Integral points of Shimura varieties: an “all or nothing” principle

Haohao Liu
email: kyung@mail.ustc.edu.cn

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1 Introduction

A complex space $X$ is called Brody hyperbolic if every morphism $\mathbb{C} \to X$ is constant (see, e.g., [Liu18, p.5341]). For example, a genus $g$ compact Riemann surface is Brody hyperbolic if and only if $g \geq 2$ ([Cos05, p.78]). Conjecture 1.1 predicts that a geometric property, hyperbolicity, restricts the behavior of rational points.

**Conjecture 1.1** (Lang, [Lan74, (1.3)], [Lan86, p.160]). Let $X$ be a projective variety over a number field $k(\subset \mathbb{C})$. If $X^{an}$ is Brody hyperbolic, then the set of rational points $X(k)$ is finite.

Fact 1.2 is derived from [Nad89, Thm. 0.2] in the paragraph following [UY10, Thm. 2.1]

**Fact 1.2** (Nadel). Let $S$ be a Shimura variety associated with a triple $(G, X, K)^1$. Then there is an open subgroup $K' \leq K$ such that the analytification $S^{*,an}$ of the Baily-Borel compactification $S^{*}$ of the corresponding finite étale cover $S'$ over $S$ is Brody hyperbolic.

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^1 in the sense of page 7
As $S'$ is defined on a number field $F$ (depending on $K'$), Conjecture 1.1 predicts $S'(F')$ to be finite for every finite extension $F'/F$. Ullmo and Yafaev [UY10] introduced the notion of “Lang locus” (Example 2.1) for algebraic varieties over $\mathbb{Q}^a$ that measures the failure of Conjecture 1.1. For Shimura varieties, they proved an alternative principle.

**Fact 1.3** ([UY10, Thm. 1.1]). Let $S$ be a Shimura variety of sufficiently high level. Then its Lang locus is either full $S$ or empty.

As Ullmo and Yafaev put it, Fact 4.3 roughly means that for Shimura varieties, Conjecture 1.1 is either true or very false.

As Shimura varieties are not proper in general, it is equally natural to consider integral points. Conjecture 1.1 predicts that $S'$ has only finitely many integral points (compare [Lan91, Conjecture 5.1]). This note aims at deriving an analogue of Fact 4.3, with integral points at the place of rational points. We define a notion of “integral Lang locus” (Definition 5.1) for algebraic varieties over $\mathbb{Q}^a$ that measures the failure of finiteness of integral points.

**Main Result** (Theorem 5.12). The integral Lang locus of a Shimura variety $S$ (associated with a triple $(G, X, K)$) is either empty or full $S$.

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**2 Preliminaries**

Let $\mathbb{Q}^a$ be the algebraic closure of $\mathbb{Q}$ in $\mathbb{C}$. Unless otherwise specified, an (algebraic) variety means a finite type separated (geometrically) reduced scheme over a field. A component of the variety refers to an irreducible component. The closure of a subset of the variety is taken in the Zariski topology. A subvariety is closed unless otherwise specified. A Zariski-closed subset of a variety is endowed with the reduced scheme structure, hence a subvariety.

the field of definition of such a variety or a morphism between varieties is a number field (subfield of $\mathbb{Q}^a$). By an étale cover $X \to Y$, we mean a finite étale morphism between integral varieties (hence a surjection). It is called Galois if $\text{Aut}(X/Y)$ acts transitively on each fiber. For a topological space $X$, we write $X^{>0}$ for the union of components of positive Krull dimension. Note that if $X \subset Y$ is a subspace then $X^{>0} \subset Y^{>0}$.

We shall show that an alternative principle (Corollary 4.4) for an abstract locus is a formal consequence of some axioms. We begin with these axioms.

Suppose that for each integral variety $X/\mathbb{Q}^a$, we define a subvariety $X^L \subset X$. For the decomposition into components $Z = \bigcup_{i=1}^a Z_i$ of a reducible variety $Z/\mathbb{Q}^a$,
we define $Z^L = \bigcup_{i=1}^{n} Z_{i}^{L}$. Assume that the formation satisfies the following four axioms: For any integral varieties $X, Y/Q$:

1. (Dimension) If $X^L \neq \emptyset$, then every component of $X^L$ has positive dimension;

2. (Inheritance) If $i : X \to Y$ is a closed immersion over $Q$, then $i(X^L) \subset Y^L$;

3. (Cover) For any étale cover $f : X \to Y$ over $Q$, we have $f(X^L) \subset Y^L$;

4. (Iteration) $X^L \subset (X^L)^L$.

By Axiom 2, $X^L \supset (X^L)^L$. From Axiom 4, $X^L = (X^L)^L$.

**Example 2.1.** The Lang locus defined in [UY10, Sec. 2.2] satisfy the four axioms, which we recall now. For each integral variety $X/Q$, by [Gro65, Prop. 4.8.13] there exist a number field $F$ and an integral variety $X_F/F$ with an isomorphism $X_F \otimes_F Q \to X$. For each finite extension $M/F$, let $X(X_F, M)$ be the image of the natural injection\(^2\) $X_F(M) \to X(Q)$. The Lang locus of $X$ relative to $X_F$ is defined to be the Zariski closure of

$$\bigcup_{M} \overline{X(X_F, M)^{>0}}$$

in $X$, where $M$ runs through all finite extensions of $F$. (Notice that the closures are taken inside $X$ rather than $X_F$!) The Lang locus depends only on $X$ by Lemma 2.2 and satisfies the axioms above by [UY10, Lemmes 2.3, 2.5]. It measures the failure of finiteness of rational points, since $X^L = \emptyset$ if and only if $X_F(M)$ is finite for every finite extension $M/F$.

**Lemma 2.2.** The Lang locus of $X$ is independent of the choice of $X_F$.

**Proof.** Take another such $X_{F'}/F'$. There is a $Q$-isomorphism $X_F \otimes_F Q \to X_{F'} \otimes_{F'} Q$. By [Gro65, Prop. 4.8.13], it is defined over a number field $F''$ including both $F$ and $F'$. For any finite extension $M/F$, there is a number field $M'$ containing $M$ and $F''$. Then $X(X_F, M) \subset X(X_{F'}, M')$, so $X(X_F, M)^{>0} \subset \overline{X(X_{F'}, M')^{>0}}$ and hence the Lang locus relative to $X_F$ is contained in that relative to $X_{F'}$. The reverse inclusion follows by symmetry. □

**Remark 2.3.** The Lang locus $X^L$ in Example 2.1 is slightly different from the “lieu de Lang” $X^L_F$ (a closed subset of $X_F$) defined by [UY10, (1)]. Let $\phi : X \to X_F$ be the natural morphism of schemes. For every finite extension $M/F$, let $X_F[M]$ be the image of the natural map $X_F(M) \to X_F$, then $\phi(X(X_F, M)) = X_F[M]$. Because $\phi$ is integral and surjective integral morphisms preserve the dimension, $\phi(X(X_F, M)) = X_F[M]$ and hence $\phi(\overline{X(X_F, M)^{>0}}) = \overline{X_F[M]}^{>0}$. Therefore, $\phi(X^L) = X^L_F$.

We gather some properties of the so defined locus.

\(^2\)The natural map $X_F(M) \to X_F$ is not injective in general.
Lemma 2.4. Let $X/Q^n$ be a variety.

1. (Union) If $X = \bigcup_{i=1}^r Z_i$, where each $Z_i$ is a subvariety of $X$, then $X^L = \bigcup_{i=1}^r Z_i^L$.

2. (Component) If $Z$ is a component of $X^L$, then $Z^L = Z$.

3. (Galois) If $f: X \rightarrow Y$ is Galois cover over $Q^n$, then $f^{-1}(f(X^L)) = X^L$. If $Z \subset Y$ is an irreducible subvariety, and $Z'$ is a component of $f^{-1}(Z)$, then $f(f^{-1}(Z)) = f(Z')$.

Proof.

- By Axiom 2, $\bigcup_{i=1}^r Z_i^L \subset X^L$. If $Y$ is a component of $X$, then there exists an index $i$ such that $Y \subset Z_i$. By Axiom 2, $Y^L \subset Z_i^L$. By definition, $X^L \subset \bigcup_{i=1}^r Z_i^L$.

- Write $X^L = \bigcup_{i=1}^n Z_i$ for the decomposition into components with $Z_1 = Z$. By Axiom 4,

\[ Z \subset X^L = (X^L)^L = \bigcup_{i=1}^n Z_i^L. \]

As $Z$ is irreducible, there is an index $i$ such that $Z \subset Z_i^L \subset Z_i$. Then $i = 1$ as $Z = Z_1$ is a component, so $Z = Z^L$.

- If $x \in f^{-1}(f(X^L))$, there is $x' \in X^L$ such that $f(x') = f(x)$. Let $\Theta$ be the Galois group of $f$. There is $\theta \in \Theta$ with $\theta(x') = x$, so $x \in X^L$ by Axiom 2. Therefore, $f^{-1}(f(X^L)) = X^L$. Since $\Theta$ permutes transitively the components of $f^{-1}(Z)$, one has $f^{-1}(Z) = \Theta \cdot Z'$, so $f^{-1}(Z^L) = \Theta \cdot Z_i^L$ by 1. Then apply $f$ to both sides.

\[ \square \]

In general, given an étale cover $f: X \rightarrow Y$, the induced map $X^L \rightarrow Y^L$ is not surjective. We introduce a sublocus that lifts along any étale cover.

For an integral variety $X/Q^n$, define its \textit{locus at infinite level} by

\[ X^{L_{\infty}} := \bigcap_{f: T \rightarrow X} f(T^L), \]

where $f: T \rightarrow X$ runs through all étale covers of $X$. By Axiom 3, $X^{L_{\infty}}$ is a subvariety of $X$. As $X$ is topologically noetherian, and $f(T^L) \subset X$ is closed for each $f: T \rightarrow X$, there exists a particular cover $f_1: X_1 \rightarrow X$ such that $f_1(X_1^L) = X^{L_{\infty}}$. For any étale cover $X_2 \rightarrow X_1$, the composition $X_2^L \rightarrow X_1^L \rightarrow X^{L_{\infty}}$ is still surjective. If $X^{L_{\infty}} \neq \emptyset$, then its components are \textit{positive dimensional} by Axiom 1. For a reducible $Q^n$-variety $Y = \bigcup_{i=1}^n Y_i$ decomposed into components, define $Y^{L_{\infty}} = \bigcup_{i=1}^n Y_i^{L_{\infty}}$, which is a subvariety of $Y^L$.

Lemma 2.5.

1. If $f: T \rightarrow S$ is an étale cover over $Q^n$, then $f^{-1}(S^{L_{\infty}}) = T^{L_{\infty}}$. In particular, $T^{L_{\infty}} = T$ if and only if $S^{L_{\infty}} = S$.
2. Let \( X/Q^a \) be a variety with \( X^{L_{\infty}} = X \), \( Y \subset X \) be a component. Then \( Y^{L_{\infty}} = Y \).

**Proof.**

- We show \( T^{L_{\infty}} \subset f^{-1}(S^{L_{\infty}}) \). For \( t \in T^{L_{\infty}} \), set \( s = f(t) \). For any étale cover \( g : S' \to S \), there is a diagram

\[
\begin{array}{ccc}
T' & \xleftarrow{f'} & S' \\
\downarrow{g'} & & \downarrow{g} \\
T & \xrightarrow{f} & S \\
\end{array}
\]

where each arrow is an étale cover. There is \( t' \in T'^{L_{\infty}} \) lying above \( t \), then \( s' = f'(t') \in S'^{L_{\infty}} \) by Axiom 3, so \( s = g(s') \in \text{Im}(S'^{L_{\infty}} \to S) \). Thus, \( s \in S^{L_{\infty}} \).

Conversely, if \( t \in f^{-1}(S^{L_{\infty}}) \), then \( s = f(t) \in S^{L_{\infty}} \). For any étale cover \( Z \to T \), there is \( N \to Z \) an étale cover such that \( N \to S \) is a Galois cover:

\[N \to Z \to T \to S.\]

One has

\[(N \to T)^{-1}(t) \subset (N \to S)^{-1}(s) \subset (N \to S)^{-1}(S^{L_{\infty}})
\]

\[\subset \text{Lem} 2.4 \ 3 \ (N \to S)^{-1}((N \to S)(N^{L})) = N^{L}.
\]

Therefore, \( (Z \to T)^{-1}(t) \subset (N \to Z)(N^{L}) \subset Z^{L} \) and \( t \in \text{Im}(Z^{L} \to T) \), so \( t \in T^{L_{\infty}} \). This shows that \( f^{-1}(S^{L_{\infty}}) = T^{L_{\infty}} \).

- Let \( X = \bigcup_{i=1}^{n} Y_i \) be the decomposition into components, where \( Y = Y_1 \). Then

\[Y \subset X = X^{L_{\infty}} = \bigcup_{i=1}^{n} Y_i^{L_{\infty}}.\]

There is \( 1 \leq i \leq n \) with \( Y \subset Y_i^{L_{\infty}} \subset Y_i \). As \( Y \) is a component of \( X \), \( i = 1 \) and \( Y = Y_i^{L_{\infty}} \).

\[\square\]

### 3 Shimura varieties and special subvarieties

We review some basic facts about Shimura varieties, the main objects of interest in this note.

Let \( G/Q \) be an affine algebraic group.
Definition 3.1 ([Pin90, Sec. 0.1, p.13]). For every prime \( p \), choose an embedding \( \bar{\mathbb{Q}} \to \bar{\mathbb{Q}}_p \).

- For an element \( g = (g_p)_p \in \text{GL}_n(\mathbb{A}_f) \), let \( \Gamma_p \leq \bar{\mathbb{Q}}^*_p \) be the subgroup generated by all eigenvalues of \( g_p \in \text{GL}_n(\mathbb{Q}_p) \). If
  \[
  \cap_p(\bar{\mathbb{Q}}^* \cap \Gamma_p)_{\text{tor}} = \{1\},
  \]
  then \( g \) is called neat.

- An element of \( G(\mathbb{A}_f) \) is called neat if its image in some faithful algebraic representation of \( G \to \text{GL}_n/\mathbb{Q} \) is neat.

- A subgroup of \( G(\mathbb{A}_f) \) is called neat if all its elements are neat.

Fact 3.2 ([Pin90, p.13]).

1. Let \( K \leq G(\mathbb{A}_f) \) be a compact open subgroup. Then there is an open normal subgroup \( K' \leq K \) that is neat.

2. If \( K \leq G(\mathbb{A}_f) \) is a neat subgroup, then \( K \cap G(\mathbb{Q}) \) is a neat subgroup of \( G(\mathbb{Q}) \) in the sense of [Mil17, p.34].

From now on, let \( (G, X) \) be a Shimura datum in the sense of [Mil17, Def. 5.5], \( K \leq G(\mathbb{A}_f) \) be a compact open subgroup. Then \( X \) is naturally a finite disjoint union of hermitian symmetric domains by [Mil17, Prop. 5.9]. In view of Fact 3.2 1, we always make the mild assumption that \( K \) is neat. Then by [Pin90, Prop. 3.3 (b)], \( \text{Sh}_K(G, X) \) is naturally a complex manifold. Let \( X^+ \) be a connected component of \( X \). The set \( G(\mathbb{R}) \) is naturally a (real) Lie group. For a Lie group \( L \), let \( L^+ \) be its identity component. Let \( G^{\text{ad}} \) be the quotient of \( G \) by its center. Set \( G(\mathbb{R})_+ \) to be the preimage of \( G^{\text{ad}}(\mathbb{R})^+ \) under the natural morphism \( G(\mathbb{R}) \to G^{\text{ad}}(\mathbb{R}) \) of Lie groups, and

\[
G(\mathbb{Q})_+ := G(\mathbb{Q}) \cap G(\mathbb{R})_+.
\]

Then \( G(\mathbb{Q})_+ \) is a finite-index subgroup of \( G(\mathbb{Q}) \). For each \( g \in G(\mathbb{A}_f) \), put \( \Gamma_g := gKg^{-1} \cap G(\mathbb{Q})_+ \) and \( S_g := \Gamma_g \backslash X^+ \). By Fact 3.2 2 and [Mil17, Prop. 4.1], \( \Gamma_g \) is a neat (hence torsion-free) arithmetic subgroup of \( G(\mathbb{Q}) \) in the sense of [Mil17, p.33]. From [Mil17, Prop. 3.1], \( S_g \) is naturally a connected complex manifold. Let \( C \) be a set of representatives for the double coset space \( G(\mathbb{Q})_+ \backslash G(\mathbb{A}_f)/K \). From [Mil17, Lem. 5.13], the complex manifold \( \text{Sh}_K(G, X) = \sqcup_{g \in C} S_g \).

By [Mil17, Thm. 3.12; Cor. 3.16], the complex manifold \( S_g \) has a unique canonical structure of a complex algebraic variety. The algebraic variety \( S_g \) is an irreducible, smooth arithmetic locally symmetric variety ([Mil17, p.58]). It is Zariski-open in its Baily-Borel compactification ([Mil17, p.40]), which is a projective variety. Thus, \( \text{Sh}_K(G, X) \) is also a smooth quasi-projective (reducible) complex algebraic variety. Let \( E(G, X) \subset \mathbb{Q}^* \) be the reflex field of the Shimura datum \( (G, X) \) ([Mil17, Def. 12.2]). By [Mil17, Rk. 12.3 (a)], \( E(G, X) \) is a number field. From [Mil99, Rk. 2.4 (b)] and [Mil17, p.128],
Sh\(_{K}(G, X)\) admits a unique (up to a unique isomorphism) *canonical model* over \(E(G, X)\) in the sense of [Mil17, Def. 12.8]. Hence a *smooth* quasi-projective variety over the \(E(G, X)\).

Each connected component of \(\text{Sh}_{K}(G, X)\) and the inclusion to the Baily-Borel compactification is defined over a finite abelian extension of \(E(G, X)\). ([GN20, Remark (3), p.56], [Moo98b, p.282]) A component algebraic variety is called a Shimura variety associated with \((G, X, K)\).\(^3\)

For each open subgroup \(K' \leq K\), the natural morphism \(\text{Sh}_{K'}(G, X) \to \text{Sh}_{K}(G, X)\) is finite étale and defined over \(E(G, X)\). There is a component \(S'\) of \(\text{Sh}_{K'}(G, X)\) such that \(S' \to S\) is an étale cover defined over a finite extension of \(E(G, X)\). When \(K'\) is normal in \(K\), this étale cover is Galois of group \(K/K'\). ([CK16, p.1901]).

**Definition 3.3.** [Moo98a, Definition 2.5] An irreducible subvariety \(Z \subset \text{Sh}_{K}(G, X)\)\(_{\mathbb{C}}\) is called special if there exist an algebraic subgroup of \(E\) \(H\) of \(G\), \(X\), \(K\) called a Shimura variety associated with \((G, X, K)\).\(^2\)

Therefore, for an irreducible subvariety \(Y \subset \text{Sh}_{K}(G, X)\)\(_{\mathbb{C}}\), if each component of any intersection of special subvarieties of \(\text{Sh}_{K}(G, X)\) is again special ([Moo98a, Sec. 2.9]). Therefore, for an irreducible subvariety \(Y \subset \text{Sh}_{K}(G, X)\)\(_{\mathbb{C}}\), there is a smallest special subvariety \(Z_{Y} \subset \text{Sh}_{K}(G, X)\) containing \(Y\). If \(Y\) is defined over \(\mathbb{Q}^{a}\), then so is \(Z_{Y}\). We say that \(Y\) is *Hodge generic* in \(Z_{Y}\). The set of \(\mathbb{C}\)-points that are not Hodge generic in \(\text{Sh}_{K}(X, G)\)\(_{\mathbb{C}}\) is a strict countable union of analytic subspaces ([CDK95]). Fact 3.4 is a characterization of special subvarieties.

**Fact 3.4** ([UY10, Lem. 2.7], [UY14, Lem. 2.1]). Let \((H, X_{H}) \subset (G, X)\) be a Shimura subdatum\(^4\) with \(H\) being the generic Mumford-Tate group\(^5\) on \(X_{H}\). Let \(X_{H}^{+}\) be a connected component of \(X_{H}\) contained in \(X^{+}\). Set \(\Gamma_{H,g} = gKg^{-1} \cap H(\mathbb{Q})^{+}\) and \(\tilde{Z} = \tilde{Z}_{g} := \Gamma_{H,g}\backslash X_{H}^{+}\) (a component of \(\text{Sh}_{gKg^{-1}\cap H(\mathbb{A})}(H, X_{H})\)). Then the image \(Z_{g}\) of \(\tilde{Z}_{g}\) under the \(\mathbb{C}\)-morphism

\[
\text{Sh}_{gKg^{-1}\cap H(\mathbb{A})}(H, X_{H}) \to \text{Sh}_{K}(G, X), \quad [x, h] \mapsto [x, hg]
\]

is a special subvariety of \(S_{\mathbb{C}}\) and the induced morphism \(\pi : \tilde{Z} \to Z\) is finite birational. Conversely, every special subvariety of \(S_{\mathbb{C}}\) arises in this way, so is defined over a number field.

\(^3\)It is a connected Shimura variety in the sense of [Mil17, Def. 4.10] when \(K\) is sufficiently small by [Moo98b, Prop. 2.9] and [Mil17, Thm. 5.17].

\(^2\)in the sense of [CLZ16, p.894]

\(^5\)in the sense of [Moo17, p.286] and [Moo04, p.16]
Given an open subgroup $K'$ of $K$, let $f : S' \to S$ be the corresponding étale cover of Shimura varieties over $\mathbb{C}$. Then any component $\tilde{Z}'$ of $\tilde{Z} \times_{S} S'$ is an étale cover of $\tilde{Z}$. The image $Z'$ of $\tilde{Z}'$ in $S'$ is a special subvariety which is an étale cover of $Z$ and a component of $f^{-1}(Z)$.

We may replace $S$ by an étale cover for our purpose. Consider the open subgroup $K' := K \cap gKg^{-1}$ of $K$, and set $\Gamma' = K' \cap G(\mathbb{Q})_{+}$ which induces an étale cover $S' \to S$ with $S' = \Gamma' \backslash X^+$. Thus up to replace $S$ (resp. $K$, resp. $\Gamma$) by $S'$ (resp. $K'$, resp. $\Gamma'$), we may assume $g = 1$. Then $\tilde{Z}_g$ is a connected component of $\text{Sh}_{K \cap H, \mathbb{A}'}(H, X_H)$.

Lemma 3.5 is a synthesis of [UY10, Lemmas 2.8–2.10]. In Section 4, Lemma 3.5 forces certain subset of the original locus to live inside the infinite level locus, i.e., can lift along covers.

**Lemma 3.5.** Let $Z \subset S$ be a subvariety over $\mathbb{Q}^a$ such that $Z_{\mathbb{C}}$ is a finite union of special subvarieties, then:

1. There is a Galois cover $f : S' \to S$ over $\mathbb{Q}^a$ such that $f(f^{-1}(Z)^L) = Z^{L_\infty}$ and $f(S'^L) = S^{L_\infty}$.

2. If $Z$ is irreducible, one may further require that any component $Z' \subset f^{-1}(Z)$ is an étale cover of $Z$ and satisfies $Z'^L = Z'^{L_\infty}$.

3. If $S^{L_\infty} \subset Z$, then $S^{L_\infty} = Z^{L_\infty}$.

**Proof.**

- We prove 1 and 2. By Lemma 2.4 1, we may replace $Z$ by one of its components and take a normal closure of the finitely many covers corresponding to each component of $Z$. Now that $Z$ is irreducible, $Z_{\mathbb{C}}$ is a special subvariety of $S_{\mathbb{C}}$. We may assume that $g = 1$.

As $S$ is topologically noetherian, there is an open subgroup $K' \leq K$ such that the induced étale cover $f_1 : S_1 \to S$ over $\mathbb{Q}^a$ satisfies $f(S_1^L) = S^{L_\infty}$. Similarly, there is an étale cover $g : Z_1 \to Z$ over $\mathbb{Q}^a$ such that $g(Z_1^L) = Z^{L_\infty}$.

Let $(H, X_H) \subset (G, X)$ be a Shimura subdatum and a component $\tilde{Z} \subset \text{Sh}_{K \cap H, \mathbb{A}'}(H, X_H)$ corresponding to $Z$ via Fact 3.4. Replacing $Z_1$ by an étale cover of $Z_1$, one may assume that there is an open subgroup $K'_H \leq K \cap H(\mathbb{A}_f)$ such that the induced morphism

$$\text{Sh}_{K'_H}(H, X_H) \to \text{Sh}_{K \cap H, \mathbb{A}'}(H, X_H)$$

makes a component $\tilde{Z}' \subset \text{Sh}_{K'_H}(H, X_H)$ an étale cover of $\tilde{Z}$, and the morphism $\tilde{Z}' \to Z_1$ given by [Sza09, Prop. 5.5.5] is birational. Hence a commutative diagram of algebraic varieties over $\mathbb{Q}^a$:
Take an open normal subgroup $K_N \leq K$ such that $K_N \subset K'$ and $K_N \cap H(H_f) \subset K'_H$. It induces an étale cover $S' \to S_1$ over $\mathbb{Q}^a$ such that the composition $f : S' \to S$ is a Galois cover of group $\Theta := K/K_N$. Here $S'$ is a quotient of $X^+$ and a component of $\text{Sh}_{K_N}(G, X)$. Then $f(S'^L) = S'^L_{\infty}$.

Put $\Gamma' = K_N \cap H(\mathbb{Q})_\flat$. Then $\tilde{Z}' := \Gamma' \backslash X_H^+$ is a component of $\text{Sh}_{K_N \cap H(H_f)}(H, X_H)$ mapped inside $\tilde{Z}'$ under the morphism

$$
\text{Sh}_{K_N \cap H(H_f)}(H, X_H) \to \text{Sh}_{K'_H}(H, X_H).
$$

Its image $Z'$ under the morphism

$$
\text{Sh}_{K_N \cap H(H_f)}(H, X_H) \to \text{Sh}_{K_N}(G, X)
$$

is a component of $f^{-1}(Z)$, which is an étale cover of $Z_1$. By Lemma 2.5 1, $Z'^L = Z'^L_{\infty}$. Since $\Theta$ permutes transitively the components of $f^{-1}(Z)$, it remains true for any component. Point 2 is proved.

Now $f(Z'^L) = Z'^L_{\infty}$. By Lemma 2.4 3, $f(f^{-1}(Z)^L) = Z'^L_{\infty}$. Therefore, 1 is proved.

- By hypothesis and 1, $S'^L \subset f^{-1}(S'^L_{\infty}) \subset f^{-1}(Z) \subset S'$, so $S'^L = f^{-1}(Z)^L$ by Axioms 4 and 2. Thus, $S'^L_{\infty} = f(S'^L) = f(f^{-1}(Z)^L) = Z'^L_{\infty}$ by 1.

\[\square\]
4 Proof of the formal alternative

We show that an alternative (Theorem 4.3) results from the formal axioms. Let $S$ be a Shimura variety associated with $(G, X, K)$ as defined in Section 3. Proposition 4.1 is related to [UY10, Prop. 3.4].

**Proposition 4.1.** If $S^{L_{\infty}} \neq \emptyset$ and contains a component $Z$ that is Hodge generic in $S$, then $S^{L_{\infty}} = S$.

**Proof.** We may replace $S$ by an étale cover induced by an open subgroup of $K$. Indeed, for an étale cover $f : S_1 \to S$, by Lemma 2.5 1, $S^{L_{\infty}} = f^{-1}(S^{L_{\infty}}) \supset f^{-1}(Z)$, and $f^{-1}(Z)$ has a component that is Hodge generic in $S_1$. Thus, the condition holds for $S_1$. If the statement $S^{L_{\infty}}_1 = S_1$ is true, then $S^{L_{\infty}} = S$.

Thus, one may assume $g = 1$.

Consider the Hecke correspondence. For $q \in G(\mathbb{Q})^+$, put $\Gamma_q = \Gamma_1 \cap q^{-1}\Gamma_1 q$ and $S_q = \Gamma_q \backslash X^+$. Then the morphisms $\alpha_q : S_q \to S$ (resp. $\beta_q : S_q \to S$) induced by $\text{Id}_{X^+}$ (resp. $X^+ \to X^+, x \mapsto q \cdot x$) are étale covers over $\mathbb{Q}$ ([UY10, p.700]):

\[
\begin{array}{ccc}
S_q & \alpha_q & S \\
\downarrow & & \downarrow \beta_q \\
S & & S
\end{array}
\]

The Hecke operator $T_q = \beta_q \alpha_q^*$ acting on cycles of $S$. By Lemma 2.5 1, $\alpha_q(S^{L_{\infty}}_q) = \beta_q(S^{L_{\infty}}_q) = S^{L_{\infty}}$, so

$$S^{L_{\infty}} \subset T_q S^{L_{\infty}}.$$ 

As noted in Section 2, $\dim(Z) > 0$. The proof is completed by Theorem 4.2 below with $Y = S^{L_{\infty}}$.

**Theorem 4.2** (Ullmo-Yafaev). Let $Y \subset S_\mathbb{C}$ be a subvariety with one positive dimensional component that is Hodge generic in $S$. If $Y \subset T_q Y$ for all $q \in G(\mathbb{Q})^+$, then $Y = S$.

**Proof.** Write $Y = Y_1 \cup Y_2$, where $Y_1$ is the union of components of $Y$ that are Hodge generic in $S$, and $Y_2$ is the union of other components. By assumption, $\dim Y_1 > 0$ and $Y_1 \subset T_q Y = T_q Y_1 \cup T_q Y_2$. Each component of $T_q Y_2$ is not Hodge generic in $S$, so $Y_1 \subset T_q Y_1$. By [UY10, Théorème 3.6], $Y_1 = S$ and $Y = S$.

**Theorem 4.3** (Ullmo-Yafaev alternative). Either $S^{L_{\infty}} = \emptyset$ or $S^{L_{\infty}} = S$.

**Proof.** By Lemma 2.5 1, we may replace $S$ by its étale cover induced by an open subgroup of $K$. We may therefore assume $S^L = S^{L_{\infty}} \neq \emptyset$. For each component $Z \subset S^L$, $\dim(Z) > 0$ and $Z^L = Z$.

We prove that $Z_\mathbb{C}$ is a special subvariety of $S_\mathbb{C}$. Let $S_{M, \mathbb{C}}$ be the smallest special subvariety of $S_\mathbb{C}$ containing $Z_\mathbb{C}$. By Fact 3.4, there is a subvariety $S_M \subset S$ containing $Z$ whose base change to $\mathbb{C}$ is $S_{M, \mathbb{C}}$. 


We claim that $Z \subset S^L_M\cap S$. In fact, take $f : S' \to S$ given by Lemma 3.5 for $S_M \subset S$. Since $Z$ is a component of $S^L$, any component $T \subset f^{-1}(Z)$ is a component of

$$f^{-1}(S^L) = f^{-1}(S^L\cap S) = S^L$$

by Axiom 3. By Lemma 2.4, $T = T'$. There is a component $S'_M \subset f^{-1}(S_M)$ containing $T$ and

$$f|_{S'_M} : S'_M \to S_M$$

is an étale cover. From the choice of $f$,

$$T = T' \subset S'_M = S^L_M.$$

Therefore, $f(T) \subset f(S^L_M) \subset S^L_M$ and $Z = f(f^{-1}(Z)) \subset S^L_M$ follows. The claims is proved.

The claim shows that $Z$ is a component of $S^L_M$. By Proposition 4.1, $S_M = S^L_M = S^L_M \subset S^L$. The irreducible subset $S_M$ of $S^L$ contains one component $Z$ of $S^L$, so $Z = S_M$ is a special subvariety of $S$.

Now that $S^L$ is a finite union of special subvarieties, by Lemma 3.5, $(S^L)_M = S^L_M = S^L$. By Lemma 2.5, $Z^L = S$. This property is preserved up and down along étale covers (Lemma 2.5 3), so also preserved by Hecke correspondences. Explicitly, let $Z' \subset T_q Z$ be a component, then there is a diagram

$$
\begin{array}{ccc}
Z & \xleftarrow{\alpha_q} & Z' \\
\downarrow{\beta_q} & & \downarrow{\beta_q} \\
Z & \xrightarrow{\beta_q} & Z'
\end{array}
$$

where $Z_1 \subset \alpha_q^{-1}(Z)$ is a component and both arrows are étale covers. In particular, $Z' = (Z')^L = Z^L \subset S^L$, so $T_q Z \subset S^L$ and hence $T_q S^L \subset S^L$. From the last paragraph in [UY10, p.705], $S^L = S$.

Corollary 4.4 ([UY10, Thm. 1.1]). If a Shimura variety $S/Q^a$ is of sufficiently high level, then either $S^L = \emptyset$ or $S^L = S$.

Proof. As the level is high, we have $S^L = S^L$. The proof is completed by Theorem 4.3. □

5 “All or nothing” principle for integral points

We define an locus concerning integral points, an analog of Lang locus. We check the axioms for this locus and then an alternative principle follows.

Let $X/Q^a$ be an integral variety. From Section 2, there is a number field $F \subset Q^a$ and an integral variety $X_F/F$ and an isomorphism $X_F \otimes_F Q^a \to X$. There exists a finite set $\Sigma$ of places of $F$ including all archimedean ones and an integral scheme $\mathcal{X}$ that is finite type and separated (hence flat) over the ring $O_{F, \Sigma}$ of $\Sigma$-integers, whose generic fiber is $X_F$. We call $\mathcal{X}$ an integral model for $X$. 

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relative to \((F, \Sigma)\). For any finite extension \(M/F\) and a finite set \(\Omega\) of places of \(M\) containing all the places above \(\Sigma\), which we denote by \((M, \Omega)/(F, \Sigma)\) informally, we define \(X(X, M, \Omega)\) to be the image of the injection \(X(M, \Omega) \to X(Q^a)\).

**Definition 5.1.** Let \(X^I\) be the Zariski closure of
\[
\bigcup_{(M, \Omega)/(F, \Sigma)} X(X, M, \Omega) > 0
\]
inside \(X\) and call it the integral Lang locus of \(X\) relative to \((X, F, \Sigma)\).

The integral Lang locus \(X^I\) is a subvariety of the Lang locus of \(X\).

**Lemma 5.2.** Given two models \(X_1/O_{F_1, \Sigma_1}\) and \(X_2/O_{F_2, \Sigma_2}\) for \(X\), we have \(X^I_1 = X^I_2\).

**Proof.** By [Gro66, Corollaire 8.8.2.5], there is a number field \(F_3\) containing \(F_1, F_2\) with a finite set of places \(\Sigma_3 \supset \Sigma_1, \Sigma_2\) such that there is an \(O_{F_3, \Sigma_3}\)-isomorphism
\[
X_1 \otimes_{O_{F_1, \Sigma_1}} O_{F_3, \Sigma_3} \to X_2 \otimes_{O_{F_2, \Sigma_2}} O_{F_3, \Sigma_3}
\]
extending the isomorphism between the generic fibers. For any \((M_1, \Omega_1)/(F_1, \Sigma_1)\), there is a pair \((M_2, \Omega_2)\) over \((F_3, \Sigma_3)\) and \((M_1, \Omega_1)\). Then
\[
X_1(O_{M_1, \Omega_1}) \subset X_1(O_{M_2, \Omega_2}) = X_2(O_{M_2, \Omega_2}),
\]
so \(X(X_1, M_1, \Omega_1) \subset X(X_2, M_2, \Omega_2)\). Therefore,
\[
X(X_1, M_1, \Omega_1) > 0 \subset X(X_2, M_2, \Omega_2) > 0 \subset X^I_2,
\]
so \(X^I_1 \subset X^I_2\). The reverse inclusion is similar.

By Lemma 5.2, we may safely use the notation \(X^I\) for \(X^I\) and call it integral Lang locus of \(X\). We extend the definition to reducible varieties as in Section 2.

**Example 5.3.** Let \(X = P^1 - \{0, 1, \infty\} = Y(2)\) be a modular curve over \(Q^a\). Its compactification is \(X^* = P^1\) and the Lang locus of \(X\) is full but \(X\) is arithmetically hyperbolic by S-unit theorem. An elliptic curve with one puncture provides another such example by Siegel’s theorem on integral points.

**Remark 5.4.** If \(X\) is proper over \(Q^a\), then there is an integral model \((X, F, \Sigma)\) of \(X\) such that \(X\) is proper over \(O_{F, \Sigma}\). By [Poo17, Thm. 3.2.13 (ii)], \(X^I\) coincides with the Lang locus of \(X\).

**Definition 5.5.** [Ull04, Définition 2.3] A variety \(X/Q^a\) with \(X^I = \emptyset\) is called arithmetically hyperbolic.

A variety \(X/Q^a\) is arithmetically hyperbolic if and only if for one (hence for every by Lemma 5.2) model \(X/O_{F, \Sigma}\), the set of integral points \(X(O_{M, \Omega})\) is finite for any \((M, \Omega)/(F, \Sigma)\), so [Ull04, Lem. 2.4] follows from Lemma 5.2.

**Conjecture 5.6** ([Ull04, Conjecture 2.5]). If \(X/Q^a\) is an irreducible quasi-projective variety such that \(X^{an}\) is hyperbolic, then \(X\) is arithmetically hyperbolic.
Fact 5.7 is an evidence of Conjecture 5.6 and relies on Faltings’ solution \cite{Fal83, Satz 6} to Shafarevich’s conjecture.

**Fact 5.7** ([Ull04, Thm. 3.2 (a)]). Let $(G, X)$ be an adjoint Shimura datum of abelian type,\(^6\) and let $K \leq G(\mathbb{A}_f)$ be a neat compact open subgroup. Then each component of $\text{Sh}_K(G, X)_{\mathbb{Q}^a}$ is arithmetically hyperbolic.\(^7\)

We prove that Ullmo-Yafaev’s alternative principle holds for integral points on general Shimura varieties. For this, we check the four axioms listed in Section 2 for integral Lang loci. The Axiom 1 holds, since a component of $X^I$ with dimension 0 is an isolated point. Lemma 5.8 verifies Axioms 3 and 2.

**Lemma 5.8.** Let $f : Z_1 \to Z_2$ be a $\mathbb{Q}^a$-morphism between integral varieties. If $f$ has finite geometric fibers, then $f(Z_1^I) \subset Z_2^I$.

**Proof.** We may choose a finite set $\Sigma$ of places of $F$, a model $Z_i/O_{F, \Sigma}$ for $Z_i$ and an $O_{F, \Sigma}$-morphism $f' : Z_1 \to Z_2$ whose base change to $F$ is $f$. Then for any $(M, \Omega)/(F, \Sigma)$, $f'(Z_1(1, M, \Omega)) \subset Z_2(1, Z_2, \Omega)$, so $f(Z_1(1, M, \Omega)) \subset Z_2(1, M, \Omega)$. Thus,

$$f(Z_1(1, M, \Omega)) \subset Z_2(1, M, \Omega).$$

Let $C \subset Z_1(1, M, \Omega)$ be a component of positive dimension, then $f(C)$ is irreducible but not isolated (for otherwise, $C$ is a finite set by assumption, which is a contradiction), so

$$f(C) \subset Z_2(1, M, \Omega)^0 \subset Z_2^I.$$

Therefore, $f(Z_1(1, M, \Omega)^0) \subset Z_2^I$ and $f(Z_1^I) \subset Z_2^I$. \(\Box\)

**Corollary 5.9** ([Ull04, Prop. 2.6]). A locally closed subvariety of an arithmetically hyperbolic variety is also arithmetically hyperbolic.

**Proof.** It follows from Lemma 5.8. \(\Box\)

Lemma 5.10 verifies Axiom 4 for integral Lang loci.

**Lemma 5.10.** If $X/\mathbb{Q}^a$ is an integral variety, then $X^I \subset (X^I)^I$.

**Proof.** Write $X^I = \bigcup_{i=1}^n Y_i$ as the union of components. Take a model $\mathcal{X}/O_{F, \Sigma}$ for $X$ and let $Y_i$ be the scheme-theoretic image of the composition $Y_i \to X \to \mathcal{X}$, which is model of $Y_i$ relative to $(F, \Sigma)$. For any $(M, \Omega)/(F, \Sigma)$, any $x \in \mathcal{X}(O_{M, \Omega})$, i.e., a section $x : \text{Spec}(O_{M, \Omega}) \to \mathcal{X}$, we have

$$x|_{\text{Spec} M} \in X_M(M) \subset \mathcal{X}(\mathcal{X}, M, \Omega) = \mathcal{X}(\mathcal{X}, M, \Omega)^0 \cup \{p_1, \ldots, p_t\},$$

\(^6\)In the sense of [Ull04, p.4118]

\(^7\)The model over $\mathbb{Q}^a$ defined by Faltings used in [Ull04, Thm. 3.2 (a)] is the base change of the canonical model ([Moc98b, 2.17]).
where \( p_i \in X(X, M, \Omega) \) are isolated points. If \( x|_{\text{Spec} \mathbb{M}/\mathbb{Q}} \notin \{p_1, \ldots, p_t\} \), then
\[
x|_{\text{Spec} \mathbb{M}/\mathbb{Q}} \in X(X, M, \Omega)^{>0} \subset X^I.
\]
Thus, there exists an index \( i \) such that \( x|_{\text{Spec} \mathbb{M}/\mathbb{Q}} \in Y_i \). The section \( x \) factors through \( Y_i \), i.e. \( x \in Y_i(O_{M, \Omega}) \). Therefore,
\[
X(X, M, \Omega) \subset \bigcup_{i=1}^n Y_i \subset \bigcup_{i=1}^n Y_i(O_{M, \Omega}) \cup \{p_1, \ldots, p_t\}.
\]

Then
\[
X(X, M, \Omega)^{>0} \subset \bigcup_{i=1}^n Y_i(O_{M, \Omega})^{>0} \subset \bigcup_{i=1}^n Y_i^I = (X^I)^I,
\]
so \( X^I \subset (X^I)^I \).

Proposition 5.11 implies [Ull04, Prop. 2.8].

**Proposition 5.11** (Chevalley-Weil). If \( f : X \to Y \) is an étale cover over \( \mathbb{Q}^a \), then \( f(X^I) = Y^I \). In particular, \( X^{I_{\infty}} = X^I \) and the property that “integral Lang locus is full/empty” lifts and descends along étale covers.

**Proof.** By Lemma 5.8, \( f(X^I) \subset Y^I \). There is a number field \( F \), a finite set \( \Sigma \) of places of \( F \) containing all the archimedean ones and \( f' : \mathcal{X} \to \mathcal{Y} \) a finite étale \( O_{F, \Sigma} \)-morphism between models whose restriction to generic fiber recovers \( f \). From the proof in [SBW89, Section 4.2] we see that for any \((M, \Omega)/(F, \Sigma)\) there is \((M', \Omega')/(M, \Omega)\) such that \( Y(\mathcal{Y}, M', \Omega') \subset f(X(\mathcal{X}, M', \Omega')) \). Recall that zero dimensional scheme is discrete,
\[
\overline{Y(\mathcal{Y}, M, \Omega)}^{>0} \subset \overline{f(X(\mathcal{X}, M', \Omega'))^{>0}} \subset f(X^I)
\]
and we get \( Y^I \subset f(X^I) \).

**Theorem 5.12.** The integral Lang locus of a Shimura variety \( S \) is either empty or whole \( S \).

**Proof.** Now that the formation of the integral Lang locus \((\cdot)^I\) satisfies the four axioms by Lemmas 5.8 and 5.10, the result is a combination of Theorem 4.3 and Proposition 5.11.

**References**


