{\partial}$-manifolds.

**Fact 4.3.1.2** ($\partial \bar{\partial}$-lemma, [DGMS75, 5.14, 5.21], [Var86, Prop. 3.4], [Huy05, Cor. 3.2.10]). Assume that $X$ is compact. Then $X$ is regular if and only if for every $d$-closed smooth $(p,q)$-form $\eta$ on $X$, the following conditions are equivalent:

1. $\eta$ is $d$-exact;
2. $\eta$ is $\partial$-exact;
3. $\eta$ is $\bar{\partial}$-exact;
4. $\eta$ is $\partial \bar{\partial}$-exact.

If the above conditions hold and $\eta$ is real, then there is a real smooth $(p-1, q-1)$-form $\rho$ on $X$ with $\eta = i \partial \bar{\partial} \rho$.

**Remark 4.3.1.3.** For Fact 4.3.1.2, it is important that the decomposition (4.6) is induced by the filtration (4.4). In fact, [COUV16, Prop. 4.3] constructs a non-regular compact complex manifold $X$ of dimension 3 such that the spectral sequence (4.5) for $\mathcal{E} = \mathbb{C}_X$ degenerates at page $E_1$, with numerical Hodge symmetry $h^{p,q} (X) = h^{q,p} (X)$ for any two integers $p, q \geq 0$. In this case, there is a non canonical decomposition of the form (4.6).

For the rest of Section 4.3.1, we assume that $X$ is a regular manifold. For every integer $k \geq 0$ (resp. any two integers $p, q \geq 0$), the space of global $\partial$-closed, $\bar{\partial}$-closed smooth $k$ (resp. $(p,q)$) forms on $X$ is denoted by $Z^k (X)$ (resp. $Z^{p,q} (X)$). For any two integers $p, q \geq 0$, the Dolbeault cohomology group $H^q (X, \Omega_X^p)$ is denoted by $H^{p,q} (X)$.

**Corollary 4.3.1.4.** For any integers $p, q \geq 0$ and $k := p+q$, there is a canonical commutative diagram
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\[ \begin{array}{c}
Z^{p,q}(X) \\
\downarrow \\
H^{p,q}(X) \\
\downarrow \\
Z^k(X) \\
\end{array} \]

where the first row is the natural inclusion and each vertical map is surjective. Moreover,

\[ H^k(X, \mathbb{C}) = \oplus_{p+q=k} \text{im}(i^{p,q}) , \]

where each \( \text{im}(i^{p,q}) \) can be identified with \( H^{p,q}(X) \). The complex conjugation map \( Z^{p,q}(X) \to Z^{q,p}(X) \) descends to a \( \mathbb{C} \)-antilinear isomorphism \( H^{p,q}(X) \to H^{q,p}(X) \) (Hodge symmetry).

**Proof.** For each \( \bar{\partial} \)-closed \((p, q)\)-form \( \eta \) on \( X \), \( \partial \eta \) is a \( d \)-closed, \( \partial \)-exact \((p + 1, q)\)-form. By Fact 4.3.1.2, there is a \((p, q - 1)\)-form \( \rho \) on \( X \) with \( \partial \eta + \bar{\partial} \rho = 0 \), then the \((p, q)\)-form \( \eta + \bar{\partial} \rho \) is in \( Z^{p,q}(X) \). Therefore, the map taking Dolbeault cohomology class \( Z^{p,q}(X) \to H^{p,q}(X) \) is surjective.

Note that \( \eta + \bar{\partial} \rho \) is \( d \)-closed. Its de Rham cohomology class is independent of the choice of \( \rho \). Indeed, if \( \rho' \) is another \((p, q - 1)\)-form with \( \eta + \bar{\partial} \rho' \) also \( d \)-closed, then \( \partial(\rho - \rho') \) is \( d \)-closed and \( \partial \)-exact. By Fact 4.3.1.2, it is \( d \)-exact.

Thus the map

\[ i^{p,q} : H^q(X, \Omega^p_X) \to H^{p+q}_{\text{dR}}(X, \mathbb{C}), \quad [\eta] \to [\eta + \bar{\partial} \rho] \]

is a well-defined \( \mathbb{C} \)-linear map. By a third application of Fact 4.3.1.2, the map \( i^{p,q} \) is injective. Thus, \( H^{p,q} \) is identified with \( \text{im}(i^{p,q}) \).

We claim that the sum \( \sum_{p+q=k} \text{im}(i^{p,q}) \) is direct. In fact, if \( \alpha^{p,q} \in Z^{p,q}(X) \) for each pair \((p, q)\) with \( p + q = k \) and the de Rham class of \( \sum_{p+q=k} \alpha^{p,q} = 0 \) in \( H^k_{\text{dR}}(X, \mathbb{C}) \), then there is a \((k - 1)\)-form \( \beta \) on \( X \) with \( d\beta = \sum_{p+q=k} \alpha^{p,q} \). Thus,

\[ \alpha^{p,q} = \partial(\beta^{p-1,q}) + \bar{\partial}(\beta^{p,q-1}) \]

The \( \partial \)-exact form \( \partial(\beta^{p-1,q}) \) is thereby \( \bar{\partial} \)-closed, so \( d \)-closed. By Fact 4.3.1.2 again, \( \partial(\beta^{p-1,q}) \) is \( \bar{\partial} \)-exact, hence \( [\alpha^{p,q}] = 0 \) in \( H^{p,q}(X) \) for every \((p, q)\). The claim is proved.

By assumption,

\[ \dim \mathbb{C} H^k(X, \mathbb{C}) = \sum_{p+q=k} \dim \mathbb{C} H^{p,q}(X), \]

hence the decomposition (4.7). In particular, the map taking de Rham cohomology class \( Z^k(X) \to H^k(X, \mathbb{C}) \) is surjective. The complex conjugate of \( Z^{p,q}(X) \) is exactly \( Z^{q,p}(X) \), the Hodge symmetry follows. □

**Lemma 4.3.1.5.** For every integer \( k \geq 0 \), the map \( H^k(X, \mathbb{C}) \to H^k(X, O_X) \) induced by the inclusion \( \mathbb{C} \to O_X \) coincides with the projection \( H^k(X, \mathbb{C}) \to H^0,^k(X) \) given by the Hodge decomposition (4.7).
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Proof. Consider the following commutative diagram

$$
\begin{array}{cccc}
\mathbb{C}_X & \longrightarrow & A^0_X & \longrightarrow & A^1_X & \longrightarrow & \ldots \\
\downarrow & & \downarrow p^{0,0} & & \downarrow p^{0,1} & & \\
O_X & \longrightarrow & A^{0,0}_X & \longrightarrow & A^{0,1}_X & \longrightarrow & \ldots
\end{array}
$$

The first row is an acyclic resolution of $\mathbb{C}_X$ by (smooth) Poincaré lemma, and the second row is the Dolbeault resolution. The first vertical map is the inclusion and each $p^{0,j}: A^j_X \to A^{0,j}_X$ is taking the $(0,j)$-part of a $j$-form. It is a morphism of complexes. Taking global sections, the induced map on $k$-th cohomology groups is the first map in the statement.

For a class $[\alpha] \in H^k(X, \mathbb{C})$, we may assume that the representative $k$-form $\alpha$ is $\partial$-closed and $\bar{\partial}$-closed by Corollary 4.3.1.4. Then its image under the first map $H^k(X, \mathbb{C}) \to H^k(X, O_X)$ is represented by the $(0,k)$-part of $\alpha$, which is still $\partial$-closed and $\bar{\partial}$-closed. This describes exactly the projection induced by the Hodge decomposition (4.7).

4.3.2 Jacobian

For a connected compact complex manifold $X$, let $b_1(X) := \dim_{\mathbb{C}} H^1(X, \mathbb{C})$ be its first Betti number. The exponential short exact sequence

$$0 \to \mathbb{Z} \to O_X \xrightarrow{f \mapsto \exp(2\pi i f)} O_X^* \to 1$$

induces a long exact sequence

$$H^0(X, O_X) \xrightarrow{f \mapsto \exp(2\pi i f)} H^0(X, O_X^*) \to H^1(X, \mathbb{Z}) \to H^1(X, O_X) \xrightarrow{\delta} H^2(X, \mathbb{Z}).$$

(4.8)

Set $\text{Pic}(X) := H^1(X, O_X^*)$ for the Picard group, $\text{NS}(X) := \text{im}(\delta)$ for the Néron-Severi group, $\text{Pic}^0(X) := \ker(\delta)$ and $\text{Pic}^\tau(X) := \delta^{-1}(H^2(X, \mathbb{Z})_{tor})$. As $X$ is compact connected, one has $H^0(X, O_X) = \mathbb{C}$, $H^0(X, O_X^*) = \mathbb{C}^*$, and the first map in (4.8) is surjective. Accordingly, the third map $H^1(X, \mathbb{Z}) \to H^1(X, O_X)$ is injective and

$$\text{Pic}^0(X) = \frac{H^1(X, O_X)}{H^1(X, \mathbb{Z})}.$$  

(4.9)

If $X$ is a complex torus, then $H^2(X, \mathbb{Z})$ is torsion free and

$$\text{Pic}^0(X) = \text{Pic}^\tau(X).$$  

(4.10)

For general $X$, let $\text{Loc}^1(X)$ (resp. $\text{Loc}^{u,1}(X)$) be the set of isomorphism classes of rank-1 (resp. and unitary) local systems on $X$. Then $\text{Loc}^1(X)$ is a group under tensor product and $\text{Loc}^{u,1}(X)$ is a subgroup. For each $\mathcal{L} \in \text{Loc}^1(X)$, $L := \mathcal{L} \otimes_{\mathbb{C}} O_X$ is a flat line bundle on $X$. By [Dem12, Ch. V, § 9], $L \in \text{Pic}^\tau(X)$, whence a group morphism

$$\text{Loc}^1(X) \to \text{Pic}^\tau(X), \quad \mathcal{L} \mapsto \mathcal{L} \otimes_{\mathbb{C}} O_X.$$  

(4.11)
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Remark 4.3.2.1. Theorem 4.2.3.1 implies that a line bundle on $X$ is flat unitary if and only if its class in $\text{Pic}(X)$ lies in the image of the restriction of (4.11):

$$\text{Loc}^{u,1}(X) \to \text{Pic}^r(X).$$

(4.12)

The image of (4.12) may not be contained in $\text{Pic}^0(X)$. For instance, let $X$ be an Enriques surface, then $\pi_1(X, x_0) = \mathbb{Z}/2$, $\# \text{Loc}^{u,1}(X) = \mathbb{Z}/2$. By Corollary 4.4.2.2 below, the map (4.12) is an isomorphism, while $\text{Pic}^0(X)$ is trivial.

Corollary 4.3.2.2. Assume that $X$ is regular. Then $\text{Pic}^r(X)$ has a natural structure of compact complex Lie group with identity component $\text{Pic}^0(X)$ that is a complex torus of dimension $b_1(X)/2$. Moreover, $\pi_0(\text{Pic}^r(X)) = \text{NS}(X)_{\text{tor}}$.

Proof. The inclusion $\mathbb{R} \subset O_X$ induces an $\mathbb{R}$-linear map

$$\phi : H^1(X, \mathbb{R}) \to H^1(X, O_X).$$

(4.13)

Because of Lemma 4.3.1.5 and the Hodge symmetry in Corollary 4.3.1.4, taking complex conjugate inside $H^1(X, \mathbb{C})$ induces an $\mathbb{R}$-linear map

$$\bar{\phi} : H^1(X, \mathbb{R}) \to H^0(X, \Omega^1_X).$$

(4.14)

If $\xi \in \ker(\phi)$, then the image of $\xi$ under the injection $H^1(X, \mathbb{R}) \to H^1(X, \mathbb{C})$ is $\phi(\xi) + \bar{\phi}(\xi) = 0$, so $\xi = 0$. This shows that $\phi$ is injective. But $\dim_{\mathbb{R}} H^1(X, \mathbb{R}) = \dim_{\mathbb{R}} H^1(X, O_X) = b_1(X)$, so $\phi$ is a linear isomorphism.

The map $H^1(X, \mathbb{Z}) \to H^1(X, O_X)$ in (4.8) factors through $\phi$. Since $H^1(X, \mathbb{Z})$ is a full lattice of $H^1(X, \mathbb{R})$, it remains a full lattice in $H^1(X, O_X)$. Therefore, the quotient $\text{Pic}^0(X)$ is a complex torus of dimension $b_1(X)/2$. The $\mathbb{Z}$-module $\text{Pic}^0(X)$ is divisible, so the short exact sequence

$$0 \to \text{Pic}^0(X) \to \text{Pic}^r(X) \to \text{NS}(X)_{\text{tor}} \to 0$$

spits. Therefore, there is a natural structure of compact complex Lie group on $\text{Pic}^r(X)$ satisfying the stated properties. \hfill \square

The complex torus $\text{Pic}^0(X)$ in Corollary 4.3.2.2 is called the Jacobian of the regular manifold $X$.

Example 4.3.2.3. Here are two examples showing how Corollary 4.3.2.2 fails for non-regular compact complex manifolds.

1. Let $X$ be a Hopf surface ([Huy05, Example 3.3.2]). The Betti number $b_1(X) = 1$, $H^1(X, \mathbb{Z}) = \mathbb{Z}$ and $H^1(X, O_X) = \mathbb{C}$, so the complex manifold $\text{Pic}^0(X) = \mathbb{C}/\mathbb{Z}$ is not compact. However, by [Kod64], the Frölicher spectral sequence of $\mathbb{C}^\infty_X$ degenerates.

2. Let $Y$ be a Calabi-Eckmann manifold ([BS17, Sec 1.2]). Then $H^1(Y, O_Y) = \mathbb{C}$ and $Y$ is simply connected, so $H^1(Y, \mathbb{Z}) = 0$ and $b_1(Y) = 0$, but $\text{Pic}^0(Y) = \mathbb{C}$ is not compact and $b_1(Y)/2 < \dim \text{Pic}^0(Y)$.
4.4 Albanese torus

We turn to the conception of Albanese torus and Albanese map. They help to reduce some problems about general complex manifolds to those about complex tori. They are also tools to study the Jacobian. Again, Section 4.4 conveys the fact that Hodge theory guarantees the usual properties of the Albanese torus and Albanese map.

Fix a connected regular manifold $X$ and a base point $x_0 \in X$.

4.4.1 Basics of Albanese torus

From [Uen06, Cor. 9.5, p. 101], every element of $H^0(X, \Omega^1_X)$ is $d$-closed, so there is a well-defined natural map

$$\iota : H^1(X, \mathbb{Z}) \to H^0(X, \Omega^1_X)^\ast, \quad [\gamma] \mapsto (\beta \mapsto \int_{\gamma} \beta),$$

where $\gamma$ runs through closed paths on $X$. Set

$$\text{Alb}(X) = H^0(X, \Omega^1_X)^\ast / \text{im}(\iota).$$

(4.16)

Lemma 4.4.1.1. On $\text{Alb}(X)$, there is a natural structure of $h^{1,0}(X)$-dimensional complex torus with $H_1(\text{Alb}(X), \mathbb{Z}) = \text{im}(\iota)$.

Proof. Using the $\mathbb{R}$-linear isomorphism (4.14) and de Rham isomorphism

$$H^1_{\text{dR}}(X, \mathbb{R}) \to H^1(X, \mathbb{R}),$$

the map (4.15) is identified with the natural map $H_1(X, \mathbb{Z}) \to H^1_{\text{dR}}(X, \mathbb{R})^\ast$. The latter extends to an $\mathbb{R}$-linear isomorphism $H_1(X, \mathbb{R}) \to H^1_{\text{dR}}(X, \mathbb{R})^\ast$ by Poincaré duality. Therefore,

$$\ker(\iota) = H_1(X, \mathbb{Z})_{\text{tor}},$$

(4.17)

and $\text{im}(\iota)$ is a full lattice in $H^0(X, \Omega^1_X)^\ast$ isomorphic to $H_1(X, \mathbb{Z})_{\text{free}}$. Thus, the quotient $\text{Alb}(X)$ is a complex torus with the stated properties. $\square$

The complex torus $\text{Alb}(X)$ in Lemma 4.4.1.1 is called the Albanese torus of $X$. For each $x \in X$, choose two paths $\gamma_x$, $\gamma'_x$ connecting $x_0$ to $x$. Then the composition $\gamma$ of $\gamma_x$ followed by the reverse of $\gamma'_x$ is a closed path on $X$ and

$$\int_{\gamma_x} \bullet - \int_{\gamma'_x} \bullet = \int_{\gamma} \bullet = \iota([\gamma])$$

belongs to $\text{im}(\iota)$. Therefore, $[\int_{\gamma_x} \bullet] = [\int_{\gamma'_x} \bullet]$ in $\text{Alb}(X)$. As $[\int_{\gamma_x} \bullet]$ is independent of the choice of $\gamma_x$, we write it as $\int_{x_0}^x \bullet$. For the fixed base point $x_0 \in X$, the associated Albanese map is

$$\alpha_{X, x_0} : X \to \text{Alb}(X), \quad x \mapsto \int_{x_0}^x \bullet.$$ 

(4.18)

The subscripts $X$ and $x_0$ are omitted when they are clear from the context.
Proposition 4.4.1.2.

1. The Albanese map $\alpha_{X,x_0} : X \rightarrow \text{Alb}(X)$ is a morphism of complex manifolds and the formation of Albanese map is functorial for the pair $(X, x_0)$.

2. The induced morphism $\alpha_{x_0,*} : H_1(X, \mathbb{Z}) \rightarrow H_1(\text{Alb}(X), \mathbb{Z})$ is surjective with kernel $H_1(X, \mathbb{Z})\text{tor}$.

3. The morphism $\alpha_{x_0}$ satisfies the following universal property: every morphism of pointed complex manifolds $(X, x_0) \rightarrow (A, 0)$ with $A$ a complex torus factors uniquely through a morphism of complex tori $\text{Alb}(X) \rightarrow A$. In particular, the complex subtorus of $\text{Alb}(X)$ generated by $\alpha_{x_0}(X)$ is $\text{Alb}(X)$.

4. The pullback morphism $\alpha_{x_0,*} : H^1(\text{Alb}(X), \mathbb{Z}) \rightarrow H^1(X, \mathbb{Z})$ is an isomorphism of weight 1 $\mathbb{Z}$-Hodge structures independent of the choice of $x_0$.

5. The pullback $\alpha_{x_0,*} : \text{Pic}^0(\text{Alb}(X)) \rightarrow \text{Pic}^0(X)$ is an isomorphism of complex tori independent of the choice of $x_0$. In particular, the complex tori $\text{Alb}(X)$ and $\text{Pic}^0(X)$ are dual to each other.\(^8\)

Proof.

1. When $X$ is Kähler, it is proved in [Huy05, Prop. 3.3.8]. The general case is similar.

2. By Lemma 4.4.1.1, $H_1(\text{Alb}(X), \mathbb{Z}) = \text{im}(\iota)$. Let $\gamma : [0, 1] \rightarrow X$ be a closed path on $X$ based at $x_0$. It defines a path

$\zeta : [0, 1] \rightarrow H^0(X, \Omega^1_X)^\vee, \quad \zeta(t) = \int_{\gamma(0)}^{\gamma(t)} \bullet$,

where the integral is along a part of $\gamma$. Then

$\zeta \pmod{\text{im}(\iota)} = \alpha_{x_0} \circ \gamma : [0, 1] \rightarrow \text{Alb}(X)$.

Therefore, $\alpha_{x_0,*}[\gamma] = \zeta(1) - \zeta(0) = \int_{\gamma} \bullet = \iota([\gamma])$. Hence a commutative triangle

$\begin{array}{ccc}
H_1(X, \mathbb{Z}) & \xrightarrow{\alpha_{x_0,*}} & H^0(X, \Omega^1_X)^\vee \\
\downarrow{\iota} & & \downarrow{\iota} \\
\text{im}(\iota) & = & H_1(\text{Alb}(X), \mathbb{Z})
\end{array}$

Therefore, $\alpha_{x_0,*}$ is surjective and $\ker(\alpha_{x_0,*}) = \ker(\iota) = H_1(X, \mathbb{Z})\text{tor}$, where the last equality uses (4.17).

3. The universal property follows from Point 1. Let $T$ be the complex subtorus of $\text{Alb}(X)$ generated by $\alpha_{x_0}(X)$. Then the pointed morphism $\alpha_{x_0} : (X, x_0) \rightarrow (T, 0)$ factors through $\alpha_{x_0} : (X, x_0) \rightarrow (\text{Alb}(X), 0)$, so $T = \text{Alb}(X)$.

\(^8\)in the sense of [BL04, p.34]
4. From [BL04, Thm. 1.4.1 b)], the map $\alpha_{x_0}^*: H^1(\text{Alb}(X), \mathbb{Z}) \to H^1(X, \mathbb{Z})$ is a $\mathbb{C}$-linear isomorphism. By [BL04, Sec 1.3, p.13], $H^1(\text{Alb}(X), \mathbb{Z})$ is naturally isomorphic to $\text{Hom}(\text{im}(i), \mathbb{Z})$. By Poincaré duality, the latter is identified with $H^1(X, \mathbb{Z})$, so

$$\alpha_{x_0}^*: H^1(\text{Alb}(X), \mathbb{Z}) \to H^1(X, \mathbb{Z})$$

is an isomorphism of weight 1 $\mathbb{Z}$-Hodge structures. Up to translation, different base points give rise to the same Albanese map. More precisely, for $x \in X$, $T_{\alpha_{x_0}} \circ \alpha_{x_0} = \alpha_x$, where

$$T_a : \text{Alb}(X) \to \text{Alb}(X), \quad u \mapsto u + a$$

is the translation by $a$ on $\text{Alb}(X)$. The independence stated in Point 4 follows.

5. As the isomorphism (4.9) is functorial in $X$, there is a commutative diagram with exact rows

$$\begin{array}{cccccc}
H^1(\text{Alb}(X), \mathbb{Z}) & \longrightarrow & H^1(\text{Alb}(X), O_{\text{Alb}(X)}) & \longrightarrow & \text{Pic}^0(\text{Alb}(X)) & \longrightarrow & 0 \\
\downarrow \alpha_{x_0}^* & \quad & \downarrow \alpha_{x_0}^* & \quad & \downarrow \alpha_{x_0}^* & \quad & \\
H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, O_X) & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & 0.
\end{array}$$

By Point 4, the left two vertical maps are isomorphisms independent of $x_0$. Therefore, the right vertical map is an isomorphism independent of $x_0$. As $\text{Alb}(X)$ is a complex torus, by [BL04, Proposition 2.4.1], $\text{Pic}^0(\text{Alb}(X))$ is the dual torus of $\text{Alb}(X)$. As $\alpha_{x_0}^*: \text{Pic}^0(\text{Alb}(X)) \to \text{Pic}^0(X)$ is an isomorphism, $\text{Pic}^0(X)$ is dual to $\text{Alb}(X)$.

Remark 4.4.1.3. By [Uen06, Cor. 9.5, p.101], for every connected regular manifold $X$ the formation of $\text{Alb}(X)$ and $\alpha_{x_0}$ agrees with the construction in [Bla56, §2]. Then [Bla56, p.163] gives another proof of the universal property stated in Proposition 4.4.1.2 3.

Example 4.3.2.3 1 (continued). If $X$ were a Hopf surface, then $H_1(X, \mathbb{Z}) = \mathbb{Z}$ and $H^0(X, \Omega_X^1) = 0$. Equation (4.16) would define a point and Proposition 4.4.1.2 2 would fail.

4.4.2 Back to Jacobian

Albanese torus helps to understand the Jacobian. Corollary 4.4.2.1 is used to show the jumping loci are analytic subsets.
4.4. ALBANESE TORUS

Corollary 4.4.2.1 (Universal line bundle). There exists a unique (up to isomorphism) line bundle $L$ on $X \times \text{Pic}^0(X)$ such that its pullback module to $\{x_0\} \times \text{Pic}^0(X)$ is trivial and for every point $y \in \text{Pic}^0(X)$, the isomorphism class of the pullback line bundle $L|_{X \times \{y\}}$ in $\text{Pic}(X)$ is $y$.

Proof. Consider the map

$$f = \alpha_{x_0} \times \text{Id}_{\text{Pic}^0(X)} : X \times \text{Pic}^0(X) \to \text{Alb}(X) \times \text{Pic}^0(X).$$

By Proposition 4.4.1.25 and [GH78, Lemma, p.328], there is a holomorphic line bundle $P$ on $\text{Alb}(X) \times \text{Pic}^0(X)$ that is trivial on $\{0\} \times \text{Pic}^0(X)$ such that for every $y \in \text{Pic}^0(X)$, the line bundle $P|_{\text{Alb}(X) \times \{y\}}$ is of class $y$ in $\text{Pic}^0(\text{Alb}(X))$.

Let $L = f^*P$, then $L|_{\{x_0\} \times \text{Pic}^0(X)} = f^*(P|_{\{0\} \times \text{Pic}^0(X)})$ is trivial. For every $y \in \text{Pic}^0(X)$, the line bundle

$$L|_{X \times \{y\}} = f^*(P|_{\text{Alb}(X) \times \{y\}}) = \alpha_{x_0}^*(P|_{\text{Alb}(X) \times \{y\}})$$

is of class $y$ in $\text{Pic}^0(X)$. The existence is proved. The uniqueness follows from [BL04, Cor. A.9].

Let

$$\text{Char}(X) = \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{C}^*)$$

be the group of characters of the first homology of $X$. By [Hat05, Cor. A.8, A.9], the abelian group $H_1(X, \mathbb{Z})$ is finitely generated. From [Mil17a, Ch. 12 b.], $\text{Char}(X)$ has a natural structure of diagonalizable algebraic group over $\mathbb{C}$, with identity component $\text{Char}^a(X)$ isomorphic to $\mathbb{G}_{\mathbb{m}}^b(X)$. Moreover, $\text{Char}^a(X) := \text{Hom}(H_1(X, \mathbb{Z}), S^1)$ is a real Lie subgroup of $\text{Char}(X)$ of dimension $b_1(X)$. There is a canonical group isomorphism by taking character sheaves

$$\text{Char}^a(X) \to \text{Loc}^{a,1}(X), \quad \chi \mapsto \mathcal{L}_\chi. \quad (4.19)$$

Set $T(X) := \text{Hom}(H_1(X, \mathbb{Z})_{\text{free}}, S^1)$. Then $T(X)$ is the identity component of $\text{Char}^a(X)$. From Corollary 4.3.2.2, composing the isomorphism (4.19) and the map (4.12) gives a morphism of real Lie groups

$$T(X) \to \text{Pic}^0(X). \quad (4.20)$$

In Corollary 4.4.2.2, the isomorphism allows one to identify certain characters with topologically trivial line bundles. This identification is used in the proof of Theorem 4.7.1.3. When $X$ is in Fujiki Class $C$ (resp. Kähler), Corollary 4.4.2.2 is also in [Ara90, Lem. 2] (resp. the proof of [Bot16, Cor. 1.4]).

Corollary 4.4.2.2.

1. The morphism (4.20) is an isomorphism of real Lie groups.

2. The map (4.12) is a group isomorphism and $\text{NS}(X)_{\text{tor}} = H^2(X, \mathbb{Z})_{\text{tor}}$. In particular, every element of $\text{Pic}^+(X)$ is a flat unitary line bundle.
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Proof.

1. Lemma 4.4.1.1 gives an identification $H_1(X, \mathbb{Z})_{\text{free}} = \text{im}(\iota)$. By [BL04, Prop. 2.2.2], the natural group morphism

$$\text{Hom}(\text{im}(\iota), S^1) \to \text{Pic}^0(\text{Alb}(X))$$

(4.21)

defined via factors of automorphy ([BL04, p.30]) is an isomorphism. The map (4.20) is the composition of (4.21) with the isomorphism $\alpha^*_0 : \text{Pic}^0(\text{Alb}(X)) \to \text{Pic}^0(X)$ in Proposition 4.4.1.2 5.

To sum it up:

$$\text{Char}^u(\text{Alb}(X)) = \text{Hom}(\text{im}(\iota), S^1) \sim T(X) \longrightarrow \text{Char}^u(X) = \text{Loc}^{u,1}(X)$$

(4.21)

$$\text{Pic}^0(\text{Alb}(X)) \sim \text{Pic}^0(X) \longrightarrow \text{Pic}^1(X)$$

(4.20)

(4.12)

2. The commutative diagram of abelian sheaves on $X$

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{R} & \longrightarrow & S^1 & \longrightarrow & 0 \\
& & \downarrow{\text{Id}} & \searrow & \downarrow{\text{exp}(2\pi i)} & \searrow & \downarrow{\text{Id}} & \searrow & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & O_X^{\text{reg}} & \longrightarrow & O_X & \longrightarrow & 0
\end{array}
$$

has exact rows. Moreover, the $\mathbb{Z}$-module $\mathbb{R}$ is injective. Therefore, there is a commutative diagram with exact rows

$$
\begin{array}{cccccc}
0 & \longrightarrow & \text{Hom}(H_1(X, \mathbb{Z})_{\text{tor}}, S^1) & \cong & \text{Ext}^1_\mathbb{Z}(H_1(X, \mathbb{Z})_{\text{tor}}, \mathbb{Z}) & \longrightarrow & 0 \\
& & \uparrow{r} & & \uparrow{\cong} & & \\
0 & \longrightarrow & \text{Hom}(H_1(X, \mathbb{Z}), \mathbb{Z}) & \longrightarrow & \text{Char}^u(X) & \longrightarrow & \text{Ext}^1_\mathbb{Z}(H_1(X, \mathbb{Z}), \mathbb{Z}) & \longrightarrow & 0 \\
& & \uparrow{\cong} & & \uparrow{\cong} & & \uparrow{\cong} & & \\
0 & \longrightarrow & H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, \mathbb{R}) & \longrightarrow & H^1(X, S^1) & \longrightarrow & H^2(X, \mathbb{Z}) \\
& & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & \\
0 & \longrightarrow & H^1(X, \mathbb{Z}) & \longrightarrow & H^1(X, O_X) & \longrightarrow & H^1(X, O_X^{\text{reg}}) & \delta & \longrightarrow & H^2(X, \mathbb{Z})
\end{array}
$$

where $r$ is the restriction, the vertical morphisms in the middle are from [Hat05, Thm. 3.2] and $\text{im}(\xi) = H^2(X, \mathbb{Z})_{\text{tor}}$ by [Hat05, p.196]. Hence an isomorphism $\psi : \text{Hom}(H_1(X, \mathbb{Z})_{\text{tor}}, S^1) \to H^2(X, \mathbb{Z})_{\text{tor}}$ fitting into a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & T(X) & \longrightarrow & \text{Loc}^{u,1}(X) & \overset{r}{\longrightarrow} & \text{Hom}(H_1(X, \mathbb{Z})_{\text{tor}}, S^1) & \longrightarrow & 0 \\
& & \downarrow{4.20} & & \downarrow{4.12} & & \downarrow{\psi} & & \\
0 & \longrightarrow & \text{Pic}^0(X) & \longrightarrow & \text{Pic}^1(X) & \overset{\delta}{\longrightarrow} & H^2(X, \mathbb{Z})_{\text{tor}} & \longrightarrow & 0,
\end{array}
$$
where the first row is exact. Thus, $\delta$ is surjective and the second row is also exact. By the five lemma, the middle vertical map (4.12) is an isomorphism.

\[ \square \]

4.5 Defect of semismallness

In this section, we review the defect of semismallness of a morphism, an invariant introduced by de Cataldo and Migliorini that plays a crucial role in the decomposition theorem and Lefschetz’s theorem. It appears in Fact 4.1.1.6 and Theorem 4.7.1.3. Its main property that we need is Proposition 4.5.3.2.

4.5.1 Stratifications and constructible sheaves

We refer to [BF84, Sec. 2.1] for the definitions of constructible stratifications and Whitney stratifications of a complex analytic space.

Theorem 4.5.1.1 is about the semicontinuity of fiber dimension. Although it is well-known, a short proof is included due to the lack of reference. Its analogue in algebraic geometry is a celebrated theorem of Chevalley [Gro66, Cor. 13.1.5].

**Theorem 4.5.1.1** (Analytic Chevalley theorem). Let $f : X \to Y$ be a proper morphism of reduced complex analytic spaces. For every integer $n \geq 0$, let $Y_n = \{ y \in Y : \dim f^{-1}(y) = n \}$ and $Y_{\geq n} = \bigcup_{m \geq n} Y_m$. Then $Y_{\geq n}$ is an analytic subset of $Y$. In particular, $\{ Y_n \}_{n \in \mathbb{N}}$ is a constructible stratification of $Y$.

**Proof.** Let $F_n := \{ x \in X : \dim_x f^{-1}(f(x)) \geq n \}$. By [Fis06, Thm. 3.6, p.137], $F_n$ is an analytic subset of $X$. By the definition of global dimension [GR84, p.94], one has $Y_{\geq n} = f(F_n)$. By Remmert theorem (see, e.g., [Whi72, Thm. 4A, p.150]), the subset $Y_{\geq n}$ is analytic in $Y$. \[ \square \]

**Definition 4.5.1.2.** ([BF84, p.125]) Let $f : X \to Y$ be a morphism of complex analytic spaces. If two Whitney stratifications $\mathcal{X} : X = \bigcup_{\alpha} X_\alpha$ and $\mathcal{Y} : Y = \bigcup_{\lambda} Y_\lambda$ satisfy that:

1. For each $\alpha$, there is $\lambda$ with $f(X_\alpha) \subset Y_\lambda$;
2. For each pair $(\alpha, \lambda)$ with $f(X_\alpha) \subset Y_\lambda$, the restricted morphism $f : X_\alpha \to Y_\lambda$ is smooth.

Then such a pair $(\mathcal{X}, \mathcal{Y})$ is called a Whitney stratification of $f$.

**Fact 4.5.1.3** ([Hir77, Thm. 1], [BF84, Lem. 2.4], [GM88, Thm. p.43]). Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Suppose that $\mathcal{X}$ (resp. $\mathcal{Y}$) is a constructible stratification of $X$ (resp. $Y$), then there exists a Whitney stratification $(\mathcal{X}', \mathcal{Y}')$ of $f$ such that $\mathcal{X}'$ (resp. $\mathcal{Y}'$) refines $\mathcal{X}$ (resp. $\mathcal{Y}$).

Corollary 4.5.1.4 is useful but implicit in the literature.
Corollary 4.5.1.4. Let $X$ be a complex analytic space. For finitely many constructible stratifications of $X$, there exists a Whitney stratification of $X$ refining all of them.

Proof. It suffices to consider the case of two constructible stratifications $\mathcal{X}_1$ and $\mathcal{X}_2$ of $X$. By Fact 4.5.1.3, there is a Whitney stratification $(\mathcal{X}, \mathcal{X}')$ of $\text{Id}_X$ such that $\mathcal{X}$ (resp. $\mathcal{X}'$) refines $\mathcal{X}_1$ (resp. $\mathcal{X}_2$). Moreover, $\mathcal{X}$ refines $\mathcal{X}'$ by Definition 4.5.1.2. Hence a Whitney stratification $\mathcal{X}$ refining both $\mathcal{X}_1$ and $\mathcal{X}_2$.  

For a complex analytic space $X$, using analytic constructible stratifications, one can define constructible sheaves. Let $\mathcal{D}^b_c(X)$ be the triangulated category of complexes of sheaves of $\mathbb{C}$-vector spaces whose cohomology is bounded and constructible (see, e.g., [Dim04, p.82]).

Fact 4.5.1.5 ([KS13, Prop. 8.5.7 (b)], [Dim04, Thm. 4.1.5 (b)]). Let $f : X \to Y$ be a morphism of complex analytic spaces and $K \in \mathcal{D}^b_c(X)$. If $f$ is proper on $\text{Supp}(K)$, then $Rf_* K \in \mathcal{D}^b_c(Y)$.

Corollary 4.5.1.6. Let $f : X \to Y$ be a proper morphism of complex analytic spaces and $K \in \mathcal{D}^b_c(X)$. Then there exists a Whitney stratification $(\mathcal{X}, \mathcal{Y})$ of $f$ such that for every integer $i$ and every stratum $S$ of $\mathcal{Y}$, the restriction $\mathcal{H}^i(Rf_* K)|_S$ is a local system on $S$.

Proof. By Fact 4.5.1.5, $Rf_* K \in \mathcal{D}^b_c(Y)$. In particular, there are only finitely many $j \in \mathbb{Z}$ with $\mathcal{H}^j(Rf_* K) \neq 0$. For each such $j$, there is an admissible partition (in the sense of [Dim04, p.81]) $\mathcal{P}_j$ on $Y$ such that the restriction of $\mathcal{H}^j(Rf_* K)$ to each stratum of $\mathcal{P}_j$ is a local system. By Corollary 4.5.1.4, there exists a Whitney stratification $\mathcal{Y}^0$ of $Y$ refining the finitely many $\mathcal{P}_j$. By Fact 4.5.1.3, there is a Whitney stratification $(\mathcal{X}, \mathcal{Y})$ of $f$ satisfying the properties.  

4.5.2 Equivalent definitions

The defect of semismallness measures how far a morphism of complex manifolds is from being semismall (see, e.g., [KW01, Def. 7.3, p.156]). However, in the literature there exist multiple seemingly different definitions. We review some of them and show that they are equivalent.

Definition 4.5.2.1. Let $f : X \to Y$ be a proper morphism of complex manifolds with $\dim X = n$.

- ([EV89, Definition 1.1]) Define
  \[ r_1(f) = \max_Z (\dim Z - \dim f(Z) - \text{codim}_X(Z)), \quad (4.22) \]
  where $Z$ runs through all irreducible analytic subsets of $X$.

- ([Max19, Definition 9.3.7]) For a Whitney stratification $(X = \sqcup S_\alpha, Y = \sqcup T_\lambda)$ of $f$, we choose a point $y_\lambda \in T_\lambda$ in each stratum, and define
  \[ r_2(f) = \max_\lambda \{ 2 \dim f^{-1}(y_\lambda) + \dim T_\lambda - n \}. \quad (4.23) \]

(By convention, the empty space has dimension $-\infty$.)
4.5. DEFECT OF SEMISMALLNESS

- ([dCM05, Definition 4.7.2]) For each integer \( i \geq 0 \), let \( Y_i = \{ y \in Y : \dim f^{-1}(y) = i \} \). Define
  \[
  r_3(f) = \max_{i \geq 0} (2i + \dim Y_i - n).
  \]

- ([PS13, Definition 2.8]) For each integer \( i \geq 0 \), let \( Y_{\geq i} = \{ y \in Y : \dim f^{-1}(y) \geq i \} \) for each \( i \geq 0 \). Define
  \[
  r_4(f) = \max_{i \geq 0} (2i + \dim Y_{\geq i} - n).
  \]

- ([dCM09, Sec. 3.3.2, part 2]) Define
  \[
  r_5(f) = \dim X \times_Y X - n. \tag{4.24}
  \]

- ([Wil16, Sec 3.2]) Define
  \[
  r_6(f) = \max \{ i \in \mathbb{Z} : p^H_i(Rf_*C_X[n]) \neq 0 \}.
  \]

Proposition 4.5.2.2. The first five numbers in Definition 4.5.2.1 are all equal.

This common integer is called the defect of semismallness of \( f \) and denoted by \( r(f) \). We shall show \( r(f) = r_6(f) \) in Proposition 4.5.3.22.

Proof.

- \( r_3(f) = r_4(f) \): As each \( Y_i \) is a subset of \( Y_{\geq i} \), one has \( r_3(f) \leq r_4(f) \). There are only finitely many integers \( i \geq 0 \) with \( Y_{\geq i} \) nonempty, so the maximum defining \( r_4(f) \) is attained at some \( i_0(\geq 0) \). Then
  \[
  2(i_0 + 1) + \dim Y_{\geq i_0 + 1} \leq 2i_0 + \dim Y_{\geq i_0}.
  \]
  Since \( Y_{\geq i_0} = Y_{\geq i_0 + 1} \cup Y_{i_0} \), one has \( \dim Y_{\geq i_0} = \dim Y_{i_0} \). Then
  \[
  r_4(f) = 2i_0 + \dim Y_{\geq i_0} - n \leq r_3(f).
  \]
  Therefore, \( r_3(f) = r_4(f) \).

- \( r_2(f) = r_6(f) \): By Thom’s first isotopy lemma (see, e.g., [Mat12, Prop. 11.1]), for every \( \lambda \), the restriction \( f|_{f^{-1}(T_\lambda)} : f^{-1}(T_\lambda) \to T_\lambda \) is a topologically locally trivial fibration. Therefore, \( \dim f^{-1}(y_\lambda) \) is independent of \( y_\lambda \in T_\lambda \) and
  \[
  \dim f^{-1}(T_\lambda) \times_{T_\lambda} f^{-1}(T_\lambda) = \dim T_\lambda + 2\dim f^{-1}(y_\lambda). \tag{4.25}
  \]
  As \( \{ f^{-1}(T_\lambda) \times_{T_\lambda} f^{-1}(T_\lambda) \}_\lambda \) is a locally finite partition of \( X \times_Y X \) into locally closed subsets (in the analytic Zariski topology), one has
  \[
  \dim X \times_Y X = \max_\lambda [\dim f^{-1}(T_\lambda) \times_{T_\lambda} f^{-1}(T_\lambda)]. \tag{4.26}
  \]
  Plugging (4.25) into (4.26) we get \( r_5(f) = r_2(f) \). In particular, \( r_2(f) \) is independent of the choice of the stratifications.
CHAPTER 4. GENERIC VANISHING THEOREM FOR FUJIKI CLASS \( \mathcal{C} \)

- \( r_1(f) \leq r_2(f) \): For every irreducible analytic subset \( Z \subset X \), \( f(Z) \) is an irreducible analytic subset of \( Y \). Then \( \{Y \setminus f(Z), f(Z)\} \) is a constructible stratification of \( Y \). Fact 4.5.1.3 yields a Whitney stratification \( (X = \cup S_{\alpha}, Y = \cup T_{\lambda}) \) of \( f \) with \( Y = \cup T_{\lambda} \) refining \( \{Y \setminus f(Z), f(Z)\} \). There exists \( \lambda_0 \) such that \( T_{\lambda_0} \) is an open subset of \( f(Z) \), hence \( \dim T_{\lambda_0} \leq \dim f(Z) \). Then \( f^{-1}(T_{\lambda_0}) \cap Z \) is a nonempty open subset of \( Z \). Therefore,

\[
\dim Z = \dim(f^{-1}(T_{\lambda_0}) \cap Z) \leq \dim f^{-1}(T_{\lambda_0}).
\]

Then

\[
2 \dim Z - \dim f(Z) \leq 2 \dim f^{-1}(T_{\lambda_0}) - \dim T_{\lambda_0} = 2 \dim f^{-1}(y_{\lambda_0}) + \dim T_{\lambda_0}.
\]

This shows \( r_1(f) \leq r_2(f) \). In particular, the maximum in (4.22) is indeed attained.

- \( r_2(f) \leq r_1(f) \): Fix a Whitney stratification \( Y = \cup T_{\lambda} \) defining \( r_2(f) \). For every \( \lambda \) with \( f^{-1}(y_{\lambda}) \) nonempty, \( T_{\lambda} \) is an analytic subset of \( Y \) of dimension \( \dim T_{\lambda} \). Then \( f^{-1}(T_{\lambda}) \) is a nonempty analytic subset of \( X \). Let \( Z_0 \) be an irreducible component of \( f^{-1}(T_{\lambda}) \) with \( \dim Z_0 = \dim f^{-1}(T_{\lambda}) \). Then \( f(Z_0) \subset T_{\lambda} \) and \( \dim f(Z_0) \leq \dim T_{\lambda} \). Therefore,

\[
2 \dim f^{-1}(y_{\lambda}) + \dim T_{\lambda} = 2 \dim f^{-1}(T_{\lambda}) - \dim T_{\lambda} \leq 2 \dim Z_0 - \dim f(Z_0).
\]

This shows \( r_2(f) \leq r_1(f) \).

- \( r_2(f) \leq r_3(f) \): By Theorem 4.5.1.1, \( \{Y_i\} \) is a constructible stratification of \( Y \). By Fact 4.5.1.3, there is a Whitney stratification \( (X = \cup S_{\alpha}, Y = \cup T_{\lambda}) \) of \( f \) such that the stratification \( Y = \cup T_{\lambda} \) refines \( Y = \cup Y_i \). For every \( \lambda \), there is \( \lambda_0 \) with \( T_{\lambda} \subset Y_{i_0} \). In particular, for every \( y_{\lambda} \in T_{\lambda} \), one has \( \dim f^{-1}(y_{\lambda}) = i_0 \), so

\[
2 \dim f^{-1}(y_{\lambda}) + \dim T_{\lambda} \leq 2i_0 + \dim Y_{i_0}.
\]

This shows \( r_2(f) \leq r_3(f) \).

- \( r_3(f) \leq r_2(f) \): For every integer \( i \geq 0 \) with \( Y_i \) nonempty, \( Y_i = \cup T_{\lambda} \) is a constructible stratification, so there is an index \( \lambda_0 \) with \( \dim(Y_i \cap T_{\lambda_0}) = \dim Y_i \). Then \( \dim Y_i \leq \dim T_{\lambda_0} \). One may take \( y_{\lambda_0} \in Y_i \cap T_{\lambda_0} \). Then

\[
2i + \dim Y_i \leq 2 \dim f^{-1}(y_{\lambda_0}) + \dim T_{\lambda_0},
\]

which shows \( r_3(f) \leq r_2(f) \).

From the diagonal inclusion \( X \to X \times_Y X \), one gets \( \dim X \leq \dim X \times_Y X \), so \( r(f) = r_3(f) \geq 0 \). If \( r(f) = 0 \), then \( f \) is said to be semismall.

**Example 4.5.2.3.**
4.5. DEFECT OF SEMISMALLNESS

1. If \( f : X \to Y \) is a proper morphism of complex manifolds that is flat of relative dimension \( r \), then \( r(f) = r \).

2. Let \( X \) be projective manifold such that \(-K_X\) is nef and \( \alpha : X \to \text{Alb}(X) \) be the Albanese map associated with some base point. Then \( r(\alpha) = \dim X - \dim \alpha(X) \) by [LTZZ10, Theorem].

4.5.3 Direct image of local systems

Defect of semismallness is an important invariant appearing in the decomposition of direct image of perverse sheaves. Proposition 4.5.3.2 is an elementary instance. We begin with a well-known estimation of cohomological dimension of a complex analytic space, used in the proof of Proposition 4.5.3.2. An analogue for topological manifolds is [KS13, Prop. 3.2.2 (iv)].

**Lemma 4.5.3.1.** Let \( X \) be a paracompact complex analytic space of complex dimension \( n \). Then \( H^q(X, F) = 0 \) for every abelian sheaf \( F \) on \( X \) and every integer \( q > 2n \).

**Proof.** By [GR84, Prop., p.94], there is an open covering \( \{U_\alpha\}_\alpha \) of \( X \) such that for each \( \alpha \), there is a finite morphism \( f_\alpha : U_\alpha \to B_\alpha \) of complex analytic spaces to an open ball \( B_\alpha \subset \mathbb{C}^n \). As \( X \) is Hausdorff paracompact, by [Mun00, Lemma 41.6], there exists a locally finite open covering \( \{V_\alpha\} \) on \( X \) such that \( V_\alpha \subset U_\alpha \) for each \( \alpha \).

From [Mun00, p.314], for every \( \alpha \), the topological dimension ([Mun00, Def., p.305]) \( \text{covdim}(B_\alpha) = 2n \). By [KK11, Prop. 51 A.2], the topological space \( X \) is metrizable. From [Mun00, Thm. 32.2], each \( U_\alpha \) is normal. Therefore, by [Eng95, Thm. 3.3.10, p.200], \( \text{covdim}(U_\alpha) \leq 2n \). By [Eng95, Theorem 3.1.3, p.169], \( \text{covdim}(V_\alpha) \leq 2n \). Similarly, \( X \) is normal, so \( \text{covdim}(X) \leq 2n \) by [Eng95, Thm. 3.1.10, p.172]. By Alexandroff theorem (see, e.g., [Bre12, p.122]), the cohomological dimension ([Eng95, p.75]) \( \dim_{\mathbb{Z}} X \leq 2n \).

The category \( D^b_{\mathbb{C}}(X) \) has a natural perverse t-structure (\( p \) being the middle perversity)

\[
(p \mathcal{D}_{\leq 0}(X), p \mathcal{D}_{\geq 0}(X)),
\]

whose heart \( \text{Perv}(X) \) is a \( \mathbb{C} \)-linear abelian category ([BBDG82], see also [HT07, Thm. 8.1.27]). An object of \( \text{Perv}(X) \) is called a perverse sheaf on \( X \). For every integer \( i \), the functor taking the \( i \)-th perverse cohomology sheaf is denoted by \( p^i : D^b_{\mathbb{C}}(X) \to \text{Perv}(X) \). For any two integers \( a \leq b \), set

\[
\begin{align*}
p^{[a,b]}(X) &:= \{ \mathcal{K} \in D^b_{\mathbb{C}}(X) : p^i(\mathcal{K}) = 0, \forall i \notin [a,b] \}; \\
p^{[a,b]}(X) &:= \{ \mathcal{K} \in D^b_{\mathbb{C}}(X) : H^i(\mathcal{K}) = 0, \forall i \notin [a,b] \}.
\end{align*}
\]

Verdier duality \( D_X : D^b_{\mathbb{C}}(X) \to D^b_{\mathbb{C}}(X) \) is a contravariant auto-equivalence that interchanges \( p \mathcal{D}_{\leq 0}(X) \) and \( p \mathcal{D}_{\geq 0}(X) \) (see, e.g., [HT07, p.192]).
**Proposition 4.5.3.2.** Let $f : X \to Y$ be a proper morphism of complex manifolds, where $X$ is of pure dimension $n$. Let $\mathcal{L}$ a local system on $X$. Then:

1. $R^i_f(L[n]) \in pD^{[−r(f),r(f)]}(Y)$. In particular, $R^i_f[n] \in \text{Perv}(Y)$ when $f$ is moreover semismall.

2. When $\mathcal{L} = \mathbb{C}_X$, for $j = ±r(f)$, one has $\mathcal{H}^j(R^i_f\mathbb{C}_X[n]) \neq 0$. In particular, $r(f) = r_0(f)$.

**Proof.** From Corollary 4.5.1.6, there exists a Whitney stratifications $(X = \bigsqcup \alpha X_\alpha, Y = \bigsqcup \beta Y_\beta)$ of $f$ such that for every $\lambda$, every integer $j$, the restriction $\mathcal{H}^j(R^i_f\mathcal{L}[n]|_{Y_\lambda})$ is a local system. For each $\lambda$, choose a point $y_\lambda \in Y_\lambda$.

1. First, we show that $R^i_f\mathcal{L}[n] \in pD^{≤r(f)}(Y)$. Fix an integer $i$. If $\dim Y_\lambda > r(f) - i$, then by (4.23), one has $i + n > 2 \dim f^{-1}(y_\lambda)$. Since the fiber $f^{-1}(y_\lambda)$ is a compact complex analytic space, by Lemma 4.3.3.1,

$$H^{i+n}(f^{-1}(y_\lambda), \mathcal{L}|_{f^{-1}(y_\lambda)}) = 0.$$  

By proper base change theorem (see, e.g., [Mil13, Thm. 17.2]),

$$\mathcal{H}^i(R^i_f\mathcal{L}[n]|_{Y_\lambda}) = H^{i+n}(f^{-1}(y_\lambda), \mathcal{L}|_{f^{-1}(y_\lambda)}).$$

So $\mathcal{H}^i(R^i_f\mathcal{L}[n]) = 0$ on every stratum $Y_\lambda$ with $\dim Y_\lambda > r(f) - i$. Therefore, $\dim \text{Supp} \mathcal{H}^i(R^i_f\mathcal{L}[n]) \leq r(f) - i$ and hence $R^i_f\mathcal{L}[n] \in pD^{≤r(f)}(Y)$.

It remains to show $R^i_f\mathcal{L}[n] \in pD^{≥r(f)}(Y)$. By what we have proved, $R^i_f\mathcal{L}[n] \in pD^{≥r(f)}(Y)$. Since $D_X(\mathcal{L}[n]) = \mathcal{L}^\vee[n]$, one has

$$R^i_f\mathcal{L}^\vee[n] = R^i_fD_X(\mathcal{L}[n]) = D_Y(R^i_f\mathcal{L}[n]).$$

The last equality uses Verdier’s duality (see, e.g., [Max19, Prop. 5.3.9]). This shows $R^i_f\mathcal{L}[n] \in pD^{≥r(f)}(Y)$.

2. By (4.23), there exists $\lambda_0$ with $r(f) = 2 \dim f^{-1}(y_{\lambda_0}) + \dim Y_{\lambda_0} - n$. In particular, $f^{-1}(y_{\lambda_0})$ is nonempty. Let $i_0 = r(f) - \dim Y_{\lambda_0}$, then $i_0 + n = 2 \dim f^{-1}(y_{\lambda_0})$. By proper base change theorem again,

$$\mathcal{H}^{i_0}(R^i_f\mathbb{C}[n]|_{Y_\lambda}) = H^{i_0+n}(f^{-1}(y_\lambda), \mathbb{C}) \neq 0.$$  

Therefore, $Y_{\lambda_0} \subset \text{Supp} \mathcal{H}^{i_0}(R^i_f\mathbb{C}[n])$ and hence

$$\dim \text{Supp} \mathcal{H}^{i_0}(R^i_f\mathbb{C}[n]) \geq \dim Y_{\lambda_0} = r(f) - i_0.$$  

Then $R^i_f\mathbb{C}[n] \notin pD^{≥r(f)−1}(Y)$. Together with Point 1, this shows $\mathcal{H}^{r(f)}(R^i_f\mathbb{C}_X[n]) \neq 0$.

The other part follows from Verdier’s duality. \qed
4.6 Generic vanishing for constructible sheaves

In Section 4.6, we review the generic vanishing theorem for (complexes of) constructible sheaves on a complex torus. The case of abelian varieties is treated in [KW15b] and the general case in [BSS18]. We shall reduce the generic vanishing problem on a manifold in Fujiki class \( C \) to results on its Albanese torus.

4.6.1 Thin subsets

To state Krämer-Weissauer's theorem, we recall the terminology “thin subset” introduced in [KW15b, p.532 and p.536].

Fix a complex torus \( A \). Then \( \text{Char}(A) \) has a natural structure of algebraic torus over \( \mathbb{C} \) of dimension \( 2 \dim A \) and \( T(A) = \text{Char}^u(A) \) is a real Lie subgroup of \( \text{Char}(A) \). For each complex subtorus \( B \subset A \), let \( K(B) \) be the kernel of the morphism of algebraic tori \( \text{Char}(A) \to \text{Char}(B) \) induced by functoriality. The induced morphism \( \pi_1(B,0) \to \pi_1(A,0) \) is injective with torsion-free cokernel of rank \( 2 \dim A - 2 \dim B \), so \( K(B) \) is an algebraic subtorus of \( \text{Char}(A) \).

Definition 4.6.1.1. A thin subset of \( \text{Char}(A) \) is a finite union of translates \( \chi_i \cdot K(A_i) \) for certain characters \( \chi_i \in \text{Char}(A) \) and certain nonzero complex subtori \( A_i \subset A \). If every \( \chi_i \) can be chosen to be a torsion point of \( \text{Char}(A) \), then such a thin subset is called arithmetic.

A thin subset of \( \text{Char}(A) \) is strict and Zariski closed. If the complex torus \( A \) is nonzero and simple, then a subset of \( \text{Char}(A) \) is thin if and only if it is finite.

For each complex subtorus \( B \subset A \), we have a functorial commutative diagram

\[
\begin{array}{cccc}
\text{Char}^u(A) & \xrightarrow{(4.19)} & \text{Loc}^u(A) & \xrightarrow{(4.12)} & \text{Pic}^\tau(A) & \xleftarrow{(4.10)} & \text{Pic}^0(A) \\
\downarrow & & \downarrow & & \downarrow & & \\
\text{Char}^u(B) & \rightarrow & \text{Loc}^u(B) & \rightarrow & \text{Pic}^\tau(B) & \leftarrow & \text{Pic}^0(B)
\end{array}
\]

(4.28)

where all the horizontal maps are isomorphisms by Corollary 4.4.2.2.

A subset of \( \text{Pic}^0(A) \) is called (arithmetic and) thin, if it is the intersection of \( \text{Char}^u(A) \) with a (arithmetic and) thin subset of \( \text{Char}(A) \) when \( \text{Pic}^0(A) \) is identified with \( \text{Char}^u(A) \) via the diagram (4.28).

Lemma 4.6.1.2. Every thin subset of \( \text{Pic}^0(A) \) is a finite union of translates of strict complex subtori.

Proof. Let \( B \) be a subtorus of \( A \). As the induced morphism \( \pi_1(B,0) \to \pi_1(A,0) \) is injective with torsion-free cokernel of rank \( 2(\dim A - \dim B) \), the restriction morphism \( \phi : \text{Char}^u(A) \to \text{Char}^u(B) \) in (4.28) is surjective, and its kernel \( K(B) \cap \text{Char}^u(A) \) is the group of unitary characters of \( \pi_1(A,0)/\pi_1(B,0) \). Therefore, the kernel of the morphism \( \psi : \text{Pic}^0(A) \to \text{Pic}^0(B) \) in (4.28) is a complex subtorus of dimension \( \dim A - \dim B \). \( \square \)
For a connected regular manifold $X$, let $\alpha : X \to \text{Alb}(X)$ be its Albanese morphism corresponding to some base point. Then $\alpha$ induces a morphism $\alpha^* : \text{Char}((\text{Alb}(X)) \to \text{Char}(X)$ of algebraic groups. By Proposition 4.4.1.2, this map identifies $\text{Char}((\text{Alb}(X))$ with the identity component $\text{Char}^0(X)$ of $\text{Char}(X)$. Thus we can define thin subsets of $\text{Char}^0(X)$. By Proposition 4.4.1.2, $\text{Pic}^0(X)$ is naturally identified with $\text{Pic}^0((\text{Alb}(X))$, thus we can define (arithmetic and) thin subsets of $\text{Pic}^0(X)$.

4.6.2 Generic vanishing result on regular manifolds

Roughly speaking, Krämer-Weissauer’s theorem controls the failure of vanishing for perverse sheaves on complex tori, measured by the following loci.

Let $X$ be a compact complex manifold of dimension $d$. For any integers $k \geq 0$, $i$ and for every $K \in \mathcal{D}^b_c(X)$, consider the cohomology support locus

$$\Sigma^i(X, K) := \{ \chi \in \text{Char}(X) : H^i(X, \mathcal{L}_\chi \otimes K) \neq 0 \}.$$  \hfill (4.57)

Let $\Sigma^{\neq 0}(X, K) := \cup_{i \neq j} \Sigma^i(X, K)$. Similarly, let $\Sigma^{> j}(X, K)$ for every integer $j$. By Verdier’s duality, $H^{2d-i}(X, K^\vee \otimes \mathcal{L}_{\chi^{-1}})$ is the $\mathbb{C}$-linear dual of $H^i(X, \mathcal{K} \otimes \mathcal{L}_{\chi}^\vee)$. Therefore,

$$\Sigma^{2d-i}(X, K^\vee) = \{ \chi^{-1} : \chi \in \Sigma^j(X, K) \}. \hfill (4.29)$$

**Fact 4.6.2.1.** Let $X$ be a compact Kähler manifold, and let $K \in \mathcal{D}^b_c(X)$. Then:

1. ([Bot16, p.547]) For every integer $i$, the subset $\Sigma^i(X, K)$ of $\text{Char}(X)$ is Zariski closed.
2. ([BSS18, Thm. 1.1]) If $X$ is a complex torus, and if $K \in \text{Perv}(X)$, then $\Sigma^{\neq 0}(X, K)$ is a strict subset of $\text{Char}(X)$.
3. ([KW15b, Thm. 1.1 and Lem. 11.2 (c)]) If further $X$ is a complex abelian variety, then $\Sigma^{\neq 0}(X, K)$ is contained in a thin (and arithmetic when $K$ is semisimple of geometric origin) subset of $\text{Char}(X)$.

**Corollary 4.6.2.2.** Let $X$ be a compact Kähler manifold, and let $K \in \mathcal{D}^b_c(X)$. Then:

1. There are only finitely many integers $i$ such that $\Sigma^i(X, K) \neq \emptyset$. In particular, $\Sigma^{\neq 0}(X, K)$ and for every integer $j$, $\Sigma^{> j}(X, K)$ are Zariski closed in $\text{Char}(X)$.
2. If $X$ is a complex torus, and if $K \in \mathcal{P}D^{\leq m}(X)$ for some integer $m$, then $\Sigma^{> m}(X, K) \neq \text{Char}(X)$.
3. If $X$ is a complex abelian variety, and $K \in \mathcal{P}D^{\leq m}(X)$ for some integer $m$, then $\Sigma^{> m}(X, K)$ is contained in a thin (and arithmetic when $K$ is semisimple of geometric origin) subset of $\text{Char}(X)$.

**Proof.** The proof is sketched in [KW15b, p.533].
1. There exist two integers $c < d$ such that $\mathcal{K} \in D^{[c,d]}(X)$. Applying [KS13, Proposition 10.2.12] to the proper morphism $X \to p$, where $p$ is a point, one gets two integers $a < b$ such that $Rf_!(D^{[a,b]}(X)) \subset D^{[a,b]}(p)$. For every character sheaf $\mathcal{L}$ on $X$, the functor $\ast \otimes L : D^b_b(X) \to D^b_b(X)$ is t-exact with respect to the standard t-structure. Consequently, $\mathcal{K} \otimes L \in D^{[c,d]}(X)$ and hence $Rf_!(\mathcal{K} \otimes L \mathcal{L}) \in D^{[a,b]}(p)$. For all integers $i \notin [a,b]$, $\Sigma^i(X, \mathcal{K}) = \emptyset$. This shows the first part of the assertion. The second part of the assertion follows from Fact 4.6.2.1 1.

2. By shifting degree, one may assume $m = 0$. For every character sheaf $\mathcal{L}$ on $X$, the functor $\ast \otimes L : D^b_b(X) \to D^b_b(X)$ is t-exact with respect to the perverse t-structure ([KW15b, Prop. 4.1]). Hence for every integer $j$, $p\mathcal{H}^j(\mathcal{K} \otimes L \mathcal{L}) = p\mathcal{H}^j(\mathcal{K}) \otimes L \mathcal{L}$. Consider the subset

$$W = \bigcup_{j \in \mathbb{Z}^\neq 0}(X, p\mathcal{H}^j(\mathcal{K}))$$

(4.30)

of $\text{Char}(X)$. It is in fact a finite union, because by [Dim04, Remark 5.1.19], $p\mathcal{H}^j(\mathcal{K}) \neq 0$ for only finitely many integers $j$. By Fact 4.6.2.1 2, $W \neq \text{Char}(X)$.

For every $\chi \in \text{Char}(X) \setminus W$, consider the Grothendieck spectral sequence from [dCM09, p.543]

$$E_2^{i,j} = H^i(X, p\mathcal{H}^j(\mathcal{K}) \otimes L \mathcal{L}_\chi) \Rightarrow H^{i+j}(X, \mathcal{K} \otimes L \mathcal{L}_\chi).$$

(4.31)

For any integers $i \neq 0$ and $j$, one has $H^i(X, p\mathcal{H}^j(\mathcal{K}) \otimes L \mathcal{L}_\chi) = 0$, so the spectral sequence (4.31) degenerates\(^\text{10}\) at page $E_2$ and hence

$$H^j(X, \mathcal{K} \otimes L \mathcal{L}_\chi) = H^0(X, p\mathcal{H}^j(\mathcal{K}) \otimes L \mathcal{L}_\chi) = H^0(X, p\mathcal{H}^j(\mathcal{K}) \otimes L \mathcal{L}_\chi)$$

for every integer $j$. Now that $\mathcal{K} \in pD^{\leq 0}(X)$, for every $i > 0$ one has $p\mathcal{H}^i(\mathcal{K}) = 0$ and hence $H^i(X, \mathcal{K} \otimes L \mathcal{L}_\chi) = 0$. This shows $\chi \notin \Sigma^\geq 0(X, \mathcal{K})$. One concludes that $\Sigma^\geq 0(X, \mathcal{K}) \subset W$.

3. As $p\mathcal{H}^j(\mathcal{K}) \neq 0$ for only finitely many integers $j$, by Fact 4.6.2.1 3, the subset $W$ defined by (4.30) is contained in a thin (and arithmetic when $\mathcal{K}$ is semisimple of geometric origin) subset of $\text{Char}(X)$.

\[\square\]

Theorem 4.6.2.3 is a generic vanishing result for local systems on a manifold admitting Hodge theory. When $X$ is a projective manifold, [PS13, Theorem 1.5] gives a dimension estimate of $\Sigma^k(X, \mathcal{L})$.

**Theorem 4.6.2.3.** Let $X$ be a connected regular manifold of dimension $n$. Let $\alpha : X \to \text{Alb}(X)$ be the Albanese map associated with some base point and $\mathcal{E}$ be a local system on $X$. Let $k$ be an integer either $< n - r(\alpha)$ or $> n + r(\alpha)$. Then:

\(^{10}\)in the sense of [Sta23, Tag 0110 (2)]
1. $\Sigma^k(X, E) \cap \text{Char}^\circ(X)$ is a strict Zariski closed subset of $\text{Char}^\circ(X)$.

2. If furthermore $\text{Alb}(X)$ is algebraic, then $\Sigma^k(X, E) \cap \text{Char}^\circ(X)$ is contained in a thin subset of $\text{Char}^\circ(X)$.

**Proof.** In view of (4.29), one may assume $k > d + r(\alpha)$. Set $K := R\alpha_* E[d + r(\alpha)]$.

We first prove

$$\Sigma^k(X, E) \cap \text{Char}^\circ(X) = \Sigma^{k-d-r(\alpha)}(\text{Alb}(X), K) \subset \Sigma^{>0}(\text{Alb}(X), K). \quad (4.32)$$

This is used in the proof of both 1 and 2.

Indeed, by Proposition 4.5.3.2, the complex of sheaves $K$ lies in $p^\perp D^{<0}(\text{Alb}(X))$. For every $\chi \in \text{Char}^\circ(X)$, let $D_\chi$ (resp. $L_\chi$) be the corresponding character sheaf on $\text{Alb}(X)$ (resp. on $X$). Then $\alpha^* D_\chi = L_\chi$. By [KW01, Cor. 7.5 (g), p.109], $R\alpha_* (E \otimes L_\chi) = (R\alpha_* E) \otimes L D_\chi$ in $D^b_c(\text{Alb}(X))$. It follows that

$$H^k(X, E \otimes L_\chi) = H^k(\text{Alb}(X), (R\alpha_* E) \otimes L D_\chi) = H^{k-d-r(\alpha)}(\text{Alb}(X), K \otimes L D_\chi),$$

whence (4.32). Now Point 1 follows from Fact 4.6.2.1 and Corollary 4.6.2.2, and Point 2 follows from Corollary 4.6.2.3. \hfill \Box

### 4.7 Generic vanishing result for manifolds in Fujiki class $C$

In Section 4.7.1, we recall the definition of Fujiki class $C$, the object of central interest in this note. Then we restrict mainly to algebraic varieties in Section 4.7.2.

#### 4.7.1 Fujiki class $C$

**Definition 4.7.1.1** (Fujiki class $C$, [Uen80, Def. 1]). A compact complex manifold is called in Fujiki class $C$ if it is the meromorphic image of a compact Kähler manifold.

Every compact Kähler manifold is in Fujiki class $C$. The reason why Fujiki class $C$ is interesting is two-fold. For one thing, this class is large enough in practice. For another, in this class there is a Hodge theory with unitary local systems as coefficients.

**Fact 4.7.1.2** ([Tim87, Cor. 5.3], [Ara90, Thm. 1, Thm. 2, Cor. 2]). Let $X$ be a complex manifold in Fujiki class $C$. Then, for every unitary local system $E$ on $X$, $X$ is $E$-regular.

In particular, from Fact 4.7.1.2, every manifold in Fujiki class $C$ is regular. As is explained in Section 4.3 and Section 4.4, the Jacobian and Albanese of a complex manifold in Fujiki class $C$ behave well.
4.7. GENERIC VANISHING RESULT FOR MANIFOLDS IN FUJIKI CLASS C

Theorem 4.7.1.3. Let $X$ be an $n$-dimensional complex manifold in Fujiki class $C$, and let $\alpha : X \to \text{Alb}(X)$ be the Albanese map associated with some base point. Let $E \to X$ be a flat unitary holomorphic vector bundle. Then for any integers $p, q \geq 0$, one has:

1. The locus $S^{p,q}(X, E)$ is an analytic subset of $\text{Pic}^0(X)$.
2. $S^{n-p,n-q}(X, E^\vee) = \{L \in \text{Pic}^0(X) | L^\vee \in S^{p,q}(X, E)\}$.
3. If $p + q < n - r(\alpha)$ or $p + q > n + r(\alpha)$, then $S^{p,q}(X, E)$ is contained in a strict (and thin when $\text{Alb}(X)$ is algebraic) subset of $\text{Pic}^0(X)$.

Proof.

1. The projection $p_2 : X \times \text{Pic}^0(X) \to \text{Pic}^0(X)$ is a regular family in the sense of [GR84, p.207]. Let $p_1 : X \times \text{Pic}^0(X) \to X$ be the other projection. Let $\mathcal{P}$ be the universal line bundle on $X \times \text{Pic}^0(X)$ given by Corollary 4.4.2.1. Applying the upper semi-continuity theorem ([GR84, p.210]) to the vector bundle $\mathcal{P} \otimes p_1^*\Omega_X^p$ and the regular family $p_2$, one gets that $S^{p,q}(X, E)$ is an analytic subset of $\text{Pic}^0(X)$.

2. By Serre duality (see, e.g., [Huy05, Prop. 4.1.15]), for every $L \in \text{Pic}(X)$, there is a perfect pairing

$$H^q(X, \Omega_X^p \otimes_{\mathcal{O}_X} L \otimes_{\mathcal{O}_X} E) \times H^{n-q}(X, \Omega_X^{n-p} \otimes_{\mathcal{O}_X} L^\vee \otimes_{\mathcal{O}_X} E^\vee) \to \mathbb{C},$$

so $L \in S^{p,q}(X, E)$ if and only if $L^\vee \in S^{n-p,n-q}(X, E^\vee)$.

3. By Theorem 4.2.3.1, there is a unitary local system $\mathcal{E}$ on $X$ such that $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{O}_X$ is isomorphic to $E$. For each $\chi \in \text{Char}(X)$, let $L_\chi := \mathcal{L}_\chi \otimes_{\mathcal{O}_X} \mathcal{O}_X$. Then the isomorphism (4.20) of real Lie groups is given by $\chi \mapsto L_\chi$. Moreover, the Hodge decomposition (4.6) for $\mathcal{E} \otimes_{\mathcal{O}_X} L_\chi$ provided by Fact 4.7.1.2 is

$$H^k(X, \mathcal{E} \otimes_{\mathcal{O}_X} L_\chi) = \oplus_{p+q=k} H^q(X, \Omega_X^p \otimes_{\mathcal{O}_X} \mathcal{E} \otimes_{\mathcal{O}_X} L_\chi) = H^q(X, \Omega_X^p \otimes_{\mathcal{O}_X} E \otimes_{\mathcal{O}_X} L_\chi).$$

Therefore, under the isomorphism (4.20), one has

$$\Sigma^k(X, \mathcal{E}) \cap T(X) = \cup_{p+q=k} S^{p,q}(X, E). \quad (4.33)$$

The result follows from Theorem 4.6.2.3.

Remark 4.7.1.4. Theorem 4.7.1.3 extends Fact 4.1.1.6 from Kähler manifolds to Fujiki class $C$. As $\dim X - r(\alpha) \leq \dim \alpha(X)$, the numerical hypothesis in Theorem 4.7.1.3 is more restrictive than that in Fact 4.1.1.1. An example from [GL87, Remark, p.401] is reconsidered in the last paragraph of [KW15b, Sec. 3], to show that the bound $p + q < \dim X - r(\alpha)$ is optimal for Fact 4.1.1.6.
4.7.2 Moishezon manifolds

Moishezon manifolds are examples of manifolds in Fujiki class $C$.

**Definition 4.7.2.1** (Moishezon manifold, [MM07, Def. 2.2.12]). A connected compact complex manifold $X$ is called Moishezon if it has $\dim X$ algebraically independent meromorphic functions.

In fact, according to [MM07, Thm. 2.2.16], for every Moishezon manifold $X$, there is a proper modification $\pi : X' \to X$ with $X'$ a projective manifold. In particular, $X$ is the meromorphic image of a projective manifold, hence in Fujiki class $C$. Conversely, a connected compact complex manifold that is the meromorphic image of a projective manifold is Moishezon by the proof of [Voi02, Cor. 12.12]. For more references, see [JM22, Sec. 1].

The intersection of the two subclasses, Kähler and Moishezon, is exactly the class of projective manifolds. More precisely, Moishezon’s theorem (see, e.g., [Voi02, Thm. 12.13]) asserts that a Moishezon manifold is Kähler if and only if it is projective. A Moishezon manifold may not be homotopy equivalent to a Kähler manifold ([Oga94, Thm. 1]). Kodaira-Spencer stability theorem (see, e.g., [Voi02, Thm. 9.1]) shows that small deformations of a Kähler manifold are Kähler. Similarly, small deformations of a regular manifold are regular ([AT13, Cor. 3.7]). By contrast, there is a small deformation of a Moishezon manifold that is not in Fujiki class $C$ ([Cam91, Sec. 0]). In particular, there exists a regular manifold that is not in Fujiki class $C$.

Moishezon manifolds are abundant. For example, for every smooth proper algebraic variety $X/\mathbb{C}$, its analytification $X^{an}$ is a Moishezon manifold ([Har77, p.442]). Hironaka ([Hir60], see also [Har77, p.443]) gives examples of Moishezon manifolds that are not algebraic, and smooth proper algebraic varieties that are not projective. The situation is depicted below. Every inclusion in this graph is strict.
We need Proposition 4.7.2.2 on the algebraicity of Picard torus and Albanese torus to compare them with the Picard variety and Jacobian variety of an algebraic variety.

**Proposition 4.7.2.2.** If \( X \) is a Moishezon manifold, then \( \text{Alb}(X) \) and \( \text{Pic}^0(X) \) are complex abelian varieties dual to each other.

**Proof.** By [MM07, Thm. 2.2.16], \( X \) admits a proper modification \( \pi : X' \to X \) with \( X' \) a projective manifold. By [Voi02, Prop. 7.16], the Jacobian \( \text{Pic}^0(X') \) is projective. From Proposition 4.4.1.25, the torus \( \text{Alb}(X') \) is dual to \( \text{Pic}^0(X') \), so \( \text{Alb}(X') \) is algebraic. By [Uen06, Prop. 9.12, p.107], the morphism \( \pi_* : \text{Alb}(X') \to \text{Alb}(X) \) given by Proposition 4.4.1.21 is an isomorphism.

**Remark 4.7.2.3.** By [BL04, p.70], the analytic dual torus of a complex abelian variety is an abelian variety. Moreover, by [MRM74, p.86], the (algebraic) dual abelian variety (defined in [MRM74, p.78]) of a complex abelian variety coincides with its analytic dual torus, so we do not distinguish the two duals in this case.

From now on, let \( X/\mathbb{C} \) be a smooth proper algebraic variety of dimension \( n \) with a base point \( x_0 \in X(\mathbb{C}) \), and let \( \text{Sch}/\mathbb{C} \) (resp. Sets) be the category of \( \mathbb{C} \)-schemes (resp. sets). The fppf-sheaf associated to the functor

\[
P_{X/\mathbb{C}} : (\text{Sch}/\mathbb{C})^{\text{op}} \to \text{Sets}, \quad T \mapsto \text{Pic}(X \times_{\mathbb{C}} T)
\]

is called the relative Picard functor of \( X \) ([BLR12, Def. 2, p.201]). From [BLR12, p.211, p.231 and p.233], the relative Picard functor of \( X \) is represented by a smooth group scheme \( \text{Pic}_X/\mathbb{C} \) over \( \mathbb{C} \). In particular, the group \( \text{Pic}_X/\mathbb{C}(\mathbb{C}) = \text{Pic}(X) \). By [BLR12, Thm. 3, p.232], the identity component \( \text{Pic}_X^{0}/\mathbb{C} \) of \( \text{Pic}_X/\mathbb{C} \) is proper over \( \mathbb{C} \), hence a complex abelian variety called the Picard variety of \( X \).
From [Ser58, Thm. 5], there is an abelian variety $\text{Alb}(X)/\mathbb{C}$ with a $\mathbb{C}$-morphism $\alpha_{X,x_0} : (X, x_0) \to (\text{Alb}(X), 0)$ of pointed varieties satisfying the following universal property:\footnote{11} every $\mathbb{C}$-morphism of pointed varieties $(X, x_0) \to (A, 0)$ with $A/\mathbb{C}$ an abelian variety factors uniquely through a morphism of abelian varieties $\text{Alb}(X) \to A$. Such morphism $\alpha_{x_0}$ is unique up to a unique isomorphism. We call $\text{Alb}(X)$ the \textit{algebraic Albanese variety} of $X$ and $\alpha_{X,x_0} : (X, x_0) \to (\text{Alb}(X), 0)$ the \textit{algebraic Albanese morphism} corresponding to $x_0$.

For every $O_X$-module $F$, let $F^{\text{an}}$ be the corresponding $O_X^{\text{an}}$-module defined in [GR71, Exp. XII, 1.3]. Hence a functor

$$\text{Mod}(O_X) \to \text{Mod}(O_X^{\text{an}}), \quad F \mapsto F^{\text{an}}.$$ 

By Serre’s GAGA [GR71, Exp. XII, Thm. 4.4], the natural group morphism

$$\text{Pic}(X) \to \text{Pic}(X^{\text{an}}), \quad L \mapsto L^{\text{an}}$$

is an isomorphism. Corollary 4.7.2.4 2 of GAGA type compares the algebraic Picard variety and the analytic Jacobian. Once again, it is well-known, but a proof is given for the lack of reference.

**Corollary 4.7.2.4.**

1. The analytification of $\text{Alb}(X)$ (resp. $\alpha_{X,x_0} : X \to \text{Alb}(X)$) is $\text{Alb}(X^{\text{an}})$ (resp. $\alpha_{X^{\text{an}},x_0} : X^{\text{an}} \to \text{Alb}(X^{\text{an}})$).

2. The analytification of $\text{Pic}^0_{X/\mathbb{C}}$ is $\text{Pic}^0(X^{\text{an}})$.

**Proof.**

1. Since $X^{\text{an}}$ is a Moishezon manifold, by Proposition 4.7.2.2, its Albanese torus $\text{Alb}(X^{\text{an}})$ is projective. By Chow’s theorem [BL04, Cor. A.4], the map $\alpha_{X^{\text{an}},x_0}$ is algebraic. By Proposition 4.4.1.2 3, every algebraic morphism $(X, x_0) \to (A, 0)$ to an abelian variety $A/\mathbb{C}$ factors uniquely through an analytic (hence algebraic by Chow’s theorem again) morphism of complex tori $\text{Alb}(X^{\text{an}}) \to A^{\text{an}}$. The result follows.

2. By [Moc12, Prop. A.6], the (algebraic) dual abelian variety of $\text{Pic}^0_{X/\mathbb{C}}$ is $\text{Alb}(X)$. By Proposition 4.4.1.2 5, $\text{Pic}^0(X^{\text{an}})$ is the (analytic) dual torus of $\text{Alb}(X^{\text{an}}) = \text{Alb}(X)^{\text{an}}$, so $\text{Pic}^0(X^{\text{an}})$ is the analytification of $\text{Pic}^0_{X/\mathbb{C}}$.

Identifying $\text{Pic}^0_{X/\mathbb{C}}$ with $\text{Pic}^0(X^{\text{an}})$ via Corollary 4.7.2.4 2, one can define thin subsets of $\text{Pic}^0_{X/\mathbb{C}}$. Define the defect of semismallness of a proper morphism $f : M \to N$ between complex algebraic varieties by $r(f) = r(f^{\text{an}})$. With this terminology, we get the following generic vanishing result for smooth proper algebraic varieties.
Corollary 4.7.2.5. Let $E$ be a unitary local system on $X^{an}$, and let $E = \mathcal{E} \otimes \mathbb{C}$ $
abla_X^{an}$ be the corresponding holomorphic vector bundle. Then for any integers $p, q \geq 0$ with $p + q > n + r(\alpha)$ or $p + q < n - r(\alpha)$, the locus $S^{p,q}(X^{an}, E)$ is contained in a thin (and arithmetic when $E$ is semisimple of geometric origin in $D^b_c(X^{an})$) subset of $Pic^0_{X/C}$.

Proof. By Corollary 4.7.2.4 1, the analytification $\alpha_{X,x_0}^{an} : X^{an} \to Alb(X)^{an}$ coincides with $\alpha_{X^{an},x_0} : X^{an} \to Alb(X^{an})$, and by definition, $r(\alpha) = r(\alpha^{an})$.

From Theorem 4.7.1.3 3, the locus $S^{p,q}(X^{an}, E)$ is contained in a thin subset of $Pic^0(X)$.

What remains to show is the assertion in the parentheses. Assume that $E$ is semisimple of geometric origin. By the decomposition theorem [BBDG82, Thm. 6.2.5], $K := R\alpha_*\mathcal{E}[n + r(\alpha)]$ is semisimple of geometric origin in $D^b_c(Alb(X^{an}))$.

By Theorem 4.7.1.3 2, one may assume that $p + q > n + r(\alpha)$, so that

$$S^{p,q}(X^{an}, E) \subset \Sigma^{p+q}(X^{an}, E) \cap T(X) \subset \Sigma^{>0}(Alb(X), K),$$

where the first inclusion follows from (4.33) and the second from (4.32). From Corollary 4.6.2.2 3, $\Sigma^{>0}(Alb(X), K)$ is contained in an arithmetic thin subset of $Pic^0_{X/C}$. 

Remark 4.7.2.6. By Chow’s theorem, every analytic subset of $X^{an}$ is algebraic. Therefore, $D^b_c(X^{an})$ coincides with $D^b_c(X(C), \mathbb{C})$ defined in [BBDG82, p.66] using algebraic Whitney stratifications.
CHAPTER 4. GENERIC VANISHING THEOREM FOR FUJIKI CLASS C
Chapter 5

Fourier-Mukai transform on complex tori, revisited

5.1 Introduction

For a ringed space \((Z, O_Z)\), let \(D(Z)\) be the derived category of the abelian category of \(O_Z\)-modules. A scheme of finite type and separated over a field is called an algebraic variety. For two algebraic varieties (resp. complex analytic spaces) \(M, N\) and an object \(K \in D(M \times N)\), the integral transform \(\phi_{K}^{[M \to N]} : D(M) \to D(N)\) with integral kernel \(K\) is defined as

\[
\phi_{K}^{[M \to N]} = R\pi_{N, *}(K \otimes L^{p_{M}^{*}}M), \tag{5.1}
\]

where \(p_{M} : M \times N \to M\) and \(p_{N} : M \times N \to N\) are the two projections.

When \(Z\) is a complex analytic space, let \(D_{\text{good}}(Z) \subset D(Z)\) be the full subcategory consisting of complexes whose cohomology sheaves are good (Definition A.1.4.1). Roughly speaking, an analytic sheaf of modules is good if it can be approximated by coherent submodules. For a complex torus \(X\) of dimension \(g\), let \(\hat{X}\) be the dual complex torus. Let \(P\) be the normalized\(^1\) Poincaré line bundle on \(X \times \hat{X}\). Define functors \(RS : D(\hat{X}) \to D(X)\) and \(R\hat{S} : D(X) \to D(\hat{X})\) by

\[
RS = \phi_{P}^{[\hat{X} \to X]}, \quad R\hat{S} = \phi_{P}^{[X \to \hat{X}]} \tag{5.2}
\]

The pair \((RS, R\hat{S})\) is called the Fourier-Mukai transform of \(X\). Theorem 5.1.0.1 establishes an analog of the Fourier inversion formula for this pair.

Theorem 5.1.0.1 (Theorem 5.4.1.1). The functor \(R\hat{S}\) (resp. \(RS\)) restricts to a functor \(D_{\text{good}}(X) \to D_{\text{good}}(\hat{X})\) (resp. \(D_{\text{good}}(\hat{X}) \to D_{\text{good}}(X)\)). Moreover, there are natural isomorphisms of functors

\[
RS \circ R\hat{S} = [-1]_{X}^{*}[-g] : D_{\text{good}}(X) \to D_{\text{good}}(X),
\]

\[
R\hat{S} \circ RS = [-1]_{\hat{X}}^{*}[-g] : D_{\text{good}}(\hat{X}) \to D_{\text{good}}(\hat{X}),
\]

\(^{1}\)i.e., both pullback modules \(P|_{X \times 0}\) and \(P|_{0 \times \hat{X}}\) are trivial
where \([-g]\) denotes degree shift.

Theorem 5.1.0.1 is a variant of [Muk81, Thm. 2.2] (Statement 5.2.0.7). This statement has a minor problem for lack of quasi-coherence condition. For complex tori, a parallel false assertion is made as [BBBP07, Thm. 2.1] (Statement 5.2.0.8). Theorem 5.1.0.1 shows that "good sheaves" on complex manifolds serve as substitutes for "quasi-coherent sheaves" on algebraic varieties in this case. As an application of Theorem 5.1.0.1, we give a proof of Theorem 5.5.3.6 due to Matsushima and Morimoto. It classifies homogeneous vector bundles (Definition 5.5.3.1) on complex tori.

**Notation**

For a topological space \(M\), the category of abelian sheaves on \(M\) is denoted by \(\text{Ab}(M)\). The category of ringed spaces is denoted by \(\text{RingS}\). For a ringed space \((X, O_X)\), let \(\text{Mod}(O_X)\) be the category of \(O_X\)-modules. The full subcategory of \(\text{Mod}(O_X)\) comprised of quasi-coherent (resp. coherent) \(O_X\)-modules is denoted by \(\text{Qch}(X)\) (resp. \(\text{Coh}(X)\)). For a closed subset \(Z \subset X\), let \(\text{Coh}_Z(X) \subset \text{Coh}(X)\) be the full subcategory consisting of modules with support contained in \(Z\). Given a symbol \(* \in \{\emptyset, +, -, b\}\), the notation \(D^*(X)\) refers to the unbounded/bounded below/bounded above/bounded derived category of \(\text{Mod}(O_X)\) in order. The full subcategory of \(D^*(X)\) consisting of the complexes whose cohomologies are coherent (resp. quasi-coherent) (Definition A.1.1.1) is denoted by \(D^c(X)\) (resp. \(D^qc(X)\)). Denote by \(R\text{Hom}_X : D(X)^{op} \times D(X) \to D(X)\) the internal hom constructed in [Sta23, Tag 08DH].

For a locally ringed space \(X\) and \(x \in X\), let \(i_x : (x, O_{X,x}) \to (X, O_X)\) be the canonical morphism of locally ringed spaces. For an \(O_{X,x}\)-module \(M\), the \(O_X\)-module \((i_x)_*M\) is denoted by \(M_x\).

All complex analytic spaces (in the sense of [KK11, Def. 43.2]) are assumed to be paracompact. Let \(\text{An}\) be the category of complex analytic spaces. The dimension of a complex manifold always refers to the complex dimension, which is assumed to be finite.

When \(X\) is an abelian variety (resp. complex torus), its dual abelian variety (resp. complex torus) is denoted by \(\hat{X}\). The normalized Poincaré bundle on \(X \times \hat{X}\) is denoted by \(\mathcal{P}\). For \(y \in \hat{X}\) (resp. \(x \in X\)), let \(P_y\) (resp. \(P_x\)) denote the line bundle \(\mathcal{P}|_{X \times y}\) (resp. \(\mathcal{P}|_{x \times \hat{X}}\)).

### 5.2 Fourier-Mukai transform

Complex tori are generalization of complex abelian varieties. Every complex torus of dimension 1 is an abelian variety. By contrast, for every integer \(g \geq 2\), a very general complex torus of dimension \(g\) is not\(^2\) an abelian variety (see, e.g.,

\(^2\)To the contrary, it is incorrectly implied in [BBR94, p.151] that every complex torus of dimension 2 admits a compatible structure of algebraic complex surface. In fact, it fails for each 2-dimensional complex torus \(X\) that is not a projective manifold. For otherwise, assume
The Fourier-Mukai transform is an analog of the classical Fourier transform. It is proposed by Mukai [Muk81] on abelian varieties and complex tori. Let $k$ be an algebraically closed field. Let $X/k$ be an abelian variety (resp. $X$ be a complex torus) of dimension $g$. Write $RS$ and $R\hat{S}$ for $\varphi^{[X \to X]}_P$ and $\varphi^{[X \to \hat{X}]}_P$ respectively. The pair $(RS, R\hat{S})$ is called the Fourier-Mukai transform of $X$. The functor $RS$ (resp. $R\hat{S}$) restricts to a functor $D^b(\hat{X}) \to D^b(X)$ (resp. $D^b(X) \to D^b(\hat{X})$).

**Lemma 5.2.0.1** ([thr]). If the functor $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence of categories, then $g = 0$.

**Proof.** When $X$ is a complex torus, let $k = \mathbb{C}$. In both cases, let $F = k_{0}^{N}$ be the product of a countable infinite family of $k_0$ in $\text{Mod}(O_X)$. Since $k^{N} = k^{\oplus I}$ as a $k$-module for some index set $I$, the direct sum sheaf $k_{0}^{\oplus I}$ is isomorphic to $F$. Therefore, by [Sta23, Tag 07D9 (2)], $F$ is the direct sum of $I$ copies of $k_0$ in $D^b(\hat{X})$. We claim that $F$ is the product of $\mathbb{N}$ copies of $k_0$ in $D^b(\hat{X})$.

By [Gro57b, p.129], the abelian category $\text{Mod}(O_{\hat{X},0})$ satisfies the $AB$ 4*’ axiom. From [Sta23, Tag 07KC (2)], the inclusion $\text{Mod}(O_{\hat{X},0}) \to D^b(\text{Mod}(O_{\hat{X},0}))$ commutes with countable products. Let $i : 0 \to \hat{X}$ be the closed immersion. Since $i_* : \text{Mod}(O_{\hat{X},0}) \to \text{Mod}(O_X)$ is exact, there is a commutative square

$$
\begin{array}{ccc}
\text{Mod}(O_{\hat{X},0}) & \xrightarrow{i_*} & \text{Mod}(O_X) \\
\downarrow & & \downarrow \\
D^b(\text{Mod}(O_{\hat{X},0})) & \xrightarrow{Ri_*} & D^b(\hat{X}).
\end{array}
$$

Since $Ri_* : D^b(\text{Mod}(O_{\hat{X},0})) \to D^b(\hat{X})$ has a left adjoint, it commutes with products. As $F = i_*(k^N)$, the claim is proved.

As $RS : D^b(\hat{X}) \to D^b(X)$ is an equivalence, inside $D^b(X)$, the object $RS(F)$ is the direct sum of $I$ copies of $RS(k_0)$ and the product of $\mathbb{N}$ copies of $RS(k_0)$. By Example 5.2.0.5 (when $X$ is an abelian variety) and Lemma 5.2.0.3 (when $\hat{X}$ is a complex torus), one has $RS(k_0) = O_X$. Therefore, $RS(F)$ is isomorphic to $O_X^{\oplus I}$ and to $O_X^{\mathbb{N}}$ in $\text{Mod}(O_X)$.

Assume the contrary that $g > 0$. Then there is a nonempty connected open subset $V \subset X$, such that $O_X(V)$ is an integral domain but not a field. In particular, the ring $O_X(V)$ is not Artinian. By [Har77, II, Exercise 1.11] (when $X$ is an abelian variety) and Corollary A.1.5.4 (when $X$ is a complex torus), the $O_X(V)$-module $\Gamma(V, RS(F))$ is isomorphic to $O_X(V)^{\oplus I}$ and to $O_X(V)^{\mathbb{N}}$. However, this contradicts Fact 5.2.0.2.

**Fact 5.2.0.2** ([Len68, Thm, p.211]). If $A$ is a commutative ring such that $A^N$ is a free $A$-module, then $A$ is Artinian.

If $X$ is a projective manifold, a contradiction.

There is an algebraic surface $V/\mathbb{C}$ with $V^{an} = X$. Then $V$ is proper by [GR71, XII, Prop. 3.2 (v)]. In consequence, the algebraic variety $V$ is projective by [Har77, p.357]. Thus, $X$ is a projective manifold, a contradiction.
For algebraic varieties, the analog of Lemma 5.2.0.3 follows from the flat base change theorem and the projective formula.

**Lemma 5.2.0.3.** Let $X, Y$ be two complex analytic spaces, let $K \in D(X \times Y)$, and let $x \in X$. Then $\phi_K^{[X \to Y]}(C_x) = Li_x^*K$, where the closed embedding $i_x : Y \to X \times Y$ is defined by $y \mapsto (x, y)$.

**Proof.** Let $p : X \times Y \to X$, $q : X \times Y \to Y$ be the two projections. Denote the inclusion $x \to X$ by $j$. The cartesian square

$$
\begin{array}{ccc}
Y & \xrightarrow{p_0} & X \\
\downarrow{i_x} & & \downarrow{j} \\
X \times Y & \xrightarrow{p} & X
\end{array}
$$

in $\text{An}$ induces a natural morphism

$$
\phi : p^*C_x \to R(i_x)_*O_Y
$$

in $\text{Mod}(O_{X \times Y})$. Both sheaves are supported on $\{x\} \times Y$.

For two (Hausdorff) topological vector spaces $E, F$ over $\mathbb{C}$ that are locally convex, the completed projective topological tensor product $E \hat{\otimes}_\mathbb{C} F$ is defined in [Gro55, Ch. I, Déf. 2, p.32]. For every $y \in Y$, by [GR84, p.27], the stalk $O_{X \times Y, (x,y)} = O_{X,x} \hat{\otimes}_\mathbb{C} O_{Y,y}$. Then

$$(p^*C_x)_{x,y} = C \otimes_{O_{X,x}} O_{X \times Y, (x,y)} = O_{Y,y}.$$ 

Therefore, $\phi_{(x,y)} : (p^*C_x)_{(x,y)} \to (i_*O_Y)_{(x,y)}$ is an isomorphism. Thus, $\phi$ is an isomorphism.

By [Sta23, Tag 0B55], the natural morphism $(Ri_{x,*}O_Y) \otimes^L K \to Ri_{x,*}(Li_x^*K)$ is an isomorphism. Then

$$
\phi_K^{[X \to Y]}(C_x) = Rq_*(p^*C_x \otimes^L K) \cong Rq_*(Ri_{x,*}O_Y \otimes^L K) \\
\cong Rq_*(Ri_{x,*}(Li_x^*K)) = R(qi_x)_*(Li_x^*K) = Li_x^*K.
$$

Let $X/k$ be an abelian variety. The usual exchange of translation and time shifting (resp. multiplication and convolution) of Fourier transform finds analog for Fourier-Mukai transform, namely the exchange of translation and line bundle twisting (resp. tensor product and Pontrjagin product) in [Muk81, (3.1) (resp. (3.7))]. Moreover, Mukai proves a duality theorem similar to the classical Fourier inversion formula.

**Fact 5.2.0.4 (Algebraic Mukai duality).** There are canonical isomorphisms of functors

$$
RS \circ R\hat{S} \cong [-1]_X^*[-g] : D_{qc}(X) \to D_{qc}(X); \\
R\hat{S} \circ RS \cong [-1]_X^*[-g] : D_{qc}(\hat{X}) \to D_{qc}(X).
$$

In particular, the functor $RS : D_{qc}(\hat{X}) \to D_{qc}(X)$ is an equivalence of categories, with a quasi-inverse $[-1]_X^* \circ R\hat{S}[g]$. 

Example 5.2.0.5 ([Muk81, Eg. 2.6]). For every \( y \in \hat{X}(k) \), one has \( RS(k_y) = P_y \) and \( RS(P_y) = k_{-y\left[ -g \right]} \).

Remark 5.2.0.6. Combining Fact 5.2.0.4, the natural equivalence \( D(Qch(X)) \to D_{qc}(X) \) ([BN93, Cor. 5.5]) with the compatibility of derived direct images [TT07, Cor. B.9], one gets [Rot96, Mukai’s Theorem, p.569] stated for \( D_b(Qch(\ast)) \) instead of \( D_{qc}(\ast) \). The quasi-coherence restriction is essential for Čech resolution with respect to affine covers in [Rot96, p.571].

The proof of Fact 5.2.0.4 uses projection formula and the flat base change theorem ([Lip60, Prop. 3.9.4; Prop. 3.9.5]). Compared with Fact 5.2.0.4, the original statement (Statement 5.2.0.7) has no quasi-coherence restriction.

Statement 5.2.0.7 ([Muk81, Thm. 2.2]). The functor \( RS \) gives an equivalence of categories between \( D(\hat{X}) \) and \( D(X) \), and its quasi-inverse is \([-1]_X \circ R\hat{S}[g] \).

In [BBBP07, Thm. 2.1], an assertion similar to Statement 5.2.0.7 is made for complex tori.

Statement 5.2.0.8. Let \( X \) be a complex torus. Then the integral transform \( RS : D_b(\hat{X}) \to D_b(X) \) is an equivalence of triangulated categories.

However, Lemma 5.2.0.1 shows that Statement 5.2.0.7 (resp. Statement 5.2.0.8) holds if and only if \( g = 0 \). The error of Statement 5.2.0.7 occurs in the proof of [Muk81, Prop. 1.3], when the flat base change theorem [Har66, Prop. 5.12] stated for objects of \( D_{qc}(\ast) \) is applied to objects in \( D_-(\ast) \). Similarly, the error of Statement 5.2.0.8 originates from a lack of certain analytic quasi-coherence in the wrong Statement 5.2.0.9 (a counterpart of [Muk81, Prop. 1.3]).

A modification of Statement 5.2.0.9 is Proposition 5.4.2.3.

Statement 5.2.0.9 ([BBBP07, p.427]). If \( M, N, \) and \( P \) are compact complex manifolds and \( K \in D_b(M \times N) \) and \( L \in D_b(N \times P) \), then one has a natural isomorphism of functors from \( D_b(M) \) to \( D_b(P) \):

\[
\phi_{[N \to P]}^L \circ \phi_{[M \to N]}^M \cong \phi_{[K \to L]}^{[M \to P]},
\]

where

\[
K \ast L = R_{p_M \times P^*} p_{M \times N}^* K \otimes L \in D_b(M \times P),
\]

and \( p_{M \times N}, p_{M \times P}, p_{N \times P} \) are the natural projections \( M \times N \times P \to M \times N, \) etc.

When \( X \) is an abelian variety of positive dimension, by Fact 5.2.0.4, \( RS(F) \) is the product of \( \mathbb{N} \) copies of \( O_X \) in \( Qch(X) \). It is not isomorphic to \( O_X^n \) by Lemma 5.2.0.10.

Lemma 5.2.0.10. Let \( X \) be an integral scheme with generic point \( \eta \). If the \( O_X \)-module \( O_X^n \) is quasi-coherent, then the natural morphism \( \eta \to X \) is an isomorphism.

Proof. Consider an arbitrary affine open \( U = \text{Spec}(A) \subset X \). Then \( A \) is an integral domain of fraction field \( \kappa(\eta) \). We show that the natural inclusion \( A \to \kappa(\eta) \) is an isomorphism.
For otherwise, there exists $f \in A \setminus (A^* \cup \{0\})$. Let $D_f \subset U$ be the corresponding standard open subset. Note $\Gamma(U, \mathcal{O}_X^N) = A^N$ and $\Gamma(D_f, \mathcal{O}_X^N) = (A_f)^N$. As $\mathcal{O}_X^N \in \text{Qch}(X)$, the natural $A_f$-module morphism $\Gamma(U, \mathcal{O}_X^N)_{f} \to \Gamma(D_f, \mathcal{O}_X^N)$ is an isomorphism. Or equivalently, the natural map $\phi : (A^N)_{f} \to (A_f)^N$ is an isomorphism.

In particular, there exists $a = (a_0, a_1, \ldots) \in A^N$ and an integer $m \geq 0$ such that $a/m = (1/f^i)_{i \geq 0}$. Then $a_{m+1} f = f^{-1}$ in $A_f$. There exists an integer $n \geq 0$ such that $(a_{m+1} f - 1)m^n = 0$ in $A$. Since $A$ is a domain, $a_{m+1} f - 1 = 0$ in $A$. This contradicts the fact that $f \notin A^*$.

Therefore, the natural morphism $\eta : U \to V$ is an isomorphism. The proof is completed as $U$ is taken arbitrarily. \hfill \Box

Lemma 5.2.0.11 computes the derived restriction of a relatively flat module, which is a partial converse to [Huy06, Lemma 3.31] in the analytic setting.

**Lemma 5.2.0.11.** Let $f : S \to X$ be a flat morphism of complex analytic spaces, and let $K$ be an $O_{S}$-module flat over $X$. For $x \in f(S)$, let $i_{x} : S_{x} \to S$ be the inclusion of the fiber over $x$. Then $\text{Li}_{x}^{*}K = \mathcal{I}_{x}^{*}K$.

**Proof.** To simplify the notation, we denote $i_{x}$ by $i$. By [Sta23, Tag 0B55], the natural morphism

$$ Ri_{x}O_{S_{x}} \otimes^{L}_{O_{S}} K \to Ri_{x}(\text{Li}^{*}K) $$

(5.3)

is an isomorphism. They are supported on $S_{x}$, since for every integer $n$ one has

$$ H^{n}(Ri_{x}(\text{Li}^{*}K)) = Ri_{x} H^{n}(\text{Li}^{*}K) = i_{x} H^{n}(\text{Li}^{*}K). $$

For every $s \in S_{x}$, the morphism $j : (s, O_{S,s}) \to S$ of ringed spaces is flat and $j^{*} : \text{Mod}(O_{S}) \to \text{Mod}(O_{S,s})$ is taking the stalk at $s$. Let $m_{s}$ be the maximal ideal of $O_{X,x}$ As the ring map $f_{x}^{\#} : O_{X,x} \to O_{S,s}$ is flat, one has

$$ O_{S,s} \otimes^{L}_{O_{S}} O_{S,s} = (O_{X,x}/m_{x}) \otimes^{L}_{O_{X,x}} O_{S,s} = (O_{X,x}/m_{x}) \otimes^{L}_{O_{S,s}} O_{S,s}.$$  

(5.4)

By [Sta23, Tag 079U], one has

$$ \text{Li}_{x}^{*}(Ri_{x}O_{S_{x}} \otimes^{L}_{O_{S}} K) = \text{Li}_{x}^{*} Ri_{x}O_{S_{x}} \otimes^{L}_{O_{S,s}} \text{Li}_{x}^{*} K $$

$$ = O_{S_{x}} \otimes^{L}_{O_{S,s}} K = (O_{X,x}/m_{x}) \otimes^{L}_{O_{X,x}} O_{S,s} \otimes^{L}_{O_{S,s}} K $$

$$ = (O_{X,x}/m_{x}) \otimes^{L}_{O_{S,s}} (O_{S,s} \otimes^{L}_{O_{S,s}} K) $$

$$ = (O_{X,x}/m_{x}) \otimes^{L}_{O_{S,s}} K = (O_{X,x}/m_{x}) \otimes^{L}_{O_{X,x}} K, $$

(5.5)

where the third (resp. fourth, resp. last) equality uses (5.4) (resp. Lemma 5.4.21, resp. the flatness of the $O_{X,x}$-module $K_{s}$).

Then for every integer $n \neq 0$, every $s \in S_{x}$, the stalk

$$ [H^{n}(Ri_{x}O_{S_{x}} \otimes^{L}_{O_{S}} K)]_{s} = H^{n}[\text{Li}_{x}^{*}(Ri_{x}O_{S_{x}} \otimes^{L}_{O_{S}} K)] = H^{n}((O_{X,x}/m_{x}) \otimes^{L}_{O_{X,x}} K) = 0, $$

where the second equality uses (5.5). Hence

$$ i_{x} H^{n}(\text{Li}^{*}K) = H^{n}[Ri_{x}(\text{Li}^{*}K)] \cong H^{n}(Ri_{x}O_{S_{x}} \otimes^{L}_{O_{S}} K) = 0, $$

where the first equality uses (5.4) and the second equality uses (5.5).
5.3. **GOOD MODULES**

where the second equality uses (5.3). Thus, for every integer \( n \neq 0 \), \( H^n(Li^*K) = 0 \) in \( \operatorname{Mod}(O_{S_x}) \).

Remark 5.2.0.12. Lemmas 5.2.0.3 and 5.2.0.11 yield an analytic version of [Huy06, Eg. 5.4 vi]: Let \( X, Y \) be two complex analytic spaces. Let \( x \in X \) and \( K \) be an \( O_{x}\times Y \)-module flat over \( X \). Then \( \phi_K^{[X\rightarrow Y]}(C_x) = K|_{x\times Y} \).

Remark 5.2.0.13. Here is an example showing the necessity of the flatness of \( f \) in Lemma 5.2.0.11.

Let \( A = \mathbb{C}[t] \) and \( B = \mathbb{C}[x, y]/xy \). Then the \( B \)-module \( xB \) (resp. \( yB \)) is isomorphic to \( B/y \) (resp. \( B/x \)). Let \( S = \operatorname{Spec}(B) \) and \( X = \operatorname{Spec}(A) = \mathbb{A}^1_{\mathbb{C}} \). The morphism \( A \rightarrow B \) of \( k \)-algebras defined by \( t \mapsto x \) induces a morphism \( f : S \rightarrow X \) of schemes. Let \( K \) be the coherent \( O_S \)-module corresponding to the \( B \)-module \( B/y \). Then \( K \) is flat over \( X \), because the ring map composition \( A \rightarrow B \rightarrow B/y \) is an isomorphism. Let \( i : S_0 \rightarrow S \) be the inclusion of the fiber over \( 0 \in X(\mathbb{C}) \). Then \( i \) is a closed immersion defined by ideal \( xB \subset B \), so \( Li^*K \) is induced by \( K \otimes_B^L (B/x) \). By [Osbl12, Exercise 9, b), p.72],

\[
\operatorname{Tor}^B_2(B/y, B/x) = (yB) \otimes_B (xB) \cong (B/x) \otimes_B (B/y) = B/(x, y) = \mathbb{C}.
\]

In particular, \( Li^*K \neq i^*K \). Taking analytification one gets \( L(i^{an})^*K^{an} \neq (i^{an})^*K^{an} \).

Corollary 5.2.0.14 follows from Lemma 5.2.0.11, and it is an analytic counterpart of [Huy06, Example 5.4 vi]]).

**Corollary 5.2.0.14.** In Lemma 5.2.0.3, if \( K \in \operatorname{Mod}(O_{X\times Y}) \) is flat over \( X \), then \( \phi_K^{[X\rightarrow Y]}(C_x) = i^*K \).

By Corollary 5.2.0.14 and Theorem 5.4.1.1, Example 5.2.0.5 remains true when \( X \) is a complex torus.

5.3 **Good modules**

As Section 5.2 explains, to obtain an analytic analogue of Fact 5.2.0.4, it is necessary to find a substitute for quasi-coherence on complex manifolds. We show that goodness introduced by Kashiwara (Definition A.1.4.1) can be used as such.

5.3.1 **Functoriality**

In Section 5.3.1, we prove in Corollary 5.3.1.14 that goodness is preserved by integral transforms. To prove this, we show that goodness is preserved by the operations involved in (5.1).

**Example 5.3.1.1.** [Har66, Example 1., p.68] Let \( f : X \rightarrow Y \) be a morphism of ringed spaces. Then the functor \( Lf^* : D(Y) \rightarrow D(X) \) is bounded above and \( Rf_* : D(X) \rightarrow D(Y) \) is bounded below.
The weak dimension \( \text{wgd}(R) \) of a commutative ring \( R \) is defined to be the supremum of flat dimension of all \( R \)-modules.

**Lemma 5.3.1.2** (Serre). Let \( R \) be a commutative Noetherian regular local ring. Then \( \text{wgd}(R) \) coincides with the Krull dimension of \( R \), hence finite.

*Proof.* From [Osb12, Cor. 4.21], the weak dimension coincides with the global dimension of \( R \). By Serre’s theorem (see, e.g., [Osb12, p.332]), the global dimension equals the Krull dimension, which is finite. \( \square \)

Let \( \text{Ch}(\text{Mod}(O_X)) \) be the category of chain complexes over \( \text{Mod}(O_X) \).

**Proposition 5.3.1.3** (Pullback). Let \( f : X \to Y \) be a morphism of complex analytic spaces. Then the derived pullback (constructed in [Spa88, Prop. 6.7 (a)] and [Sta23, Tag 06YI]) \( Lf^* : D(Y) \to D(X) \) restricts to a functor

1. \( D^b_c(Y) \to D^b_c(X) \) when \( Y \) is a complex manifold or \( f \) is flat;

2. \( D_{\text{good}}(Y) \to D_{\text{good}}(X) \).

*Proof.*

1. Because \( Y \) is smooth or \( f \) is flat, by Lemma 5.3.1.4, the morphism \( f \) has finite tor-dimension. Thus, \( Lf^* \) restricts to a functor \( D^b_c(Y) \to D^b_c(X) \).

Consider \( F \in D^b_c(Y) \). To prove that \( Lf^*F \in D^b_c(X) \), by [Har66, I, Prop. 7.3 (i)], one may assume \( F \in \text{Coh}(Y) \). This case is proved by Lemma A.1.3.3.

2. Let \( G \in D_{\text{good}}(Y) \). By Lemma A.1.4.3 and a dual of [Har66, Prop. 7.3 (ii)], to prove \( Lf^*G \in D_{\text{good}}(X) \), one may assume \( G \in \text{Good}(Y) \). Let \( U \) be a relatively compact open subset of \( X \). Then \( f(U) \) is compact subset of \( Y \), so contained in a relatively compact open subset \( V \) of \( Y \). Since \( G \) is good, its restriction \( G|_V = \sum_{i \in I} G_i \) is the sum of a directed family of coherent \( O_V \)-submodules of \( G|_V \). Let \( g : f^{-1}(V) \to V \) be the restriction of \( f \). As \( Lf^* \) commutes with colimits, one has

\[
(Lf^*G)|_{f^{-1}(V)} = \text{colim}_i Lg^*G_i.
\]

For every integer \( n \), in \( \text{Mod}(O_{f^{-1}(V)}) \) one has

\[
H^n(Lf^*G)|_{f^{-1}(V)} = H^n([Lg^*G]|_{f^{-1}(V)})
\]

\[
= H^n(\text{colim}_i Lg^*G_i) = \text{colim}_i H^n(Lg^*G_i).
\]

Since \( G_i \) is coherent and \( g \) is a morphism of complex analytic spaces, by Lemma A.1.3.3, the \( O_{f^{-1}(V)} \)-module \( H^n(Lg^*G_i) \) is coherent. By Lemma A.1.4.3, the \( O_{f^{-1}(V)} \)-module \( H^n(Lf^*G)|_{f^{-1}(V)} \) is good. Since \( U \) is a compact subset of \( f^{-1}(V) \), the subset \( U \) is relatively compact in \( f^{-1}(V) \). Hence, \( H^n(Lf^*G)|_U \) is the sum of a directed family of coherent submodules. This proves \( Lf^*G \in D_{\text{good}}(X) \).
Then consider the general case $C \in D_{\text{good}}(Y)$. For every integer $m \geq 0$, the $m$-th canonical truncation ([Spa88, Tag 0118 (4)]) $C_m := \tau_{\leq m}C$ is in $D_{\text{good}}(Y)$. From the proof of [Lip60, Prop. 2.5.5], there is a bounded above complex of flat $O_Y$-modules $Q_m$ with a quasi-isomorphism $Q_m \to C_m$ that is functorial in $C_m$. Moreover, the complex $Q := \text{colim}_m Q_m$ is K-flat (in the sense of [Spa88, Def. 5.1]) and the canonical morphism $Q \to C$ is a quasi-isomorphism. Because $Lf^* : D(Y) \to D(X)$ admits a right adjoint, it commutes with colimits. Thus, the resulting morphisms

$$\text{colim}_m Lf^* Q_m \to Lf^* Q \to Lf^* C$$

are isomorphisms in $D(X)$. The directed set $\mathbb{N}$ can be seen naturally as a category. Define a functor

$$\mathbb{N} \to \text{Ch}(\text{Mod}(O_X)), m \mapsto f^* Q_m.$$ 

Because $\text{Mod}(O_X)$ is a Grothendieck abelian category, for every integer $n$, by [Hov01, Lem. 1.5], the natural morphism

$$\text{colim}_m H^n(f^* Q_m) \to H^n(\text{colim}_m f^* Q_m)$$

in $\text{Mod}(O_X)$ is an isomorphism. Hence an isomorphism $H^n(Lf^* C) \cong \text{colim}_m H^n(Lf^* Q_m)$ in $\text{Mod}(O_X)$. Since $Q_m \in D_{\text{good}}(Y)$, the $O_X$-module $H^n(Lf^* Q_m)$ is good. By Lemma A.1.4.33, so is the $O_X$-module $H^n(Lf^* C)$.

\hfill \Box

The tor-dimension tor-dim $f$ of a morphism $f : X \to Y$ of ringed spaces is defined to be the lower dimension (in the sense of [Lip60, 1.11.1]) of the functor $Lf^* : D^{-}(Y) \to D(X)$. If $f$ is flat, then tor-dim $f = 0$. If $f$ has finite tor-dimension, then $Lf^* : D^{-}(Y) \to D(X)$ restricts to a functor $D^b(Y) \to D^b(X)$.

**Lemma 5.3.1.4.** Let $f : X \to Y$ be a morphism of complex analytic spaces, with $Y$ a complex manifold. Then $f$ has finite tor-dimension.

**Proof.** From [Lip60, 2.7.6.4], one only needs to show that for every $x \in X$, the flat dimension of the $O_{Y,f(x)}$-module $O_{X,x}$ is uniformly bounded. By definition, the flat dimension of every $O_{Y,f(x)}$-module is bounded by the weak dimension of the ring $O_{Y,f(x)}$. Because $Y$ is a complex manifold, the local ring $O_{Y,f(x)}$ is Noetherian regular. By Lemma 5.3.1.2, $\text{wgd}O_{Y,f(x)}$ is the Krull dimension of $O_{Y,f(x)}$, which coincides with the dimension of the complex manifold $Y$ near $x$. \hfill \Box

**Proposition 5.3.1.5** (Tensor product). Let $X$ be a complex analytic space. Then the bifunctor (constructed in [Spa88, Thm A. (ii)]) $\otimes^L : D(X) \times D(X) \to D(X)$ restricts to a bifunctor

1. $D^b(X) \times D^b(X) \to D^b(X)$ (resp. $D^b_c(X) \times D^b_c(X) \to D^b_c(X)$) when $X$ is a complex manifold;
2. $D_{\text{good}}(X) \times D_{\text{good}}(X) \rightarrow D_{\text{good}}(X)$.

Proof:

1. The weak dimension of a ringed space $(M, O_M)$ is defined to be $\sup_{x \in M} \text{wgld}(O_{M,x})$.

By [HT07, (C.2.20)], to prove the preservation of $D^b(X)$, it suffices to bound the weak dimension of $X$. As $X$ is smooth, for every $x \in X$, the stalk $O_{X,x}$ is a Noetherian regular local ring. Thus, by Lemma 5.3.1.2, its weak dimension $\text{wgld}(O_{X,x})$ is equal to the Krull dimension of $O_{X,x}$. They coincide with the dimension of the complex manifold $X$ near $x$. Therefore, the weak dimension of $X$ is not greater than $\dim X$.

Consider any $F, G \in D^b(X)$. To prove that $F \otimes^L G \in D^b(X)$, by [Har66, I, Prop. 7.3 (i)], one may assume $F, G \in \text{Coh}(X)$. Then the conclusion follows from [GH78, 4., p.700].

2. Take $F, G \in D_{\text{good}}(X)$. To prove that $F \otimes^L G \in D_{\text{good}}(X)$, as in the proof of Proposition 5.3.1.3 2, one may assume that $F, G \in \text{Good}(X)$. By a dual of [Har66, I, Prop. 7.3 (ii)], one may assume that $F, G \in \text{Good}(X)$. Let $U$ be a relatively compact open subset of $X$.

For every integer $n$, we claim that the $O_U$-module $H^n(F \otimes^L_X G)|_U$ is good. By assumption, the restrictions can be written as sum of directed family of coherent submodules: $F|_U = \sum_{i \in I} F_i$ and $G|_U = \sum_{j \in J} G_j$. By [Sta23, Tag 08DJ], the functor

$$\otimes^L_{O_U}(G|_U) : D(U) \rightarrow D(U)$$

has a right adjoint, so

$$(F \otimes^L G)|_U = \colim_i [F_i \otimes^L (G|_U)]. \quad (5.6)$$

By [Sta23, Tag 05NI (2)], there exists a complex $C^*$ of flat $O_U$-modules and a quasi-isomorphism $C^* \rightarrow G|_U$. Then in $D(U)$

$$F_i \otimes_{O_U} C^* = F_i \otimes^L_{O_U} G|_U. \quad (5.7)$$

Define a functor $I \rightarrow \text{Ch(Mod}(O_X))$ by $i \mapsto F_i \otimes C^*$. By [Hov01, Lem. 1.5], the natural morphism

$$\colim_i H^n(F_i \otimes C^*) \rightarrow H^n(\colim_i (F_i \otimes C^*))$$

in $\text{Mod}(O_U)$ is an isomorphism. Combining it with (5.6) and (5.7), one gets an isomorphism in $\text{Mod}(O_U)$

$$\colim_i H^n(F_i \otimes_{O_U} G|_U) \rightarrow H^n(F \otimes^L_X G)|_U.$$

Because $\text{Good}(U)$ is closed under colimits in $\text{Mod}(O_U)$ by Lemma A.1.4.3 3, one may assume that $F|_U$ is coherent. Similarly, one may assume further that $G|_U$ is coherent. Then the claim follows from Lemma A.1.3.4.
Remark 5.3.1.6. Proposition 5.3.1.5 2 can also be derived from Proposition 5.3.1.3 2 as in the proof of [Bjö93, Thm. 3.2.13 (3)].

As the proof of Theorem 5.3.1.7 is lengthy, we split it into a series of lemmas.

**Theorem 5.3.1.7** (Pushout). Let \( f : X \to Y \) be a proper morphism of complex analytic spaces. If \( \dim X \) is finite, then \( Rf_* : D(X) \to D(Y) \) restricts to a functor \( D_{\text{good}}(X) \to D_{\text{good}}(Y) \) (resp. \( D^b_{\text{good}}(X) \to D^b_{\text{good}}(Y) \)).

**Proof.** By Lemma 5.3.1.11, the functor \( Rf_* \) restricts to a functor \( D^b(X) \to D^b(Y) \). We show that \( Rf_* F \in D_{\text{good}}(Y) \) for every \( F \in D_{\text{good}}(X) \). By [Har66, I, Prop. 7.3 (iii)] and Lemma A.1.4.3 3, one may assume that \( F \in \text{Good}(X) \).

For every relatively compact open subset \( V \subset Y \), its closure \( \bar{V} \) is compact in \( Y \). As \( f \) is proper, the preimage \( f^{-1}(\bar{V}) \) is compact. Thus, \( U := f^{-1}(V) \) is a relatively compact open subset of \( X \). Since \( F \) is good, \( F|_U = \text{colim}_{i \in I} F_i \), where \( \{F_i\}_{i \in I} \) is a directed family of coherent \( O_U \)-submodules of \( F|_U \). Let \( g : U \to V \) be the base change of \( f \). Fix an integer \( q \). By Lemma 5.3.1.9, in \( \text{Mod}(O_Y) \)

\[
(R^q f_* F)|_V = R^q g_* (F|_U) = \text{colim}_i R^q g_* F_i.
\]

As a base change of \( f \), the morphism \( g \) is proper. Then by Fact 5.3.1.8, for every \( i \in I \), the \( O_Y \)-module \( R^q g_* F_i \) is coherent. By \( \text{Coh}(V) \subset \text{Good}(V) \) and Lemma A.1.4.3 3, the \( O_Y \)-module \( (R^q f_* F)|_V \) is good.

**Fact 5.3.1.8** (Grauert direct image theorem, [GR84, p.207]). Let \( f : X \to Y \) be a proper morphism of complex analytic spaces. Then \( Rf_* : D(X) \to D(Y) \) restricts to a functor \( \text{Coh}(X) \to D_c(Y) \).

**Lemma 5.3.1.9.** Let \( f : X \to Y \) be a proper map between Hausdorff locally compact spaces. Then for every integer \( n \geq 0 \), the functor \( R^n f_* : \text{Ab}(X) \to \text{Ab}(Y) \) commutes with filtrant colimits.

**Proof.** Let \( (F_i, f_{ij}) \) be a filtrant inductive system with inductive limit \( F \) in \( \text{Ab}(X) \). Since the abelian category \( \text{Ab}(Y) \) is Grothendieck, the filtrant inductive limit \( G = \text{colim} R^n f_* F_i \) exists and there is a canonical morphism \( \phi : G \to R^n f_* F \) in \( \text{Ab}(Y) \). For every \( y \in Y \), the functor \( \text{Ab}(Y) \to \text{Ab} \) taking the stalk at \( y \) commutes with colimits, so \( G_y = \text{colim}_i (R^n f_* F_i)_y \). By [Mil13, Thm. 17.2], for every \( i \) the stalk \( (R^n f_* F_i)_y = H^n(X_y, F_i|_{X_y}) \). By [God58, Thm. 4.12.1], the morphism \( \phi_y : G_y \to (R^n f_* F)_y \) is an isomorphism. Therefore, \( \phi \) is an isomorphism.

The proof of Fact 5.3.1.10 is similar to that of [KS13, Prop. 3.2.2].

**Fact 5.3.1.10.** Let \( X \) be a locally compact Hausdorff topological space which is countable at infinity. Suppose that there is an integer \( n \geq 0 \) such that every point of \( X \) has an open neighborhood homeomorphic to a locally closed subset of \( \mathbb{R}^n \). Then for every abelian sheaf \( F \) on \( X \) and every integer \( j > n \), one has \( H^j(X, F) = 0 \).
Lemma 5.3.1.11. Let $X$ be a complex analytic space of finite dimension $n$. Let $f : X \to Y$ be a proper morphism of complex analytic spaces. Then for an object $E \in D(X)$ with $H^m(E) = 0$ for every integer $m > 0$, one has $H^i(Rf_*E) = 0$ for every integer $i > 2n$. In particular, the functor $Rf_* : D(X) \to D(Y)$ is bounded.

**Proof.** For every open subset $V \subset Y$, every integer $i > 2n$ and every $O_Y$-module $M$, from Fact 5.3.1.10, $H^i(f^{-1}(V), M) = 0$. Applying Lemma 5.3.1.13 to the functor $\Gamma(f^{-1}(V), -) : \text{Mod}(O_X) \to \text{Ab}$, one gets

$$H^i(\Gamma(f^{-1}(V), E)) = H^i(\Gamma(f^{-1}(V), \tau_{\geq 1} E)) = 0.$$ 

By Lemma 5.3.1.12, the $O_Y$-module $H^i(Rf_*E) = 0$. \hfill $\square$

Lemma 5.3.1.12 is a derived version of [Har77, III, Prop. 8.1].

Lemma 5.3.1.12. Let $f : X \to Y$ be a continuous map of topological spaces.

Then for every integer $i$ and every $F \in D(\text{Ab}(\text{Ab}(X)))$, the sheaf $H^i(Rf_*F)$ on $Y$ is the sheaf associated to the abelian presheaf $V \mapsto H^i\Gamma(f^{-1}(V), F)$.

**Proof.** By [Spa88, Thm. D, p.125], there is a quasi-isomorphism $F \to I$, where $I$ is a K-injective complex of abelian sheaves on $X$. Then the canonical morphism $Rf_*F \to f_*I$ is an isomorphism in $D(\text{Ab}(Y))$. By [Mur06, Lem. 3], $H^i(Rf_*F)$ is the sheaf associated to the presheaf

$$V \mapsto H^i\Gamma(V, f_*I) = H^i\Gamma(f^{-1}(V), I) = H^i\Gamma(f^{-1}(V), F).$$

\hfill $\square$

Lemma 5.3.1.13. Let $X$ be a ringed space as in Fact 5.3.1.10. Let $F : \text{Mod}(O_X) \to \text{Ab}$ be an additive functor. Assume that $F$ commutes with countable products, and there is an integer $N \geq 0$ with $R^p F(M) = 0$ for every integer $p \geq N$ and every $M \in \text{Mod}(O_X)$. Then the right derived functor $RF : D(X) \to D(\text{Ab})$ exists. Moreover, for every $E \in D(X)$ and any integers $i \geq j$, the canonical morphism

$$H^i(RF(E)) \to H^i(RF(\tau_{\geq j - N + 1} E))$$

is an isomorphism.

**Proof.** The existence of $N$ and [Wei95, Cor. 10.5.11] show that $RF : D^+(X) \to D^+(\text{Ab})$ extends to a right derived functor $RF : D(X) \to D(\text{Ab})$.

For every integer $m$, set $E_m := \tau_{> - m} E$. Then $\{E_m\}_{m \in \mathbb{Z}}$ forms an inverse system in $D(X)$. Let $n$ be as in Fact 5.3.1.10. Then for every open subset $U \subset X$, any integers $p(> n)$ and $q$, one has $H^p(U, H^q(E)) = 0$. Then by [Sta23, Tag 0D64], the canonical morphism $E \to R\text{lim}_m E_m$ is an isomorphism in $D(X)$. Since $F$ commutes with countable products, from [Sta23, Tag 08U1],

\footnote{Because $\text{Mod}(O_X)$ is a Grothendieck abelian category, it has enough injectives. Then by [Ver66, p.338], the total right derived functor $RF : D^+(X) \to D^+(\text{Ab})$ exists.}
in $D(Ab)$ one has $RF(E) = \lim_m R^i RF(E_m)$. For every integer $i$, by [Sta23, Tag 08U5], there is a short exact sequence in $Ab$

$$0 \to R^1 \lim_m H^{i-1}(RF(E_m)) \to H^i(RF(E)) \to \lim_m H^i(RF(E_m)) \to 0. \quad (5.8)$$

we claim that $R^1 \lim_m H^{i-1}(RF(E_m)) = 0$.

For every integer $m \geq N - i$, by [Sta23, Tag 08J5], there is an exact triangle

$$H^{-m}(E)[m] \to E_m \to E_{m-1} \to H^{-m}(E)[m+1] \quad (5.9)$$

in $D(X)$. By assumption, one has

$$H^i(RF(H^{-m}(E)[m])) = R^{i+m}F((H^{-m}(E)) = 0; \quad H^i(RF(H^{-m}(E)[m+1])) = R^{i+m+1}F((H^{-m}(E)) = 0.$$

Taking the long exact sequence associated to (5.9), one concludes that the canonical morphism $H^i(RF(E_m)) \to H^i(RF(E_{m-1}))$ in $Ab$ is an isomorphism.

Since the inverse system $\{H^i RF(E_m)\}_{m \geq 1}$ is constant starting with $m = N - i - 1$, it satisfies the Mittag-Leffler condition in the sense of [Sta23, Tag 02N0]. From [Sta23, Tag 07KW (3)], one obtains

$$R^1 \lim_m H^i(RF(E_m)) = 0,$$

which proves the claim.

When $i \geq j$, one has $\lim_m H^i(RF(E_m)) = H^i[RF(E_{m-1})]$ as the inverse system is constant from $m = N - j - 1$. Then the sequence (5.8) induces an isomorphism $H^i(RF(E)) \to H^i(RF(\tau_{\geq j-N+1}E)).$

\[\square\]

**Corollary 5.3.1.14.** Let $X, Y$ be complex manifolds (resp. complex analytic spaces), where $X$ is compact. If $F$ is an object of $D^b_c(X \times Y)$ (resp. $D_{good}(X \times Y)$), then $g^{[X \to Y]} \circ f$ restricts to a functor $D^b_c(X) \to D^b_c(Y)$ (resp. $D_{good}(X) \to D_{good}(Y)$).

**Proof.** It is a combination of Proposition 5.3.1.3 Point 1 (resp. Point 2), Proposition 5.3.1.5 1 (resp. 2) and Fact 5.3.1.8 (resp. Theorem 5.3.1.7). \[\square\]

**Remark 5.3.1.15.** Although we don’t need the functors $R\text{Hom}, f_!$ and $f^!$, it is interesting to know whether they preserve goodness or not.

### 5.3.2 Smooth base change

Theorem 5.3.2.3 together with Fact 5.3.2.2 gives an analytic smooth base change result, as a replacement for the (algebraic) flat base change theorem used in Mukai’s proof of Fact 5.2.0.4.

Consider a cartesian square in the category $\text{An}$

$$X' \xrightarrow{g'} X \xrightarrow{f} S' \xrightarrow{g} S. \quad (5.10)$$
Then [Sta23, Tag 08HY] gives a natural transformation of functors $D(X) \to D(S')$
\[ Lg^* Rf_* \to Rf'_* Lg'^*, \] (5.11)
coming from the adjunction in [Sta23, Tag 079W].

**Definition 5.3.2.1.** A morphism $g : S' \to S$ is called locally product, if for every $s' \in S'$, there is an open neighborhood $U$ of $s' \in S'$ and a complex analytic space $Z$, such that $g(U)$ is open in $S$ and there is a $g(U)$-isomorphism $U \to g(U) \times Z$.

By [CD94, II, Cor. 2.7], a locally product morphism is flat.

**Fact 5.3.2.2.** [Gro61b, Thm. 3.1] A morphism of complex analytic spaces is smooth (in the sense of [Gro61b, Déf. 3.2]) if and only if it is a submersion (in the sense of [Fis06, p.100]). In particular, a smooth morphism is locally product.

**Theorem 5.3.2.3.** Consider the square (5.10) with both $\dim X$ and $\dim X'$ finite, $f : X \to S$ proper and $g : S' \to S$ locally product. Then (5.11) induces an isomorphism of functors $D_{\text{good}}(X) \to D_{\text{good}}(S')$.

**Definition 5.3.2.4.** A morphism of complex analytic spaces $g : S' \to S$ is said to satisfy property $Q_S$ if for every proper morphism $f : X \to S$ of complex analytic spaces, every coherent $O_X$-module $F$ and every integer $i \geq 0$, the base change morphism $g^* R^i f_* F \to R^i f'_*(g'^* F)$ induced by (5.10) is an isomorphism in $\text{Mod}(O_{S'})$.

Lemma 5.3.2.5 shows that the property $Q$ is local on the source and the target.

**Lemma 5.3.2.5.** Let $g : S' \to S$ and be a morphism of complex analytic spaces.

1. Let $h : S'' \to S'$ be another morphism of complex analytic spaces. If $g$ and $h$ satisfy $Q_S$ and $Q_{S'}$ respectively, then $gh$ satisfies $Q_S$.

2. Assume that $\{S'_{i} \}_{i \in I}$ (resp. $\{S_{j} \}_{j \in J}$) is an open covering of $S'$ (resp. $S$) such that for every $i \in I$ (resp. $j \in J$), the morphism $g|_{S_i} : S'_i \to S$ (resp. $g^{-1}(S_j) \to S_j$) satisfies $Q_S$ (resp. $Q_{S_j}$). Then $g$ satisfies $Q_S$.

3. If $g$ is an open embedding of complex analytic spaces, then $g$ satisfies $Q_S$.

**Proof.**

1. The proof is similar to that of [Day23, Lem. 2.13 (2)].

2. It follows from the local nature of sheaves.

3. The proof is similar to that of [Har77, Cor. 8.2, p.251].

We begin with the special case of products (Corollary 5.3.2.9).
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**Lemma 5.3.2.6.** Let \( f : X \to S \) be a proper morphism of complex analytic spaces, with \( S \) Stein. Then for every coherent \( O_X \)-module \( F \) and every integer \( n \geq 0 \), one has \( H^n(X, F) = H^0(S, R^n f_* F) \).

**Proof.** By properness of \( f \) and Fact 5.3.1.8, the \( O_S \)-module \( R^n f_* F \) is coherent. As \( S \) is Stein, from Cartan’s Theorem B (see, e.g., [KK11, Sec. 52, Thm. B]), for every integer \( m > 0 \) one has \( H^m(S, R^n f_* F) = 0 \). The conclusion follows from [Sta23, Tag 01F4 (2)]. \( \square \)

**Remark 5.3.2.7.** As an application of Lemma 5.3.2.6, we give an enhancement of Lemma 5.3.1.11 for good modules. Let \( f : X \to Y \) be a proper morphism of complex analytic spaces with \( \dim X \) finite. Then for every good \( O_X \)-module \( G \) and every integer \( n > \dim X \), one has \( R^n f_* G = 0 \).

Assume first that \( G \) is coherent. For every Stein open subset \( V \subset Y \), from Cartan’s Theorem A (see, e.g., [GR13, Theorem A, p.XVIII]), the restriction \( R^n f_* G|_V \) is generated by sections \( H^0(V, R^n f_* G|_V) \). By Lemma 5.3.2.6, one has

\[
H^0(V, R^n f_* G|_V) = H^n(f^{-1}(V), G|_{f^{-1}(V)}),
\]
which vanishes by [Rei64, Cor. p.2333]. Thus, \( R^n f_* G|_V = 0 \). Therefore, \( R^n f_* G = 0 \).

Assume now that \( G \in \text{Good}(X) \) is arbitrary. For every relatively compact open subset \( W \subset Y \), the open subset \( f^{-1}(W) \) of \( X \) is relatively compact. Then there is a directed family of coherent submodules \( \{G_i\}_{i \in I} \) of \( G|_{f^{-1}(W)} \) such that \( G|_{f^{-1}(W)} = \colim G_i \). By Lemma 5.3.1.9, one gets \( (R^n f_* G)|_W = \colim (R^n f_* G)|_{f^{-1}(W)} \). Thus, \( R^n f_* G = 0 \).

**Lemma 5.3.2.8.** Let \( X, Y \) be complex analytic spaces, with \( Y \) Stein. Let \( p : X \times Y \to X \) be the projection. Then for every coherent \( O_X \)-module \( F \) and every integer \( i \geq 0 \), the natural morphism \( H^i(X, F) \otimes_{O_Y} O_Y \to H^i(X \times Y, p^* F) \) of locally convex topological vector spaces is an isomorphism.

**Proof.** Choose a Stein covering \( \mathcal{U} \) of \( X \). Let \( C^* \) be the corresponding Čech complex of \( F \). Then \( H^i(C^*) = H^i(X, F) \). For every integer \( q \), the \( q \)-th term \( C^q \) of the complex \( C^* \) is a Fréchet space by [EP+96, Prop. 4.1.5]. Moreover, \( \{U \times Y : U \in \mathcal{U}\} \) forms a Stein covering of \( X \times Y \). By [EP+96, Prop. 4.2.3; Thm. 4.2.4], the Čech complex corresponding to this Stein covering and \( p^* F \) is \( C^* \otimes_{O_Y} O_Y \). Therefore, \( H^i(C^* \otimes_{O_Y} O_Y) = H^i(X \times Y, p^* F) \). By [EP+96, Prop. 4.1.5], \( O(Y) \) is a unital Fréchet nuclear algebra, so from [EP+96, Thm. A1.6 (d)], the functor \( \otimes_{O_Y} O_Y \) preserves exact sequences, hence commutes with taking cohomology groups of the Čech complexes. \( \square \)

**Corollary 5.3.2.9.** Let \( S, Z \) be two complex analytic spaces. Then the projection \( S \times Z \to S \) satisfies \( Q_S \).

**Proof.** Fix a proper morphism \( X \to S \) of complex analytic spaces and a coherent \( O_X \)-module \( F \). By Lemma 5.3.2.5, we may assume that \( S, Z \) are Stein spaces. Then the result follows from Lemma 5.3.2.6, Lemma 5.3.2.8 and [EP+96, Prop. 4.2.3; Thm. 4.2.4]. \( \square \)
Corollary 5.3.2.10. Every locally product morphism \( g : S' \to S \) of complex analytic spaces satisfies \( Q_S \).

Proof. Fix \( s' \in S' \), and let \( s = g(s') \). Since \( g \) is locally product, there are open neighborhoods \( U \) and \( V \) of \( s' \in S' \) and \( s \in S \) respectively, a complex analytic space \( Z \) and an isomorphism \( \psi : U \to Z \times V \) of complex analytic spaces such that the diagram

\[
\begin{array}{ccc}
U & \xrightarrow{\psi} & Z \times V \\
\downarrow^{g|U} & & \downarrow^{p_2} \\
V & \leftarrow & \\
\end{array}
\]

commutes, where \( p_2 \) is the projection to the second factor. By Corollary 5.3.2.9, \( p|U : U \to V \) satisfies \( Q_V \). By Lemma 5.3.2.5, the morphism \( g : S' \to S \) satisfies \( Q_S \). \qed

Proof of Theorem 5.3.2.3. The morphism \( f' \) is a base change of \( f \), hence a proper morphism. Because \( \dim X, \dim X' \) are finite, by Theorem 3.3.17 and Proposition 3.3.13, the functors \( Lg^*Rf_* \) and \( Rf'_*Lg^* \) restrict to functors \( D_{\text{good}}(X) \to D_{\text{good}}(S') \).

For every \( K \in D_{\text{good}}(X) \), we prove that the base change morphism \( Lg^*Rf_*K \to Rf'_*Lg^*K \) in \( D(S') \) is an isomorphism. By Lemma 5.3.1.11, the functors \( Rf_* : D(X) \to D(S) \) and \( Rf'_* : D(X') \to D(S') \) are bounded. From [Har66, I, Prop. 7.1 (iii)] and Lemma A.1.4.3, one may assume that \( K \in \text{Good}(X) \). For every \( s' \in S' \), there is a relatively compact open subset \( V \subset S \) of \( g(s') \). The preimage \( f^{-1}(V) \) is a relatively compact open subset of \( X \). Let \( u : f^{-1}(V) \to V \) (resp. \( v : g^{-1}(V) \to V \)) be the restriction of \( f \) (resp. \( g \)), and let \( u' \) (resp. \( v' \)) be the base change of \( u \) (resp. \( v \)). Because \( g \) is locally product, so is \( v \). Then one can write \( K|_{f^{-1}(V)} = \text{colim}_i G_i \), where \( \{ G_i \}_{i \in I} \) is a directed family of coherent submodules of \( K|_{f^{-1}(V)} \). By Lemma 5.3.1.9, the natural morphism

\[
(g^*R^if_*K)|_{g^{-1}(V)} \to R^if'_*(g'^*K)|_{g^{-1}(V)}
\]

in \( \text{Mod}(O_{g^{-1}(V)}) \) is the colimit of the morphisms

\[
v^*R^iu_*G_i \to R^iu'_*v'^*G_i,
\]

which for every \( i \in I \) is an isomorphism by Corollary 5.3.2.10. Then (5.12) is an isomorphism. \qed

Remark 5.3.2.11. In the proof of [BBR94, Lem. 5], an analytic flat base change result is applied without further justification. In [MS08, p.153], a flat base change theorem for cartesian squares in the category of complex manifolds is stated, referring to [Spa88] for the proof. However, the cited result [Spa88, Prop. 6.20] is for cartesian squares in the category \( \text{RingS} \). In general, a cartesian square in the category of complex manifolds is not cartesian in \( \text{RingS} \). For
example, the complex vector space $\mathbb{C}^2$ is the product of two copies of $\mathbb{C}$ in the category of complex manifolds, but is not the product even in the subcategory $\text{LRS} \subset \text{RingS}$ of locally ringed spaces.\footnote{By contrast, every cartesian square in the category of schemes remains cartesian in LRS [Sta23, Tag 01JN].}

In fact, the product $E$ of two copies of $\mathbb{C}$ in LRS exists by [Gil11, Cor. 5]. By the universal property of $E$, there is a unique morphism $f : \mathbb{C}^2 \to E$ in LRS induced by the two projections $p_i : \mathbb{C}^2 \to \mathbb{C}$. Let $a = f(0) \in E$. We claim that the local ring $O_{E,a}$ is not Noetherian.

Note that $A := O_{\mathbb{C},0} = \mathbb{C}(z)$ is the ring of convergent power series. Let $B = A \otimes_\mathbb{C} A$. Let $\epsilon : B \to A$ be the surjective (diagonal) morphism defined by $\epsilon(f \otimes g) = f g$ and $I = \ker(\epsilon)$. Let $c : A \to \mathbb{C}$ be the ring map taking the constant term. Then $c : B \to \mathbb{C}$ is surjective, so $m = \ker(\epsilon c)$ is a maximal ideal of $B$ containing $I$. With this notation, $O_{E,a} = B_m = S^{-1}B$, where $S = B \setminus m$. From [Tu97, p.367], $I/I^2$ is a free $B/I$-module of infinite rank. Thus, $S^{-1}(I/I^2) = (S^{-1}I)/(S^{-1}I^2)$ is a free $S^{-1}(B/I) = (S^{-1}B)/(S^{-1}I)$-module of infinite rank. In particular, the ideal $S^{-1}I$ of $S^{-1}B$ is not finitely generated. The claim is proved.

By [GH78, p.679], the ring $\mathbb{C}[x, y]$ is Noetherian. Thus, the local morphism $f_0^* : O_{E,a} \to O_{\mathbb{C},0} = \mathbb{C}[x, y]$ is not an isomorphism. Hence, $f$ is not an isomorphism in LRS.

**Remark 5.3.2.12.** A base change theorem for algebraic varieties may not have a direct generalization to complex analytic spaces. For example, the affine base change theorem [Sta23, Tag 02KG] fails for morphism of Stein manifolds. In the cartesian square (5.10), assume that $S = \text{Specan} \mathbb{C}$ is a point, $X = \mathbb{C}$ and $S$ is a positive-dimensional complex manifold. Then there is an open subset $U \subset S'$ isomorphic to an open ball in $\mathbb{C}^n$ with $n > 0$. On the one hand, by Cartan's Theorem B, $Rf_*O_X = f_*O_X = O_{\mathbb{C}}(\mathbb{C})$. Thus, $g^*Rf_*O_X$ is a free $O_{S'}$-module of infinite rank $\dim_{\mathbb{C}} O_{\mathbb{C}}(\mathbb{C})$. From Corollary A.1.5.4, $\Gamma(U, g^*Rf_*O_X) = O_{S'}(U) \otimes_{\mathbb{C}} O_{\mathbb{C}}(\mathbb{C})$.

On the other hand, one has $f^{-1}(U) = U \times \mathbb{C}$ and $g^*O_X = O_X$, so

$$\Gamma(U, f^\#_1g^*O_X) = \Gamma(f^{-1}(U), O_{X'})$$

\[= \Gamma(U \times \mathbb{C}, O_{U \times \mathbb{C}}) = O_U(U) \otimes_{\mathbb{C}} O_{\mathbb{C}}(\mathbb{C}), \] (a)

where (a) uses [EP+06, p.75]. The morphism $\Gamma(U, g^*Rf_*O_X) \to \Gamma(U, f^\#_1g^*O_X)$ is not an isomorphism, so the base change morphism $g^*f_*O_X \to f^\#_1g^*O_X$ is not an isomorphism.

**Lemma 5.3.2.13 (Base change).** Consider the cartesian square (5.10 with $\dim X, \dim S'$ finite and $f$ flat proper. Then (5.11) induces an isomorphism $Lg^*Rf_* \to Rf'_*Lg'^*$ of functors $D_{\text{good}}(X) \to D_{\text{good}}(S')$.

**Proof.** By Theorem 5.3.1.7 and Proposition 5.3.1.3 2, the functor $Lg^*Rf_* : D(X) \to D(S')$ restricts to a functor $D_{\text{good}}(X) \to D_{\text{good}}(S')$. Consider the following commutative diagram

$$\begin{array}{ccc}
Lg^*Rf_*
& \longrightarrow
& Rf'_*Lg'^*
\\
\downarrow
& & \downarrow
\\
D(X)
& \longrightarrow
& D(S')
\end{array}$$
Lemma 5.3.2.14. In the cartesian square (5.10), assume that $i$ is the closed embedding of an analytic subspace. Consider functors $D_{\text{good}}(X) \rightarrow D(S')$. Because $\varphi$ is locally proper, by Theorem 5.3.2.3, the natural transformation $Lp^*RF_* \rightarrow R(\text{Id}_{S'} \times f)_*Lp^*$ is an isomorphism. Because $f$ is flat proper, so is $\text{Id}_{S'} \times f$. Moreover, $\dim S' \times X = \dim S' + \dim X$ is finite. Thus, there is an isomorphism

$$Lg^*RF_* = Li^*Lp^*RF_* \sim Rf'_*Lg'^*,$$

where the isomorphism (a) uses Lemma 5.3.2.14. By [Sta23, Tag 0E47], the isomorphism (5.13) of functors $D_{\text{good}}(X) \rightarrow D_{\text{good}}(S')$ is induced by (5.11).

Theorem 5.3.2.14. In the cartesian square (5.10), assume that $g$ is the closed embedding of an analytic subspace. Then:

1. The base change morphism $f^*g_*O_{S'} \rightarrow g'_*O_X^*$ in $\text{Mod}(O_X)$ is an isomorphism.

2. If $f$ is flat proper and $X$ has finite dimension, then (5.11) is an isomorphism.

Proof. 1. Let $I$ be the kernel of the canonical surjection $O_S \rightarrow g_*O_{S'}$ in $\text{Mod}(O_S)$. Since $f^* : \text{Mod}(O_S) \rightarrow \text{Mod}(O_X)$ is left exact, the sequence

$$f^*I \rightarrow O_X \rightarrow f^*g_*O_{S'} \rightarrow 0$$

is exact in $\text{Mod}(O_X)$. Because $g$ is a closed embedding, by [Gro61a, Remarque 2.10], the square (5.10) is cartesian in the category $\text{RingS}$. From [Gro61a, 9-05], the cokernel of the morphism $f^*I \rightarrow O_X$ in $\text{Mod}(O_X)$ is $g'_*O_X^*$. Therefore, the morphism $f^*g_*O_{S'} \rightarrow g'_*O_X^*$ is an isomorphism.

2. As $g$ is a closed embedding, the functor $g_* : \text{Ab}(S') \rightarrow \text{Ab}(S)$ is exact and $i^{-1}i_* = \text{Id}_{\text{Ab}(S')}$. Therefore, the functor $Rg_* = g_* : D(S') \rightarrow D(S)$ is conservative in the sense of [Rie17, p.180]. Thus, it suffices to show that the natural transformation

$$Rg_*Lg^*RF_*E \rightarrow Rg_*Rf'_*Lg'^*E \sim Rf'_*(Rg'_*Lg'^*)E$$

of functors $D(X) \rightarrow D(S)$ is an isomorphism. By [Sta23, Tag 0B55], the natural morphisms $(Rg_*O_{S'}) \otimes_{O_X}^{L} RF_*E \rightarrow Rg_*Lg^*RF_*E$ and $(Rg'_*O_X^*) \otimes_{O_X}^{L} E \rightarrow Rg'_*Lg'^*E$ are isomorphisms. One has

$$(a) \quad Rg_*O_X^* = g'_*O_X^* = f^*g_*O_{S'},
(b) \quad f^*g_*O_{S'} = Lf^*Rg_*O_{S'},$$
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where (a) uses Point 1 and (b) uses the flatness of \( f \). Thus, the morphism (5.14) becomes the natural transformation

\[
(R_g O_{S'}) \otimes_{O_X} Rf_*(Lf^* Rg_* O_{S'}) \otimes_{O_Y} E).
\]

It is an isomorphism by the finiteness of \( \dim X \), the properness of \( f \), and Fact 5.3.2.15.

From Fact 5.3.1.10, one gets Fact 5.3.2.15 as a special case of [Spa88, Prop. 6.18]. A slight variant can also be derived from [KS13, Prop. 2.6.6] and Lemma 5.4.2.1.

**Fact 5.3.2.15** (Projection formula). *Let* \( f : X \to Y \) *be a morphism of complex analytic spaces. If* \( \dim X \) *is finite, then there is a canonical isomorphism* \( (Rf_! -) \otimes_{O_Y} + \to Rf_!( - \otimes_{O_X} Lf^* + ) \) *of bifunctors* \( D(X) \times D(Y) \to D(Y) \).

5.3.3 **GAGA**

Let \( X \) be a complex algebraic variety.

**Review**

We recall the work of Serre [Ser56] (known as “GAGA”), which gives an equivalence of algebraic coherent modules and analytic coherent modules on complex projective varieties. It is extended to proper complex algebraic varieties in [GR71, Exp. XII].

Let \( \Psi_X \) be the functor \( \text{An} \to \text{Sets} \) sending a complex analytic space \( Y \) to the set \( \text{Hom}_{\text{C}}(Y, X) \) of morphisms of spaces with a sheaf of \( \mathbb{C} \)-algebras. By [GR71, Exp. XII, Thm. 1.1], the functor \( \Psi_X \) is represented by a complex analytic space\(^5\) and a flat morphism \( \psi_X \in \text{Hom}_{\text{C}}(X^{an}, X) \), called the analytification of \( X \). Because \( X \) is of finite type over \( \mathbb{C} \), from [GR71, Exp. XII, Prop. 2.1 (viii)], the dimension of \( X^{an} \) is finite.

The pullback functor

\[
\psi_X^*: \text{Mod}(O_X) \to \text{Mod}(O_{X^{an}}), \quad F \mapsto F^{an}
\]

is exact and admits a right adjoint, so it commutes with colimits. From [GR71, Exp. XII, 1.3], it restricts to a functor \( \text{Coh}(X) \to \text{Coh}(X^{an}) \).

**Lemma 5.3.3.1.** The functor (5.15) restricts to a functor

\[
\text{Qch}(X) \to \text{Good}(X^{an})
\]

and induces a functor

\[
D_{qc}(X) \to D_{\text{good}}(X^{an}).
\]

\(^5\)Strictly speaking, complex analytic spaces are allowed to be non-Hausdorff in [GR71, Exp. XII]. In our case, \( X \) is assumed to be separated over \( \mathbb{C} \), by [GR71, Exp. XII, Prop. 3.1 (viii)], the topology of \( X^{an} \) is Hausdorff.
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Proof. For every quasi-coherent \( O_X \)-module \( F \), by Fact 5.3.3.2,
\[
F = \sum_i F_i
\]
(5.18)
is the sum of a direct family of coherent \( O_X \)-submodules. As \( \psi_X \) commutes with colimits, one has
\[
F^\text{an} = \text{colim}_i F^\text{an}_i
\]
(5.19)
in the category \( \text{Mod}(O_{X^\text{an}}) \). Since \( \psi_X \) is exact, each \( F^\text{an}_i \) is a coherent \( O_{X^\text{an}} \)-submodule. Therefore, the \( O_{X^\text{an}} \)-module \( F^\text{an} \) is good.

Fact 5.3.3.2 ([Gro60, Cor. 9.4.9], [Sta23, Tag 01PG]). Every quasi-coherent \( O_X \)-module is the sum of the directed family of all coherent submodules.

By [GR71, Exp. XII, 1.2], for every morphism \( f : X \to Y \) of complex algebraic varieties, there is a commutative square
\[
\begin{array}{ccc}
X^\text{an} & \xrightarrow{\psi_X} & X \\
\downarrow f^\text{an} & & \downarrow f \\
Y^\text{an} & \xrightarrow{\psi_Y} & Y
\end{array}
\]
(5.20)
in the category \( \text{RingS} \). In other words, the analytification induces a functor \( (-)^{\text{an}} \) from the category of complex algebraic varieties to \( \text{An} \).

GAGA for quasi-coherent modules

Using Fact 5.3.3.2 and that \( \psi_X \) commutes with colimits, we extend GAGA from coherent \( O_X \)-modules to quasi-coherent \( O_X \)-modules.

Proposition 5.3.3.3. Let \( f : X \to Y \) be a proper morphism of algebraic varieties over \( \mathbb{C} \). Then the base change natural transformation \( (Rf_* \bullet)^{\text{an}} \to Rf^{\text{an}}_* (\bullet^{\text{an}}) \) (induced by the commutative square (5.20)) induces an isomorphism of functors \( D_{\text{qc}}(X) \to D_{\text{good}}(Y^\text{an}) \).

Proof. For every \( F \in D_{\text{qc}}(X) \), by [Lip60, Prop. 3.9.2], one has \( Rf_* F \in D_{\text{qc}}(Y) \). By Lemma 5.3.3.1, one has \( F^{\text{an}} \in D_{\text{good}}(X^\text{an}) \) and \( (Rf_* F)^{\text{an}} \in D_{\text{good}}(Y^\text{an}) \). Since \( f \) is proper, from [GR71, Exp. XII, Prop. 3.2 (v)], the morphism \( f^{\text{an}} : X^\text{an} \to Y^\text{an} \) is proper. As \( X^\text{an} \) has finite dimension, by Theorem 5.3.1.7, \( Rf^{\text{an}}_* F^{\text{an}} \in D_{\text{good}}(Y^\text{an}) \). Therefore, both the functors \( (Rf_* \bullet)^{\text{an}} \) and \( Rf^{\text{an}}_* (\bullet^{\text{an}}) \) restrict to functors \( D_{\text{qc}}(X) \to D_{\text{good}}(Y^\text{an}) \).

We prove that the morphism \( (Rf_* F)^{\text{an}} \to Rf^{\text{an}}_* F^{\text{an}} \) is an isomorphism. By Lemma 5.3.1.11 (resp. [Lip60, Prop. 3.9.2]), the functor \( Rf^{\text{an}}_* : D(X^\text{an}) \to D(Y^\text{an}) \) (resp. \( Rf_* : D_{\text{qc}}(X) \to D_{\text{qc}}(Y) \)) is bounded above and below. From [Sta23, Tag 06YZ], the inclusion functor \( \text{Qch}(X) \to \text{Mod}(O_X) \) exhibits \( \text{Qch}(X) \).
as a weak Serre subcategory of $\text{Mod}(\mathcal{O}_X)$. Then by (way-out argument) [Har66, I, Prop. 7.1 (iii)], one may assume $F \in \text{Qch}(X)$. By [KS06, Prop. 13.1.5 (ii), p.320], it suffices to check that for every integer $n \geq 0$, the natural morphism $(R^n f_* F)^{\text{an}} \to R^n f_\text{an}^*(F^{\text{an}})$ in $\text{Mod}(\mathcal{O}_Y^{\text{an}})$ is an isomorphism.

By Fact 5.3.3.2, $F = \sum_{i \in I} F_i$ is the sum of a direct family of coherent $\mathcal{O}_X$-submodules of $F$. By [Sta23, Tag 07TB], one has

$$R^n f_* F = \text{colim}_i R^n f_* F_i.$$ 

The analytification is the pullback of modules along the natural morphism $X^{\text{an}} \to X$ of ringed space, so it commutes with colimits and hence

$$(R^n f_* F)^{\text{an}} = \text{colim}_i (R^n f_* F_i)^{\text{an}}.$$ 

By [GR71, XII, Thm. 4.2], the natural morphism $(R^n f_* F_i)^{\text{an}} \to R^n f_\text{an}^*(F_i^{\text{an}})$ is an isomorphism for every $i \in I$. By Lemma 5.3.1.9, the natural morphism

$$\text{colim}_i R^n f_\text{an}^*(F_i^{\text{an}}) \to R^n f_\text{an}^*(F^{\text{an}})$$

is an isomorphism. 

Proposition 5.3.3.4 shows that goodness on complex analytic spaces is an analytic counterpart of quasi-coherence on complex algebraic varieties.

**Proposition 5.3.3.4.** Suppose that the complex algebraic variety $X$ is proper. Then (5.16) is an equivalence of abelian categories.

**Proof.**

- The functor (5.16) is essentially surjective: Indeed, because $X$ is proper over $\mathbb{C}$, by [GR71, Exp. XII, Prop. 3.2 (v)], the complex analytic space $X^{\text{an}}$ is compact. Then for every good $\mathcal{O}_{X^{\text{an}}}$-module $G$, one can write $G = \sum_{i \in I} G_i$ as the sum of a directed family of coherent $\mathcal{O}_{X^{\text{an}}}$-submodules. From the equivalence $\psi_X^* : \text{Coh}(X) \to \text{Coh}(X^{\text{an}})$ ([GR71, XII, Thm. 4.4]), there is a filtered inductive system $\{H_i\}_{i \in I}$ in $\text{Coh}(X)$ whose analytification is the filtered inductive system $\{G_i\}$. By [Sta23, Tag 01LA (4)], the colimit $H$ of $\{H_i\}$ in $\text{Mod}(\mathcal{O}_X)$ exists and lies in Qch($X$). Because $\psi_X^*$ commutes with colimits, one has $H^{\text{an}} = \text{colim}_i G_i$. In particular, $H^{\text{an}}$ is isomorphic to $G$ in Good($X^{\text{an}}$).

- The functor (5.16) is fully faithful: For any quasi-coherent $\mathcal{O}_X$-modules $F, G$, we have to show that the canonical morphism

$$\text{Hom}_{\mathcal{O}_X}(F, G) \to \text{Hom}_{\mathcal{O}_{X^{\text{an}}}}(F^{\text{an}}, G^{\text{an}})$$

(5.21)

is an isomorphism. Assume first that $F$ is coherent.

- From [GW20, Exercise 7.20 (b)], one has

$$[\text{Hom}_{\mathcal{O}_X}(F, G)]^{\text{an}} = \text{Hom}_{\mathcal{O}_{X^{\text{an}}}}(F^{\text{an}}, G^{\text{an}}).$$

- As $F$ is of finite presentation, the $\mathcal{O}_X$-module $\text{Hom}_{\mathcal{O}_X}(F, G)$ is quasi-coherent.
Therefore, by Proposition 5.3.3.3, the canonical morphism

\[ H^0(X, \mathcal{H}om_{\mathcal{O}_X}(F, G)) \to H^0(X^{an}, \mathcal{H}om_{\mathcal{O}_{X^{an}}}(F^{an}, G^{an})) \]

is an isomorphism, which is exactly (5.21).

By (5.18) and (5.19), the general case follows. \(\square\)

With quasi-coherence condition, the algebraic and analytic integral transforms are compatible.

**Corollary 5.3.3.5.** Let \(X, Y\) be two complex algebraic varieties, with \(X\) proper. Then for every \(K \in D_{qc}(X \times Y)\), the natural square

\[
\begin{array}{ccc}
D(X) & \xrightarrow{\phi_{K}^{[X \to Y]}} & D(Y) \\
\downarrow_{\psi_X^{[X^{an} \to Y^{an}]}} & & \downarrow_{\psi_Y^{[Y^{an}]}} \\
D(X^{an})^{[X^{an} \to Y^{an}]} & \xrightarrow{\phi_{K}^{[X^{an} \to Y^{an}]}} & D(Y^{an}),
\end{array}
\]

restricts to a commutative square

\[
\begin{array}{ccc}
D_{qc}(X) & \xrightarrow{\phi_{K}^{[X \to Y]}} & D_{qc}(Y) \\
\downarrow_{\psi_X^{[X^{an} \to Y^{an}]}} & & \downarrow_{\psi_Y^{[Y^{an}]}} \\
D_{good}(X^{an})^{[X^{an} \to Y^{an}]} & \xrightarrow{\phi_{K}^{[X^{an} \to Y^{an}]}} & D_{good}(Y^{an}).
\end{array}
\tag{5.22}
\]

**Proof.** From [Sta23, Tag 08DW (1)], [Sta23, Tag 08DX (1)] and [Sta23, Tag 08D5 (1)], the functor \(\phi_{K}^{[X \to Y]}\) restricts to a functor \(D_{qc}(X) \to D_{qc}(Y)\). By Corollary 5.3.1.14, the functor \(\phi_{K}^{[X^{an} \to Y^{an}]}\) restricts to a functor \(D_{good}(X^{an}) \to D_{good}(Y^{an})\). By Lemma 5.3.3.1, the functor \(\psi_X^{[X^{an} \to Y^{an}]}\) restricts to a functor \(D_{qc}(X) \to D_{good}(X^{an})\) (resp. \(D_{qc}(Y) \to D_{good}(Y^{an})\))

By [Sta23, Tag 0D5S] (resp. [Sta23, Tag 079U]), analytification commutes with derived pullback (resp. tensor product). As \(X/\mathbb{C}\) is proper, the projection \(p_Y : X \times Y \to Y\) is proper. By Proposition 5.3.3.3, analytification commutes with derived direct image. Thus, the square (5.22) is commutative. \(\square\)

Fact 5.3.3.6 can be retracted from Remark 1.1 and the proof of Theorem A in [Hal23].

**Fact 5.3.3.6.** If the complex algebraic variety \(X\) is proper, then the functor (5.15) induces a \(t\)-exact equivalence \(D^b_c(X) \to D^b_c(X^{an})\).

**Question 5.3.3.7.** Suppose that the complex algebraic variety \(X\) is smooth projective. Is the functor (5.17) an equivalence of categories?

**Remark 5.3.3.8.** Fact 5.3.3.6 proves Corollary 5.4.1.2 below for complex tori that are algebraic. Indeed, if \(X/\mathbb{C}\) is an abelian variety, then every functor in the following square is an equivalence.
5.4. ANALYTIC MUKAI DUALITY

\[
\begin{array}{c}
D^b_c(X) \xrightarrow{RS} D^b_c(\hat{X}) \\
\downarrow_{\psi_X} & \quad \downarrow_{\psi_{\hat{X}^\text{an}}} \\
D^b_c(X^\text{an}) \xrightarrow{RS} D^b_c(\hat{X}^\text{an})
\end{array}
\]

In fact, using [Huy06, Def. 5.1] and the natural equivalence \( D^b(\text{Coh}(X)) \to D^b_c(X) \) in [FJJ +06, Exp. II, Cor. 2.2.2.1], one gets that the functor \( RS \) restricts to a functor \( D^b_c(X) \to D^b_c(\hat{X}) \). The functor on the top of the square is an equivalence by Fact 5.2.0.4. From Fact 5.3.3.6, the vertical functors are also equivalences. From Corollary 5.3.1.14, the functor \( RS \) restricts to a functor \( D^b_c(X^\text{an}) \to D^b_c(\hat{X}^\text{an}) \). The commutativity of the square follows from Corollary 5.3.3.5.

5.4 Analytic Mukai duality

5.4.1 Statement

Let \( X \) be a complex torus of dimension \( g \).

Theorem 5.4.1.1 (Mukai, Ben-Bassat, Block, Pantev). There are natural isomorphisms of functors

\[
RS \circ RS \cong [-1]_X [-g] : D_{\text{good}}(X) \to D_{\text{good}}(X);
\]

\[
RS \circ RS \cong [-1]_{\hat{X}} [-g] : D_{\text{good}}(\hat{X}) \to D_{\text{good}}(\hat{X}).
\]

In particular, \( RS : D_{\text{good}}(\hat{X}) \to D_{\text{good}}(X) \) is an equivalence of categories, with a quasi-inverse \( [-1]_{\hat{X}} R\hat{S}[-g] \).

Corollary 5.4.1.2. The functors \( RS : D^b_c(\hat{X}) \to D^b_c(X) \) and \( R\hat{S} : D^b_c(X) \to D^b_c(\hat{X}) \) are equivalences of triangulated categories.

Proof. It follows from Corollary 5.3.1.14 and Theorem 5.4.1.1.

Remark 5.4.1.3. A Mukai duality for complex tori similar to Corollary 5.4.1.2 is stated in [Blo10, p.314], with \( D^b(\text{Coh}(\ast)) \) at the place of \( D^b_c(\ast) \). However, Prof. Jonathan Block told the author that here we should stick to \( D^b_c(\ast) \). In fact, in general the abelian category \( \text{Coh}(X) \) does not have enough injectives, so it is unclear how to define the derived direct image involved in [Blo10, p.314]. Moreover, recently Prof. A. Bondal announced\(^7\) that for a generic complex torus \( X \) of dimension > 2, the natural functor \( D^b(\text{Coh}(X)) \to D^b_c(X) \) is not an equivalence.

5.4.2 Proof

We follow the strategy of [BBBF07, Thm. 2.1] to prove Theorem 5.4.1.1.

---

\(^{6}\)[PPS17, Thm. 13.1] relies on Statement 5.2.0.8.

\(^{7}\)https://www.mathnet.ru/eng/present35371
Preliminaries

Lemma 5.4.2.1 (Associativity). Let $A, B$ be two sheaves of rings on a topological space $X$. For $M \in D(\text{Mod}(A))$, $N \in D(\text{BiMod}(A, B))$, and $K \in D(\text{Mod}(B))$, there is a canonical isomorphism $M \otimes_A^L (N \otimes_B^L K) = (M \otimes_A^L N) \otimes_B^L K$ in $D(\text{BiMod}(A, B))$.

Proof. By [Sta23, Tag 06YF], there exists a quasi-isomorphism $M' \to M$ (resp. $K' \to K$) in $D(\text{Mod}(A))$ (resp. $D(\text{Mod}(B))$), where $M'$ (resp. $K'$) is a K-flat complex of $A$ (resp. $B$) modules. From [Sta23, Tag 06YH], one has

$$M \otimes_A^L (N \otimes_B^L K) = M' \otimes_A^L (N \otimes_B^L K) = (M' \otimes_A^L N) \otimes_B^L K' = (M \otimes_A^L N) \otimes_B^L K.$$ 

Lemma 5.4.2.2, an analytic analog of [Muk81, Example 1.2], exhibits the derived pullback and direct image as particular examples of integral transforms.

Lemma 5.4.2.2. Let $f : X \to Y$ be a morphism of complex analytic spaces, and let $i : \Gamma_f \to X \times Y$ be the inclusion of its graph. Set $F = i_* O_{\Gamma_f} \in \text{Mod}(O_{X \times Y})$. Then there are canonical isomorphism of functors

$$\phi_F^{[X \to Y]} \sim \to Rf_* : D(X) \to D(Y);$$

$$\phi_F^{[Y \to X]} \sim \to Lf^* : D(Y) \to D(X).$$

Proof. Let $g : \Gamma_f \to X$ be the projection. Since $g$ is an isomorphism of complex analytic spaces, one has a canonical isomorphism

$$Lg^* \sim \to R(g^{-1})^*$$

of functors $D(X) \to D(\Gamma_f)$. Consider the following diagram

$$\begin{array}{ccc}
\Gamma_f & \xrightarrow{i} & X \times Y \\
\downarrow g & & \downarrow p_Y \\
X & \xleftarrow{p_X} & Y.
\end{array}$$

As $i$ is a closed embedding of complex analytic spaces, by [Sta23, Tag 0B55], the natural transformation

$$Ri_* O_{\Gamma_f} \otimes^L Lp_X^*(\cdot) \to Ri_* Li^* Lp_X^*(\cdot)$$

8Here, $\text{BiMod}(A, B)$ denotes the category of sheaves of $(A, B)$-bimodules.
is an isomorphism of functors $D(X) \to D(X \times Y)$. One has

\[
\phi_F^{[X \to Y]} := Rp_Y^* (F \otimes^L p_X^*) = Rp_Y^* (Ri_* O_Y \otimes^L Lp_Y^*)
\]

\[(a)\] \[\sim \] \[\to \] \[\sim \] \[\to \] \[\sim \] \[(b)\]

\[
Rp_Y^* Ri_* Li^* Lp^* \sim Rp_Y^* Ri_* Lg^*
\]

\[(c)\] \[\sim \] \[\to \] \[\sim \] \[\to \] \[\sim \] \[(d)\]

\[
Rp_Y^* Ri_* R(g^{-1})^* \sim Rf^* .
\]

where (a) (resp. (c)) uses (5.26) (resp. (5.25)), and the equalities (b) and (d) are from [Spa88, Thm. A (iii)].

Thus, (5.23) is proved. The proof of (5.24) is similar. \hfill \Box

Proposition 5.4.2.3 is the first ingredient of the proof of Theorem 5.4.1.1, which expresses the composition of two integral transforms as another integral transform.

**Proposition 5.4.2.3.** Let $M, N, P$ be finite dimensional complex analytic spaces, with $M, N$ compact. Let $p_{ij}$ be the projections of the product $M \times N \times P$. For $K \in D_{\text{good}}(M \times N)$ and $L \in D(N \times P)$, set

\[
H = Rp_{13}^* (p_{12}^* K \otimes^L p_{23}^* L) (\in D(M \times P)).
\]

Then there is a natural isomorphism $\phi_K^{[N \to P]} \circ \phi_{M \to N} \sim \phi_H^{[M \to P]}$ of functors $D_{\text{good}}(M) \to D(P)$.

**Proof.** Let

\[
a : M \times N \to M, \quad b : N \times P \to P,
\]

\[
p : M \times N \to N, \quad q : N \times P \to N,
\]

\[
u : M \times P \to M, \quad v : M \times P \to P
\]

be projections.

The morphism $q$ is locally product. Properness of $p$ follows from the compactness of $M$. By Propositions 5.3.1.3 2 and 5.3.1.5 2, the functor $K \otimes^L a^* : D(M) \to D(M \times N)$ restricts to a functor $D_{\text{good}}(M) \to D_{\text{good}}(M \times N)$. Then one can apply Theorem 5.3.2.3 to the cartesian square

\[
\begin{array}{ccc}
M \times N \times P & \xrightarrow{p_{12}} & M \times N \\
\downarrow p_{23} & & \downarrow p \\
N \times P & \xrightarrow{q} & N,
\end{array}
\]

so the base change natural transformation induces an isomorphism

\[
q^* Rp_* (K \otimes^L a^*) \to Rp_{23}^* p_{12}^* (K \otimes^L a^*)
\]

(5.27)
of functors $D_{\text{good}}(M) \to D_{\text{good}}(N \times P)$. Thus, one has isomorphisms
\[
\phi_L^{[N \to P]} \phi_K^{[M \to N]} = Rb_* [L \otimes^L \quad q^* R_{p*}(K \otimes^L a^*)] \\
\sim Rb_* [L \otimes^L \quad R_{p23*}p_{12}^*(K \otimes^L a^*)] \\
\sim Rb_* R_{p23*} [p_{23}^* L \otimes^L p_{12}^*(K \otimes^L a^*)] \\
= R_{p*} [p_{23}^* L \otimes^L p_{12}^*(K \otimes^L a^*)] \\
= Rv_* R_{p13*} (p_{12}^* K \otimes^L p_{23}^* L \otimes^L p_{12}^*).
\]
\[
\sim Rv_* [H \otimes^L u^*] = \phi_H^{[M \to P]},
\]

of functors $D_{\text{good}}(M) \to D(P)$ where (a) uses (5.27), and (b) (resp. (c)) use the compactness of $M$ (resp. $N$) and Fact 5.3.2.15.

The other ingredient of the proof of Theorem 5.4.1.1, Fact 5.4.2.4, calculates the cohomology of the Poincaré bundle.

**Fact 5.4.2.4** ([Kem91, Thm. 3.15]). Let $X$ be a complex torus of dimension $g$. Let $p : X \times \tilde{X} \to X$, $q : X \times X \to \tilde{X}$ be the two projections. Then for the Poincaré bundle $\mathcal{P}$, one has $R_p \mathcal{P} = \mathbb{C}_0[-g]$ in $D^b(X)$ and $Rq_* \mathcal{P} = \mathbb{C}_0[-g]$ in $D^b(\tilde{X})$.

**Proof of Theorem 5.4.1.1**

By Corollary 5.3.1.14, the functor $RS$ (resp. $R\tilde{S}$) restricts to a functor $D_{\text{good}}(\tilde{X}) \to D_{\text{good}}(X)$ (resp. $D_{\text{good}}(X) \to D_{\text{good}}(\tilde{X})$). Let $p_{ij}$ be the projections of $X \times X \times \tilde{X}$. Set
\[
H = R_{p12*} (p_{13}^* \mathcal{P} \otimes^L p_{23}^* \mathcal{P}).
\]

By Propositions 5.3.1.3 and 5.3.1.5, Fact 5.3.1.8 and Lemma 5.3.1.11, one has $H \in D^b(X \times X \times \tilde{X})$. By Proposition 5.4.2.3, one has an isomorphism of $RS \circ R\tilde{S} \sim \phi_H^{[X \times X \times \tilde{X}]}$ of functors $D_{\text{good}}(X) \to D_{\text{good}}(X)$. Let $m : X \times X \to X$ be the group law. Since the $O_{X \times X \times \tilde{X}}$-module $p_{13}^* \mathcal{P}$ is flat, one has $p_{13}^* \mathcal{P} \otimes^L p_{23}^* \mathcal{P} = p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}$. By [BL04, Lem. 14.1.7], the $O_{X \times X \times \tilde{X}}$-module $p_{13}^* \mathcal{P} \otimes p_{23}^* \mathcal{P}$ is isomorphic to $(m \times \text{Id}_X)^* \mathcal{P}$. Then $H \sim R_{p12*} (m \times \text{Id}_X)^* \mathcal{P}$. Because the morphism $m$ is smooth, applying Theorem 5.3.2.3 to the cartesian square

\[
\begin{array}{ccc}
X \times X \times \tilde{X} & \overset{m \times \text{Id}_X}{\longrightarrow} & X \times \tilde{X} \\
\downarrow p_{12} & & \downarrow p_X \\
X \times X & \overset{m}{\longrightarrow} & X
\end{array}
\]

\[\text{It is stated for abelian varieties, but its proof works for complex tori.}\]
in the category An, one has an isomorphism
\[ H \sim m^* R p_{X,*} \mathcal{P} \]
in \( D^b(X \times X) \). Let \( i : \Gamma_{[-1]} \to X \times X \) be the inclusion of the graph of \([-1]_X : X \to X\). From Fact 5.4.2.4, one has \( H \sim m^* \mathbb{C}_0[-g] = i_* \mathcal{O}_{\Gamma_{[-1]}[-g]} \).

By Lemma 5.4.2.2, there is an isomorphism \( H \sim m^* C_0^{[X]}[-g] = i^* \mathcal{O}_{\Gamma[-1]} [-g] \).

By Lemma 5.4.2.2, there is an isomorphism \( \phi^X \to X \) of functors \( D(\hat{X}) \to D(X) \), which shows the isomorphism \( R_S \circ R_{\hat{S}} \sim [-1]_X[-g] \) of functors \( D_{\text{good}}(X) \to D_{\text{good}}(X) \). The proof of the second isomorphism is similar.

## 5.5 Properties of Fourier-Mukai transform

For later reference purposes, we check that each result starting from Theorem 2.2 to (3.12’) in [Muk81] has an analytic version. We only indicate the necessary modifications in statements and proofs.

For a complex torus \( X \), let \( g_X \) be its dimension. Let \((R_S X, R_{\hat{S}} X)\) be the Fourier-Mukai transform of \( X \). The subscripts are omitted when there is only one complex torus in context. For a morphism \( \phi : X \to Y \) of complex tori, let \( \hat{\phi} : \hat{Y} \to \hat{X} \) be the dual morphism.

### 5.5.1 Functoriality

Exchange of translations and twists

For every point \( x \) of the complex torus \( X \), let \( T_x : X \to X, \quad x' \mapsto x' + x \) be the translation by \( x \).

**Proposition 5.5.1.1.** For every \( x \in X \) and every \( y \in \hat{X} \), there are canonical isomorphisms
\[
RS \circ T^*_x \cong (\cdot \otimes_{O_X} P_{-\hat{y}}) \circ RS, \tag{5.28}
\]
\[
RS \circ (\cdot \otimes_{O_{\hat{X}}} P_{x}) \cong T_x^* \circ RS \tag{5.29}
\]
of functors \( D(\hat{X}) \to D(X) \).

**Proof.** We prove (5.28). From [BL04, Cor. A.9], one gets
\[
T^*_x \mathcal{P} \sim \mathcal{P} \otimes_{O_{X \times X}} P_X P_{-\hat{y}}; \tag{5.30}
\]
\[
T^*_x \mathcal{P} \sim \mathcal{P} \otimes_{O_{X \times X}} P_{\hat{X}} P_x. \tag{5.31}
\]
Then there are isomorphisms
\[ RS(T^*_X \cdot) = Rp_X^*(P \otimes_{O_{X \times \hat{X}}} p_X^* T^*_X) \]
\[ = Rp_X^*(P \otimes_{O_{X \times \hat{X}}} T^*_{(0,\hat{z})} p_X^*) \]
\[ = Rp_X^*(T^*_{(0,\hat{z})}(T^*_X)^* P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ \sim_{\sim} R p_X^*(T^*_{(0,\hat{z})} P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ = R p_X^*(T^*_{(0,\hat{z})} P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ \sim_{\sim} R p_X^*(p_X^* P_{-\hat{z}} \otimes P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ = P_{-\hat{z}} \otimes R P_X^*(P \otimes_{O_{X \times \hat{X}}} p_X^*) \]

of functors \( D(\hat{X}) \to D(X) \), where (a) (resp. (b)) uses (5.30) (resp. Fact 5.3.2.15).

We prove (5.29) as follows:
\[ RS(P_x \otimes \cdot) = Rp_X^*(P \otimes_{O_{X \times \hat{X}}} p_X^*(P_x \otimes \cdot)) \]
\[ = Rp_X^*(P \otimes_{O_{X \times \hat{X}}} p_X^* P_x \otimes p_X^*) \]
\[ \sim_{\sim} R p_X^*(T^*{(x,0)} P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ = R p_X^*(T^*{(x,0)} P \otimes_{O_{X \times \hat{X}}} T^*{(-x,0)} p_X^*) \]
\[ \sim_{\sim} R p_X^*(T^*{(-x,0)} P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ = R(T_{-x}) R p_X^*(P \otimes_{O_{X \times \hat{X}}} p_X^*) \]
\[ = T^*_X R \]

where (a) uses (5.31). \( \square \)

**Exchange of the direct image and the inverse image**

A result similar to Proposition 5.5.1.2 is stated as [Lau96, Prop. 1.3.1]. As Laumon omits its proof, we give one. Fourier-Mukai transform is functorial, as Proposition 5.5.1.2 shows.

**Proposition 5.5.1.2.** For a morphism \( \phi : Y \to X \) of complex tori, there are canonical isomorphisms of functors
\[ L \phi^* \circ R S_X \cong R S_Y \circ R \phi_* : D_{\text{good}}(\hat{X}) \to D_{\text{good}}(Y), \quad (5.32) \]
\[ R \phi_* \circ R S_Y \cong R S_X \circ L \phi^* (\cdot)[g - g'] : D_{\text{good}}(\hat{Y}) \to D_{\text{good}}(X). \quad (5.33) \]

([Muk81, (3.4)]) In particular, if \( \phi \) is an isogeny, then
\[ \phi^* \circ R S_X \cong R S_Y \circ \phi_* : D_{\text{good}}(\hat{X}) \to D_{\text{good}}(Y); \]
\[ \phi_* \circ R S_Y \cong R S_X \circ \phi^* : D_{\text{good}}(\hat{Y}) \to D_{\text{good}}(X). \]
5.5. PROPERTIES OF FOURIER-MUKAI TRANSFORM

Proof. The isomorphism (5.33) follows from (5.32) as follows. There are isomorphisms

\[
\begin{align*}
(a) & \quad R\phi_* R\mathcal{S}_Y \cong [-1]_X R\mathcal{S}_X R\tilde{\mathcal{S}}_X R\phi_* R\mathcal{S}_Y(\cdot)\cdot g_X \\
(b) & \quad \cong [-1]_X R\mathcal{S}_X L\phi^* R\tilde{\mathcal{S}}_Y R\mathcal{S}_Y(\cdot)\cdot g_X \\
(c) & \quad \cong [-1]_X R\mathcal{S}_X L\phi^* [-1]_Y(\cdot)\cdot [g_X - g_Y] \\
& \quad = R\mathcal{S}_X L\phi^* (\cdot)\cdot [g_X - g_Y]
\end{align*}
\]

of functors $D_{\text{good}}(\hat{Y}) \rightarrow D_{\text{good}}(X)$, where (a) and (c) use Theorem 5.4.1.1, and (b) uses (5.32).

To prove (5.32), we show

\[
(\phi \times \text{Id}_\hat{X})^* \mathcal{P}_X \cong (\text{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y.
\] (5.34)

Set $L := (\phi \times \text{Id}_\hat{X})^* \mathcal{P}_X \otimes (\text{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y^{-1}$. By definition, on the one hand for every $\hat{x} \in \hat{X}$, one has $L|_{Y \times \hat{x}} \cong \phi^* P_2 \otimes P_2^{-1} \sim O_Y$; on the other hand, one has $L|_{0 \times \hat{X}} \sim \hat{\phi}^* O_Y \sim O_X$. By the seesaw principle [BL04, Cor. A.9], these imply $L \sim O_{Y \times \hat{X}}$.

By applying Theorem 5.3.2.3 to the cartesian square

\[
\begin{array}{ccc}
Y \times \hat{X} & \xrightarrow{p_2} & \hat{X} \\
\| & \downarrow \phi & \downarrow \hat{\phi} \\
Y \times \hat{Y} & \xrightarrow{p_Y} & \hat{Y}
\end{array}
\]

in the category $\text{An}$, the base change natural transformation

\[
p_Y^* R\phi_* \rightarrow R(\text{Id}_Y \times \hat{\phi})_* p_2^*
\] (5.35)

induces an isomorphism of functors $D_{\text{good}}(\hat{X}) \rightarrow D_{\text{good}}(Y \times \hat{Y})$. By Propositions 5.3.1.3 2 and 5.3.1.5 2, the functor $\mathcal{P}_X \otimes p_X^* : D(\hat{X}) \rightarrow D(X \times \hat{X})$ restricts to a functor $D_{\text{good}}(\hat{X}) \rightarrow D_{\text{good}}(X \times \hat{X})$. Because $p_X$ is smooth proper, by applying Lemma 5.3.2.13 to the cartesian square

\[
\begin{array}{ccc}
Y \times \hat{X} & \xrightarrow{\phi \times \text{Id}_\hat{X}} & X \times \hat{X} \\
p_{1} & \downarrow \square & \downarrow p_X \\
Y & \xrightarrow{\phi} & X
\end{array}
\]

in the category $\text{An}$, the base change natural transformation induces an isomorphism

\[
L\phi^* Rp_X^* (\mathcal{P}_X \otimes p_X^* \cdot) \rightarrow Rp_1^* L(\phi \times \text{Id}_\hat{X})^* (\mathcal{P}_X \otimes p_X^* \cdot)
\] (5.36)

of functors $D_{\text{good}}(\hat{X}) \rightarrow D_{\text{good}}(Y)$.
There are isomorphisms
\[
L\phi^* \circ R\Sigma_X = L\phi^* R\Sigma_X (\mathcal{P}_X \otimes p_X^*)
\]
\[
(a) \quad \sim R\Sigma_1, L(\phi \times \text{Id}_X)^* (\mathcal{P}_X \otimes p_X^*)
\]
\[
= R\Sigma_1 [L(\phi \times \text{Id}_X)^* \mathcal{P}_X \otimes L(\phi \times \text{Id}_X)^* p_X^*]
\]
\[
= R\Sigma_1 [(\phi \times \text{Id}_X)^* \mathcal{P}_X \otimes p_2^*]
\]
\[
(b) \quad \sim R\Sigma_1 [(\text{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y \otimes p_2^*]
\]
\[
= R\Sigma_2 R(\text{Id}_Y \times \hat{\phi})_2 [L(\text{Id}_Y \times \hat{\phi})^* \mathcal{P}_Y \otimes p_2^*]
\]
\[
(c) \quad \sim R\Sigma_2 [\mathcal{P}_Y \otimes R(\text{Id}_Y \times \hat{\phi})_2 p_2^*]
\]
\[
(d) \quad \sim R\Sigma_2 [\mathcal{P}_Y \otimes p_2^* R\hat{\phi}_2]
\]
\[
= R\Sigma_Y R\hat{\phi}_2
\]
of functors \(D_{\text{good}}(\hat{X}) \rightarrow D_{\text{good}}(Y)\), where the (a) (resp. (b), resp. (c), resp. (d)) line uses (5.36) (resp. (5.34), resp. Fact 5.3.2.15, resp. (5.35)). This proves (5.32). Assume now \(\phi\) is an isogeny. so \(\phi\) is finite flat and \(g_Y = g\).
By [GR13, Thm. 4, p.47], the functor \(\phi_\ast : \text{Mod}(Y) \rightarrow \text{Mod}(X)\) is exact, so \(R\phi_\ast = \phi_\ast\) as a functor \(D(Y) \rightarrow D(X)\). By the flatness, the inverse image \(\phi^* : \text{Mod}(X) \rightarrow \text{Mod}(Y)\) is exact and \(L\phi^* = \phi^*\) as a functor \(D(Y) \rightarrow D(Y)\).

Remark 5.5.1.3. In [Muk81, (3.4)], for an isogeny \(\phi : Y \rightarrow X\) of abelian varieties, the derived functor \(R\phi_\ast : D_{\text{qc}}(Y) \rightarrow D_{\text{qc}}(X)\) is also written as \(\phi_\ast\), but for a different reason [Sta23, Tag 08D7].

For the first half of [Muk81, Prop. 3.11 (4)], the result [MRM74, Sec. 23, Lem. 3] cited in its proof still holds for complex tori, with a similar (and simpler) proof.

Exchange of the Pontrjagin product and the tensor product

Let \(p_t\) be the two projections \(X \times X \rightarrow X\). Define a bifunctor \(*^R : D(X) \times D(X) \rightarrow D(X)\) by \(- *^R + = Rm_* \cdot (p_1^* \otimes \otimes L p_2^*)\). As in Corollary 5.3.1.14, the bifunctor \(*^R\) restricts to a bifunctor \(D_{\text{good}}(X) \times D_{\text{good}}(X) \rightarrow D_{\text{good}}(X)\) (resp. \(D_{\text{qc}}^b(X) \times D_{\text{qc}}^b(X) \rightarrow D_{\text{qc}}^b(X)\)).

Fact 5.5.1.4 ([Muk81, (3.7)]). For every \(F \in D_{\text{good}}(\hat{X})\), there are canonical isomorphisms
\[
RS(F *^R \cdot) \cong RS(F) \otimes L RS(\cdot),
\]
\[
RS(F \otimes \cdot) \cong RS(F) *^R RS(\cdot)[g]
\]
of functors \(D_{\text{good}}(\hat{X}) \rightarrow D_{\text{good}}(X)\).
Commutativity with external tensor product

Let $M, N$ be two complex analytic spaces. Let $p : M \times N \to M$ and $q : M \times N \to N$ be the projections. The bifunctor $D(M) \times D(N) \to D(M \times N)$, $(-,+)$ $\to (p^* -) \otimes^L (q^* +)$ is denoted by $\cdot \boxtimes^L \cdot$.

**Proposition 5.5.1.5** ([Lau96, Prop. 1.3.2]). Let $X, Y$ be two complex tori and $Z = X \times Y$. Then there is a canonical isomorphism $RS_Z(- \boxtimes^L +) = RS_X(-) \boxtimes^L RS_Y(+)$ of bifunctors $D_{\text{good}}(\hat{X}) \times D_{\text{good}}(\hat{Y}) \to D_{\text{good}}(\hat{Z})$.

**Proof.** By the seesaw principle, one has $\mathcal{P} \sim \mathcal{P} \otimes^L \mathcal{P}$. Then there are canonical isomorphisms of bifunctors

\[
RS_Z(- \boxtimes^L +) = Rp_Z(P \otimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P})
\]

\[
\sim Rp_Z((P \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P}) \boxtimes^L (\mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P}))
\]

\[
\sim R(p_X \times p_Y)_*(P \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P})
\]

\[
\sim Rp_{X,Y}((P \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P}) \boxtimes^L (\mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P}))
\]

(5.37) (a) $\sim \leftarrow Rp_X(P \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P})$ (5.38)

\[
RS_X(-) \boxtimes^L RS_Y(+)
\]

(5.39) (5.40) $\sim \leftarrow \leftarrow Rp_X(P \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P})$ (5.41)

$D_{\text{good}}(\hat{X}) \times D_{\text{good}}(\hat{Y}) \to D_{\text{good}}(\hat{Z})$, where (a) uses Lemma 5.5.1.6 2. \hfill $\square$

**Lemma 5.5.1.6.**

1. Let $X,Y,T$ be complex analytic spaces, with $X,T$ finite dimensional. Let $f : X \to Y$ be a proper morphism. Then there is a canonical isomorphism

\[
Rf_*(-) \boxtimes^L (+) \to R(f \times 1_T)_*(- \boxtimes^L +)
\]

of bifunctors $D_{\text{good}}(X) \times D(T) \to D(Y \times T)$.

2. Let $f_i : X_i \to Y_i$ $(i = 1,2)$ be proper morphism of complex analytic spaces. If $X_1, X_2$ and $Y_1$ are finite dimensional, then there is a canonical isomorphism

\[
(Rf_{1*}-) \boxtimes^L (Rf_{2*}+) \to R(f_1 \times f_2)_*(- \boxtimes^L +)
\]

of bifunctors $D_{\text{good}}(X_1) \times D_{\text{good}}(X_2) \to D_{\text{good}}(Y_1 \times Y_2)$.

**Proof.**

1. Consider the notation in the commutative diagram

\[
\begin{array}{ccc}
X \times T & \xrightarrow{u} & X \\
\downarrow & & \downarrow \\
T & \xleftarrow{v} & Y \times T
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{u} & X \\
\downarrow & & \downarrow \\
Y & \xrightarrow{p} & Y,
\end{array}
\]

\[
Rf_*(-) \boxtimes^L (+) \to R(f \times 1_T)_*(- \boxtimes^L +)
\]

(5.42) (5.43) $\sim \leftarrow \leftarrow Rp_X(P \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P} \boxtimes^L \mathcal{P})$ (5.44)
where \( u, v, p \) and \( q \) are projections. Since \( v = q \circ (f \times \text{Id}_T) \), there is a canonical isomorphism \( v^* \cong L(f \times \text{Id}_T)^* q^* \) of functors \( D(T) \to D(X \times T) \). As \( f \times \text{Id}_T \) is a base change of \( f \), it is also proper. As \( \dim X \times T \) is finite, by Fact 5.3.2.15, the canonical morphism

\[
[R(f \times \text{Id}_T)_* u^* -] \otimes^L q^* + \to R(f \times \text{Id}_T)_*[u^* - \otimes^L v^* +] \tag{5.42}
\]

of bifunctors \( D(X) \times D(T) \to D(Y \times T) \) is an isomorphism.

By Theorem 5.3.2.3, one has a canonical isomorphisms

\[
p^* Rf_* \to R(f \times \text{Id}_T)_* u^*: D_{\text{good}}(X) \to D_{\text{good}}(Y \times T). \tag{5.43}
\]

Therefore

\[
Rf_* - \boxtimes^L + = p^* Rf_* - \otimes^L q^* +
\]

\[(a) \quad \cong [R(f \times \text{Id}_T)_* u^* -] \otimes^L q^* +
\]

\[(b) \quad \cong R(f \times \text{Id}_T)_*[p^* - \otimes^L u^* +]
\]

\[= R(f \times \text{Id}_T)_*(- \boxtimes^L +),
\]

where \( (a) \) (resp. \( (b) \)) uses (5.43) (resp. (5.42)).

2. As \( \dim X_1 \times X_2 \) is finite, as in Corollary 5.3.1.14, the bifunctor \( R(f_1 \times f_2)_* (- \boxtimes^L +) \) restricts to a bifunctor \( D_{\text{good}}(X_1) \times D_{\text{good}}(X_2) \to D_{\text{good}}(Y_1 \times Y_2) \).

As \( \dim Y_1, \dim X_2 \) are finite, by Point 1, there are canonical isomorphisms

\[
Rf_{1*} - \boxtimes^L + \to R(f_1 \times \text{Id}_{X_2})_* (- \boxtimes^L +): D_{\text{good}}(X_1) \times D(X_2) \to D(Y_1 \times X_2),
\tag{5.44}
\]

\[
(Rf_{1*} -) \boxtimes^L (Rf_{2*} +) \to R(\text{Id}_{Y_1} \times f_2)_*[ (Rf_{1*} -) \boxtimes^L + ] : D(X_1) \times D_{\text{good}}(X_2) \to D(Y_1 \times Y_2),
\tag{5.45}
\]

Then there is a canonical isomorphism

\[
(Rf_{1*} -) \boxtimes^L (Rf_{2*} +) \to R(\text{Id}_{Y_1} \times f_2)_*[ (Rf_{1*} -) \boxtimes^L + ] \rightarrow R(\text{Id}_{Y_1} \times f_2)_* R(f_1 \times \text{Id}_{X_2})_* (- \boxtimes^L +)
\]

\[\rightarrow R(f_1 \times f_2)_* (- \boxtimes^L +) : D_{\text{good}}(X_1) \times D_{\text{good}}(X_2) \to D_{\text{good}}(Y_1 \times Y_2).
\]

\( \square \)

**Skew commutativity with duality**

We summarize classical facts about the duality theory on complex manifolds.

**Fact 5.5.1.7.** Let \( X \) be a complex manifold of pure dimension \( n \), and let \( \omega_X = \bigwedge^n \Omega_X \) be the canonical line bundle.
1. ([RR70, p.81; p.90]) The dualizing functor $D_X = R\text{Hom}_X(\cdot, \omega_X)[n] : D(X) \to D(X)$ restricts to a functor $D_c(X) \to D_c(X)$ and the natural transformation $\text{Id} \to D_X \circ D_X : D_c(X) \to D_c(X)$ is an isomorphism. If $X$ is compact, then $D_X$ exchanges $D^b_c(X)$ and $D^c(X)$ and induces an equivalence $D^b_c(X) \to D^b_c(X)$.

2. ([RRV71, p.264]) There is a canonical isomorphism $R\text{Hom}_X(\cdot, +) \to D_X(- \otimes D_X^+) \circ \text{Functors} D_c(X) \times D^+_c(X) \to D(X)$.

3. ([RRV71, p.264], [Bjö93, p.122]) Let $f : X \to Y$ be a proper morphism of complex manifolds. Then there is a canonical isomorphism of functors $Rf_* D_X \to D_Y Rf_* : D_c(X) \to D(Y)$.

**Proposition 5.5.1.8** ([Muk81, (3.8)]). There are canonical isomorphisms of functors

\[
D_X \circ RS \xrightarrow{\sim} (-1)^X \circ RS \circ D_X[g] : D^+_c(\hat{X}) \to D^-_c(\hat{X});
\]

\[
D_X \circ R\hat{S} \xrightarrow{\sim} (-1)^X \circ R\hat{S} \circ D_X[g] : D^+_c(\hat{X}) \to D^-_c(\hat{X}).
\]

We make some preparation for the proof of Proposition 5.5.1.8. Lemma 5.5.1.9 is an adaption of [Har66, Ch.II, Prop. 5.8] and [Sta23, Tag 0C6I].

**Lemma 5.5.1.9.** Let $f : X \to Y$ be a flat morphism of complex analytic spaces. Then:

1. There is a natural transformation of bifunctors

\[
f^* R\text{Hom}_Y(\cdot, +) \to R\text{Hom}_X(f^*, f^*+) : D(Y) \times D(Y) \to D(X).
\]

(5.46)

2. The natural transformation (5.46) restricts to an isomorphism of bifunctors $D^+_c(Y) \times D(Y) \to D(X)$.

**Proof.** Set $G \in D(Y)$.

1. By [Spa88, Thm. D, p.125], there is a functorial quasi-isomorphism $G \to G'$, where $G'$ is a K-injective complex over $\text{Mod}(O_Y)$. There are natural transformations of functors $D(Y) \to D(X)$

\[
f^* R\text{Hom}_Y(\cdot, G) \to f^* \text{Hom}_Y(\cdot, G') \to \text{Hom}_X(f^*, f^*G') \to R\text{Hom}_X(f^*, f^*G') \xrightarrow{\sim} R\text{Hom}_X(f^*, f^*G).
\]

2. By [Har66, Examples 1., p.68], the (contravariant) functors

\[
f^* R\text{Hom}_Y(\cdot, G), R\text{Hom}_X(f^*, f^*G) : D(Y) \to D(X)
\]

are bounded below. Set $F \in D^-_c(Y)$. To show the natural transformation

\[
f^* R\text{Hom}_Y(F, G) \to R\text{Hom}_X(f^*F, f^*G) : D^-_c(Y) \to D(X)
\]

is an isomorphism,

\[\text{By [FS13, p.497]}, in general the functor } R\text{Hom}_X(\cdot, \omega_X) : D(X) \to D(X) \text{ does not exchange } D^{b, \leq 0}_c(X) \text{ and } D^{b, \geq 0}_c(X).\]
by [Har66, I, Prop. 7.1 (ii)], one may assume \(F \in \text{Coh}(Y)\). By [Sta23, Tag 08DL], one may shrink \(Y\) to open subsets. Thus, from Lemma A.1.3.1, one may assume that there is a quasi-isomorphism \(K \to F\), where \(K\) is a complex of finite free \(O_Y\)-modules. Then \(f^*K \to f^*F \to 0\) is a globally free resolution of \(f^*F\). The morphism (5.46) is identified with 

\[ f^*\text{Hom}_Y(K, G) \to \text{Hom}_X(f^*K, f^*G), \]

which is an isomorphism.

\[ \square \]

Lemma 5.5.1.10. Let \(E \to X\) be a holomorphic vector bundle on a complex manifold, and let \(E'\) be the dual vector bundle. Then there is an isomorphism \(D_X(E \otimes -) = E' \otimes D_X\cdot\) of functors \(D(X) \to D(X)\).

Proof. Since \(E\) is a vector bundle, one has isomorphisms

\[ E \otimes - \xrightarrow{\sim} \text{Hom}_X(E', \cdot) \xrightarrow{\sim} R\text{Hom}_X(E', \cdot) \]

of functors \(D(X) \to D(X)\). Then

\[ D_X(E \otimes -) = R\text{Hom}_X(R\text{Hom}_X(E', \cdot), \omega_X)[\dim X]. \]

As \(E'\) is a perfect object of \(D(X)\) (in the sense of [Sta23, Tag 08CM]), by [Sta23, Tag 0G40], one has \(D_X(E \otimes -) = R\text{Hom}_X(\cdot, \omega_X)[\dim X] \otimes^E E' = E' \otimes D_X\cdot\). \(\square\)

Corollary 5.5.1.11. Let \(f : X \to Y\) be a flat morphism of complex manifolds of relative dimension \(n\). Set \(\omega_f = \omega_X \otimes f^*\omega_Y\) to be the relative dualizing line bundle. Then there is a canonical isomorphism of functors \(D_X f^*D_Y \to \omega_f \otimes_{O_X} f^*(\cdot)[n] : D^+_X(Y) \to D^+_X(X)\).

Proof. One has

\[ D_X f^*D_Y = D_X f^*R\text{Hom}_Y(O_Y, \omega_Y)[\dim Y] = D_X(f^*\omega_Y)[\dim Y] \]

\[ = R\text{Hom}_X(f^*\omega_Y, \omega_X)[\dim X - \dim Y] = \text{Hom}_X(f^*\omega_Y, \omega_X)[n] \]

\[ = f^*\omega_Y \otimes_{O_X} \omega_X[n] = \omega_f[n], \]

(5.47)

where (a) uses that \(f^*\omega_Y\) is a line bundle on \(X\).

By Fact 5.5.1.7 1 and 2, there is an isomorphism \(D_Y \xrightarrow{\sim} R\text{Hom}_Y(\cdot, D_YO_Y)\) of functors \(D^+_Y(Y) \to D^+_X(Y)\). From Lemma 5.5.1.9 2, there are isomorphisms

\[ f^*D_Y \xrightarrow{\sim} f^*R\text{Hom}_Y(\cdot, D_YO_Y) \xrightarrow{\sim} R\text{Hom}_X(f^*\cdot, f^*D_YO_Y) \]

of functors \(D^+_Y(Y) \to D^+_X(X)\). Then by Fact 5.5.1.7 1 and 2 again, there are isomorphisms

\[ D_X f^*D_Y = f^* \cdot \otimes^L D_X f^*D_Y = f^* \cdot \otimes_{O_X} \omega_f[n] = f^* \cdot \otimes_{O_X} \omega_f[n] \]

of functors \(D^+_Y(Y) \to D^+_X(X)\), where the second (resp. last) equality uses (5.47) (resp. the fact that \(\omega_f\) is locally free). \(\square\)
Corollary 5.5.1.11 yields an isomorphism of functors $D$.

Lemma 5.5.1.12. There is an isomorphism $R_p X_\ast (P^{-1} \otimes L p_X^\ast \cdot) = [-1]^X RS$ of functors $D(X) \to D(X)$.

Proof. Using [BL04, Cor. A.9], one can prove that $P^{-1} \cong ([[-1]^X \times [1]^X]^\ast P)$. Since $p_X \circ ([[-1]^X \times [1]^X]) = p_X$, there are isomorphisms

$$R_p X_\ast (P^{-1} \otimes L p_X^\ast \cdot) \cong R_p X_\ast ([[-1]^X \times [1]^X]^\ast (P \otimes L p_X^\ast \cdot))$$

Thus, there is an isomorphism $[-1]^X R_p X_\ast (P \otimes L p_X^\ast \cdot) = [-1]^X RS$ of functors $D(X) \to D(X)$. □

Proof of Proposition 5.5.1.8. By Fact 5.5.1.7, there are isomorphisms

$$D_X \circ RS = D_X R_p X_\ast (P \otimes L p_X^\ast \cdot) \cong R_p X_\ast D_X \times X (P \otimes L p_X^\ast \cdot)$$

of functors $D_c^+ (\hat{X}) \to D^+ (X)$. From Lemma 5.5.1.10, there is an isomorphism $D_X \times X (P \otimes L p_X^\ast \cdot) = P^{-1} \otimes L D_X \times X p_X^\ast \cdot$ of functors $D(\hat{X}) \to D(X \times \hat{X})$. By Fact 5.5.1.7, the functor $D_X$ restricts to a functor $D_c^+ (\hat{X}) \to D_c^+ (X)$, whence Corollary 5.5.1.11 yields an isomorphism $D_X \times X p_X^\ast \cdot \cong \left( p_X^\ast D_X \right)^g$ of functors $D_c^+ (\hat{X}) \to D_c^+ (X \times \hat{X})$. Therefore, there are isomorphisms

$$D_X \circ RS = R_p X_\ast (P^{-1} \otimes L p_X^\ast \cdot D_X \times X) \cong [-1]^X RS (D_X \times X)$$

of functors $D_c^+ (\hat{X}) \to D_c^+ (X)$, where (a) uses Lemma 5.5.1.12.

The second isomorphism follows from the first by swapping $X$ and $\hat{X}$. □

5.5.2 Unipotent vector bundles

Definition 5.5.2.1 ([Muk81, Def. 2.3]). We say that W.I.T. (weak index theorem) holds for a coherent $O_X$-module $F$ if there is an integer $i(F)$ such that $H^i \hat{R}S F = 0$ for every integer $i \neq i(F)$. In that case, the integer $i(F)$ is called the index of $F$ and the coherent module $\hat{F} := H^{i(F)} \hat{R}S F$ on $X$ is called the Fourier transform of $F$. We say that I.T. (index theorem) holds for $F$ if there is an integer $i_0$ such that for every $L \in \text{Pic}^0(X)$ and every integer $i \neq i_0$, one has $H^i(X, F \otimes O_X L) = 0$.

Fact 5.5.2.2 ([Nak94, p.80]). Let $F$ be a coherent $O_X$-module, then I.T. holds for $F$ if and only if W.I.T holds for $F$ and $\hat{F}$ is locally free on $X$.

Example 5.5.2.4 show that that the word "Artinian" in Statement 5.5.2.3 is a typo. It should be "finite length" ([Muk78, Thm. 4.12 (1)]).

Statement 5.5.2.3 ([Muk81, Eg. 2.9]). Let $X$ be an abelian variety. Let $\text{Mod}^{\text{Art}}(O_{\hat{X},0}) \subset \text{Mod}(O_{\hat{X},0})$ be the full subcategory comprised of Artinian $O_{\hat{X},0}$-modules. Then the functor $\text{Mod}(O_X) \to \text{Mod}(O_{\hat{X},0})$ taking the stalk at $0$ restricts to an equivalence $\text{Coh}_0(\hat{X}) \to \text{Mod}^{\text{Art}}(O_{\hat{X},0})$ of categories.
Example 5.5.2.4. When \( \dim X = 1 \), the ring \( O_{X,0} \) is a discrete valuation ring (DVR). Let \( \mathbb{C}(\hat{X}) \) be the fraction field of \( O_{X,0} \) (or equivalently, the field of rational functions on \( \hat{X} \)). By Lemma 5.5.2.5, the \( O_{X,0} \)-module \( \mathbb{C}(\hat{X})/O_{X,0} \) is Artinian but not finitely generated, so cannot be the stalk at \( 0 \) of any coherent \( O_X \)-module.

Lemma 5.5.2.5. Let \( R \) be a DVR with a uniformizer \( \pi \) and fraction field \( K \), then:

1. For every non-zero proper \( R \)-submodule \( M \subseteq K \), there is an integer \( n \) such that \( M = \pi^n R \).

2. The \( R \)-module \( K/R \) is Artinian but not finitely generated.

Definition 5.5.2.6. A vector bundle \( U \) on a complex analytic space \( M \) is called unipotent if it has a filtration by vector subbundles

\[ 0 = U_0 \subseteq U_1 \subseteq \cdots \subseteq U_{n-1} \subseteq U_n = U \]

such that \( U_i/U_{i-1} \cong O_M \) for \( 1 \leq i \leq n \). Denote the full subcategory of \( \text{Coh}(M) \) consisting of unipotent vector bundles by \( \text{Uni}(M) \).

By [FL14, Lem. 5.1], every unipotent vector bundle on a complex torus admits a flat holomorphic connection whose underlying local system is unipotent.

Proposition 5.5.2.7. W.I.T. with index \( g \) holds for every unipotent vector bundle on \( X \). The functor \( H^g R\hat{S} : \text{Mod}(O_X) \to \text{Mod}(O_{\hat{X}}) \) restricts to an equivalence \( \text{Uni}(X) \to \text{Coh}_0(\hat{X}) \), with a quasi-inverse \( H^0 R\hat{S} = RS : \text{Coh}_0(\hat{X}) \to \text{Uni}(X) \). Moreover, for every unipotent vector bundle \( U \to X \) and every integer \( i \geq 0 \), one has \( H^i(X, U) = \text{Ext}^i_{O_{\hat{X}}}(\mathbb{C}, \hat{U}) \).

Proof. Because \( R\hat{S} \) is a triangulated functor, the full subcategory of \( \text{Coh}(X) \) comprised of modules satisfying W.I.T. of a fixed index is closed under extensions. By Lemma 5.2.0.3 and Theorem 5.4.1.1, one has \( R\hat{S}(O_X) = R\hat{S}RS(\mathbb{C}_0) \to \mathbb{C}_0[-g] \). Then W.I.T. with index \( g \) holds for \( O_X \), so it holds for every unipotent vector bundle on \( X \). Then \( H^g R\hat{S} \to R\hat{S}[g] : \text{Uni}(X) \to \text{Mod}(O_{\hat{X}}) \). Similarly, the full subcategory of \( \text{Mod}(O_X) \) comprised of modules \( F \) with \( \text{Supp} H^g R\hat{S}(F) \subset \{0\} \) is closed under extensions and contains \( O_X \), so it contains \( \text{Uni}_X \). Since \( \text{Uni}(X) \subset \text{Coh}(X) \), the functor \( H^g R\hat{S} : \text{Mod}(O_X) \to \text{Mod}(O_{\hat{X}}) \) restricts to a functor \( \text{Uni}(X) \to \text{Coh}_0(\hat{X}) \).

For every \( F \in \text{Coh}_0(\hat{X}) \), the projection \( \text{Supp}(p_{\hat{X}}^* F \otimes \mathcal{P}) \to X \) is finite. By [GR13, Thm. 4, p.47], one has \( RS(F) = H^0 R\hat{S}(F) \). By Lemma 5.5.2.8, \( F \) has a filtration with successive quotients isomorphic to \( \mathbb{C}_0 \). Then \( RS(F) \) has a filtration with successive quotients isomorphic to \( RS(\mathbb{C}_0) = O_X \). By [Gro60, Ch. 0, 5.4.9], every term of the filtration of \( O_X \)-modules is finite locally free. Therefore, \( RS(F) \in \text{Uni}(X) \) and \( RS \) restricts to a functor \( \text{Coh}_0(\hat{X}) \to \text{Uni}(X) \).

By Theorem 5.4.1.1, \( H^g R\hat{S} : \text{Uni}(X) \to \text{Coh}_0(\hat{X}) \) is an equivalence with a quasi-inverse \( RS \). The equality follows from [Muk81, Prop. 2.7]. \( \square \)
5.5. PROPERTIES OF FOURIER-MUKAI TRANSFORM

For a commutative ring $R$, let $\text{Mod}_f(R) \subset \text{Mod}(R)$ be the full subcategory comprised of $R$-modules of finite length. Lemma 5.5.2.8 confirms a guess in [Gro61a, 9-12] for complex field.

**Lemma 5.5.2.8.** Let $X$ be a complex analytic space. Let $x \in X$.

1. Let $F$ be a coherent $O_X$-module. If $\text{Supp}(F) \subset \{x\}$, then the stalk $F_x$ is a finite length $O_{X,x}$-module. In particular, if $X$ is a singleton, then $\dim \mathbb{C}O_X$ is finite.

2. If $M$ is a finite length $O_{X,x}$-module, then $M_x$ is a coherent $O_X$-module.

3. The functor $\text{Coh}_x(X) \to \text{Mod}_f(O_{X,x})$ taking the stalk at $x$ is an equivalence.

**Proof.**

1. We may assume that $F_x \neq 0$. Then $\text{Supp}_{O_{X,x}}(F_x)$ is nonempty. As $F$ is a finite type $O_X$-module, its stalk $F_x$ is a finite $O_{X,x}$-module. Let $m_x$ be the maximal ideal of $O_{X,x}$. For every $f \in m_x$, there is an open neighborhood $U$ of $x \in X$ such that $f$ is the stalk of some $\bar{f} \in O_X(U)$. Then $\bar{f}$ vanishes on $\text{Supp}(F)$. By the Rücks Nullstellensatz (see, e.g., [GR84, p.67]), there is an integer $n \geq 1$ such that $f^n F = 0$ near $x$. In particular, $f \in \sqrt{\text{Ann}_{O_{X,x}}(F_x)}$. Therefore,

$$m_x \subset \sqrt{\text{Ann}_{O_{X,x}}(F_x)}.$$

By [GR84, Corollary, p.44], the ideal $m_x$ is finitely generated, so there is an integer $N \geq 1$ such that $m_x^N \subset \text{Ann}_{O_{X,x}}(F_x)$. By [Sta23, Tag 00L6], $\text{Supp}_{O_{X,x}}(F_x)$ is the unique closed point of $\text{Spec}(O_{X,x})$. By [Sta23, Tag 00L5], $F_x$ is a finite length $O_{X,x}$-module. The second statement follows from Fact 5.5.2.9.

2. Up to isomorphism, the only simple $O_{X,x}$-module is the residue field $\mathbb{C}$. As $M$ has finite length, $M$ has a composite series with successive quotients isomorphic to $\mathbb{C}$. Thus, $M_x$ has a filtration with successive quotients isomorphic to $\mathbb{C}_x$. Since $\mathbb{C}_x$ is coherent, by [Sta23, Tag 01BY (4)], $M_x$ is coherent.

3. Let $i_x : (x, O_{X,x}) \to (X, O_X)$ be the canonical morphism of locally ringed spaces. There is a canonical isomorphism $i_x^*(\mathcal{O}) \simeq \text{Id}_{\text{Mod}(O_{X,x})}$ of functors $\text{Mod}(O_{X,x}) \to \text{Mod}(O_{X})$. Therefore, $(i_x)_* : \text{Mod}(O_{X,x}) \to \text{Mod}(O_X)$ is fully faithful. By Point 2, $(i_x)_*$ restricts to a functor $\text{Mod}_f(O_{X,x}) \to \text{Coh}_x(O_X)$. For every object $F$ of $\text{Coh}_x(O_X)$, by Point 1, $F_x$ is an object of $\text{Mod}_f(O_{X,x})$. The adjunction morphism $F \to (i_x)_*(F_x)$ is an isomorphism. Thus, $(i_x)_* : \text{Mod}_f(O_{X,x}) \to \text{Coh}_x(O_X)$ is essentially surjective and hence an equivalence. Therefore, the functor $i_x^* : \text{Coh}_x(O_X) \to \text{Mod}_f(O_{X,x})$ taking the stalk at $x$ is an equivalence.

**Lemma 5.5.2.9.** Let $F \to A$ be a ring map, where $F$ is a field and $(A,m)$ is an Artinian local ring. If $\dim_F A/m$ is finite, then $\dim_F A$ is finite.
Proof. Because \( A \) is an Artinian local ring, by [Ati69, Prop. 8.4], there is an integer \( n > 0 \) with \( m^n = 0 \). For every integer \( i \geq 0 \), the \( A \)-module \( m^i \) is finitely generated, so the \( A/m \)-module \( m^i/m^{i+1} \) is finitely generated. Thus, \( \dim_F m^i/m^{i+1} = \dim_F A/m \cdot \dim_F m^i/m^{i+1} \) is finite. Then \( \dim_F A = \sum_{i=0}^{n} \dim_F m^i/m^{i+1} \) is finite. \( \square \)

### 5.5.3 Homogeneous vector bundles

**Definition 5.5.3.1.** A vector bundle \( E \to X \) is called homogeneous if for every \( x \in X \), one has \( T^*_x E \cong E \). Let \( H(X) \subset \text{Coh}(X) \) be the full subcategory of homogeneous vector bundles.

For a complex analytic space \( M \), let \( \text{Coh}_f(M) \subset \text{Coh}(M) \) be the full subcategory consisting of objects with finite support.

**Proposition 5.5.3.2.**

1. For every integer \( i \), the functor \( H^i R\hat{S} : \text{Mod}(O_X) \to \text{Mod}(\hat{O}_X) \) restricts to a functor \( H(X) \to \text{Coh}_f(\hat{X}) \).

2. \( W.I.T. \) holds for every homogeneous vector bundle on \( X \) with index \( g \).

3. The functor \( H^g R\hat{S} : \text{Mod}(O_X) \to \text{Mod}(\hat{O}_X) \) restricts to an equivalence of categories \( H(X) \to \text{Coh}_f(\hat{X}) \).

**Proof.**

1. Let \( E \) be a homogeneous vector bundle on \( X \). For every \( x \in X \), by Proposition 5.5.1.1, one has \( R\hat{S}(E) \xrightarrow{\sim} R\hat{S}(T^*_x E) \xrightarrow{\sim} P^*_x \otimes R\hat{S}(E) \), so \( H^i R\hat{S}(E) \xrightarrow{\sim} P^*_x \otimes H^i R\hat{S}(E) \). By Corollary 5.3.1.14, the \( O_X \)-module \( H^i R\hat{S}(E) \) is coherent. From Lemma 5.5.3.4, it has finite support.

2. For every integer \( i \neq g \), one has

\[
0 = H^{i-g}([-1]_X E) \\
= H^i([-1]_X E)[-g]) \\
\xrightarrow{(a)} H^i R\hat{S} \circ R\hat{S}(E) \\
= H^i R\hat{S}(P \otimes^L p^*_X R\hat{S}(E)) \\
\xrightarrow{(b)} H^0 R\hat{S}(P \otimes^L p^*_X H^i R\hat{S}(E)) \\
= H^0 R\hat{S}(H^i R\hat{S}(E)),
\]

where (a) (resp. (b)) uses Theorem 5.4.1.1 (resp. Point 1 and [GR13, Thm. 4, p.47]).

It remains to prove that for every \( F \in \text{Coh}_f(\hat{X}) \) with \( H^0 R\hat{S}(F) = 0 \), one has \( F = 0 \). Since \( F \) is the direct sum of finitely many coherent submodules whose supports are singletons, one may assume that \( \text{Supp}(F) \) is a singleton. By Proposition 5.5.1.1, one may assume that \( F \in \text{Coh}_0(\hat{X}) \). From Proposition 5.5.2.7, \( F = 0 \).
3. By Point 1, the functor $H^0R\hat{S} : \text{Mod}(O_X) \to \text{Mod}(O_{\hat{X}})$ restricts to a functor $H(X) \to \text{Coh}_f(\hat{X})$. From Point 2, one has $H^0R\hat{S} = R\hat{S}[g] : H(X) \to \text{Coh}_f(\hat{X})$.

By Propositions 5.5.1.1 and 5.5.2.7, the functor $H^0RS : \text{Mod}(O_X) \to \text{Mod}(O_X)$ restricts to a functor $H^0RS = RS : \text{Coh}_f(\hat{X}) \to H(X)$. By Theorem 5.4.1.1, the functor $H^0RS : H(X) \to \text{Coh}_f(\hat{X})$ is an equivalence with a quasi-inverse $H^0RS$.

For a complex analytic space $M$ and an $O_M$-module $F$, set $T(M)$ to be the torsion part of $M$ ([CD94, p.60]).

**Lemma 5.5.3.3.** Let $X$ be a compact Kähler manifold. Let $C$ be an irreducible component of $\text{Supp}(F)$. Then for every coherent $O_X$-module $F$, there is a connected compact Kähler manifold $Z$ and a morphism $h : Z \to X$, such that $h(Z) = C$ and $h^*F/T(h^*F)$ is a vector bundle on $Z$ of positive rank.

**Proof.** By [GR84, p.76], $\text{Supp}(F)$ is an analytic set in $X$. Because $X$ is a Kähler manifold, with the induced reduced complex structure, the subspace $C$ is a Kähler space in the sense of [Var89, II, 1.3]. Let $i : C \to X$ be the inclusion and

$$D = \{x \in C : i^*F \text{ is not locally free at } x\}.$$ 

From [Ros68, Prop. 3.1], $D$ is a proper analytic subset of $C$. By Rossi’s theorem (see, e.g. [Rie71, Thm. 2]), there is a reduced irreducible complex analytic space $W$ and a proper modification $f : W \to C$, such that $W \setminus f^{-1}(D) \to C \setminus D$ is biholomorphic and $E := N/T(N)$ is a vector bundle on $W$, where $N = f^*i^*M$. From [GD71, Cor. 5.24.1], one has $\text{Supp}(N) = W$. From [CD94, I, Thm. 9.12], one gets $\text{Supp}(T(N)) \neq W$. Therefore, the rank $r$ of the vector bundle $E$ on $W$ is positive.

Since $f$ is bimeromorphic, the space $W$ is in the Fujiki class $\mathcal{C}$ (defined in [Fuj78, p.34]). By [Fuj78, Lem. 4.6, 1], there is a connected compact Kähler manifold $Z$ with a surjective morphism $g : Z \to W$. Let $h : Z \to X$ be the composition $ifg$. Then $h(Z) = C$. As $E$ is flat over $O_W$, by [Sta23, Tag 05NJ], applying $g^*$ to the natural short exact sequence

$$0 \to T(N) \to N \to E \to 0$$

in $\text{Mod}(O_W)$, one gets a short exact sequence in $\text{Mod}(O_Z)$:

$$0 \to g^*T(N) \to h^*F \to g^*E \to 0.$$ 

As $g^*E$ is torsion free, $g^*T(N) \supset T(h^*F)$. One has $g^*T(N) \subset T(g^*N) = T(h^*F)$. Therefore, $T(h^*F) = g^*T(N)$ and $h^*F/T(h^*F) = g^*E$ is a vector bundle on $Z$ of rank $r$.

**Lemma 5.5.3.4.** Let $M$ be a coherent sheaf on the complex torus $X$. If $M \otimes P \cong M$ for all $P \in \text{Pic}^0(X)$, then $\text{Supp}(M)$ is finite.
Proof. Suppose the contrary that $\text{Supp}(M)$ is infinite. With the induced reduced complex structure, the complex subspace $\text{Supp}(M)$ has positive dimension. Let $C$ be an irreducible component of $\text{Supp}(M)$ of maximal dimension. Take a morphism $h: Z \to X$ provided by Lemma 5.5.3.3. Then the rank $r$ of the vector bundle $E := h^*M/T(h^*M)$ is positive. As $h(Z) = C$, the morphism of complex tori $h^*: \text{Pic}^0(X) \to \text{Pic}^0(Z)$ is nonzero. In particular, there is $L \in \text{Pic}^0(X)$ such that the line bundle $(h^*L)^r$ is non trivial.

On the other hand, we claim that the line bundle $(h^*L)^r$ is trivial. Indeed, by assumption $M \otimes L \cong M$, so $h^*M \otimes h^*L \cong h^*M$. Since $T(h^*M \otimes h^*L) = T(h^*M) \otimes h^*L$, one gets $E \otimes h^*L \cong E$. Taking the determinant of both sides, one has $\det(E) \otimes (h^*L)^r \cong \det(E)$. As $\det(E)$ is an invertible sheaf, the line bundle $(h^*L)^r$ on $Z$ is trivial. The claim is proved, which gives a contradiction. \[\Box\]

Remark 5.5.3.5. The proof of [Muk81, Lem. 3.3] (the algebraic counterpart of Lemma 5.5.3.4) relies on the following fact: Every positive dimensional projective variety contains a projective curve. By contrast, every simple non-algebraic complex torus contains no 1-dimensional analytic subset ([Pil00, Lem. 4.3]).

The classification of homogeneous vector bundles on complex tori is due to Matsushima [Mat59] and Morimoto [Mor59]. Using the Fourier-Mukai transform, Mukai [Muk81, p.159] proves an analog for abelian varieties. We can similarly recover Matsushima-Morimoto’s theorem.

Theorem 5.5.3.6. A vector bundle $F$ on the complex torus $X$ is homogeneous if and only if there is an integer $n \geq 0$, unipotent vector bundles $U_1, \ldots, U_n$ on $X$ and $P_1, \ldots, P_n \in \text{Pic}^0(X)$, such that $F$ is isomorphic to $\oplus_{i=1}^n P_i \otimes U_i$.

Proof. It follows from Propositions 5.5.1.1, 5.5.2.7 and 5.5.3.2. \[\Box\]
Chapter 6
Sheaves with connection on complex tori

6.1 Introduction

For a (not necessarily commutative) ringed space $(Z, \mathcal{R})$, let $\text{Mod}(\mathcal{R})$ be the category of left $\mathcal{R}$-modules. Given a symbol $* \in \{\emptyset, +, -, b\}$, the notation $D^*_c(\mathcal{R})$ refers to the unbounded/bounded below/bounded above/bounded derived category of $\text{Mod}(\mathcal{R})$ in order. Let $D^*_c(\mathcal{R}) \subset D^*(\mathcal{R})$ be the full subcategory of objects whose cohomologies are $\mathcal{R}$-coherent in the sense of [Sta23, Tag 01BV].

Let $k$ be an algebraically closed field and $A/k$ be an abelian variety with $\dim A = g$. Let $B/k$ be the abelian variety dual to $A$ and $P$ be the Poincaré line bundle on $A \times B := A \times_k B$. The $O$-module Fourier-Mukai transform introduced in [Muk81, Sec. 2] is (up to a sign convention):

$$
\begin{align*}
RS_1 &= R_{p_A*}(\mathcal{P} \otimes^L p_B^*?) : D(O_B) \to D(O_A); \\
RS_2 &= R_{p_B*}(\mathcal{P}^{-1} \otimes^L p_A^*?) : D(O_A) \to D(O_B),
\end{align*}
$$

where $p_A$ (resp. $p_B$) denote the projection from $A \times B$ to $A$ (resp. $B$). Its fundamental property is an analog of Fourier inversion formula, recalled in Fact 6.1.0.1. For an algebraic variety $V/k$, let $D_{qc}(O_V) \subset D(O_V)$ be the full subcategory comprised of complexes whose cohomologies are quasi-coherent as $O_V$-modules.

**Fact 6.1.0.1** (Mukai, [Muk81, Thm. 2.2], [Rot96, p.569]). 1. There are natural isomorphisms of functors $RS_1 \circ RS_2 \cong T^{-g}$ on $D_{qc}(O_A)$ and $RS_2 \circ RS_1 \cong T^{-g}$ on $D_{qc}(O_B)$, where $T$ denotes the respective degree shift automorphism on the derived categories. In particular, $RS_1 : D_{qc}(O_B) \to$

\[1\] Notice that for an algebraic variety $V$, the notation $\text{Mod}(O_V)$ in [Rot96, Rot97] refers to the category of quasi-coherent $O_V$-modules, a convention inherited from [Rot96, p.565].

\[2\] Algebraic variety refers to an integral separated scheme of finite type over an algebraically closed field.
Chapter 6. Sheaves with Connection on Complex Tori

$D_{qc}(O_A)$ is an equivalence of triangulated categories, with a quasi-inverse $T^gR_S$.

2. The functor $RS_1$ sends $D^b_c(O_B)$ to $D^b_c(O_A)$, and the restriction $RS_1 : D^b_c(O_B) \rightarrow D^b_c(O_A)$ is also an equivalence.

Let $0 \rightarrow G^0 \rightarrow B^2 \xrightarrow{g} B \rightarrow 0$ be the universal vectorial extension of $B$ constructed in [Ros58, Prop. 11]. Laumon and Rothstein independently lift the $O$-module Fourier-transform to $D$-modules and establish a duality result similar to Fact 6.1.0.1. Let $D_{qc}(D_A)$ be the full subcategory of $D(D_A)$ consisting of complexes whose cohomologies are quasi-coherent as $O_A$-modules.

**Fact 6.1.0.2** (Laumon, Rothstein).

1. There are functors $RS_1 : D(O_B)$ to $D(D_A)$ and $RS_2 : D(D_A) \rightarrow D(O_B)$ fitting into the following commutative squares

\[
\begin{array}{c}
D_{qc}(O_B) \xrightarrow{RS_1} D_{qc}(O_A) \\
\downarrow{R_{p_*}} & \downarrow{F_{forA}} \\
D_{qc}(O_B) & \xrightarrow{RS_2} D_{qc}(O_A) \\
\end{array}
\]  

(6.2)

\[
\begin{array}{c}
D_{qc}(O_B) \xleftarrow{RS_2} D_{qc}(O_A) \\
\downarrow{R_{p_*}} & \downarrow{F_{forA}} \\
D_{qc}(O_B) & \xleftarrow{RS_1} D_{qc}(O_A) \\
\end{array}
\]  

(6.3)

2. ([Lau96, Thm. 3.2.1], [Rot96, Thm. 4.5], [Rot97], [Vig21, Thm. 2.2.21], Remark D.1.0.12) $RS_1RS_2 = T^{-g}$ on $D_{qc}(D_A)$ and $RS_2RS_1 = T^{-g}$ on $D_{qc}(O_B)$, hence an equivalence $RS_1 : D_{qc}(O_B) \rightarrow D_{qc}(D_A)$.

3. ([Lau96, Cor. 3.1.3], [Rot96, Thm. 6.2]) The functor $RS_1$ sends $D^b_c(O_B)$ to $D^b_c(D_A)$, and the restriction $RS_1 : D^b_c(O_B) \rightarrow D^b_c(D_A)$ is also an equivalence.

**Remark 6.1.0.3.** Laumon and Rothstein uses apparently different definitions for the functors on $D$-modules. Laumon does not mention the commutativity of (6.2) and (6.3) in [Lau96], while Rothstein implicitly uses this compatibility when deriving [Rot96, (2.25)]. For Rothstein’s version, the compatibility can be proved as in Proposition 6.3.0.2. We sketch why the two definitions agree.

Recall $\mathcal{F} : D^b_{qc}(D_A) \rightarrow D^b_{qc}(O_B)$ and $\mathcal{F}^2 : D^b_{qc}(O_B) \rightarrow D^b_{qc}(D_A)$ from [Lau96, p.14] in the notation of [Vig21, Sec. 2.1.1].

\[
\begin{array}{c}
D^b_{qc}(D_A) \xrightarrow{\mathcal{F}} D^b_{qc}(O_B) \\
\downarrow{\tilde{p}^{b,(O_B)}} & \downarrow{\tilde{p}^{b,(O_B)}} \\
D^b_{qc}(D_B \times A/B) & \xrightarrow{\mathcal{F}^2} D^b_{qc}(D_A) \\
\end{array}
\]  

\[
\begin{array}{c}
D^b_{qc}(D_B \times A/B) \xrightarrow{\tilde{p}^{b,(D_B \times A/B)}} D^b_{qc}(D_B \times A/B) \\
\downarrow{\tilde{p}^{b,(D_B \times A/B)}} & \downarrow{\tilde{p}^{b,(D_B \times A/B)}} \\
D^b_{qc}(D_A \times B/B) & \xrightarrow{\mathcal{F}^2} D^b_{qc}(D_A) \\
\end{array}
\]  

(6.3)

(6.4)

---

\[\text{Attention that it is denoted by } A^2 \text{ in [Lau96].}\]
Applying projection formula (to $\text{Id}_{A \times p} : A \times B^\sharp \to A \times B$) and flat base change to the cartesian square

\[
\begin{array}{ccc}
A \times B^\sharp & \xrightarrow{\text{Id}_{A \times p}} & A \times B \\
\downarrow{p_{B^\sharp}} & & \downarrow{p_B} \\
B^\sharp & \to & B,
\end{array}
\]

one can prove that for $\widetilde{\mathcal{F}}^\sharp = F'^{*}p_{*}$ as functors $D^b_{\text{qc}}(O_{B^\sharp}) \to D^b_{\text{qc}}(O_A)$. This shows the compatibility with Fourier-Mukai transform as well as that $[-1]_{A}^{*}\widetilde{\mathcal{F}}^\sharp$ is the restriction of $[\text{Rot96}, (4.16)]$ to $D^b_{\text{qc}}(O_{B^\sharp})$. (The sign is due to different conventions.)

**Remark 6.1.0.4.** The bifunctor $\otimes_O$ on relative $D$-modules is compatible with that on the underlying $O$-modules, see [Sch14, p.97] and the square in [HT07, p.38]. However, the two following triangles

\[
\begin{align*}
D^b_{\text{qc}}(D_{A \times B^\sharp / B^\sharp}) & \xrightarrow{(\tilde{\mathcal{P}}, \tilde{\nabla}) \otimes^L_{D_{A \times B^\sharp}} \cdot} D^b_{\text{qc}}(D_{A \times B^\sharp / B^\sharp}), \\
& \xrightarrow{\text{for}} D^b_{\text{qc}}(O_{A \times B^\sharp}), \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \ quad
In particular, $RS_1 : D_{\text{good}}(O_X) \to D_{\text{good}}(O_Y)$ is an equivalence of categories with a quasi-inverse $T^0 RS_2$.

2. ([PPS17, Thm. 13.1]) The functors $RS_i$ preserve $D^b_c(O)$.

In this text, we lift the $O$-module Fourier-Mukai transform on complex tori to $D$-modules and give an analytic analog of Fact 6.1.0.2. The main result is summarized in Theorem 6.1.0.6, where $D$-good modules are reviewed in Section 6.6.1. Let $D_{O-\text{good}}(*) \subset D(*)$ be the full subcategory of objects whose cohomologies are $O$-good.

**Theorem 6.1.0.6.**

- (Prop. 6.5.0.2) There is a canonical sheaf of rings $A$ on $X$, such that the transforms $RS_1$ and $RS_2$ lift naturally to $RS_1 : D(A) \to D(D_Y)$ and $RS_2 : D(D_Y) \to D(A)$ respectively.
- (Thm. 6.5.0.3) The restrictions $RS_1 : D_{O-\text{good}}(A) \to D_{O-\text{good}}(D_Y)$ and $RS_2 : D_{O-\text{good}}(D_Y) \to D_{O-\text{good}}(A)$ are equivalences.
- (Thm. 6.6.3.1) Let $D^b_{\text{good}}(D_Y) \subset D^b(D_Y)$ be the full subcategory of objects whose cohomologies are $D_Y$-good. Then the composition $RS_1 RS_2$ preserves $D^b_{\text{good}}(D_Y)$.

**Remark 6.1.0.7.** By Proposition E.5.4.7, the complex Lie group $(A^*)^m$ is isomorphic to $(\mathbb{C}^*)^{2g}$. In light of [Fav12, Thm. 3], an analytic version of Fact 6.1.0.2 needs a modification. The proof of Fact 6.1.0.2 due to Laumon [Lau96] and that of Rothstein [Rot97] are different. As an immediate step, Laumon proves in [Lau96, Thm. 2.4.1] that the adjunction morphism $O_{\text{Spec}(k)} \to R\pi_*^1 O_{A^*}$ is an isomorphism in $D^b_{\text{qc}}(O_{\text{Spec}(k)})$, where $\pi^1 : A^* \to \text{Spec}(k)$ is the structural morphism. However, it is no longer true in the analytic context. When $k = \mathbb{C}$, the adjunction morphism $O_{\text{Specan}(\mathbb{C})} \to R(\pi^1)^* O_{(A^*)^m}$ is not an isomorphism.

Still, the proof of [Rot97] works for complex tori. We follow4 it closely, except

---

4We should notice four misprints therein.

- [Rot97, (2.11)] should be
  \[ \bar{\pi}_{12}^* P = [(1_X \times \bar{m})^* P] \oplus_{O_X \times Y \times Y} [\bar{\pi}_{13}^* P]^{1-1}, \]
  where $\bar{\pi}_{ij}$ denotes the projections on $X \times Y \times Y$ and $\bar{Y}$ is the first-order neighborhood of $0 \in Y$.
- [Rot97, (2.23)] should be
  \[ \pi_{13}^* P -1 \otimes \pi_{23}^* P = (\epsilon_X \times 1_Y)^* P, \]
  where $\pi_{ij}$ denotes the projections on $X \times X \times Y$.
- [Rot97, (2.24)], the starting equation should be
  \[ \bar{\pi}_{12}^* O_{\Delta} \otimes \bar{\pi}_{13}^* P -1 \otimes \bar{\pi}_{23}^* P, \]
- [Rot97, Prop. 2.4], the notation $\text{Mod}(X \times X)_{(-1,1)-sp}$ should be $\text{Mod}(X \times X)_{(1,-1)-sp}$.
that the underived Fourier-Mukai transforms [Rot97, (2.14), (2.15)] are ignored. Instead, we define the corresponding functors on the derived categories directly.

6.2 Preliminaries

For the convenience of the reader, we recall the notation of [Rot97, Sec. 2.1].

6.2.1 Categories of splittings

For a (holomorphic) vector bundle $M \to Z$ on a complex manifold, by [Har77, III, Prop. 6.3 (c)], we have $H^1(Z, M) = \text{Ext}^1(O_Z, M)$. Thus, every $\alpha \in H^1(Z, M)$ determines an exact sequence

$$0 \to M \to E_\alpha \xrightarrow{\mu_\alpha} O_Z \to 0. \quad (6.6)$$

Since $O_Z$ is a flat $O_Z$-module, by [Sta23, Tag 05NJ], for every $F \in \text{Mod}(O_Z)$, the sequence (6.6) remains exact after tensored with $F$:

$$0 \to M \otimes F \to E_\alpha \otimes F \xrightarrow{\mu_\alpha \otimes \text{Id}_F} F \to 0. \quad (6.7)$$

**Definition 6.2.1.1.** Define a category $\text{Mod}(O_Z)_{\alpha - sp}$ as follows: the objects are pairs $(F, \psi)$, where $F \in \text{Mod}(O_Z)$ and $\psi : F \to E_\alpha \otimes O_Z F$ is an $\alpha$-splitting on $F$, i.e., an $O_Z$-linear splitting of $\mu_\alpha \otimes \text{Id}_F$. The morphisms in $\text{Mod}(O_Z)_{\alpha - sp}$ are required to be compatible with the splittings.

**Example 6.2.1.2.** When $\alpha = 0$, the sequence (6.6) identifies $E_0$ with $M \oplus O_Z$.

There is a natural functor $\text{Mod}(O_Z) \to \text{Mod}(O_Z)_{\alpha - sp}$ defined by $F \mapsto (F, \psi)$, where $\psi : F \to E_0 \otimes F = (M \otimes F) \oplus F$ is the canonical injection to the second factor.

**Lemma 6.2.1.3.** For an $O_Z$-module $F$, there is an $\alpha$-splitting on $F$ if and only if the map $i_* : H^1(Z, M) \to H^1(Z, M \otimes O_Z \text{End}(F))$ (induced by the natural morphism $O_Z \to \text{End}(F)$) sends $\alpha$ to 0. In that case, the set of $\alpha$-splittings on $F$ has a natural simple transitive action of the abelian group $\text{Hom}_{O_Z}(F, M \otimes F)$.

**Proof.** The natural morphism $O_Z \to \text{End}(F)$ induces a morphism $i : M \to \text{Hom}(F, M \otimes F)$ defined by $i(m)(f) = m \otimes f$. There is a canonical evaluation morphism $e : \text{Hom}(F, M \otimes F) \otimes F \to M \otimes F$ defined by $e(\phi \otimes f) = \phi(f)$.

The five-term exact sequence of the spectral sequence

$$E^3_{ij} = \text{Ext}^i(O_Z, \text{Ext}^j(F, M \otimes F)) \implies \text{Ext}^{i+j}(F, M \otimes F)$$

gives an injection $i : \text{Ext}^i(O_Z, \text{Hom}(F, M \otimes F)) \to \text{Ext}^1(F, M \otimes F)$, which is $e_v \circ (\cdot \otimes F)$.

Note that $e_v \circ (i \otimes \text{Id}_F)(m \otimes f) = e_v(i(m) \otimes f) = i(m)(f) = m \otimes f$, i.e., $e_v \circ (i \otimes \text{Id}_F) = \text{Id}_{M \otimes F}$. Therefore, the following diagram is commutative:
\[ \text{Ext}^1(F, M \otimes F) \cong \text{Ext}^1(OZ, M) \] 
\[ \cong \text{Ext}^1(F, \text{Hom}(F, M \otimes F) \otimes F) \]
\[ \cong \text{Ext}^1(OZ, \text{Hom}(F, M \otimes F)). \]

Then \( F \) admits an \( \alpha \)-splitting if \( \alpha \otimes F = 0 \) iff \( \iota_\alpha(\alpha) = 0 \). Any two \( \alpha \)-splittings on \( F \) differ by a unique element of \( \text{Hom}(F, M \otimes F) \).

To each object \( (F, \psi) \in \text{Mod}(OZ)_{\alpha-\text{sp}} \), we associate an element
\[ [\psi, \psi] \in \Gamma(Z, (\bigwedge M) \otimes_{OZ} \text{End}(F)) \]
as follows.

The sequence (6.6) induces a short exact sequence
\[ 0 \to \bigwedge M \to \bigwedge F \to \bigwedge F \to 0, \]
where
\[ \omega_\alpha(\rho_1 \wedge \rho_2) = \mu_\alpha(\rho_1)\rho_2 - \mu_\alpha(\rho_2)\rho_1. \]
The flatness of \( M \) ensures the exactness when tensoring with each \( OZ \)-module \( F \):
\[ 0 \to (\bigwedge M) \otimes F \to (\bigwedge F) \otimes F \to \bigwedge F \to 0. \]

For every object \( (F, \psi) \in \text{Mod}(OZ)_{\alpha-\text{sp}} \), let \( \psi^1 \) be the composition
\[ \bigwedge F \xrightarrow{\text{Id}_\bigwedge} \bigwedge F \xrightarrow{\psi} \bigwedge F \xrightarrow{\text{Id}_\bigwedge} \bigwedge F, \]
where the middle isomorphism is from the associativity of tensor product and \( a : \bigwedge F \to \bigwedge F \) is defined by \( e \otimes e' \mapsto e \wedge e' \).

**Lemma 6.2.1.4.** If \( \psi \) is an \( \alpha \)-splitting on an \( OZ \)-module \( F \), then \( (\omega_\alpha \otimes \text{Id}_F) \circ \psi^1 \psi = 0. \)

**Proof.** Locally, the vector bundle \( \bigwedge F \) has a (holomorphic) frame \( \{e_1, \ldots, e_r\} \).

For a local section \( f \in F \), write \( \psi(f) = \sum_{i=1}^r e_i \otimes f_i \), where \( f_i \) are local sections of \( F \). Similarly, \( \psi(f_i) = \sum_{j=1}^r e_j \otimes f_i^{(j)} \), where \( f_i^{(j)} \) are local sections of \( F \). As \( \psi \) is a section to \( \mu_\alpha \otimes \text{Id}_F \), we have
\[ f = (\mu_\alpha \otimes \text{Id}_F) \psi(f) = \sum_{i=1}^r \mu_\alpha(e_i)f_i; \]
\[ f_i = (\mu_\alpha \otimes \text{Id}_F) \psi(f_i) = \sum_{j=1}^r \mu_\alpha(e_j)f_i^{(j)}. \]
Thus,
\[ \psi(f) = \mu_\alpha(e_i) \psi(f_i). \]

By construction, \( \psi \psi(f) = \sum_{i,j=1}^r (e_i \wedge e_j) \otimes f_j^{(i)} \). Then
\[
(\omega_\alpha \otimes \text{Id}_F) \psi \psi(f) = \sum_{i,j=1}^r [\mu_\alpha(e_i)e_j - \mu_\alpha(e_j)e_i] \otimes f_j^{(i)}
\]
\[
= \sum_{i=1}^r \mu_\alpha(e_i) \sum_{j=1}^r e_j \otimes f_j^{(i)} - \sum_{i=1}^r e_i \otimes \left[ \sum_{j=1}^r \mu_\alpha(e_j) f_j^{(i)} \right]
\]
\[
(6.11) \sum_{i=1}^r \mu_\alpha(e_i) \psi(f_i) - \sum_{i=1}^r e_i \otimes f_i
\]
\[
(6.12) \psi(f) - \psi(f) = 0.
\]

From Lemma 6.2.1.4 and (6.9), we have \( \psi \psi(F) \subset (\wedge^2 M) \otimes F \), hence an element \([\psi, \psi] \in \Gamma(Z, (\wedge^2 M) \otimes \text{O}_Z \text{End}(F))\).

**Example 6.2.1.5.** For the complex torus \( X \), set \( g = H^1(X, \text{O}_X) \). Then
\[
H^1(X, g^* \otimes \text{O}_X) = g^* \otimes \text{g} = \text{End}(g).
\]
Hence a category \( \text{Mod}(\text{O}_X)_{T-sp} \) for each \( T \in \text{End}(g) \). The identity element \( 1 \in \text{End}(g) \) corresponds to the tautological exact sequence [Rot96, (1.3)]:
\[
0 \rightarrow g^* \otimes \text{O}_X \rightarrow E \rightarrow \text{O}_X \rightarrow 0.
\]

We also write \( \text{Mod}(\text{O}_X)_{sp} \) for \( \text{Mod}(\text{O}_X)_{1-sp} \). For \( (F, \psi) \in \text{Mod}(\text{O}_X)_{sp} \), the element \([\psi, \psi] \) lies in
\[
\Gamma(X, \bigwedge^2 g^* \otimes \text{O}_X \otimes \text{O}_X \text{End}(F)) = \bigwedge^2 g^* \otimes \text{End}(F)
\]
and we recover [Rot96, (4.8)]. Similarly, \( H^1(X \times X, g^* \otimes O_{X \times X}) = \text{End}(g) \oplus \text{End}(g) \), so for every pair \( T_1, T_2 \in \text{End}(g) \), the category \( \text{Mod}(O_{X \times X})(T_1, T_2)_{-sp} \) is defined.

**Example 6.2.1.6.** When \( M = \Omega_{Z}^1 \), and \( \alpha = 0 \), then an \( \alpha \)-splitting \( \phi \) on a holomorphic vector bundle \( E \rightarrow Z \) is exactly a holomorphic 1-form on \( Z \) with values in \( \text{End}(E) \). The pair \( (E, \phi) \) is a Higgs bundle in the sense of [BGPG07, p.980] if and only if \([\phi, \phi] = 0\).
6.2.2 Categories of twisted connection

We continue to review the twisted (relative) connection introduced in [Rot97, p.206]. Consider a smooth morphism of complex manifolds $f : Z \to S$, with relative cotangent sheaf $\Omega^1_f$. As $f$ is smooth, the sheaf $\Omega^1_f$ is a vector bundle on $Z$. Let $d_f : O_Z \to \Omega^1_f$ denote the differential relative to $f$. An element $\alpha \in H^1(Z, \Omega^1_f)$ determines an extension
\[
0 \to \Omega^1_f \to \mathcal{E}_\alpha \xrightarrow{\mu} O_Z \to 0. \tag{6.14}
\]

**Definition 6.2.2.1.** On an $O_Z$-module $G$, an $\alpha$-connection is an $f^{-1}(O_S)$-linear splitting $\nabla : G \to \mathcal{E}_\alpha \otimes_{O_Z} G$ to $\mu_\alpha \otimes \text{Id}_G$, satisfying the Leibniz rule
\[
\nabla(h\phi) = h\nabla(\phi) + df(h) \otimes \phi, \tag{6.15}
\]
where $h$ and $\phi$ are local sections of $O_Z$ and $G$ respectively. Let $\text{Mod}(O_Z)_{f,\alpha-\text{cxn}}$ be the category of pairs $(G, \nabla)$, where $G \in \text{Mod}(O_Z)$ and $\nabla$ is an $\alpha$-connection on $G$.

**Example 6.2.2.2.** If $\alpha = 0$, then $\alpha$-connection are exactly $f$-relative connection. Define a sheaf $\mathcal{D}_{Z/S}$ of noncommutative $O_Z$-algebras by gluing the following local data. If $\{\xi_1, \ldots, \xi_n\}$ is a local frame of $(\Omega^1)^\vee_f$ (the vector bundle dual to $\Omega^1_f$) on an open subset $U \subset Z$, then a multiplication law on $O_U: \{\xi_1, \ldots, \xi_n\}$ is introduced by imposing the commutation relation $[\xi_i, h] = \xi_i(h)$ for local sections $h$ of $O_Z$. Let it be $\mathcal{D}_{Z/S}|_U$. Then $\text{Mod}(Z)_{f,0-\text{cxn}} = \text{Mod}(\mathcal{D}_{Z/S})$. The category $\text{Mod}(O_Z)_{f,0-\text{cxn}}$ is denoted by $\text{Mod}(O_Z)_{\text{cxn}}$ when $f$ is the structure morphism $Z \to \text{Specan}(\mathbb{C})$.

**Remark 6.2.2.3.** In fact, a twisted connection is a particular splitting. Define another extension
\[
0 \to \Omega^1_f \to \mathcal{E}_{\alpha'} \to O_Z \to 0 \tag{6.16}
\]
in $\text{Mod}(O_Z)$ as follows. As an extension of abelian sheaves, (6.16) is same as (6.14). Let $h$ (resp. $s'$) be a local section of $O_Z$ (resp. $\mathcal{E}_{\alpha'}$) and $s$ denote the local section of $\mathcal{E}_\alpha$ induced by $s'$. The $O_Z$-module structure on $\mathcal{E}_{\alpha'}$ is defined such that the local section $hs + \mu_\alpha(s)dfh$ of $\mathcal{E}_\alpha$ induces the local section $hs'$ of $\mathcal{E}_{\alpha'}$.

We claim this indeed defines an $O_Z$-module structure on $\mathcal{E}_{\alpha'}$. For local sections $h_1, h_2$ of $O_Z$, let $t$ be the local section of $\mathcal{E}_{\alpha}$ induced by $h_2s'$. Then $t = h_2s + \mu_\alpha(s)dfh_2$, so $\mu_\alpha(t) = h_2\mu_\alpha(s)$. Thus, the local section of $\mathcal{E}_{\alpha}$ corresponding to $h_1(h_2s')$ is
\[
h_1t + \mu_\alpha(t)dfh_1 = h_1h_2s + h_1\mu_\alpha(s)dfh_2 + h_2\mu_\alpha(s)dfh_1 = (h_1h_2)s + \mu_\alpha(s)df(h_1h_2).
\]
Therefore, $h_1(h_2s') = (h_1h_2)s'$. The claim is proved.

By construction, the morphisms in (6.16) are $O_Z$-linear. Then (6.16) is indeed an extension in $\text{Mod}(O_Z)$, hence a new extension class $\alpha' \in \text{Ext}(O_Z, \Omega^1_f)$. An $\alpha$-connection on a module $G \in \text{Mod}(O_Z)$ is equivalent to an $\alpha'$-splitting on $G$. Hence an equivalence of categories
\[
\text{Mod}(O_Z)_{f,\alpha-\text{cxn}} \to \text{Mod}(O_Z)_{\alpha'-\text{sp}}.
\]
There is a notion of integrable $\alpha$-connection ([Rot97, Remark, p.206]). Let $\text{Mod}(O_Z)_{f,\alpha-\text{cxn},\#}$ be the full subcategory of $\text{Mod}(O_Z)_{f,\alpha-\text{cxn}}$ comprised of objects whose connection are integrable. Then $\text{Mod}(O_Z)_{f,\alpha-\text{cxn},\#}$ coincides with MIC$(f)$ defined in [ABC20, 4.3.7], which is further equivalent to $\text{Mod}(D_{Z/S})$. Here $D_{Z/S}$ is the sheaf of ring of relative differential operators on $Z/S$ defined in [SS94, p.9].

**Example 6.2.2.4.** Consider the projection $p_X : X \times Y \to X$. Since $\Omega^1_{p_X} = p_X^*(g^* \otimes_C O_X)$, there is a natural morphism

$$p_X^* : \text{End}(g) = H^1(X, g^* \otimes_C O_X) \to H^1(X \times Y, \Omega^1_{p_X}).$$

For every $T \in \text{End}(g)$, the category $\text{Mod}(O_{X\times Y})_{p_X^*p_XT-\text{cxn}}$ (resp. $\text{Mod}(O_{X\times Y})_{p_X^*p_XT-\text{cxn},\#}$) is also written as $\text{Mod}(O_{X\times Y})_{T-\text{cxn}}$ (resp. $\text{Mod}(O_{X\times Y})_{T-\text{cxn},\#}$).

**Fact 6.2.2.5.** The Poincaré bundle $P$ is naturally an object of $\text{Mod}(O_{X\times Y})_{-1-\text{cxn},\#}$.

In local coordinates, the $p_X^*(-1)$-connection on $P$ is explained in [Rot96, (1.10) and p.575ff] (except that we use a Stein open cover of $X$ at the place of Rothstein’s affine open cover).

### 6.2.3 Functors between them

Recall that the Fourier-Mukai transform (6.1) is the composition of the pullback, the tensor product with $P$ as well as the derived direct image. Rothstein’s lift of Fourier-Mukai transform on modules with connection keeps an extra track of the splittings and connection.

**Remark 6.2.3.1.** Combining [Rot97, (2.21)] with the fact that twisted relative connection are kinds of splittings (Remark 6.2.2.3), the categories under consideration ($\text{Mod}(O_X)_{T-\text{sp}}$, $\text{Mod}(O_{X\times Y})_{T-\text{cxn}}$ etc.) are equivalent to categories of modules over sheaves of certain noncommutative flat $O$-algebras. In particular, each of them is a Grothendieck abelian category. Each has enough K-injectives ([Sta23, Tag 079P]) and enough objects flat over $O$ ([HT07, Lem. 1.5.2 (ii)]), cf. [Rot97, Cor. 2.3]. Thus, all the (left exact) direct image functors involved below admit right derived functors on the unbounded derived categories (see [Sta23, Tag 070K] and [Sta23, Tag 079P])$^5$.

**From splittings to connection**

Given $T \in \text{End}(g)$ and $(F, \psi) \in \text{Mod}(O_X)_{T-\text{sp}}$, the induced map

$$p_X^{-1}\psi : p_X^{-1}F \to p_X^{-1}E \otimes_{p_X^{-1}O_X} p_X^{-1}F$$

$^5$Unlike [Rot97, Sec. 2.2], we do not use Čech resolutions, since we are dealing with all $O$-modules.
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is $\pi^{-1}O_X$-linear. By Example 6.2.2.4, the sequence (6.13) induces a short exact sequence

$$0 \to \Omega^1_{pX} \to p_X^*\mathcal{E} \to O_{X \times Y} \to 0$$

in $\text{Mod}(O_{X \times Y})$. Its extension class is $p_X^*T \in H^1(X \times Y, \Omega^1_{pX})$. Define another $\pi^{-1}O_X$-linear map

$$\nabla_\psi : p_X^*F = (O_{X \times Y} \otimes_{p_X^{-1}O_X} p_X^{-1}F) \to p_X^*\mathcal{E} \otimes_{O_{X \times Y}} p_X^{-1}F = p_X^*\mathcal{E} \otimes_{p_X^{-1}O_X} p_X^{-1}F$$

by

$$\nabla_\psi(h \otimes s) = d_{pX}(h) \otimes s + h \otimes [(p_X^{-1}\psi)(s)],$$

where $h$ (resp. $s$) is a local section of $O_{X \times Y}$ (resp. $p_X^{-1}F$). By construction, $\nabla_\psi$ satisfies the Leibniz rule (6.15) and hence a $p_X^*T$-connection on $p_X^*F$. Thus, we get the exact functor in [Rot97, (2.5)]:

$$p_X^* : \text{Mod}(O_X)_{T-\text{sp}} \to \text{Mod}(O_{X \times Y})_{T-\text{cxn}}. \quad (6.17)$$

Tensoring with Poincaré bundle

By Fact 6.2.2.5 and [Rot97, (2.10)], the functor

$$\cdot \otimes_{O_{X \times Y}} \mathcal{P} : \text{Mod}(O_{X \times Y})_{1-\text{cxn}} \to \text{Mod}(O_{X \times Y})_{0-\text{cxn}} \quad (6.18)$$

sends the subcategory $\text{Mod}(O_{X \times Y})_{1-\text{cxn}, \text{ft}}$ to $\text{Mod}(O_{X \times Y})_{0-\text{cxn}, \text{ft}} \cong \text{Mod}(D_{X \times Y/X})$. The functor (6.18) is an equivalence of abelian categories, with a quasi-inverse $\cdot \otimes_{O_{X \times Y}} \mathcal{P}^{-1}$.

From connection to splittings

For every $(H, \nabla) \in \text{Mod}(O_{X \times Y})_{1-\text{cxn}}$, the map

$$\nabla : H \to p_X^*\mathcal{E} \otimes_{O_{X \times Y}} H = p_X^{-1}\mathcal{E} \otimes_{p_X^{-1}O_X} H$$

is a $p_X^{-1}O_X$-splitting to $(p_X^{-1}\mu_1) \otimes \text{Id}_H$. By projection formula (see e.g., [KS13, Prop. 2.6.6]), the induced map

$$p_{X*}\nabla : p_{X*}H \to \mathcal{E} \otimes_{O_X} p_{X*}H$$

is an $O_X$-linear splitting to $\mu_1 \otimes_{O_X} \text{Id}_{p_{X*}H}$. Hence $(p_{X*}H, p_{X*}\nabla) \in \text{Mod}(X)_{\text{sp}}$. Therefore, we get a left exact functor (a special case of [Rot97, (2.13)]):

$$p_{X*} : \text{Mod}(O_{X \times Y})_{1-\text{cxn}} \to \text{Mod}(X)_{\text{sp}}. \quad (6.19)$$

If $(H, \nabla)$ is integrable, then $[p_{X*}\nabla, p_{X*}\nabla]$ defined in (6.8) is zero.
6.2. PRELIMINARIES

Between connection

We define the inverse image and the direct image of relative connection on changing bases. Consider a cartesian square of complex manifolds

$$
\begin{array}{ccc}
W & \to & Z \\
\downarrow^{f'} & & \downarrow^{f} \\
T & \to & S,
\end{array}
$$

where $f$ is smooth. For every $(G, \nabla) \in \text{Mod}(O_Z)_{f,0-\text{cxn}}$, by [ABC20, Sec. 4.2], the relative connection $\nabla$ is equivalent to a splitting to the natural projection $P^*_f \otimes_{O_Z} G \to G$, where $P^*_f$ denotes the sheaf of first order jets defined in [ABC20, Sec. 4.1.2]. Applying $g^*\nabla$ to the induced splitting, we get a splitting to the natural projection $P^*_f \otimes_{O_W} g^*G \to g^*G$. This is equivalent to an $f'$-connection on $g^*G$. Hence an inverse image functor

$$
g^* : \text{Mod}(O_Z)_{f,0-\text{cxn}} \to \text{Mod}(O_W)_{f',0-\text{cxn}} \tag{6.21}
$$

that is right exact. Moreover, the connection induced by $\nabla$ is integrable provided that $\nabla$ is so, cf. [ABC20, Sec. 5.1].

Now for direct image. Fix $\alpha \in H^1(Z, \Omega^1_Z)$. For every

$$(H, \nabla) \in \text{Mod}(O_W)_{f',g^*\alpha-\text{cxn}},$$

by projection formula (see e.g., [Har77, II, Ex. 5.1 (d)]), we have

$$g^*_\alpha(H \otimes_{O_W} g^*\mathcal{E}_\alpha) = (g^*_\alpha H) \otimes_{O_Z} \mathcal{E}_\alpha.$$  

Then the induced morphism

$$g^*_\alpha \nabla : g^*_\alpha H \to (g^*_\alpha H) \otimes_{O_Z} \mathcal{E}_\alpha$$

is $f^{-1}(O_S)$-linear. Since $d_{f'} : O_W \to \Omega^1_{f'}$ and $d_f : O_Z \to \Omega^1_Z$ are related by $g^*d_f = d_{f'}$, the induced map $g^*_\alpha \nabla$ satisfies the Leibniz rule (6.15). Hence, the pair $(g^*_\alpha H, g^*_\alpha \nabla) \in \text{Mod}(O_Z)_{f,\alpha-\text{cxn}}$. In this manner, we get a left exact functor

$$g^*_\alpha : \text{Mod}(O_W)_{f',g^*\alpha-\text{cxn}} \to \text{Mod}(O_Z)_{f,\alpha-\text{cxn}}. \tag{6.22}$$

When $\alpha = 0$, the functor (6.22) sends MIC($f'$) to MIC($f$).

Example 6.2.3.2. If (6.20) is

$$
\begin{array}{ccc}
X \times Y & \to & Y \\
\downarrow^{p_Y} & & \downarrow \\
X & \to & \text{Specan}(\mathbb{C}),
\end{array}
$$
then \( p^* \): Mod\((O_{X \times Y})_{0-\text{cxn}}\) sits on the left of the diagram [Rot97, (2.15)] and

\[
p_Y^* : \text{Mod}(O_{X \times Y})_{0-\text{cxn}} \to \text{Mod}(Y)_{\text{cxn}} \tag{6.23}
\]
is [Rot97, (2.12)]. There are also restrictions

\[
p_Y^* : \text{MIC}(p_X) \to \text{Mod}(D_Y); \tag{6.24}
p_X^* : \text{Mod}(D_Y) \to \text{MIC}(p_X). \tag{6.25}
\]

**Remark 6.2.3.3.** Fix \( \alpha = 0 \in H^1(Z, \Omega^1_Z) \). From another point of view, the morphism \( O_Z \to g'_1 O_W \) between sheaves of rings extends to a morphism \( \tilde{D}_{Z/S} \to g'_1 D_{W/T} \). Then (6.21) and (6.22) are respectively the pullback and the pushout along the induced morphism \((W, \tilde{D}_{W/T}) \to (Z, \tilde{D}_{Z/S})\) of ringed spaces. By [Sta23, Tag 0096], the functor (6.21) is the left adjoint to (6.22). Then from [Sta23, Tag 09T3], the derived functor

\[
Lg'^* : \text{D}(\text{Mod}(Z)_{f,0-\text{cxn}}) \to \text{D}(\text{Mod}(W)_{f',0-\text{cxn}})
\]
is the left adjoint to

\[
Rg'_* : \text{D}(\text{Mod}(W)_{f',0-\text{cxn}}) \to \text{D}(\text{Mod}(Z)_{f,0-\text{cxn}}).
\]

### 6.3 Rothstein transform on modules with connection

**Definition 6.3.0.1.** Define \( R\mathcal{S}_1 : \text{D}(\text{Mod}(O_X)_{\text{sp}}) \to \text{D}(\text{Mod}(O_Y)_{\text{cxn}}) \) by

\[
R\mathcal{S}_1(?) = Rp_Y^*(\mathcal{P} \otimes_{O_{X \times Y}} p_X^* ?),
\]

and \( R\mathcal{S}_2 : \text{D}(\text{Mod}(O_Y)_{\text{cxn}}) \to \text{D}(\text{Mod}(O_X)_{\text{sp}}) \) by

\[
R\mathcal{S}_2(?) = Rp_X^*(\mathcal{P}^{-1} \otimes_{O_{X \times Y}} p_Y^* ?).
\]

Here \( Rp_Y^* \) (resp. \( Rp_X^* \)) is the right derived functor of (6.23) (resp. (6.19)). The pair \((R\mathcal{S}_1, R\mathcal{S}_2)\) is called Rothstein transform.

Let \( D_{O-\text{good}}(\text{Mod}(O_Y)_{\text{cxn}}) \subset \text{D}(\text{Mod}(O_Y)_{\text{cxn}}) \) (resp. \( D_{O-\text{good}}(\text{Mod}(O_X)_{\text{sp}}) \subset \text{D}(\text{Mod}(O_X)_{\text{sp}})\)) be the full subcategory comprised of objects whose cohomologies are good as \( O \)-module in the sense of [Kas03, Def. 4.22]. In view of Proposition 6.3.0.2, Rothstein transform is compatible with Fourier-Mukai transform.

**Proposition 6.3.0.2.** There are commutative squares

\[
\begin{array}{ccc}
\text{D}(\text{Mod}(O_X)_{\text{sp}}) & \xrightarrow{R\mathcal{S}_1} & \text{D}(\text{Mod}(O_Y)_{\text{cxn}}) \\
\downarrow & & \downarrow \\
\text{D}(O_X) & \xrightarrow{R\mathcal{S}_1} & \text{D}(O_Y)
\end{array}
\]

and
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\[
\begin{align*}
D(\text{Mod}(O_Y)_{\text{cxn}}) & \xrightarrow{R\mathcal{S}_2} D(\text{Mod}(O_X)_{\text{sp}}) \\
\downarrow & \downarrow \\
D(O_Y) & \xrightarrow{R\mathcal{S}_2} D(O_X),
\end{align*}
\]

where the vertical functors are forgetful. In particular, \(R\mathcal{S}_1\) (resp. \(R\mathcal{S}_2\)) sends \(D_{O-\text{good}}(\text{Mod}(O_X)_{\text{sp}})\) (resp. \(D_{O-\text{good}}(\text{Mod}(O_Y)_{\text{cxn}})\)) to \(D_{O-\text{good}}(\text{Mod}(O_Y)_{\text{cxn}})\) (resp. \(D_{O-\text{good}}(\text{Mod}(O_X)_{\text{sp}})\)).

**Proof.** All the functors \(p_X^* : \text{Mod}(O_X) \to \text{Mod}(O_X \times Y)\), (6.17), (6.18) and

\[
\mathcal{P} \otimes_{O_X \times Y} : \text{Mod}(O_X \times Y) \to \text{Mod}(O_X \times Y)
\]

are exact. To prove (6.26) is commutative, it remains to show the square

\[
\begin{align*}
D(\text{Mod}(O_X \times Y)_{0-\text{cxn}}) & \xrightarrow{Rp_Y^*} D(\text{Mod}(O_Y)_{\text{cxn}}) \\
\downarrow_{\text{for}_{X \times Y}} & \downarrow_{\text{for}_Y} \\
D(O_X \times Y) & \xrightarrow{Rp_Y^*} D(O_Y).
\end{align*}
\]

is so. Since the forgetful functor \(\text{for}_Y : \text{Mod}(O_Y)_{\text{cxn}} \to \text{Mod}(O_Y)\) is exact, the composition \(\text{for}_Y Rp_Y^* : D(\text{Mod}(O_X \times Y)_{0-\text{cxn}}) \to D(O_Y)\) is the right derived functor of

\[
\forall_{X \times Y} \circ p_Y^* : \text{Mod}(O_X \times Y)_{0-\text{cxn}} \to \text{Mod}(O_Y).
\]

From Remark 6.2.3.1, [Sta23, Tag 0006] and [Sta23, Tag 08BJ], the functor \(\forall_{X \times Y} : \text{Mod}(O_X \times Y)_{0-\text{cxn}} \to \text{Mod}(O_X \times Y)\) preserves K-injective complexes. By Lemma D.1.0.11, the composition \(Rp_Y^* \forall_{X \times Y} : D(\text{Mod}(O_X \times Y)_{0-\text{cxn}}) \to D(O_Y)\) is the right derived functor of

\[
p_Y^* \forall_{X \times Y} : \text{Mod}(O_X \times Y)_{0-\text{cxn}} \to \text{Mod}(O_Y).
\]

Since \(\forall_{X \times Y} \circ p_Y^* = p_Y^* \circ \forall_{X \times Y}\), the square (6.27) is indeed commutative.

By the commutativity of (6.26) and Corollary 5.3.1.14, the transform \(R\mathcal{S}_1\) preserves \(O\)-goodness. The other half about \(R\mathcal{S}_2\) is similar. \(\square\)

**Theorem 6.3.0.3** (Rothstein). We have \(R\mathcal{S}_1 R\mathcal{S}_2 = T^{-g}\) on \(D_{O-\text{good}}(\text{Mod}(O_Y)_{\text{cxn}})\) and \(R\mathcal{S}_2 R\mathcal{S}_1 = T^{-g}\) on \(D_{O-\text{good}}(\text{Mod}(O_X)_{\text{sp}})\).

We begin the proof of Theorem 6.3.0.3 with Proposition 6.3.0.4, a direct adaption of [Rot97, Prop. 2.4] for complex tori.

**Proposition 6.3.0.4.** Let \(\Delta \subset X \times X\) be the diagonal. Define a homomorphism \(\epsilon_X : X \times X \to X\) by \((x_1, x_2) \mapsto x_2 - x_1\). Then

\[
Rp_{12}^* (\epsilon_X \times 1_Y)^* \mathcal{P} \cong O_\Delta[-g]
\]

in \(D^b(\text{Mod}(O_X \times X)_{(1,-1)-\text{sp}})\), where \(p_{12} : X \times X \times Y \to X \times X\) is the projection.
For a complex manifold $Z$ and $z \in Z$, let $i_z : (z, \mathbb{C}) \to (Z, O_Z)$ be the closed embedding of complex manifolds. Set $C_z := (i_z)_* \mathbb{C}$, which is a coherent $O_Z$-module.

**Proof.** The identification $R^pX_*\mathcal{P} = C_0[-g]$ in $D^b(O_X)$ from [Kem91, Thm. 3.15] can be lifted to an isomorphism in $D^b(\text{Mod}(O_X)_{(-1,-sp)})$. As stated in the last sentence of the proof of [Vig21, Prop. 2.1.21], a morphism of modules with splittings (or connection) is an isomorphism, provided that the underlying morphism of $O$-modules is so. Then apply Theorem 5.3.2.3 to the cartesian square

$$
\begin{array}{ccc}
X \times X \times Y & \xrightarrow{\epsilon_X \times 1_Y} & X \times Y \\
\downarrow \rho_{12} & & \downarrow \rho_X \\
X \times X & \xrightarrow{\epsilon_X} & X.
\end{array}
$$

Arguing as in Proposition 6.3.0.4, we can prove the analytic version of [Rot97, Prop. 2.5; Prop. 3.1]. These three results are used in the proof of Theorem 6.3.0.3 below.

**Proof of Theorem 6.3.0.3.** Repeat the proof of [Rot97, Thm. 3.2], which requires the projection formula and smooth base change theorem for modules with connection. For this, we first construct the corresponding comparison morphism that is compatible with the underlying $O$-module comparison morphism. The construction reduces to the adjunction between derived inverse image and derived direct image of relative connection in Remark 6.2.3.3.

The compatibility with $O$-module comparison morphism can be proved in a way similar to Proposition 6.3.0.2. On the level of $O$-modules, the comparison morphism is an isomorphism by Fact 5.3.2.15 and Theorem 5.3.2.3. (This type of arguments can also be found in the proof of [Vig21, Prop. 2.1.21; Thm. 2.1.33].)

**Remark 6.3.0.5.** Rothstein’s first proof ([Rot96, Thm. 2.2]) is based on a problematic lemma [Rot96, Lem. 2.3]. To save his first proof, one may attempt to replace this “lemma” by its close variant, the bounded way-out lemma (see e.g., [Lip60, Lem. 1.11.3 (i)]). The difficulty is that, a priori, there is no canonical choice of a natural transformation between the two functors to be compared, which is required by way-out argument. For instance, in the end of the proof of Proposition 5.4.2.3, there are isomorphism arrows of opposite directions.

A holomorphic vector bundle $H \to Y$ is called homogeneous if $T^*_y H$ is isomorphic to $H$ for all $y \in Y$, where $T^*_y : Y \to Y$ is the addition by $y$. The first half of Theorem 6.3.0.6 is a special case of [Mat59, Thm. 1].

---

\[6\]the problem is explained in [Rot97, Sec. 1].
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**Theorem 6.3.0.6** (Matsushima). Let $E$ be a coherent $O_Y$-module with a connection $\nabla$. Then $E$ is a homogeneous vector bundle and the pair $(E, \nabla)$ is translation invariant.

**Proof.** By Proposition 6.3.0.2, for every integer $i$, the coherent $O_X$-module $H^iRS_2(E)$ admits a 1-splitting. By Lemma 6.3.0.7, $H^iRS_2(E)$ is finitely supported. Consequently, $RS_2(E)$ is isomorphic to $\oplus_{i\in\mathbb{Z}}T^{-i}H^iRS_2(E)$ in $D'_c(O_X)$. From Proposition 5.5.3.2 and Fact 6.1.0.5, $T^{-i}E$ is isomorphic to $\oplus_{i\in\mathbb{Z}}T^{-i}H^0RS_1(H^iRS_2(E))$ in $D'_c(O_Y)$ and each $H^0RS_1(H^iRS_2(E))$ is a homogeneous vector bundle on $Y$. Then $E$ is isomorphic to $H^0RS_1(H^0RS_2(E))$, hence a homogeneous vector bundle.

We adopt the argument in [BK09, Footnote (6), p.388]. For every $y \in Y$, $T_y^*\nabla$ is a connection on $T_y^*E \to E$ and $T_y^*\nabla = \nabla$. The map

$$Y \to H^0(Y, \Omega^1_Y \otimes End(E)), \quad y \mapsto T_y^*\nabla - \nabla$$

is holomorphic. It is constantly 0 since $Y$ is compact and $H^0(Y, \Omega^1_Y \otimes End(E))$ is a finite-dimensional vector space (Cartan-Serre’s theorem). Thus, $T_y^*(E, \nabla) = (E, \nabla)$ for every $y \in Y$. \hfill \Box

**Lemma 6.3.0.7** ([Rot96, Lem. 3.1]). Let $X$ be a complex torus and $F$ be a coherent $O_X$-module with a 1-splitting, then $F$ is finitely supported.

**Proof.** Suppose the contrary that $\text{Supp}(F)$ is infinite. By [GR84, p.76], $\text{Supp}(F)$ is an analytic set in $X$. Then $\dim \text{Supp}(F) \geq 1$. Let $C$ be an irreducible component of $\text{Supp}(F)$ of maximal dimension and $i : C \to X$ be the inclusion. Take a morphism $h : Z \to X$ provided by Lemma 5.5.3.3. Then $h(Z) = C$ and $F' := F'/T(F')$ is a vector bundle on $Z$ of positive rank $r$, where $F' = h^*F$ and $T(*)$ denotes the torsion part of a sheaf of modules. In consequence, the homomorphism of complex tori $h^* : \text{Pic}^0(X) \to \text{Pic}^0(Z)$ is nonzero. We claim that its tangent map at origin $h^* : g \to H^1(Z, O_Z)$ is zero.

Let $E' = h^*(E)$. Because $O_X$ is flat over itself, pulling back (6.13) to $Y$ and tensoring with $F''$, by [Sta23, Tag 05NJ] we get an exact sequence

$$0 \to g^* \otimes_{\mathbb{C}} F'' \to E' \otimes F'' \to F'' \to 0. \quad (6.28)$$

Since $E'$ is a vector bundle on $Z$, we have

$$\frac{E' \otimes F'}{T(E' \otimes F')} = E' \otimes F'',$$

thus the splitting on $F$ induces a splitting $F'' \cong E' \otimes F''$ of (6.28). Let $\beta$ be the natural morphism $\beta : O_Z \to \text{End}(F'')$. By Lemma 6.2.1.3, the composition

$$\text{End}(g) \xrightarrow{Id_g \otimes h^*} g^* \otimes_{\mathbb{C}} H^1(Z, O_Z) \xrightarrow{Id_g \otimes \beta} g^* \otimes_{\mathbb{C}} H^1(Z, \text{End}(F''))$$

sends $1 \in \text{End}(g)$ to 0. Therefore, the map $\beta h^* : g \to H^1(Z, \text{End}(F''))$ is zero. Taking trace, we get a morphism $\tau : \text{End}(F'') \to O_Z$ with $\tau \beta = r \cdot \text{Id}_{O_Z}$. Then $h^* = \frac{1}{r} \tau \beta h^* = 0$ as a map $g \to H^1(Z, O_Z)$ and hence the claim. The claim gives a contradiction. \hfill \Box
Corollary 6.3.0.8. Every local system (of finite dimensional $\mathbb{C}$-vector spaces) on a complex torus is translation invariant.

Proof. Let $L$ be a local system on $Y$. By Theorem 6.3.0.6, the pair $(L \otimes \mathcal{O}_Y, \text{Id}_L \otimes d)$ is translation invariant. The result follows from the Riemann-Hilbert correspondence [Del70, I, Thm. 2.17].

6.4 Laumon-Rothstein sheaf of algebras

To lift Fourier-Mukai transform to $D$-modules, we recall (Definition 6.4.0.1) the sheaf $A = A_X$ from [Rot96, p.576]. Fix a $\mathbb{C}$-basis $\{\omega^1, \ldots, \omega^g\}$ of the $\mathbb{C}$-vector space

$$g^* = H^0(Y, \Omega^1_Y) = \Gamma(X, g^* \otimes \mathcal{O}_X) \subset \Gamma(X, \mathcal{E}).$$

For each Stein open subset $U \subset X$, by [KK11, Sec. 52, Thm. B] we have

$$H^1(U, g^* \otimes \mathcal{O}_X) = 0.$$ 

Hence, there is $\rho \in \mathcal{E}(U)$ with $\mu(\rho) = 1 \in \mathcal{O}_X(U)$. For two such pairs $(U, \rho)$ and $(\tilde{U}, \tilde{\rho})$ with $U \cap \tilde{U} \neq \emptyset$, we have $\mu(\tilde{\rho} - \rho) = 0 \in \mathcal{O}_X(U \cap \tilde{U})$, so $\tilde{\rho} - \rho \in g^* \otimes \mathcal{O}_X(U \cap \tilde{U})$. There exists a unique tuple $f_1, \ldots, f_g \in \mathcal{O}_X(U \cap \tilde{U})$ such that

$$\tilde{\rho} - \rho = \sum_{i=1}^g \omega^i \otimes f_i$$

on $U \cap \tilde{U}$.

Definition 6.4.0.1. For each chosen pair $(U, \rho)$ as above, introduce independent variables $x_1^\rho, \ldots, x_g^\rho$ and put

$$A|_U = \mathcal{O}_U[x_1^\rho, \ldots, x_g^\rho].$$

For another choice $(\tilde{U}, \tilde{\rho})$ with the tuple $(f_1, \ldots, f_g)$ as above, we glue $A|_U$ and $A|_{\tilde{U}}$ by the rule that

$$x_i^\rho - x_i^{\tilde{\rho}}|_{U \cap \tilde{U}} = f_i.$$ (6.29)

The resulting sheaf $A$ is a sheaf of commutative $\mathcal{O}_X$-algebra.

In coordinate-free terms, $A$ is the $\mathcal{O}_X$-subalgebra of $\pi_* O_{X^\natural}$ consisting of sections whose restriction to each fiber of $\pi$ are polynomials, where

$$0 \to g^* \to X^\natural \to \pi_* X \to 0$$ (6.30)

is the universal vectorial extension\(^7\) of $X$ constructed in (E.22). For every integer $m \geq 0$, set $O_{X^\natural}(m)$ to be the subsheaf of $\mathcal{O}_{X^\natural}$ consisting of sections whose restriction to the fibers of $\pi$ are homogeneous polynomials of degree $m$. Similar to [Bjö93, Def 1.6.1], there exists a sheaf of graded rings on $X^\natural$: $O_{X^\natural} := \oplus_{m \geq 0} \mathcal{O}_{X^\natural}(m)(\subset \mathcal{O}_{X^\natural})$. Then $A = \pi_* O_{X^\natural}$ and $\Gamma(X, A) = \mathbb{C}$.

\(^7\)By [Rot96, p.567], it is also the $g^*$-principal bundle associated to the tautological extension (6.13).
Remark 6.4.0.2. Unlike the analytic case, if \( X \) is an abelian variety, then the notation \( \mathcal{A} \) in [Rot96, p.576] is the algebraic direct image \( \pi_* O_X \). Morally, such difference also lies between algebraic and analytic \( D \)-modules. For a complex manifold or a smooth algebraic variety \( V \), let \( p : T^* V \to V \) be the natural projection of the cotangent bundle. For the degree filtration \( F \) on \( D_V \), denote the associated graded ring by \( \text{gr}^F D_V \). Then \( \text{gr}^F D_V = p_* O_{T^* V} \) in the algebraic case ([HT07, p.57]). By contrast, in the analytic case, \( \text{gr}^F D_V \) is the \( O_V \)-submodule of \( p_* O_{T^* V} \) consisting of sections whose restriction to each fiber of \( p \) are polynomials.

Remark 6.4.0.3. The sheaf of rings \( \mathcal{A}_X \) is functorial in \( X \) in the following sense. Let \( \phi : X' \to X \) be a homomorphism of complex tori. By Proposition E.5.4.7, it induces a homomorphism \( \tilde{\phi}^\flat : X'^\flat \to X^\flat \) of complex Lie groups fitting into a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \to & H^0(Y', \Omega^1_{Y'}) & \to & X'^\flat & \xrightarrow{\pi'^*} & X' & \to & 0 \\
\downarrow{\hat{\phi}^*} & & \downarrow{\phi^*} & & \downarrow{\phi} & & \downarrow{\phi} & & \\
0 & \to & H^0(Y, \Omega^1_Y) & \to & X^\flat & \xrightarrow{\pi} & X & \to & 0.
\end{array}
\]

Here \( \hat{\phi} : Y \to Y' \) is the homomorphism dual to \( \phi \). For each local section of \( O_{[X']} \), its \( \hat{\phi}^\flat \)-pullback (a local section of \( O_{X'^\flat} \)) restricts to a polynomial on each fiber of \( \pi' \). Indeed, this restriction is the \( \hat{\phi}^* \)-pullback of a restriction to a fiber of \( \pi \). Therefore, the natural morphism \( O_{X^\flat} \to \hat{\phi}^\flat O_{X'^\flat} \) sends \( O_{[X']} \) to \( O_{[X^\flat]} \). The corresponding morphism of ringed spaces \( (X'^\flat, O_{X'^\flat}) \to (X^\flat, O_{X^\flat}) \) descends to another morphism of ringed spaces

\[
\hat{\phi} : (X', \mathcal{A}_{X'}) \to (X, \mathcal{A}_X), \tag{6.31}
\]

which is compatible with \( \phi \). In particular, the following square

\[
\begin{array}{ccc}
D(A_{X'}) & \xrightarrow{R\tilde{\phi}^*} & D(A_X) \\
\downarrow & & \downarrow \\
D(O_{X'}) & \xrightarrow{R\hat{\phi}^*} & D(O_X)
\end{array}
\]  
\quad \tag{6.32}

is commutative, where the vertical functors are forgetful. If \( M \) is an \( O_X \)-module, then

\[
L\tilde{\phi}^*(A_X \otimes_{O_X} M) = A_{X'} \otimes_{O_{X'}} L\phi^* M. \tag{6.33}
\]

When \( \phi \) is the diagonal homomorphism \( X \to X^2 \), then as bifunctors \( D(A_X) \times D(A_X) \to D(A_X) \) we have

\[
L\tilde{\phi}^* [(\cdot) \otimes_{\mathcal{A}} (\cdot)] = (\cdot) \otimes_{\mathcal{A}_X} (\cdot). \tag{6.34}
\]

Here the bifunctor \( \otimes_{\mathcal{A}} \) is defined in (6.54).
Notice that \( \mathcal{A} \) has a natural degree filtration \( \{ \mathcal{A}_m \}_{m \in \mathbb{Z}} \), where
\[
\mathcal{A}_m = \pi_* \left( \bigoplus_{j=0}^m \mathcal{O}_X(j) \right)
\]
is the \( \mathcal{O}_X \)-submodule of \( \mathcal{A} \) consisting of polynomials of degree at most \( m \), see [Rot96, Sec. 5.3] and the end of [Lau96, p.10]. Then \( \mathcal{A}_0 = \mathcal{O}_X \), \( \mathcal{A}_1 = \mathcal{E}^\gamma \) (cf. the start of [Lau96, p.10]), and every \( \mathcal{A}_m \) is a locally free \( \mathcal{O}_X \)-module of finite rank. Moreover, for every \( l, m \in \mathbb{N} \), we have
\[
\mathcal{A}_l \mathcal{A}_m = \mathcal{A}_{l+m}, \tag{6.35}
\]
so \( \mathcal{A} \) is a sheaf of positively filtered rings (in the sense of [Bjö93, p.459; p.464]) on the complex torus \( X \).

We review some terminology from [Bjö93, A:III]. A coherent sheaf of rings on a locally compact Hausdorff space is called noetherian if every increasing sequence of ideal sheaves is stationary over relatively compact subsets ([Bjö93, 2.24, p.470]). Let \( R \) be a commutative filtered ring. If the subring \( \bigoplus_{v \in \mathbb{Z}} R_v T^v \) of \( R[T, T^{-1}] \) is a noetherian ring, then \( R \) is called a noetherian filtered ring.

**Definition 6.4.0.4.** [Bjö93, A:III, 1.7; Def. 1.11; 1.19] A filtration on an \( R \)-module \( M \) is a family of additive subgroups \( \{ M_v \}_{v \in \mathbb{Z}} \) such that \( M_v \subset M_{v+1} \); \( R_k M_v \subset M_{k+v} \); \( \bigcup_v M_v = M \). This filtration is called *separated* if \( \bigcap_{v \in \mathbb{Z}} M_v = 0 \), and called *good* if \( \bigoplus_{v \in \mathbb{Z}} M_v T^v \) is a finitely generated \( \bigoplus_{v \in \mathbb{Z}} R_v T^v \)-module.

A zariskian filtered ring is a noetherian filtered ring such that good filtrations on finitely generated modules are separated. A filtered sheaf of rings is called stalkwise zariskian if every stalk is a zariskian filtered ring [Bjö93, Def. 2.6, p.465].

**Lemma 6.4.0.5.** The sheaf of rings \( \mathcal{A} \) is coherent and noetherian. The sheaf of filtered rings \( \mathcal{A} \) is stalkwise zariskian.

**Proof.** By (6.29) or [Rot96, (7.2)], the graded ring associated to the degree filtration of \( \mathcal{A} \) is
\[
G\mathcal{A} := \bigoplus_{m \geq 0} \mathcal{A}_m / \mathcal{A}_{m-1} = \text{Sym}(g) \otimes \mathcal{O}_X = \mathcal{O}_X[x_1, \ldots, x_g]. \tag{6.36}
\]
Here for each chosen pair \((U, \rho)\) as above, \( x_i|_U \in \Gamma(U, \mathcal{A}_1 / \mathcal{A}_0) \subset \Gamma(U, G\mathcal{A}) \) is the image of \( x_i \in \Gamma(U, \mathcal{A}_1) \). From [Bjö79, Thm. 1.26, p.460], \( \mathcal{A} \) is stalkwise zariskian. The other part follows from [Bjö79, Prop. 1.27, p.460; Thm. 2.7, p.465]. (See also the proof of [Bjö93, Thm. 1.2.5].)

In view of the difference mentioned in Remark 6.4.0.2, the statement of [Rot96, Prop. 4.4] is slightly modified as Fact 6.4.0.6. For every \( \mathcal{A} \)-module \( F \), each chosen pair \((U, \rho)\) as above, define \( \psi^\rho_U : F(U) \to \mathcal{E}(U) \otimes \mathcal{O}_X(U) \mathcal{F}(U) \) by
\[
\psi^\rho_U(s) = \rho \otimes s + \sum_{i=1}^g \omega_i|_U \otimes (x_i^\rho s).
\]
Then \((\mu_1 \otimes \text{Id}_F)(\psi_U^F(s)) = s\). In light of (6.29), the family \(\{\psi_U^F(\phi)\}_U\) glue to a 1-splitting \(\psi\) on \(F\). By the commutativity of \(A\) and [Rot96, (4.9)], one has \([\psi, \psi] = 0\).

**Fact 6.4.0.6.** The resulting functor \(\text{Mod}(A) \to \text{Mod}(O_X)\) (defined by \(F \mapsto (F, \psi)\)) induces an equivalence from \(\text{Mod}(A)\) to the full subcategory of \(\text{Mod}(O_X)\) comprised of objects \((F, \psi)\) with \([\psi, \psi] = 0\).

From Fact 6.4.0.6 and the proof of [Rot96, Prop. 4.1], the functor (6.17) restricts to an exact functor \(p_X^* : \text{Mod}(A) \to \text{Mod}(O_X \times Y)_{1-\text{cxn, fl}}\). Similarly by [Rot96, Prop. 4.2], the functor (6.19) restricts to \(p_Y^* : \text{Mod}(O_X \times Y)_{1-\text{cxn, fl}} \to \text{Mod}(A)\).

(6.37)

### 6.5 Laumon-Rothstein transform

**Definition 6.5.0.1.** Define
\[
RS_1(\cdot) = R\gamma_Y^*(P \otimes_{O_X \times Y} P_X^*) : D(A_X) \to D(D_Y);
\]
\[
RS_2(\cdot) = R\gamma_X^*(P^{-1} \otimes_{O_X \times Y} P_Y^*) : D(D_Y) \to D(A_X),
\]
where \(R\gamma_Y^* : D(\text{MIC}(p_Y)) \to D(D_Y)\) (resp. \(R\gamma_X^* : D(\text{Mod}(O_X \times Y)_{1-\text{cxn, fl}}) \to D(A_X)\)) is the right derived functor of (6.24) (resp. (6.37)). The pair is called the Laumon-Rothstein transform.

The situation is depicted below.

\[
\begin{array}{ccc}
\text{Mod}(A) & \xrightarrow{H^0RS_1} & \text{Mod}(D_Y) \\
\downarrow^{p_X^*} & & \downarrow^{p_Y^*} \\
\text{Mod}(O_X \times Y)_{1-\text{cxn, fl}} & \xrightarrow{\otimes P} & \text{Mod}(O_X \times Y)_{0-\text{cxn, fl}} \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{Mod}(D_Y) & \xrightarrow{H^0RS_2} & \text{Mod}(A) \\
\downarrow^{p_Y^*} & & \downarrow^{p_X^*} \\
\text{Mod}(O_X \times Y)_{0-\text{cxn, fl}} & \xrightarrow{\otimes P^{-1}} & \text{Mod}(O_X \times Y)_{1-\text{cxn, fl}}.
\end{array}
\]

**Proposition 6.5.0.2.** There are commutative squares
\[
\begin{array}{ccc}
D(A_X) & \xrightarrow{RS_1} & D(D_Y) \\
\downarrow & & \downarrow \\
D(O_X) & \xrightarrow{RS_1} & D(O_Y);
\end{array}
\]
\[
\begin{array}{ccc}
D(D_Y) & \xrightarrow{RS_2} & D(A_X) \\
\downarrow & & \downarrow \\
D(O_Y) & \xrightarrow{RS_2} & D(O_X),
\end{array}
\]

where the vertical functors are forgetful. In particular, \(RS_1\) (resp. \(RS_2\)) sends \(D_{O-\text{good}}(A)\) (resp. \(D_{O-\text{good}}(D_Y)\)) to \(D_{O-\text{good}}(D_Y)\) (resp. \(D_{O-\text{good}}(A)\)).

**Proof.** The proof is similar to that of Proposition 6.3.0.2, as \(A_X\) (resp. \(D_Y\)) is flat over \(O_X\) (resp. \(O_Y\)). \(\Box\)

With Proposition 6.5.0.2, Theorem 6.5.0.3 can be proved in a way similar to Theorem 6.3.0.3.
Theorem 6.5.0.3 (Laumon, Rothstein). We have $RS_1 RS_2 = T^{-g}$ on $DO_{\text{good}}(DY)$ and $RS_2 RS_1 = T^{-g}$ on $DO_{\text{good}}(A)$.

Proposition 6.5.0.4 follows from Proposition 6.5.0.2, Theorem 6.5.0.3 and Fact 6.1.0.11 as in the proof of [Rot96, Thm. 6.1], cf. [Lau96, Prop. 3.1.2; Cor. 3.2.4].

Proposition 6.5.0.4. We have

$RS_2(DY \otimes_{O_Y} \cdot) = A \otimes_{O_X} RS_2(\cdot) : D_{\text{good}}(O_Y) \to DO_{\text{good}}(A)$;

$RS_1(A \otimes_{O_X} \cdot) = D_Y \otimes_{O_Y} RS_1(\cdot) : D_{\text{good}}(O_X) \to DO_{\text{good}}(DY)$.

For $x \in X$ (resp. $y \in Y$), let $P_x = \mathcal{P}|_{x \times Y}$ (resp. $P_y = \mathcal{P}|_{X \times y}$) be the pullback line bundle on $Y$ (resp. $X$).

Corollary 6.5.0.5. For $x \in X$ and $y \in Y$, let $i_y : y \to Y$ be the inclusion. We have:

$RS_2(DY \otimes_{O_Y} C_y) = A_X \otimes_{O_X} P_{-y};$

$T^g RS_1(A \otimes_{O_X} P_{-y}) = D_Y \otimes_{O_Y} C_y = i_y + C = B_{(y)|Y};$

$RS_1(A \otimes_{O_X} C_x) = D_Y \otimes_{O_Y} P_x;$

$T^g RS_2(DY \otimes_{O_Y} P_x) = A \otimes_{O_X} C_x.$

Proof. It follows from Theorem 6.5.0.3, Proposition 6.5.0.4, Fact 6.1.0.5 and Lemma 5.2.0.3. The fact that $D_Y \otimes_{O_Y} C_y = B_{(y)|Y}$ is contained in [HT07, Example 1.6.4].

Proposition 6.5.0.6, due to Matsushima [Mat59, Thm. 1] and Morimoto [Mor59, Thm. 2], is a converse to Theorem 6.3.0.6. For abelian varieties, Nakayashiki [Nak94, Prop. 5.9] gives a proof using the Fourier-Mukai transform.

Proposition 6.5.0.6. A homogeneous vector bundle on a complex torus admits an integrable connection.

Proof. Let $E \to Y$ be a homogeneous vector bundle and $\hat{E} = H^0 RS_2(E)$. According to Proposition 5.5.3.2 and Fact 6.1.0.1, we have $E = H^0 RS_1(\hat{E})$ and $\text{Supp}(\hat{E})$ is finite. By Lemma 6.5.0.7, $\hat{E}$ has an $A$-module structure. By Proposition 6.5.0.2, the $O_Y$-module underlying $H^0 RS_1(\hat{E})$ is $E$. Note that the $D_Y$-module $H^0 RS_1(\hat{E})$ carries naturally an integrable connection.

The proof of Proposition 6.5.0.6 needs Lemma 6.5.0.7, a converse to Lemma 6.3.0.7.

Lemma 6.5.0.7. If $F$ is an $O_X$-module with finite support on the complex torus $X$, then $F$ admits a 1-splitting $\psi$ with $[\psi, \psi] = 0$.

8The notation $B_\ast$ is defined in [Kas03, (3.30), p.51].
Proof. There is a decomposition $F = \oplus_{i=1}^{m} F_i$, where $\text{Supp}(F_i)$ is a singleton for each $i$. Thus, we may assume $\text{Supp}(F)$ is a singleton. Thus, there exists a morphism of complex manifolds $s : U \to X$ that is a local section to (6.30), on an open neighborhood $U$ of $\text{Supp}(F)$. Let $i : U \to X$ be the inclusion. Applying $\pi_*$ to the morphism of sheaves of rings $O_{X^\#} \to s_* O_U$, we get $\pi_* O_{X^\#} \to i_* O_U$. This gives $i_* O_U$ an $A$-module structure$^9$ as $A$ is a subalgebra of $\pi_* O_{X^\#}$. Since the canonical $O_{X^*}$-morphism $\text{Id}_F \otimes i^\# : F \to F \otimes_{O_{X^*}} i_* U$ is an isomorphism, $F$ also obtains an $A$-module structure. This induces such a splitting by Fact 6.4.0.6. \hfill \qed

Proposition 6.5.0.6, together with Theorem 6.3.0.6, yields (a slight generalization of) Morimoto's theorem $^{[Mor59, \text{Thm. 2, p.91}]}$.

Corollary 6.5.0.8 (Morimoto). A coherent module admitting a connection on a complex torus is a vector bundle admitting an integrable connection.

6.6 Good modules

6.6.1 Definition

In Section 6.6.1, we define good $A_X$-modules. We also review several definitions of good $D$-modules and show that they are equivalent.

Definition 6.6.1.1. $^{[Bjö93, \text{2.5, p.465}]}$ Let $R$ be a positively filtered sheaf of rings on a complex manifold $Z$ such that the associated graded ring $GR$ is coherent. Let $M$ be a coherent left $R$-module. A filtration on $M$ is an increasing sequence of subsheaves $\{M_v\}_{v \in \mathbb{Z}}$ satisfying $\bigcup_{v \in \mathbb{Z}} M_v = M$ and $R_k M_v \subset M_{k+v}$ for all $k \in \mathbb{N}, v \in \mathbb{Z}$. This filtration is called

- B-good if for every $x \in Z$, there exists an open neighborhood $U$, a finite set $\{m_1, \ldots, m_s\} \subset \Gamma(U, M)$ and $k_1, \ldots, k_s \in \mathbb{Z}$ such that $M_{v+1} |_U = \sum_{i=1}^{s} R_{v-k_i} m_i$ for all $v \in \mathbb{Z}$ ($^{[Bjö93, \text{Remark 2.16, p.467}]}$).

- locally good$^{10}$ if each $M_v$ is $O_Z$-coherent and for every $x \in Z$, there is an open neighborhood $U$ of $x$ and $k_0 \in \mathbb{N}$ such that $R_{m} M_{k_0} = M_{m+k_0}$ on $U$ for all $m \in \mathbb{N}$.

Proposition 6.6.1.2 is similar to $^{[HT07, \text{Prop. 2.1.1; Def. 2.1.2}]}$ and can be proved in the same way.

Proposition 6.6.1.2. Let $F = (M_v)_{v \in \mathbb{Z}}$ be a filtration on a coherent $A_X$-module $M$, then $F$ is $B$-good if and only if $F$ is locally good. (In that case, we call $F$ a good filtration on $M$.)

Proof. • If $F$ is a $B$-good filtration, then $\oplus_{v \in \mathbb{Z}} M_v / M_{v-1}$ is coherent $G_A$-module by Lemma 6.4.0.5 and $^{[Bjö93, \text{Thm. 2.17, p.467}]}$. Because of


$^9$This example shows that Lemma 6.3.0.7 fails without coherent condition.

$^{10}$terminology from $^{[Meb89, \text{Prop. 2.1.12 (i)}]}$
(6.36) and the proof of [Bjö93, Prop. 1.4.5], every homogeneous component $M_v/M_{v-1}$ is $O_X$-coherent. The filtration $F$ is locally bounded blow by

[5]

$D$ gives a coherent $O$-module admits a globally good filtration. For every $x \in X$, by definition, there is an open neighborhood $U$ of $x$, sections $m_1, \ldots, m_s \in \Gamma(U, M)$ and integers $k_1, \ldots, k_s$ such that $M_{k_i}|U = \sum_{v=1}^{s} A_{v-k_i} m_i$ for all $v \in Z$. Put $k_0 = \max_{j=1}^{s} k_j$. For every $k \in \mathbb{N}$, by Definition 6.6.1.1, $A_k M_{k_0} \subset M_{k+k_0}$. Conversely,

$$M_{k+k_0}|U = \sum_{i=1}^{s} A_{k+k_0-k_i} m_i \subset \sum_{i=1}^{s} A_k A_{k_0-k_i} m_i \subset A_k M_{k_0}.$$ 

Therefore, $A_k M_{k_0} = M_{k+k_0}$ on $U$ for all $k \in \mathbb{N}$.

- In the other direction, let $F$ be a locally good filtration. For a fixed $x \in X$, take $U$ and $k_0$ provided by the definition of local goodness. Since $M_{k_0}$ is $O_X$-coherent, by shrinking $U$, we may assume the $O_U$-module $M_{k_0}|U$ is generated by sections $s_1, \ldots, s_m \in \Gamma(U, M_{k_0})$. Define $\phi : A^m|U \to M_{k_0}|U$ by $(f_1, \ldots, f_m) \mapsto \sum_{j=1}^{m} f_j s_j$. Then $\phi$ is $A$-linear. Since $F$ is a filtration, $A_{v-k_0} M_{k_0} \subset M_v$ for all $v \in Z$, so $\phi(\oplus_{j=1}^{m} A_{v-k_0}) \subset M_v$. By construction, $\phi(\oplus_{j=1}^{m} A_0) = M_{k_0}|U$. For every $k \geq k_0$, on $U$ we have

$$M_k = A_{k-k_0} M_{k_0} = A_{k-k_0} \phi(\oplus_{j=1}^{m} A_0) \subset \phi(\oplus_{j=1}^{m} A_{k-k_0}).$$

Hence, the filtration $F$ is $B$-good.

On a smooth algebraic variety, a coherent $D$-module admits a globally defined good filtration ([HT07, Thm. 2.1.3 (i)]). By contrast, Malgrange [Ma104, p.405] gives a coherent $D$-module on the complex manifold $\mathbb{C}^* \times P^1 \mathbb{C}$ that does not admit any global good filtration.

**Definition 6.6.1.3.** Let $Z$ be a complex manifold. An $O_Z$-module $F$ is called

- countably quasi-good ([KS97, p.942]) if every compact subset of $Z$ has an open neighborhood $U$ such that $F|U$ is the union of an increasing sequence of coherent $O_U$-submodules.

- $O$-quasi-good ([KS16, p.12]) if for any relatively compact open subset $U \subset Z$, the restriction $F|U$ is a sum of coherent $O_U$-submodules.

A $D_Z$-module $M$ is called

- good coherent if for any relatively compact open subset $U$ of $Z$, there is a finite filtration \( \{M_k\}_{k \in Z} \) of $M|U$ such that each quotient $M_k/M_{k-1}$ is a coherent $D_U$-module admitting a good filtration. (See [Sai89, p.369], [SS94, p.10] and [KS96, p.43].)
• S-quasi-good ([KS96, p.43]) if for every relatively compact open subset $U$ of $Z$, $M|_U$ admits a filtration $\{M_v\}_{v \in \mathbb{Z}}$ by coherent $D_U$-submodule such that each quotient $M_v/M_{v-1}$ admits a good filtration and $M_v = 0$ for $v \ll 0$.

**Proposition 6.6.1.4.** Let $Z$ be a complex manifold, $M$ be a coherent $D_Z$-module, then the following are equivalent:

1. For each relatively compact open subset $U$ of $Z$, there is an $O_U$-coherent submodule $F \subset M|_U$ such that $D_U F = M$.
2. For each relatively compact open subset $U$ of $Z$, the $D_U$-module $M|_U$ admits a good filtration.
3. The $D_Z$-module $M$ is good coherent.
4. The $D_Z$-module $M$ is S-quasi-good.
5. The $O_Z$-module $M$ is countably quasi-good.
6. $M$ is $O_Z$-good.
7. $M$ is $O_Z$-quasi-good.

**Proof.** We follow the circular chain.

1 implies 2 See [Bjö93, 1.4.10].

2 implies 3 For every relatively compact open subset $U$ of $Z$, define a finite filtration of $M|_U$ by $M_0 = 0$ and $M_1 = M|_U$. Then the graded piece $M_1/M_0$ admits a good filtration over $U$.

3 implies 4 For every relatively compact open subset $U$ of $Z$, consider the filtration $\{M_k\}$ in the definition. By induction on $k$, it can be proved that each $M_k$ is $D_U$-coherent.

4 implies 5 Every quotient $M_v/M_{v-1}$ admits a good filtration, then by [Bjö93, Cor. 1.4.6], $M_v/M_{v-1}$ is countably quasi-good. By induction on $v$ and using [KS97, Lem. 2.1.1], we can prove that every $M_v$ is countably quasi-good. Therefore, for every $v \in \mathbb{Z}$, there is an increasing sequence $\{M^k_v\}_{k \geq 1}$ of coherent $O_U$-submodules of $M_v$ such that $M_v = \bigcup_{k \geq 1} M^k_v$. For every $k \geq 1$, let $M^k := \sum_{i \leq k, v < k} M^i_v$. By [Sta23, Tag 01BY], $M^k$ is an $O_U$-coherent submodule of $M$. Note that

$$M = \bigcup_{v \in \mathbb{Z}} M_v = \bigcup_{v \in \mathbb{Z}} \bigcup_{i \geq 1} M^i_v = \bigcup_{k \geq 1} M^k,$$

so $M$ is countably quasi-good.

5 implies 6 An increasing sequence forms a directed family.

6 implies 7 By definition.
7 implies 1 Let $U$ be a relatively compact open subset of $Z$. Because $M$ is a finite type $D_Z$-module, for every $x \in U$, there is a relatively compact open neighborhood $U(x) \subset Z$ of $x$ and finitely many

$$\{s_i^x\}_{1 \leq i \leq n(x)} \subset \Gamma(U(x), M)$$

generating the $D_{U(x)}$-module $M|_{U(x)}$. By compactness of $\bar{U}$, the open cover $\{U(x)\}_{x \in \bar{U}}$ of $\bar{U}$ has a finite subcover $\{U(x_j)\}_{1 \leq j \leq r}$. Then $V = \bigcup_{j=1}^{r} U(x_j)$ is a relatively compact open subset of $Z$ containing $U$. By condition 7, we can write $M|_{V} = \sum_{i \in I} G_{i}$, where $I$ is an index set, and each $G_i$ is a coherent $O_V$-submodule of $M|_{V}$.

For every $x \in \bar{U}$, there is an open neighborhood $V(x) \subset U(x)$ of $x$, such that for each $1 \leq i \leq n(x)$, the restriction $s_i^x|_{V(x)}$ of $s_i^x$ to $V(x)$ is in $\Gamma(V(x), G_{\alpha(x,i)})$ for certain index $\alpha(x,i) \in I$. By compactness of $\bar{U}$ again, the open cover $\{V(x)\}_{x \in \bar{U}}$ has a finite subcover $\{V(x_k')\}_{1 \leq k \leq m}$. Then

$$F := \sum_{1 \leq k \leq m, 1 \leq i \leq n(x_k')} G_{\alpha(x_k',i)}$$

is a finite type $O_V$-submodule of $M|_{V}$. By Lemma 6.6.2.7, $F$ is $O_V$-coherent. Moreover, $D_U(F|_{\bar{U}}) = M|_{\bar{U}}$.

\[\square\]

Similar to Proposition 6.6.1.4, Proposition 6.6.1.5 holds.

**Proposition 6.6.1.5.** Let $M$ be a coherent $A_X$-module on the complex torus. Then $M$ is $O_X$-good if and only if there is an $O_X$-coherent submodule $F \subset M$ such that $A_X F = M$.

Hereafter, let the sheaf of rings $\mathcal{R}$ be either $D_Z$ on a complex manifold $Z$ or $A_X$ on the fixed complex torus $X$ respectively.

**Definition 6.6.1.6.** [Kas03, Def. 4.24] A coherent $\mathcal{R}$-module is called $\mathcal{R}$-good if it is a good $O$-module.

For example, $\mathcal{R}$ is a good left $\mathcal{R}$-module by Lemma 6.4.0.5 and [Bjö93, Thm. 1.2.3]. Let $\text{Good}(\mathcal{R}) \subset \text{Coh}(\mathcal{R})$ (resp. $D^b_{\text{good}}(\mathcal{R}) \subset D^b_{\text{O-good}}(\mathcal{R})$) be the full subcategory comprised of $\mathcal{R}$-good modules (resp. complexes whose cohomologies are $\mathcal{R}$-good). By Proposition 6.6.1.4, the category $D^b_{\text{good}}(D_Z)$ is what Björk denotes by $D^b_{\text{coh}}(D_Z)_f$ in [Bjö93, p.119].

**Fact 6.6.1.7** (GAGA),

- ([HK84, Thm. 1.1 (2)]) Let $V$ be a smooth proper complex algebraic variety. Then the analytification functor induces an equivalence $\text{Coh}(D_V) \to \text{Good}(D_{V^{\text{an}}})$.

- Let $A$ be a complex abelian variety. Then the analytification functor induces an equivalence $\text{Coh}(A_A) \to \text{Good}(A_{A^{\text{an}}})$. 

6.6. GOOD MODULES

On a complex manifold \( Z \), a coherent \( D_Z \)-module is called \textit{holonomic} if its characteristic variety is of (minimal) dimension \( \dim Z \) ([Bjö93, Def. 3.1.1]). Malgrange ([Mal94, p.35], [Mal96, p.367], [Sab11, Thm. 4.3.4 (2)]) proved that for every holonomic \( D_Z \)-module is generated by a coherent \( O_Z \)-submodule, so it is a good \( D_Z \)-module. Let \( D^b_c(D_Z) \subset D^b(D_Z) \) be the full subcategory of objects with holonomic cohomologies.

6.6.2 Basic properties

**Lemma 6.6.2.1** (Induced modules). The functor

\[
\mathcal{R} \otimes_{O_Z} : \text{Mod}(O_Z) \to \text{Mod}(\mathcal{R})
\]

is exact. For every coherent \( O_Z \)-module \( F \), \( \mathcal{R} \otimes_{O_Z} F \) is a good \( \mathcal{R} \)-module. Hence an exact functor \( \mathcal{R} \otimes_{O_Z} : \text{Coh}(Z) \to \text{Good}(\mathcal{R}) \) and a \( t \)-exact functor \( \mathcal{R} \otimes_{O_Z} : D^b_c(O_Z) \to D^b_{\text{good}}(\mathcal{R}). \)

**Proof.** As \( \mathcal{R} \) is flat over \( O_Z \), the functor (6.40) is exact. Consider the degree filtration \( \{ \mathcal{R}(m) \}_{m \geq 0} \) of \( \mathcal{R} \), where \( \mathcal{R}(m) \subset \mathcal{R} \) is the \( O_Z \)-submodule of polynomials of degree at most \( m \). Each \( \mathcal{R}(m) \) is vector bundle on \( Z \) and \( \mathcal{R} = \text{colim}_m \mathcal{R}(m) \). Therefore, \( \mathcal{R} \) is \( O_Z \)-good. Then so is \( \mathcal{R} \otimes_{O_Z} F \) by Proposition 5.3.1.5 2. Because \( F \) is an \( O_Z \)-module of finite presentation, \( \mathcal{R} \otimes_{O_Z} F \) is an \( \mathcal{R} \)-module of finite presentation. Then it is \( \mathcal{R} \)-coherent by [Bjö93, Thm. 1.2.5] and Lemma 6.4.0.5. The other part follows.

**Lemma 6.6.2.2.** The category \( \text{Good}(\mathcal{R}) \) is a weak Serre subcategory of \( \text{Mod}(\mathcal{R}) \) and \( D^b_{\text{good}}(\mathcal{R}) \) is a triangulated subcategory of \( D^b(\mathcal{R}). \)

**Proof.** The first half is a combination of [Kas03, Prop. 4.23], [Sta23, Tag 01BY] and [Sta23, Tag 0754]. The second half follows by [Yek19, Prop. 7.4.5]. □

For a morphism of complex manifolds \( h : M \to N \), the direct image of \( D \)-modules \( h_+ : D(M) \to D(N) \) is constructed in [Bjö93, 2.3.12].

**Fact 6.6.2.3** ([Bjö93, Thm. 2.8.1, 2.8.7]). Let \( f : W \to Z \) be a morphism of complex manifolds and \( M \in D^b_{\text{good}}(D_W) \). If \( f|_{\text{Supp}(M)} : \text{Supp}(M) \to Z \) is proper, then \( f_+ M \in D^b_{\text{good}}(D_Z) \).

**Proposition 6.6.2.4.** Let \( f : W \to Z \) be a proper morphism of complex manifolds. Then the direct image functor \( f_+ : D(W) \to D(Z) \) preserves \( D_{O-Z} \)-good.

**Proof.** Take \( M \in D_{O-Z} \)-good \( (D_W) \). By [Sab11, Remark 3.3.4 (4)], the functor \( f_+ \) has finite cohomological dimension. To prove \( f_+ M \in D_{O-Z} \)-good \( (D_Z) \), by [Har66, I, Prop. 7.3 (iii)], one may assume that \( M \in \text{Mod}(D_W) \) is \( O_W \)-good. Let \( i : \Gamma \to W \times Z \) be the inclusion of the graph of \( f \), and let \( q : W \times Z \to Z \) be the projection. Then \( f_+ = q_+ i_* \) by [Sab11, Thm. 3.3.6 (1)]. The restriction \( q|_{\Gamma} : \Gamma \to Z \) is proper. By [Bjö93, Prop. 2.4.8], \( f_+ M = Rq_* [DR_{W \times Z/G}(i_+ M)] [\dim Z] \). As each term of the (relative) de Rham complex \( DR_{W \times Z/G}(i_+ M) \) is \( O_{W \times Z} \)-good
and supported on $\Gamma$, by Theorem 5.3.1.7, the object $Rq_*[DR_{W \times Z/G}(i + M)]$ is in $D_{\text{good}}(O_Z)$. The proof is completed. \hfill \Box

For a closed embedding $i : M \to N$ of complex manifolds, the inverse image $i^* : \text{Mod}(D_N) \to \text{Mod}(D_M)$ does not preserve $D$-coherence in general ([HT07, Rk. 1.5.10]). Still, Fact 6.6.2.5 can be proved by applying [Kas03, Thm. 4.7] or repeating the proof of [HT07, Prop. 1.5.10(ii)].

**Fact 6.6.2.5.** Let $f : M \to N$ be a smooth morphism of complex manifolds, then $Lf^* : D^b(D_N) \to D^b(D_M)$ sends $D^b_c(D_N)$ (resp. $D^b_{\text{good}}(D_N)$) to $D^b_c(D_M)$ (resp. $D^b_{\text{good}}(D_M)$).

Proposition 6.6.2.6 concerns the local existence of good filtrations on a coherent $A$-module.

**Proposition 6.6.2.6.** Let $M$ be a coherent $A$-module on $X$. For every $x \in X$ there exists an open neighborhood $U$ of $x$ and a positive good filtration on $M|_U$.

**Proof.** Let $A^0|_U \xrightarrow{\phi} A^p|_U \to 0$ be a local presentation of $M$ on a relatively compact open neighborhood $U$ of $x$. For every $v \in \mathbb{Z}$, set $M_v = \phi(A^v)$, which is an $O_U$-submodule of $M|_U$. Then $M_v = 0$ when $v < 0$. Moreover, $\cup_{v \in \mathbb{Z}} M_v = M|_U$ and $A_m M_k \subset M_{k+m}$ for every $m, k \in \mathbb{N}$. Hence, $\{M_v\}_{v \in \mathbb{Z}}$ is a positive filtration of $M|_U$. Note that for every $k \in \mathbb{N}$, $A_k M_0 = M_k$. It remains to prove that each $M_k$ is $O_U$-coherent.

Fix $k \in \mathbb{N}$. We claim that $\phi(A^0_m) \cap A^p_k$ is $O_U$-coherent for each $m \in \mathbb{N}$. In fact, for every $y \in U$, there is an integer $s \geq \max(0, k - m)$ such that $\phi(A^0_m) \subset A^p_{m+s}$ near $y$. In side the $O$-coherent module $A^p_{m+s}$, the two $O$-submodules $\phi(A^0_m)$ and $A^p_k$ are finite type. By [Sta23, Tag 01BY], their intersection $\phi(A^0_m) \cap A^p_k$ is $O$-coherent near $y$. The claim is proved.

Because $A^p_k$ is a noetherian $O_X$-module, the increasing sequence $\{\phi(A^0_m) \cap A^p_k\}_{m \geq 0}$ of submodules is stationary on $U$. Therefore, the union $\phi(A^0) \cap A^p_k = \ker(\phi) \cap A^p_k$ is $O_U$-coherent. Since the sequence

$$0 \to \ker(\phi) \cap A^p_k \to A^p_k \to M_k|_U \to 0$$

is exact in $\text{Mod}(O_U)$, the restriction $M_k|_U$ is $O_U$-coherent. The constructed filtration is therefore good. \hfill \Box

When $R = D_Z$, Lemma 6.6.2.7 is [Sab11, Exercise E.2.4 (4)]. On a complex manifold $Z$, an $O_Z$-module $F$ is called pseudo-coherent if for every open subset $U$ of $X$, every finite type $O_U$-submodule of $F|_U$ has a finite presentation ([Kas03, Def A.5]).

**Lemma 6.6.2.7.** If $M$ is a coherent $R$-module, then $M$ is pseudo-coherent over $O_Z$.

**Proof.** Let $F \subset M$ be a finite type $O_X$-submodule. For every $x \in Z$, by [Meb89, Prop. 2.1.9] (in the case $R = D_Z$) and Proposition 6.6.2.6 (in the case $R = A$), there exists an open neighborhood $U$ of $x$ and a good filtration on $M|_U$. By
Then, by [Sta23, Tag 01BY (1)], \( M = \mathbb{Z} \) of the categories.

**Proof of Theorem 6.6.3.1.**

The proof of Theorem 6.6.3.1 needs a cohomological dimension estimation.

**Lemma 6.6.2.8.** Let \( M \) be a good \( \mathcal{R} \)-module, \( N \) be a finite type \( \mathcal{R} \)-submodule of \( M \), then \( N \) is also \( \mathcal{R} \)-good.

**Proof.** By [KS13, Prop. 3.2.2 (iv)], \( M \) is \( \mathcal{R} \)-coherent. For every relatively compact open subset \( U \) of \( X \), every \( x \in U \), there is an open neighborhood \( U(x) \subset X \) of \( x \) and finitely many sections \( \{ s_i(x) \}_{i=1}^{n(x)} \subset \Gamma(U(x), F) \) generating the \( \mathcal{R}|_{U(x)} \)-module \( F|_{U(x)} \). The open cover \( \{ U(x) \}_{x \in U} \) of \( U \) has a finite subcover \( \{ U(x_j) \}_{j=1}^{m} \). Let \( F_0 \) be the \( \mathcal{R}_U \)-submodule of \( F|_U \) generated by the finitely many local sections

\[ \{ s_i(x_j) \}_{1 \leq j \leq m, 1 \leq i \leq n(x_j)} \]

Then \( F_0 \) is a finite type \( \mathcal{R}_U \)-module. Because \( M|_U \) is good over \( \mathcal{R}|_U \), by Lemma 6.6.2.7 \( F_0 \) is \( \mathcal{R}_U \)-coherent. By choice, \( RF_0 = F|_U \). Therefore, the \( \mathcal{R} \)-module \( N \) is good by Proposition 6.6.1.4 and Proposition 6.6.1.5. \( \square \)

**6.6.3 Preservation of goodness**

**Theorem 6.6.3.1.** The functors \( RS_1 \) and \( RS_2 \) restrict to equivalences between the categories \( D^b_{\text{good}}(\mathcal{A}) \) and \( D^b_{\text{good}}(D_Y) \).

The proof of Theorem 6.6.3.1 needs a cohomological dimension estimation.

**Lemma 6.6.3.2.** For an \( O_Y \)-module \( F \), we have \( RS_1(F) \in D^{[0,2g]}(O_Y) \). Similarly, for an \( O_Y \)-module \( G \), we have \( RS_2(G) \in D^{[0,2g]}(O_X) \).

**Proof.** By the left exactness of \( p_{Y,*} : \text{Mod}(O_X \times Y) \to \text{Mod}(O_Y) \), we get \( R^iS_1(F) = 0 \) when \( i < 0 \). For every \( y \in Y \), by proper base change theorem (see e.g., [Mil13, Thm. 17.2]), the stalk \( R^iS_1(F)_y = H^i(X \times y, M) \), where \( M \) is the restriction of the abelian sheaf \( \mathcal{F} \otimes p_{X,Y}^! F \) to \( X \times y \). When \( j > 2g \), \( H^j(X \times y, M) = 0 \) by [KS13, Prop. 3.2.2 (iv)]]. Therefore, \( R^jS_1(F) = 0 \). The other part is similar. \( \square \)

**Proof of Theorem 6.6.3.1.** For every coherent \( O_Y \)-module \( F \), by Theorem 6.5.0.3, \( RS_2(D_Y \otimes L_{O_Y}^b F) = A \otimes L_{O_X}^b RS_2(F) \). By Fact 6.1.0.5 2, \( RS_2(F) \in D^b(\mathcal{O}_X) \). From Lemma 6.6.2.1, \( A \otimes L_{O_X}^b RS_2(F) \in D^b_{\text{good}}(A) \).

We claim that \( H^iRS_2(M) \in \text{Good}(\mathcal{A}) \) for every \( M \in \text{Good}(D_Y) \), every \( i \in \mathbb{Z} \). We use descending induction on \( i \in \mathbb{Z} \).

When \( i > 2g \), the \( \mathcal{O}_X \)-module underlying \( H^iRS_2(M) \) is \( H^iRS_2(M) \), which is zero by Lemma 6.6.3.2. Hence \( H^iRS_2(M) = 0 \) is \( \mathcal{A} \)-good. Assume the statement for \( i + 1 \). By Proposition 6.6.1.4, there is a coherent \( O_Y \)-submodule \( F \subset M \) with \( D_Y F = M \). Let \( M' \) be the kernel of the natural epimorphism \( D_Y \otimes_{O_Y} F \to M \). Then

\[ 0 \to M' \to D_Y \otimes_{O_Y} F \to M \to 0 \]

(6.41)
is an exact sequence in \( \text{Mod}(D_Y) \). By Lemma 6.6.2.1, the \( D_Y \)-module \( D_Y \otimes_{O_Y} F \) is good. By Lemma 6.6.2.2, the \( D_Y \)-module \( M \) is good. From (6.41) we obtain an exact sequence in \( \text{Mod}(A) \):

\[
H^iRS_2(M') \to H^iRS_2(D_Y \otimes_{O_Y} F) \to H^iRS_2(M) \to H^{i+1}RS_2(M') \to H^{i+1}RS_2(D_Y \otimes_{O_Y} F).
\]

(6.42)

Form first paragraph, \( H^iRS_2(D_Y \otimes_{O_Y} F) \in \text{Good}(A) \) for \( j \in \{i, i+1\} \). By the inductive hypothesis, \( H^{i+1}RS_2(M') \in \text{Good}(A) \).

Let \( G = \ker[H^{i+1}RS_2(M') \to H^{i+1}RS_2(D_Y \otimes_{O_Y} F)] \). By Lemma 6.6.2.2, \( G \) is good (and hence finite type) over \( A \). The sequence (6.42) yields an exact sequence

\[
H^iRS_2(D_Y \otimes_{O_Y} F) \to H^iRS_2(M) \to G \to 0,
\]

so \( H^iRS_2(M) \) is a finite type \( A \)-module for every coherent \( D_Y \)-module \( M \). In particular, \( H^iRS_2(M') \) is a finite type \( A \)-module.

Let \( N = \text{im}(H^iRS_2(M') \to H^iRS_2(D_Y \otimes_{O_Y} F)) \). Then \( N \) is a finite type \( A \)-submodule of a good \( A \)-module. By Lemma 6.6.2.8, \( N \) is also a good \( A \)-module. The sequence (6.42) yields an exact sequence

\[
0 \to N \to H^iRS_2(D_Y \otimes_{O_Y} F) \to H^iRS_2(M) \to H^{i+1}RS_2(M') \to H^{i+1}RS_2(D_Y \otimes_{O_Y} F).
\]

By Lemma 6.6.2.2, the \( A \)-module \( H^iRS_2(M) \) is good. The induction is completed.

From the claim, Lemma 6.6.2.2 and [Har66, I, Prop. 7.3 (i)], the functor \( RS_2 \) sends \( D^b_{\text{good}}(D_Y) \) to \( D^b_{\text{good}}(A) \). Similarly, using Proposition 6.6.1.5 one can prove \( RS_1(K) \in D^b_{\text{good}}(D_Y) \) for every \( K \in D^b_{\text{good}}(A) \). The proof is completed by Theorem 6.5.0.3. 

\[ \square \]

### 6.7 Relations with other functors

The properties [Muk81, (3.1), (3.4), (3.8)] of Fourier-Mukai transform have analog for Laumon-Rothstein transform.

#### 6.7.1 Exchange of translation and multiplication

For every \( y \in Y \), we view \( P_y \) as an object of \( \text{Mod}(O_Y)_{0-\text{sp}} \) via Example 6.2.1.2. There is a canonical isomorphism \( T_{(0,y)}^\bullet \mathcal{P} \cong \mathcal{P} \otimes_{O_{X \times Y}} p_X^* P_y \) in \( \text{Mod}(X \times Y)_{-1-\text{cxn}} \), where \( p_X^* : \text{Mod}(O_X)_{0-\text{sp}} \to \text{Mod}(O_{X \times Y})_{0-\text{cxn}} \) is defined in (6.17) and the functor

\[
\mathcal{P} \otimes_{O_{X \times Y}} (\cdot) : \text{Mod}(O_{X \times Y})_{0-\text{cxn}} \to \text{Mod}(O_{X \times Y})_{-1-\text{cxn}}
\]

is from [Rot97, (2.10)]. Arguing as in [Muk81, (3.1)], we get Proposition 6.7.1.1 from projection formula.
Proposition 6.7.1.1.

\[
RS_2 \circ T'^*_y \cong (\otimes_{O_y} P_y) \circ RS_2 : D(D_Y) \to D(A);
\]

\[
RS_2 \circ (\otimes_{O_y} P_x) \cong T^*_{x} \circ RS_2 : D(D_Y) \to D(A);
\]

\[
RS_1 \circ (\otimes_{O_x} P_y) \cong T'^*_y \circ RS_1 : D(A) \to D(D_Y);
\]

\[
RS_1 \circ T^*_{x} \cong (\otimes_{O_x} P_{-x}) \circ RS_1 : D(A) \to D(D_Y).
\]

Similar results hold for \(R_1\) and \(R_2\).

Remark 6.7.1.2. Goodness over \(O\) is not necessary in Proposition 6.7.1.1, as the proof does not use smooth base change.

6.7.2 Duality

Let \(Z\) be a complex manifold. The duality (contravariant) functor on \(O_Z\)-modules \(\Delta^{O_Z} : D^b_c(O_Z) \to D^b_c(O_Z)\) is defined by

\[
\Delta^{O_Z} F = R\text{Hom}_{O_Z}(F, \omega^{-1})[\dim Z].
\]

The duality functor on \(D_Z\)-modules \(\Delta^{D_Z} : D(D_Z) \to D(D_Z)\) is defined by \(\Delta^{D_Z} F = G[\dim Z]\), where \(G\) is the complex of left \(D_Z\)-modules associated to the complex \(R\text{Hom}_{D_Z}(F, D_Z)\) of right \(D_Z\)-modules. Then the natural transform \(\text{Id} \to \Delta^{D_Z} \circ \Delta^{D_Z}\) of functors on \(D^b_c(D_Z)\) is an isomorphism, see [Bjö93, Def. 2.11.1].

Lemma 6.7.2.1 ([KS16, p.16]). The functor \(\Delta^{D_Z}\) sends \(D^b_{\text{good}}(D_Z)\) to itself.

Proof. Suppose \(F\) is a coherent \(O_Z\)-module and \(N = D_Z \otimes_{O_Z} F\), then by [Bjö93, (ii), p.122], there is \(G \in D^b_c(O_Z)\) with \(\Delta^{D_Z} N = D_Z \otimes_{O_Z} G\). By Lemma 6.6.2.1, \(\Delta^{D_Z} N \in D^b_{\text{good}}(D_Z)\).

Assume that \(M \in \text{Good}(D_Z)\). For every relatively compact open subset \(U \subset Z\), by [Bjö93, Thm. 1.58] and Proposition 6.6.1.4, there is a finite length exact sequence in \(\text{Mod}(D_Z)\):

\[
0 \to D_U \otimes_{O_U} F^{-n} \to \cdots \to D_U \otimes_{O_U} F^0 \to M|_U \to 0,
\]

where each \(F^i\) is a coherent \(O_U\)-module. From last paragraph, \(\Delta^{D_U}(D_U \otimes_{O_U} F^i) \in D^b_{\text{good}}(D_U)\) for each \(i\). Thus, \((\Delta^{D_Z} M)|_U = \Delta^{D_U}(M|_U) \in D^b_{\text{good}}(D_U)\) by Lemma 6.6.2.2. Therefore, \(\Delta^{D_Z} M \in D^b_{\text{good}}(D_Z)\). The general case follows from [Har66, I, Prop. 7.3 (i)].

For algebraic varieties, an analogue of Fact 6.7.2.2 is stated as [HT07, Cor. 2.6.8 (iii), Prop. 3.2.1]. And all of the arguments in [HT07, Sec. 2.6] are valid for analytic \(D\)-modules ([HT07, p.101]).

Fact 6.7.2.2.

1. The contravariant functor \(\Delta^{D_Z} : D^b_c(D_Z) \to D^b_c(D_Z)\) an equivalence.
2. Let $M$ be a coherent $D_Z$-module, then $M$ is holonomic if and only if $H^i(\Delta^Z M) = 0$ for all $i \in \mathbb{Z}\setminus\{0\}$.

**Fact 6.7.2.3.** Let $f : M \to N$ be a morphism of complex manifolds, then:

1. [Bjö93, Thm. 3.2.13 (1)] The inverse image $L f^* : D^b(D_N) \to D^b(D_M)$ preserves $D^b_k$.

2. [Sab11, Thm. 4.4.1] Assume $F \in D^b_k(D_M)$ and $f|\text{Supp}(F)$ is proper, then $f_* F \in D^b_k(D_N)$.

3. [Bjö93, Thm. 3.2.13 (3)] For every pair $F, F' \in D^b_k(D_M)$, one has $F \otimes^L_{D_M} F' \in D^b_k(D_M)$.

Restricted to the complex torus $Y$, [Bjö93, (ii), p.122] becomes [Rot96, (6.12)]:

$$\Delta^D_Y(D_Y \otimes^L_{D_Y} \cdot) = D_Y \otimes^L_{O_Y} \Delta^O_Y : D^b_k(O_Y) \to D^b_k(D_Y).$$

Likewise, define the duality (contravariant) functor on $A$-modules $\Delta^A : D^b(A) \to D^b(A)$ as

$$\Delta^A? = T^g R\text{Hom}_A(?, A).$$

It preserves $D^b_k(A)$ and similar to Lemma 6.7.2.1, it sends $D^b_{\text{good}}(A)$ to $D^b_{\text{good}}(A)$.

**Theorem 6.7.2.4** (Rothstein). Let $F \in D^b_{\text{good}}(A)$ be an object such that $RS_1(F)$ is concentrated in a single degree $i \in \mathbb{Z}$. Then $H^i RS_1(F)$ is holonomic if and only if $RS_1 \Delta^A F$ is concentrated in degree $g - i$.

Proposition 6.7.2.5 can be deduced from Corollary 6.7.2.7, Proposition 6.5.0.4 and Proposition 5.5.1.8, in the same way that [Rot96, Prop. 6.3] is proved\(^{11}\).

**Proposition 6.7.2.5.**

$$RS_2 \Delta^D_Y = [-1] \chi T^{-g} \Delta^A RS_2 : D^b_{\text{good}}(D_Y) \to D^b_{\text{good}}(A);$$

$$\Delta^D_Y RS_1 = [-1] \chi T^g RS_1 \Delta^A : D^b_{\text{good}}(A) \to D^b_{\text{good}}(D_Y).$$

**Lemma 6.7.2.6** ([Huy06, (3.13)]). Let $Z$ be a complex manifold, $K, L \in D(O_Z)$ and $M \in D^{-}_{c}(O_Z)$. Then the natural morphism\(^{12}\)

$$K \otimes^L_{O_Z} R\text{Hom}_{O_Z}(M, L) \to R\text{Hom}_{O_Z}(M, K \otimes^L_{O_Z} L)$$

is an isomorphism in $D(O_Z)$.

---

\(^{11}\)Both [Rot96, (6.13), (6.14)] miss a factor $[-1]^{\ast}$, due to a missing $[-1]^{\ast} \chi$ in (6.15) loc.cit. Still, this sign does not affect the statement of [Rot96, Thm. 6.5].

\(^{12}\)provided by [Sta23, Tag 0BYS]
Proof. By [Har66, I, Prop. 7.1 (ii)], we may assume that $M \in \text{Coh}(O_Z)$. By [Sta23, Tag 08DL] and [GH78, p.696], we may shrink $Z$ so that $M$ admits a globally free resolution $F \to M$, where the complex $F$ is

$$0 \to O_Z^k \to \cdots \to O_Z^1 \to O_Z^0 \to 0$$

with $O_Z^i$ placed in degree $-i$. The morphism (6.45) becomes

$$K \otimes_{O_Z} \text{Hom}_{O_Z}(F, L) \to \text{Hom}_{O_Z}(F, K \otimes_{O_Z} L),$$

which is an isomorphism. \hfill \square

Corollary 6.7.2.7 proves (the analytic counterpart of) [Rot96, (6.12)].

Corollary 6.7.2.7. As a functor $D^b_c(O_X) \to D^b_c(A)$, we have $\Delta^A(A \otimes^L_{O_X} \bullet) = A \otimes^L_{O_X} \Delta^{O_X} \bullet$.

Proof. For an object or a morphism $? \in D^b_c(O_X)$, by [Rot96, (6.2)] we have

$$\Delta^A(A \otimes^L_{O_X} ?) = T^g R\text{Hom}_A(A \otimes^L_{O_X} ?, A) = T^g R\text{Hom}_{O_X}(?, A).$$

By Lemma 6.7.2.6, it equals to $T^g R\text{Hom}_{O_X}(?, O_X) \otimes^L_{O_X} A = A \otimes^L_{O_X} \Delta^{O_X} ?$. \hfill \square

Example 6.7.2.8. If $F = T^g A$, then $R\text{S}_1(F) = D_Y \otimes_{O_Y} \mathbb{C}_0$ by Corollary 6.5.0.5. Since $\Delta^A F = A$ and $R\text{S}_1 \Delta^A F$ is concentrated in degree $g$, the $D_Y$-module $D_Y \otimes_{O_Y} \mathbb{C}_0$ is holonomic.

Example 6.7.2.8 leads to a question: When is an induced $D$-module holonomic? A full answer is in Proposition D 2.0.1.

13 There is seemingly a paradox. Suppose $g = 1$ and let $i: 0 \to Y$ be the closed inclusion. Then the $O_Y$-module pullback $i^*\mathbb{C}_0 = \mathbb{C}$ and $i^*(D_Y \otimes_{O_Y} \mathbb{C}_0) = (i^*D_Y) \otimes_{\mathbb{C}} (i^*\mathbb{C}_0) = i^*D_Y$ is the fiber of $D_Y$ at 0, which is nonzero. On the other hand, by [Bjö93, p.87] the derived inverse image $i^!(D_Y \otimes_{O_Y} \mathbb{C}_0)$ is a complex of $D_0 = \mathbb{C}$-modules concentrated in degree $-1$. From [Bjö93, 2.3.7], the underlying complex of $O_Y = \mathbb{C}$-modules is $i^!(D_Y \otimes_{O_Y} \mathbb{C}_0)$. Its 0-th cohomology $i^!(D_Y \otimes_{O_Y} \mathbb{C}_0)$ is zero, a contradiction! We suggest catching the mistake.

In fact, on a complex manifold $V$, $D_Y$ has two different structures of $O_Y$-modules. Consider local sections $h$ (resp. $\delta$) of $O_V$ (resp. $D_Y$). One module structure defines $h \cdot \delta$ as $h \delta$, the product in $D_Y$. This $O_Y$-module is denoted by $D_Y^h$. Let $\mathbb{F} = \text{Mod}(D_Y) \to \text{Mod}(O_X)$ be the forgetful functor. Then $\mathbb{F}(D_Y) = D_Y^h$.

The other $O_Y$-module structure defines $h \cdot \delta$ as the reversed product $h \delta$ in $D_Y$. We denote it $D_Y^\delta$. Given an $O_Y$-module $F$, which one is used in the tensor product defining the induced left $O_Y$-module $D_Y \otimes_{O_Y} F$? In fact, it relies on the $(D_Y, O_Y)$-bimodule structure on $D_Y^h$. But $M := \mathbb{F}(D_Y \otimes_{O_Y} F)$ is NOT the $O_Y$-module tensor product of $F$ with neither $D_Y^h$ nor $D_Y^\delta$.

Return to the special case that $F = \mathbb{C}_0 = O_Y/I$ on $V = Y$, where $I \subset O_Y$ is the coherent ideal sheaf corresponding to $i$. Take a local coordinate $z$ around $0 \in Y$ such that $O_{Y,0} = \mathbb{C}(z)$ and the maximal ideal $m_0 = (z) \subset O_{Y,0}$. Let $\partial$ be the corresponding local vector field near $0 \in Y$. Then $M = \mathbb{F}(D_Y/D_Y I)$, the $O_Y$-module for $(H^0 \mathbb{I} (D_Y \otimes_{O_Y} \mathbb{C}_0)) = i^! M = M_0/m_0 M_0$. The $O_Y$-action is defined by $z \cdot M_0 = 0$. By $[\partial, z] = 1$, for each $k \in \mathbb{N}$, we have $\partial^k = \frac{1}{k!} (\partial z + z \partial^k) \in D_Y, m_0 + m_0 D_Y$. So, $M_0/m_0 M_0 = 0$, even though the $O$-module pullback to 0 of both $D_Y^h \otimes_{O_Y} \mathbb{C}_0$ and $D_Y^\delta \otimes_{O_Y} \mathbb{C}_0$ are nonzero.
6.7.3 Pullback and pushout

**Proposition 6.7.3.1.** [Lau96, Prop. 3.3.2] Let \( f : X' \to X \) be a homomorphism of complex tori, with \( \dim X' = g' \). Let \( \hat{f} : (X', A_{X'}) \to (X, A_X) \) be the induced morphism (6.31) and \( \hat{f} : Y \to Y' \) be the dual homomorphism. Then

1. \[
L \hat{f}^* RS'_1 = RS_1 R \hat{f}_* : DO_{good}(A_{X'}) \to DO_{good}(D_Y); \quad (6.46)
\]
\[
R \hat{f}_* RS'_2 = T^{g-g'} RS_2 L \hat{f}^* : DO_{good}(D_Y) \to DO_{good}(A_X). \quad (6.47)
\]

2. \[
RS'_2 \hat{f}^+ = L \hat{f}^* RS_2 : D^{b}_{good}(D_Y) \to D^{b}_{good}(A_X'); \quad (6.48)
\]
\[
\hat{f}^+ RS_1 = T^{g'-g} RS'_2 L \hat{f}^* : D^{b}_{good}(A_X) \to D^{b}_{good}(D_{Y'}). \quad (6.49)
\]

**Proof.**

1. The isomorphism (6.47) follows from (6.46) as follows:

\[
R \hat{f}_* RS'_2 \overset{\text{Thm. 6.5.0.3}}{=} T^g RS_2 L \hat{f}_* RS'_2 \overset{(6.46)}{=} T^g RS_2 L \hat{f}^* RS'_1 RS'_2 \overset{\text{Thm. 6.5.0.3}}{=} T^{g-g'} RS_2 L \hat{f}^*.
\]

Then we prove (6.46).

By (6.32) (resp. the proof of [HT07, Prop. 1.5.8]), the derived direct image (resp. inverse image) functor of \( \mathcal{A} \)-modules (resp. \( D \)-modules) regards that of the underlying \( \mathcal{O} \)-modules. From Proposition 5.3.1.3 2, the functor \( \mathcal{P}' \otimes_{O_{X' \times Y'}}^L p_{X'}^* : D(A_{X'}) \to D(O_{X' \times Y'}) \) restricts to a functor \( D_{O_{good}}(A_{X'}) \to D_{good}(O_{X' \times Y'}). \) An application of Lemma 5.3.2.13 to the cartesian square

\[
\begin{array}{ccc}
X' \times Y & \xrightarrow{1_{X' \times Y} \times \hat{f}} & X' \times Y' \\
\downarrow p_2 & & \downarrow p_{Y'} \\
Y & \xrightarrow{\hat{f}} & Y'
\end{array}
\]

yields a natural isomorphism in \( D(D_Y) \):

\[
L \hat{f}^* R p_{Y'}(\mathcal{P}' \otimes_{O_{X' \times Y'}}^L p_{X'}^*) \to R p_{2*} L(1_{X' \times Y} \times \hat{f})^* (\mathcal{P}' \otimes_{O_{X' \times Y'}}^L p_{X'}^*). \quad (6.50)
\]

Applying Theorem 5.3.2.3 to the cartesian square

\[
\begin{array}{ccc}
X' \times Y & \xrightarrow{p_1} & X' \\
\downarrow f \times 1_{Y} & & \downarrow f \\
X \times Y & \xrightarrow{p_X} & X,
\end{array}
\]

yields a natural isomorphism in \( D(D_Y) \):

\[
R \hat{f}_* RS' \overset{\text{Thm. 6.5.0.3}}{=} T^g RS_2 L \hat{f}_* RS'_2 \overset{(6.46)}{=} T^g RS_2 L \hat{f}^* RS'_1 RS'_2 \overset{\text{Thm. 6.5.0.3}}{=} T^{g-g'} RS_2 L \hat{f}^*.
\]
of complex manifolds, one gets a natural isomorphism

$$p_X^* R\hat{f}_* \rightarrow R(f \times 1_Y)_* p_Y^*$$  \hfill (6.51)

of functors $D_{O\text{--good}}(\mathcal{A}_{X'}) \rightarrow D(\text{Mod}(O_{X \times Y})_{1\text{--cxn,fl}})$. Then

$$L\hat{f}_* RS_1 = L\hat{f}_* Rp_Y^*(\mathcal{P}' \otimes_{O_{X' \times Y}}^L p_Y^*)$$

\begin{align*}
\text{(a)} & \quad \cong Rp_2_2_2(L(1_{X'} \times \hat{f})^*(\mathcal{P}' \otimes_{O_{X' \times Y}}^L p_Y^*)) \\
\text{(b)} & \quad \cong Rp_2_2_2_2_2_2[(f \times 1_Y)^* \mathcal{P} \otimes_{O_{X' \times Y}}^L p_Y^*]
\end{align*}

\begin{align*}
\text{(c)} & \quad \cong Rp_Y R(f \times 1_Y)_*[(f \times 1_Y)^* \mathcal{P} \otimes_{O_{X' \times Y}}^L p_Y^*] \\
\text{(d)} & \quad \cong Rp_Y [\mathcal{P} \otimes_{O_{X' \times Y}}^L R(f \times 1_Y)_* p_Y^*]
\end{align*}

where (a) (resp. (b), resp. (c), resp. (d)) uses (6.50) (resp. (5.34), resp. Fact 5.3.2.15, resp. (6.51)). This proves (6.46).

2. The isomorphism (6.49) follows from (6.48) as follows:

$$\hat{f}_+ RS_1 \cong \hat{f}_+ RS_1 RS_1 RS_1$$

\begin{align*}
\text{(a)} & \quad \cong Tg' RS_1 RS_1 RS_1 \\
\text{(b)} & \quad \cong Tg' RS_1 \hat{f}_+ RS_1 \\
\text{(c)} & \quad \cong Tg'^{-g} RS_1 \hat{f}_+
\end{align*}

where (a) and (c) use Theorem 6.6.3.1, and (b) uses (6.48). Then we prove (6.48).

Using (6.33), we can prove that the functor $L\hat{f}_*$ sends $D_{\text{good}}^b(\mathcal{A}_{X})$ to $D_{\text{good}}^b(\mathcal{A}_{X'})$. From Fact 6.6.2.3, the direct image functor $\hat{f}_+ : D^b(D_Y) \rightarrow D^b(D_{Y'})$ sends $D_{\text{good}}^b(D_Y)$ to $D_{\text{good}}^b(D_{Y'})$. For an object or a morphism $U$ (resp. an object $V$) of $D_{\text{good}}^b(D_Y)$ (resp. $D_{\text{good}}^b(\mathcal{A}_{X'})$), we have canonical
identifications
\[ \text{Hom}_{D^b_{\text{good}}(A_X)}(RS_2^t\tilde{f}^*U, V) \overset{\text{Thm 6.6.3.1}}{=} \text{Hom}_{D^b_{\text{good}}(D_V)}(\tilde{f}^*T^gRS_1^tV) \]
\[ = \text{Hom}_{D^b_{\text{good}}(D_V)}(U, T^gL\tilde{f}^*RS_1^tV) \]
\[ = \text{Hom}_{D^b_{\text{good}}(D_V)}(U, T^gRS_1^tR\tilde{f}^*V) \]
\[ = \text{Hom}_{D^b_{\text{good}}(A_X)}(RS_2^t\tilde{f}^*U, V) \]

From Yoneda lemma, we get \( RS_2^t\tilde{f}^*U = L\tilde{f}^*RS_2^tU \) in \( D^b_{\text{good}}(A_X) \). The proof is completed.

\[ \square \]

### 6.7.4 External tensor product

For two complex manifolds \( U, V \), recall the (exact) external tensor product bifunctor

\( (\cdot) \boxtimes_O (\cdot) : \text{Mod}(D(U)) \times \text{Mod}(D(V)) \to \text{Mod}(D(U \times V)) \) \tag{6.52}

defined in [Bjö93, 2.4.4]. By exactness, it descends to

\[ D(D(U)) \times D(D(V)) \to D(D(U \times V)). \] \tag{6.53}

**Remark 6.7.4.1.** The bifunctor (6.52) sends \( \text{Coh}(D(U)) \times \text{Coh}(D(V)) \to \text{Coh}(D(U \times V)) \) and \( \text{Good}(D(U)) \times \text{Good}(D(V)) \to \text{Good}(D(U \times V)) \), see [Bjö93, 2.4.13]. By [Har66, I, Prop. 7.3 (i)], the bifunctor (6.53) sends \( D^b_c(D(U)) \times D^b_c(D(V)) \to D^b_c(D(U \times V)) \) and \( D^b_{\text{good}}(D(U)) \times D^b_{\text{good}}(D(V)) \to D^b_{\text{good}}(D(U \times V)) \). By [Bjö93, p.139], it also preserves \( D^b_h(D) \).

Using Lemma 5.5.1.6 (at the place of [HT07, Lem. 1.5.31]), Proposition 6.6.2.4 and [Sab11, Thm. 3.3.6 (1)], one can argue as in [HT07, Prop. 1.5.30] to get Fact 6.7.4.2.

**Fact 6.7.4.2.**

1. Let \( M, N, T \) be complex manifolds, \( f : M \to N \) be a proper morphism, \( F \in D_{O-\text{good}}(D_M) \) and \( G \in D(D_T) \). Then the canonical morphism

\[ f_+ F \boxtimes_O G \to (f \times \text{Id}_T)_+(F \boxtimes_O G) \]

is an isomorphism.

2. Let \( f_i : U_i \to V_i \) (\( i = 1, 2 \)) be two proper morphisms of complex manifolds. For \( M_i \in D_{O-\text{good}}(D_{U_i}) \), the canonical morphism

\[ f_1 + M_1 \boxtimes_O f_2 + M_2 \to (f_1 \times f_2)_+(M_1 \boxtimes_O M_2) \]

is an isomorphism.
For two complex tori $X, X'$, set $u : X \times X' \to X$ (resp. $u' : X \times X' \to X'$) for the projection. For an $A_X$-module $F$ and an $A_{X'}$-module $G$, denote $\tilde{u}^*F \otimes_{A_{X \times X'}} \tilde{u}'^*G$ by $F \boxtimes_A G$. As

$$F \boxtimes_A G = u^{-1}F \otimes_{u^{-1}A_X} A_{X \times X'} \otimes_{u'^{-1}A_{X'}} u'^{-1}G,$$

and $A_{X \times X'}$ is flat over $u^{-1}A_X$ and over $u'^{-1}A_{X'}$, we get that the exactness of the bifunctor

$$\cdot \boxtimes_A (\cdot) : \text{Mod}(A_X) \times \text{Mod}(A_{X'}) \to \text{Mod}(A_{X \times X'}). \quad (6.54)$$

Although the tensor product of two $A_X$-modules is different from the tensor product of the underlying $O_X$-module, external products do agree. Lemma 6.7.4.3 is used in the proof of Lemma 6.7.4.5.

**Lemma 6.7.4.3.** Let $X, X'$ be two complex tori. For an $A_X$-module $F$ and an $A_{X'}$-module $G$, we have $F \boxtimes_A G = F \boxtimes_O G$ in the category $\text{Mod}(O_{X \times X'})$.

**Proof.** By construction, we have

$$A_{X \times X'} = A_X \boxtimes_O A_{X'} = u^{-1}A_X \otimes_{u^{-1}O_X} u'^{*}A_{X'}.$$  \hfill (6.55)

Then

$$\tilde{u}^*F := u^{-1}F \otimes_{u^{-1}A_X} A_{X \times X'}$$

$$= (u^{-1}F \otimes_{u^{-1}A_X} (u^{-1}A_X \otimes_{u^{-1}O_X} u'^{*}A_{X'}))$$

$$= u^{-1}F \otimes_{u^{-1}O_X} u'^{*}A_{X'}$$

$$= (u^{-1}F \otimes_{u^{-1}O_X} O_{X \times X'}) \otimes_{O_{X \times X'}} u'^*A_{X'}$$

$$= u'^* F \otimes_{O_{X \times X'}} u'^*A_{X'}.$$  \hfill (6.55)

Similarly, we have $\tilde{u}'^*G = u'^* A_X \otimes_{O_{X \times X'}} u'^*G$. Then

$$F \boxtimes_A G := \tilde{u}^*F \otimes_{A_{X \times X'}} \tilde{u}'^*G$$

$$= (u'^* F \otimes_{O_{X \times X'}} u'^*A_{X'}) \otimes_{u'^*A_X \otimes_{O_{X \times X'}} u'^*A_{X'}} (u'^* A_X \otimes_{O_{X \times X'}} u'^*G)$$

$$= u'^* F \otimes_{O_{X \times X'}} u'^*G$$

$$:= F \boxtimes_O G.$$
Remark 6.7.4.4. We can reprove (6.55) as follows:

\[(a)\] \(\mathcal{A}_X \boxtimes O \mathcal{A}_Y = \text{for}(RS_2(D_Y \otimes_{O_Y} C_0)) \boxtimes_O \text{for}(RS_2'(D_Y' \otimes_{O_Y} C_0))\]

\[(b)\] \(= RS_2(D_Y \otimes_{O_Y} C_0) \boxtimes_O RS_2'(D_Y' \otimes_{O_Y} C_0)\)

\[(c)\] \(= RS_2''((D_Y \otimes_{O_Y} C_0)) \boxtimes_O (D_Y' \otimes_{O_Y} C_0)\)

\[(d)\] \(= RS_2'''((D_Y' \otimes_{O_Y} D_Y') \otimes_{O_Y} (C_0 \otimes O C_0))\)

\[(e)\] \(= \text{for}(RS_2''(D_Y'' \otimes C_0))\)

\[(f)\] \(= \mathcal{A}_Y''\)

where \(X'' = X \times X'\), the complex torus \(Y''\) is dual to \(X''\), (a) and (f) use Corollary 6.5.0.5, (b) and (e) use Proposition 6.5.0.2, and (c) (resp. (d)) uses Proposition 5.5.1.5 (resp. [Bjö93, 2.4.4, (i)]).

Lemma 6.7.4.5. Let \(X'\) be another complex torus and \(X'' = X \times X'\). Then

\[RS_2''_[(\cdot) \boxtimes O (\cdot)] = RS_2(\cdot) \boxtimes_A RS_2'(\cdot) : D_{O\text{-good}}(D_Y) \times D_{O\text{-good}}(D_Y') \to D_{O\text{-good}}(\mathcal{A}_Y''');\]

\[RS_2'''_[(\cdot) \boxtimes_A (\cdot)] = RS_1(\cdot) \boxtimes_O RS_1'(\cdot) : D_{O\text{-good}}(\mathcal{A}_X) \times D_{O\text{-good}}(\mathcal{A}_X) \to D_{O\text{-good}}(D_Y'').\]

Proof. It follows from Proposition 5.5.1.5, Lemma 6.7.4.3 and Proposition 6.5.0.2.

\[\square\]

### 6.7.5 Convolution and tensor product

**Definition 6.7.5.1** (Convolution). [Lau96, p.22] On the complex torus \(Y\) (resp. \(X\)), define a bifunctor

\[(\cdot) *_D (\cdot) : D(D_Y) \times D(D_Y) \to D(D_Y)\]

(resp.

\[(\cdot) *_A (\cdot) : D(A_X) \times D(A_X) \to D(A)\]

by \((\cdot) *_D (\cdot) = \mu_+ \circ [\cdot \otimes O (\cdot)]\) (resp. \((\cdot) *_A (\cdot) = R\mu_* \circ [\cdot \otimes A (\cdot)]\)), where \(\mu : Y^2 \to Y\) (resp. \(m : X^2 \to X\)) is the group law.

As \(\mu\) is proper, the direct image \(\mu_+\) sends \(D_{O\text{-good}}(D_Y)\) (resp. \(D_{O\text{-good}}(D_Y')\)), resp. \(D_{b_{O\text{-good}}}(D_Y)\) to \(D_{b_{O\text{-good}}}(D_Y)\) (resp. \(D_{b_{O\text{-good}}}(D_Y')\)), resp. \(\mathcal{D}_{b}(D_Y)\)) by Fact 6.6.2.3 (resp. Proposition 6.6.2.4, resp. Fact 6.7.2.3 2). Together with Remark 6.7.4.1, this implies the bifunctor \(*_D\) preserves \(D_{b_{O\text{-good}}}(D)\) (resp. \(D_{O\text{-good}}(D),\) resp. \(D_{b}(D)).\)
The pair \((D(Y), *_D)\) is a symmetric tensor triangulated category\(^\text{14}\) with unit \(D_Y \otimes_{O_Y} C_0\).

**Proof.** Let \(i : \text{Specan}(\mathbb{C}) \rightarrow Y\) be the inclusion of \(0 \in Y\). Then \(D_Y \otimes_{O_Y} C_0 = \mathbb{C}\). For an object or a morphism \(\phi\) of \(D(Y)\), we have

\[
(i + \mathbb{C}) *_D \phi := \mu_+[(i + \mathbb{C}) \otimes \mathcal{O}(\mathbb{C})] \quad \mu := \mu_+[(i + \mathbb{C}) \otimes \mathcal{O}(\mathbb{C})] \\
\text{Fact 6.7.4.2} \quad \mu_+(i \otimes \mathcal{O}(\mathbb{C})) \otimes \mathcal{O}(\mathbb{C}) = \mu_+[(i + \mathbb{C}) \otimes \mathcal{O}(\mathbb{C})] \\
\text{[Lab10, Def. 3]} \quad \mu := \mu_+[(i + \mathbb{C}) \otimes \mathcal{O}(\mathbb{C})].
\]

Therefore, \(D_Y \otimes_{O_Y} C_0\) is the unit. The other axioms can be verified as in [Wei07, pp. 10-11].

**Proposition 6.7.5.3** (Weissauer). [Wei11] For every \(M \in D^b_{\text{good}}(D_Y)\), the functor \(\bullet *_D M\) on \(D^b_{\text{good}}(D_Y)\) admits a right adjoint \(([-1]^Y \Delta^D M) *_D \bullet\).

**Proof.** Define an automorphism \(f : Y^2 \rightarrow Y^2\) of the complex torus \(Y^2\) by \(f(a,b) = (a+b,-a)\). Then \(p_1f = \mu, p_2f = [-1]_Y p_1\) and \(\mu f = p_2\). Note that \(L^f Y^2 = O_{Y^2}\) in \(D^b(Y^2)\).

For every \(F, G \in D^b_{\text{good}}(D_Y)\), we have canonical bijections

\[
\text{Hom}_{D^b_{\text{good}}(D_Y)}(F * M, G) := \text{Hom}_{D^b_{\text{good}}(D_Y)}(\mu_+(F \otimes \mathcal{O} M), G) \\
\text{[Bjö93, Thm. 2.11.8]} \quad \text{Hom}_{D(D_Y)}(F \otimes \mathcal{O} M, T^g\mu^* G) \\
\text{[Kas03, (3.13)]} \quad \text{Hom}_{D(D_Y)}(O_{Y^2}, \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
\text{Prop. 6.7.5.4} \quad \text{Hom}_{D(D_Y)}(O_{Y^2}, \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
:= \text{Hom}_{D(D_Y)}(O_{Y^2}, \mu_+ \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
:= \text{Hom}_{D(D_Y)}(f! O_{Y^2}, \mu_+ \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
:= \text{Hom}_{D(D_Y)}(([-1]^Y \Delta^D M) \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
\text{[Kas03, Thm. 4.12]} \quad \text{Hom}_{D(D_Y)}(O_{Y^2}, T^g \Delta^D Y^2 (\mu^* F) \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
\text{[Kas03, (3.13)]} \quad \text{Hom}_{D(D_Y)}(\mu_+ \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
\text{[Kas03, Thm. 4.40]} \quad \text{Hom}_{D(D_Y)}(F, \mu_+ \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G) \\
\text{Lem. 6.7.2.1} \quad \text{Hom}_{D^b_{\text{good}}(D_Y)}(F, \mu_+ \Delta^D Y^2 F \otimes \mathcal{O} M \otimes T^g\mu^* G)
\]

As the bijections are functorial in \(F\) and \(G\), the proof is completed. \(\square\)

\(^{14}\)in the sense of [Lab10, Def. 3]
In the proof of Proposition 6.7.5.3, we need the commutativity of duality with external tensor product for \( D \)-modules.

**Proposition 6.7.5.4.** \(^{15}\) Let \( Z_i \) (\( i = 1, 2 \)) be two complex manifolds, \( M_i \in D^b_c(D_{Z_i}) \), then there is a canonical isomorphism

\[
\Delta^{D_{Z_1}}(M_1) \boxtimes \Delta^{D_{Z_2}}(M_2) \to \Delta^{D_{Z_1 \times Z_2}}(M_1 \boxtimes O M_2)
\]

in \( D^b_c(D_{Z_1 \times Z_2}) \).

For a complex manifold \( Z \), \( D_{Z} \otimes_{\mathbb{C}_Z} D_{Z}^{op} \) is naturally a sheaf of \( \mathbb{C}_Z \)-algebra and \( D_{Z} \) is naturally a left \( D_{Z} \otimes_{\mathbb{C}_Z} D_{Z}^{op} \)-module.

**Proof.** For \( N_1 \in D(D_{Z_1}^{op}) \), we have

\[
N_1 \boxtimes O N_2 = (N_1 \boxtimes O N_2) \otimes_{D_{Z_1} \boxtimes D_{Z_2}} D_{Z_1 \times Z_2} \quad (6.58)
\]

in \( D(D_{Z_1 \times Z_2}^{op}) \), cf. \([HT07, p.30]\).

There is a natural morphism in \( D^b(D_{Z_1} \boxtimes D_{Z_2})^{op})\):

\[
R\text{Hom}_{D_{Z_1}}(M_1, D_{Z_1}) \boxtimes \mathbb{C} \to \text{RHom}_{D_{Z_2}}(M_2, D_{Z_2}) \quad (6.59)
\]

\[
\to \text{RHom}_{D_{Z_1} \boxtimes D_{Z_2}}(M_1 \boxtimes \mathbb{C} M_2, D_{Z_1} \boxtimes \mathbb{C} D_{Z_2}).
\]

In fact, take a \( D_{Z_i} \otimes_{\mathbb{C}} D_{Z_i}^{op} \)-injective resolution \( D_{Z_i} \to I_i^* \). Then \( I_i^* \boxtimes \mathbb{C} I_i^* \) is a complex of modules over

\[
(D_{Z_1} \otimes_{\mathbb{C}} D_{Z_1}^{op}) \boxtimes \mathbb{C} (D_{Z_2} \otimes_{\mathbb{C}} D_{Z_2}^{op}) = (D_{Z_1} \boxtimes \mathbb{C} D_{Z_2}) \otimes_{\mathbb{C}} (D_{Z_1} \boxtimes \mathbb{C} D_{Z_2})^{op}. \quad (6.60)
\]

By \([\text{Sta23, Tag 013K (2)}]\), there exists an injective resolution \( I_i^* \boxtimes \mathbb{C} I_i^* \to I_i^* \) (hence an induced injective resolution \( D_{Z_i} \boxtimes \mathbb{C} D_{Z_2} \to I_i^* \)) over \( (6.60) \). Note that the natural morphism \( D_{Z_i} \to D_{Z_i} \otimes_{\mathbb{C}} D_{Z_i}^{op} \) is flat, so each injective \( D_{Z_i} \otimes_{\mathbb{C}} D_{Z_i}^{op} \)-module is injective over \( D_{Z_i} \). Likewise, each term of \( I_i^* \) is injective over \( D_{Z_i} \boxtimes \mathbb{C} D_{Z_2} \). Then \( (6.59) \) is defined to be the composition of the natural morphisms

\[
\text{Hom}_{D_{Z_1}}(M_1, I_i^*) \boxtimes \mathbb{C} \to \text{Hom}_{D_{Z_2}}(M_2, I_i^*) \to \text{Hom}_{D_{Z_1} \boxtimes D_{Z_2}}(M_1 \boxtimes \mathbb{C} M_2, I_i^*).
\]

Similarly, there is a natural morphism in \( D^b(D_{Z_1 \times Z_2}^{op})\):

\[
R\text{Hom}_{D_{Z_1} \boxtimes D_{Z_2}}(M_1 \boxtimes \mathbb{C} M_2, D_{Z_1} \boxtimes \mathbb{C} D_{Z_2}) \to \text{RHom}_{D_{Z_1} \boxtimes D_{Z_2}}(M_1 \boxtimes \mathbb{C} M_2, D_{Z_1} \boxtimes \mathbb{C} D_{Z_2}). \quad (6.61)
\]

In fact, take an injective resolution \( D_{Z_1} \boxtimes \mathbb{C} D_{Z_2} \to J_i^* \) over \( (6.60) \). By \([\text{Sta23, Tag 013K (2)}]\), there exists an injective resolution \( J_i^* \otimes_{\mathbb{C}} D_{Z_2} \xrightarrow{} K_i^* \)

\(^{15}\)Proposition 6.7.5.4 and its proof, both due to Claude Sabbah, are explained to me by Gabriel Ribeiro.
over $(D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}) \otimes_{\mathcal{O}} D_{Z_1 \times Z_2}^{op}$. Then (6.61) is defined to be the composition of the natural morphisms

$$\text{Hom}_{D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}}(M_1 \boxtimes_{\mathcal{O}} M_2, J^*) \otimes_{D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}} D_{Z_1 \times Z_2} \rightarrow \text{Hom}_{D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}}(M_1 \boxtimes_{\mathcal{O}} M_2, J'^* \otimes_{D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}} D_{Z_1 \times Z_2}) \rightarrow \text{Hom}_{D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}}(M_1 \boxtimes_{\mathcal{O}} M_2, K'^*)$$

Again, there is a natural morphism in $D^b(D_{Z_1 \times Z_2}^{op})$:

$$\text{RHom}_{D_{Z_1} \boxtimes_{\mathcal{O}} D_{Z_2}}(M_1 \boxtimes_{\mathcal{O}} M_2, D_{Z_1 \times Z_2}) \rightarrow \text{RHom}_{D_{Z_1 \times Z_2}^{op}}(M_1 \boxtimes_{\mathcal{O}} M_2, D_{Z_1 \times Z_2}) \rightarrow \text{RHom}_{D_{Z_1 \times Z_2}^{op}}(M_1 \boxtimes_{\mathcal{O}} M_2, D_{Z_1 \times Z_2})$$

which can be defined by taking a $D_{Z_1 \times Z_2} \otimes_{\mathcal{O}} D_{Z_1 \times Z_2}^{op}$-injective resolution of $D_{Z_1 \times Z_2}$.

Composing the morphisms (6.58), (6.59), (6.61) and (6.62) in order, we get a natural morphism in $D^b(D_{Z_1 \times Z_2}^{op})$:

$$\text{RHom}_{D_{Z_1}}(M_1, D_{Z_1} \boxtimes_{\mathcal{O}} \text{RHom}_{D_{Z_2}}(M_2, D_{Z_2})) \rightarrow \text{RHom}_{D_{Z_1 \times Z_2}^{op}}(M_1 \boxtimes_{\mathcal{O}} M_2, D_{Z_1 \times Z_2})$$

To show (6.63) is an isomorphism, we may assume $M_i \in \text{Coh}(D_{Z_i})$ by [Har66, I, Prop. 7.1 (i)]. By shrinking $Z_i$ and using [KS13, Prop. 11.2.6], we may find a bounded resolution of $M_i$ by free $D_{Z_i}$-modules of finite rank. Then we may further assume that $M_i = D_{Z_i}$, in which case (6.63) is indeed an isomorphism. Since $\omega_{Z_1 \times Z_2} = \omega_{Z_1} \boxtimes_{\mathcal{O}} \omega_{Z_2}$ in $\text{Mod}(D_{Z_1 \times Z_2}^{op})$, the proof is completed by [HT07, Eg. 2.6.3].

**Corollary 6.7.5.5 ([Laub96, Cor. 3.3.3]).** The equivalence $RS_2 : (D^b_{\text{good}}(D_Y), \ast_D) \rightarrow (D^b_{\text{good}}(A_X), \otimes_{A_X}^L)$ is a strong monoidal functor. In fact,

$$RS_2[\ast_D \cdot] = RS_2 \cdot \otimes_{A_X}^L RS_2 : D^b_{\text{good}}(D_Y) \times D^b_{\text{good}}(D_Y) \rightarrow D^b_{\text{good}}(A_X);$$

$$RS_1 \cdot \ast_D RS_1 = T^{-} RS_2[\otimes_{A_X}^L \cdot] : D^b_{\text{good}}(A_X) \times D^b_{\text{good}}(A_X) \rightarrow D^b_{\text{good}}(D_Y);$$

$$RS_1[\ast_A \cdot] = RS_1 \cdot \otimes_{\mathcal{O}}^L RS_1 : D_{\text{O-good}}(A_X) \times D_{\text{O-good}}(A_X) \rightarrow D_{\text{O-good}}(D_Y);$$

$$RS_2 \cdot \ast_A RS_2 = T^{-} RS_2[\otimes_{\mathcal{O}}^L \cdot] : D_{\text{O-good}}(D_Y) \times D_{\text{O-good}}(D_Y) \rightarrow D_{\text{O-good}}(A_X).$$

**Proof.** Let $f : X \rightarrow X^2 = X'$ be the diagonal homomorphism. Its dual homomorphism is $\mu$. We have

$$RS_2[\ast_D \cdot] = RS_2 \mu_*[\otimes_{\mathcal{O}}^L \cdot]$$

$$= Lf^* RS_2[\otimes_{\mathcal{O}}^L \cdot]$$

$$= Lf^* [RS_2(\cdot) \otimes_{A_X}^L RS_2(\cdot)]$$

$$= RS_2(\cdot) \otimes_{A_X}^L RS_2(\cdot).$$
which shows \((6.64)\). Corollary 6.5.0.5 shows \(RS_2\) preserves units, so it is strong monoidal. In addition, \((6.65)\) follows:

\[
RS_1(\cdot) *_D RS_1(\cdot) \overset{\text{Thm } 6.6.3.1}{=} T^g RS_1 RS_2[RS_1(\cdot) *_D RS_1(\cdot)]
\]

\[
\overset{(6.64)}{=} T^g RS_1[RS_2 RS_1(\cdot) *_D RS_2 RS_1(\cdot)]
\]

\[
\overset{\text{Thm } 6.6.3.1}{=} T^g RS_1[T^{-g}(\cdot) *_D T^{-g}(\cdot)]
\]

\[
= T^{-g} RS_1[(\cdot) \otimes_{\Delta X}(\cdot)].
\]

Similarly, because the diagonal homomorphism \(\delta_Y : Y \to Y^2\) is dual to \(m : X' = X^2 \to X\), we have

\[
RS_1[(\cdot) *_A (\cdot)] = RS_1 R\delta_*[(\cdot) \boxtimes_A (\cdot)]
\]

\[
\overset{(6.46)}{=} L\delta_Y^* RS_1'[(\cdot) \boxtimes_A (\cdot)]
\]

\[
\overset{(6.57)}{=} L\delta_Y^* [RS_1(\cdot) \boxtimes O RS_1(\cdot)]
\]

\[
\overset{[HT07, p.39]}{=} RS_1(\cdot) \otimes_{O} RS_1(\cdot).
\]

This demonstrates \((6.66)\). Then \((6.67)\) follows:

\[
RS_2(\cdot) *_A RS_2(\cdot) \overset{\text{Thm. 6.5.0.3}}{=} T^g RS_2 RS_1[RS_2(\cdot) *_A RS_2(\cdot)]
\]

\[
\overset{(6.66)}{=} T^g RS_2[RS_1 RS_2(\cdot) \otimes_{O} RS_1 RS_2(\cdot)]
\]

\[
\overset{\text{Thm. 6.5.0.3}}{=} T^g RS_2[T^{-g}(\cdot) \otimes_{O} T^{-g}(\cdot)]
\]

\[
= T^{-g} RS_2[(\cdot) \otimes_{O} (\cdot)].
\]

Remark 6.7.5.6. We reprove Proposition 6.7.5.3 using Laumon-Rothstein transform as follows. By [Sta23, Tag 08DJ], for every \(M \in D^b_{\text{good}}(\mathcal{A}_X)\) the functor \(\bullet \otimes_{\mathcal{A}_X} M\) on \(D^b_{\text{good}}(\mathcal{A}_X)\) admits a right adjoint

\[
R\mathcal{H}om_{\mathcal{A}_X}(M, \bullet) \overset{[Hay06, p.84]}{=} T^{-g} \Delta^A(M) \otimes_{\mathcal{A}_X} \bullet.
\]

Combining Proposition 6.7.2.5 with Corollary 6.7.5.5, we get Proposition 6.7.5.3.
Appendix A

Sheaves of modules

A.1 Sheaves of modules

We recall some facts about sheaves of modules. Let \((X, O_X)\) be a ringed space.

A.1.1 Generalities

Definition A.1.1.1. An \(O_X\)-module \(F\) is called

1. ([Sta23, Tag 01B5]) of \textit{finite type} if every \(x \in X\) admits an open neighborhood \(U\) such that \(F|_U\) is generated by finitely many sections;

2. ([Sta23, Tag 01BN]) of \textit{finite presentation} if for every \(x \in X\), there is an open neighborhood \(U \subset X\), integers \(n, m \geq 0\) and an exact sequence of \(O_U\)-modules
   \[ O^n_U \rightarrow O^m_U \rightarrow F|_U \rightarrow 0; \]

3. ([Gro60, 5.1.3]) \textit{quasi-coherent} if for every \(x \in X\), there is an open neighborhood \(U \subset X\), two sets \(I, J\) and a morphism \(O^\oplus_I U \rightarrow O^\oplus_J U\) whose cokernel is isomorphic to \(F|_U\);

4. ([Kas03, Def. A.5 (1)]) \textit{pseudo-coherent} if for every open subset \(U \subset X\), every finite type \(O_U\)-submodule of \(F|_U\) is of finite presentation. Let \(\text{PCoh}(X) \subset \text{Mod}(O_X)\) be full subcategory of pseudo-coherent modules;

5. ([Kas03, Def. A.5 (2)]) \textit{K-coherent} if \(F\) is pseudo-coherent and of finite type;

6. ([Sta23, Tag 01BV]) \textit{S-coherent} if \(F\) is of finite type and for every open subset \(U \subset X\) and every finite collection \(\{s_i\}_{1 \leq i \leq n}\) in \(F(U)\), the kernel of the associated morphism \(O^n_U \rightarrow F|_U\) is of finite type over \(O_U\).

Every property in Definition A.1.1.1 is local, in the sense that it restricts to every open subset, and if it holds on each member of an open covering of \(X\), then it holds on \(X\).
Let \( 0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0 \) be an exact sequence in \( \text{Mod}(\mathcal{O}_X) \).

**Lemma A.1.1.2.** If \( F, H \) are of finite presentation, then so is \( G \).

**Proof.** For every \( x \in X \), by [Sta23, Tag 01B8], there is an open neighborhood \( U \) of \( x \) such that the sequence \( G(U) \xrightarrow{f|_U} H(U) \to 0 \) is exact. Up to shrinking \( U \), there exist integers \( m, n, p, q \geq 0 \) and two exact sequences

\[
\begin{align*}
O^m_U &\to O^n_U \xrightarrow{f|_U} F|_U \to 0, & O^p_U &\to O^q_U \xrightarrow{h|_U} H|_U \to 0.
\end{align*}
\]

The morphism \( h \) is defined by \( q \) elements \( s_1, \ldots, s_q \) of \( H(U) \). For each \( 1 \leq i \leq q \), choose a preimage \( t_i \in G(U) \) of \( s_i \). Define a morphism \( \phi : O^{n+q}_U \to G|_U \) by \( if(e_1), \ldots, if(e_n), t_1, \ldots, t_q \in G(U) \). Hence a commutative diagram with two exact middle rows

\[
\begin{array}{ccccccc}
0 & \to & O^m_U & \xrightarrow{f} & \ker(\phi) & \to & O^n_U \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & O^p_U & \xrightarrow{g} & O^{n+q}_U & \to & O^q_U & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F|_U & \xrightarrow{f|_U} & G|_U & \to & H|_U & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & \text{coker}(\phi) & \to & 0.
\end{array}
\]

By the snake lemma, \( \phi \) is surjective and \( \ker(\phi) \) is finite type. Shrinking \( U \) again, one may find an integer \( k \geq 0 \) and a surjection \( O^k_U \to \ker(\phi) \). The induced sequence \( O^k_U \to O^{n+q}_U \to G|_U \to 0 \) is exact. Therefore, \( G \) is of finite presentation. \( \square \)

### A.1.2 Pseudo-coherent modules

**Lemma A.1.2.1.**

1. Let \( 0 \to F \xrightarrow{f} G \xrightarrow{g} H \to 0 \) be a short exact sequence in \( \text{Mod}(\mathcal{O}_X) \). If \( F, H \) are pseudo-coherent, then so is \( G \).

2. Let \( I \) be a directed set and let \( (M_i, f_{ij}) \) be a system over \( I \) consisting of pseudo-coherent \( \mathcal{O}_X \)-modules. Then \( M := \text{colim}_{i \in I} M_i \) in \( \text{Mod}(\mathcal{O}_X) \) is pseudo-coherent.

3. If \( \{M_\alpha\}_{\alpha \in A} \) is a family of pseudo-coherent \( \mathcal{O}_X \)-modules, then \( S := \oplus_{\alpha \in A} M_\alpha \) is also pseudo-coherent.

**Proof.** Let \( U \) be an open subset of \( X \).
1. Let \( M \) be a finite type submodule of \( G|_U \). Then the kernel of \( g|_M : M \to H|_U \) is \((F|_U) \cap M\). Thus, \( g|_M \) induces an injection \( M/(F|_U \cap M) \to H|_U \). As \( H \) is pseudo-coherent, the finite type \( O_U \)-submodule \( M/(F|_U \cap M) \) is of finite presentation. By [Sta23, Tag 01BP (2)], \( F|_U \cap M \) is of finite type. As \( F \) is pseudo-coherent, \( F|_U \cap M \) is of finite presentation. Applying Lemma A.1.1.2 to the exact sequence \( 0 \to F|_U \cap M \to M \to M/(F|_U \cap M) \to 0 \), one gets that \( M \) is of finite presentation. Thus, \( G \) is pseudo-coherent.

2. Let \( N \) be a finite type submodule of \( M|_U \). For every \( x \in U \), from the first three lines of the proof of [Sta23, Tag 01BB], there is an open neighborhood \( V \subset U \) of \( x \) and \( i \in I \) such that \( N|_V \subset F_i|_V \). Since \( F_i \) is pseudo-coherent, \( N|_V \) is of finite presentation. As finite presentation is a local property, \( N \) is of finite presentation. Thus, \( M \) is pseudo-coherent.

3. Let \( I \) be the set of all finite subsets of \( A \) with the inclusion order. Then \( I \) is a directed set. For \( B \in I \), set \( F_B = \bigoplus_{a \in B} M_a \). By Point 1, \( F_B \) is pseudo-coherent. For \( B \leq B' \) in \( I \), set \( f_{B,B'} : F_B \to F_{B'} \) to be the inclusion. Hence a system \((F_B, f_{B,B'})\) over \( I \). By Point 2, \( S = \text{colim}_{B \in I} F_B \) is pseudo-coherent.

\[\square\]

**Lemma A.1.2.2.** An \( O_X \)-module is \( K \)-coherent if and only if \( S \)-coherent.

**Proof.** Let \( U \subset X \) be an open subset. Assume that \( F \) is a \( K \)-coherent module. Let \((s_i)_{1 \leq i \leq n}\) be a finite collection in \( F(U) \), and let \( f : O_U^n \to F|_U \) be the associated morphism. Then \( \text{im} f \) is a finite type submodule of \( F|_U \). Because \( F \) is pseudo-coherent, \( \text{im} f \) is of finite presentation over \( O_U \). From [Sta23, Tag 01BP (2)], \( \ker f \) is of finite type over \( O_U \). Therefore, \( F \) is \( S \)-coherent.

Conversely, assume that \( F \) is an \( S \)-coherent module. Let \( M \) be a finite type submodule of \( F|_U \). By [Sta23, Tag 01BY (1)], \( M \) is \( S \)-coherent over \( O_U \). From [Sta23, Tag 01BW], \( M \) is of finite presentation. Thus, \( F \) is pseudo-coherent and hence \( K \)-coherent.

The two equivalent notions in Lemma A.1.2.2 are called "coherent". The module \( O_X \) is quasi-coherent, but in general not pseudo-coherent. If it is pseudo-coherent, then it is called a coherent sheaf of rings ([Kas03, p.214], [Bjö93, A:II, Def. 6.29]).

**Lemma A.1.2.3.** If \( X \) is a locally Noetherian scheme, then every quasi-coherent module is pseudo-coherent.

**Proof.** By [Gro60, Cor. 9.4.9], a quasi-coherent module is a directed limit of coherent modules, hence pseudo-coherent by Lemma A.1.2.2.

**Example A.1.2.4.** Let \( X = A^1 \) be the affine line over a field. Let \( U = X \setminus \{0\} \), and let \( j : U \to X \) be the inclusion. By [Har77, II, Example 5.2.3], the \( O_X \)-module \( j_! O_U \) is not quasi-coherent. From [Har77, II, Exercise 1.19 (c)], it is a submodule of the coherent module \( O_X \). Hence, \( j_! O_U \) is pseudo-coherent.
Definition A.1.2.5 defines a local property. It is weaker than [Bjö93, A:III, 2.24] and [Kas03, Def. A.7].

**Definition A.1.2.5.** Assume that $O_X$ is a coherent sheaf of rings. If for every open subset $U \subset X$, every family of coherent ideal sheaves $\{I_i\}_i$ in $O_U$, the ideal sheaf $\sum_i I_i$ is $O_U$-coherent, then $O_X$ is called a quasi-Noetherian sheaf of rings.

**Example A.1.2.6.** If $(X, O_X)$ is a locally Noetherian scheme, then $O_X$ is quasi-Noetherian. If $(X, O_X)$ is a complex analytic space, then by the Oka-Cartan theorem (see, e.g., [Kas03, Thm. A.12]), $O_X$ is also quasi-Noetherian.

### A.1.3 Coherent modules

Let $X$ be a complex analytic space. We show that a coherent $O_X$-module admits a local free resolution, from which we deduce that coherence is preserved by derived pullback. An analog of Lemma A.1.3.1 for algebraic varieties is [Har77, III, Example 6.5.1]. By local syzygies [GH78, p.696], on complex manifolds, every coherent module admits a finite-length local free resolution.

**Lemma A.1.3.1.** Every $x \in X$ admits an open neighborhood $U$, such that for every coherent $O_X$-module $F$, there is a (possibly infinite-length) resolution

$$
\cdots \rightarrow O_{U_2}^{n_2} \rightarrow O_{U_1}^{n_1} \rightarrow F|_U \rightarrow 0,
$$

where $n_i \geq 0$ are integers.

**Proof.** Shrinking $X$ to an open neighborhood of $x$, one may assume that $X$ is Stein. By [GR13, Thm. 8, p.108], there is a compact neighborhood $K \subset X$ of $x$, such that Theorem B is valid on $K$ in the sense of [GR13, Def. 1, p.100]. Let $U = K^\circ$.

For a coherent $O_X$-module $F$, we construct inductively a sequence of morphisms.

From [GR13, Cor. p.101], there is an integer $n_0 \geq 0$ and a morphism $f_0 : O_{U_0}^{n_0} \rightarrow F|_{U_0}$ in $\text{Mod}(O_{U_0})$ such that $f_0|_U$ is an epimorphism in $\text{Mod}(O_U)$. Set $\ker(f_{j-1})|_{U_0} = F|_{U_0}$. Given such a morphism $f_j : O_{U_j}^{n_j} \rightarrow \ker(f_{j-1})|_{U_j}$ for some integer $j \geq 0$ and an open neighborhood $U_j$ of $K$, by [Sta23, Tag 01BY (3)] the $O_{U_j}$-module $\ker(f_j)$ is coherent. By [GR13, Cor. p.101], there is an open neighborhood $U_{j+1} \subset U_j$ of $K$, an integer $n_{j+1} \geq 0$ and a morphism $f_{j+1} : O_{U_{j+1}}^{n_{j+1}} \rightarrow \ker(f_j)|_{U_{j+1}}$ in $\text{Mod}(O_{U_{j+1}})$ such that $f_{j+1}|_U$ is an epimorphism.

Thus, one gets a sequence

$$
\cdots \rightarrow O_{U_2}^{n_2} \xrightarrow{f_2|_U} O_{U_1}^{n_1} \xrightarrow{f_1|_U} O_{U_0}^{n_0} \xrightarrow{f_0|_U} F|_U \rightarrow 0
$$

in $\text{Mod}(O_U)$. By construction, it is exact, hence a resolution. \hfill $\square$

**Example A.1.3.2.** Assume that $x \in X$ is a singular point. Then $F := \mathbb{C}_x$ is a coherent $O_X$-module, but for every open neighborhood $U \subset X$ of $x$, there is no finite-length resolution of $F|_U$ by finite locally free $O_U$-modules. Otherwise,
such a resolution induces a finite-length free resolution of the $O_{X,x}$-module $F_x = \mathbb{C} = O_{X,x}/m_x$. From [Osib12, Ch. 4, Prop. 4.4], the projective dimension $\text{pd}_{O_{X,x}} O_{X,x}/m_x$ is finite. By [Mat87, Lem. 1, p.154] and [Osib12, Prop. 4.9], the global dimension of the ring $O_{X,x}$ is finite. By Serre’s theorem (see, e.g., [Osib12, p.332]), the local ring $O_{X,x}$ is regular. From [Ser56, p.6], $x$ is a smooth point of $X$, a contradiction.

Therefore, Lemma A.1.3.1 fails if one consider only finite-length resolutions. See also [EP96, Thm. 4.1.2].

**Lemma A.1.3.3.** Let $f : X \to Y$ be a morphism of complex analytic spaces. Then for every coherent $O_Y$-module $F$, the derived pullback $Lf^*F \in D_c(X)$.

**Proof.** For every $x \in X$, by Lemma A.1.3.1, there is an open neighborhood $V$ of $f(x) \in Y$, such that there is a resolution $E_* \to F|_V \to 0$ by finite free $O_V$-modules. Let $g : f^{-1}(V) \to V$ be the restriction of $f$. Then the morphism $g^*E_* \to (Lf^*F)|_V$ in $D(V)$ is an isomorphism. For every integer $j \geq 0$, $g^*E_j$ is a finite free $O_{f^{-1}(V)}$-module. Thus, the $O_{f^{-1}(V)}$-module $(H^{-j}Lf^*F)|_{f^{-1}(V)}$ is coherent. Since coherence is a local property, the $O_X$-module $H^{-j}Lf^*F$ is coherent.

**Lemma A.1.3.4.** For coherent $O_X$-modules $F,G$, one has $F \otimes^l_{O_X} G \in D_c(X)$.

**Proof.** For every $x \in X$, by Lemma A.1.3.1, there is an open neighborhood $U \subset X$ of $x$ and a resolution $E_* \to F|_U \to 0$ by finite free $O_U$-modules. The natural morphism $E_* \otimes_{O_U} G|_U \to F|_U \otimes_{O_U} G|_U$ in $D(U)$ is an isomorphism. For every integer $n$, the $O_U$-module $H^n(E_* \otimes_{O_U} G|_U) = H^n(E_* \otimes_{O_U} G|_U)$ is coherent. Therefore, the $O_U$-module $H^n(F \otimes_{O_X} G)|_U = H^n(F|_U \otimes_{O_U} G|_U)$ is coherent. Since coherence is a local property, the $O_X$-module $H^n(F \otimes_{O_X} G)$ is coherent.

### A.1.4 Good modules

Assume that $X$ is locally compact Hausdorff.

**Definition A.1.4.1.** [Kas03, Def. 4.22] An $O_X$-module $F$ is called *good* if for every relatively compact open subset $U \subset X$, there exists a directed family $\{G_{\alpha}\}_\alpha$ of coherent $O_U$-submodules of $F|_U$ such that $F|_U = \sum_{\alpha} G_{\alpha}$, where $\{G_{\alpha}\}$ being a directed family means that for any $\alpha$ and $\alpha'$ there is $\alpha''$ with $G_{\alpha} + G_{\alpha'} \subset G_{\alpha''}$ (and hence $F|_U = \text{colim}_\alpha G_{\alpha}$). The full subcategory of Mod$(O_X)$ consisting of good $O_X$-modules is denoted by $\text{Good}(X)$.

**Lemma A.1.4.2** (Goodness vs. pseudo-coherence).

1. ([Kas03, p.77]) $\text{Coh}(X) \subset \text{Good}(X) \subset \text{PCoh}(X)$.

2. Let $E$ be a pseudo-coherent $O_X$-module. If on each relatively compact open subset $U \subset X$, $E|_U$ is the sum of its finite type $O_U$-submodules, then $E$ is good.
Proof.

1. By definition, every coherent $O_X$-module is good. Let $E$ be a good $O_X$-module. Let $W$ be an open subset of $X$, and let $F \subset E|_W$ be a finite type $O_W$-submodule. We show that $F$ is of finite presentation over $O_W$. Replacing $(X,E)$ with $(W,E|_W)$, one may assume that $W = X$. For every $x \in X$, there exists a relatively compact open neighborhood $U \subset X$ of $x$ and finitely many sections $s_1, \ldots, s_n \in F(U)$ generating $F|_U$. As $E$ is good, $E|_U = \sum \alpha G_\alpha$ is the sum of a directed family of coherent submodules. There exists $\alpha_0$ and an open neighborhood $V$ of $x \in U$ with $s_i|_V \in G_{\alpha_0}(V)$ for all $1 \leq i \leq n$, then $F|_V$ is a finite type submodule of $G_{\alpha_0}|_V$. By [Sta23, Tag 01BY (1)], $F|_V$ is $O_V$-coherent. As coherence is a local property, the $O_X$-module $F$ is coherent. From [Sta23, Tag 01BW], $F$ is of finite presentation.

2. The family of finite type submodules of $E|_U$ is directed, since the sum of two finite type submodules is of finite type. For every relatively compact open subset $U \subset X$, as $E$ is pseudo-coherent, every finite type submodule of $E|_U$ is pseudo-coherent and hence coherent. Thus, $E$ is good. 

Basic properties of good modules (similar to those of quasi-coherent modules on algebraic varieties) are recapped in Lemma A.1.4.3, among which Point 3 should be compared to [Con06, Lemma 2.1.8 (1)].

Lemma A.1.4.3.

1. For every family of objects $\{F_i\}_{i \in I}$ in $\text{Good}(X)$, the direct sum $\oplus_{i \in I} F_i$ in $\text{Mod}(O_X)$ is good.

2. The category $D_{\text{good}}(X)$ has arbitrary direct sums. Moreover, the inclusion functor $\text{Good}(X) \to D_{\text{good}}(X)$ commutes with direct sums.

Suppose that $O_X$ is quasi-Noetherian. Then:

3. The subcategory $\text{Good}(X) \subset \text{Mod}(O_X)$ is weak Serre (in the sense of [Sta23, Tag 02MO]) and closed under filtered colimits in $\text{Mod}(O_X)$. In particular, $\text{Good}(X)$ is a locally Noetherian category (in the sense of [Gab62, p.356]).

4. The inclusion functor $D_{\text{good}}(X) \to D(X)$ is a triangulated subcategory.

Proof.

1. Over each relatively compact open subset $U$ of $X$, the direct sum $(\oplus_{i \in I} F_i)|_U$ is the sum of its coherent $O_U$-submodules. By Lemma A.1.2.1 3, the $O_X$-module $\oplus_{i \in I} F_i$ is pseudo-coherent. By Lemma A.1.4.2 2, it is good.
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2. By [Sta23, Tag 07D9], the category $D(X)$ has arbitrary direct sums and they are computed by taking termwise direct sums of any representative complexes. Therefore, for every integer $q$, the functor $H^q : D(X) \to \text{Mod}(O_X)$ commutes with direct sums. The result follows from Point 1.

3. As $O_X$ is quasi-Noetherian, by the proof of [Kas03, Prop. 4.23] and [Sta23, Tag 0754], $\text{Good}(X)$ is a weak Serre subcategory of $\text{Mod}(O_X)$. From [KS06, Thm. 18.1.6 (v)], the category $\text{Mod}(O_X)$ is a Grothendieck abelian category. By Point 1 and [Sta23, Tag 002P], the filtered colimits in $\text{Good}(X)$ exist and agree with the filtered colimits in $\text{Mod}(O_X)$. Thus, filtered colimits in $\text{Good}(X)$ are exact.

Because of [Sta23, Tag 01BC], there is a set of coherent $O_X$-modules $\{F_i\}_{i \in I}$ such that each coherent $O_X$-module is isomorphic to exactly one of the $F_i$. Then $\{F_i\}$ is a family of Noetherian generators of $\text{Good}(X)$ by definition. Therefore, the category $\text{Good}(X)$ is locally Noetherian.

4. It follows from [Yek19, Prop. 7.4.5] and Point 3.

Lemma A.1.4.4. A good module on a complex analytic space is quasi-coherent.

Proof. Let $F$ be a good module on a complex analytic space $X$. From [Fri67, Thm. I, 9; Rem. I, 10], every $x \in X$ admits a neighborhood $K$ that is a Noetherian Stein compactum. There is a relative compact open subset $U$ of $X$ containing $K$. As $F$ is good, the $O_U$-module $F|_U = \sum_{i \in I} F_i$ is the sum of a directed family of coherent subsheaves. Applying the functor $\Gamma(K, \cdot)$ to the directed family $\{F_i\}_{i \in I}$ in $\text{Coh}(U)$, by [Tay02, Prop. 11.9.2], one gets a directed family of finitely generated $\Gamma(K, O_K)$-submodule $\{M_i\}_{i \in I}$ of $\Gamma(K, F)$, whose associated family in $\text{Mod}(O_K)$ is $\{F_i|_K\}_{i \in I}$. Let $M$ be $\text{colim}_{i \in I} M_i$ in $\text{Mod}(\Gamma(K, O_K))$. Since the localization functor $\text{Mod}(\Gamma(K, O_K)) \to \text{Mod}(O_K)$ is left adjoint to $\Gamma(K, \cdot) : \text{Mod}(O_K) \to \text{Mod}(\Gamma(K, O_K))$, it preserves colimits. Then $F|_K$ is associated to $M$. By Lemma B.2.0.5, $F$ is quasi-coherent.

Remark A.1.4.5. The restriction of a good $O_X$-module to an open subset $U$ is a good $O_U$-module. Unlike quasi-coherence on schemes, goodness is not a local property. In fact, by Lemma A.1.4.3 3 every free module on a complex manifold is good, while Gabber [Con06, Eg. 2.1.6] gives a locally free (hence quasi-coherent and pseudo-coherent) module that is not good. (In particular, the converse of Lemma A.1.4.4 is wrong.) Still, given an $O_X$-module $F$, if for every relatively compact open subset $U \subset X$, $F|_U$ is $O_U$-good, then $F$ is $O_X$-good.

A.1.5 Sections of direct sum of sheaves

By [Har77, II, Exercise 1.11], on a Noetherian topological space, taking section commutes with (possibly infinite) direct sum of sheaves. This fails on complex manifolds, as Example A.1.5.1 shows.
Example A.1.5.1. Let \( X = \mathbb{C} \). Let \( F \) be the \( O_X \)-module \( \oplus n \geq 0 \mathcal{C}_n \). Then there is a section \( s \in \Gamma(X, F^\oplus \mathbb{N}) \), whose stalk \( s_n \in (F^\oplus \mathbb{N})_n = (F_n)^\oplus \mathbb{N} \) at an integer \( n \geq 0 \) is \((1, 1, \ldots, 1, 0, 0, \ldots)\), where the first \( n + 1 \) entries are 1 and all the other entries are 0. Then \( s \) has no preimage under the canonical map \( \Gamma(X, F)^\oplus \mathbb{N} \rightarrow \Gamma(X, F^\oplus \mathbb{N}) \). For otherwise, let \((t^n)_{n \geq 0} \in \Gamma(X, F)^\oplus \mathbb{N}\) be a preimage of \( s \). Then there are only finitely many integers \( n \geq 0 \) with \( t^n \neq 0 \). Every \( t^n \) has only finitely many nonzero stalks. However, \( s \) has infinitely many nonzero stalks, which is a contradiction.

Let \( X \) be a complex manifold. An \( O_X \)-module is said to have property \( P \) if for every connected open subset \( U \subset X \) and every \( x \in U \), the map \( \Gamma(U, F) \rightarrow F_x \) taking the stalk at \( x \) is injective. By the identity theorem (see, e.g., [GH78, p.7]), \( O_X \) has property \( P \).

Lemma A.1.5.2. Assume that \( X \) is connected. Let \( \{F_i\}_{i \in I} \) be a family of \( O_X \)-modules having property \( P \). Then the canonical map \( \oplus_{i \in I} \Gamma(X, F_i) \rightarrow \Gamma(X, \oplus_{i \in I} F_i) \) is bijective.

Proof. Let \( P \) be the presheaf direct sum of \( \{F_i\}_{i \in I} \). Let \( \theta : P \rightarrow \oplus_{i \in I} F_i \) be the sheafification morphism. Then \( P(X) = \oplus_{i \in I} \Gamma(X, F_i) \) and \( \theta_X : \oplus_{i \in I} \Gamma(X, F_i) \rightarrow \Gamma(X, \oplus_{i \in I} F_i) \) is the colimit of

\[
\theta^{(j)}_X : \oplus_{i \in J} \Gamma(X, F_i) \rightarrow \Gamma(X, \oplus_{i \in I} F_i),
\]

where \( J \) runs through the finite subsets of \( I \). For every such \( J \), by [Sta23, Tag 01AH (4)], the presheaf direct sum of \( \{F_i\}_{i \in J} \) is a subsheaf of \( \oplus_{i \in I} F_i \), so the map \( \theta^{(J)}_X \) is injective. Therefore, their limit map \( \theta_X \) is also injective. We prove that \( \theta_X \) is surjective.

By construction of sheafification in [Har77, p.64], for every \( s \in \Gamma(X, \oplus_{i \in I} F_i) \), there is a covering \( \{U_\alpha\}_{\alpha \in A} \) of \( X \) by nonempty connected open subsets and an element \( t_\alpha \in \Gamma(U_\alpha, P) \) for each \( \alpha \in A \) such that \( s_x = t_{\alpha,x} \in (\oplus_{i \in I} F_i)_x = \oplus_{i \in I} F_i \) for every \( x \in U_\alpha \).

Fix \( x_0 \in X \) and \( \alpha_0 \in A \) with \( x_0 \in U_{\alpha_0} \). Then there is a finite subset \( I_0 \subset I \) such that \( t_{\alpha_0} \in \Gamma(X, \oplus_{i \in I_0} F_i) \subset \Gamma(X, P) \). Let \( B \subset A \) be the subset of indices \( \alpha \) with \( t_\alpha \notin \Gamma(U_\alpha, \oplus_{i \in I_0} F_i) \) and \( V = \cup_{\alpha \in B} U_\alpha \). Then \( V \) is open in \( X \) and its complement

\[
X \setminus V \subset \cup_{\alpha \in A \setminus B} U_\alpha. \tag{A.1}
\]

For every \( \alpha \in A \setminus B \), we claim that \( U_\alpha \subset X \setminus V \).

In fact, for every \( y \in U_\alpha \), every \( \beta \in A \) with \( y \in U_\beta \) and every \( i \in I \setminus I_0 \), the stalk \( s_i^\beta \|_{y} = s_i^{\alpha,y} = 0 \) in \( F_i \). Since \( F_i \) has property \( P \) and \( U_\beta \) is connected, the map \( \Gamma(U_\beta, F_i) \rightarrow F_{i,y} \) is injective. Thus, \( t_i^{\beta,y} = 0 \) in \( \Gamma(U_\beta, F_i) \). Therefore, \( t_{\beta} \in \Gamma(X, \oplus_{i \in I_0} F_i) \), i.e., \( t_{\beta} \notin B \). Hence \( y \notin V \).

From the claim and (A.1), the subset \( X \setminus V = \cup_{\alpha \in A \setminus B} U_\alpha \) is also open in \( X \) and contains \( U_{\alpha_0} \). Since \( X \) is connected, \( V = B = \emptyset \). Consequently, \( t_\alpha \in \Gamma(X, \oplus_{i \in I_0} F_i) \) for every \( \alpha \in A \). Then the family \( \{t_\alpha\}_{\alpha \in A} \) glues to a preimage of \( s \) in \( \Gamma(X, \oplus_{i \in I_0} F_i) \subset \Gamma(X, P) \). Thus, \( \theta_X \) is surjective and hence a group isomorphism. \( \square \)
Corollary A.1.5.3. If $F$ is a locally free (not necessarily of finite rank) $O_X$-module, then $F$ has property $\mathcal{P}$.

Proof. Let $U$ be a connected open subset of $X$. Fix $x_0 \in U$. We prove that the map $\Gamma(U, F) \to F_{x_0}$ is injective. Take $s \in \Gamma(U, F)$ with $s_{x_0} = 0$. By [Har77, II, Exercise 1.14], the set $Z := \{ x \in U : s_x = 0 \}$ is open in $U$.

We claim that $Z$ is closed in $U$. Let $\{x_n\}_{n \geq 1}$ be a sequence of points in $Z$ converging to $y \in U$. Because $F$ is locally free, there is a connected open neighborhood $V \subset U$ of $y$, a set $I$ and an isomorphism $\phi : F|_V \to O_V^{\oplus I}$ of $O_V$-modules. There is an integer $N > 0$ with $x_N \in V$. Because $O_V$ has property $\mathcal{P}$, from Lemma A.1.5.2, the map on the bottom of the commutative square

$$
\begin{array}{ccc}
\Gamma(V, F) & \longrightarrow & F_{x_N} \\
\downarrow \phi_V & & \downarrow \phi_{x_N} \\
\Gamma(V, O_V^{\oplus I}) & \longrightarrow & O_V^{\oplus I}
\end{array}
$$

is injective. Then so is the map on the top. Since $s_{x_N} = 0$, one has $s|_V = 0$ and $s_y = 0$. Thus, $y \in Z$. The claim is proved.

Because $U$ is connected and $x_0 \in Z$, by claim one has $Z = U$. Therefore, $s = 0$ in $\Gamma(U, F)$.

Corollary A.1.5.4. Let $X$ be a connected complex manifold. Let $\{F_i\}_{i \in I}$ be a family of locally free $O_X$-modules. Then the canonical map $\oplus_{i \in I} \Gamma(X, F_i) \to \Gamma(X, \oplus_{i \in I} F_i)$ is bijective.

Proof. It follows from Lemma A.1.5.2 and Corollary A.1.5.3.

A.2 Gabber’s example

We present an example of a locally free module on the open unit disc that is not good. It illustrates that goodness on complex manifolds, unlike quasi-coherence on algebraic varieties, is not a local property. This construction is already exhibited in the context of rigid geometry by [Con06, Example 2.1.6], which attributes the originality to Gabber. Furthermore, in [Con06, p.1058] it is mentioned that Gabber’s example makes sense in complex-analytic geometry as well. We reproduce this construction with a few extra details.

Lemma A.2.0.1. Let $X$ be a complex manifold, $U$ be a dense open subset of $X$. If $F$ is a locally free $O_X$-module, then the restriction map $r : \Gamma(X, F) \to \Gamma(U, F)$ is injective.

Proof. Every $x \in X$ admits a connected open neighborhood $V$ such that $F|_V$ is free $O_V$-module. Then $\Gamma(V, F)$ is a free $\Gamma(V, O_X)$-module by Corollary A.1.5.4. By density of $U$, $V \cap U$ is nonempty. For every $s \in \ker(r)$, $s|_{V \cap U} = 0$. As $F|_V$ is free and the map $\Gamma(V, O_X) \to \Gamma(V \cap U, O_X)$ is injective, the restriction $s|_V = 0$. By local nature of sheaves, $s = 0$. 

\[\square\]
Lemma A.2.0.2. Let $X$ be a Hausdorff locally compact space, $K$ be a compact subspace and $j : K \to X$ be the inclusion. Then for every $q \in \mathbb{Z}$ and every $F \in \text{Ab}(X)$, the canonical morphism $\psi^q : \text{colim}_U H^q(U,F) \to H^q(K,j^{-1}F)$ is an isomorphism, where $U$ ranges through the family of open neighborhoods of $K$ in $X$. The two groups are written as $H^q(K,F)$.

Proof. We prove that both sides are the $q$-th right derived functor applied to $F$ of a same functor.

Define a category $I$ as follows. The objects are the open subsets of $X$ containing $K$. For every $U, V \in I$, if $U \supset V$, then $\text{Hom}_I(U, V)$ is a singleton; else $\text{Hom}_I(U, V) = \emptyset$. Thus, $I$ is a small category. Let $\text{Ab}^I$ be the category of functors from $I$ to $\text{Ab}$. By [Wei95, Exercise 2.3.7], $\text{Ab}^I$ is an abelian category with enough injectives. Recall that $\text{Ab}$ is a Grothendieck abelian category, so $\text{colim}_I : \text{Ab}^I \to \text{Ab}$ is exact. By [KS13, Proposition 2.5.1], the composition of the functor $\Phi : \text{Ab}(X) \to \text{Ab}^I$ defined by $\Phi(F)(U) = \Gamma(U, F)$ with $\text{colim}_I : \text{Ab}^I \to \text{Ab}$ is $\Gamma(K, j^{-1} \cdot) : \text{Ab}(X) \to \text{Ab}$. Therefore, the $q$-th right derived functor of $\Gamma(K, j^{-1} \cdot)$ is $\text{colim}_I \circ R^q\Phi = \text{colim}_I H^q(U, \cdot)$.

The functor $\Gamma(K, j^{-1} \cdot) : \text{Ab}(X) \to \text{Ab}$ is the composition of $j^{-1} : \text{Ab}(X) \to \text{Ab}(K)$ with $\Gamma(K, \cdot) : \text{Ab}(K) \to \text{Ab}(X)$. Every injective object $G$ of $\text{Ab}(X)$ is $j^{-1}G$ is $c$-soft by [KS13, Proposition 2.5.7 (i)]. By [KS13, Proposition 2.5.10], $j^{-1}G$ is right acyclic for $\Gamma(K, \cdot)$. By [Sta23, Tag 015M], $H^q(K, j^{-1} \cdot)$ is also the $q$-th right derived functor of $\Gamma(K, j^{-1} \cdot)$. We conclude that $\psi^q$ is an isomorphism.

Definition A.2.0.3 (Compact Stein set). [Con06, p.1053] Let $K$ be a compact subset of a complex manifold $X$. If $H^q(K, F) = 0$ for every open neighborhood $U$ of $K \subset X$, every coherent $O_U$-module and every $q \in \mathbb{N}^*$, then $K$ is called a compact Stein set in $X$.

Lemma A.2.0.4 ([Con06, p.1058]). Let $K$ be a compact compact Stein set in a complex manifold $X$, $F$ be a good $O_X$-module, then $H^q(K,F) = 0$ for all $q \in \mathbb{N}^*$.

Proof. There is a relative compact open subset $U \subset X$ containing $K$. By definition, $F|_U = \text{colim}_i F_i$, where $\{F_i\}$ is a direct family of coherent $O_U$-submodules of $F|_U$. By [God58, II, Thm. 4.12.1], $H^q(K,F) = \text{colim}_i H^q(K,F_i) = 0$.

Example A.2.0.5 (Gabber). Let $\Delta$ be the open unit disc in $\mathbb{C}$ and let $K = \{ z \in \mathbb{C} : |z| \leq 1/2 \}$. Then $B(0, 2/3)$ is a relatively compact open subset of $\Delta$ containing $K$. By [Dou66, Thm. 3 (B), p.51; (a) p.53], $K$ is a compact Stein set in $\Delta$.

Let $x', x''$ be two distinct points of the interior of $K$. Let $U' = \Delta \setminus \{ x' \}$, $U'' = \Delta \setminus \{ x'' \}$ and define $U = U' \cap U''$. Let $F' = \oplus_{n \in \mathbb{Z}} O_{U'} e'_n$, $F'' = \oplus_{n \in \mathbb{Z}} O_{U''} e''_n$ be two free sheaves with countably infinite rank on $U'$ and $U''$ respectively.
A.2. GABBER’S EXAMPLE

We glue $F'$ and $F''$ to define a locally free $O_{\Delta}$-module $F$ as follows. Define $h \in O_{\Delta}(U)$ by

$$h(z) = e^{z + 1/z''}, \quad \forall z \in U.$$ 

Then $h$ has essential singularities at $x'$ and $x''$. Define $F$ by identifying $F'|_U$ and $F''|_U$ with the free sheaf $\oplus_{n \in \mathbb{Z}} O_U e_n$ via the conditions

$$e_{2m} = e_{2m}'|_U = e_{2m}''|_U + h e_{2m+1}''|_U,$$

$$e_{2m+1} = e_{2m+1}''|_U = e_{2m+1}'|_U + h e_{2m+2}''|_U$$

for every $m \in \mathbb{Z}$ respectively.

We prove that $\Gamma(K, F) = 0$. For every $s \in \Gamma(K, F)$, by Lemma A.2.0.2, there is an open subset $W$ of $\Delta$ containing $K$ such that $s$ lifts to an element of $\Gamma(W, F)$. By Corollary A.1.5.4, $\Gamma(U, F) = \oplus_{n \in \mathbb{Z}} \Gamma(U, O_{\Delta}) e_n$. So, $s|_{U \cap W} = \sum_{n \in \mathbb{Z}} f_n e_n$ with $f_n \in O_{\Delta}(U \cap W)$ that vanish for all but finitely many $n$. Note that

$$s|_{U''} = \sum_{n \in \mathbb{Z}} (f_{2n} e_{2n}'' + (f_{2n} h + f_{2n+1} e_{2n+1}'')|_{U''}.$$ 

Therefore, $f_{2n}$ and $f_{2n} h + f_{2n+1}$ are holomorphic near $x'$ for all $n \in \mathbb{Z}$. Similarly, $f_{2n+1}$ and $f_{2n+1} h + f_{2n+2}$ are holomorphic near $x''$ for all $n \in \mathbb{Z}$.

We claim that $s|_{U \cap W} = 0$. Otherwise, let $n_0$ be the maximum with $f_{n_0} \neq 0$. If $n_0$ is odd (resp. even), $f_{n_0}$ and $h f_{n_0}$ are holomorphic near $x''$ (resp. $x'$). The ratio $h = h f_{n_0}/f_{n_0}$ is meromorphic near $x''$ (resp. $x'$). It contradicts the choice of $h$. The claim is proved.

By Lemma A.2.0.1, the restriction map $\Gamma(W, F) \to \Gamma(W \cap U, F)$ is injective, so $s = 0$.

We prove that $F$ is not good. Let $t$ be the standard coordinate on $\Delta$, then $0 \to F \xrightarrow{t} F \to F/tF \to 0$ is a short exact sequence in $\text{Mod}(O_{\Delta})$. The associated cohomology sequence induces an injection $H^0(K, F/tF) \to H^1(K, F)$ by Lemma A.2.0.2. As $F/tF$ is the skyscraper supported at the origin, we have $H^0(K, F/tF) \neq 0$ and hence $H^1(K, F) \neq 0$. By Lemma A.2.0.4, the $O_{\Delta}$-module $F$ is not good. And $F|_K$ is not induced by a $\Gamma(K, O_K)$-module. In particular, $F$ is not quasi-coherent in the sense of last paragraph of [BBBP07, p.443]. Nevertheless, $F$ is quasi-coherent in the sense of [Gro60, 5.1.3] since it is locally free.
Appendix B

Quasi-coherent sheaves on complex analytic spaces

B.1 Introduction

Let \((X, O_X)\) be a ringed space. The category of \(O_X\)-modules is denoted by \(\text{Mod}(O_X)\).

**Definition B.1.0.1.** An \(O_X\)-module \(F\) is called *quasi-coherent* if for every \(x \in X\), there is an open neighborhood \(U \subset X\), two sets \(I, J\) and a morphism \(O^\oplus_U \to O^\oplus_U\) whose cokernel is isomorphic to \(F|_U\). The full subcategory of \(\text{Mod}(O_X)\) comprised of quasi-coherent modules is denoted by \(\text{Qch}(X)\).

According to [Sta23, Tag 01BD], in general \(\text{Qch}(X)\) is not an abelian category. If \(X\) is a scheme, then by [Sta23, Tag 06YZ], \(\text{Qch}(X)\) is a weak Serre subcategory (in the sense of [Sta23, Tag 02MO (2)]) of \(\text{Mod}(O_X)\). We show a complex analytic analog of this result.

**Theorem B.1.0.2.** If \(X\) is a complex analytic space, then the subcategory \(\text{Qch}(X) \subset \text{Mod}(O_X)\) is weak Serre.

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B.2 Preliminaries

**Example B.2.0.1.** [Sta23, Tag 01BI] Let \(f : (X, O_X) \to (\{\ast\}, O_X(X))\) be the morphism of ringed spaces with \(f : X \to \{\ast\}\) the unique map and with \(f^\#: O_X(X) \to O_X(X)\) the identity. For an \(O_X(X)\)-module \(M\), its pullback \(f^*M\) is called the sheaf associated to \(M\). This \(O_X\)-module is quasi-coherent. The
functor $f^* : \text{Mod}(O_X(X)) \to \text{Mod}(O_X)$ is called the localization and denoted by $\tilde{\cdot}$.

From [Gro60, 4.1.1], on a scheme the direct sum of any family of quasi-coherent modules is quasi-coherent. It fails for complex manifolds, shown by Example B.2.0.2.

**Example B.2.0.2.** [hs] Let $X \subset \mathbb{C}$ be the unit open disk. For every integer $n \geq 2$, Gabber ([Conf06, Eg. 2.1.6], see also Example A.2.0.5) constructs a locally free (hence quasi-coherent) $O_X$-module $F_n$ of infinite rank, such that for every open subset $U \subset X$ containing $\{\pm 1/n\}$, one has $\Gamma(U, F_n) = 0$. We claim that $F := \oplus_{n \geq 2} F_n$ is not quasi-coherent.

Assume the contrary. Then there is an open neighborhood $V$ of $0 \in X$, a set $I$ and a quotient morphism $q : O_V^\oplus I \to F|_V$. There is an integer $N \geq 2$ with $\{\pm 1/N\} \subset V$. Let $p : F|_V \to F_N|_V$ be the quotient morphism. Because $\text{Hom}_{\text{Mod}(O_V)}(O_V, F_N|_V) = \Gamma(V, F_N) = 0$, the morphism $pq = 0$. However, $F_N|_V \neq 0$, a contradiction. The claim is proved.

Let $X$ be a complex analytic space in the sense of [GR13, p.18]. For an inclusion $i : K \to X$ of a compact subset, let $O_K = i^{-1}O_X$. Then $O_K$ is naturally a sheaf of rings on $K$.

**Definition B.2.0.3.** A compact subset $K \subset X$ is called a Stein compactum if $K$ has a fundamental system of open neighborhoods that are Stein subspaces of $X$. A Stein compactum $K$ is called Noetherian if $O_K(K)$ is a Noetherian ring.

**Fact B.2.0.4** ([Fri67, Thm. 1.9; Rem. 1.10]). Every $x \in X$ admits a neighborhood which is a Noetherian Stein compactum in $X$.

**Lemma B.2.0.5.** Let $F$ be an $O_X$-module. Then the following conditions are equivalent:

1. ([BBBP07, Def. 5.1]) Every $x \in X$ admits a neighborhood $K$ which is a Noetherian Stein compactum such that $F|_K$ is associated to a $\Gamma(K, O_K)$-module.

2. The $O_X$-module $F$ is quasi-coherent.

**Proof.**

- Assume Condition 1. For every $x \in X$, take such a $K$ and suppose that $F|_K$ is associated to a $\Gamma(K, O_K)$-module $M$. There is an exact sequence $\Gamma(K, O_K)^{\oplus I} \to \Gamma(K, O_K)^{\oplus J} \to M \to 0$ in the category of $\Gamma(K, O_K)$-modules. By [Sta23, Tag 01BH], it induces an exact sequence $O_K^{\oplus J} \to O_K^{\oplus J} \to F|_K \to 0$ in $\text{Mod}(O_K)$. Then the $O_K$-module $F|_K$ is quasi-coherent. Thus, Condition 2 is proved.

- Assume Condition 2. Because $X$ is locally compact Hausdorff, for every $x \in X$, by [Sta23, Tag 01BK], there is an open neighborhood $U \subset X$ of $x$ such that $F|_U$ is associated to a $\Gamma(U, O_X)$-module. From Fact
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B.2.0.4, there is a neighborhood $K$ of $x \in U$ which is a Noetherian Stein compactum. By [Sta23, Tag 01BJ] applied to the morphism $(K, O_K) \to (U, O_U)$ of ringed spaces, $F|_K$ is associated to a $\Gamma(K, O_K)$-module. Thus, Condition 1 is proved.

Lemma B.2.0.6. Let $K$ be a Noetherian Stein compactum in $X$.

1. The natural transformation $\text{Id} \to \Gamma(K, \hat{\bullet})$ of functors $\text{Mod}(\Gamma(K, O_K)) \to \text{Mod}(\Gamma(K, O_K))$ is an isomorphism.

2. The localization functor $\hat{\bullet} : \text{Mod}(O_K(K)) \to \text{Mod}(O_K)$ is exact, fully faithful.

3. For every $O_K(K)$-module $M$ and every integer $q > 0$, one has $H^q(K, \tilde{M}) = 0$.

Proof.

1. Let $M$ be a $\Gamma(K, O_K)$-module. We prove that the morphism $M \to \Gamma(K, \tilde{M})$ is an isomorphism. Assume first that $M$ is finitely generated. Then the result follows from [Tay02, p.299]. Assume now that $M$ is arbitrary. Let $\{M_i\}_{i \in I}$ be the family of all finitely generated submodules of $M$. This family is directed in the inclusion relation and

$$M = \sum_{i \in I} M_i. \tag{B.1}$$

By [Sta23, Tag 01BH (4)], the localization functor preserves colimits. Therefore,

$$\tilde{M} = \text{colim}_{i \in I} \tilde{M}_i. \tag{B.2}$$

By [God58, Thm. 4.12.1], one has

$$\Gamma(K, \tilde{M}) = \text{colim}_{i \in I} \Gamma(K, \tilde{M}_i) = \text{colim}_{i \in I} M_i = M.$$

2. The exactness is proved in [Tay02, Prop. 11.9.3 (ii)]. For any $M, N \in \text{Mod}(O(K))$, we prove that the natural morphism

$$\text{Hom}_{O(K)}(M, N) \to \text{Hom}_{O_K}(\tilde{M}, \tilde{N}) \tag{B.3}$$

is an isomorphism.

Assume first that $M$ is finitely generated. As the ring $O(K)$ is Noetherian, the $O(K)$-module $M$ is of finite presentation. Then by [GW20, Exercise 7.20 (b)], one has $\text{Hom}_{O(K)}(M, N) = \tilde{\text{Hom}}_{O_K}(\tilde{M}, \tilde{N})$. By Point 1, the morphism (B.3) is an isomorphism. Assume now that $M$ is arbitrary. By (B.1) and (B.2), the morphism (B.3) is the inverse limit of the morphisms $\text{Hom}_{O(K)}(M_i, N) \to \text{Hom}_{O_K}(\tilde{M}_i, \tilde{N})$, each of which is an isomorphism.
3. When $M$ is finitely generated, it follows from [Tay02, Prop. 11.9.2] and [Car57, Thm. 1 (B)]. Assume now that $M$ is arbitrary. By (B.2) and [God58, Thm. 4.12.1], one has $H^q(K, M) = \colim_i H^q(K, M_i) = 0$.

\[\square\]

### B.3 Proof of Theorem B.1.0.2

For every morphism $f : F \to G$ in $Qch(X)$, we prove that $\ker(f), \text{coker}(f)$ in $\text{Mod}(O_X)$ lie in $\text{Qch}(X)$.

For every $x \in X$, by Lemma B.2.0.5, there is a neighborhood $A$ (resp. $B$) of $x \in X$ which is a Noetherian Stein compactum and an $O_A(A)$-module $M$ (resp. $O_B(B)$-module $N$), such that $F|_A$ (resp. $G|_B$) is associated to $M$ (resp. $N$). By Fact B.2.0.4, there is a neighborhood $C$ of $x \in A^\circ \cap B^\circ$ which is a Noetherian Stein compactum. From [Sta23, Tag 01BJ], $F|_C$ (resp. $G|_C$) is associated to $M \otimes_{O_A(A)} O_C(C)$ (resp. $N \otimes_{O_B(B)} O_C(C)$). By Lemma B.2.0.6 2, there is a morphism $\phi : M \otimes_{O_A(A)} O_C(C) \to N \otimes_{O_B(B)} O_C(C)$ in $\text{Mod}(O_C(C))$ whose localization is $f|_C : F|_C \to G|_C$. The restriction functor $\text{Mod}(O_X) \to \text{Mod}(O_{C^\circ})$ is exact, so $\ker(f)|_{C^\circ}$ (resp. $\text{coker}(f)|_{C^\circ}$) is the localization of $\ker(\phi \otimes_{O_C(C)} \text{Id}_{O_X(C^\circ)})$ (resp. $\text{coker}(\phi \otimes_{O_C(C)} \text{Id}_{O_X(C^\circ)})$) in $\text{Mod}(O_X(C^\circ))$. Therefore, the $O_X$-modules $\ker(f), \text{coker}(f)$ are quasi-coherent.

Let

$$0 \to F' \to F \to F'' \to 0$$

be a short exact sequence in $\text{Mod}(O_X)$, with $F', F''$ quasi-coherent. We prove that $F$ is quasi-coherent. For every $x \in X$, there is a neighborhood $K'$ (resp. $K''$) of $x$ which is a Noetherian Stein compactum and an $O_{K'}(K')$-module $M'$ (resp. $O_{K''}(K'')$-module $M''$) whose localization is $F'|_{K'}$ (resp. $F''|_{K''}$). By Fact B.2.0.4, there is a neighborhood $K$ of $x \in K'^\circ \cap K''^\circ$ which is a Noetherian Stein compactum. From [Sta23, Tag 01BJ], $F'|_K$ (resp. $F''|_K$) is associated to $M' \otimes_{O_{K'}(K')} O_K(K)$ (resp. $M'' \otimes_{O_{K''}(K'')} O_K(K)$).

Let $P = \Gamma(K, M)$. By Lemma B.2.0.6 1 and 3, the sequence (B.4) induces a short exact sequence in $\text{Mod}(O_K)$:

$$0 \to M' \otimes_{O_{K'}(K')} O_K(K) \to P \to M'' \otimes_{O_{K''}(K'')} O_K(K) \to 0.$$ 

From Lemma B.2.0.6 2, its localization induces a short exact sequence in $\text{Mod}(O_K)$:

$$0 \to M' \otimes_{O_{K'}(K')} O_K(K) \to \tilde{P} \to M'' \otimes_{O_{K''}(K'')} O_K(K) \to 0.$$ 

By restriction to $K^\circ$ and [Sta23, Tag 01BJ], one has a commutative diagram

$$
\begin{array}{cccccc}
0 & \to & M' \otimes_{O_{K'}(K')} O_X(K^\circ) & \to & P \otimes_{O_K(K)} O_X(K^\circ) & \to & M'' \otimes_{O_{K''}(K'')} O_X(K^\circ) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & F'|_{K^\circ} & \to & F|_{K^\circ} & \to & F''|_{K^\circ} & \to & 0
\end{array}
$$
in $\text{Mod}(O_{K^\circ})$, where the vertical morphisms are given by the adjunction of
$\bullet : \text{Mod}(O_X(K^\circ)) \to \text{Mod}(O_{K^\circ})$ and $\Gamma(K^\circ, \bullet) : \text{Mod}(O_{K^\circ}) \to \text{Mod}(O_X(K^\circ))$. The rows are exact and the two outside vertical arrows are isomorphisms. By the five lemma, the middle vertical morphism is an isomorphism. Therefore, $F|_{K^\circ}$ is quasi-coherent. Consequently, $F$ is quasi-coherent.

By [Sta23, Tag 0754], $\text{Qch}(X)$ is a weak Serre subcategory of $\text{Mod}(O_X)$. 
APPENDIX B. QUASI-COHERENT SHEAVES ON COMPLEX ANALYTIC SPACES
Appendix C

Complex analytic geometry

C.1 Dimension of the fiber product

Section C.1 aims at understanding the dimension of the fiber product of algebraic varieties/complex analytic spaces. The proof in the analytic case is inspired by that in the algebraic case. Therefore, we begin with the algebraic situation.

C.1.1 Algebraic case

Fix a field \( k \). Under flatness condition, the dimension of the fiber product behaves well.

Lemma C.1.1.1. Let \( X, Y, Z \) be three schemes of finite type over \( k \) and \( f : X \to Z, \ g : Y \to Z \) be \( k \)-morphisms. Assume that the schemes \( X, Z \) are irreducible, \( Y \) is equidimensional, and \( g \) is flat. Put \( W = X \times_Z Y \). If \( W \) is nonempty, then \( W \) is equidimensional of dimension \( \dim X + \dim Y - \dim Z \).

Proof. Applying [Har77, Ch. III, Corollary 9.6] to the flat morphism \( g \), we find that \( g \) is of relative dimension \( \dim Y - \dim Z \). By virtue of [Sta23, Tag 02NK], its base change \( W \to X \) is also flat of relative dimension \( \dim Y - \dim Z \). Then the reverse direction of the cited [Har77, Ch. III, Corollary 9.6] shows that \( W \) is equidimensional of dimension \( \dim Y - \dim Z + \dim X \).

In the proof of Proposition C.1.1.2, the general case is reduced to the case of a flat morphism.

Proposition C.1.1.2. Let \( X, Y, Z/k \) be three finite type schemes, \( f : X \to Z, \ g : Y \to Z \) be dominant \( k \)-morphisms. Assume that \( X, Z \) are irreducible and \( Y \) is equidimensional, and put \( W = X \times_Z Y \), then \( \dim W + \dim Z \geq \dim X + \dim Y \).

Proof. Since the reduction \( \text{Z}_{\text{red}} \to Z \) is a universal homeomorphism, we may assume that \( Z \) is an integral scheme. By generic flatness [Sta23, Tag 052A], there is a nonempty affine open subset \( U \subset Z \) such that the restriction \( g^{-1}(U) \to U \)
is flat. By [Sta23, Tag 01UA], the morphism $g^{-1}(U) \to U$ is open. By shrinking $U$, we may assume further that $g^{-1}(U) \to U$ is surjective.

Because $f$ is dominant, $f^{-1}(U)$ is a nonempty open subset of $X$. Therefore, by [Har77, Ch. II, Exercise 3.20 (e)] we have $\dim U = \dim Z$, $\dim f^{-1}(U) = \dim X$ and $g^{-1}(U)$ is equidimensional of dimension $\dim Y$. Hence, we may base change everything along $U \to Z$ which does not increase $\dim W$. In particular, we can assume that $g$ is flat surjective. Then $W \to X$ is also flat surjective. In particular, $W \neq \emptyset$. We conclude by Lemma C.1.1.1.

Example C.1.1.3 shows that the inequality in Proposition C.1.1.2 can be strict.

Example C.1.1.3. If $f : X \to P^3_k$ is the blow up at a point $p \in P^3(k)$, then the morphism $f$ is projective surjective, $\dim X = 3$, $\dim X \times_{P^3_k} X = 4$ and the defect of semismallness $r(f) = 1$.

Corollary C.1.1.4. Let $X, Y/k$ be two finite type schemes and $f : X \to Y$ be a $k$-morphism. If the scheme $X$ is irreducible, then $\dim X \times_Y X \geq 2 \dim X - \dim f(X)$, where $f(X)$ is the Zariski closure of $f(X)$ in $Y$.

Proof. Because the reduction $X_{\text{red}} \to X$ is a universal homeomorphism, we may assume that $X$ is reduced. Let $Z \to Y$ be the scheme theoretic image of $f$. By [Har77, Ch. II, Exercise 3.11 (d)], the induced morphism $X \to Z$ is dominant and the underlying topological space of $Z$ is $f(X)$. Therefore, $Z$ is also irreducible. By magic square [Vak23, 1.3.S], the natural morphism $X \times_Z X \to X \times_Y X$ is the base change of the diagonal isomorphism $Z \to Z \times_Y Z$, hence also an isomorphism. By Proposition C.1.1.2, $\dim X \times_Y X = \dim X \times_Z X \geq 2 \dim X - \dim Z$.

C.1.2 Analytic case

The contents of this section is parallel to those of Section C.1.1. Lemma C.1.2.1 is an analogue of [Har77, III, Corollary 9.6], whose proof is also a direct adaptation. A complex analytic space is called equidimensional if every irreducible component is of same dimension.

Lemma C.1.2.1. Let $f : X \to Y$ be a flat morphism of complex analytic spaces, and assume that $Y$ is irreducible. Then the following conditions are equivalent:

1. $X$ is equidimensional of dimension $n + \dim Y$;
2. for every $y \in f(X)$, the fiber $X_y$ is equidimensional of dimension $n$.

In that case, we say $f$ is flat of relative dimension $n$.

Proof. Assume 1. Given $y \in f(X)$, let $Z$ be an irreducible component of $X_y$. Because the set of irreducible components of a complex analytic space is locally finite, there is $x \in Z$ which is not in any other irreducible component of $X_y$. Applying [CD94, Proposition 2.11, p.113], we have $\dim_x Z + \dim_y Y = \dim_x X$. 


As \( Y, Z \) are irreducible hence pure dimensional, we have \( \dim_y Y = \dim Y \) and \( \dim_x Z = \dim Z \). Now that \( \dim_x X = \dim Y + n \), we have \( \dim Z = n \).

Conversely, assume 2. Let \( W \) be an irreducible component of \( X \). Let \( x \in W \) be a point which is not contained in any other irreducible component of \( X \) and \( y = f(x) \). Then we have \( \dim_x X = \dim W \) and \( \dim_y Y = \dim Y \). Applying [CD94, Proposition 2.11, p.113], we obtain

\[
\dim_x(X_y) + \dim_y Y = \dim_x X.
\]

By assumption, \( \dim_x(X_y) = n \). Thus \( \dim W = \dim Y + n \) as required. \( \square \)

Lemma C.1.2.2 is similar to Lemma C.1.1.1.

**Lemma C.1.2.2.** Let \( f : X \to Z, g : Y \to Z \) be complex analytic space morphisms. Assume that \( X, Z \) are irreducible, \( Y \) is equidimensional, and \( g \) is flat. Put \( W = X \times_Z Y \). If \( W \) is nonempty, then \( W \) is equidimensional of dimension \( \dim X + \dim Y - \dim Z \).

Proposition C.1.2.3 is the main result of Section C.1.

**Proposition C.1.2.3.** Let \( X, Y, Z \) be irreducible complex analytic spaces. Let \( f : X \to Z, g : Y \to Z \) be morphisms and put \( W = X \times_Z Y \). If \( f \) is surjective and the (Euclidean) topology of \( X \) is second-countable, then \( \dim W + \dim Z \geq \dim X + \dim Y \).

**Proof.** Because reduction does not change the dimension [GR84, p.96], we may assume that \( X, Y, Z \) are reduced. Let \( A = \{ x \in X : f \) is not flat at \( x \} \). By Frisch’s theorem [CD94, Theorem 2.8, p.112], \( A \) is an analytic subset of \( X \) and \( f(A) \neq Z \). Then \( X \setminus f^{-1}(f(A)) \to Z \setminus f(A) \) is a surjective flat morphism. By shrinking \( X, Y, Z \) suitably, we may assume further that \( f \) is flat surjective. Then \( W \) is nonempty and we conclude by Lemma C.1.2.2. \( \square \)

The invariant \( \dim X \times_Y X \) considered in Corollary C.1.2.4 appears in the definition of defect of semismallness (4.24).

**Corollary C.1.2.4.** Let \( f : X \to Y \) be a proper morphism of irreducible complex analytic spaces. If the (Euclidean) topology of \( X \) is second-countable, then \( \dim X \times_Y X \geq 2 \dim X - \dim f(X) \).

**Proof.** The image \( Z := f(X) \) is an analytic subset of \( Y \). Endow \( Z \) with the reduced structure of complex analytic space. Then \( Z \) is also irreducible and the morphism \( f : X \to Z \) is surjective. The natural morphism \( X \times_Z X \to X \times_Y X \) is an isomorphism. Then we conclude by Proposition C.1.2.3. \( \square \)

### C.2 Connection on line bundles

The purpose of Section C.2 is to show Lemma C.2.0.4. For one thing, it is closely related to Corollary 4.4.2.2. For another, it implies the possibility to
extend the Donaldson-Uhlenbeck-Yau theorem and nonabelian Hodge theory to manifolds more general than Kähler ones (Remarks C.2.0.6 and C.2.0.7). For work towards this direction, see [BD23], which extends nonabelian Hodge theory to Fujiki class C manifolds.

We begin the proof with a variation of the classical maximum principle.

**Proposition C.2.0.1.** Let \( U \subset \mathbb{R}^n \) be a nonempty connected open subset, \( f : U \rightarrow \mathbb{C} \) be a harmonic function. If \( |f| \) attains its maximum in \( U \), then \( f \) is constant.

**Lemma C.2.0.2** concerns the uniqueness of solution to \( \bar{\partial} \partial \)-equation.

**Lemma C.2.0.2.** Let \( f : X^n \rightarrow \mathbb{C} \) be a smooth function on a compact connected complex manifold \( X \) with \( \bar{\partial} \partial f = 0 \), then \( f \) is constant.

**Proof.** Since \( X \) is compact, the subset \( A = \{ x \in X : |f(x)| = \max_{t \in X} |f(t)| \} \) is nonempty closed in \( X \). For any \( p \in A \), there exists a local holomorphic coordinate \((U; z_1, \ldots, z_n)\), where \( U \) is a connected open neighborhood of \( p \) in \( X \). With this chart, we identify \( U \) as an open subset of \( \mathbb{C}^n \). Since \( \bar{\partial} \partial f = 0 \), we have \( \frac{\partial^2 f}{\partial z_j \partial \bar{z}_l} = 0 \) for all \( 1 \leq j, l \leq n \). In particular, \( \sum_{j=1}^n \frac{\partial^2 f}{\partial z_j \partial \bar{z}_j} = 0 \), or equivalently, \( f \) is a harmonic function on \( U \). By Proposition C.2.0.1, \( f \) is constant on \( U \) and so \( U \subset A \). Therefore, \( A \) is open in \( X \). By connectedness of \( X \), \( A = X \). So for any \( p \in X \), \( f \) is locally constant near \( p \). By connectedness of \( X \) again, \( f \) is constant. \( \square \)

Let \( X \) be a regular manifold for the rest of Section C.2.

We need a comparison between the Atiyah class ([Huy05, Def. 4.2.18]) and the first Chern class. For Kähler manifolds, it is [Ati57a, Prop. 12].

**Lemma C.2.0.3.** Let \( X \) be a regular manifold. Let \( L \rightarrow X \) be a holomorphic line bundle. Let \( A(L) \in H^1(X, \Omega^1_X) \) be the Atiyah class of \( L \). Then

\[
\frac{i}{2\pi} A(L) = c^\text{R}_1(L)
\]

in \( H^2(X, \mathbb{R}) \). In particular, \( L \) admits a holomorphic connection if and only if \( L \in \text{Pic}^\text{c}(X) \).

**Proof.** By Corollary 4.3.1.4, we have a commutative diagram

\[
\begin{array}{ccc}
Z^{1,1}(X) & \longrightarrow & Z^2(X) \\
\phi \downarrow & & \psi \downarrow \\
H^1(X, \Omega^1_X) & \longrightarrow & H^2_{dR}(X; \mathbb{C}),
\end{array}
\]

where \( \phi \) is taking Dolbeault cohomology class and \( \psi \) is taking de Rham cohomology class. Take a hermitian metric \( h \) on \( L \). Let \( R \) be the corresponding Chern curvature form. By [Huy05, Corollary 4.4.5], \( R \in Z^{1,1}(X) \). Then by [Huy05, Proposition 4.3.10], \( A(L) = \phi(R) \) and \( c^\text{R}_1(L) = \frac{i}{2\pi} \psi(R) \). The equality follows. The second part follows from [Huy05, Proposition 4.2.19]. \( \square \)
**Lemma C.2.0.4.** Let $X$ be a regular manifold, $L \in \text{Pic}^{\tau}(X)$, then:

1. $L$ admits a unique (up to a positive scalar) hermitian metric whose Chern connection is flat;
2. Every holomorphic connection on $L$ is flat.

**Proof.**

1. We begin with the existence. By Corollary 4.4.2.2, there is a unitary local system $\mathcal{L} \in \text{Loc}^{u,1}(X)$ on $X$ with $\mathcal{L} \otimes \mathcal{O}_X = L$. Applying Theorem 4.2.3.1 the existence of such metric follows.

Now for uniqueness. Let $h, h'$ be two hermitian metrics whose respective Chern connections $\nabla, \nabla'$ are flat holomorphic connections. By Theorem 4.2.3.1, $\ker(\nabla), \ker(\nabla') \in \text{Loc}^{u,1}(X)$ have the same induced line bundle. By Corollary 4.4.2.2, $\ker(\nabla) = \ker(\nabla')$ in $\text{Loc}^{u,1}(X)$. The hermitian metrics $h, h'$ restrict to two monodromy invariant hermitian forms on the common local system $\ker(\nabla)$. Moreover, by Theorem 4.2.3.1 one can recover the hermitian metric on the line bundle $L$ from the restricted hermitian form on the local system. Since this local system is of rank 1, at one stalk these two hermitian forms differ by a scalar. Globally they differ by this scalar as they are monodromy invariant. Then the metrics $h, h'$ also differ by a scalar.

2. By Lemma C.2.0.3 and [Huy05, Prop. 4.2.19], $L$ admits a holomorphic connection. We show that the curvature forms (which are global holomorphic 2 forms) of different holomorphic connections on $L$ are the same. In fact, for two such connections $D, D'$ on $L$, by [Huy05, p.179], $D' - D \in H^0(X, \Omega^2_X)$. This form is $d$-closed by [Uen06, Corollary 9.5, p.101]. By [Huy05, Lemma 4.3.4], the curvature of $D'$ equals that of $D$.

We adopt the argument in [BK09, Footnote (6), p.388]. By Cartan-Serre theorem [Car53, Théorème], the complex vector space $H^0(X, \Omega^2_X)$ is finite dimensional. Taking the curvature form of one (hence every) holomorphic connection on elements of $\text{Pic}^0(X)$, we get a holomorphic map $\text{Pic}^0(X) \to H^0(X, \Omega^2_X)$. As the complex torus $\text{Pic}^0(X)$ is compact connected, this map is constant. The canonical connection on the trivial line bundle $\mathcal{O}_X (\in \text{Pic}^0(X))$ is flat, so this map is constantly zero. In other words, for every $K \in \text{Pic}^0(X)$, any holomorphic connection on $K$ is flat.

As $L \in \text{Pic}^\tau(X)$, there is an integer $n \geq 1$ such that $L^\otimes n \in \text{Pic}^0(X)$. Take a holomorphic connection on $L$ of curvature form $R$, then it induces a holomorphic connection on $L^\otimes n$ of curvature form $nR$. As $nR = 0$, it holds that $R = 0$. The flatness follows from the first paragraph and the existence in Point 1.

**Remark C.2.0.5.** Here is a second proof of Lemma C.2.0.4 1. Take a hermitian metric $h$ on $L$. Locally its Chern curvature is given by $\nabla = d + h^{-1} \partial h$. More precisely, let $s$ be a local holomorphic frame for $L$, and by abuse of notation let $h$ be the local (smooth positive) function $h(s, s)$. Then $\nabla(s) = (h^{-1} \partial h) \otimes s$ and
the Chern curvature form $R = \bar{\partial}(h^{-1}\partial h)$ is a $d$-closed smooth $(1,1)$-form whose de Rham class is 0. Moreover $iR$ is a real form. (This is part of Chern-Weil theory, see [Huy05, Proposition 4.3.8 (iii); 4.3.10 and p.196].) Therefore, by Fact 4.3.1.2, there is a smooth function $f : X \to \mathbb{R}$ with

$$ R + \bar{\partial}\partial f = 0. \quad (C.1) $$

Define a new hermitian metric $h'$ by

$$ h'(s, s) = e^f h(s, s). \quad (C.2) $$

Then the new Chern connection is given by $\nabla'(s) = \nabla(s) + (\partial f) \otimes s$. The new curvature form $R' = R + \bar{\partial}\partial f = 0$, i.e., the new Chern connection is flat and compatible with the holomorphic structure, hence a holomorphic connection.

So far we have established the existence of such metric. As for uniqueness, any hermitian metric $h'$ with flat Chern connection is in the form of (C.2) where $f$ is a solution to (C.1). Lemma C.2.0.2 shows that such a solution $f$ is unique up to addition by constant. So such metric $h'$ is unique up to a positive scalar.

**Remark C.2.0.6.** When $X$ is a compact Kähler manifold, Lemma C.2.0.4 is a consequence of known results. In fact [Kob14, Proposition 5.7.7 (a)] shows a holomorphic line bundle is slope stable. By Donaldson-Uhlenbeck-Yau theorem [UY86, Corollary 8.1, p.292], there is $L \in \text{Loc}^{n-1}(X)$ such that $L = L^{\otimes \mathcal{O}_X}$, and $L$ induces such a metric via Theorem 4.2.3.1. For any such hermitian metric, its Chern connection is a Hermitian-Yang-Mills connection. The uniqueness of such metric is mentioned in [Bea92, (3.2) c)] and follows from [UY86, Theorem, p.262] and [Che22, Corollary 2.18].

**Remark C.2.0.7.** Lemma C.2.0.4 can be viewed as a step toward nonabelian Hodge theory on regular manifolds. In fact, a semisimple local system on a compact Kähler manifold is unitary if and only if the associated Higgs bundle $(E, \theta)$ has $\theta = 0$ ([Sim92, Example p.21]). The metric given by Lemma C.2.0.4 is exactly the harmonic metric provided by Corlette Theorem [GRR15, Theorem 1, p.151].

### C.3 Jacobi inversion theorem

In this section, we give a refinement of Proposition 4.4.1.23.

**Lemma C.3.0.1.** For a pointed regular manifold $(X, x_0)$, for every $n \geq h^{1,0}(X)$, the holomorphic map $f_n : X^n \to \text{Alb}(X)$ defined by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n \alpha_{x_0}(x_i)$ is surjective.

When $X$ is a compact Riemann surface, then Lemma C.3.0.1 reduces to (part of) Jacobi inversion theorem in [GH78, p.235].

Two proofs are provided. They are inspired by [Voi02, Lemma 12.11] and [BL04, Proposition 11.11.8] respectively, but with an extra attention to the feasible range of $n$. The first proof is shorter, while the second proof provides a stronger result, Lemma C.3.2.1.
C.3. JACOBI INVERSION THEOREM

C.3.1 First proof

Lemma C.3.1.1. Let $X$ be a compact complex manifold. Then there there is subset $S \subset X$ with $\#S \leq h^{1,0}(X)$ such that, for any $\eta \in H^0(X, \Omega^1_X)$ with $\eta(x) = 0$ in the (holomorphic) cotangent space $(T^*_X)^\vee$, we have $\eta = 0$.

Proof. For every $x \in X$, let $V_x$ be the subspace $\{ \eta \in H^0(X, \Omega^1_X) : \eta(x) = 0 \}$ of $H^0(X, \Omega^1_X)$. Then $\cap_{x \in X} V_x = \{0\}$. Hence, there is a subset $S \subset X$ with $\#S \leq h^{1,0}(X)$ and $\cap_{x \in S} V_x = \{0\}$.

Here is the first proof.

First proof of Lemma C.3.0.1. Consider the cotangent map $(d_pf)^* : (T^h_{f_p}(\text{Alb}(X)))^\vee \to (T^h_p X^n)^\vee$ at $p = (p_1, \ldots, p_n) \in X^n$. Since the cotangent bundle $\Omega^1_{\text{Alb}(X)}$ is trivial, this map is identified with the composition

$$H^0(\text{Alb}(X), \Omega^1_{\text{Alb}(X)}) \to (T^h_{f_p}(\text{Alb}(X)))^\vee \to \prod_{i=1}^n (T^h_{p_i} X)^\vee.$$  

By Proposition 4.4.1.2 4, it is further identified with the natural map

$$H^0(X, \Omega^1_X) \to \prod_{i=1}^n (T^h_{p_i} X)^\vee. \quad (C.3)$$

By Lemma C.3.1.1, there exist $n_0 \leq h^{1,0}(X)$ and $x = (x_1, \ldots, x_{n_0}) \in X^{n_0}$ such that for any $\eta \in H^0(X, \Omega^1_X)$ with $\eta(x_i) = 0$ for all $i$, we have $\eta = 0$. Then for every $n \geq n_0$, the map (C.3) is injective when $p = (x, x_0, \ldots, x_0)$. Or equivalently, $f_n$ is a submersion of smooth manifolds near $p$. From local normal form theorem, the image $f_n(X^n)$ contains a nonempty open subset of $\text{Alb}(X)$. By Remmert theorem [Whi72, Theorem 4A, p.150], $f_n(X^n)$ is an analytic set of $\text{Alb}(X)$. By [GR84, Theorem, p.168], $f_n(X^n) = \text{Alb}(X)$, i.e., $f_n$ is surjective.

C.3.2 Second proof

To certain extent, Lemma C.3.2.1 shows that a generating subset of a complex torus generates the complex torus “uniformly”.

Lemma C.3.2.1. Let $A$ be a $g$-dimensional commutative complex Lie group. Let $M$ be a compact irreducible analytic subset of $A$ containing $0$. If the complex Lie subgroup of $A$ generated by $M$ is $A$, then for every integer $n \geq g$, the map $f_n : M^n \to A$ defined by $(x_1, \ldots, x_n) \mapsto \sum_{i=1}^n x_i$ is surjective. In particular, $A$ is a complex torus.

Proof. Since $M$ is connected, the identity component of $A$ contains $M$. Therefore, $A$ is connected.

The statement is true when $g = 0$. So we assume $g > 0$, then $M \neq \{0\}$ and hence $\dim M \geq 1$. For every $n \geq 1$, let $A_n = f_n(M^n)$, which is an analytic
subset of $A$ by Remmert theorem [Whi72, Theorem 4A, p.150]. Since $f_1 : M \to A$ is the inclusion, we find $A_1 = M \ni 0$. For every $x \in M^n$, $f_{n+1}(x,0) = f_n(x)$, so $A_n \subset A_{n+1}$, hence an increasing sequence of analytic subsets of $A$:

$$A_1 \subset A_2 \subset \ldots$$

Consider the integer sequence of analytic dimensions $\{\dim_0 A_n\}_{n \geq 1}$. By [GR84, p.96], this sequence is non-decreasing and bounded above by $\dim_0 A = g$. Therefore, there is $n_0 \leq g$ such that $\dim_0 A_{n_0} = \dim_0 A_{n_0+1}$.

By assumption, $M^n$ is an irreducible complex analytic space. By [CD94, (14.14), p.89], the complex analytic space $A_n$ is irreducible and pure dimensional for every $n \geq 1$ and $A_1 = A_{n_0+1}$.

We claim that for every $m > n_0$, $A_{n_0} = A_m$.

We prove the claim by induction on $m$. It holds when $m = n_0 + 1$. If it is true for $m - 1$ with $m \geq n_0 + 2$, then for every $(x_1, \ldots, x_m) \in M^m$, $\sum_{i=1}^{m-1} x_i \in A_{m-1} = A_{n_0}$, so there is $(p_1, \ldots, p_{n_0}) \in M^{n_0}$ with $\sum_{j=1}^{m-1} p_j = \sum_{i=1}^{m-1} x_i$. Then

$$\sum_{i=1}^{m} x_i = x_m + \sum_{j=1}^{n_0} p_j \in A_{n_0+1} = A_{n_0}.$$ 

Therefore, $A_m = A_{n_0}$. The induction is completed.

For every $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_m) \in M^n$, we have $f_n(x) + f_n(y) = f_{2n}(x,y)$, so $A_n + A_n \subset A_{2n}$. In particular, $A_{n_0} + A_{n_0} \subset A_{2n_0} = A_{n_0}$. This shows $A_{n_0}$ is closed under addition.

We are going to show that $-A_{n_0} = A_{n_0}$ and then $A_{n_0}$ would be a subgroup of $A$.

As $A$ is commutative connected, by [AK01, Proposition 1.1.2], its universal covering is in the form of $\pi : \mathbb{C}^g \to A$ and the lattice $\ker(\pi)$ is identified with the fundamental group $\pi_1(A,0)$. As $\pi$ is locally biholomorphic and every $A_n$ is irreducible, the preimage $\pi^{-1}(A_n)$ is an analytic subset of $\mathbb{C}^g$, every irreducible component of whom is of dimension $\dim A_n$. Any two different irreducible components are disjoint and differ by a translation by an element of $\ker(\pi)$.

Let $V_n$ be the unique irreducible component of $\pi^{-1}(A_n)$ containing 0, then $\pi(V_n) = A_n$. Fix an integer $k \geq 1$ and let $[k] : A \to A$ be the multiplication by $k$. As $A_{n_0}$ is closed under addition, we get $[k]A_{n_0} \subset A_{n_0}$. As $\pi$ is a group morphism, we have $k \cdot \pi^{-1}(A_{n_0}) \subset \pi^{-1}(A_{n_0})$. As $k : \mathbb{C}^g \to \mathbb{C}^g$ is biholomorphic, $kV_{n_0}$ is an irreducible analytic subset of $\mathbb{C}^g$ isomorphic to $V_{n_0}$. As $0 \in kV_n$, we have $kV_{n_0} \subset V_{n_0}$. As $\dim kV_{n_0} = \dim V_{n_0}$, by [CD94, (14.14), p.89] again, we get $kV_{n_0} = V_{n_0}$, i.e., the morphism $k : V_{n_0} \to V_{n_0}$ is biholomorphic.

For every $x \neq 0 \in A_{n_0}$, we check that $-x \in A_{n_0}$. In fact, take $v \in V_{n_0} \cap \pi^{-1}(x)$. Then $v \neq 0$. By last paragraph, $v/k \in V_{n_0}$ for every $k \geq 1$. Let $l$ be the complex line in $\mathbb{C}^g$ spanned by $v$. By the identity theorem for holomorphic functions on $l$, the smallest analytic subset of $l$ containing $\{v/k\}_{k \geq 1}$ is $l$. Now that $V_{n_0} \cap l$ is an analytic subset of $l$ containing $\{v/k\}_{k \geq 1}$, we get $l = V_{n_0} \cap l \subset V_{n_0}$. In particular, $-v \in V_{n_0}$ and then $-x \in A_{n_0}$ as desired.
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So far we have shown that $A_{n_0}$ is a subgroup of $A$ that is a complex analytic subset. By Corollary E.2.0.5, $A_{n_0}$ is an embedded complex Lie subgroup of $A$. By assumption, $A_{n_0} = A$. From the claim we get the surjectivity of $f_n$ for every $n \geq n_0$. In particular, $A$ is compact, hence a complex torus.

Example C.3.2.2. In Lemma C.3.2.1, we cannot remove the condition that $0 \in M$. For example, consider $A = \mathbb{C}^*$ and $M = \{2\}$. The irreducibility of $M$ is also necessary. For instance, take $A$ to be the elliptic curve $\mathbb{C}/\mathbb{Z}[i]$, $M = \{0, x\}$, where $x \in A \setminus A_{\text{tor}}$. Then $f_n$ is not surjective for all integers $n \geq 1$.

Second proof of Lemma C.3.0.1. Let $M = a_{x_0}(X)$, which is an irreducible analytic subset of $\text{Alb}(X)$ by Remmert theorem [Whi72, Theorem 4A, p.150] and [CD94, (14.14), p.89]. In addition, $0 \in M$. The proof is completed by citing Proposition 4.4.1.23 and Lemma C.3.2.1.

\[\square\]
Appendix D

\textbf{D-modules}

\section*{D.1 Unbounded Bernstein’s equivalence}

In Section D.1, let \(X\) be a smooth algebraic variety over an algebraically closed field \(k\) of characteristic 0, unless otherwise specified.

\textbf{Fact D.1.0.1} (J. Bernstein), [B+87, VI, Thm. 2.10] The natural functor

\[ \iota': D^b(\text{Mod}_{\text{qc}}(D_X)) \to D^b_{\text{qc}}(D_X) \]

is an equivalence.

We remark that the first sentence of the proof in [B+87] needs (implicitly) [Mur07, Remark 64] and Fact D.1.0.2. Fact D.1.0.2 can be proved as [B+87, I, Prop. 12.8; VI, Prop. 1.14].

\textbf{Fact D.1.0.2.} Let \(B\) be an weak Serre subcategory of an abelian category \(A\). Then the full class \(\text{Ob}(B)\) of objects in \(B\) is a generating class of \(D^b_{\text{qc}}(A)\) (defined in [Sta23, Tag 06UJ]) in the sense of [B+87, I, Def. 12.4].

Following the strategy pointed out in [gh], we prove an unbounded generalization of Fact D.1.0.1. We do not claim originality here.

\textbf{Theorem D.1.0.3}. The natural functor

\[ \iota': D(\text{Mod}_{\text{qc}}(D_X)) \to D_{\text{qc}}(D_X) \]

induced by the inclusion \(\text{Mod}_{\text{qc}}(D_X) \to \text{Mod}(D_X)\) is an equivalence of categories.

We need a series of lemmas for the proof of Theorem D.1.0.3.

\textbf{Lemma D.1.0.4}. Every object of \(\text{Mod}_{\text{qc}}(D_X)\) is the inductive limit of its \(D_X\)-coherent submodules.
Proof. Let $F$ be such an object. Then the family of $D_X$-coherent submodules of $F$ is directed. In fact, if $G_1, G_2$ are $D_X$-coherent submodules of $F$, then both are finite type over $D_X$. Their sum $G_1 + G_2$ is of finite type over $D_X$. As $\text{Qch}(O_X)$ is an abelian subcategory of $\text{Mod}(O_X)$, the image $G_1 + G_2$ of the natural morphism $G_1 + G_2 \to F$ is $O_X$-quasi-coherent. By [HT07, Prop. 1.4.9 (ii)], the $D_X$-submodule $G_1 + G_2$ of $F$ is coherent.

We prove that $F$ is the union of its coherent $D_X$-submodules. (It is stated as [HT07, Cor. 1.4.17 (iii)], whose proof is omitted.) Let $U \subset X$ be an affine open, $s \in \Gamma(U, F)$ be a section, and $G \subset F|_U$ be the $D_U$-submodule generated by $s$. Then $G$ is $D_U$-coherent by [HT07, Prop. 1.4.3; 1.4.4 and 1.4.13]. By [Meb89, Prop. 2.5.7], there is a $D_X$-coherent submodule $G' \subset F$ with $G'|_U = G$. Since $X$ has a basis for the Zariski topology consisting of affine opens, we conclude that every local section of $F$ is locally contained in a $D_X$-coherent submodule. This finishes the proof. \hfill \ensuremath{\Box}

For an open immersion $j : U \to X$, we have a natural morphism of ringed spaces $j : (U, D_U) \to (X, D_X)$. The functor $j_* : D(D_U) \to D(D_X)$ is the right derived functor of the corresponding (left exact) direct image $j_* : \text{Mod}(D_U) \to \text{Mod}(D_X)$ ([B'87, VI, 5.2]). By [Ber83, 2, p.12] and [Sta23, Tag 0096], the inverse image $j^* : \text{Mod}(D_X) \to \text{Mod}(D_U)$ is left adjoint to $j_*$. Lemma D.1.0.5 2 helps to construct a quasi-inverse to (D.1).

Lemma D.1.0.5.

1. The category $\text{Mod}_{qc}(D_X)$ is locally noetherian.

2. The inclusion functor $i' : \text{Mod}_{qc}(D_X) \to \text{Mod}(D_X)$ admits a right adjoint

$Q' = Q'_X : \text{Mod}(D_X) \to \text{Mod}_{qc}(D_X)$. And the unit natural transform $\eta' : \text{Id}_{\text{Mod}_{qc}(D_X)} \to Q'i'$ is an isomorphism.

Remark D.1.0.6. For an affine (possibly singular) variety $X$, the abelian category $\text{Mod}_{qc}(D_X)$ is still Grothendieck, see [Yan22, p.6] and [GR14, 4.7.1; 5.5].

Proof. Note that $\text{Mod}_{qc}(D_X)$ is an abelian subcategory of $\text{Mod}(D_X)$ closed under colimits, as $\text{Qch}(O_X)$ is so in $\text{Mod}(O_X)$ by [Sta23, Tag 01LA (4)].

1. When $X$ is affine, the abelian category $\text{Mod}_{qc}(D_X)$ is equivalent to $\text{Mod}(D_X(X))$ by [HT07, Prop. 1.4.4 (ii)]. As the ring $D_X(X)$ is left noetherian, the category $\text{Mod}(D_X(X))$ is locally noetherian by the last paragraph of [Gab62, p.402].

For a general $X$, we may assume that there exists an open covering $X = U \cup V$, such that the statement holds for $U$ and $V$. Arguing as in [Gab62, Prop. 2, p.441], we obtain that $\text{Mod}_{qc}(D_X)$ is the gluing of $\text{Mod}_{qc}(D_U)$ and $\text{Mod}_{qc}(D_V)$ along $\text{Mod}_{qc}(D_{U \cap V})$ in the sense of [Gab62, VI. 1]. Let $j : U \to X$ be the inclusion. Then

$\quad \quad \quad \quad j^* : \text{Mod}_{qc}(D_X) \to \text{Mod}_{qc}(D_U)$
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is exact and admits a right adjoint

\[ j_+ : \text{Mod}_{qc}(D_X) \to \text{Mod}_{qc}(D_U), \]

see [HT07, Prop. 1.5.29]. The (counit) natural transformation \( \epsilon : j^* j_+ \to \text{Id}_{\text{Mod}_{qc}(D_U)} \) is an isomorphism. From [Gab62, Prop. 5, p.374], the subcategory \( \ker(j^*) \) is localizing in \( \text{Mod}_{qc}(D_X) \) (in the sense of [Gab62, p372]) and \( j^* \) induces an equivalence

\[ \text{Mod}_{qc}(D_X)/\ker(j^*) \to \text{Mod}_{qc}(D_U). \]

A similar result holds for \( V \). Therefore, the gluing category \( \text{Mod}_{qc}(D_X) \) is locally noetherian by [Gab62, Lem. 2, p.442].

2. It follows from 1 and Lemma D.1.0.7.

\[ \square \]

Lemma D.1.0.7. Let \( F : A \to B \) be a functor of categories preserving all colimits, where \( A \) is a Grothendieck abelian category, then

1. \( F \) admits a right adjoint \( G : B \to A \).

2. Furthermore, if \( F \) is fully faithful, then the unit natural transformation \( \eta : \text{Id}_A \to GF \) is an isomorphism.

Proof. 1. Let \( \text{Sets} \) be the category of sets. For each object \( Y \in B \), consider the functor

\[ A^{\text{op}} \to \text{Sets}, X \mapsto \text{Hom}_B(F(X), Y). \]

It transforms colimits into limits, so representable by [Sta23, Tag 07D7]. From [ML13, Cor. 2, p.85], the functor \( F \) admits a right adjoint.

2. If follows from Yoneda lemma.

\[ \square \]

To reduce the problem to the study of \( O_X \)-modules, we consider the following square.

\[
\begin{array}{ccc}
\text{Mod}(D_X) & \xrightarrow{Q_X} & \text{Mod}_{qc}(D_X) \\
\downarrow_{\text{for } X} & & \downarrow_{\text{for } X} \\
\text{Mod}(O_X) & \xrightarrow{Q_X} & Q\text{ch}(O_X)
\end{array}
\]  \hspace{1cm} (D.2)

where \( Q_X = Q \) is the coherator\(^1\) of \( X \).

Lemma D.1.0.8. Suppose \( X \) is affine and let \( R = \Gamma(X, D_X) \). Then:

1. The functor \( \bullet := D_X \otimes_R \bullet : \text{Mod}(R) \to \text{Mod}(D_X) \) is left adjoint to the global section functor \( \Gamma(X, \cdot) : \text{Mod}(D_X) \to \text{Mod}(R) \);

\(^1\)the right adjoint to the inclusion \( Q\text{ch}(O_X) \to \text{Mod}(O_X) \), see [Sta23, Tag 077P (2)].
2. The square (D.2) is commutative.

Proof.

1. Let \((X, D_X) \to \{\ast\}, R)\) be the morphism of ringed spaces with \(\sigma : X \to \{\ast\}\) the unique map and with \(\sigma^\#\) given by \(\text{Id}_R\). Then \(\Gamma(X, \cdot) = \sigma_\ast\). By [Sta23, Tag 01BH], the functor \(\bullet = \sigma^*\). We conclude by [Sta23, Tag 0096].

2. From 1 and [HT07, Prop. 1.4.4 (ii)], the functor \(Q' : \text{Mod}(D_X) \to \text{Mod}_{qc}(D_X)\) is the composition of \(\Gamma(X, \cdot) : \text{Mod}(D_X) \to \text{Mod}(R)\) with \(\bullet\). The largest rectangle in the following diagram

\[
\begin{array}{ccccccc}
\text{Mod}(D_X) & \xrightarrow{\Gamma(X, \bullet)} & \text{Mod}(R) & \xrightarrow{D_X \otimes_R \bullet} & \text{Mod}_{qc}(D_X) & \xrightarrow{\Gamma(X, \bullet)} & \text{Mod}(R) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\text{Mod}(O_X) & \xrightarrow{\Gamma(X, \bullet)} & \text{Mod}(O_X(X)) & \xrightarrow{O_X \otimes_{O_X(X)} \bullet} & \text{Qch}(O_X) & \xrightarrow{\Gamma(X, \bullet)} & \text{Mod}(O_X(X)) \\
\end{array}
\]

is same as the small square on the left, hence commutative. Moreover, the two horizontal functors \(\Gamma(X, \bullet)\) on the right are equivalences, so \(Q'\) is compatible with \(Q\).

\[\square\]

Lemma D.1.0.9. Suppose \(X/k\) is a smooth algebraic variety, then:

1. The square (D.2) is commutative.

2. The square

\[
\begin{array}{c}
\text{D}(D_X) \xrightarrow{RQ'_X} \text{D}(\text{Mod}_{qc}(D_X)) \\
\downarrow_{\text{for } X} \quad \downarrow_{\text{for } X} \\
\text{D}(O_X) \xrightarrow{RQ_X} \text{D}(\text{Qch}(O_X)),
\end{array}
\]

is commutative, where \(RQ'_X\) (resp. \(RQ_X\)) is the derived functor of the left exact functor \(Q'_X\) (resp. \(Q_X\)).

Proof.

1. We deduce a formula for \(Q'_X\). Let \(\{U_\alpha\}\) be a finite cover of \(X\) by affine opens. Each \(U_\alpha \cap U_\beta\) is quasi-compact, so we can choose a finite cover \(\{U_{\alpha \beta \gamma}\}\) of \(U_\alpha \cap U_\beta\) by affine opens. We denote all the various open
immersions $U_{\alpha\beta\gamma} \to X$ and $U_\alpha \to X$ as $j$. For every $F \in \text{Mod}(D_X)$, the sheaf axiom gives an equalizer diagram in $\text{Mod}(D_X)$:

$$0 \to F \to \oplus \alpha j_*(F|_{U_\alpha}) \rightrightarrows \oplus (\alpha,\beta,\gamma) j_*(F|_{U_{\alpha\beta\gamma}}),$$

where the two right maps are induced by $U_{\alpha\beta\gamma} \to U_\alpha \cap U_\beta \to U_\alpha$ and $U_{\alpha\beta\gamma} \to U_\alpha \cap U_\beta \to U_\beta$. By Lemma D.1.0.10, we have another equalizer diagram in $\text{Mod}_{\text{qc}}(D_X)$:

$$0 \to Q'_XF \to \oplus \alpha j_*[Q'_{U_\alpha}(F|_{U_\alpha})] \rightrightarrows \oplus (\alpha,\beta,\gamma) j_*[Q'_{U_{\alpha\beta\gamma}}(F|_{U_{\alpha\beta\gamma}})]. \quad \text{(D.3)}$$

There is a natural transformation $\iota Q'_X \to \text{Id}_{\text{Mod}(D_X)}$ between functors on $\text{Mod}(D_X)$. Applying for $\text{Mod}(D_X) \to \text{Mod}(O_X)$, we get a natural transformation for $\circ \iota' \circ Q'_X \to$ for between functors on $\text{Mod}(D_X)$. It induces a natural transformation

$$\mu_X : \text{for}_X \circ Q'_X \to Q_X \circ \text{for}_X$$

between functors $\text{Mod}(D_X) \to \text{Qch}(O_X)$. If $X$ is affine, then $\mu_X$ is an isomorphism by Lemma D.1.0.8 2.

For a general $X$ and an object or a morphism $? \in \text{Mod}(D_X)$, by (D.3) and [TT07, (B.14.2)], there is a commutative diagram in $\text{Qch}(O_X)$:

$$
\begin{array}{cccc}
0 & \longrightarrow & \text{for}_X(Q'_X?) & \longrightarrow & \oplus \alpha j_*\text{for}_{U_\alpha}Q'_{U_\alpha}(?|_{U_\alpha}) & \longrightarrow & \oplus (\alpha,\beta,\gamma) j_*\text{for}_{U_{\alpha\beta\gamma}}Q'_{U_{\alpha\beta\gamma}}(?|_{U_{\alpha\beta\gamma}}) \\
& & \downarrow \mu_X & & \downarrow & & \downarrow \\
0 & \longrightarrow & Q_X\text{for}_X(?) & \longrightarrow & \oplus \alpha j_*Q_{U_\alpha}\text{for}_{U_\alpha}(?|_{U_\alpha}) & \longrightarrow & \oplus (\alpha,\beta,\gamma) j_*Q_{U_{\alpha\beta\gamma}}\text{for}_{U_{\alpha\beta\gamma}}(?|_{U_{\alpha\beta\gamma}}),
\end{array}
$$

where the two vertical arrows on the right are isomorphisms, so is the leftmost vertical arrow. Hence $\mu_X$ is still an isomorphism.

2. The morphism $(X,D_X) \to (X,O_X)$ of ringed spaces is flat, and the corresponding direct image $\text{Mod}(D_X) \to \text{Mod}(O_X)$ is the forgetful functor $\text{for}_X$. By [Sta23, Tag 08BJ], it preserves K-injective complexes. The conclusion follows from Point 1, Lemma D.1.0.11 and [Sta23, Tag 070K].

\[ \square \]

**Lemma D.1.0.10.** Let $j : U \to X$ be an open immersion. Then $Q'_X \circ j_* = j^* \circ Q'_U$ as functors $\text{Mod}(D_U) \to \text{Mod}_{\text{qc}}(D_X)$.

**Proof.** As $j^* : \text{Mod}(D_X) \to \text{Mod}(D_U)$ preserves $\text{Mod}_{\text{qc}}$, we have $t'_U j^* = j^* t'_X$ as functors $\text{Mod}_{\text{qc}}(D_X) \to \text{Mod}(D_U)$. The functor $j_* : \text{Mod}(D_U) \to \text{Mod}(D_X)$ regards the direct image $\text{Mod}(O_U) \to \text{Mod}(O_X)$, so it also preserves $\text{Mod}_{\text{qc}}$. As $Q'$ is right adjoint to $t'$ and $j_*$ is right adjoint to $j^*$, it follows that $Q'_X \circ j_* = j^* \circ Q'_U$. \[ \square \]
Lemma D.1.0.11. Let $F : A \to B$ and $G : B \to C$ be left exact functors between abelian categories. Assume $A, B$ are Grothendieck. If the natural morphism $GF(I) \to RG(F(I))$ is an isomorphism in $D(C)$ for every $K$-injective complex $I$ over $A$, then the canonical natural transformation $\epsilon : R(G \circ F) \to RG \circ RF$ is an isomorphism of functors from $D(A)$ to $D(C)$.

Proof. Let $A$ be a complex over $A$. By [Sta23, Tag 079P], there is a quasi-isomorphism $A \to I$ such that $I$ is a $K$-injective complex. By [Sta23, Tag 070K], the morphism $\epsilon_A$ is the composition of isomorphisms $R(G \circ F)(A) = GF(I) \to RG(F(I)) = RG(RF(A))$.

Proof of Theorem D.1.0.3. Note that $\text{Mod}(D_X)$ is a Grothendieck abelian category. By [Sta23, Tag 079P] and [Sta23, Tag 070K], the functor $Q' : \text{Mod}(D_X) \to \text{Mod}_{qc}(D_X)$ admits a right derived functor $RQ' : D(D_X) \to D(\text{Mod}_{qc}(D_X))$. By [Sta23, Tag 09T3], the functor $RQ'$ is a right adjoint to $\iota' : D(\text{Mod}_{qc}(D_X)) \to D(D_X)$.

Let $\Psi' : D_{qc}(D_X) \to D(\text{Mod}_{qc}(D_X))$ be the restriction of $RQ'$. There are two natural commutative squares

$$
\begin{array}{ccc}
D(\text{Mod}_{qc}(D_X)) & \xrightarrow{\Psi'} & D(\text{Mod}_{qc}(D_X)) \\
\downarrow \text{for} & & \downarrow \text{for} \\
D(Qch(O_X)) & \xrightarrow{\iota} & D_{qc}(O_X)
\end{array}
$$

and

$$
\begin{array}{ccc}
D_{qc}(D_X) & \xrightarrow{\Psi'} & D(\text{Mod}_{qc}(D_X)) \\
\downarrow \text{for} & & \downarrow \text{for} \\
D_{qc}(O_X) & \xrightarrow{\Psi} & D(Qch(O_X)),
\end{array}
$$

where $\iota$ is induced by the inclusion $Qch(O_X) \to \text{Mod}(O_X)$ and $\Psi$ is the restriction of the derived functor $RQ_X : D(O_X) \to D(Qch(O_X))$ constructed in [Sta23, Tag 08D6].

By Lemma D.1.0.9 2 and that $\Psi$ is right adjoint to $\iota$, the counit $\epsilon' : \iota'\Psi' \to \text{Id}_{D_{qc}(D_X)}$ (resp. unit $\eta' : \text{Id}_{D(\text{Mod}_{qc}(D_X))} \to \Psi\iota'$) is compatible with the counit $\epsilon : \iota\Psi \to \text{Id}_{D_{qc}(O_X)}$ (resp. unit $\eta : \text{Id}_{D(Qch(O_X))} \to \Psi\iota$). Note that a morphism in $D(D_X)$ is an isomorphism, as long as the underlying morphism in $D(O_X)$ is so, cf. the proof of Proposition 6.3.0.4. By [Sta23, Tag 09T4], the counit $\epsilon$ and the unit $\eta$ are isomorphisms, so are the counit $\epsilon'$ and the unit $\eta'$. In particular, the functor (D.1) is an equivalence with a quasi-inverse $\Psi'$. $\square$
Remark D.1.0.12. For a smooth algebraic variety $V/k$, let $\text{Qch}(O_V) \subset \text{Mod}(O_V)$ (resp. $\text{Mod}_{qc}(D_V) \subset \text{Mod}(D_V)$) be the full subcategory of $O_V$-quasi-coherent modules. Strictly speaking, [Rot96] and [Rot97] demonstrate that the functor

$$R(qc)S_1 : D(\text{Qch}(O_{B^1})) \to D(\text{Mod}_{qc}(D_A))$$

(D.4)

defined by

$$? \mapsto R\rho_A^\ast(P \otimes_{O_{A \times B}} p_B^\ast \pi^\ast ?)$$

is an equivalence. In comparison, Laumon's result [Lau96, Thm. 3.2.1] is stated for bounded derived categories $D^b_{qc}$ and needs the characteristic of $k$ to be 0.

We sketch how to get Fact 6.1.0.2 2 from Rothstein's original statement. For every algebraic variety $V/k$, by [Sta23, Tag 077P (1)], the abelian category $\text{Qch}(O_V)$ has enough injectives. Furthermore, from [Con00, Lem. 2.1.3], the inclusion $\text{Qch}(O_V) \to \text{Mod}(O_V)$ preserves injectives. Let $\text{Mod}(O_B)_{sp}$ be as in Example 6.2.1.5 (resp. $\text{Mod}(O_{A \times B})_{-1-cxn}$ be $\text{Mod}(O_{A \times B})_{-1-cxn}$), and $\text{Qch}(O_B)_{sp}$ (resp. $\text{Qch}(O_{A \times B})_{-1-cxn}$) be the full subcategory of quasi-coherent objects.

Then the exact functor $\pi_* : \text{Qch}(O_B) \to \text{Qch}(O_B)_{sp}$ is the restriction of $R\pi_* : D(O_B) \to D(\text{Mod}(O_B)_{sp})$. Using [Lip60, Prop. 3.9.2] and [Har66, I, Prop. 7.1 (iii)], one proves that the canonical square

$$
\begin{array}{ccc}
D(\text{Qch}(O_{B^1})) & \rightarrow & D_{qc}(O_{B^1}) \\
\downarrow^{\pi_*} & & \downarrow^{R\pi_*} \\
D(\text{Qch}(O_B)_{sp}) & \rightarrow & D_{qc}(\text{Mod}(O_B)_{sp})
\end{array}
$$

is commutative. Similarly, using [Kas04, Remark 3.2], one proves that the canonical square

$$
\begin{array}{ccc}
D(\text{Qch}(O_{A \times B})_{-1-cxn}) & \rightarrow & D_{qc}(\text{Mod}(O_{A \times B})_{-1-cxn}) \\
\downarrow^{p_A^\ast} & & \downarrow^{Rp_A^\ast} \\
D(\text{Mod}_{qc}(D_A)) & \rightarrow & D_{qc}(D_A)
\end{array}
$$

is commutative. Therefore, the following square is commutative

$$
\begin{array}{ccc}
D(\text{Qch}(O_{B^1})) & \xrightarrow{R(qc)S_1} & D(\text{Mod}_{qc}(D_A)) \\
\downarrow & & \downarrow \\
D_{qc}(O_{B^1}) & \xrightarrow{RS_1} & D_{qc}(D_A)
\end{array}
$$

(D.5)

By [Sta23, Tag 077P] and Theorem D.1.0.3, the two vertical functors in (D.5) are equivalences. As $R(qc)S_1$ is an equivalence, so is the bottom row.
D.2 When is an induced $D$-module holonomic?

**Proposition D.2.0.1.** Let $X$ be a complex manifold, $F$ be an $O_X$-module. Then the following conditions are equivalent:

1. the induced module $D_X \otimes_{O_X} F$ is holonomic;
2. $F$ is coherent with $\text{Supp}(F)$ discrete.

Lemma D.2.0.2 and Lemma D.2.0.3 are needed for the proof of Proposition D.2.0.1.

**Lemma D.2.0.2.** Let $A$ be a Gorenstein local ring\(^4\) of Krull dimension $n$, $M$ be a finite $A$-module, then the following conditions are equivalent:

1. $\text{Ext}^i(M, A) = 0$ for all $i \neq n$
2. the module $M$ has finite length.

**Proof.** Let $k$ be the residue field of $A$.

- We assume condition 1. To prove 2, we may assume that $M \neq 0$. By definition, $A[0]$ is a dualizing complex of $A$. By [Mat87, Thm. 18.1, p.141], one has $\text{RHom}_A(k, A[n]) = k[0]$, so $A[n]$ is the normalized dualizing complex of $A$ (in the sense of [Sta23, Tag 0A7M]). By [Sta23, Tag 0B5A], the module $M$ is Cohen-Macaulay and

\[
M = \text{Ext}_A^{n-d}(\text{Ext}_A^{n-d}(M, A), A),
\]

where $d$ is the depth of $M$. Thus, $\text{Ext}_A^{n-d}(M, A) \neq 0$ and $n - d = n$. Therefore, $\dim \text{Supp}(M) = d = 0$. By [Ati69, Exercise 19 v), p.46], $\dim A/\text{Ann}(M) = 0$, so $A/\text{Ann}(M)$ is an artinian ring. From [Eis13, Cor. 2.17], we deduce that $M$ has finite length.

- We assume condition 2. Induction on the length $l(M)$ of $M$. When $l(M) = 0$, we have $M = 0$ and Condition 1 holds. Now assume $l(M) > 0$ and the statement holds for all modules of length less than $l(M)$. There is a submodule $N$ of $M$ such that $M/N$ is a simple module and $l(N) < l(M)$. By [Sta23, Tag 00J2], the module $M/N$ is isomorphic to $k$. For every $i \neq n$, from the short exact sequence $0 \to N \to M \to M/N \to 0$, we get an exact sequence $\text{Ext}^i(M/N, A) \to \text{Ext}^i(M, A) \to \text{Ext}^i(N, A)$. By inductive hypothesis, $\text{Ext}^i(N, A) = 0$. By [Mat87, Thm. 18.1, p.141], $\text{Ext}^i(M/N, A) = 0$. Therefore, $\text{Ext}^i(M, A) = 0$.

\[\square\]

**Lemma D.2.0.3.** Let $X$ be a complex space, and $F$ be a coherent $O_X$-module. Then the length of the $O_{X,x}$-module $F_x$ is finite for every $x \in X$ if and only if the subspace $\text{Supp}(F) \subset X$ is discrete.

\(^4\)defined in [Sta23, Tag 0DW7 (1)]
D.2. WHEN IS AN INDUCED D-MODULE HOLonomic?

Proof. The “if” part follows from Lemma 5.5.2.8 1. We prove the “only if” part. By [GR84, p.76], \( \text{Supp}(F) \) is an analytic set in \( X \). Assume the contrary that \( \text{Supp}(F) \) is not discrete, then \( \dim \text{Supp}(F) > 0 \). Let \( C \) be an irreducible component of \( \text{Supp}(F) \) of maximal dimension and \( i : C \to X \) be the inclusion.

For every \( x \in C \), the stalk \( F_x \) is a natural \( O_{C,x} \)-module, so the length of the \( O_{C,x} \)-module \((i^*F)_x = F_x \) is finite by [Sta23, Tag 00IX].

Replacing \((X, F)\) by \((C, i^*F)\), we may assume that \( X \) is irreducible with \( \dim X > 0 \) and \( \text{Supp}(F) = X \). From [RS17, p.238], \( F \) is not a torsion sheaf. By [Ros68, Prop. 3.1 and p.69], there is \( x \in X \) such that \( F_x \) is a free \( O_{X,x} \)-module of positive rank. Thus, the ring \( O_{X,x} \) is a finite length module over itself, hence an artinian ring. Dimension formula in [GR84, p.96] and [CD94, (14.14), p.89] yield \( \dim X = \dim_x X = \dim O_{X,x} = 0 \), a contradiction. \( \square \)

Proof of Proposition D.2.0.1. Let \( M = D_X \otimes_{O_X} F \) and \( \hat{F} = R\text{Hom}_{O_X}(F, O_X) \). By [Sta23, Tag 08DJ], we get

\[
\text{Hom}_{O_X}(\omega_X, \hat{F}) = R\text{Hom}_{O_X}(\omega_X \otimes_{O_X} F, O_X). \quad \text{(D.6)}
\]

Provided that \( F \) is coherent, from [Bjö93, (ii) p.122] we have

\[
\Delta^{D_X} M = D_X \otimes_{O_X} \text{Hom}_{O_X}(\omega_X, \hat{F})[\dim X], \quad \text{(D.7)}
\]

Plugging (D.6) into (D.7), we get

\[
\Delta^{D_X} M = D_X \otimes_{O_X} R\text{Hom}_{O_X}(\omega_X \otimes_{O_X} F, O_X)[\dim X].
\]

For every \( i \in \mathbb{Z} \setminus \{0\} \), one has

\[
\mathcal{H}^i(\Delta^{D_X} M) = D_X \otimes_{O_X} \text{Ext}^{i+\dim X}_{O_X}(\omega_X \otimes_{O_X} F, O_X).
\]

Its stalk at \( x \in X \) is

\[
D_{X,x} \otimes_{O_{X,x}} \text{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x})
\]

by [Sta23, Tag 01CB] and [GH78, 1. p.700].

- Assume Condition 2. Then \( M \) is \( D_X \)-coherent, see [Bjö93, 1.5.1]. By Lemma D.2.0.3, the \( O_{X,x} \)-module \( F_x \) has finite length. As \( O_{X,x} \) is a noetherian regular local ring of Krull dimension \( \dim_x X \), by Lemma D.2.0.2, the group

\[
\text{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}) = 0
\]

for all \( x \in X \) and hence \( \mathcal{H}^i(\Delta^{D_X} M) = 0 \). From Fact 6.7.2.2, \( M \) is holonomic.

- Assume Condition 1. From [SS94, p.55], the \( O_X \)-module \( F \) is coherent. From Fact 6.7.2.2, for all \( i \in \mathbb{Z} \setminus \{0\} \) we have \( \mathcal{H}^i(\Delta^{D_X} M) = 0 \). As \( D_{X,x} \) is a nonzero free \( O_{X,x} \)-module, we get \( \text{Ext}^{i+\dim_x X}_{O_{X,x}}(F_x, O_{X,x}) = 0 \) for all \( x \in X \). By Lemma D.2.0.2, the \( O_{X,x} \)-module \( F_x \) has finite length for every \( x \in X \). Thus, Lemma D.2.0.3 proves that \( \text{Supp}(F) \) is discrete.
An algebraic version, Proposition D.2.0.4, can be proved by arguing as in Proposition D.2.0.1.

**Proposition D.2.0.4.** Let $X$ be a smooth algebraic variety over an algebraically closed field of characteristic 0, $F$ be an $O_X$-module. Then the following conditions are equivalent:

1. the induced module $D_X \otimes_{O_X} F$ is holonomic;

2. $F$ is coherent with $\text{Supp}(F)$ finite.
Appendix E

Group extensions of complex Lie groups

E.1 Introduction

In the history of cohomology theory of abelian varieties over positive characteristic fields, the study of group extension problem played an important role. For instance, Rosenlicht obtains Fact E.1.0.1 through considering vectorial extensions of abelian varieties. Let $k$ be an algebraically closed field and $A/k$ be an abelian variety with $\dim A = g$. The dual abelian variety of $A$ is denoted by $A^\vee$.

**Fact E.1.0.1.** [Ros58, Theorem 1 and 2] The dimension of the $k$-vector space $H^1(A, O_A)$ is $g$.

A notable byproduct of Rosenlicht’s work is the existence of the following object, the so-called universal vectorial extension.

**Fact E.1.0.2.** [Ros58, Prop. 11] There is a short exact sequence of commutative algebraic groups over $k$: $0 \to \mathbb{G}_a \to A^3 \to A \to 0$, where $A^3$ is the moduli space of line bundles equipped with an integrable connection on $A^\vee$.

In [Rot96, (1.17)] and [Lau96, Thm. 3.2.1], it is proved that the Fourier-Mukai transform $D^b(\text{Qch}(O_A)) \to D^b(\text{Qch}(O_{A^\vee}))$ lifts to an equivalence $D^b(\text{Qch}(O_A)) \to D^b(\text{Qch}(D_{A^\vee})))$, where for a smooth algebraic variety $M/k$, $\text{Qch}(O_M)$ (resp. $\text{Qch}(D_M)$) refers to the category of $O_M$ (resp. left $D_M$) modules that are $O_M$-quasi-coherent.

The cohomology theory of complex analytic analogue of abelian varieties, namely complex tori, is elementary. By contrast, as far as we know, the existence of universal vectorial extension in the analytic setting is not covered in the literature, though admittedly easier and should be known. The main results are summarized as follows.

\footnote{in the sense of [Ros58, Sec. 2, p.691]}
**Proposition E.1.0.3** (Proposition E.4.3.1). For two commutative complex Lie groups $A, B$, the commutative extensions of $A$ by $B$ are classified by the abelian group

$$\text{Ext}^1_{\mathbb{Z}}(\pi_0(A), \pi_0(B)) \oplus \text{Hom}_{\mathbb{A}b}(\pi_1(A_0), \pi_0(B_0)) \oplus \text{coker}(s).$$

Here $s$ is the restriction morphism $\text{Hom}_{\text{Vec}}(L(A), L(B_0)) \to \text{Hom}_{\mathbb{A}b}(\pi_1(A_0), B_0)$, $A_0$ (resp. $B_0$) signifies the identity component of $A$ (resp. $B$), the notation $\pi_1(*)$ refers to the fundamental group, and $A/A_0 = \pi_0(A)$ denotes the 0-th homotopy group of $A$ and similar for $B$.

**Theorem E.1.0.4.** Let $A$ be a complex torus of dimension $g$. Then:

- (Theorem E.5.2.4 (resp. Theorem E.5.3.2)) The dual torus $\text{Pic}^0(A)$ (resp. tangent space $T_0 A = H^1(A, O_A)$) naturally classifies the extensions of $A$ by the multiplicative group $\mathbb{C}^*$ (resp. additive group $\mathbb{C}$).

- (Proposition E.5.4.5 1, Proposition E.5.4.7) There is an extension

$$0 \to H^0(A^\vee, \Omega^1_{A^\vee}) \to (\mathbb{C}^*)^{2g} \to A \to 0$$

that is universal among all vectorial extensions of $A$.

We emphasize some differences between the analytic case and the algebraic case. For a complex torus $A$, let $\text{Div}(A)$ be the group of analytic divisors on $A$ modulo linear equivalence. Let $\text{Pic}(A)$ be the group of isomorphic classes of line bundles on $A$. The natural map $\text{Div}(A) \to \text{Pic}(A)$ is surjective if and only if $A$ is an abelian variety ([Deb05, Sec. 4.3, Cor. 4]). This is why the Picard group is used in Theorem E.5.2.4 while divisor group appears in its algebraic analogue ([Wei49, no. 2], [Ser12, Thm. 6]). Discrete groups like $\mathbb{Z}$ are not (finite type) algebraic groups, but there is no reason to exclude them as complex Lie groups. Plenty of important analytic morphisms are not algebraic, like the universal covering (exponential map) $\exp: \mathbb{C} \to \mathbb{C}^*$.

The organization is as follows. The main goal of this text is to classify extensions of complex Lie groups. Section E.2 contains preliminaries about complex Lie groups. In Section E.3 we define complex Lie group extensions and give several first results about the classification. Then we focus on commutative extensions in Section E.4. Commutative extensions of complex tori deserve extra attention, and they are discussed in Section E.5. Some extensions with complex-tori base are automatically commutative, as Section E.6 shows. Noncommutative extensions are treated superficially in Section E.7.

**Convention and notation**

A statement about Lie groups is understood to hold for both real and complex Lie groups. The topology underlying a Lie group is always assumed to be second countable.\(^2\)

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\(^2\)A partial reason for such restriction is that, in this case, Condition (2) of [Hoc51b, Definition 1.1] is implied by Condition (1), showed in p.542 loc.cit.
For every Lie group $G$, the identity component of $G$ is denoted by $G_0$. The Lie algebra of $G$ is written as $L(G)$. And $Z(G)$ denotes the center of $G$. The automorphism group of $G$ is denoted by $\text{Aut}(G)$. Let $\text{Inn} : G \to \text{Aut}(G)$ be the group morphism defined by taking conjugation $g \mapsto g \cdot g^{-1}$. Then the subgroup $\text{Inn}(G)$ of inner automorphisms is normal in $\text{Aut}(G)$. Let $\text{Out}(G) = \text{Aut}(G)/\text{Inn}(G)$ be the group of outer automorphisms. Let $G^{\text{op}}$ be the Lie group opposite to $G$. (If $G$ is complex, then so is $G^{\text{op}}$.) There is a natural identification of real/complex manifolds $G \to G^{\text{op}}$ denoted by $g \mapsto g^*$. If $G$ is connected, then the universal covering group of $G$ is denoted by $\tilde{G}$ and the fundamental group of $G$ with the identity $e_G$ as the base point is denoted by $\pi_1(G)$.

Complex Lie subgroups refer to embedded closed complex Lie subgroups. If $G$ is a complex Lie group and $S \subset G$ is a subset, by [HN11, Exercise 15.1.3 (b)] there is a smallest complex Lie subgroup of $G$ containing $S$, called the complex Lie subgroup generated by $S$.

Let $\text{Vec}$ (resp. $\text{Ab}$, resp. $\text{C}$, resp. $\text{Sets}$) be the category of finite dimensional complex vector spaces (resp. abelian groups, resp. commutative complex Lie groups, resp. sets). For a complex manifold $X$ and a commutative complex Lie group $B$, let $B_X$ be the abelian sheaf on $X$ of germs of holomorphic maps from $X$ to $B$.

E.2 Generalities on complex Lie groups

Two fundamental facts about complex Lie groups are recalled.

**Fact E.2.0.1** ([Bou72, Ch. III, §3, no.8, Prop. 28]). Let $f : G \to H$ be a morphism of complex Lie groups. Then:

1. $\ker(f)$ is a normal complex Lie subgroup of $G$ and $L(\ker(f)) = \ker(d_e f : L(G) \to L(H))$.

2. If $f(G)$ is closed in $H$, then $f(G)$ is a complex Lie subgroup of $H$, and $f$ induces a complex Lie group isomorphism $G/\ker(f) \to f(G)$. In particular, if $f$ is surjective, then $d_e f : L(G) \to L(H)$ is surjective. If $f$ bijective, then $f$ is an isomorphism.

**Remark E.2.0.2.** Fact E.2.0.1 2 fails if the topology of $G$ is not assumed to be second countable. For example, let $\tau$ (resp. $\tau'$) be the discrete topology (resp. the Euclidean topology) of $\mathbb{C}$, then $\text{Id} : (\mathbb{C}, \tau) \to (\mathbb{C}, \tau')$ is a bijective morphism but not open.

Right principal bundle is defined in [Bou07, 6.2.1]. Left principal bundle can be defined similarly.

**Fact E.2.0.3** ([HBS66, Thm. 3.4.3], [Bou72, Ch. III, §1, Propositions 10 and 11]). Suppose $G$ is a complex Lie group and $K$ is a normal complex Lie subgroup of $G$. Then the group $G/K$ has a unique structure of complex manifold, such that the quotient map $\pi : G \to G/K$ is a submersion.\(^3\) With this structure, $G/K$

\(^3\)in the sense of [Bou07, 5.9.1]
is a complex Lie group and \( p \) is a left principal \( K \)-bundle under the natural left group action \( K \times G \to G \) defined by \((k, g) \mapsto kg\). In particular, every surjective morphism of complex Lie groups is open.

We recall that principal bundles are classified by the first sheaf cohomology, in the following way. Let \( X \) (resp. \( B \)) be a complex manifold (resp. commutative complex Lie group). Let \( S \) be the set of isomorphism classes of principal \( B \)-bundles\(^4\) over \( X \). Define a map

\[
\Psi : S \to H^1(X, B_X)
\]

as follows. For every \([p : P \to X] \in S\), there exists an open cover \( \{U_i\}_{i \in I} \) of \( X \) and a family of local trivializations \( f_i : U_i \times B \to p^{-1}(U_i) \) for every \( i \in I \). For any indices \( i, j \in I \) and every \( x \in U_i \cap U_j \), there exists a unique element \( b_{ij}(x) \in B \) such that \( b_{ij}(x) \cdot f_i(y) = f_j(y) \) for all \( y \in p^{-1}(x) \). Hence a morphism \( b_{ij} : U_i \cap U_j \to B \) of complex manifolds. Moreover, for any indices \( i, j, k \in I \) and every \( x \in U_i \cap U_j \cap U_k \), they satisfy the 1-cocycle relation \( b_{ij}(x) + b_{jk}(x) + b_{ki}(x) = 0 \). Thus, the family \( \{b_{ij}\}_{i, j \in I} \) defines an element \( \Psi(p) \) of \( H^1(X, B_X) \).

As per [HBS66, 3.2 b), p.41], the map \( \Psi \) is bijective. The structure of abelian group on \( H^1(X, B_X) \) is translated to \( S \) via \( \Psi \). The zero element of \( S \) is the class of the trivial principal \( B \)-bundle. For every pair \([p_1 : P_1 \to X] \) and \([p_2 : P_2 \to X] \) in \( S \), by taking a family of trivialization for each \( p_i \), we can define a morphism \( \phi : P_1 \times_X P_2 \to P_1 + P_2 \) of principal \( B \)-bundles on \( X \) such that for every \( b, b' \in B \), \( u \in P_1 \), \( v \in P_2 \) with \( p_1(u) = p_2(v) \), one has

\[
\phi(b \cdot u, b' \cdot v) = (b + b') \cdot \phi(u, v).
\]

In particular, \( \phi \) is surjective. Restricted to the fiber at some \( x \in X \), \( \phi \) is induced by the group law of \( B \) and the chosen trivializations.

We need a complex version of Cartan’s subgroup theorem. Notice that a real analytic closed subgroup of a complex Lie group may not be a complex analytic subset.

**Lemma E.2.0.4** ([Bjö93, p.513]). Let \( X \) be a complex manifold, \( Y \subset X \) be a complex analytic subset. If \( p \in Y \) is a smooth point of \( Y \), then near \( p \), the subset \( Y \) is an embedded complex submanifold of \( X \).

**Proof.** As the problem is local, we may assume that \( X \) is an open subset \( C^n \), there exist \( f_1, \ldots, f_m \in O_X(X) \) with \( O_{X,p}/(f_1, \ldots, f_m) = O_{Y,p} \) and \( Y = Z(f_1, \ldots, f_m) \). Let \( r = \text{rank}_p(f_1, \ldots, f_m) \). By reordering subscripts, one may assume

\[
\det(\frac{\partial f_i}{\partial z_j})_{1 \leq i, j \leq r} \neq 0.
\]

Then \((f_1, \ldots, f_r) : X \to C^r \) is a holomorphic submersion near \( p \). Therefore, near \( p \), the subset \( Z(f_1, \ldots, f_r) \) is an embedded complex submanifold of \( X \).

---

\(^4\)Here \( B \) is commutative, so it is unnecessary to specify the principal bundle to be left or right.
of dimension $n - r$. By the Jacobian criterion (see, e.g., [GR84, p.114]), one has $\text{emb}_p Y = n - r$. By the criterion of smoothness ([GR84, p.116]), one has $\text{dim}_p Y = n - r$. Now that $Y \subset Z(f_1, \ldots, f_r)$, near $p$ the subset $Y$ is an irreducible component of $Z(f_1, \ldots, f_r)$, hence also an embedded complex submanifold of $X$.

Corollary E.2.0.5 contains [Lee01, Prop. 1.23] as a special case.

**Corollary E.2.0.5** (Complex Cartan subgroup theorem). Let $G$ be a complex Lie group, and let $H$ be a subgroup that is a complex analytic subset of $G$. Then $H$ is a complex Lie subgroup of $G$.

**Proof.** Endow $H$ with the induced structure of reduced complex analytic space. By [GR84, p.117], the complex analytic space $H$ has a smooth point $p$. For every $q \in H$, the left multiplication by $q^{-1}$ gives a biholomorphic map $G \to G$, which sends $H$ to $H$ and maps $p$ to $q$. Therefore, $q$ is also a smooth point of $H$. By Lemma E.2.0.4, $H$ is a complex submanifold of $G$ near $q$ for all $q \in H$. Thus, $H$ is a complex submanifold of $G$ and hence a complex Lie subgroup. □

In Lemma E.2.0.6, if $G$ is furthermore connected, then the result of is contained in [Bou72, Ch.III, Sec. 6, no. 4, Cor. 4].

**Lemma E.2.0.6.** Let $G$ be a complex Lie group. Then the center $Z(G)$ is a complex Lie subgroup of $G$.

**Proof.** The holomorphic map $G \times G \to G$ defined by $(x,y) \mapsto yxy^{-1}$ is a group action of $G$ on itself. By [Bou72, Ch. III, Sec. 1, no. 7, Prop. 14], for every $x \in G$, the stabilizer $C_G(x)$ of $x \in G$ is a complex Lie subgroup of $G$. Therefore, so is $Z(G) = \cap_{x \in G} C_G(x)$ by [HN11, Exercise 15.1.3 (a)]. □

A complex Lie group isomorphic to a complex Lie subgroup of $\text{GL}_n(C)$ for some integer $n \geq 1$ is called linear. Proposition E.2.0.7, due to Matsushima and Morimoto, is a characterization of commutative linear complex Lie groups.

**Proposition E.2.0.7.** Let $B$ be a connected commutative complex Lie group. Then the following conditions are equivalent:

1. $B$ is isomorphic to $\mathbb{C}^m \times (\mathbb{C}^*)^n$ for some integers $m,n \geq 0$;

2. the complex Lie group $B$ is linear;

3. $B$ is a Stein group (i.e., the underlying complex manifold is a Stein manifold).

In that case, the pair $(m,n)$ is unique.

**Proof.** See [HN11, Exercise 15.3.1] for the fact that 1 implies 2. Since $\text{GL}_n(C)$ is a Stein manifold, 2 implies 3. As per [MM60, Proposition 4], 3 implies 1. The uniqueness is contained in the Remmert-Morimoto decomposition (see, e.g., [AK01, Thm. 1.1.3]). □
Remark E.2.0.8. The commutativity of $B$ in Proposition E.2.0.7 is important. In fact, there is a connected Stein group that is not linear ([Ari19, Sec. 1]). This differs from the algebraic case where every algebraic group that is an affine variety is linear ([Mil17a, Cor. 4.10]).

In some sense, Definition E.2.0.9 is an antipode to Stein groups.

Definition E.2.0.9. A connected complex Lie group on which every holomorphic function is constant is called a toroidal group.\(^5\)

Complex tori are toroidal groups, but there exist toroidal groups that are not compact ([AK01, p.1]). Every toroidal group is a semi-torus in the sense of [NW13, Def. 5.1.5].

By [AK01, 1.1.5], every connected commutative complex Lie group $G$ is uniquely isomorphic to $G_l \times (\mathbb{C}^*)^m \times X$ with a toroidal group $X$. In particular, $G$ can be presented as an extension of a complex torus by a connected linear group. (From [NW13, pp.169-170], a semi-torus can admit nonequivalent presentations, while semiabelian varieties admit exactly one algebraic presentation.)

E.3 Group extensions

Given a surjective Lie group morphism $p : E \to Q$, by Fact E.2.0.1, $K := \ker(p)$ is a normal Lie subgroup of $E$ and the induced morphism $E/K \to Q$ is an isomorphism. We write it as

$$1 \to K \overset{i}{\to} E \overset{p}{\to} Q \to 1 \quad \text{(E.3)}$$

and call it a short exact sequence. In that case, $E$ is called an extension of the base $Q$ by the extension kernel $K$. Moreover, $d_p : L(E) \to L(Q)$ is surjective of kernel $L(K)$, hence an extension of Lie algebras

$$0 \to L(K) \to L(E) \overset{d_p}{\to} L(Q) \to 0.$$ 

When $K \subset Z(E)$, such an extension is called central. If (E.3) is a central extension with $Q$ commutative, as in [MRM74, p.222], using Fact E.2.0.3 one can construct a skew-symmetric bimorphism

$$e : Q \times Q \to K, \quad \text{(E.4)}$$

to measure the deviation of $E$ from commutativity. Indeed, the group $E$ is commutative if and only if $e$ is constant.

Several topological properties of Lie groups are preserved by extensions.

Fact E.3.0.1. If $K, Q$ in (E.3) are compact (resp. connected, resp. discrete), then so is $E$.\(^5\)

\(^5\)also known as a Cousin group
E.3. GROUP EXTENSIONS

Proof. The statement concerning connectedness is in [Che46, Prop. 2, p.36].
The others are consequences of Fact E.2.0.3. \qed

Fact E.3.0.2 ([HN11, Cor. 16.3.9]). If (E.3) is a central extension of complex
Lie groups, where $K$ is finite and $E$ is connected, then $Q$ is linear if and only
if $E$ is linear.

The finiteness of $K$ in Fact E.3.0.2 is necessary. Consider the exact sequence
$0 \rightarrow \mathbb{Z}^2 \rightarrow C \rightarrow A \rightarrow 0$ defining a complex torus $A$. Here $\mathbb{Z}^2$ and $C$ are linear,
while $A$ is not.

Similarly, an extension $E$ of a finite group $Q$ by a linear group $K$ is linear.
Indeed, let $\rho : K \rightarrow \text{GL}_n(\mathbb{C})$ be a faithful representation, then the induced
representation $\text{Ind}_K^E \rho : E \rightarrow \text{GL}_m(\mathbb{C})$ is also faithful, where $m = \#Q$. Again,
the finiteness of $Q$ is essential here. Example E.3.0.3 shows the statement fails
when $Q$ is only discrete and linear but infinite.

Example E.3.0.3. Work of Deligne [Del78] (see also [KRW20, p.470]) shows
that for any integers $g \geq 2, n \geq 3$, there is a central extension $1 \rightarrow \mathbb{Z}/n \rightarrow G \rightarrow \text{Sp}_{2g}(\mathbb{Z}) \rightarrow 1$ for which $G$ is not residually finite. By Malcev’s theorem ([Mal40,
Thm. VIII]; see also [Nic13, p.1]), the discrete complex Lie group $G$ is not linear,
even though $\mathbb{Z}/n$ and $\text{Sp}_{2g}(\mathbb{Z})$ are linear.

We turn to the classification of extensions. Two Lie group extensions $C$ and
$C'$ of $B$ by $A$ are called equivalent if there exists a morphism $f : C \rightarrow C'$ making
a commutative diagram

\[
\begin{array}{ccccccccc}
0 & \rightarrow & B & \rightarrow & C & \rightarrow & A & \rightarrow & 0 \\
& & \downarrow{\text{Id}} & & \downarrow{f} & & \downarrow{\text{Id}} & & \\
0 & \rightarrow & B & \rightarrow & C' & \rightarrow & A & \rightarrow & 0.
\end{array}
\]

In this case, $f$ is bijective, hence an isomorphism by Fact E.2.0.1. The trivial
extension of $Q$ by $K$ refers to the equivalence class of the obvious sequence
$1 \rightarrow K \rightarrow K \times Q \rightarrow Q \rightarrow 1$.

Fact E.3.0.4 ([Bou72, Ch.III, no.4, Prop. 8]). The Lie group extension (E.3)
is trivial if and only if there is a morphism $r : E \rightarrow K$ with $ri = \text{Id}_K$. The extension is a semidirect product if and only if there is a morphism $s : Q \rightarrow E$ with $ps = \text{Id}_Q$.

The extension (E.3) defines a group morphism $\psi : Q \rightarrow \text{Out}(K)$, called the outer action corresponding to the extension. We call $(K, \psi)$ the extension kernel of (E.3). Equivalent extensions induce the same outer action. For two complex Lie groups $Q, K$ and a group morphism $\psi : Q \rightarrow \text{Out}(K)$, denote by $\text{Ext}(Q, K, \psi)$ the set of equivalence classes of extensions of $Q$ by $K$ with outer action $\psi$.

Since the center $Z(K)$ is a characteristic complex Lie subgroup of $K$ by Lemma E.2.0.6, there is a canonical group morphism $\text{Aut}(K) \rightarrow \text{Aut}(Z(K))$...
which passes to another group morphism \( \text{Out}(K) \to \text{Aut}(Z(K)) \). Hence a group morphism

\[
\psi_0 : Q \to \text{Aut}(Z(K))
\]  

(E.5)

induced by \( \psi \). When \( K \) is commutative, \( \psi = \psi_0 \) and the construction of Baer sum (\((E.42)\) and [FLa19, p.444]) makes \( \text{Ext}(Q, K, \psi) \) an abelian group.

### E.3.1 Pullback and pushout

Extensions can be pulled back.

**Example E.3.1.1 (Pullback).** Given a morphism \( g : Q' \to Q \) of complex Lie groups, pulling \((E.3)\) back along \( g \) gives an extension of \( Q' \) by \( K \) as follows.

The map \( E \times Q' \to Q \) defined by \( (x, h') \mapsto p(x)^{-1}g(h') \) is holomorphic, so the preimage \( E' \) of the identity element \( e_Q \in Q \) is an analytic subset of \( E \times Q' \). As \( E' = \{(x, h') \in E \times Q' : p(x) = g(h')\} \) is a subgroup of \( E \times Q', \) by Corollary E.2.0.5, \( E' \) is a complex Lie subgroup of \( E \times Q' \) (which is the extension group).

Let \( p' : E' \to Q' \) and \( \epsilon : E' \to E \) be the projections. Then the triple \((E', \epsilon, p')\) is the fiber product \( E \times_Q Q' \) in the category of complex Lie groups.

For every \( h' \in Q' \), by surjectivity of \( p \), there is \( x \in E \) with \( p(x) = g(h') \).

Then \( (x, h') \in E' \) with \( p'(x, h') = h' \). Hence \( p' \) is surjective.

Define a morphism \( i' : K \to E' \) by \( i'(k) = (k, e_{Q'}) \). Then \( i' \) is injective. Since \( p'i' \) is trivial, \( i'(K) \subset \ker(p') \). Conversely, for every \( (x, h') \in \ker(p') \), \( h' = e_{Q'} \) and \( p(x) = g(e_{Q'}) = e_Q \). Thus, \( x \in K \) and \( (x, h') = i'(x) \in i'(K) \).

Hence a commutative diagram with exact rows

\[
\begin{array}{ccc}
1 & \longrightarrow & K \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
1 & \longrightarrow & K
\end{array}
\quad
\begin{array}{cc}
E' & \longrightarrow & Q' \\
\downarrow \epsilon & & \downarrow g \\
E & \longrightarrow & Q \\
\downarrow p & & \downarrow \text{Id} \\
1 & \longrightarrow & 1
\end{array}
\]

The first row is called the pullback extension of \((E.3)\) along \( g \). Its outer action is \( \psi g : Q' \to \text{Out}(K) \). Hence a map \( \text{Ext}(Q, K, \psi) \to \text{Ext}(Q', K, \psi g) \). It is a group morphism when \( K \) is commutative ([Hoc51a, p.99]).

The universal property of pullback shows that the first row of every such commutative diagram is determined by the second row and \( g : Q' \to Q \). By construction, the pullback of a central extension is also central.

A pushout extension along a morphism \( f : K \to K' \) of complex Lie groups may not exist. When it exists, it satisfies a universal property.

**Lemma E.3.1.2.** Consider a commutative diagram of complex Lie groups, where each row is exact

\[
\begin{array}{ccc}
1 & \longrightarrow & K \\
\downarrow \text{Id} & & \downarrow \text{Id} \\
1 & \longrightarrow & K'
\end{array}
\quad
\begin{array}{ccc}
E & \longrightarrow & Q \\
\downarrow \pi & & \downarrow \text{Id} \\
E' & \longrightarrow & Q \\
\downarrow \epsilon & & \downarrow \text{Id} \\
1 & \longrightarrow & 1
\end{array}
\]

(E.6)
Then the triple \((E', m, \iota)\) has the following universal property: For every commutative diagram of complex Lie groups
\[
\begin{array}{ccc}
K & \xrightarrow{i} & E \\
\downarrow f & & \downarrow m \\
K' & \xrightarrow{\iota} & E'
\end{array}
\]
with \(\psi(m(c)^{-1}bn(c)) = \phi(c)^{-1}\psi(b)\phi(c)\) for every \(c \in E\) and \(b \in K'\), there exists a unique morphism \(\eta: E' \to H\) keeping the diagram commutative.

In particular, up to a unique equivalence, the second row of (E.6) has at most one choice when the first row and \(f: K \to K'\) are given.

\textbf{Proof.} We construct a map \(\eta: E' \to H\) as follows. For every \(c' \in E'\), there exists \(c \in E\) with \(p(c) = \pi(c')\). Let \(b' = m(c)^{-1}c'\). Then \(\pi(b') = p(c)^{-1}\pi(c') = e_Q\), so \(b' \in K'\). Define \(\eta(c') = \phi(c)\psi(b')\).

To show that \(\eta\) is well-defined, we claim that \(\eta(c')\) is independent of the choice of \(c\). Indeed, take another \(c_1 \in E\) with \(p(c_1) = \pi(c')\), then \(p(c^{-1}c_1) = e_Q\), hence \(c^{-1}c_1 \in K\). This time the element in \(K'\) is \(b'_1 = m(c_1)^{-1}c'\), so \(b' = f(c^{-1}c_1)b'_1\) in \(K'\) and hence \(\psi(b') = \phi(c^{-1}c_1)\psi(b'_1)\). Therefore, \(\phi(c)\psi(b') = \phi(c_1)\psi(b'_1)\) in \(H\) as claimed.

We check that \(\eta\) is holomorphic near \(c' \in E'\). Indeed, by Fact E.2.0.3, there is an open neighborhood \(U\) of \(\pi(c')\in Q\), and a holomorphic map \(s: U \to E\) with \(ps = \text{Id}_U\). The map
\[
\pi^{-1}(U) \to U \times K', \quad x \mapsto (\pi(x), [ms\pi(x)]^{-1}x)
\]
is biholomorphic. The map
\[
U \times K' \to H, \quad (u, b') \mapsto \phi(s(u))\psi(b')
\]
is holomorphic. The composition is exactly \(\eta|_{\pi^{-1}(U)}\).

We check that \(\eta\) is a group morphism. For \(c'_i \in E'\) (\(i = 1, 2\)), choose \(c_i \in E\) with \(p(c_i) = \pi(c'_i)\). Then for \(c'_1c'_2\) we can choose \(c_1c_2\). Let \(b'_1 = m(c_1)^{-1}c'_1\) and \(b'_2 = m(c_2)^{-1}c'_2\). Then
\[
b' := m(c_1c_2)^{-1}c'_1c'_2 = m(c_2)^{-1}b'_1m(c_2)b'_2.
\]
By the construction of \(\eta\), one has
\[
\eta(c'_1c'_2) = \phi(c_1c_2)\psi(b') = \phi(c_1)\phi(c_2)\psi(m(c_2)^{-1}b'_1m(c_2))\psi(b'_2) = \phi(c_1)\psi(b'_1)\phi(c_2)\psi(b'_2) = \eta(c'_1)\eta(c'_2).
\]

Then \(\eta\) is a morphism of complex Lie groups. By construction, \(\eta\) is the unique group morphism keeping the diagram commutative. \(\square\)
Example E.3.1.3. Assume that \( Q \) is connected. As the map \( p : E \to Q \) in (E.3) is open by Fact E.2.0.3, \( p(E_0) \) is a nonempty open subgroup of \( Q \) and hence \( p(E_0) = Q \) by the connectedness of \( Q \). Then the following diagram is commutative and each row is exact

\[
\begin{array}{cccccc}
1 & \longrightarrow & K \cap E_0 & \longrightarrow & E_0 & \longrightarrow & Q & \longrightarrow & 1 \\
\downarrow & & \downarrow & & \downarrow & & \Id & \\
1 & \longrightarrow & K & \longrightarrow & E & \longrightarrow & Q & \longrightarrow & 1 \\
\end{array}
\]

By Lemma E.3.1.2, the second row is determined by the inclusion \( K \cap E_0 \to K \) (an open normal subgroup) and the first row.

E.3.2 Rudimentary classification

Let \( K, Q \) be complex Lie groups, where \( Q \) is discrete. Consider an abstract group extension \( 1 \to K \to E \to Q \to 1 \). Then as a set \( E = \sqcup_x xK \), where \( x \) runs through a set of left representatives of \( E/K \). Thus \( E \) admits a unique complex manifold structure making the maps holomorphic. However, the group law of \( E \) needs not to be holomorphic in this complex structure. The semidirect product sequence \( 1 \to C \to C \rtimes \mathbb{Z}/2 \to \mathbb{Z}/2 \to 1 \) serves as an example, where \( \mathbb{Z}/2 \) acts on \( C \) by complex conjugation. But when the base is discrete and the outer action is trivial, the Lie group extension problem reduces to the abstract group extension problem.

Proposition E.3.2.1. Let \( K, Q \) be complex Lie groups. If \( Q \) is discrete, then the natural forgetful map \( \phi : \text{Ext}(Q,K,1) \to \text{Ext}_{\text{Abs}}(Q,K,1) \) is bijective, where \( \text{Ext}_{\text{Abs}}(Q,K,1) \) denotes the set of isomorphism classes of abstract group extensions of \( Q \) by \( K \) with trivial outer action. In fact, for every abstract group extension \( 1 \to K \to E \to Q \to 1 \), \( E \) admits a unique complex manifold structure making the sequence an extension of complex Lie groups.

Proof. We prove that \( \phi \) is injective. Consider \( E_1, E_2 \in \text{Ext}(Q,K,1) \) with \( \phi(E_1) = \phi(E_2) \). Then there is an abstract group isomorphism \( f : E_1 \to E_2 \) making a commutative diagram

\[
\begin{array}{ccc}
K & \longrightarrow & E_1 \\
\downarrow & \nearrow & \downarrow f \\
& E_2 & \\
\end{array}
\]

For every \( x \in E_1 \), the restriction \( xK \to f(x)K \) of \( f \) is holomorphic, since the left multiplication \( K \to xK \) (resp. \( K \to f(x)K \)) by \( x \) (resp. \( f(x) \)) in \( E_1 \) (resp. \( E_2 \)) is biholomorphic. Thus, \( f \) is holomorphic and hence an equivalence of complex Lie group extensions.

We prove that \( \phi \) is surjective. Given an abstract group extension \( 1 \to K \to E \to Q \to 1 \) in \( \text{Ext}_{\text{Abs}}(Q,K,1) \), we endow \( E \) with the complex structure making the maps holomorphic. We show the group law \( m : E \times E \to E \) is holomorphic.
E.3. GROUP EXTENSIONS

Choose a set-theoretic section \( s : Q \rightarrow E \). Then the map \( K \times Q \rightarrow E \) defined by \( (a, b) \mapsto as(b) \) is biholomorphic. With this identification, \( m \) becomes the map

\[
\mu : K \times Q \times K \rightarrow K \times Q, \quad (a, b, a', b') \mapsto (as(b)a's(b')s(bb')^{-1}, bb') = (a\rho(a')s(b)s(b')s(bb')^{-1}, bb'),
\]

where \( \rho : K \rightarrow K \) is \( x \mapsto s(b)x\bar{s}(b)^{-1} \). Since the outer action is trivial, \( \rho \in \text{Im}(K) \). Therefore, the map \( K \times K \rightarrow K \) defined by \( (a, a') \mapsto a\rho(a') \) is holomorphic. Because \( Q \) is discrete, \( \mu \) (and hence \( m \)) is holomorphic. Then \( E \) is a complex Lie group and the abstract extension lifts to \( \text{Ext}(Q, K, 1) \).

Corollary E.7.2.6 below is a result about discrete base with nontrivial outer action. We turn to two other simple cases.

**Proposition E.3.2.2.** Every extension of \( C \) is a semidirect product. In particular, every central extension of \( C \) trivial.

**Proof.** Let \( 0 \rightarrow B \rightarrow C \xrightarrow{\rho} \mathbb{C} \rightarrow 0 \) be an extension. Then \( 0 \rightarrow L(B) \rightarrow L(C) \xrightarrow{d} L(C) \rightarrow 0 \) is an exact sequence of Lie algebras. Take a \( \mathbb{C} \)-linear map \( ds : L(\mathbb{C}) \rightarrow L(C) \) with \( d_{x}p \circ ds = \text{Id}_{L(C)} \). Because \( \text{dim}_{\mathbb{C}} L(\mathbb{C}) = 1 \), \( ds \) is a Lie algebra morphism. As \( C \) is simply connected, there is a unique morphism \( s : C \rightarrow C \) with \( d_{x}s = ds \). Since \( d_{x}(ps) = \text{Id}_{L(\mathbb{C})} \), one has \( ps = \text{Id}_{\mathbb{C}} \). Therefore, this extension is a semidirect product by Fact E.3.0.4.

**Proposition E.3.2.3.** Let \( B \) be a connected commutative complex Lie group. Then every central extension of \( \mathbb{C}^{*} \) by \( B \) is trivial.

**Proof.** Let \( C \) be a central extension of \( \mathbb{C}^{*} \) by \( B \). Consider the pullback extension along \( \exp(2\pi i \bullet) : \mathbb{C} \rightarrow \mathbb{C}^{*} \). By Proposition E.3.2.2, there is a morphism \( \rho : \mathbb{C} \rightarrow C' \) with \( \rho'\rho = \text{Id}_{\mathbb{C}} \). Then \( p\rho(1) = \exp(2\pi i) = 1 \), so \( \rho(1) \in B \). As \( B \) is connected commutative, its exponential map \( \exp_{B} : L(B) \rightarrow B \) is surjective. Take \( v \in L(B) \) with \( \exp_{B}(-v) = \epsilon\rho(1) \).

\[
\begin{array}{cccccc}
Z & \downarrow & & & \downarrow \\
1 & \rightarrow & B & \rightarrow & C' & \xrightarrow{\rho} & \mathbb{C} & \rightarrow & 0 \\
\downarrow \rho' \circ \exp(2\pi i \bullet) & & & & & & \downarrow \text{id} & & \downarrow \rho' \circ \exp(2\pi i \bullet) \\
1 & \rightarrow & B & \rightarrow & C & \xrightarrow{p} & \mathbb{C}^{*} & \rightarrow & 1
\end{array}
\]

Define a holomorphic map

\[
\rho' : \mathbb{C} \rightarrow C', \quad \rho'(z) = \exp_{B}(zv)\rho(v).
\]

We check that \( \rho' \) is a group morphism. For every \( z, w \in \mathbb{C} \),

\[
\rho'(z + w) = \exp_{B}((z + w)v)\rho(z + w) = \exp_{B}(zv)\exp_{B}(wv)\rho(z)\rho(w) = \exp_{B}(zv)\rho(z)\exp_{B}(wv)\rho(w) = \rho'(z)\rho'(w),
\]

where the last but one equality uses \( B \subset \mathcal{Z}(C) \).
Therefore, $\rho'$ is a complex Lie group morphism. Moreover, $\rho'(1) = \exp_B(v)\rho(1) = \epsilon\rho(-1)\rho(1)$. Then $\epsilon\rho'(1) = e_C$. Therefore, $\rho'(\mathbb{Z}) \subseteq \ker(\epsilon)$. Thus, $\rho'$ induces a morphism $s : \mathbb{C}^* \to C$ making a commutative diagram

$$
\begin{array}{ccc}
\mathbb{C} & \xrightarrow{\rho'} & C' & \xrightarrow{p'} & C \\
\downarrow{\exp(2\pi i \cdot)} & & \downarrow{\exp(2\pi i \cdot)} \\
\mathbb{C}^* & \xrightarrow{s} & C & \xrightarrow{p} & \mathbb{C}^*
\end{array}
$$

Since $p'p = \text{Id}_{\mathbb{C}^*}$ and $\exp(2\pi i \cdot) : \mathbb{C} \to \mathbb{C}^*$ is surjective, $ps = \text{Id}_{\mathbb{C}^*}$. From Fact E.3.0.4, the extension $C$ is trivial. \hfill \Box

Example E.7.1.7 gives a result about non-central extensions of $\mathbb{C}^*$. Now assume that the Lie group $K$ is discrete and commutative. We recall results\textsuperscript{6} from [Hoch51b, Sec. 3].

**Fact E.3.2.4** ([Hoch51b, p.545]). Let $K, Q$ be Lie groups. If $K$ is discrete commutative and $Q$ is connected, then the extension (E.3) of Lie groups is central.

**Corollary E.3.2.5.** Let $K, Q$ be commutative Lie groups. If $Q$ is connected and $K$ is discrete, then every extension of $Q$ by $K$ is commutative.

**Proof.** Let (E.3) be such an extension. By Fact E.3.2.4, this extension is central. Then consider the induced continuous map (E.4). Since $Q$ is connected and $K$ is discrete, this map is constant, or equivalently, $E$ is commutative. \hfill \Box

Let $\text{Ab}_c$ be the abelian category of abelian groups that are at most countable. Fact E.3.2.6 shows that the universal cover of a connected Lie group is "universal" among all the extensions with discrete commutative kernels.

**Fact E.3.2.6** (Hochschild, [Hoch51b, Thm. 3.2 and Cor.]). Let $Q$ be a connected Lie group. Then the functor $\text{Ext}(Q, \cdot, 1) : \text{Ab}_c \to \text{Ab}$ is represented by $\pi_1(Q)$ and the class of the universal cover sequence $1 \to \pi_1(Q) \to \tilde{Q} \to Q \to 1$ in $\text{Ext}(Q, \pi_1(Q), 1)$. Hence an isomorphism $\Gamma_K : \text{Ext}(Q, K, 1) \to \text{Hom}_{\text{Ab}}(\pi_1(Q), K)$ functorial in $K \in \text{Ab}_c$. Moreover, $E \in \text{Ext}(Q, K, 1)$ is connected if and only if $\Gamma_K(E)$ is surjective.

\section*{E.4 Commutative Extensions}

### E.4.1 Generalities

**Lemma E.4.1.1.** The category $C$ is naturally additive with finite direct products.

**Proof.** The Hom sets are commutative groups, and composition of morphisms is bilinear. Moreover, the product $G_1 \times G_2$ of two commutative complex Lie groups is both a product and a coproduct of $G_1$ and $G_2$ in $C$. \hfill \Box

\textsuperscript{6}They are stated for real Lie groups, but the proofs extend to the complex setting.
Although the category $\text{Alg}$ of commutative complex algebraic groups is an abelian category ([Mil17a, Thm. 5.62]), as Example E.4.1.2 and Example E.4.1.3 show, $\mathcal{C}$ is NOT an abelian category.

**Example E.4.1.2.** The map $i : \mathbb{Z}^2 \to \mathbb{C}$ defined by $(a, b) \mapsto a + b\sqrt{2}$ is injective. The image is not closed in $\mathbb{C}$ as it is dense in $\mathbb{R}$. For every morphism $f : \mathbb{C} \to X$ in $\mathcal{C}$, with $fi = 0$, we have $f = 0$ by identity theorem for holomorphic maps. Thus $i$ is a monomorphism and epimorphism in $\mathcal{C}$, but not an isomorphism.

**Example E.4.1.3.** Let $p : \mathbb{C}^2 \to \mathbb{C}^2/\mathbb{Z}^4$ be the natural projection. Let $i : \mathbb{C} \to \mathbb{C}^2$ be the closed embedding defined by $z \mapsto (z, \sqrt{2}z)$. Then the composition $pi : \mathbb{C} \to \mathbb{C}^2/\mathbb{Z}^4$ is an injective morphism (hence a monomorphism) in $\mathcal{C}$. By [Lee13, Example 7.19], $pi(C)$ is a connected dense subset of $\mathbb{C}^2/\mathbb{Z}^4$. In particular, $pi$ is an epimorphism in $\mathcal{C}$. The cokernel of $pi$ is the zero morphism $\mathbb{C}^2/\mathbb{Z}^4 \to 0$. However, $pi$ is not an isomorphism in $\mathcal{C}$.

**Proposition E.4.1.4.** is a special case of [Con14, Prop. D.2.1]. An elementary proof is given.

**Proposition E.4.1.4.**

1. $\text{Hom}_\mathcal{C}(\mathbb{C}^*, \mathbb{C}) = 0$.

2. For $A \in \mathcal{C}$, the map

$$\text{Hom}_\mathcal{C}(\mathbb{C}^n, A) \to \text{Hom}_{\text{Vec}}(L(\mathbb{C}^n), L(A)), \quad f \mapsto df$$

is a group isomorphism.

3. Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a morphism in $\mathcal{C}$. Then there is an integer $k$ such that $f(z) = z^k$ for every $z \in \mathbb{C}^*$. Hence an isomorphism $\mathbb{Z} = \text{Hom}_\mathcal{C}(\mathbb{C}^*, \mathbb{C}^*)$.

**Proof.** The Lie algebra of $\mathbb{C}^*$ is $\mathbb{C}$. The exponential map $\exp : \mathbb{C} \to \mathbb{C}^*$ is normalized as $w \mapsto e^{2\pi iw}$.

1. Let $f : \mathbb{C}^* \to \mathbb{C}$ be a morphism. Then $df : \mathbb{C}^* \to \mathbb{C}$ is linear. There is $a \in \mathbb{C}$ with $df(1) = aw$ for all $v \in \mathbb{C}$. Since $1 \in \mathbb{C} = L(\mathbb{C}^*)$ is mapped to $1 \in \mathbb{C}^*$ under the exponential map $\exp(2\pi i)$, one has $0 = f(1) = df(1) = a$. Then $df(1) = 0$ and $f = 0$.

2. It follows from the fact that $\mathbb{C}^n$ is simply connected and both groups are commutative.

3. Consider the induced linear map on Lie algebras $df : \mathbb{C} \to \mathbb{C}$. There is a unique complex number $k$ such that $df(w) = kw$ for all $w \in \mathbb{C}$. Then

$$e^{2\pi ik} = \exp(df(1)) = f \exp(1) = f(1)^k = 1.$$

Therefore, $k$ is an integer. For every $z \in \mathbb{C}^*$, there is $w \in \mathbb{C}$ with $\exp(w) = z$. Then $f(z) = f(\exp(w)) = \exp df(w) = \exp(kw) = z^k$. 
For $A, B \in C$, the set of isomorphism classes of commutative extensions of $A$ by $B$ is denoted by $\text{Ext}(A, B)$.

**Proposition E.4.1.5.**

1. $\text{Ext}(\bullet, \bullet) : C^{\text{op}} \times C \to \text{Sets}$ is a covariant functor.

2. Let $\mathcal{E}$ be the collection of extensions in $C$. Then the pair $(C, \mathcal{E})$ is an exact category.$^7$

**Proof.**

1. Fix $A, B \in C$ and an element of $\text{Ext}(A, B)$: $0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0$.

(a) If $f : B \to B'$ is a morphism in $C$, then

\[ g : B \to C \times B', \quad b \mapsto (-b, f(b)) \]

is a morphism in $C$. It is injective and the (set-theoretic) image is closed in $C \times B'$. By Fact E.2.0.1 2, $g$ identifies $B$ as a complex Lie subgroup of $C \times B'$. Let $f_* C$ be the quotient $(C \times B')/B$ provided by Fact E.2.0.3. The canonical map $B' \to C \times B'$ induces an injective morphism $f_* i : B' \to f_* C$ since $B \cap \{0\} \times B' = \{0\}$. Moreover, $B$ is in the kernel of the composition $C \times B' \to A$ by $(c, \beta) \mapsto p(c)$, hence a surjective morphism $f_* p : f_* C \to A$.

Note that $f_* p \circ f_* i = 0$, so $f_* i(B') \subseteq \ker(f_* p)$. For every element $x$ of $\ker(f_* p)$, take a representative $(c, \beta) \in C \times B'$. As $p(c) = 0$, $c \in B$. Then $(0, \beta + f(c)) - (c, \beta) = g(c)$. Therefore,

\[ x = [(0, \beta + f(c))] = f_* i(\beta + f(c)) \in f_* (B'). \]

Thus, $f_* i(B') = \ker(f_* p)$.

Therefore, the sequence

\[ 0 \to B' \xrightarrow{f_* i} f_* C \xrightarrow{f_* p} A \to 0 \]

is exact and $f_* C \in \text{Ext}(A, B')$. Hence a morphism $f_* : \text{Ext}(A, B) \to \text{Ext}(A, B')$ in the category $\text{Sets}$.

Let $F$ be the canonical morphism $C \to f_* C$. By construction, the extension $f_* C \in \text{Ext}(A, B')$ has the following universal property: for every morphism $h : A \to A'$ in $C$, every $C' \in \text{Ext}(A', B')$, every morphism $G : C \to C'$ making the diagram commutative

\[
\begin{array}{ccc}
0 & \longrightarrow & B' \\
\downarrow & & \downarrow \text{id} \\
0 & \longrightarrow & f_* C \\
\downarrow \text{id} & & \downarrow \text{id} \\
0 & \longrightarrow & C' \\
\downarrow h & & \downarrow h \\
0 & \longrightarrow & A' \\
\end{array}
\]

(E.7)

$^7$see [Sta23, Tag 05SF]
there exists a unique morphism \( u : f_\ast C \to C' \) keeping the diagram commutative.

(b) If \( h : A' \to A \) is a morphism in \( C \), by Example E.3.1.1, we get a morphism \( h^\ast : \text{Ext}(A, B) \to \text{Ext}(A', B) \) in the category \( \text{Sets} \). Let \( F \) be the canonical projection \( h^\ast C \to C \). By construction, the extension \( g^\ast C \) has the following universal property: for every morphism \( g : B' \to B \), every extension \( C' \in \text{Ext}(A', B') \), every morphism \( G : C' \to C \) making the following diagram commutative

\[
\begin{array}{ccc}
0 & \longrightarrow & B' \\
\downarrow & & \downarrow g \\
B & \longrightarrow & h^\ast C \\
\downarrow & & \downarrow h \\
0 & \longrightarrow & A'
\end{array}
\quad \text{(E.8)}
\]

there exists a unique morphism \( v : C' \to h^\ast C \) keeping the diagram commutative.

(c) Let \( f : B \to B' \), \( g : A \to A' \) be morphisms in \( C \), \( C \in \text{Ext}(A, B) \), and \( C' \in \text{Ext}(A', B') \). Then the relation \( f_\ast C = g^\ast C' \) in \( \text{Ext}(A, B') \) is equivalent to the existence of a morphism \( F : C \to C' \) making a commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & B \\
\downarrow & & \downarrow f \\
C & \longrightarrow & A \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & A'
\end{array}
\]

Indeed, it follows from the universal properties in Points (1a) and (1b). For every \( X \in \text{Ext}(A', B) \), in view of the diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & B \\
\downarrow & & \downarrow f \\
B' & \longrightarrow & X \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & A'
\end{array}
\]

one has \( f_\ast g^\ast X = g^\ast f_\ast X \). This completes the proof.

2. It follows from Point 1 and Lemma E.4.1.1.

\[\square\]

**Example E.4.1.6.** If \( A \) is a complex torus with \( \dim A = g \), \( B \) is the discrete group \( \mathbb{Q}/\mathbb{Z} \), then \( \text{Ext}(A, B) \) is isomorphic to \( B^{2g} \) by Fact E.3.2.6. Even though \( B \) is an injective object of \( \text{Ab} \), the functor \( \text{Ext}(\cdot, B) : C^{\text{op}} \to \text{Ab} \) is nonzero.
Example E.4.1.7. For an extension \(0 \to B \xrightarrow{i} C \xrightarrow{p} A \to 0\) in \(C\), the pushout \(i_*C \in \text{Ext}(A, C)\) is the trivial extension. In fact, \(i_*C = C \times C/B\) with the embedding
\[
B \to C \times C, \quad b \mapsto (-b, b).
\]
The group law \(C \times C \to C\) descends to a morphism \(r : i_*C \to C\). Then \(r \circ i_*(i) = \text{Id}_C\). By Fact E.3.0.4, \(i_*C\) is trivial.

Similarly, as the diagonal inclusion \(C \to C \times C\) factors through a morphism \(s : C \to p^*C\) and \(p^*(p) \circ s = \text{Id}_C\), the pullback \(p^*C \in \text{Ext}(C, B)\) is also trivial.

Fact E.4.1.8 ([Ros58, Prop. 5], [Ser12, Prop. 1, p.163]). 1. For every \(A, B \in C\), under the Baer sum \(\text{Ext}(A, B)\) is an abelian subgroup of \(\text{Ext}(A, B, 1)\).

2. The functor \(\text{Ext}(\bullet, \bullet) : C^{op} \times C \to \text{Ab}\) is an additive bifunctor.

3. For any \(C, C' \in \text{Ext}(A, B)\), the product \(C \times C'\) is naturally an element of \(\text{Ext}(A \times A, B \times B)\).

4. Let \(d : A \to A \times A\) the diagonal map of \(A\) and \(s : B \times B \to B\) the group law of \(B\). Then \(C + C' = d^*s_*(C \times C')\) in \(\text{Ext}(A, B)\).

Corollary E.4.1.9. For every commutative complex Lie group \(A\), the restriction \(\text{Ext}(A, \cdot) : \text{Vec} \to \text{Ab}\) factors through a functor from \text{Vec} to the category of all complex vector spaces.

By Example E.4.3.3 below, for every \(V \in \text{Vec}\), \(\dim_C \text{Ext}(A, V)\) is finite. Hence an additive functor \(\text{Ext}(A, \cdot) : \text{Vec} \to \text{Vec}\).

Example E.4.1.10. Endowing each object of \(\text{Ab}\), the discrete topology gives a functor \(\text{Ab}_d \to C\), identifying \(\text{Ab}_d\) as a full subcategory of \(C\). The subcategory \(\text{Ab}_d\) is closed under extension by Fact E.3.0.1. From Proposition E.3.2.1, the forgetful morphism \(\text{Ext}(A, B) \to \text{Ext}_d(A, B)\) is an isomorphism for every \(A \in \text{Ab}\) and every \(B \in C\).

Example E.4.1.11. Analytification functor \((\bullet)_{an} : \text{Alg} \to C\) identifies \(\text{Alg}\) as a subcategory of \(C\) (which is not full). The extension problem within the subcategory \(\text{Alg}\) is discussed by Rosenlicht [Ros58] and Serre [Ser12, Ch. VII].

They define a similar additive functor \(\text{Ext}_{\text{Alg}} : \text{Alg}^{op} \times \text{Alg} \to \text{Ab}\). For every \(A, B \in \text{Alg}\), there is a natural morphism \(\text{Ext}_{\text{Alg}}(A, B) \to \text{Ext}(A_{an}, B_{an})\).

In general, this morphism is neither injective nor surjective.

For example, when \(A/C\) is an abelian variety, \(\text{Ext}_{\text{Alg}}(G_m, A) = 0\) while \(\text{Ext}_{\text{Alg}}(G_m, A)\) is non-canonically isomorphic to the torsion subgroup \(A_{tor}\) of \(A\) ([MM66, Introduction, 1.]). But \(\text{Ext}(C^*, A_{an}) = 0\) by Proposition E.3.2.3, so the natural morphism \(\text{Ext}_{\text{Alg}}(G_m, A) \to \text{Ext}(C^*, A_{an})\) is not injective.

For any two abelian varieties \(X_1/C (i = 1, 2)\) of positive dimension, \(\text{Ext}_{\text{Alg}}(X_2, X_1)\) is countable while \(\text{Ext}(X_2^{an}, X_1^{an})\) is uncountable. In fact, the natural morphism \(\text{Ext}_{\text{Alg}}(X_2, X_1) \to \text{Ext}(X_2^{an}, X_1^{an})\) is an embedding onto the torsion subgroup of \(\text{Ext}(X_2^{an}, X_1^{an})\) ([BL99, Ch. 1; Prop. 6.1, Cor. 6.3]).
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Lemma E.4.1.12 is mentioned at the bottom of [Hoc51b, p.546].

Lemma E.4.1.12. If $G$ is a commutative connected Lie group, then $G$ is a divisible $\mathbb{Z}$-module.

Proof. The exponential map $\exp : L(G) \to G$ is surjective. For every $x \in G$, there is $v \in L(G)$ with $\exp(v) = x$. For every integer $n \geq 1$, $\exp(v/n) \in G$ and $n(\exp(v/n)) = x$.

Corollary E.4.1.13. An extension $0 \to B \to C \to A \to 0$ in $\mathcal{C}$ with $B$ connected and $A$ discrete is trivial. In particular, for every $G \in \mathcal{C}$, the natural exact sequence

$$0 \to G_0 \to G \to G/G_0 \to 0$$

is a trivial extension, hence a non-canonical isomorphism $G \to G_0 \times G/G_0$ in $\mathcal{C}$.

Proof. By Lemma E.4.1.12, the $\mathbb{Z}$-module $B$ is divisible, so the functor $\text{Ext}_1^{\mathbb{Z}}(\cdot, B) : \text{Ab} \to \text{Ab}$ is zero. Since $A$ is discrete, the result follows from Example E.4.1.10.

Example E.4.1.14. The abelian group underlying a complex torus $B$ is divisible by Lemma E.4.1.12, hence an injective object of $\text{Ab}$ and $\text{Ext}_1^{\mathbb{Z}}(\cdot, B) : \text{Ab} \to \text{Ab}$ is zero. However, $\text{Ext}(\cdot, B) : \text{C}^{\text{op}} \to \text{Ab}$ can be nonzero. In fact, [BL04, (8) b), p.68] gives an example of a nontrivial exact sequence of complex tori

$$0 \to B \to C \to A \to 0$$

with $\dim A = \dim B = 1$.

E.4.2 Exact sequences of $\text{Ext}$

Let $0 \to A' \to A \to A'' \to 0$ be an exact sequence in $\mathcal{C}$, i.e., $A \in \text{Ext}(A'', A')$. For $f \in \text{Hom}(A', B)$, there is $f_* A \in \text{Ext}(A'', B)$. Hence a map

$$d : \text{Hom}(A', B) \to \text{Ext}(A'', B), \quad d(f) = f_* A.$$

Then $d$ is a group morphism. The formation of $d$ is functorial in $B$.

Proposition E.4.2.1. Let $B \in \mathcal{C}$. The sequence in $\text{Ab}$ with obvious morphisms

$$0 \to \text{Hom}_\mathcal{C}(A'', B) \to \text{Hom}_\mathcal{C}(A, B) \to \text{Hom}_\mathcal{C}(A', B) \xrightarrow{d} \text{Ext}(A'', B) \xrightarrow{E} \text{Ext}(A, B) \to \text{Ext}(A', B)$$

is exact and functorial in $B$.

Proof.

- Exactness at $\text{Hom}(A, B)$ follows from Fact E.2.0.1.
• Exactness at Hom($A', B$): By Example E.4.1.7, the composition

$$\text{Hom}(A, B) \xrightarrow{i} \text{Hom}(A', B) \to \text{Ext}(A'', B)$$

is zero. Now take $\phi \in \ker(d)$. By Fact E.3.0.4, there is a morphism $r: \phi_* A \to B$ with $r\phi_*(i) = \text{Id}_B$. Let $F: A \to \phi_* A$ be the canonical morphism. Then $rFi = r\phi_*(i)\phi = \phi$. Hence $\phi \in \text{im}(i_*)$.

• Exactness at $\text{Ext}(A'', B)$: By Example E.4.1.7, for every $\phi \in \text{Hom}(A', B)$, $p^*d\phi = p^*\phi_* A = \phi_* p^* A = 0$, i.e., the composition

$$\text{Hom}(A', B) \xrightarrow{d} \text{Ext}(A'', B) \xrightarrow{p} \text{Ext}(A, B)$$

is zero.

Now take $C \in \ker(p^*) \subset \text{Ext}(A'', B)$ with connecting morphisms $f: B \to C$ and $g: C \to A''$. By Fact E.3.0.4, there is a morphism $s: A \to p^* C$ with $p^*(p) \circ s = \text{Id}_A$. For every $a' \in A'$, the image of $s(a')$ in $A''$ is $p(a') = 0$, so the image of $s(a')$ in $C$ lies in $B$. Thus, the restriction of $s$ to $A'$ is a morphism $\phi: A' \to B$. By construction, the extension group of $d(\phi) = \phi_* A \in \text{Ext}(A'', B)$ is $A \times B/D$, where $D = \{(-a', \phi(a')): a' \in A'\}$. Define $\psi: A \to C$ by $\psi = F \circ s$. Define

$$A \times B \to C, \quad (a, b) \mapsto \psi(a) + f(b).$$

For every $a' \in A'$, $\psi(-a') + f(s(a')) = 0$, hence a factorization $\phi_* A \to C$ in the middle keeping the diagram commutative:

$$
\begin{array}{ccccccc}
0 & \longrightarrow & A' & \xrightarrow{i} & A & \xrightarrow{p} & A'' & \longrightarrow & 0 \\
& & \downarrow{\phi} & & \downarrow{\text{Id}} & & & \\
0 & \longrightarrow & B & \xrightarrow{\phi_* i} & \phi_* A & \longrightarrow & A'' & \longrightarrow & 0 \\
& & \downarrow{\text{Id}} & & \downarrow{\text{Id}} & & & \\
0 & \longrightarrow & B & \xrightarrow{f} & C & \xrightarrow{g} & A'' & \longrightarrow & 0 \\
& & \downarrow{\text{Id}} & & \downarrow{F} & & \downarrow{p} & & \downarrow{p} \\
0 & \longrightarrow & B & \xrightarrow{p^* C} & A & \longrightarrow & 0.
\end{array}
$$

Then $C = \phi_* A = d\phi$ in $\text{Ext}(A'', B)$. Therefore, $\ker(p^*) = \text{im}(d)$.

• Exactness at $\text{Ext}(A, B)$: As the composition $i^* A' \to A \to A''$ is zero and $\text{Ext}(\bullet, B): C^{\text{op}} \to \text{Ab}$ is an additive functor, the composition $\text{Ext}(A'', B) \to \text{Ext}(A, B) \to \text{Ext}(A', B)$ is zero.

Conversely, if $C_1 \in \text{Ext}(A, B)$ with $i^* C_1 = 0$ in $\text{Ext}(A', B)$, then there is a morphism $s: A' \to i^* C_1$ with $i^* g \circ s = \text{Id}_{A'}$. The composition $\phi: A' \to C_1$ is injective. Indeed, if $a' \in \ker(\phi)$, then $s(a') = (a', 0)$ in $A' \times C_1$. Thus,
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\( i(a') = 0 \) by the construction of pullback extension. Since \( i \) is injective, \( a' = 0 \).

Let \( C_1 \rightarrow C = C_1/\phi(A') \) be the quotient morphism. Let \( f_0 : B \rightarrow C \) be
the induced morphism. Then \( f_0 \) is injective. Indeed, if \( b \in \ker(f_0) \), then
\( f(b) = \phi(a') \) for some \( a' \in A' \). Then \( (a', f(b)) \in i^*C_1 \), so \( i(a') = gf(b) = 0 \).
Hence \( a' = 0 \) and \( f(b) = 0 \). Therefore, \( b = 0 \).

Because \( pg\phi = p \circ i = 0 \), the morphism \( pg : C_1 \rightarrow A'' \) descends to a
surjective morphism \( g_0 : C \rightarrow A'' \). We prove that the bottom row of the
following diagram is exact:

\[
\begin{array}{ccccccc}
0 & \rightarrow & B & \rightarrow & i^*C_1 & \rightarrow & A' & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{i^*g} & & \downarrow{i} & & \\
0 & \rightarrow & B & \rightarrow & C_1 & \rightarrow & A & \rightarrow & 0 \\
\downarrow{id} & & \downarrow{g} & & \downarrow{p} & & \\
0 & \rightarrow & B & \rightarrow & C & \rightarrow & A'' & \rightarrow & 0.
\end{array}
\]

Since \( gf = 0 \), one has \( g_0f_0 = 0 \). Therefore, \( f_0(B) \subset \ker(g_0) \). Conversely, for \( c \in \ker(g_0) \), there is \( c_1 \in C_1 \) with \([c_1] = c\). Since \( pg(c_1) = g_0(c) = 0 \),
one gets \( g(c_1) \in A' \). Then \( g\phi g(c_1) = gc_1 \). So \( c_1 - \phi g(c_1) \in \ker(g) = B \)
and
\[ f_0(c_1 - \phi g(c_1)) = [c_1 - \phi g(c_1)] = c. \]

Therefore, \( \ker(g_0) = f_0(B) \). In particular, the bottom row is exact, \( \text{i.e.,} \)
\( C \in \Ext(A'', B) \). By the universal property showed in the diagram (E.8),
\( C_1 = p^*C \).

\[ \square \]

**Example E.4.2.2.** Let \( A \) be a complex torus, and let \( B \) be a finite abelian
group. Then \( \Hom_\mathbb{C}(A, B) = 0 \). Let integer \( n(\geq 1) \) be a multiple of \( \#B \).
Applying Proposition E.4.2.1 to the exact sequence in \( \mathbb{C} \)

\[ 0 \rightarrow A[n] \rightarrow A \xrightarrow{[n]} A \rightarrow 0, \]

one gets an exact sequence in \( \text{Ab} \):

\[ 0 \rightarrow \Hom(A[n], B) \rightarrow \Ext(A, B) \xrightarrow{f} \Ext(A, B). \]

Since the morphism \([n]_B : B \rightarrow B\) is zero in \( \mathbb{C} \), by Fact E.4.1.8, \( f = ([n]_B)_* = 0 \).
Hence an isomorphism \( \Hom(A[n], B) \rightarrow \Ext(A, B) \) that is functorial in \( B \), which
is also confirmed by Fact E.3.2.6.

Let \( 0 \rightarrow B' \rightarrow B \rightarrow B'' \rightarrow 0 \) be an exact sequence in \( \mathbb{C} \). If \( A \in \mathbb{C} \) and \( \phi \in \Hom(A, B'') \), then \( \phi^*B \in \Ext(A', B) \). Define a map \( d : \Hom(A, B'') \rightarrow \Ext(A, B') \) by \( d(\phi) = \phi^*B \).
Appendix E. Group Extensions of Complex Lie Groups

Proposition E.4.2.3. Let $0 \to B' \to B \to B'' \to 0$ be an exact sequence in $\mathcal{C}$ and $A \in \mathcal{C}$. Then the sequence

$$0 \to \text{Hom}(A, B') \to \text{Hom}(A, B) \to \text{Hom}(A, B'') \xrightarrow{d} \text{Ext}(A, B') \to \text{Ext}(A, B) \to \text{Ext}(A, B'')$$

in $\text{Ab}$ is exact and functorial in $A$.

The proof is analogous to that of Proposition E.4.2.1 and is thereby omitted.

Consider the extension problem with connected bases. Corollary E.4.2.4 should be compared to [Sha49, Thm. 1]: for two compact connected real Lie groups $G, H$, the cokernel of the restriction morphism $\text{Hom}(\tilde{H}, \mathbb{Z}(G)) \to \text{Hom}(\pi_1(H), \mathbb{Z}(G))$ is isomorphic to the group of extensions of $H$ by $G$.

Corollary E.4.2.4. Let $A, B$ be commutative complex Lie groups. Assume that $A$ is connected with universal cover $\omega: \tilde{A} \to A$. Then there is a canonical exact sequence in $\text{Ab}$:

$$0 \to \text{Hom}_C(A, B) \xrightarrow{\omega_*} \text{Hom}_C(\tilde{A}, B) \xrightarrow{r} \text{Hom}_{\text{Ab}}(\pi_1(A), B) \to \text{Ext}(A, B) \to 0, \quad (E.9)$$

where $r$ is induced by restriction.

Proof. By Proposition E.3.2.2, Fact E.3.2.6 and Corollary E.4.1.13, the functor $\text{Ext}(\mathcal{C}, \bullet): \mathcal{C} \to \text{Ab}$ is zero. By Fact E.4.1.8,

$$\text{Ext}(\mathcal{C}^n, \bullet) = 0. \quad (E.10)$$

The proof is concluded by Proposition E.4.2.1. $\square$

Example E.4.2.5. In Corollary E.4.2.4, if $B$ discrete, then $\text{Hom}_C(\tilde{A}, B) = 0$ and the natural morphism $\text{Hom}(\pi_1(A), B) \to \text{Ext}(A, B)$ is an isomorphism, which agrees with Fact E.3.2.6.

E.4.3 Determination of commutative extension group

The commutative extension problem of complex Lie groups is answered by Proposition E.4.3.1. Fix two commutative complex Lie groups $A, B$.

Proposition E.4.3.1. There is a non-canonical isomorphism in $\text{Ab}$:

$$\text{Ext}(A, B) \to \text{Ext}^1(A/A_0, B/B_0) \oplus \text{Hom}_{\text{Ab}}(\pi_1(A_0), B/B_0) \oplus \text{Ext}(A_0, B_0),$$

and $\text{Ext}(A_0, B_0)$ is the cokernel of the natural restriction morphism

$$s: \text{Hom}_{\text{Vec}}(L(A), L(B)) \to \text{Hom}_{\text{Ab}}(\pi_1(A_0), B_0).$$

Proof. By Corollary E.4.1.13, there are non-canonical isomorphisms in $\mathcal{C}$: $A \to A/A_0 \times A_0$ and $B \to B/B_0 \times B_0$. Using Fact E.4.1.8, one gets an isomorphism in $\text{Ab}$:

$$\text{Ext}(A, B) \to \text{Ext}(A/A_0, B_0) \oplus \text{Ext}(A/A_0, B/B_0) \oplus \text{Ext}(A_0, B_0) \oplus \text{Ext}(A_0, B_0).$$
The first factor $\Ext(A/A_0, B_0) = 0$ by Corollary E.4.1.13. By Example E.4.1.10, the natural morphism $\Ext(A/A_0, B/B_0) \to \Ext^1_2(A/A_0, B/B_0)$ is an isomorphism. Fact E.3.2.6 gives a natural isomorphism $\Hom_{\Ab} (\pi_1 (A_0), B/B_0) \to \Ext(A_0, B/B_0)$. Corollary E.4.2.4 identifies $\Ext(A_0, B_0)$ with the cokernel of the restriction map $r: \Hom_{\C} (\tilde{A}_0, B_0) \to \Hom_{\Ab} (\pi_1 (A_0), B_0)$. By Proposition E.4.1.42, the group morphism

$$t: \Hom_{\C} (\tilde{A}_0, B_0) \to \Hom_{\Vec} (L(A), L(B)), \quad \phi \mapsto d_\omega \phi$$

is an isomorphism. The proof is finished by setting $s = rt^{-1}$. \hfill $\Box$

For every $C \in \Ext(A, B)$, by Fact E.2.0.3, the morphism $C \to A$ is a principal $B$-bundle. The bijection (E.1) gives rise to a canonical map

$$\pi: \Ext(A, B) \to H^1(A, B_A). \tag{E.11}$$

Fact E.4.3.2 is taken from [Ros58, pp.698-699] and the proof of [Ser12, Ch. VII, no. 5, Prop. 5].

**Fact E.4.3.2.** The map (E.11) is a group morphism and the formation of $\pi$ is functorial, in the sense that it commutes with the morphisms $f_* : \Ext(A, B) \to \Ext(A, B')$ defined by $f: B \to B'$ and $g^* : \Ext(A, B) \to \Ext(A', B)$ defined by $g: A' \to A$. When $B$ is a vector group, the map $\pi$ is $\C$-linear.

**Example E.4.3.3.** Let $X$ be a toroidal group, and let $\omega: \tilde{X} \to X$ be the universal covering of kernel $F$. Then $F$ is a discrete subgroup of the vector space $\tilde{X}$. By Proposition E.4.2.1,

$$\Hom_{\C} (X, \C) \to \Hom_{\C} (\tilde{X}, \C) \to \Hom_{\C} (F, \C) \to \Ext(X, \C) \to \Ext(\tilde{X}, \C)$$

is an exact sequence in $\Ab$. From Definition E.2.0.9, $\Hom_{\C} (X, \C) = 0$. By Proposition E.10, $\Ext(\tilde{X}, \C) = 0$. Hence the first exact row of Diagram (E.12).

According to [AK01, p.48], there is a $\C$-linear isomorphism $\Hom_{\C} (\tilde{X}, \C) \to H^0(X, \Omega^1_X)$ and every global holomorphic 1-form on $X$ is $d$-closed. So taking de Rham cohomology class results in a linear map $H^0(X, \Omega^1_X) \to H^1(X, \C)$. The inclusion $\C_X \to O_X$ induces a linear map $H^1(X, \C) \to H^1(X, O_X)$. By universal coefficient theorem (see, e.g., [Hat05, Thm. 3.2]), the natural morphism $\Hom_{\C} (F, \C) \to H^1(X, \C)$ is an isomorphism. Hence a commutative diagram

$$\begin{CD}
0 @>>> \Hom_{\C} (\tilde{X}, \C) @>>> \Hom_{\C} (F, \C) @>>> \Ext(X, \C) @>>> 0 \\
@. @VV\cong V @VV\cong V @V{(E.11)} VV \\
0 @>>> H^0(X, \Omega^1_X) @>>> H^1(X, \C) @>>> H^1(X, O_X). \tag{E.12}
\end{CD}$$

Let $b_1(X) := \dim_{\C} H^1(X, \C)$ be the first Betti number of $X$, i.e., the $\Z$-rank of $F$. From [AK01, p.48], as a $\C$-vector space

$$\Ext(X, \C) = \frac{H^1(X, \C)}{H^0(X, \Omega^1_X)} \tag{E.13}$$
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is of dimension $b_1(X) - \dim X$.

If $X$ is a toroidal theta group, then $\pi : \text{Ext}(X, \mathbb{C}) \rightarrow H^1(X, O_X)$ is a $\mathbb{C}$-linear isomorphism by [AK01, Thm. 2.2.6 b)]. Otherwise, $X$ is a toroidal wild group and $H^1(X, O_X)$ is infinite dimensional by [AK01, Prop. 2.2.7].

A seemingly different way to compute the last factor in Proposition E.4.3.1, i.e., the group of commutative extensions of two connected commutative complex Lie groups, is given in Example E.4.3.4.

Example E.4.3.4. Start by the special case that $X$ is a toroidal group and $B$ is a connected commutative complex Lie group. Denote the kernel of the universal cover of $B$ (resp. $X$) by $\iota : K \rightarrow \hat{B}$ (resp. $F \rightarrow \hat{X}$). By (E.10) and Proposition E.4.1.4 2, the sequence

$$0 \rightarrow \text{Hom}_\mathbb{C}(\hat{X}, K) \rightarrow \text{Hom}_\mathbb{C}(\hat{X}, \hat{B}) \rightarrow \text{Hom}_\mathbb{C}(\hat{X}, B) \rightarrow 0$$

is exact in $\text{Ab}$. As $F$ is a free $\mathbb{Z}$-module,

$$0 \rightarrow \text{Hom}_\mathbb{C}(F, K) \rightarrow \text{Hom}_\mathbb{C}(F, \hat{B}) \rightarrow \text{Hom}_\mathbb{C}(F, B) \rightarrow 0$$

in $\text{Ab}$ is also exact. Applying Proposition E.4.2.1 and the snake lemma to the commutative diagram

$$\begin{array}{cccccc}
0 & \rightarrow & \text{Hom}_\mathbb{C}(\hat{X}, K) & \rightarrow & \text{Hom}_\mathbb{C}(\hat{X}, \hat{B}) & \rightarrow & \text{Hom}_\mathbb{C}(\hat{X}, B) & \rightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \rightarrow & \text{Hom}_\mathbb{C}(F, K) & \rightarrow & \text{Hom}_\mathbb{C}(F, \hat{B}) & \rightarrow & \text{Hom}_\mathbb{C}(F, B) & \rightarrow & 0,
\end{array}$$

one gets an exact sequence in $\text{Ab}$

$$0 \rightarrow \text{Hom}_\mathbb{C}(X, B) \overset{j}{\rightarrow} \text{Ext}(X, K) \overset{\iota^*}{\rightarrow} \text{Ext}(X, \hat{B}) \rightarrow \text{Ext}(X, B) \rightarrow 0. \quad (E.14)$$

Since $K$ is a free $\mathbb{Z}$-module, by Fact E.3.2.6, $\text{Ext}(X, K) = H^1(X, \mathbb{Z}) \otimes_\mathbb{Z} K$. By Fact E.4.1.8 and (E.13), one has

$$\text{Ext}(X, \hat{B}) = \frac{H^1(X, \mathbb{C})}{H^0(X, \Omega^1_X)} \otimes_\mathbb{C} \hat{B}.$$ 

The group morphism $\iota_*$ is induced by the $\mathbb{Z}$-bilinear map

$$H^1(X, \mathbb{Z}) \times K \rightarrow \left( \frac{H^1(X, \mathbb{C})}{H^0(X, \Omega^1_X)} \otimes_\mathbb{C} \hat{B} \right), \quad (\eta, x) \mapsto [\eta] \otimes \iota(x).$$

Thus we can compute $\text{Ext}(X, B)$ from (E.14).

For a general connected commutative complex Lie group $A$, by [AK01, 1.1.5],

$$A = \mathbb{C}^l \times (\mathbb{C}^*)^m \times X_0$$

for some integers $l, m \geq 0$ and a toroidal group $X_0$. By Fact E.4.1.8, Proposition E.3.2.2 and Proposition E.3.2.3, $\text{Ext}(A, B) = \text{Ext}(X_0, B)$, reducing to the previous case.

---

8 in the sense of [AK01, Def. 2.2.1]
E.5 Commutative extensions of complex tori

E.5.1 Primitive cohomology classes

Every central extension of a compact real Lie group by a vector group is trivial, shown by Fact E.5.1.1.

Fact E.5.1.1 (Iwasawa, [Iwa49, Lem. 3.7], [Hoc51a, Footnote 10, p.107]). Let (E.3) be an exact sequence of real Lie groups. If K is a vector group and Q is compact, then this extension is a semidirect product. In particular, if this extension is central, then it is trivial.

Con contrad to the real case, Example E.5.1.2 shows a commutative extension of a complex torus by a vector group can be nontrivial.

Example E.5.1.2 ([MM60, p.145, Exemple], [lH76, Sec. I.3]). Set $C = \mathbb{C}^* \times \mathbb{C}^*$. Then $B = \{(e^z, e^{iz}) : z \in \mathbb{C}\}$ is a complex Lie subgroup of $C$ (but not an algebraic subgroup of $\mathbb{G}_m \times \mathbb{G}_m$) isomorphic to $\mathbb{C}$. The quotient $A = C/B$ is an elliptic curve. The exact sequence $0 \to B \to C \to A \to 0$ is a non trivial extension, as $C$ is not biholomorphic to $B \times A$.

In the remainder of Section E.5, unless otherwise specified, let $A$ be a complex torus of dimension $g$ and $B$ be a commutative complex Lie group.

Let $s_A : A \times A \to A$ be the group law of $A$. The dual of $A$ is $A^\vee = \text{Pic}_0(A)$. The analogue of Proposition E.5.1.3 for abelian varieties is [Ros58, Prop. 9].

Proposition E.5.1.3. The morphism (E.11) is injective.

Proof. Let $C \in \ker(\pi)$. The principal bundle $C \to A$ is trivial, so there is a morphism $s : A \to C$ of complex manifolds with $ps = \text{Id}_A$. Then there exists a unique $b \in B$ with $b \cdot s(e_A) = e_C$, where dot signifies the action of $B$ on the fiber $p^{-1}(e_A)$. Define

$$s' : A \to C, \quad s(a) = b \cdot s(a).$$

Then $s'$ is a complex manifold morphism with $ps' = \text{Id}_A$. Replacing $s$ by $s'$, we may suppose that $s(e_A) = e_C$. By [NW13, Thm. 5.1.36], $s$ is a morphism in $C$. By Fact E.3.0.4, $C = 0$ in $\text{Ext}(A, B)$. Therefore, $\pi$ is injective. \hfill $\Box$

We propose to determine the image of (E.11). Let $\text{Mfd}$ be the category of complex manifolds. Define a functor

$$T : \text{Mfd}^{op} \to \text{Ab}, \quad T(X) = H^1(X, \mathcal{E}_X).$$

When $X$ is a point, $T(X) = 0$. Let $X_1, X_2 \in \text{Mfd}$, and let $p_i : X_1 \times X_2 \to X_i$ ($i = 1, 2$) be the projection to the $i$-th factor. There is a morphism $p_1^* \oplus p_2^* : T(X_1) \times T(X_2) \to T(X_1 \times X_2)$.

Definition E.5.1.4. [Ser12, (29), no.14, Ch. VII] For $A \in \mathcal{C}$, an element $x \in T(A) = H^1(A, \mathcal{E}_A)$ is called primitive if $s_A(x) = p_1^*(x) + p_2^*(x)$ in $T(A \times A)$. Denote by $\text{PT}(A)$ the subgroup of $T(A)$ formed by the primitive elements.
Fact E.5.1.5. [Ser12, Lem. 8, p. 181] The functor $\mathbb{PT} : \mathbb{C}^{op} \to \mathbb{Ab}$ is additive.

Theorem E.5.1.6. Assume that $B_0$ is linear. Then the image of the morphism (E.11) is the set of primitive elements of $H^1(A, B_A)$.

Proof. Take $C \in \text{Ext}(A, B)$ and put $x = \pi(C)$. By Facts E.4.1.8 and E.4.3.2,

$$s_A^*(x) = s_A^*\pi(C) = \pi s_A^*(C) = \pi(p_1^*C + p_2^*C) = p_1^*\pi(C) + p_2^*\pi(C) = p_1^*x + p_2^*x,$$

so $x$ is primitive.

Conversely, let $x \in H^1(A, B_A)$ be a primitive element and let $p : C \to A$ be the corresponding principal $B$-bundle. We show that there exists a structure of commutative complex Lie group on $C$ which makes it an extension of $A$ by $B$.

By Corollary E.4.1.13, every morphism of complex manifolds $A \to B$ is constant. Let $C' \to A \times A$ be the pull-back of $C \to A$ along $s_A : A \times A \to A$. As $x$ is primitive, $C' = p_1^*C + p_2^*C$ in $T(A \times A)$. Choose a surjection $p_1^*C \times_{A \times A} p_2^*C \to C'$ satisfying (E.2). Since $p_1^*C = C \times A$ and $p_2^*C = A \times C$, as a complex manifold $p_1^*C \times_{A \times A} p_2^*C$ is isomorphic to $C \times C$. Hence a morphism $g : C \times C \to C$ of complex manifolds:

$$
\begin{align*}
C \times C = p_1^*C \times_{A \times A} p_2^*C & \quad \xrightarrow{\quad g \quad} \quad p_1^*C + p_2^*C = C' \quad \xrightarrow{\quad p \quad} \quad C \\
A \times A & \quad \xrightarrow{\quad s_A \quad} \quad A.
\end{align*}
$$

(E.15)

By construction, it satisfies

$$g(b \cdot c, b' \cdot c') = (b + b') \cdot g(c, c')$$

(E.16)

for every $c, c' \in C$ and $b, b' \in B$.

Choose a point $e \in p^{-1}(e_A)$. Since $p(g(e, e)) = s_A(e_A, e_A) = e_A$, there exists a unique $b \in B$ with $b \cdot g(e, e) = e$. Replacing $e$ by $b \cdot e$, we can suppose that

$$g(e, e) = e.$$  

(E.17)

We verify that $(C, e, g)$ is a group.

Identity According to (E.15), there is a morphism $h : C \to B$ of complex manifolds with $g(c, e) = b(c) \cdot c$ for all $c \in C$. By (E.17), $h(e) = e_B$. Furthermore, (E.16) shows that $h(b \cdot c) = h(c)$ for all $b \in B$. Therefore, $h$ factors as $C \xrightarrow{\sim} A \xrightarrow{\sim} B$. The morphism $\tilde{h}$ of complex manifolds is constant, so $g(c, e) = c$ for all $c \in C$. The formula $g(e, c) = c$ is proved similarly.
According to (E.15), there is a complex manifold morphism \( u : C \times C \times C \to B \) with
\[
g(c, g(c', c'')) = u(c, c', c'') \cdot g(g(c, c'), c''')
\]
for all \( c, c', c'' \in C \). Then \( u(e, e, e) = e_B \). Equation (E.16) shows that \( u \) factors through a morphism \( \tilde{u} : A \times A \times A \to B \) of complex manifolds. Then \( \tilde{u} \) is of constant value \( e_B \). Therefore, \( g(c, g(c', c'')) = g(g(c, c'), c''') \)
for all \( c, c', c'' \in C \).

**Inverse** Denote by \( i_A : A \to A \) (resp. \( i_B : B \to B \)) the inverse of \( A \) (resp. \( B \)). Let \( C^{-} \to A \) be the principal \( B \)-bundle corresponding to \( -x \in H^1(A, B_A) \).

There is a morphism \( f : C \to C^{-} \) of principal \( B \)-bundles over \( A \), such that for every \( b \in B, c \in C \), \( f(b \cdot c) = (-b) \cdot c \). Since \( 0_A = i_A + \text{Id}_A \), by Fact E.5.1.5, \( 0 = 0_A^* x = i_A^* x + x \), hence \( i_A^* x = -x \). In other words, the pullback of \( p : C \to A \) along \( i_A \) is \( C^{-} \to A \).

The induced morphism \( i : C \to C \) of complex manifolds is such that for every \( c \in C, b \in B \),
\[
i(b \cdot c) = (-b) \cdot i(c).
\]

Since \( i(e) \in p^{-1}(e_A) \), there is \( b \in B \) with \( b \cdot i(c) = e \). Define \( i' : C \to C \) by \( i'(x) = b \cdot i(x) \) and replace \( i \) by \( i' \). Then we may further assume that \( i(e) = e \). Because
\[
p(g(c, i(c))) = s_A(p(c), pi(c)) = s_A(p(c), i_A(p(c))) = e_A,
\]
there exists a morphism \( v : C \to B \) of complex manifolds such that \( g(c, i(c)) = v(c) \cdot e \) and \( v(c) = e_B \). By (E.16) and (E.18), \( v \) factors through \( \tilde{v} : A \to B \), which is of constant value \( e_B \). Therefore, \( g(c, i(c)) = e \) for all \( c \in C \).

In conclusion, \((C, e, g, i)\) is a complex Lie group and (E.15) shows that \( p : C \to A \) is a morphism. Define an injective map \( \iota : B \to C \) by \( b \mapsto b \cdot e \). By (E.16), then \( \iota \) is a morphism. Since \( \iota(B) = p^{-1}(e) \), the sequence
\[
0 \to B \overset{\iota}{\to} C \overset{p}{\to} A \to 0
\]
is exact. By Proposition E.6.0.2 below, \( C \) is commutative and hence \( C \in \text{Ext}(A, B) \). (The commutativity of \( C \) can also be proved using an argument of similar type.) Therefore, \( x = \pi(C) \) is in the image of \( \pi \). \( \square \)
E.5.2 The case $B = \mathbb{C}^*$

We review some basics about (holomorphic) line bundles on complex tori.

**Definition E.5.2.1.** [Wei48, Ch VIII, n.58] Let $L \rightarrow A$ be a line bundle on a complex torus. If for every $a \in A$, the pullback line bundle $T_a^*L$ is isomorphic to $L$, then we write $L \equiv O_A$. Here $T_a : A \rightarrow A$ is defined by $T_a(x) = x + a$.

By [BL04, p.36], $L$ induces a morphism

$$\phi_L : A \rightarrow A', \quad a \mapsto T_a^*L \otimes L^{-1}.$$  

Then $L \equiv O_A$ is equivalent to $\phi_L = 0$. Then [BL04, Prop. 2.5.3] becomes Fact E.5.2.2.

**Fact E.5.2.2.** Let $L \rightarrow A$ be a line bundle on a complex torus. The following conditions are equivalent:

1. $L$ is analytically equivalent to $O_A$;
2. $L \in \text{Pic}^0(A)$;
3. $L \equiv O_A$.

**Proposition E.5.2.3.** Let $L \rightarrow A$ be a line bundle on complex torus. Then $L \equiv O_A$ if and only if $s_A^*L = p_1^*L \otimes p_2^*L$.

**Proof.** If $s_A^*L = p_1^*L \otimes p_2^*L$, then for every $a \in A$, the line bundle $T_a^*L = (s_A^*L)|_{A \times a} = (p_1^*L \otimes p_2^*L)|_{A \times a} = L$, i.e., $L \equiv O_A$.

Conversely, if $L \equiv O_A$, then for every $a \in A$, $(s_A^*L)|_{A \times a} = T_a^*L = L = (p_1^*L)|_{A \times a}$. Therefore, $s^*L \otimes p_1^*L^{-1} \rightarrow A \times A$ is a line bundle, whose restriction to $A \times a$ is trivial for all $a \in A$. By seesaw theorem [BL04, A.8], there is a line bundle $M \rightarrow A$ such that $s^*L \otimes p_1^*L^{-1} = p_2^*M$. Then $s^*L = p_1^*L \otimes p_2^*M$. Hence, $L = s^*L|_{A \times A} = (p_1^*L \otimes p_2^*M)|_{A \times A} = M$. Therefore, $s^*L = p_1^*L \otimes p_2^*L$. 

Theorem E.5.2.4 is mentioned without proof in [KKN08, Sec. 1.2]. The analogue for abelian varieties is in [Wei49, no. 2].

**Theorem E.5.2.4 (Weil).** If $A$ is a complex torus, then $\pi : \text{Ext}(A, \mathbb{C}^*) \rightarrow \text{Pic}^0(A)$ is an isomorphism.

**Proof.** For $B = \mathbb{C}^*$, the sheaf $\mathcal{B}_A = O_A^*$ and $H^1(A, \mathcal{B}_A) = \text{Pic}(A)$. The class of a line bundle $L \rightarrow A$ is primitive means the line bundle $s_A^*L$ is isomorphic to $p_1^*L \otimes p_2^*L$ on $A \times A$. By Proposition E.5.2.3 and Fact E.5.2.2, it is equivalent to $[L] \in \text{Pic}^0(A)$. Then Proposition E.5.1.3 and Theorem E.5.1.6 complete the proof. 

With the identifications provided by Theorem E.5.2.4 and Proposition E.4.1.4 3, [AK01, Remark 1.1.16] can be rephrased in a coordinate-free way as follows. It is a criterion telling whether a semi-torus is a toroidal group.
Fact E.5.2.5. Let \( r \geq 1 \) be an integer, and let \( 0 \to (\mathbb{C}^*)^r \to X \to A \to 0 \) be an extension in \( \mathbb{C} \). Denote by \((L_1, \ldots, L_r) \in (A^\vee)^r \) the point corresponding to the equivalent class \([X] \in \operatorname{Ext}(A, (\mathbb{C}^*)^r)\). Then the following are equivalent:

- \( X \) is a toroidal group;
- for all \( \sigma \in \mathbb{Z}^r \setminus \{0\} \), \( \sum_{i=1}^r \sigma_i L_i \neq 0 \) in \( A^\vee \);
- for every nontrivial morphism \( f : (\mathbb{C}^*)^r \to \mathbb{C}^* \), the pushout extension \( f_*X \) of \( A \) by \( \mathbb{C}^* \) is nontrivial.

E.5.3 The case \( B = \mathbb{C} \)

When \( B = \mathbb{C} \), the sheaf \( B^A = \mathcal{O}_A \).

Fact E.5.3.1 (Künneth formula, [Men20, (3.1)]). Let \( X, Y \) be connected complex manifolds. Assume that \( Y \) is compact. Then there is a canonical decomposition

\[
H^1(X \times Y, \mathcal{O}_{X \times Y}) = H^1(X, \mathcal{O}_X) \oplus H^1(Y, \mathcal{O}_Y).
\]

The analogue of Theorem E.5.3.2 for abelian varieties is [Ros58, Theorem 1].

Theorem E.5.3.2 (Rosenlicht, Serre). If \( A \) is a complex torus, then the canonical morphism \( \pi : \operatorname{Ext}(A, \mathbb{C}) \to H^1(A, \mathcal{O}_A) \) is a \( \mathbb{C} \)-linear isomorphism. In particular,

\[
\dim \mathbb{C} \operatorname{Ext}(A, \mathbb{C}) = \dim A.
\]

Proof. Let \( m_1 \) (resp. \( m_2 \)) be the injection \( A \to A \times A \) defined by \( a \mapsto (a, 0) \) (resp. \( a \mapsto (0, a) \)). Let \( p_u : A \times A \to A (u = 1, 2) \) be the two projections. By Fact E.5.3.1, \( p_1^* \) and \( p_2^* \) identify \( T(A \times A) \) as the direct sum \( T(A) \oplus T(A) \). The projection to \( i \)th factor is \( m_i^* \). Because \( s_A \circ m_i = \operatorname{Id}_A \), one has \( s_A^*(x) = p_1^*x + p_2^*x \) for every \( x \in T(A) \), i.e., \( x \) is primitive. Then Proposition E.5.1.3 and Theorem E.5.1.6 conclude the proof.

Remark E.5.3.3. Another way to prove Theorem E.5.3.2 is to use (E.13). In this case, the diagram (E.12) can be completed into a commutative diagram with exact rows

\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \operatorname{Hom}_{\mathbb{C}}(\hat{A}, \mathbb{C}) & \longrightarrow & \operatorname{Hom}(\pi_1(A), \mathbb{C}) & \longrightarrow & \operatorname{Ext}(A, \mathbb{C}) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \pi & & \downarrow \pi & & \\
0 & \longrightarrow & H^0(A, \Omega^1_A) & \longrightarrow & H^1(A, \mathbb{C}) & \longrightarrow & H^1(A, \mathcal{O}_A) & \longrightarrow & 0.
\end{array}
\]

(E.19)

The bottom row comes from the Hodge structure on \( H^1(A, \mathbb{C}) \) ([Huy05, Lem. 3.3.1]).

Corollary E.5.3.4. Let \( A \) be a complex abelian variety, and let \( n(\geq 0) \) be an integer. Then the natural morphism \( \operatorname{Ext}_{\text{Alg}}(A, \mathbb{G}_m^n) \to \operatorname{Ext}(A^{an}, \mathbb{C}^n) \) is an isomorphism.

Proof. It is a combination of [Ser12, Thm. 7, p.185], Theorem E.5.3.2 and [Ser56, Thm. 1].

\( \square \)
E.5.4 Universal vectorial extension

**Definition E.5.4.1.** [Ros58, p.705] Let $H$ be a vector group. An extension

$$0 \to H \to G \to A \to 0 \quad (E.20)$$

in $C$ is called decomposable if there exists an extension

$$0 \to H_1 \to G_1 \to A \to 0$$

in $C$ of $A$ by a vector subgroup $H_1$ of $H$, and $H'$ is a vector subgroup of $H$ of positive dimension with an isomorphism $f : G_1 \oplus H' \to G$ such that the maps $H_1 \to H \to G$ and $H_1 \to G_1 \overset{f|_{G_1}}{\to} G$ coincide. Otherwise, the extension $G$ is called indecomposable.

**Proposition E.5.4.2.** The extension $(E.20)$ is decomposable if and only if there is a strict vector subgroup $H_1$ of $H$ and an extension $0 \to H_1 \to G_1 \to A \to 0$ with $\iota_* G_1 = G$, where $\iota : H_1 \to H$ is the inclusion.

*Proof.* If $G$ is decomposable, by definition, we can write $G = G_1 \oplus H'$, where $H' \subset H$ is a positive-dimensional vector subgroup and $0 \to H_1 \to G_1 \to A \to 0$ is an extension in $C$ of $A$ by a vector subgroup $H_1 \subset H$ making a commutative diagram

$$
\begin{array}{cccccc}
0 & & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & \text{id} & & \downarrow & \\
0 & \longrightarrow & H & \longrightarrow & G & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

By the universal property (E.7), $G = \iota_* G_1$. Moreover,

$$\dim H_1 = \dim G_1 - \dim A = \dim G - \dim H' - \dim A = \dim H - \dim H' < \dim H.$$

Conversely, assume that $\iota_* G_1 = G$. Choose a vector subspace $H'$ of $H$ with $H = H' \oplus H_1$, then $\dim H' = \dim H - \dim H_1 > 0$. The composed morphism $G_1 \oplus H' \overset{pr_1}{\to} G_1 \overset{pr_1}{\to} A$ is surjective of kernel $H_1 \oplus H' = H$, hence a commutative diagram

$$
\begin{array}{cccccc}
0 & & \longrightarrow & H_1 & \longrightarrow & G_1 & \longrightarrow & A & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & \text{id} & & \downarrow & \\
0 & \longrightarrow & H & \longrightarrow & G_1 \oplus H' & \longrightarrow & A & \longrightarrow & 0
\end{array}
$$

with exact rows. By the universal property (E.7), $G = \iota_* G_1 = G_1 \oplus H'$. This identification makes the maps $H_1 \to H \to G$ and $H_1 \to G_1 \to G$ coincide. Therefore, $G$ is decomposable.

**Proposition E.5.4.3.** Let $0 \to \mathbb{C}^n \to G \to A \to 0$ be an extension in $C$. Let $q_i : \mathbb{C}^n \to \mathbb{C}$ be the $i$-th coordinate function. Then $G$ is indecomposable if and only if the family $\{q_i_* G\}_{1 \leq i \leq n}$ of vectors in $\text{Ext}(A, \mathbb{C})$ is linearly independent.
Proof. Assume that \( \{q_i, G\} \) is linearly dependent. By changing of coordinate, one may assume that \( q_n,^\ast G = 0 \) in \( \Ext(A, C) \). By Fact E.3.0.4, there is a morphism \( r : q_n,^\ast G \to C \) with \( i_n r = \Id \) on \( q_n,^\ast G \).

Then \( i_n r \alpha i = \alpha i = i_n q_n \). Since \( i_n \) is injective, one has

\[
\alpha i = q_n. \tag{E.21}
\]

Let \( q : C^n \to C^{n-1} \) be the projection to the first \((n - 1)\) coordinates. Let \( \beta : G \to q_n G \) be the canonical morphism. Define a morphism

\[
\epsilon : G \to q_n G \oplus C, \quad g \mapsto (\beta(g), ra(g)).
\]

Then the right square of the following diagram is commutative.

\[
\begin{array}{ccc}
0 & \longrightarrow & C^n \\
\downarrow & & \downarrow r \\
0 & \longrightarrow & q_n G \oplus C \\
\end{array}
\]

By (E.21), the left square of the above diagram is commutative. Therefore, \( \epsilon \) is an equivalence of extensions and \( G = q_n G \oplus C \) is decomposable.

Conversely, assume that \( G \) is decomposable. By Proposition E.5.4.2, there is a vector subgroup \( \iota : H_1 \to C^n \) with \( \dim H_1 < n \) and an extension \( 0 \to H_1 \to G_1 \to A \to 0 \) with \( \iota G_1 = G \). There is a linear combination \( f = \sum_{i=1}^n a_i q_i : C^n \to C \), where \( a_1, \ldots, a_n \in C \) are not all zero, such that \( f \iota = 0 \). Then \( \sum_{i=1}^n a_i q_i,^\ast G = f, G = (f \iota), G_1 = 0 \). Thus, the family \( \{q_i, G\}_i \) is linearly dependent.

Corollary E.5.4.4 follows from Proposition E.5.4.3 and Theorem E.5.3.2.

**Corollary E.5.4.4.** Let \( 0 \to V \to G \to A \to 0 \) be an extension in \( C \) by a vector group \( V \). If \( \dim C V > g \), then \( G \) is decomposable.

Proposition E.5.4.5 is an analytic analogue of [Ros58, Prop. 11].

**Proposition E.5.4.5.**

1. There is a \( C \)-vector group \( H \) with \( \dim C H = g \) and an indecomposable extension

\[
0 \to H \to G \to A \to 0 \tag{E.22}
\]

such that for every \( V \in \text{Vec} \), the map

\[
\phi_V : \Hom_{\text{vec}}(H, V) \to \Ext(A, V), \quad l \mapsto l_* G \tag{E.23}
\]

is a linear isomorphism. In other words, \( H \) together with the extension (E.22) represents the functor \( \Ext(A, \bullet) : \text{Vec} \to \text{Vec} \).
2. A $G' \in \text{Ext}(A, V)$ is indecomposable if and only if the corresponding linear map $\phi_V^{-1}(G') : H \to V$ is surjective.

Proof.

1. By Theorem E.5.3.2, $\dim \mathbb{C} \text{ Ext}(A, \mathbb{C}) = g$. Take a $\mathbb{C}$-basis $\{G_1, \ldots, G_g\}$ of $\text{ Ext}(A, \mathbb{C})$. By Fact E.4.1.8, $\text{ Ext}(A, \mathbb{C}^g) = \bigoplus_{i=1}^g \text{ Ext}(A, \mathbb{C})$, so there is an element $G \in \text{ Ext}(A, \mathbb{C}^g)$ corresponding to $(G_1, \ldots, G_g) \in \bigoplus_{i=1}^g \text{ Ext}(A, \mathbb{C})$. Hence an extension $0 \to H \to G \to A \to 0$, where $H = \mathbb{C}^g$. By Proposition E.5.4.3, $G$ is indecomposable.

When $l \in H^\vee$ is taking the i-th coordinate of $H = \mathbb{C}^g$, $l_i G = G_i$. Therefore, the image of the linear map $\phi_C$ contains a basis of $\text{ Ext}(A, \mathbb{C})$. Thus, $\phi_C$ is surjective. Since $\dim \mathbb{C} H^\vee = \dim \mathbb{C} \text{ Ext}(A, \mathbb{C})$, $\phi_C$ is a linear isomorphism. Since every $V \in \text{ Vec}$ is the direct sum of finitely many copies of $\mathbb{C}$ and the formation of $\phi_V$ is functorial in $V$, $\phi_V$ is also a linear isomorphism.

2. By Proposition E.5.4.2, $G'$ is decomposable iff there is a proper linear subspace $\iota : V_1 \to V$ with $G'$ in the image of the map $\iota_* : \text{ Ext}(A, V_1) \to \text{ Ext}(A, V)$ if there is a proper linear subspace $\iota : V_1 \to V$ with $\phi_{V_1}^{-1}(G')$ in the image of the map $\iota_* : \text{ Hom}_{\text{ Vec}}(H, V_1) \to \text{ Hom}_{\text{ Vec}}(H, V)$ if $\phi_{V_1}^{-1}(G') : H \to V$ factors through a proper linear subspace $\iota : V_1 \to V$ if $\phi_{V_1}^{-1}(G') : H \to V$ is not surjective.

The extension (E.22) is called the universal vectorial extension of $A$. (As a representing object, such an extension is unique up to equivalence.) By (E.23) and Theorem E.5.3.2, $H = H^0(A^\vee, \Omega^1_A)$.

**Example E.5.1.2 (continued).** Since $\dim \text{ Ext}(A, \mathbb{C}) = 1$, this nontrivial extension is equivalent to the universal vectorial extension.

We proceed to give an explicit construction of the universal vectorial extension.

**Proposition E.5.4.6.** Let $B^{21}$ be the group of isomorphic classes of rank 1 local systems on $A$. Let $B^2$ be the group of isomorphic classes of pairs $(L, \nabla)$, where $L \to A$ is a holomorphic line bundle and $\nabla$ is a flat holomorphic connection on $L$. Then there exist natural identifications of groups

$$B^2 = B^{21} = \text{ Hom}_{\text{ Ab}}(\pi_1(A), \mathbb{C}^*) = H^1(A, \mathbb{C}^*) = \frac{H^1(A, \mathbb{C})}{H^1(A, \mathbb{Z})}.$$ 

They are isomorphic to $(\mathbb{C}^*)^{2g}$.

**Proof.** By the Riemann-Hilbert correspondence [Del70, Théorème 2.17, p.12], the map $B^2 \to B^{21}$ defined by $(L, \nabla) \mapsto \ker(\nabla)$ is a group isomorphism. By [Del70, Corollaire 1.4, p.4], there is an isomorphism $B^{21} \to \text{ Hom}_{\text{ Ab}}(\pi_1(A), \mathbb{C}^*)$. By the universal coefficient theorem [Hat05, Thm. 3.2], there is a natural isomorphism
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\[ H^1(A, \mathbb{C}^*) \rightarrow \text{Hom}(\pi_1(A), \mathbb{C}^*). \] The exact sequences \( 0 \rightarrow \mathbb{Z} \rightarrow \mathbb{C}^{\exp(2\pi i \bullet)} \rightarrow 0 \) of constant sheaves on \( A \) gives rise to an exact sequence

\[ H^0(A, \mathbb{C}) \rightarrow H^0(A, \mathbb{C}^*) \rightarrow H^1(A, \mathbb{Z}) \rightarrow H^1(A, \mathbb{C}) \rightarrow H^1(A, \mathbb{C}^*) \rightarrow H^2(A, \mathbb{Z}) \rightarrow H^2(A, \mathbb{C}). \]

Since the first map is surjective and the last map is injective, it breaks into a short exact sequence

\[ 0 \rightarrow H^1(A, \mathbb{Z}) \rightarrow H^1(A, \mathbb{C}) \rightarrow H^1(A, \mathbb{C}^*) \rightarrow 0 \]

and hence an isomorphism \( H^1(A, \mathbb{C})/H^1(A, \mathbb{Z}) \rightarrow H^1(A, \mathbb{C}^*) \) functorial in \( A \). Moreover, there is a non-canonical isomorphism \( H^1(A, \mathbb{C}^*) \rightarrow (\mathbb{C}^*)^{2g} \).

For every \((L, \nabla) \in B^2\), the line bundle \( L \in \text{Pic}^0(A) = A^\vee \) by \([\text{Dem}12, \text{Ch. V, \S 9}\). The bottom row of (E.19) induces an exact sequence in \( \mathcal{C} \):

\[ 0 \rightarrow H^0(A, \Omega^1_A) \rightarrow H^1(A, \mathbb{C}) \rightarrow H^1(A, O_A) \rightarrow H^1(A, \mathbb{Z}) \rightarrow 0. \]  \( \text{(E.24)} \)

Using the identifications \( B^2 \cong \frac{H^1(A, \mathbb{C})}{H^1(A, \mathbb{Z})} \) from Proposition E.5.4.6 and \( A^\vee = \text{Pic}^0(A) = H^1(A, O_A)/H^1(A, \mathbb{Z}) \), (E.24) is an extension of \( A^\vee \) by \( H^0(A, \Omega^1_A) \) and gives a morphism \( B^2 \rightarrow \text{Pic}^0(A) \), which sends \((L, \nabla)\) to \( L \). Hence a commutative diagram

\[
\begin{array}{ccc}
0 & \rightarrow & \text{Hom}_\mathcal{C}(\hat{A}, \mathbb{C}^*) \\
\downarrow && \downarrow \\
0 & \rightarrow & H^0(A, \Omega^1_A) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \text{Hom}_{\text{Ab}}(\pi_1(A), \mathbb{C}^*) \\
& \uparrow & \\
& & \text{Ext}(A, \mathbb{C}^*) \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \rightarrow \\
& & \rightarrow \\
0 & \rightarrow & B^2 \\
\uparrow & & u \\
\downarrow & & \rightarrow \\
0 & \rightarrow & \text{Pic}^0(A) \\
\end{array}
\]

where the first exact row is (E.9) and the second comes from (E.24). The left vertical isomorphism uses Proposition E.4.1.4.2 and the isomorphism \( L(A)^\vee \rightarrow H^0(A, \Omega^1_A) \) given by \([\text{BL}04, \text{Thm. 1.4.1 b}]\). The middle vertical isomorphism is contained in Proposition E.5.4.6.

When \( A \) is an abelian variety, it is proved in \([\text{Mes}73, \text{p. 260}]\) that (E.24) is the universal vectorial extension of \( A^\vee \). The proof is based on \([\text{Ros}58, \text{Thm. 1}]\).

In a similar manner, Proposition E.5.4.7 follows from Theorem E.5.3.2.

**Proposition E.5.4.7.** The extension (E.24) is the universal vectorial extension of \( A^\vee = \text{Pic}^0(A) \). In particular, the extension group is isomorphic to \((\mathbb{C}^*)^{2g}\) (as a complex Lie group).

**Proof.** Let \( U = H^0(A, \Omega^1_A) \). Pushing out the extension (E.24) defines a natural transformation \( \psi : \text{Hom}_{\text{Vec}}(U, \bullet) \rightarrow \text{Ext}(A^\vee, \bullet) \) between two functors on \( \text{Vec} \).

We claim that \( \psi_U \) is an isomorphism. Choose \( u \in \ker(\psi_U) \subset \text{Hom}_{\text{Vec}}(U, \mathbb{C}) \).

As the push-out along \( u \) is trivial, by Fact E.3.0.4, there is a morphism \( r : E \rightarrow \mathbb{C} \) with \( ri = \text{Id}_E \). Let \( u' : H^1(A, \mathbb{C}) \rightarrow \mathbb{C} \) be the morphism in \( \mathcal{C} \) induced by \( r \). Then \( u' = d_r u' \) is \( \mathbb{C} \)-linear. Now that \( u'(H^1(A, \mathbb{Z})) = 0 \) and \( H^1(A, \mathbb{Z}) \) contains a \( \mathbb{C} \)-basis of \( H^1(A, \mathbb{C}) \), one has \( u' = 0 \). As the diagram commutes, \( u = 0 \).
Therefore, \( \psi_C \) is injective. By Theorem E.5.3.2, \( \dim C \, \text{Ext}(A^\vee, C) = \dim C \, \text{Hom}_{\text{Vec}}(U, C) \).

Therefore, \( \psi_C \) is a linear isomorphism. Similar to the proof of Proposition E.5.4.5, \( \psi \) is a natural isomorphism of the two functors.

Another construction of the universal vectorial extension is in [Nak94, Prop. 2.4].

Remark E.5.4.8. The real Lie group extension underlying (E.24) is trivial by Fact E.5.1.1. Indeed, consider the real analytic group morphism \( A^\vee \to B \), defined by \( L \to (L, \nabla L) \), where \( \nabla L \) is the unique flat Chern connection on \( L \) given by Lemma C.2.0.4. This map is a real Lie group section to (E.24), but not holomorphic.

Remark E.5.4.9. Let \( A \) be a complex abelian variety of dimension \( g \). By Corollary E.5.3.4, the extension (E.22) is equivalent to an algebraic one. Thus, the analytification of the algebraic universal vectorial extension \( 0 \to \mathbb{C}_a^g \to E \to A \to 0 \) is exactly the analytic universal vectorial extension. From [Bri09, Prop. 2.3 (i)] and the footnote in [MRM74, p.34], the algebraic variety \( E \) is anti-affine, i.e., every morphism \( E \to A_\mathbb{C}^g \) of algebraic varieties is constant. On the other hand, by Proposition E.5.4.7, \( E^{\text{an}} \) is isomorphic to \( (\mathbb{C}^*)^{2g} \) as a complex Lie group, so \( E^{\text{an}} \) is a toroidal group. Although \( E \) is not an affine variety, \( E^{\text{an}} \) is a Stein manifold. See also Serre’s example [Har70, Example 3.2, p.232].

Remark E.5.4.10. Universal vectorial extensions can be defined for not only complex tori but also toroidal groups. Consider a toroidal group \( X \) of dimension \( n \). Similar to Proposition E.5.4.5, the functor \( \text{Ext}(X, \cdot) : \text{Vec} \to \text{Vec} \) is represented by \( \text{Ext}(X, \mathbb{C})^\vee \), which is the kernel of the natural linear map \( H_1(X, \mathbb{C}) \to H^0(X, \Omega^1_X)^\vee \) by (E.13).

An extrinsic description is possible. Choose a presentation
\[
0 \to (\mathbb{C}^*)^{n-q} \to X \to T \to 0 \quad (E.25)
\]
according to [AK01, 1.1.14], where \( T \) is a complex torus of dimension \( q \). For every \( V \in \text{Vec} \), by Proposition E.4.2.1, the induced sequence
\[
\text{Hom}_C((\mathbb{C}^*)^{n-q}, V) \to \text{Ext}(T, V) \to \text{Ext}(X, V) \to \text{Ext}((\mathbb{C}^*)^{n-q}, V)
\]
is exact in \( \text{Vec} \). By Proposition E.4.1.4.1, \( \text{Hom}_C((\mathbb{C}^*)^{n-q}, V) = 0 \). By Proposition E.3.2.3, \( \text{Ext}((\mathbb{C}^*)^{n-q}, V) = 0 \). Thus, the morphism \( \text{Ext}(T, V) \to \text{Ext}(X, V) \) is a \( \mathbb{C} \)-linear isomorphism. In other words, the natural transformation \( \text{Ext}(T, \cdot) \to \text{Ext}(X, \cdot) \) between the two functors on \( \text{Vec} \) is an isomorphism. In this way, the case of toroidal groups is reduced to the case of complex tori.

\(^9\)stated for complex abelian varieties but the proof extends to complex tori.
E.5.5 Application to the functor $\text{Ext}(A,\bullet)$

Analogue of Proposition E.5.5.1 for abelian varieties is [Ros58, Cor., p.711].

**Proposition E.5.5.1.** If $B$ is a complex Lie subgroup (not necessarily connected) of $A$, then there is a natural exact sequence in $\text{Ab}$:

$$0 \to \text{Ext}(A/B, C) \to \text{Ext}(A, C) \to \text{Ext}(B, C) \to 0.$$  

**Proof.** By Corollary E.4.1.13, there is an isomorphism $B \to B_0 \times B/B_0$ in $C$ and $\text{Ext}(B/B_0, C) = 0$. By Fact E.4.1.8, $\text{Ext}(B, C) = \text{Ext}(B_0, C)$. Since $B$ is compact and $B_0$ is open in $B$, the quotient $B/B_0$ is finite, thus $\text{Hom}_{\text{Ab}}(B/B_0, C) = 0$. By the compactness of $B_0$, $\text{Hom}_C(B_0, C) = 0$. Then $\text{Hom}(B, C) = 0$. Now that $A, B_0, A/B$ are complex tori, Theorem E.5.3.2 implies $\dim \text{Ext}(A, C) = \dim \text{Ext}(A/B, C) + \dim \text{Ext}(B, C)$. This together with Proposition E.4.2.1 proves the stated exactness. \hfill $\square$

The proof of Theorem E.5.5.2 is shorter than that of its algebraic analogue [Ser12, Thm. 12, p.195].

**Theorem E.5.5.2.** If $0 \to B' \to B \xrightarrow{\phi} B'' \to 0$ is an exact sequence in $C$, then the sequence\footnote{induced by Proposition E.4.2.3} in $\text{Ab}$

$$\text{Ext}(A, B') \to \text{Ext}(A, B) \xrightarrow{\phi_*} \text{Ext}(A, B'') \to 0$$  \hspace{1cm} (E.26)

is exact. If $B''_0$ is linear, then the first map in (E.26) is injective.

**Proof.** By Proposition E.4.2.3, it suffices to prove that $\phi_* : \text{Ext}(A, B) \to \text{Ext}(A, B'')$ is surjective. From (E.10) and Proposition E.4.2.1, one obtains a commutative square

$$\begin{array}{ccc}
\text{Hom}(\pi_1(A), B) & \longrightarrow & \text{Hom}(\pi_1(A), B'') \\
\downarrow & & \downarrow \\
\text{Ext}(A, B) & \xrightarrow{\phi_*} & \text{Ext}(A, B''),
\end{array}$$

where the vertical maps are surjective. Since $\pi_1(A)$ is a free $\mathbb{Z}$-module, the top row is surjective, then so is the bottom.

Now assume that $B''_0$ is linear, then $\text{Hom}_C(A, B'') = 0$. By Proposition E.4.2.3, the first map is injective. \hfill $\square$

**Remark E.5.5.3.** The linearity of $B''_0$ is necessary to guarantee the injectivity in Theorem E.5.5.2. For instance, let $0 \to \mathbb{C}^g \to (\mathbb{C}^*)^{2g} \to A \to 0$ be the universal vectorial extension of $A$ and assume $g \geq 1$. By Proposition E.4.2.3, the natural sequence $0 \to \text{Hom}_C(A, A) \to \text{Ext}(A, \mathbb{C}^g) \to \text{Ext}(A, (\mathbb{C}^*)^{2g})$ is exact. Thus, $\text{Id}_A$ is a nonzero element in the kernel of the first map of (E.26).
Example E.5.5.4. Applying Theorem E.5.5.2 to the exact sequence \[ 0 \to \mathbb{Z} \to \mathbb{C} \to \mathbb{C}^* \to 1, \] and using Fact E.3.2.6, Theorems E.5.2.4 and E.5.3.2, one gets an exact sequence

\[ 0 \to \text{Hom}(\pi_1(A), \mathbb{Z}) \to H^1(A, O_A) \to \text{Pic}^0(A) \to 0. \tag{E.27} \]

In particular, \( \text{Ext}(A, \cdot) \) turns the exponential map to the universal cover of the complex torus \( A^\vee \). Identifying \( \text{Hom}(\pi_1(A), \mathbb{Z}) \) with the sheaf cohomology \( H^1(A, \mathbb{Z}) \), the sequence (E.27) is also induced by the exponential sequence of sheaves on \( A \):

\[ 0 \to \mathbb{Z}_A \to O_A \to O_A^{\exp(2\pi i)} \to 1. \]

Theorem E.5.5.5 is an analytic version of [Ser12, Thm. 13, p.196]

Theorem E.5.5.5. If \( 0 \to L \to C \to A \to 0 \) is an exact sequence in \( \mathcal{C} \) with \( L \) connected and \( G \in \text{Ab}_\mathbb{C} \). Then there is a natural exact sequence

\[ 0 \to \text{Ext}(A, G) \to \text{Ext}(C, G) \xrightarrow{i^*} \text{Ext}(L, G) \to 0. \]

Proof. As \( L \) is connected and \( G \) is discrete, \( \text{Hom}_\mathcal{C}(L, G) = 0 \). By Proposition E.4.2.1, it suffices to show that \( i^* : \text{Ext}(C, G) \to \text{Ext}(L, G) \) is surjective. For every \( L' \in \text{Ext}(L, G) \), by Theorem E.5.5.2, the map \( \text{Ext}(A, L') \to \text{Ext}(A, L) \) is surjective. Thus, there exists \( C' \in \text{Ext}(A, L') \) having image \( C \in \text{Ext}(A, L) \).

\[
\begin{array}{ccccccc}
0 & \to & 0 & \to & G & \xrightarrow{\beta} & \ker(\alpha) & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L' & \to & C' & \to & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L & \xrightarrow{i} & C & \xrightarrow{\alpha} & A & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & 0 & \to & 0 & \to & 0 & & \\
\end{array}
\]

By the snake lemma, \( \alpha \) is surjective and \( \beta \) is an isomorphism. Therefore, \( C' \in \text{Ext}(C, G) \) and \( i^* C' = L' \) in \( \text{Ext}(L, G) \). \qed

In Example E.5.5.6, we give another proof of [BL99, Prop. 5.7, p.21], which computes the extension group of two complex tori.

Example E.5.5.6. Let \( X_i = \mathbb{C}^{g_i}/\Pi_i \mathbb{Z}^{2g_i} \) \((i = 1, 2)\) be two complex tori, where the chosen period matrix is of the form \( \Pi_i = (\tau_i, I_{g_i}) \) with \( \tau_i \in \text{M}_{g_i}(\mathbb{C}) \) and \( \det(\text{Im}(\tau_i)) \neq 0 \). Define \( \xi : M(2g_1 \times 2g_2, \mathbb{Z}) \to M(g_1 \times g_2, \mathbb{C}) \) by \( \xi(P) = \Pi_1 P \left( \frac{I_{g_2}}{\tau_2} \right) \).
Define a map \( \rho : M(g_1 \times g_2, \mathbb{C}) \to \text{Ext}(X_2, \tilde{X}_1) \) as follows. For every \( \alpha \in M(g_1 \times g_2, \mathbb{C}) \), let \( \alpha' = (\alpha, 0) \in M(g_1 \times 2g_2, \mathbb{C}) \). Consider the sequence

\[
0 \to \mathbb{C}^g_1 \to \mathbb{C}^{g_1+g_2}_{\{ (\alpha'v, \Pi_2v) : v \in \mathbb{Z}^{2g_2} \}} \overset{p}{\to} X_2 \to 0,
\]

where \( i \) is induced by \( \mathbb{C}^g_1 \to \mathbb{C}^{g_1+g_2}_{\{ (\alpha'v, \Pi_2v) : v \in \mathbb{Z}^{2g_2} \}} \) defined by \( x \mapsto (x, 0) \) and \( p \) is induced by the second projection \( \mathbb{C}^{g_1+g_2} \to \mathbb{C}^{g_2} \). It is an exact sequence. Denote its class by \( \rho(M) \in \text{Ext}(X_2, \tilde{X}_1) \). This sequence fits into a commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \tilde{X}_1 & \to & X_2 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & \\
0 & \to & X_1 & \to & X & \to 0,
\end{array}
\]

where the second row is \( \psi_{\Pi_1, \Pi_2}(\alpha') \in \text{Ext}(X_2, X_1) \) defined in [BL99, p.20], and

\[
X = \mathbb{C}^{g_1+g_2}_{\{ (\Pi_1u + \alpha'v, \Pi_2v) : u \in \mathbb{Z}^{2g_1}, v \in \mathbb{Z}^{2g_2} \}}.
\]

Then \( \rho \) is a linear isomorphism by Theorem E.5.3.2.

Define a map \( \phi : M(2g_1 \times 2g_2, \mathbb{Z}) \to \text{Ext}(X_2, \pi_1(X_1)) \) as follows. Given \( P = \begin{pmatrix} P_1 & P_2 \\ P_3 & P_4 \end{pmatrix} \in M(2g_1 \times 2g_2, \mathbb{Z}) \), with each \( P_i \in M(g_1 \times g_2, \mathbb{Z}) \), we set \( A = \tau_1P_2 + P_4 \in M(g_1 \times g_2, \mathbb{C}) \) and \( \alpha = \xi(P) \). The linear map \( \mathbb{C}^{g_1+g_2}_{(I, -A)} \to \mathbb{C}^{g_1} \) sends \((u, 0)\) to \( u \) for all \( u \in \mathbb{C}^{g_1} \) and sends \((\alpha'v, \Pi_2v)\) to \( \Pi_1 \begin{pmatrix} P_1 & P_2 \\ P_3 & -P_4 \end{pmatrix} \) \( v \in \Pi_1 \mathbb{Z}^{2g_1} \) for all \( v \in \mathbb{Z}^{2g_2} \). Thus it descents to the vertical morphism in the middle of the following commutative diagram

\[
\begin{array}{cccccc}
0 & \to & \mathbb{C}^{g_1} & \to & \mathbb{C}^{g_1+g_2}_{\{ (\alpha'v, \Pi_2v) : v \in \mathbb{Z}^{2g_2} \}} & \overset{(I_{g_1}, -A)}{\to} & X_2 & \to 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
0 & \to & X_1 & \to & X_1 & \to 0 & \to 0, & \end{array}
\]

(E.28)

where the first row is of class \( \rho(\alpha) = \rho(\xi(P)) \). The snake lemma gives an extension of \( X_2 \) by \( \pi_1(X_1) \), whose class is denoted by \( \phi(P) \).

The image of \( \phi(P) \) under the pushout map \( \text{Ext}(X_2, \pi_1(X_1)) \to \text{Ext}(X_2, \tilde{X}_1) \) is exactly the first row of (E.28), i.e., \( \rho(\xi(P)) \). Then \( \phi \) is a group isomorphism by Fact E.3.2.6. And there is a commutative diagram
APPENDIX E. GROUP EXTENSIONS OF COMPLEX LIE GROUPS

\[ M(2g_1 \times 2g_2, \mathbb{Z}) \xrightarrow{\xi} M(g_1 \times g_2, \mathbb{C}) \xrightarrow{\rho} \text{Im}(\xi) \xrightarrow{\psi_{\alpha_1, \alpha_2}} M(g_1 \times g_2, \mathbb{C}) \]

\[ \text{Ext}(X_2, \pi_1(X_1)) \rightarrow \text{Ext}(X_2, \tilde{X}_1) \rightarrow \text{Ext}(X_2, X_1) \rightarrow 0 \]

where the second row is from (E.14) and the induced dotted isomorphism is exactly the content of [BL99, Proposition 5.7, p.21].

To conclude Section E.5.5, we show that the groups of commutative extensions of complex tori by linear groups are naturally complex Lie groups. Let \( \mathcal{T} \) (resp. \( \mathcal{S} \)) be the full subcategory of \( \mathcal{C} \) comprised of complex tori (resp. objects whose identity component is linear). Then \( \text{Ext} : \mathcal{T}^{op} \times \mathcal{S} \rightarrow \text{Ab} \) is an additive functor by Fact E.4.1.8. Theorem E.5.5.7, an analytic analogue of [Wu86, Theorem 5], lifts this functor.

**Theorem E.5.5.7 (Wu).** There is a natural way to lift \( \text{Ext} : \mathcal{T}^{op} \times \mathcal{S} \rightarrow \text{Ab} \) to an additive functor \( \text{Ext} : \mathcal{T}^{op} \times \mathcal{S} \rightarrow \mathcal{C} \).

**Proof.** First we define a complex Lie group structure on \( \text{Ext}(A, H) \), where \( A \in \mathcal{T} \) and \( H \in \mathcal{S} \). Let \( g = \dim A \).

If there is an isomorphism \( f : H \rightarrow (\mathbb{C}^*)^n \) in \( \mathcal{S} \), then by Theorem E.5.2.4, \( f \) gives rise to an isomorphism \( \text{Ext}(A, H) \rightarrow (A^V)^n \) making \( \text{Ext}(A, H) \) a complex torus. The complex structure on \( \text{Ext}(A, H) \) is independent of the choice of the isomorphism \( f \).

If \( H \) is connected, by Proposition E.2.0.7, there is an isomorphism \( u : H \rightarrow V \times H_m \), where \( V \in \text{Vec} \) and \( H_m \) is a power of \( \mathbb{C}^* \). Then \( u_* : \text{Ext}(A, H) \rightarrow \text{Ext}(A, V) \times \text{Ext}(A, H_m) \) is an isomorphism. By Theorem E.5.3.2, the vector space \( \text{Ext}(A, V) \) is finite dimensional. Together with last paragraph, \( \text{Ext}(A, H) \) inherits a complex Lie group structure, which is independent of the choice of \( u \).

For a general object \( H \in \mathcal{S} \), the natural exact sequence \( 0 \rightarrow H_0 \rightarrow H \rightarrow H/H_0 \rightarrow 0 \) in \( \mathcal{C} \) is trivial by Corollary E.4.1.13. Thus, the resulting exact sequence \( 0 \rightarrow \text{Ext}(A, H_0) \rightarrow \text{Ext}(A, H) \rightarrow \text{Ext}(A, H/H_0) \rightarrow 0 \) in \( \text{Ab} \) is also trivial. Now that \( \text{Ext}(A, H/H_0) = \text{Hom}_{\text{Ab}}(\pi_1(A), H/H_0) \) by Fact E.3.2.6, one regards it as a discrete group. From the complex structure on \( \text{Ext}(A, H_0) \), the group \( \text{Ext}(A, H) \) has a unique complex Lie group structure, such that the identity component is \( \text{Ext}(A, H_0) \).

It remains to show:

1. If \( A \in \mathcal{T} \) is fixed, then \( \text{Ext}(A, \cdot) \) sends morphisms in \( \mathcal{S} \) to morphisms in \( \mathcal{C} \).

2. If \( H \in \mathcal{S} \) is fixed, then \( \text{Ext}(\cdot, H) \) sends morphisms in \( \mathcal{T} \) to morphisms in \( \mathcal{C} \).

To show 1, let \( h : H \rightarrow H' \) be a morphism in \( \mathcal{S} \). By decomposing \( H, H' \) according to Corollary E.4.1.13 and Proposition E.2.0.7, one may assume that each of \( H \) and \( H' \) is either discrete, \( \mathbb{C} \) or \( \mathbb{C}^* \).
• If $H$ is discrete, then so is $\text{Ext}(A, H)$, hence $\text{Ext}(A, h)$ is a morphism in $C$.

• If $H = H' = \mathbb{C}$, by Proposition E.4.1.4 2, $h$ is a linear map. By Corollary E.4.1.9, so is $\text{Ext}(A, h)$.

• If $H = \mathbb{C}$, $H' = \mathbb{C}^\ast$. By Proposition E.4.1.4 2, $h$ is the composition of a linear map $\mathbb{C} \to \mathbb{C}$ followed by the exponential map $\exp(2\pi i) : \mathbb{C} \to \mathbb{C}^\ast$. By Example E.5.5.4, $\text{Ext}(A, h)$ is the composition of a linear map $H^1(A, O_A) \to H^1(A, O_A)$ followed by the universal cover $H^1(A, O_A) \to A^\vee$. Thus, $\text{Ext}(A, h)$ is a morphism in $C$.

• If $H'$ is discrete and $H$ is connected, then $h$ is trivial and so is $\text{Ext}(A, h)$.

• If $H' = \mathbb{C}^\ast$, then $h$ is trivial by Proposition E.4.1.4 1 and so is $\text{Ext}(A, h)$.

• If $H = H' = \mathbb{C}^\ast$, then $h$ is a power map by Proposition E.4.1.4 3. Then $\text{Ext}(A, h)$ is a power map of $A^\vee$, hence a morphism in $C$.

This proves 1.

To show 2, let $g : A \to A'$ be a morphism in $T$. By decomposing $H$ again, we may divide the proof into three cases.

• $H = \mathbb{C}^\ast$. By pulling back line bundles, $g$ induces the dual morphism $g^* : \text{Pic}^0(A') \to \text{Pic}^0(A)$. It is identified with $\text{Ext}(g, H)$ by Fact E.4.3.2 and Theorem E.5.2.4.

• $H$ is discrete. Then so is $\text{Ext}(A', H)$ and thus $\text{Ext}(g, H)$ is a morphism in $C$.

• $H = \mathbb{C}$. By pulling back, $g$ induces a $\mathbb{C}$-linear map $H^1(A', O_{A'}) \to H^1(A, O_A)$. It is identified with $\text{Ext}(g, H)$ by Fact E.4.3.2 and Theorem E.5.3.2.

This proves 2.\qed

Remark E.5.5.8. In Theorem E.5.5.7, we cannot generalize from complex tori to toroidal groups, nor remove the linear restriction.

Let $X$ be a toroidal group. Then $\text{Hom}_\mathbb{C}(X, \mathbb{C}^\ast) = 0$, hence (E.14) specializes to

$$0 \to \text{Ext}(X, \mathbb{Z}) \xrightarrow{i} \text{Ext}(X, \mathbb{C}) \to \text{Ext}(X, \mathbb{C}^\ast) \to 0.$$  \hspace{1cm} (E.29)

Note that $\text{Ext}(X, \mathbb{Z}) = H^1(X, \mathbb{Z})$ (Fact E.3.2.6), and by (E.13) the injection $i$ is the composition of the inclusion $H^1(X, \mathbb{Z}) \to H^1(X, \mathbb{C})$ with the projection $H^1(X, \mathbb{C}) \to \frac{H^1(X, \mathbb{C})}{\text{im}(H^1(X, \mathbb{Z}))}$.

When $X$ is compact, the sequence (E.29) lifts to an exact sequence in $C$ by Theorem E.5.5.7. As opposed to the compact case, when $X$ is not compact and consider the presentation (E.25), one has $1 \leq q < n$, so

$$\text{rank}_{\mathbb{Z}}\text{Ext}(X, \mathbb{Z}) = n + q > 2q = \text{dim}_\mathbb{R}\text{Ext}(X, \mathbb{C}).$$
Therefore, the image of \( i \) is not closed in the vector space \( \text{Ext}(X, \mathbb{C}) \) (a phenomenon seen in Example E.4.1.2). In particular, the sequence (E.29) has no lift to an exact sequence in \( \mathcal{C} \).

Let \( A, B \) be two complex tori, \( g = \dim A, g' = \dim B \) and reconsider (E.14):

\[
0 \to \text{Hom}_\mathbb{C}(A, B) \overset{j}{\to} \text{Ext}(A, \pi_1(B)) \to \text{Ext}(A, \tilde{B}) \to \text{Ext}(A, B) \to 0.
\]

Here, \( \text{Ext}(A, \tilde{B}) \) is a \( \mathbb{C} \)-vector space of dimension \( gg' \) by Theorem E.5.3.2. Identifying \( \text{Ext}(A, \pi_1(B)) \) with \( \text{Hom}_\mathbb{A}(\pi_1(A), \pi_1(B)) \) via Fact E.3.2.6, \( j \) is the map \( \rho_r \) in [BL04, p.10]. The quotient \( \frac{\text{Ext}(A, \pi_1(B))}{\text{Hom}_\mathbb{C}(A, B)} \) is a free abelian group of rank \( 4gg' - \text{rank}_2 \text{Hom}_\mathbb{C}(A, B) \). As long as \( \text{rank}_2 \text{Hom}_\mathbb{C}(A, B) < 2gg' \) (say, when \( A = B \) is an elliptic curve without complex multiplication), then \( Z = \text{Hom}_\mathbb{C}(A, B) \), the image of the induced injection \( \frac{\text{Ext}(A, \pi_1(B))}{\text{Hom}_\mathbb{C}(A, B)} \to \text{Ext}(A, \tilde{B}) \) is not closed. In particular, \( \text{Ext}(A, B) \) has no structure of complex Lie group making this sequence exact in \( \mathcal{C} \).

### E.6 Extensions of complex tori are often commutative

In Section E.6, we prove that under suitable hypotheses, an extension of a complex torus is commutative.

**Proposition E.6.0.1.** If \( 1 \to B \to C \to A \to 1 \) is a central extension of complex Lie groups, where \( A \) is a toroidal group, then \( C \) is commutative. Or equivalently, for every \( B \in \mathcal{C} \), the natural injection \( \text{Ext}(A, B) \to \text{Ext}(A, B, 1) \) is an isomorphism.

**Proof.** Consider the holomorphic map \( A \times A \to B \) given by (E.4). By [NW13, Thm. 5.1.36], it is a group morphism, so constant. Thus, \( C \) is commutative. \( \square \)

An algebraic analogue of Proposition E.6.0.2 is [Wu86, Cor. 2, p.370].

**Proposition E.6.0.2.** Let \( 1 \to K \to E \to A \to 1 \) be an extension of complex Lie groups, where \( A \) is a complex torus.

1. If \( Z(K)_0 \) is Stein, then \( Z(K) = Z(E) \cap K \).

2. If \( K \) is commutative and \( K_0 \) is Stein, then \( E \) is commutative.

**Proof.**

1. Since \( Z(E) \cap K \subset Z(K) \), it suffices to prove that \( Z(K) \subset Z(E) \). Consider the group morphism (E.5): \( \theta : A \to \text{Aut}(Z(K)) \). For every \( x \in Z(K) \), the map

\[
\phi : A \to Z(K), \quad a \mapsto \theta_a(x)x^{-1}
\]

is continuous. Moreover, \( \phi(0) = e_K \). By the connectedness of \( A \), \( \phi(A) \subset Z(K)_0 \). As \( Z(K)_0 \) is Stein and \( A \) is compact, \( \phi(A) \) is the singleton \( \{e_K\} \). Therefore, \( \theta_a(x) = x \) for every \( x \in Z(K) \), which proves \( Z(K) \subset Z(E) \).
2. By 1, \( K \subset Z(E) \). By Proposition E.6.0.1, \( E \) is commutative. \( \square \)

In Proposition E.6.0.3, when \( B \) is isomorphic to \( \mathbb{C}^n \) for some integer \( n \geq 0 \) or to \( \mathbb{C}^* \), we recover [BZ21, Lem. 2.10].

**Proposition E.6.0.3.** Let \( 1 \to B \to C \xrightarrow{p} A \to 1 \) be an exact sequence of complex Lie groups, where \( A \) is a complex torus and \( B \) is commutative. If the group \( B/B_0 \) is torsion (i.e., every element of \( B/B_0 \) has finite order), then \( C \) is commutative.

**Proof.** Let \( Z \) be the center of \( C \). By Proposition E.6.0.1, it suffices to check \( B \subset Z \).

The outer action induces a morphism \( A \to \text{Aut}(B_0)(\leq \text{GL}(L(B))) \). It is trivial by the compactness of \( A \), i.e., \( B_0 \leq Z \). By Corollary E.4.1.13, one may assume \( B = B_0 \times D \), where \( D \) is a discrete subgroup of \( B \) isomorphic to \( B/B_0 \) and \( D \cap B_0 = \{e_B\} \). Let \( q : B \to D \) and \( r : B \to B_0 \) be the corresponding projections.

It remains to show that \( 0 \times D(\leq B) \) is contained in \( Z \). Fix \( d \in D \) and put \( b = (0, d) \in B \). The map

\[
\nu : C \to C, \quad c \mapsto cbc^{-1}
\]

is holomorphic and \( \nu(e) = b \). For every \( b' \in B \), one has

\[
\nu(cb') = cb'bb'^{-1}c^{-1} = cbc^{-1} = \nu(c).
\]

The right multiplication action of \( B \) on the complex manifold \( C \) has quotient \( A \) by Fact E.2.0.3, so \( \nu \) factors through a morphism \( u : A \to B \) of complex manifolds. Then \( qu : A \to D \) is continuous. Since \( A \) is connected, \( qu \) is constant. Since \( qu(e_A) = d \), one gets \( qu \equiv d \).

On the other hand, the map \( ru : A \to B_0 \) is holomorphic. By assumption, there is an integer \( n \geq 1 \) (depending on \( d \)) such that \( d^n = e_D \) in \( D \). Thus, \( b^n = e_B \). For every \( c \in C \), one has \( \nu(c)^n = (cbc^{-1})^n = cbb'^{-1}c^{-1} = \nu(c) \).

Therefore, \( ru(A) \) is contained in the torsion subgroup \( B_{0,\text{tor}} \) of \( B_0 \). In view of [AK01, Prop. 1.1.2], \( B_{0,\text{tor}} \) is totally disconnected. Since \( A \) is connected, \( ru \) is constant.

Since \( ru(e_A) = 0 \), one has \( ru \equiv 0 \). Therefore, \( u \equiv b \), i.e., \( b \in Z \). Therefore, \( 0 \times D \subset Z \) and the proof is completed. \( \square \)

Corollary E.6.0.4 follows immediately from Proposition E.6.0.3.

**Corollary E.6.0.4.** Given an extension

\[
0 \to (\mathbb{C}^*)^n \to G \to A \to 0 \tag{E.30}
\]

of complex Lie groups, where \( A \) is a complex torus and \( n(\geq 1) \) is an integer, then \( G \) is a semi-torus.
Corollary E.6.0.5. In Corollary E.6.0.4, if A is algebraic, then G admits a unique structure of semiabelian variety such that (E.30) defines a commutative extension of algebraic groups.

Proof. From Corollary E.6.0.4, (E.30) defines an element of $\text{Ext}(A^n, (A^*)^n)$. By [Ser12, Thm. 6, p.184] and Theorem E.5.2.4, the natural map $\text{Ext}_{\text{Alg}}(A, G_{m}^n) \to \text{Ext}(A^n, (A^*)^n)$ is identified with the analytification map $[\text{Pic}^0(A)]^n \to [\text{Pic}^0(A^n)]^n$, hence a group isomorphism. In particular, there is a unique exact sequence $0 \to G_{m}^n \to C \to A \to 0$ in Alg whose analytification is equivalent to (E.30). □

Lemma E.6.0.6 is used in the proof of Proposition E.6.0.7.

Lemma E.6.0.6. Let $G$ be a real Lie group with Lie algebra $g$.

1. If $X, Y \in g$ are such that $[X, [X, Y]] = 0$ and $[Y, [X, Y]] = 0$, then
   \[ \exp(X) \exp(Y) \exp(-X) \exp(-Y) = \exp([X, Y]). \tag{E.31} \]

2. If $X \in g$ satisfies that $\exp(X)$ commutes with every element of $G_0$ and $[X, g] \subset Z(g)$, then $X \in Z(g)$.

Proof.

1. According to Baker-Campbell-Hausdorff formula (see, e.g., [Far08, Cor. 3.4.5]), there is a symmetric open neighborhood $U$ of $0 \in g$ such that for every $A, B \in U$, $\exp(A) \exp(B) = \exp(Z)$, where
   \[ Z = Z(A, B) = A + B + [A, B]/2 + \ldots \]
   and "..." indicates terms involving higher commutators of $A$ and $B$. There is a symmetric open neighborhood $V$ of $0 \in U$ such that $Z(A, B) \in U$ for every $A, B \in V$.
   Define $f : \mathbb{R} \to G$ by
   \[ f(t) = \exp(tX) \exp(tY) \exp(-tX) \exp(-tY) \exp(-t^2[X, Y]). \]
   Then $f$ is real analytic. There is $\epsilon > 0$ such that $tX, tY \in V$ for all $t \in (-\epsilon, \epsilon)$. By assumption, $[Z(tX, tY), Z(-tX, -tY)] = 0$ and $Z(tX, tY) + Z(-tX, -tY) = t^2[X, Y]$. Then
   \[ f(t) = \exp(Z(tX, tY)) \exp(Z(-tX, -tY)) \exp(-t^2[X, Y]) = e_G \]
   for all $t \in (-\epsilon, \epsilon)$ (see [Laz54, p.144]). By [ADGK23, Cor. A.5], $f(1) = e_G$.

2. Let $D = \exp^{-1}(e_G)$. There is an open neighborhood $W$ of $0 \in g$ such that $\exp(W)$ is open in $G$ and $\exp : W \to \exp(W)$ is a diffeomorphism. Then $D \cap W = \{0\}$. For every $Y \in g$, there is $k > 0$ with $[X, Y/k] \in W$. By assumption, $[X, Y/k] \in Z(g)$, so $[X, [X, Y/k]] = 0$ and $[Y/k, [X, Y/k]] = 0$. Since $\exp(Y/k) \in G_0$, it commutes with $\exp(X)$. By 1, $\exp([X, Y/k]) = e_G$. Then $[X, Y/k] \in D \cap W$. Therefore, $[X, Y] = 0$. Thus, $X \in Z(g)$. 

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E.6. EXTENSIONS OF COMPLEX TORI ARE OFTEN COMMUTATIVE

An algebraic analogue of Proposition E.6.0.7 is [Ros56, Cor. 2, p.433].

**Proposition E.6.0.7.** Let \( 1 \to B \to C \xrightarrow{p} A \to 1 \) be an exact sequence of complex Lie groups, with \( A \) complex torus and \( B \) commutative. Then \( C_0 \) is commutative.

**Proof.** We may assume that \( C \) is connected by replacing \( C \) (resp. \( B \)) with \( C_0 \) (resp. \( B \cap C_0 \)). Let \( \omega : C^\text{u} \to A \) be the universal covering of \( A \). Denote by \( b \) (resp. \( c \)) the Lie algebra of \( B \) (resp. \( C \)). Let \( \eta : A \to \text{Aut}(B) \) be the outer action. Then \( \eta \) induces a holomorphic morphism \( \eta_0 : A \to \text{Aut}(B_0) \). Because \( \text{Aut}(B_0) \) is complex Lie subgroup of \( \text{GL}(b) \), \( \eta_0 \) is trivial.

Consider the pullback extension along \( \omega \):

\[
\begin{array}{ccccccccc}
0 & \to & \ker(\epsilon) & \xrightarrow{\pi|_{\ker(\epsilon)}} & \ker(\omega) & & \\
\downarrow & & \downarrow & & \downarrow & & \\
1 & \to & B & \to & E & \xrightarrow{\pi} & C^\text{g} & \to & 1 \\
\downarrow{\text{id}} & & {\epsilon} & & \downarrow{\omega} & & \\
1 & \to & B & \to & C & \xrightarrow{p} & A & \to & 1 \\
0 & \to & 0 & & & & \\
\end{array}
\]

By the snake lemma, \( \epsilon \) is surjective and \( \pi \) restricts to an isomorphism \( \ker(\pi) \to \ker(\omega) \). In particular, \( d_\epsilon : L(E) \to L(C) \) is an isomorphism. By Fact E.2.0.3, the morphism \( \epsilon \) is open. Since \( E_0 \) is open in \( E \), \( \epsilon(E_0) \) is an open subgroup of \( C \). By connectedness of \( C \), \( \epsilon(E_0) = C \). Similarly, \( \pi(E_0) = C^\text{g} \). By Fact E.7.2.7 below, \( B \cap E_0 \) is connected. Therefore, \( B \cap E_0 \subset B_0 \). Since \( B_0 \subset B \cap E_0 \), one has \( B_0 = B \cap E_0 \). Hence an extension \( 1 \to B_0 \to E_0 \to C^\text{g} \to 1 \). The outer action is \( \eta_0 \omega : C^\text{g} \to \text{Aut}(B_0) \), so it is a central extension. Then

\[
0 \to b \to \varepsilon \to C^\text{g} \to 0 \quad (E.32)
\]

is a central extension of Lie algebras. In particular, \( b \subset Z(\varepsilon) \). We shall prove the extension (E.32) is trivial.

We show that \( \exp_E : \epsilon \to E_0 \) is surjective. Indeed, for every \( x \in E_0 \), there is \( v \in \varepsilon \) with \( d_\epsilon p(v) = \pi(x) \). Then \( \pi(\exp_E(v)) = \pi(x) \), so \( \pi(x \exp_E(-v)) = 0 \) and hence \( x \exp_E(-v) \in B_0 \). As \( B_0 \) is connected commutative, there is \( u \in b \) with \( \exp_B(u) = x \exp_E(-v) \). Since \( u \in Z(\varepsilon) \), one gets \( x = \exp_E(u) \exp_E(v) = \exp_E(u + v) \).

By Corollary E.4.1.13, there is a decomposition \( B = B_0 \times D \), where \( D \in \text{Ab}_c \) is discrete. The natural morphism \( E_0 \times D \to E_0 \to C^\text{g} \) is surjective of kernel \( B_0 \times D \), hence the first row of the diagram
As \((E.32)\) is trivial and 

By Lemma E.3.1.2, there is an equivalence of extensions \(\phi : E \to E_0 \times D\).

Fix \(x \in \ker(\epsilon)\), let \(\phi(x) = (\phi_1(x), \phi_2(x)) \in E_0 \times D\). For every \(y \in E_0\),

\[(y, 1)\phi(x)(y, 1)^{-1} = (y\phi_1(x)y^{-1}, \phi_2(x)) \in \phi(\ker(\epsilon)).\]

Hence, \(\phi^{-1}((y\phi_1(x)y^{-1}, \phi_2(x))) \in \ker(\epsilon)\). The map

\[E_0 \to \ker(\epsilon), \quad y \mapsto \phi^{-1}((y\phi_1(x)y^{-1}, \phi_2(x)))\]

is continuous. As \(E_0\) is connected and \(\ker(\epsilon)\) is discrete, this map is constantly \(x\). Thus, \(y\phi_1(x)y^{-1} = \phi_1(x)\). Therefore, \(\phi_1(x)\) commutes with every element of \(E_0\). As \(\exp_E : \mathfrak{c} \to E_0\) is surjective, there is \(X \in \mathfrak{c}\) with \(\exp_E(X) = \phi_1(x)\). Since \(\mathbb{C}^g\) is an abelian Lie algebra, \([\mathfrak{c}, \mathfrak{c}]\) is contained in the kernel of \(d_r p : \mathfrak{c} \to \mathbb{C}^g\), which is \(\mathfrak{b}\). Then \([\mathfrak{c}, \mathfrak{c}] \subset Z(\mathfrak{c})\), i.e., \([\mathfrak{c}, [\mathfrak{c}, \mathfrak{c}]] = 0\). By Lemma E.6.0.6.2, \(X \in Z(\mathfrak{c})\).

Consider the commutative diagram

\[
\begin{array}{ccc}
\mathfrak{c} & \xrightarrow{d_r p} & \mathbb{C}^g \\
\downarrow{\exp_E} & & \downarrow{{\Id}} \\
E & \xrightarrow{\pi} & \mathbb{C}^g \\
\end{array}
\]

Then \(\pi(x) = \pi(\phi_1(x)) = d_r p(X) \in d_r p(Z(\mathfrak{c}))\). Therefore, \(\ker(\omega) = \pi(\ker(\epsilon)) \subset d_r p(Z(\mathfrak{c}))\). Since \(d_r p\) is \(\mathbb{C}\)-linear and \(\ker(\omega)\) contains a \(\mathbb{C}\)-basis of \(\mathbb{C}^g\), one has \(d_r p(Z(\mathfrak{c})) = \mathbb{C}^g\). Consequently, there is a \(\mathbb{C}\)-linear map \(s : \mathbb{C}^g \to Z(\mathfrak{c})\) with \(d_r p \circ s = \Id_{\mathbb{C}^g}\). As \(s : \mathbb{C}^g \to \mathfrak{c}\) is a Lie algebra morphism, the central extension \((E.32)\) is trivial and \(\mathfrak{c}\) is the direct sum of \(\mathfrak{b}\) and \(\mathbb{C}^g\). In particular, \(\mathfrak{c}\) is abelian. As \(C\) is connected and its Lie algebra is abelian, \(C\) is commutative.

Example E.6.0.8 shows that the the condition that \(B/B_0\) is torsion (resp. \(K_0\) is Stein) in Proposition E.6.0.3 (resp. Proposition E.6.0.2) is necessary. Moreover, in Proposition E.6.0.7, the commutativity of \(C\) fails in general.

**Example E.6.0.8.** Let \(A\) be a complex torus and \(B = A \times \mathbb{Z}\) be the product group. Consider the complex manifold morphism \(A \times B \to B\) defined by \((a, a', k) \mapsto (a' + ka, k)\). It is a non trivial group action of \(A\) on \(B\). Let \(C\) be the corresponding semidirect product (see [Bou72, Ch.III, no. 4, Prop. 7]), then the resulting complex Lie group extension \(1 \to B \to C \to A \to 1\) is not central.
E.7 Noncommutative extensions

E.7.1 Lifted extensions

The real Lie group extension problem is studied by G. Hochschild in [Hoc51a] and [Hoc51b]. As Example E.7.1.1 shows, the case of real Lie groups is different from the case of complex Lie groups.

Example E.7.1.1. Let \( G = \mathbb{C} \). The morphism of real Lie groups \( \rho : \mathbb{C} \to \mathbb{C}^* = \text{Aut}(G) \) defined by \( z \mapsto e^z \) is an action of \( G \) on itself which is real analytic but not holomorphic. Hence an exact sequence of real Lie groups \( 1 \to G \to G \rtimes_{\rho} G \to G \to 1 \) by [Bou72, Ch. III, no. 4, Prop. 7]. However, the middle term has no structure of complex Lie group making the maps holomorphic. Therefore, [Iwa49, Theorem 7] fails for complex Lie groups. Besides, this shows that the real Lie group extension problem and the complex one are different.

In Section E.7, we review Hochschild's work, but in the context of complex Lie groups. References to the original statement are given when the proofs are similar modulo slight modifications. All results in the sequel are essentially known.

In Section E.7.1, the goal is to derive Corollary E.7.1.6, a result about the extensions of a commutative group by a connected group.

Let \( L \) be a complex Lie group and \( K \subset \mathbb{C} \). For a fixed holomorphic group action \( L \times K \to K \), let \( \phi : L \to \text{Aut}(K) \) denote the induced group morphism. Let \( Z(L, K, \phi) \) denote the set of crossed morphisms, i.e., morphisms \( \rho : L \to K \) of complex manifolds such that \( \rho(l_1l_2) = \rho(l_1)\phi_l(\rho(l_2)) \) for all \( l_1, l_2 \in L \). Then \( Z(L, K, \phi) \) is an abelian group under addition. (When \( \phi \) is trivial, \( Z(L, K, \phi) = \text{Hom}(L, K) \).)

For a normal complex Lie subgroup \( H \) of \( L \), define

\[ \text{Ophom}_L(H, K, \phi) = \{ \psi \in \text{Hom}(H, K) : \psi(lhl^{-1}) = \phi_l(\psi(h)), \forall l \in L, h \in H \} \]

Then \( \text{Ophom}_L(H, K, \phi) \) is a subgroup of \( \text{Hom}(H, K) \). When \( H \subset Z(L) \), one has

\[ \text{Ophom}_L(H, K, \phi) = \text{Hom}_C(H, K^{\phi(L)}), \quad (E.33) \]

where \( K^{\phi(L)} = \bigcap_{l \in L} \{ x \in K : \phi_l(x) = x \} \) is the set of elements fixed by \( \phi(L) \) (in \( \text{Aut}(K) \)). Here \( K^{\phi(L)} \) is indeed a complex Lie subgroup of \( K \) by Corollary E.2.0.5. When \( \phi \) is trivial, \( \text{Ophom}_L(H, K, \phi) \) is the set of morphisms \( H \to K \) invariant under the conjugation action of \( L \).

Proposition E.7.1.2. Assume that \( H \) is a normal complex Lie subgroup of \( L \) contained in \( \ker(\phi) \). For every \( \rho \in Z(L, K, \phi) \), \( \rho|_H \in \text{Ophom}_L(H, K, \phi) \), hence a group morphism \( Z(L, K, \phi) \to \text{Ophom}_L(H, K, \phi) \), whose image is denoted by \( Z_H(L, K, \phi) \).

Proof. For every \( h, h' \in H \), \( \rho(hh') = \rho(h)\phi_h(\rho(h')) = \rho(h)\rho(h') \) since \( h \in \ker(\phi) \). Thus \( \rho|_H \in \text{Hom}(H, K) \). In particular, \( \rho(e_L) = e_K \). For every \( l \in L \),

\[ e_K = \rho(e_L) = \rho(ll^{-1}) = \rho(l)\phi_l(\rho(l^{-1})) \]
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so \(\rho(l)^{-1} = \phi_l(\rho(l^{-1}))\). Then

\[
\rho(lhl^{-1}) = \rho(lh)\phi_l(\rho(l^{-1})) = \rho(lh)\rho(l)^{-1} = \rho(l)\phi_l(\rho(h))\rho(l)^{-1} = \phi_l(\rho(h)).
\]

The last equality uses the commutativity of \(K\). Therefore, \(\rho|_H \in \text{Ophom}_L(H, K, \phi)\).

Let \(\omega : Q' \to Q\) be a surjective morphism of connected complex Lie groups with kernel \(F\). Let \(\eta : Q \to \text{Aut}(K)\) be a group morphism such that the induced group action \(Q \times K \to K\) is holomorphic. As \(K\) is commutative, the pulling back map \(\omega^* : \text{Ext}(Q, K, \eta) \to \text{Ext}(Q', K, \eta\omega)\) is a group morphism. Fact E.7.1.3 gives a description of \(\ker(\omega^*)\).

Define a map \(\sigma : \text{Ophom}_{Q'}(F, K, \eta\omega) \to \text{Ext}(Q, K, \eta\omega)\) as follows. As the group action defined by \(\eta\) is holomorphic, the semidirect complex Lie group \(K \rtimes_{\eta\omega} Q'\) exists by [Bou72, Ch.III, no.4, Prop. 7]. For \(\psi \in \text{Ophom}_{Q'}(F, K, \eta\omega)\), the morphism \(F \to K \rtimes_{\eta\omega} Q'\) defined by \(k \mapsto (\psi(k), k)\) identifies \(F\) as a normal complex Lie subgroup of \(K \rtimes_{\eta\omega} Q'\). Let \(E = K \rtimes_{\eta\omega} Q'/F\). The projection \(K \rtimes_{\eta\omega} Q' \to Q'\) descends to a morphism \(E \to Q\). The injection \(K \to K \rtimes_{\eta\omega} Q'\) induces a morphism \(K \to E\). Then the resulting sequence \(1 \to K \to E \to Q \to 1\) is exact with outer action \(\eta\omega\), whose equivalence class is denoted by \(\sigma(\psi)\).

**Fact E.7.1.3.** [Hoc51a, Thm. 1.1] The map \(\sigma\) is a group morphism and the sequence

\[
Z(Q', K, \eta\omega) \to \text{Ophom}_{Q'}(F, K, \eta\omega) \xrightarrow{\circ} \text{Ext}(Q, K, \eta) \xrightarrow{\omega^*} \text{Ext}(Q', K, \eta\omega)
\]

is exact.

The use of Fact E.7.1.3 is based on the existence of \(\omega : Q' \to Q\) such that every extension in \(\text{Ext}(Q, K, \eta)\) becomes a semidirect product when pulled back to \(\text{Ext}(Q', K, \eta\omega)\) along \(\omega\).

**Fact E.7.1.4.** [Hoc51a, Thm. 2.1] Let \(Q\) be a connected complex Lie group. Assume that \(\eta : Q \to \text{Aut}(K)\) is a group morphism such that the induced group action is holomorphic. Then there exists a simply connected complex Lie group \(Q'\) and a surjective morphism \(\omega : Q' \to Q\) such that the pullback morphism \(\omega^* : \text{Ext}(Q, K, \eta) \to \text{Ext}(Q', K, \eta\omega)\) is zero.

**Remark E.7.1.5.** The connectedness condition of the extension kernel in [Hoc51a, Theorems 1.1 and 2.1] is in fact unnecessary.

Corollary E.7.1.6 follows from Fact E.7.1.3 and Fact E.7.1.4.

**Corollary E.7.1.6** ([Hoc51a, Corollary 2.1]). In the notation of Fact E.7.1.4,

\[
\text{Ext}(Q, K, \eta) = \text{Ophom}_{Q'}(F, K, \eta\omega)/Z_F(Q', K, \eta\omega),
\]

where \(F = \ker(\omega)\).
Example E.7.1.7. Let $Q = \mathbb{C}^*$, $L = \mathbb{C}$ and $\omega : L \to Q$ be defined by $\omega(z) = e^{2\pi iz}$. Then $F = \ker(\omega) = \mathbb{Z}$. Let $\mathbb{C}^* \times K \to K$ be a holomorphic group action and $\eta : \mathbb{C}^* \to \text{Aut}(K)$ be the induced group morphism. Then $\text{Ophom}_L(F, K, \eta\omega) = \text{Hom}(\mathbb{Z}, K^{\eta(\mathbb{C}^*)}) = K^{\eta(\mathbb{C}^*)}$. By Proposition E.3.2.2 and Corollary E.7.1.6, one has $\text{Ext}(\mathbb{C}^*K, \eta) = K^{\eta(\mathbb{C}^*)}/\mathbb{Z}(\mathbb{C}, K, \eta\omega)$.

E.7.2 Factor systems

It is well-known that extensions of abstract groups can be classified in terms of factor systems, see [CE99, Ch. XIV, Sec. 4]. This description relies on the existence of set-theoretical cross sections. In general, nevertheless, it is not possible to find a continuous cross section to a surjective morphism of topological groups.

Consider the extension (E.3) of complex Lie groups with outer action $\psi : Q \to \text{Out}(K)$.

Example E.7.2.1. Assume that there is a cross section to (E.3), i.e., a morphism $s : Q \to E$ of complex manifolds with $ps = \text{Id}_Q$. Replacing $s$ by $s(e_Q)^{-1}s$ when necessary, one may assume that $s$ is normalized as $s(e_Q) = e_E$. Define

$$f : Q \times Q \to E, \quad f(g, h) = s(g)s(h)s(gh)^{-1}.$$  

Then $f$ is holomorphic. Since $p(f(g, h)) = e_Q$, $f(g, h) \in K$, so $f$ factors through $K$. The map $f$ measures the failure of $s$ to be a morphism. If $E$ is commutative, then additionally $f$ is symmetric in the sense of [Ser12, (16), p.166]:

$$f(x, y) = f(y, x) \quad \forall x, y \in Q. \quad (E.34)$$

Define $\phi : Q \to \text{Aut}(K)$ by $\phi_g = \text{Inn}_{s(g)}|_K$. Then $\phi$ is a map (but not necessarily a group morphism) lifting $\psi$, and the induced map

$$Q \times K \to K, \quad (g, x) \mapsto \phi_g(x) \quad (E.35)$$

is holomorphic. When $K$ is commutative, $\phi = \psi$ is a group morphism independent of the choice of $s$. When (E.3) is a central extension, $\phi$ is constantly $\text{Id}_K$.

Moreover, $f$ and $\phi$ satisfy the following relations:

$$f(e_Q, h) = f(g, e_Q) = e_K;$$

$$\phi_e = \text{Id}_K;$$

$$\phi_g\phi_h = \text{Inn}_{f(g,h)}\phi_{gh};$$

$$f(g, h)f(gh, k) = \phi_g(f(h, k))f(g, hk). \quad (E.36)$$

Example E.7.2.1 motivates Definition E.7.2.2.

Definition E.7.2.2 (Factor system). If a morphism $f : Q \times Q \to K$ of complex manifolds and a map $\phi : Q \to \text{Aut}(K)$ making (E.35) holomorphic satisfy the relations (E.36), then $f$ is called a $\phi$-factor system (and simply a factor system when $\phi$ is trivial, in which case the last relation in (E.36) is $f(g, h)f(gh, k) = f(h, k)f(g, hk)$). A factor system $f$ is called symmetric if (E.34) holds.
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When $K$ is commutative, the set of $\phi$-factor systems is an abelian group under addition.

We examine how the $\phi$-factor system $f$ induced by $s$ in Example E.7.2.1 depends on the choice of the cross section $s$.

**Example E.7.2.3.** Let $s' : Q \to E$ be another normalized cross section still inducing $\phi$. Define

$$g : Q \to E, \quad g(x) = s(x)^{-1}s'(x).$$

Then $g(e_Q) = e_E$ as $s, s'$ are normalized and $g$ is holomorphic. For every $x \in Q$, $p(g(x)) = e_Q$, so $g(x) \in K$. For every $k \in K$, $\text{Inn}_s(x)k = \phi_s(k) = \text{Inn}_{s'}(x)k$, so $g(x) \in Z(K)$, i.e., $g$ factors through $Z(K)$. Then $s' = 1$ if $s = g(x)$. Let $f'$ be the factor system induced by $s'$. Then

$$f'(x, y) = s'(x)s'(y)s'(xy)^{-1}$$
$$= s(x)g(x)s(y)g(y)[s(xy)]^{-1}$$
$$= \phi_x(g(x))s(x)s(y)g(y)g(xy)\phi_y(g(xy))^{-1}$$
$$= \phi_x(g(x))f(x, y)s(xy)g(y)g(xy)\phi_y(g(xy))^{-1}$$
$$= \phi_x(g(x))f(x, y)\phi_y(g(xy))^{-1}$$
$$= g^\phi(x, y)f(x, y),$$

where $g^\phi : Q \times Q \to K$ is a morphism of complex manifolds defined by

$$g^\phi(x, y) = \phi_x(g(x))\phi_y(g(xy))^{-1}. \quad (E.37)$$

When (E.3) is a central extension, $\phi$ is trivial, then (E.37) reduces to [Ser12, (15), p.166]: $g^\phi(x, y) = g(x)y(y)g(xy)^{-1}$.

Example E.7.2.3 motivates Definition E.7.2.4.

**Definition E.7.2.4.** Let $f, f'$ be two $\phi$-factors systems. If there is a holomorphic map $g : Q \to Z(K)$ with $g(e_Q) = e_E$ such that $f' = g^\phi f$ with $g^\phi$ defined by (E.37), then $f$ and $f'$ are called $\phi$-equivalent, denoted by $f \sim_\phi f'$.

In Definition E.7.2.4, $\sim_\phi$ is an equivalent relation on the set of $\phi$-factor systems. When $K$ is commutative, inside the group of all $\phi$-factor systems, the elements $\phi$-equivalent to the zero form a subgroup. A result similar to Proposition E.7.2.5 for algebraic groups is in [Ser12, Ch. VII, Sec. 1, no.4].

**Proposition E.7.2.5.** Let $K, Q$ be complex Lie groups with a map $\phi : Q \to \text{Aut}(K)$ such that (E.35) is holomorphic and the induced map $\psi : Q \to \text{Out}(K)$ is a group morphism. Then:

1. The set $\mathcal{F}$ of $\sim_\phi$-equivalence classes of $\phi$-factor systems is canonically identified with the subset $\mathcal{E} \subset \text{Ext}(Q, K, \psi)$ of equivalence classes of extensions of $Q$ by $K$ which admit at least one normalized cross section inducing $\phi$.
2. When $K$ is commutative, the identification in 1 is a group isomorphism.

3. If further $Q$ is also commutative and $\phi = \psi = 1$ is trivial, then the subgroup of equivalence classes of symmetric factor systems corresponds to the subgroup of equivalence classes of commutative extensions.

Proof. We only prove 1. Examples E.7.2.1 and E.7.2.3 construct a map $\Phi : \mathcal{E} \to \mathcal{F}$. (Note that equivalent extensions induces the same $\phi$-equivalence class.)

Conversely, we define a map $\Psi : \mathcal{F} \to \mathcal{E}$ by the following construction. Given a $\phi$-factor system $f$, one can construct an exact sequence $1 \to K \to E_{f,\phi} \to Q \to 1$ of complex Lie groups with a (holomorphic) normalized cross section $s : Q \to E_{f,\phi}$ as follows. Let $E_{f,\phi} = K \times Q$ as a complex manifold. Define a map

$$g : E_{f,\phi} \times E_{f,\phi} \to E_{f,\phi}, \quad g((k, x), (l, y)) = (k\phi_x(l)f(x, y), xy).$$

As $f$ and the map (E.35) are holomorphic, so is $g$. Moreover, (E.36) shows $g$ defines an associative multiplication. The pair $(1, 1) \in E_{f,\phi}$ is the identity, and the inverse of $(k, x)$ is

$$\left(\phi^{-1}_x[k^{-1}f(x, x^{-1})^{-1}], x^{-1}\right).$$

Hence $(E_{f,\phi}, g)$ is a complex Lie group. The projection $p : E_{f,\phi} \to Q$ is a surjective morphism. The map $i : K \to E_{f,\phi}$ by $k \mapsto (k, 1)$ is the kernel of $p$. Moreover, define $s : Q \to E_{f,\phi}$ by $s(g) = (1, g)$, then $s$ is normalized cross section. Put $\Psi(f) = E_{f,\phi}$.

We check that $\Psi \Phi = \text{Id}_\mathcal{E}$. Indeed, the map $E_{f,\phi} \to E$ defined by $(k, x) \mapsto ks(x)$ is an equivalence of extensions. We check that $\Phi \Psi = \text{Id}_\mathcal{F}$, or equivalently $s$ induces $f$ and $\phi$. In fact, for every $x \in Q$, $k \in K$, one has

$$\phi_x(k)s(x) = (\phi_x(k), 1)(1, x) = (\phi_x(k), x) = (1, x)(k, 1) = s(x)k;$$

so $\phi_x = \text{Inn}_{s(x)}|_K$, i.e., $s$ induces $\phi$. For every $y \in Q$,

$$s(x)s(y)s(xy)^{-1} = (1, x)(1, y)(1, xy)^{-1} = (f(x, y), xy)(\phi^{-1}_{xy}[f(xy, y^{-1}x^{-1})^{-1}, y^{-1}x^{-1}]) = (f(x, y)\phi_{xy}[f(xy, y^{-1}x^{-1})^{-1}, y^{-1}x^{-1}]) = (f(x, y), 1).$$

Therefore, $s$ induces $f$. \hfill \Box

When the base $Q$ of (E.3) is discrete, then a set-theoretic cross section is automatically holomorphic.

Corollary E.7.2.6. Let $Q$ be a discrete complex Lie groups, and let $\eta : Q \to \text{Aut}(K)$ be a group morphism. Then the group $\text{Ext}(Q, K, \eta)$ is isomorphic to the group of $\sim_\eta$-equivalence classes of $\eta$-factor systems. Furthermore, if $Q$ is also commutative, then $\text{Ext}(Q, K)$ is isomorphic to the group of $\sim$-equivalence classes of symmetric factor systems.
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Proof. Since $Q$ is discrete, the group action $Q \times K \to K$ induced by $\eta$ is holomorphic. The first (resp. second) half follows from Proposition E.7.2.5 2 (resp. 3). \hfill \qed

Another important case where a cross section exists is with simply connected bases. For this, we need a holomorphic version of Malcev’s theorem ([Ma42, (E), p.12], [Hoc51a, Lemma 3.1], [Mac60, Theorem 3.2]).

Fact E.7.2.7 (Malcev, [Bou72, Ch. III, sec 6, no.6, Prop. 14; Cor. 2]). Let $L$ be a connected complex Lie group, $N$ be a normal immersed complex Lie subgroup of $L$.

1. If $N$ is closed in $L$ and $L/N$ is simply connected, then $N$ is connected.

2. If $L$ is simply connected, $N$ is connected, then $N$ is closed in $L$ and there exists a biholomorphic map $f : L \to N \times L/N$ making a commutative diagram

$$
\begin{array}{ccc}
L & \xrightarrow{f} & N \times L/N \\
\downarrow{q} & & \downarrow{p_2} \\
L/N,
\end{array}
$$

where $p_2$ is the projection to the second factor and $q : L \to L/N$ is the quotient morphism.

In the same way that [Hoc51a, Theorem 3.1] follows from [Hoc51a, Lemma 3.1], Fact E.7.2.8 can be deduced from Fact E.7.2.7.

Fact E.7.2.8. Let (E.3) be an exact sequence of complex Lie groups, where $E$ is connected and $Q$ is simply connected. Then there exists a cross section, i.e., a holomorphic map $s : Q \to E$ with $ps = \Id_Q$. In particular, the principal $K$-bundle $p : E \to Q$ is trivial.

Example E.7.2.9. Let $A$ be a complex elliptic curve. Take a nonzero element of $A^\vee$, which induces a nontrivial extension $E$ of $A$ by $\mathbb{C}^*$ via Theorem E.5.2.4. By Proposition E.5.1.3, the principal $\mathbb{C}^*$-bundle $E \to A$ is nontrivial. Therefore, Fact E.7.2.8 fails if the base is not simply connected.

Corollary E.7.2.10 follows immediately from Fact E.7.2.8 and Proposition E.7.2.5.

Corollary E.7.2.10. Let $K, Q$ be complex Lie groups, where $K$ is connected commutative and $Q$ is simply connected. Let $\eta : Q \to \Aut(K)$ be a complex Lie group morphism\textsuperscript{11}. Then $\Ext(Q, K, \eta)$ is isomorphic to the group of $\sim_\eta$-equivalence classes of $\eta$-factor systems.

\textsuperscript{11}Here $\Aut(K)$ is a complex Lie subgroup of $\GL(L(K))$ by [Lee01, Propositions 1.26 and 1.27].
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Similar to [Hoc51a, Theorem 3.2], Fact E.7.2.11 can be proved using Fact E.7.2.7 and Fact E.7.2.8.

**Fact E.7.2.11.** Let $K, Q$ be complex Lie groups, where $K$ is connected and $Q$ is simply connected. Then the map (on the set of equivalence classes) which associates with each extension of $Q$ by $K$ the induced extension of $L(Q)$ by $L(K)$ is injective. The image is the set of classes of those extensions $0 \rightarrow L(K) \rightarrow \mathcal{E} \rightarrow L(Q) \rightarrow 0$ in which the derivation $[x, \bullet]_{|L(K)} \in \text{Der}(L(K)) = L(\text{Aut}(L(K)))$ belongs to $L(\text{Aut}(K))$ for every $x \in \mathcal{E}$. Furthermore, if $K$ is commutative and $\eta : Q \rightarrow \text{Aut}(K)$ is a morphism, then the resulting map

$$\text{Ext}(Q, K, \eta) \rightarrow \text{Ext}(L(Q), L(K), d\eta)$$

is a group isomorphism.

A connected Lie group is called semisimple if its Lie algebra is semisimple. Analogue of Fact E.7.2.12 for semisimple real Lie groups $H$ and real vector groups $G$ is contained in the proof of [Hoc51b, Theorem 5.1]. Fact E.7.2.12 can be proved in a similar way.

**Fact E.7.2.12.** Let $G, H$ be connected complex Lie groups, where $G$ is commutative and $H$ is semisimple. Let $\eta : H \rightarrow \text{Aut}(G)$ be a morphism of complex Lie groups. If $\phi \in Z(H, G, \eta)$ is a crossed morphism, then there exists $g \in G$ such that $\phi(x) = \eta_x(g)^{-1}g$ for all $x \in H$. In particular, $\phi \equiv e_G$ on $\ker(\eta)$.

Theorem E.7.2.13 is a complex version of [Hoc51a, Theorem 4.4].

**Theorem E.7.2.13.** In Fact E.7.2.12, $\text{Ext}(H, G, \eta)$ is canonically isomorphic to $\text{Hom}_{\text{Ab}}(\pi_1(H), G^{n(H)})$.

**Proof.** Let $\omega : \tilde{H} \rightarrow H$ be the universal covering of $H$. Then $\ker(\omega) = \pi_1(H)$ is a discrete subgroup of $H$. By Fact E.3.2.4, $\pi_1(H) \subset Z(\tilde{H})$. Then (E.33) gives

$$\text{Ophom}_{\tilde{H}}(\ker(\omega), G, \eta) = \text{Hom}(\pi_1(H), G^{n(H)}).$$

By Fact E.7.2.12, for every $\rho \in Z(\tilde{H}, G, \eta)$, $\rho|_{\ker(\omega)} = 1$, i.e., $Z_{\ker(\omega)}(\tilde{H}, G, \eta) = 0$. By Fact E.7.2.11, the natural map $\text{Ext}(H, G, \eta) \rightarrow \text{Ext}(L(H), L(G), d\eta)$ is a group isomorphism. Since $L(H)$ is a semisimple complex Lie algebra, Levi’s theorem [Ser64, Theorem 4.1, p.48] affirms that $\text{Ext}(L(H), L(G), d\eta) = 0$. By Fact E.7.1.3, $\text{Ext}(H, G, \eta) = \text{Hom}(\pi_1(H), G^{n(H)})$. \hfill \Box

### E.7.3 Non-abelian kernels and extensions of the center

For two complex Lie groups $K, Q$ and a group morphism $\theta : Q \rightarrow \text{Out}(K)$, if $\theta$ is induced by some extension of $Q$ by $K$, then the extension kernel $(K, \theta)$ is called extendible. The problem to determine the extendibility of a given extension
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kernel is more difficult than that for abstract groups treated in [EM47, Theorem 8.1], because of the obstruction to the existence of a cross section. For extendible kernels, Corollary E.7.3.8 shows that the problem for extensions by \( K \) can be reduced to that with an abelian kernel, namely \( Z(K) \).

Let \( 1 \to K \to E \xrightarrow{p} Q \to 1 \) and \( 1 \to K' \to E' \xrightarrow{p'} Q \to 1 \) be two extension of complex Lie groups. Denote their outer action by \( \theta : Q \to \text{Out}(K) \) and \( \theta' : Q \to \text{Out}(K') \) respectively. Assume that \( Z(K) = Z(K') := C \) and \( \theta, \theta' \) induce a common center action \( \theta_0 : Q \to \text{Aut}(C) \). Hence a commutative diagram

\[
\begin{array}{cccccc}
\text{Out}(K) & \xrightarrow{\theta} & \text{Aut}(C) \\
Q & \xrightarrow{\theta_0} & \text{Out}(K') \\
& \xleftarrow{\theta'} & \\
\end{array}
\quad (E.38)
\]

We recall the multiplication of kernels defined in [EM47, Sec. 4]. The group law \( C \times C \to C \) is holomorphic, so the subset \( C^* := \{(x, x^{-1}) : x \in C\} \) is analytic in \( C \times C \). By Lemma E.2.0.6, \( C \times C \) is an analytic subset of \( K \times K' \). As \( C^* \) is a central subgroup of \( K \times K' \), it is also a complex Lie subgroup of \( K \times K' \) by Corollary E.2.0.5. Let \( K'' = K \times K'/C^* \). From [EM47, p.328], the morphism \( C \to K'' \) by \( g \to [g, 1] \) identifies \( C \) as the center of \( K'' \).

For every \( x \in Q \), select automorphisms \( \alpha \in \theta(x)(\subset \text{Aut}(K)) \) and \( \alpha' \in \theta'(x)(\subset \text{Aut}(K')) \). Because the diagram (E.38) is commutative, \( \alpha \times \alpha' \) is an automorphism of \( K \times K' \) sending \( C^* \) into itself. It thus determines an automorphism \( \alpha'' \) of \( K'' \). The class \( [\alpha''] \in \text{Out}(K'') \) depends only on \( \theta, \theta' \), but not the choices of \( \alpha, \alpha' \). Hence a group morphism

\[
\theta'' : Q \to \text{Out}(K'') \quad (E.40)
\]

that also induces \( \theta_0 : Q \to \text{Aut}(C) \).

Definition E.7.3.1. The pair \( (K'', \theta'') \) constructed above is called the \( C \)-product of the two given extension kernels \( (K, \theta) \) and \( (K', \theta') \).

Example E.7.3.2. If \( K' = C \) is commutative, it is asserted in [EM47, (4.4)] that \( K' \) acts as an identity for the \( C \)-product. To make it explicit, we define a surjective morphism \( \phi : K \times C \to K \) of complex manifolds by \( \phi(k, k') = k'k \). Then \( \phi \) is a morphism and \( C^* = \ker(\phi) \). Thus, \( \phi \) induces an isomorphism \( \sigma : K'' \to K \) satisfying [EM47, (4.2), (4.3)].

\[\text{see (E.5)}\]
Then we review the multiplication of the given two extensions, contained in the proof of [EM47, Lem. 5.1].

As the map $E \times E' \to Q$ by $(x, x') \mapsto p'(x')p(x)^{-1}$ is holomorphic, the preimage of $e_Q$

$$D = D_{p,p'}(E, E') = \{(x, x') \in E \times E' : p(x) = p'(x')\}, \quad (E.41)$$

is analytic in $E \times E'$. Since $D$ is a subgroup of $E \times E'$, by Corollary E.2.0.5, $D$ is a complex Lie subgroup of $E \times E'$.

For every $(x, x') \in D$ with $y = p(x) = p(x')$, every $g \in C$, the element

$$(x, x')(g, g^{-1})(x^{-1}, x'^{-1}) = (\theta_0(y)(g), \theta_0(y)(g)^{-1})$$

is in $C^*$. Therefore, $C^*$ defined by (E.39) is normal in $D$.

As $C^*$ is a normal complex Lie subgroup of $D$, we can set $E'' = D/C^*$. The inclusion $K \times K' \to D$ descends to an injective morphism $K'' \to E''$. The map $D \to Q$ defined by $(x, x') \mapsto p(x)$ induces a surjective morphism $p' : E'' \to Q$ whose kernel is $K''$. Hence an extension $1 \to K'' \to E'' \to Q \to 1$. The induced outer action $Q \to \text{Out}(K'')$ is (E.40). We call $(E'', p'')$ the $C$-product of the two given extensions $(E, p)$ and $(E', p')$, written as $(E'', p'') = (E, p) \otimes (E', p')$. Thus, [EM47, Lemmas 5.1 and 5.2] hold for complex Lie groups.

**Fact E.7.3.3.** The $C$-product of two extendible kernels is extendible. The kernel of the $C$-product $(E, p) \otimes (E', p')$ of two extensions is the $C$-product of the two kernels.

**Proposition E.7.3.4.** When $K' = C$, $(E', p')$ is the semidirect product $C \rtimes_{\theta_0} Q$, then $(E'', p'')$ is naturally equivalent to $(E, p)$.

**Proof.** Consider the subgroup $D \leq E \times E' = E \times (C \rtimes_{\theta_0} Q)$ defined in (E.41). Define a map $\psi : D \to E$ by $(x, c, q) \mapsto cx$ for $x \in E$ and $(c, q) \in C \rtimes_{\theta_0} Q$. Then $\psi$ is holomorphic.

We check that $\psi$ is a group morphism. Take another $(x, c', q') \in D$. Since $\theta_{0,q}(c') = \theta_{p(x)}(c') = xc'^{-1}$, one has

$$\psi((x, c, q)(x', c', q')) = \psi(xx', c\theta_{0,q}(c'), qq') = e \theta_{0,q}(c')xx' = e \psi(x, c, q)(x', c', q').$$

For every $g \in C$, $\psi(g, g^{-1}) = e_E$, so $C^* \subset \ker \psi$. Thus, $\psi$ induces a morphism $\epsilon : E'' \to E$. Together with $\sigma$ defined in Example E.7.3.2, $\epsilon$ fits into a commutative diagram.

$$
\begin{array}{c}
1 \longrightarrow K'' \longrightarrow E'' \longrightarrow Q \longrightarrow 1 \\
\downarrow \sigma \quad \quad \downarrow \epsilon \quad \quad \downarrow \text{id} \\
1 \longrightarrow K \longrightarrow E \longrightarrow Q \longrightarrow 1
\end{array}
$$

Therefore, $\epsilon$ is an equivalence of extensions. \qed
By construction, $C$-product defines a map $\Ext(Q, K, \theta) \times \Ext(Q, K', \theta') \to \Ext(Q, K'', \theta'')$. When $K' = C$, it specializes to

$$\Ext(Q, K, \theta) \times \Ext(Q, C, \theta_0) \to \Ext(Q, K, \theta), \quad (E.42)$$

which defines an action of the abelian group $\Ext(Q, C, \theta_0)$ on the set $\Ext(Q, K, \theta)$. If further $K$ is also commutative, by [Hoc51b, p.97], $(E.42)$ is exactly the group law defined by the Baer sum on $\Ext(Q, C, \theta_0)$.

**Definition E.7.3.5.** [EM47, p.329] For every extension kernel $(K, \theta)$, let $\theta^*$ be the composition of $\theta : Q \to \Out(K)$ with the natural group isomorphism $\Out(K) \to \Out(K^{\op})$. Then the extension kernel $(K^{\op}, \theta^*)$ is called the inverse of $(K, \theta)$.

For every $(E, p) \in \Ext(Q, K, \theta)$, define $p^* : E^{\op} \to Q$ by $p^*(x^*) = p(x^{-1})$, then it is a surjective morphism. Since $\ker(p^*) = K^{\op}$, $1 \to K^{\op} \to E^{\op} \xrightarrow{p^*} Q \to 1$ is an extension. The associated outer action is $\theta^*$. Thus, we get an element $(E^{\op}, p^*) \in \Ext(Q, K^{\op}, \theta^*)$ of $(E, p)$. It is called the inverse of $(E, p)$ and its extension kernel is the inverse of $(K, \theta)$.

It is a classical result that the group action $(E.42)$ is simple transitive. For abstract groups, see [EM47, Lem. 11.2 and 11.3]. For algebraic groups, see [FLA19, Thm. 1.1]. It remains true for complex Lie groups. The first half, Fact E.7.3.6, can be proved in the same way as in [Hoc51b, Thm. 1.1], using the inverse in the group $\Ext(Q, C, \theta_0)$ and Proposition E.7.3.4.

**Fact E.7.3.6.** Let $K, Q$ be complex Lie groups, $C = Z(K)$. Let $\theta : Q \to \Out(K)$ be a group morphism that induces $\theta_0 : Q \to \Aut(C)$. Then the action of $\Ext(Q, K, \theta_0)$ on $\Ext(Q, K, \theta)$ defined by $(E.42)$ is free.

Theorem E.7.3.7 is analogue to [EM47, Lemma 11.2].

**Theorem E.7.3.7.** In the notation of Fact E.7.3.6, if $\Ext(Q, K, \theta)$ is nonempty (i.e., the extension kernel $(K, \theta)$ is extendible), then its $\Ext(Q, C, \theta_0)$-action defined by $(E.42)$ is transitive. Equivalently, for every $(E, p), (E_1, p_1) \in \Ext(Q, K, \theta)$, there exists $F \in \Ext(Q, C, \theta_0)$ with $F \otimes E$ equivalent to $E_1$.

**Proof.** Define $D_{p_1,p^*}(E_1, E^{\op})$ like $(E.41)$. Set

$$S = \{(x_1^{-1}, x^*) \in D_{p_1,p^*}(E_1, E^{\op}) : x_1 k x_1^{-1} = x k x^{-1}, \forall k \in K\}.$$  

Then $S$ is a subgroup of $E_1 \times E^{\op}$. For every $k \in K$, the map

$$\phi_k : E_1 \times E^{\op} \to K, \quad (x_1, x^*) \mapsto x_1^{-1} k x_1 x k^{-1} x^{-1}$$

is holomorphic, so $\phi_k^{-1}(e_K)$ is analytic in $E_1 \times E^{\op}$. Then $S = D_{p_1,p^*}(E_1, E^{\op}) \cap \bigcap_{k \in K} \phi_k^{-1}(e_K)$ is analytic in $E_1 \times E^{\op}$, by [Whi72, Theorem 9C, p.100]. By Corollary E.2.0.5, $S$ is a complex Lie subgroup of $E_1 \times E^{\op}$.

The map $K \times K^{\op} \to K$ by $(k, k^*) \mapsto k k^*$ is holomorphic, so $K^* = \{(k^{-1}, k^*) : k \in K\}$ is an analytic subset of $K \times K^{\op}$. It is a subgroup of $S$, hence a complex Lie subgroup of $S$ by Corollary E.2.0.5.
For every \((x_1^{-1}, x^*) \in S, k \in K\), one has
\[
(x_1^{-1}, x^*)(k^{-1}, k^*)(x_1, (x^*)^{-1}) = (x_1^{-1}k^{-1}x_1, x^*k^*(x^{-1})^*)
\]

so \(K^*\) is a normal subgroup of \(S\). Let \(F = S/K^*\) and \(\nu : S \to F\) be the quotient morphism. The map \(i : C \to F\) defined by \(c \mapsto [(c, 1)]\) is an injective morphism.

The map \(\bar{\phi} : S \to Q\) defined by \(\bar{\phi}(x_1^{-1}, x^*) = p(x^{-1})\) is a morphism with \(K^*\) contained in the kernel. We check that \(\bar{\phi}\) is surjective. For every \(h \in Q\), there exist \(x \in E\) and \(x_1 \in E_1\) with \(p(x) = p_1(x_1) = h^{-1}\). Since the two automorphisms of \(K\), \(\text{Inn}_x|_K\) and \(\text{Inn}_{x_1}|_K\) have the same class \(\theta_{h^{-1}}\) in \(\text{Out}(K)\), there exists \(k_0 \in K\) such that \(\text{Inn}_x|_K = \text{Inn}_{x_1}|_K \text{Inn}_{k_0}\). Then \((x_1^{-1}, (xk_0)^*) \in S\) and \(\bar{\phi}(x_1^{-1}, (xk_0)^*) = h\).

Thus \(\bar{\phi}\) is surjective and \(\phi : F \to Q\) with \(i(C) \supset \ker \phi\). In addition, \(\phi\) is trivial, so \(i(C) \subset \ker(\phi)\). Hence an extension \(1 \to C \xrightarrow{i} F \xrightarrow{\phi} Q \to 1\) with the induced action \(Q \to \text{Aut}(C)\) coinciding with \(\theta_0\).

It remains to show that the \(C\)-product extension \(F \otimes E\) is equivalent to \(E_1\). By construction, \(F \otimes E\) is represented by \(G = D_{\phi, p}(F, E)/C^*\), where \(C^* = \{(c, c^{-1}) \in F \times E : c \in C\}\). The pullback of \(D_{\phi, p}(F, E)\) along the natural surjection \(S \times E \to F \times E\) is \(D_{\phi, p}(S, E)\).

For every \((a, b^*, x) \in D_{\phi, p}(S, E) \subset E_1 \times E^{\text{op}} \times E\), one has \(p_1(a) = p(b^{-1}) = p(x)\), whence \(bx \in K\) and \(a \cdot (bx) \in E_1\). Define a holomorphic map \(\tau : D_{\phi, p}(S, E) \to E_1\) by \(\tau(a, b^*, x) = a \cdot (bx)\).

We check that \(\tau\) is a group morphism. For every \((a, b^*, x), (a', b'^*, x') \in D_{\phi, p}(S, E)\), since \((a', b'^*) \in S\) and \(bx \in K\), one has \(a'^{-1}(bx)a' = b'(bx)b'^{-1}\). Hence,
\[
\tau(a, b^*, x)\tau(a', b'^*, x') = [a(bx)][a'(b'x')]
= aa'[a^{-1}(bx)a'(b'x')] = aa'[b'(bx)b'^{-1}](b'x')
= aa'(b'bx'x') = \tau(aa', (b'b'^*)xx').
\]

We check that \(\tau\) is surjective. For every \(x_1 \in E_1, p_1(x_1) \in Q\). As \(\phi\nu : S \to Q\) is surjective, there is \((a, b^*) \in S\) with \(\phi\nu(a, b^*) = p_1(x_1)\). Then \(p_1(a) = p_1(x_1)\).
Thus, \(a^{-1}x_1 \in K\). Let \(x = b^{-1}(a^{-1}x_1) \in E\). Then \(p(x) = p(b^{-1}) = \phi \nu(a, b^*)\), so \((a, b^*, x) \in D_{\phi \nu, p}(S, E)\) and \(\tau(a, b^*, x) = a(bx) = a(a^{-1}x_1) = x_1\).

We check that \(\ker(\nu^*) \subset \ker(\tau)\). For every \((x_1, x^*, y) \in \ker(\nu^*) \subset E_1 \times \text{Endop} \times E\), there is \(c \in C\) with \(\{(x_1, x^*), y\} = (c, c^{-1})\) in \(F \times E\). Equivalently, \(y = c^{-1}\) in \(E\) and \(\{(x_1, x^*)\} = \{(c, 1^*)\}\) in \(F = S/K^*.\) Whence, \((x_1c^{-1}, x^*) \in K^*,\ i.e., x \in K\ and x_1 = x^{-1}c.\) Therefore, \((x_1, x^*, y) = (x^{-1}c, x^*, c^{-1})\) with \(x \in K, c \in C\). Thus, \(\tau(x_1, x^*, y) = x^{-1}c(x^{-1}c) = c E_1\) and \((x_1, x^*, y) \in \ker(\tau)\).

Conversely, we check \(\ker(\tau) \subset \ker(\nu^*)\). For every \((a, b^*, x) \in \ker(\tau)\), one has \(a(bx) = e E_1\), so \(a \in K\). Because \((a, b^*) \in D_{\phi \nu, p}(E_1, \text{Endop})\), we obtain \(p(b^{-1}) = p(a) = e Q\) and hence \(b \in K\). Since \(\text{Inn}_{a^{-1}} = \text{Inn}_b \in \text{Aut}(K)\), one has \(ab \in C\). Therefore, \([(a, b^*)] = [(ab, 1^*)] = (x^{-1}c, c^{-1}) \in C^* \subset F \times E.\) Then \((a, b^*, x) \in \ker(\nu^*)\).

Therefore, \(\ker(\tau) = \ker(\nu^*)\), so \(\tau\) induces an isomorphism \(G \to E_1\) that establishes an equivalence between the two elements of \(\text{Ext}(Q, K, \theta)\).

Fact E.7.3.6 and Theorem E.7.3.7 yield Corollary E.7.3.8.

**Corollary E.7.3.8**. Let \(K, Q\) be complex Lie groups, \(C = Z(K), \theta : Q \to \text{Out}(K)\) be a group morphism. Let \(\theta_0 : Q \to \text{Aut}(C)\) be the induced group morphism. If \(\text{Ext}(Q, K, \theta)\) is nonempty, then \(\text{Ext}(Q, K, \theta)\) is in (non-canonical) bijection with \(\text{Ext}(Q, C, \theta_0)\).

### E.8 Maximal morphisms

A result stronger than Proposition E.5.1.3 holds.

**Definition E.8.0.1.** [Ser12, Definition 1, p.125]. Let \(X\) be a complex manifold, \(A\) be a complex torus. A morphism \(f : X \to A\) is called maximal if whenever \(f\) factors as \(X \xrightarrow{g} A' \xrightarrow{h} A\), where \(A' \in C\) is connected and \(h - h(0) : A' \to A\) is a finite morphism, it holds that \(h - h(0)\) is an isomorphism.

**Proposition E.8.0.2.** If \(X\) is a regular manifold\footnote{in the sense of [Var86, p.233]}, then the Albanese morphism \(f : X \to \text{Alb}(X)\) associated to some base point \(x \in X\) is maximal.

**Proof.** Assume that \(f\) factors as \(X \xrightarrow{g} A' \xrightarrow{h} \text{Alb}(X)\), where \(A' \in C\) is a connected and \(h - h(0)\) is a finite morphism. Then \(A'\) is compact, hence a complex torus. Choosing \(g(x)\) as the new zero element of \(A'\), we get a new structure of complex torus on \(A'\), to which we stick from now on. Then \(h\) is a finite morphism. By Proposition 4.4.1.2 3, there is a morphism \(\phi : \text{Alb}(X) \to A'\) with \(\phi f = g\) and the complex Lie subgroup of \(\text{Alb}(X)\) generated by \(f(X)\) is \(\text{Alb}(X)\) itself. Then \(h \phi f = f\) and hence \(h \phi = \text{Id}_{\text{Alb}(X)}\). In particular, \(h\) is surjective. By Fact E.3.0.4, the exact sequence \(0 \to \ker(h) \to A' \xrightarrow{h} A \to 0\) defines a trivial extension, so \(A'\) is isomorphic to \(\ker(h) \times A.\) By connectedness of \(A'\), \(\ker(h) = 0\) and \(h\) is an isomorphism. \(\square\)
When \( f = \text{Id}_A \), Proposition E.8.0.3 reduces to Proposition E.5.1.3.

**Proposition E.8.0.3** ([Ser12, Prop. 14, p.188]). Let \( X \) be a connected compact complex manifold, \( A \) be a complex torus, \( B \in \mathcal{C} \). Let \( f : X \to A \) be a maximal morphism. If \( B_0 \) is linear, then the composed morphism

\[
\text{Ext}(A, B) \xrightarrow{\pi} H^1(A, B_A) \xrightarrow{f^*} H^1(X, B_X)
\]

(E.43)
is injective.

**Proof.** Let \( C \in \ker(f^* \circ \pi) \). Then the principal fiber bundle \( f^*p : f^*C \to X \) is trivial. Fix a point \( c \in f^*C \) lying over \( 0 \in C \). Then there is a morphism \( s : X \to f^*C \) with \( f^*p \circ s = \text{Id}_X \) and \( s(f^*p(c)) = c \). Let \( t : X \to C \) be the morphism induced by \( s \).

By Remmert's theorem [Whi72, Theorem 4A, p.150], \( t(X) \) is an analytic subset of \( C \). By [CD94, (14.14), p.89], the analytic space \( t(X) \) is irreducible. Moreover, \( t(X) \) is compact and \( 0 = t(f^*p(c)) \in t(X) \). Let \( A' \) be the complex Lie subgroup of \( C \) generated by \( t(X) \). By Lemma C.3.2.1, \( A' \) is a complex torus. Then \( (A' \cap B)_0 \) is a compact. As a closed complex submanifold of \( B_0 \), \( (A' \cap B)_0 \) is also a Stein manifold, hence a point. Thus, \( A' \cap B \) is discrete and compact, hence finite. Therefore, \( h : A' \to A \) is a finite morphism. As the maximal morphism \( f \) factors as \( X \xrightarrow{\lambda} A' \xrightarrow{h} A \), \( h \) is an isomorphism. Then \( h^{-1} : A \to C \) is a morphism and \( ph^{-1} = \text{Id}_A \). By Fact E.3.0.4, \( C = 0 \) in \( \text{Ext}(A, B) \).

**Example E.8.0.4.** Let \( X \) be a regular manifold, \( f : X \to A \) be the Albanese morphism associated to some base point \( x \in X \). When \( B = \mathbb{C} \), the composed morphism (E.43) is a linear isomorphism \( f^* : H^1(A, O_A) \to H^1(X, O_X) \). When \( B = \mathbb{C}^* \), it is the inclusion of the identity component \( \text{Pic}^0(A) \to \text{Pic}(X) \).

## E.9 Commutative extensions of real Lie groups

Let \( \mathcal{R} \) be the category of commutative real Lie groups. The solution to the extension problem within \( \mathcal{R} \) is summarized in Proposition E.9.0.2. Similar to Lemma E.4.1.1, the category \( \mathcal{R} \) is additive but not abelian. Parallel to the construction in Section E.4, we can define an additive functor \( \text{Ext}_\mathcal{R} : \mathcal{R}^{\text{op}} \times \mathcal{R} \to \text{Ab} \) by considering commutative extensions.

Proposition E.9.0.1 generalizes [lH76, Proposition 5, p.110] (which says that \( C \) is isomorphic to \( A \times B \)) and [HN11, Lemma 15.3.2] (which is for real tori). The similar statement for complex tori is false, shown by Example E.4.1.14.
**Proposition E.9.0.1.** Let $0 \to B \to C \to A \to 0$ be an extension of commutative real Lie groups. If $A, B$ are connected, this extension is trivial.

*Proof.* Similar to Proposition E.3.2.2, every extension of $\mathbb{R}$ is a semidirect product, hence $\text{Ext}_\mathbb{R}(\mathbb{R}, \bullet) = 0$ on $\mathbb{R}$. Similar to Proposition E.3.2.3, $\text{Ext}_\mathbb{R}(S^1, B) = 0$. According to [H76, Proposition 4, p.109], $A$ is isomorphic to $(S^1)^n \times \mathbb{R}^m$ for some $m, n \in \mathbb{N}$. As the functor $\text{Ext}_\mathbb{R}(\bullet, B) : \mathbb{R} \to \text{Ab}$ is additive, we get $\text{Ext}_\mathbb{R}(A, B) = 0$.

**Proposition E.9.0.2.** For every $A, B \in \mathcal{R}$, there is a non-canonical isomorphism in Ab:

$$\text{Ext}_\mathcal{R}(A, B) \to \text{Ext}_\mathbb{R}^1(A/A_0, B/B_0) \oplus \text{Hom}_{\text{Ab}}(\pi_1(A_0), B/B_0).$$

*Proof.* By a real version of Corollary E.4.1.13, there are non-canonical isomorphisms in $\mathcal{R}$: $A \to A/A_0 \times A_0$ and $B \to B/B_0 \times B_0$. By additivity of the bifunctor $\text{Ext}_\mathcal{R}$, we get an isomorphism in $\text{Ab}$:

$$\text{Ext}_\mathcal{R}(A, B) \to \text{Ext}_\mathcal{R}(A/A_0, B_0) \oplus \text{Ext}_\mathcal{R}(A/A_0, B/B_0) \oplus \text{Ext}_\mathcal{R}(A_0, B/B_0) \oplus \text{Ext}_\mathcal{R}(A_0, B_0).$$

Using Lemma E.4.1.12, one can prove that $\text{Ext}_\mathcal{R}(A/A_0, B_0) = 0$. Identical to Example E.4.1.10, $\text{Ext}_\mathcal{R}(A/A_0, B/B_0) = \text{Ext}_\mathbb{R}^1(A/A_0, B/B_0)$. Similar to Corollary E.3.2.5 and [Hoc51b, Thm. 3.2], $\text{Ext}_\mathcal{R}(A_0, B/B_0) = \text{Hom}_{\text{Ab}}(\pi_1(A_0), B/B_0)$. By Proposition E.9.0.1, $\text{Ext}_\mathcal{R}(A_0, B_0) = 0$. The proof is completed. \qed
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